The Online Math Open Fall Contest
November 4 – 15, 2016

Online Math Open

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.

Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2016 Fall Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams cannot communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com`. (Include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Kevin is in first grade, so his teacher asks him to calculate $20 + 1 \cdot 6 + k$, where k is a real number revealed to Kevin. However, since Kevin is rude to his Aunt Sally, he instead calculates $(20+1)\cdot(6+k)$. Surprisingly, Kevin gets the correct answer! Assuming Kevin did his computations correctly, what was his answer?

2. Yang has a standard 6-sided die, a standard 8-sided die, and a standard 10-sided die. He tosses these three dice simultaneously. The probability that the three numbers that show up form the side lengths of a right triangle can be expressed as $\frac{m}{n}$, for relatively prime positive integers m and n. Find $100m+n$.

3. In a rectangle $ABCD$, let M and N be the midpoints of sides BC and CD, respectively, such that AM is perpendicular to MN. Given that the length of AN is 60, the area of rectangle $ABCD$ is $m\sqrt{n}$ for positive integers m and n such that n is not divisible by the square of any prime. Compute $100m + n$.

4. Let $G = 10^{10^{100}}$ (a.k.a. a googolplex). Then

$$\log_{\left(\log_{(\log_{10} G)} G\right)} G$$

can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Determine the sum of the digits of $m + n$.

5. Jay notices that there are n primes that form an arithmetic sequence with common difference 12. What is the maximum possible value for n?

6. For a positive integer n, define $n? = 1^n \cdot 2^{n-1} \cdot 3^{n-2} \cdots (n-1)^2 \cdot n^1$. Find the positive integer k for which $7?9? = 5?k?$.

7. The 2016 players in the Gensokyo Tennis Club are playing Up and Down the River. The players first randomly form 1008 pairs, and each pair is assigned to a tennis court (The courts are numbered from 1 to 1008). Every day, the two players on the same court play a match against each other to determine a winner and a loser. For $2 \le i \le 1008$, the winner on court i will move to court $i-1$ the next day (and the winner on court 1 does not move). Likewise, for $1 \le j \le 1007$, the loser on court j will move to court $j+1$ the next day (and the loser on court 1008 does not move). On Day 1, Reimu is playing on court 123 and Marisa is playing on court 876. Find the smallest positive integer value of n for which it is possible that Reimu and Marisa play one another on Day n.

8. For a positive integer n, define the nth triangular number T_n to be $\frac{n(n+1)}{2}$, and define the nth square number S_n to be n^2. Find the value of

$$\sqrt{S_{62} + T_{63}\sqrt{S_{61} + T_{62}\sqrt{\cdots\sqrt{S_2 + T_3\sqrt{S_1 + T_2}}}}}.$$

9. In quadrilateral $ABCD$, $AB = 7, BC = 24, CD = 15, DA = 20$, and $AC = 25$. Let segments AC and BD intersect at E. What is the length of EC?

10. Let $a_1 < a_2 < a_3 < a_4$ be positive integers such that the following conditions hold:

   - $\gcd(a_i, a_j) > 1$ holds for all integers $1 \le i < j \le 4$.
   - $\gcd(a_i, a_j, a_k) = 1$ holds for all integers $1 \le i < j < k \le 4$.

   Find the smallest possible value of a_4.

11. Let f be a random permutation on $\{1, 2, \ldots, 100\}$ satisfying $f(1) > f(4)$ and $f(9) > f(16)$. The probability that $f(1) > f(16) > f(25)$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $100m + n$.

   Note: In other words, f is a function such that $\{f(1), f(2), \ldots, f(100)\}$ is a permutation of $\{1, 2, \ldots, 100\}$.

12. For each positive integer $n \geq 2$, define $k(n)$ to be the largest integer m such that $(n!)^m$ divides 2016!. What is the minimum possible value of $n + k(n)$?

13. Let $A_1B_1C_1$ be a triangle with $A_1B_1 = 16, B_1C_1 = 14$, and $C_1A_1 = 10$. Given a positive integer i and a triangle $A_iB_iC_i$ with circumcenter O_i, define triangle $A_{i+1}B_{i+1}C_{i+1}$ in the following way:

    (a) A_{i+1} is on side B_iC_i such that $C_iA_{i+1} = 2B_iA_{i+1}$.

    (b) $B_{i+1} \neq C_i$ is the intersection of line A_iC_i with the circumcircle of $O_iA_{i+1}C_i$.

    (c) $C_{i+1} \neq B_i$ is the intersection of line A_iB_i with the circumcircle of $O_iA_{i+1}B_i$.

    Find
    $$\left(\sum_{i=1}^{\infty}[A_iB_iC_i]\right)^2.$$

    Note: $[K]$ denotes the area of K.

14. In Yang's number theory class, Michael K, Michael M, and Michael R take a series of tests. Afterwards, Yang makes the following observations about the test scores:

    - Michael K had an average test score of 90, Michael M had an average test score of 91, and Michael R had an average test score of 92.
    - Michael K took more tests than Michael M, who in turn took more tests than Michael R.
    - Michael M got a higher total test score than Michael R, who in turn got a higher total test score than Michael K. (The total test score is the sum of the test scores over all tests)

    What is the least number of tests that Michael K, Michael M, and Michael R could have taken combined?

15. Two bored millionaires, Bilion and Trilion, decide to play a game. They each have a sufficient supply of \$1, \$2, \$5, and \$10 bills. Starting with Bilion, they take turns putting one of the bills they have into a pile. The game ends when the bills in the pile total exactly \$1,000,000, and whoever makes the last move wins the \$1,000,000 in the pile (if the pile is worth more than \$1,000,000 after a move, then the person who made the last move loses instead, and the other person wins the amount of cash in the pile). Assuming optimal play, how many dollars will the winning player gain?

16. For her zeroth project at Magic School, Emilia needs to grow six perfectly-shaped apple trees. First she plants six tree saplings at the end of Day 0. On each day afterwards, Emilia attempts to use her magic to turn each sapling into a perfectly-shaped apple tree, and for each sapling she succeeds in turning it into a perfectly-shaped apple tree that day with a probability of $\frac{1}{2}$. (Once a sapling is turned into a perfectly-shaped apple tree, it will stay a perfectly-shaped apple tree.) The expected number of days it will take Emilia to obtain six perfectly-shaped apple trees is $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

17. Let n be a positive integer. S is a set of points such that the points in S are arranged in a regular 2016-simplex grid, with an edge of the simplex having n points in S. (For example, the 2-dimensional analog would have $\dfrac{n(n+1)}{2}$ points arranged in an equilateral triangle grid). Each point in S is labeled with a real number such that the following conditions hold:

    - Not all the points in S are labeled with 0.
    - If ℓ is a line that is parallel to an edge of the simplex and that passes through at least one point in S, then the labels of all the points in S that are on ℓ add to 0.
    - The labels of the points in S are symmetric along any such line ℓ.

Find the smallest positive integer n such that this is possible.

Note: A regular 2016-simplex has 2017 vertices in 2016-dimensional space such that the distances between every pair of vertices are equal.

18. Find the smallest positive integer k such that there exist positive integers $M, O > 1$ satisfying

$$(O \cdot M \cdot O)^k = (O \cdot M) \cdot \underbrace{(N \cdot O \cdot M) \cdot (N \cdot O \cdot M) \cdot \ldots \cdot (N \cdot O \cdot M)}_{2016\ (N \cdot O \cdot M)\text{s}},$$

where $N = O^M$.

Note: This is edited from the previous text, which did not clarify that NOM represented $N \cdot O \cdot M$, for example.

19. Let S be the set of all polynomials $Q(x, y, z)$ with coefficients in $\{0, 1\}$ such that there exists a homogeneous polynomial $P(x, y, z)$ of degree 2016 with integer coefficients and a polynomial $R(x, y, z)$ with integer coefficients so that

$$P(x, y, z)Q(x, y, z) = P(yz, zx, xy) + 2R(x, y, z)$$

and $P(1, 1, 1)$ is odd. Determine the size of S.

Note: A homogeneous polynomial of degree d consists solely of terms of degree d.

20. For a positive integer k, define the sequence $\{a_n\}_{n \geq 0}$ such that $a_0 = 1$ and for all positive integers n, a_n is the smallest positive integer greater than a_{n-1} for which $a_n \equiv ka_{n-1} \pmod{2017}$. What is the number of positive integers $1 \leq k \leq 2016$ for which $a_{2016} = 1 + \binom{2017}{2}$?

21. Mark the Martian and Bark the Bartian live on planet Blok, in the year 2019. Mark and Bark decide to play a game on a 10×10 grid of cells. First, Mark randomly generates a subset S of $\{1, 2, \ldots, 2019\}$ with $|S| = 100$. Then, Bark writes each of the 100 integers in a different cell of the 10×10 grid. Afterwards, Bark constructs a solid out of this grid in the following way: for each grid cell, if the number written on it is n, then she stacks n $1 \times 1 \times 1$ blocks on top of one other in that cell. Let B be the largest possible surface area of the resulting solid, including the bottom of the solid, over all possible ways Bark could have inserted the 100 integers into the grid of cells. Find the expected value of B over all possible sets S Mark could have generated.

22. Let ABC be a triangle with $AB = 3$ and $AC = 4$. It is given that there does not exist a point D, different from A and not lying on line BC, such that the Euler line of ABC coincides with the Euler line of DBC. The square of the product of all possible lengths of BC can be expressed in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. Find $100m + 10n + p$.

Note: For this problem, consider every line passing through the center of an equilateral triangle to be an Euler line of the equilateral triangle. Hence, if D is chosen such that DBC is an equilateral triangle and the Euler line of ABC passes through the center of DBC, then consider the Euler line of ABC to coincide with "the" Euler line of DBC.

23. Let $\mathbb{N}$ denote the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that the following conditions hold:

- For any $n \in \mathbb{N}$, we have $f(n) | n^{2016}$.
- For any $a, b, c \in \mathbb{N}$ satisfying $a^2 + b^2 = c^2$, we have $f(a)f(b) = f(c)$.

Over all possible functions f, determine the number of distinct values that can be achieved by $f(2014) + f(2) - f(2016)$.

24. Let $P(x, y)$ be a polynomial such that $\deg_x(P), \deg_y(P) \leq 2020$ and

$$P(i, j) = \binom{i+j}{i}$$

over all 2021^2 ordered pairs (i, j) with $0 \leq i, j \leq 2020$. Find the remainder when $P(4040, 4040)$ is divided by 2017.

Note: $\deg_x(P)$ is the highest exponent of x in a nonzero term of $P(x, y)$. $\deg_y(P)$ is defined similarly.

25. Let $X_1X_2X_3$ be a triangle with $X_1X_2 = 4, X_2X_3 = 5, X_3X_1 = 7$, and centroid G. For all integers $n \geq 3$, define the set S_n to be the set of n^2 ordered pairs (i, j) such that $1 \leq i \leq n$ and $1 \leq j \leq n$. Then, for each integer $n \geq 3$, when given the points $X_1, X_2, \ldots, X_n$, randomly choose an element $(i, j) \in S_n$ and define X_{n+1} to be the midpoint of X_i and X_j. The value of

$$\sum_{i=0}^{\infty} \left(\mathbb{E}\left[X_{i+4}G^2\right] \left(\frac{3}{4}\right)^i \right)$$

can be expressed in the form $p + q \ln 2 + r \ln 3$ for rational numbers p, q, r. Let $|p| + |q| + |r| = \dfrac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Note: $\mathbb{E}(x)$ denotes the expected value of x.

26. Let ABC be a triangle with $BC = 9$, $CA = 8$, and $AB = 10$. Let the incenter and incircle of ABC be I and γ, respectively, and let N be the midpoint of major arc BC of the cirucmcircle of ABC. Line NI meets the circumcircle of ABC a second time at P. Let the line through I perpendicular to AI meet segments AB, AC, and AP at C_1, B_1, and Q, respectively. Let B_2 lie on segment CQ such that line B_1B_2 is tangent to γ, and let C_2 lie on segment BQ such that line C_1C_2 tangent to γ. The length of B_2C_2 can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Determine $100m + n$.

27. Compute the number of monic polynomials $q(x)$ with integer coefficients of degree 12 such that there exists an integer polynomial $p(x)$ satisfying $q(x)p(x) = q(x^2)$.

28. Let ABC be a triangle with $AB = 34, BC = 25$, and $CA = 39$. Let O, H, and ω be the circumcenter, orthocenter, and circumcircle of $\triangle ABC$, respectively. Let line AH meet ω a second time at A_1 and let the reflection of H over the perpendicular bisector of BC be H_1. Suppose the line through O perpendicular to A_1O meets ω at two points Q and R with Q on minor arc AC and R on minor arc AB. Denote $\mathcal{H}$ as the hyperbola passing through A, B, C, H, H_1, and suppose HO meets $\mathcal{H}$ again at P. Let X, Y be points with $XH \parallel AR \parallel YP, XP \parallel AQ \parallel YH$. Let P_1, P_2 be points on the tangent to $\mathcal{H}$ at P with $XP_1 \parallel OH \parallel YP_2$ and let P_3, P_4 be points on the tangent to $\mathcal{H}$ at H with $XP_3 \parallel OH \parallel YP_4$. If P_1P_4 and P_2P_3 meet at N, and ON may be written in the form $\frac{a}{b}$ where a, b are positive coprime integers, find $100a + b$.

29. Let n be a positive integer. Yang the Saltant Sanguivorous Shearling is on the side of a very steep mountain that is embedded in the coordinate plane. There is a blood river along the line $y = x$, which Yang may reach but is not permitted to go above (i.e. Yang is allowed to be located at $(2016, 2015)$ and $(2016, 2016)$, but not at $(2016, 2017)$). Yang is currently located at $(0, 0)$ and wishes to reach $(n, 0)$. Yang is permitted only to make the following moves:

- Yang may *spring*, which consists of going from a point (x, y) to the point $(x, y + 1)$.
- Yang may *stroll*, which consists of going from a point (x, y) to the point $(x + 1, y)$.
- Yang may *sink*, which consists of going from a point (x, y) to the point $(x, y - 1)$.

In addition, whenever Yang does a *sink*, he breaks his tiny little legs and may no longer do a *spring* at any time afterwards. Yang also expends a lot of energy doing a *spring* and gets bloodthirsty, so he must visit the blood river at least once afterwards to quench his bloodthirst. (So Yang may still *spring* while bloodthirsty, but he may not finish his journey while bloodthirsty.) Let there be a_n different ways for which Yang can reach $(n, 0)$, given that Yang is permitted to pass by $(n, 0)$ in the middle of his journey. Find the 2016th smallest positive integer n for which $a_n \equiv 1 \pmod 5$.

30. Let $P_1(x), P_2(x), \ldots, P_n(x)$ be monic, non-constant polynomials with integer coefficients and let $Q(x)$ be a polynomial with integer coefficients such that

$$x^{2^{2016}} + x + 1 = P_1(x)P_2(x)\ldots P_n(x) + 2Q(x).$$

Suppose that the maximum possible value of $2016n$ can be written in the form $2^{b_1} + 2^{b_2} + \cdots + 2^{b_k}$ for nonnegative integers $b_1 < b_2 < \cdots < b_k$. Find the value of $b_1 + b_2 + \cdots + b_k$.