Some combinatorics

September 3, 2020

Problem 1. *Does there exist a hexagon, which can be partitioned into four equal triangles by a straight line?*

Problem 2. *One has* 50 *stones, all of which have different integral weights, not exceeding* 100. *The sum of the weights is even. One claims that it is impossible to put some of the stones on the one side of a scale and the other ones on the other side, so that the scale is balanced. Can his words be true?*

Problem 3. *Prove that each natural* N *there exist integers* a, b, c, d *so that* $a + b = c + d$, *but* $ab = Ncd$.

Problem 4. *An* $m \times n$ *board is covered by* 1×2 *dominoes. In each domino, we choose one of the diagonals, so that no two chosen diagonals have common vertex. Prove that exactly* 2 *of the corners of the* $m \times n$ *are vertices of a domino.*

Problem 5. *Two insects move on the* 7×7 *boards. They only use the borders of the cell (let's call them paths). They started from some cell, visit all the* 64 *vertices and return to the initial cell. Find the minimal number of coinciding paths in their routes.*

Problem 6. *In each cell of an* $n \times n$ *board is written a single number. It is known, that the sum of maximal two numbers of each row is equal to* a *and the sum of maximal two numbers of each column is equal to* b. *Prove that* $a = b$.

Problem 7. *Denote by $P(n)$ the product of the positive divisors of n. Define the sequence*

$$a_1(n) = n, a_k(n) = P(a_{k-1}(n)), k \geq 2$$

Is it true that for all $S \subseteq \{1, 2, ..., 2020\}$, there exists n, so that $a_k(n)$ is a perfect square if and only if $k \in S$.

Problem 8. *Find all pairs of positive integers (a, b) so that a^{619} divides $b^{1000} + 1$ and b^{619} divides $a^{1000} + 1$.*

Problem 9. *Let a, b be integers so that $\frac{a^2+b^2}{ab-1} = k$ is an integer. Prove that $k = 5$.*