Number Theory

Winter Camp

January 6, 2018

1 Basic Techniques

1. Take mods!
2. Factorization / Algebraic Manipulation
3. Infinite descent
4. Chinese Remainder Theorem

2 Theme: Algebraic Number Theory

1. Orders mod p
2. Primitive roots and roots of unity in $\mathbb{F}_p$.
3. Cyclotomic Polynomials and primes
4. Sum of Squares Problem / Quadratic Reciprocity

3 Warning

This list is by no means complete. A lot of modern olympiad number theory problems require a lot of ad hoc manipulation. Note that this lecture doesn't really touch on polynomials or functional equations.

4 Problems

I tried to sort them in rough difficulty order.

1. How many possibilities are there for the last four digits of 2^n?

2. Let S be a of positive integers such that

   - $2 \in S$
   - If $n \in S$, then $(n+5)^2 \in S$.
   - If $n^2 \in S$, then $n \in S$.

   and S is minimal. What is S?

3. Find all integer solutions to $a^3 + 2b^3 + 4c^3 = 6abc$.

4. Do there exist integers $x_1, \ldots, x_{2018}$ such that

$$\begin{aligned} x_1 + \ldots + x_{2018} &= 123 \\ x_1^7 + \ldots + x_{2018}^7 &= 321 \end{aligned}$$

5. Determine all integer solutions to $2^a + 3^b + 5^c = n!$.

6. Let S be the set of integers that can be expressed as $a^2 + ab + b^2$ for integers a, b. Show that if $x, y \in S$ then $xy \in S$.

7. Find all integers n such that $4^n + 6^n + 9^n$ is a perfect square.

8. Show that the system of equations

$$\begin{aligned} x^6 + x^3 + x^3y + y &= 147^{157} \\ x^3 + x^3y + y^2 + y + z^9 &= 157^{147} \end{aligned}$$

   has no integer solutions.

9. Let $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-1} - a_{n-2}$. Find an odd prime factor of a_{2015}.

10. A beautiful number is an integer of the form a^n where $a \in \{3, 4, 5, 6\}$ and n is a positive intger. Prove that every integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

11. A set of postive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

12. Suppose $p > 10^9$ is a prime such that $4p+1$ is also prime. Show that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

13. (Lucas) Suppose a, b, c, d are nonnegative integers such that $b, d < p$. Show that

$$\binom{ap+b}{cp+d} \equiv \binom{a}{c}\binom{b}{d} \pmod{p}$$

14. (Wolstenhomlme) Show that for all positive integers a, b,

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}$$

15. Let p be an odd prime number. For every integer a, define the number $S_a = \sum_{j=1}^{p-1} \frac{a^j}{j}$. Let $m, n \in \mathbb{Z}$, such that $S_3 + S_4 - 3S_2 = \frac{m}{n}$. Prove that $p \mid m$.

16. Find all positive integers (a, b, c) such that $ab - c, bc - a, ca - b$ are all powers of 2.

17. Suppose $x, y > 1$ are integers such that $(x-1)(y-1), xy, (x+1)(y+1)$ are all perfect squares. Show that $x = y$.

18. Show that there are infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n + \sqrt{2n}$.

19. Suppose $a > b > 1$ are positive integers. Show that

$$\left\lfloor \frac{(a-b)^2 - 1}{ab} \right\rfloor = \left\lfloor \frac{(a-b)^2 - 1}{ab - 1} \right\rfloor$$

20. Show that
$$\frac{y^7-1}{y-1} = n^5 - 1$$
Has no solution in the integers.

21. Let p be a prime number and x be a positive integer. Let $N(p,x)$ be the number of ordered pairs (a,b) such that $p \mid a^2+b^2-x$. Give a closed form expression for $N(p,x)$ in terms of p and x. (Hint: Show that as long as $p \nmid x$, the value of $N(p,x)$ does not depend on x.)

22. Find all integer solutions to $a^3+2b^3+4c^3 = 6abc+1$.

23. Determine whether or not there exists a positive integer k such that $p = 6k+1$ is prime and
$$\binom{3k}{k} \equiv 1 \pmod{p}$$

24. Modulus doesn't always work!
    (a) Show that $n^3 - n - 3 = x^2$ has no integer solutions
    (b) Show that for every positive integer N, there are integers a and b such that $a^3 - a - 3 \equiv b^2 \pmod{N}$.