Day 4

Problem 10:

There are 2019 plates placed around a round table and on each of them there is one coin. Alice and Bob are playing a game that proceeds in rounds indefinitely as follows. In each round, Alice first chooses a plate on which there is at least one coin. Then Bob moves one coin from this plate to one of the two adjacent plates, chosen by him. Determine whether it is possible for Bob to select his moves so that, no matter how Alice selects her moves, there are never more than two coins on any plate.

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**Answer.** Yes, it is possible.

**Solution.** We provide a suitable strategy for Bob. Given a configuration of coins on the plates, let a *block* be any inclusion-wise maximal contiguous interval consisting of non-empty plates. The idea of Bob's strategy is to maintain the following invariant throughout the game: in every block, all plates except at most one contain exactly one coin, while the remaining one contains two coins. Since the total number of coins is always equal to the total number of plates, this is equivalent to the following condition: either every plate contains exactly one coin (as in the initial configuration), or in every block there is exactly one plate with two coins and all the other plates of the block contain one coin each, and moreover the blocks are delimited by single plates containing zero coins.

It now suffices to show that in any configuration $C$ satisfying the invariant, regardless of which plate Alice picks, Bob can always select his move so that the invariant is maintained after the move. We consider two cases: either Alice picks a plate with two coins, or with one coin.

Suppose first that Alice picks a plate with two coins. If any of the adjacent plates contains one coin, then Bob moves a coin to this plate and the invariant is maintained — the set of blocks remains unchanged and only within one block the plate with two coins has moved. If both of the adjacent plates contain zero coins, then Bob moves a coin to any of them. Thus, one single-plate block disappears and some other block gets extended with two plates with one coin each; hence, the invariant is maintained.

Suppose now that Alice picks a plate with one coin. If the configuration is as the initial one — every plate contains one coin — then any move of Bob maintains the invariant. Otherwise, within the block $B$ containing the plate $P$ chosen by Alice there is another plate $P'$ containing two coins, and $B$ does not contain all the plates. Without loss of generality, suppose that in order to get from $P'$ to $P$ within $B$ one needs to go in the clockwise direction. Then the move of Bob is to move the coin from $P$ also in the clockwise direction. Then either $P$ is the clockwise endpoint of $B$, and we just move one plate with one coin from $B$ to the next block in the clockwise direction, or $P$ is not the clockwise endpoint of $B$, and the move results in

This is the general solution for cases a) and b).
For case when there are 2022 plates there is an easy solution. Pair all plates (1,2), (3,4), (2021,2022).
Strategy is to move coins inside the pair, so each pair always has 2 coins, so no plate has more than 2 coins.

Problem 11. Find all positive integers $n$ with the following property: the $k$ positive divisors of $n$ have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$ the number $d_1 + d_2 + \ldots + d_i$ is a perfect square.

**Answer:** $n = 1$ and $n = 3$.

**Solution.** For $i = 1, 2, \ldots, k$ let $d_1 + \ldots + d_i = s_i^2$, and define $s_0 = 0$ as well. Obviously $0 = s_0 < s_1 < s_2 < \ldots < s_k$, so

$$s_i \geqslant i \quad \text{and} \quad d_i = s_i^2 - s_{i-1}^2 = (s_i + s_{i-1})(s_i - s_{i-1}) \geqslant s_i + s_{i-1} \geqslant 2i - 1. \tag{1}$$

The number 1 is one of the divisors $d_1, \ldots, d_k$ but, due to $d_i \geqslant 2i - 1$, the only possibility is $d_1 = 1$.

Now consider $d_2$ and $s_2 \geqslant 2$. By definition, $d_2 = s_2^2 - 1 = (s_2 - 1)(s_2 + 1)$, so the numbers $s_2 - 1$ and $s_2 + 1$ are divisors of $n$. In particular, there is some index $j$ such that $d_j = s_2 + 1$.

Notice that

$$s_2 + s_1 = s_2 + 1 = d_j \geqslant s_j + s_{j-1}; \tag{2}$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, the index $j$ cannot be greater than 2. Hence, the divisors $s_2 - 1$ and $s_2 + 1$ are listed among $d_1$ and $d_2$. That means $s_2 - 1 = d_1 = 1$ and $s_2 + 1 = d_2$; therefore $s_2 = 2$ and $d_2 = 3$.

We can repeat the above process in general.

*Claim.* $d_i = 2i - 1$ and $s_i = i$ for $i = 1, 2, \ldots, k$.

*Proof.* Apply induction on $i$. The Claim has been proved for $i = 1, 2$. Suppose that we have already proved $d = 1$, $d_2 = 3$, $\ldots$, $d_i = 2i - 1$, and consider the next divisor $d_{i+1}$:

$$d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2 = (s_{i+1} - i)(s_{i+1} + i).$$

The number $s_{i+1} + i$ is a divisor of $n$, so there is some index $j$ such that $d_j = s_{i+1} + i$.

Similarly to (2), by (1) we have

$$s_{i+1} + s_i = s_{i+1} + i = d_j \geqslant s_j + s_{j-1}; \tag{3}$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, (3) forces $j \leqslant i + 1$. On the other hand, $d_j = s_{i+1} + i > 2i > d_i > d_{i-1} > \ldots > d_1$, so $j \leqslant i$ is not possible. The only possibility is $j = i + 1$.

Hence,

$$s_{i+1} + i = d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2;$$
$$s_{i+1}^2 - s_{i+1} = i(i + 1).$$

By solving this equation we get $s_{i+1} = i + 1$ and $d_{i+1} = 2i + 1$, that finishes the proof. $\square$

Now we know that the positive divisors of the number $n$ are $1, 3, 5, \ldots, n-2, n$. The greatest divisor is $d_k = 2k - 1 = n$ itself, so $n$ must be odd. The second greatest divisor is $d_{k-1} = n - 2$; then $n - 2$ divides $n = (n - 2) + 2$, so $n - 2$ divides 2. Therefore, $n$ must be 1 or 3.

The numbers $n = 1$ and $n = 3$ obviously satisfy the requirements: for $n = 1$ we have $k = 1$ and $d_1 = 1^2$; for $n = 3$ we have $k = 2$, $d_1 = 1^2$ and $d_1 + d_2 = 1 + 3 = 2^2$.

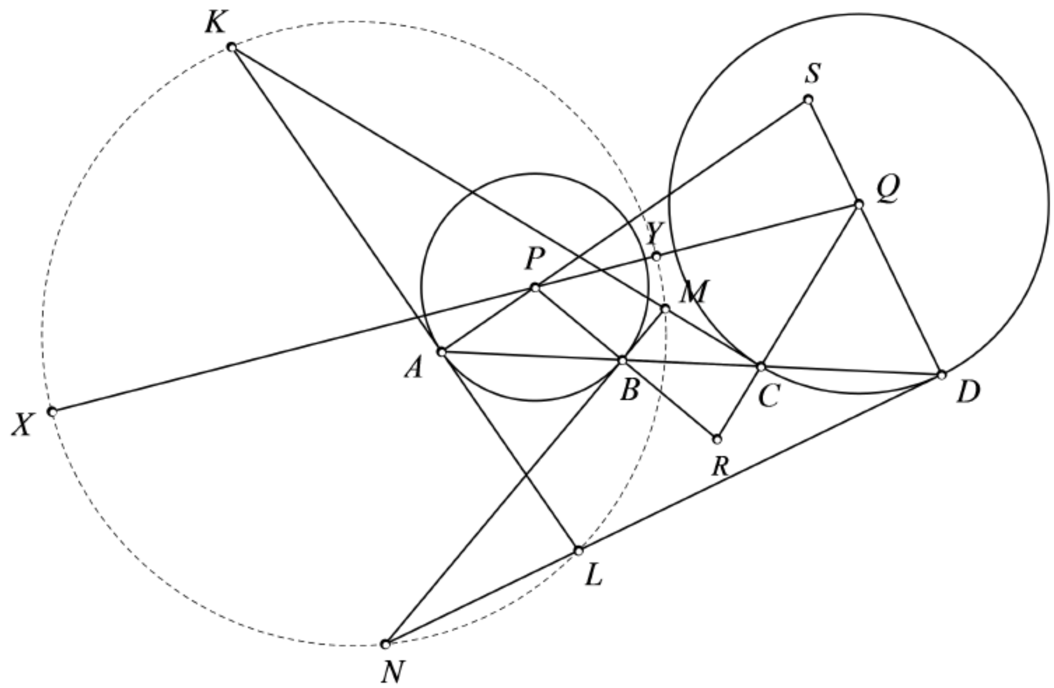
Problem 12. Let $A, B, C, D$ be points on the line $d$ in that order and $AB = CD$. Denote $(P)$ as some circle that passes through $A, B$ with its tangent lines at $A, B$ are $a, b$. Denote $(Q)$ as some circle that passes through $C, D$ with its tangent lines at $C, D$ are $c, d$. Suppose that $a$ cuts $c, d$ at $K, L$ respectively; and $b$ cuts $c, d$ at $M, N$ respectively. Prove that four points $K, L, M, N$ belong to a same circle $(\omega)$ and the common external tangent lines of circles $(P), (Q)$ meet on $(\omega)$.

*Solution.* Consider the points that arranged as following figure, the other cases will be proved similarly.

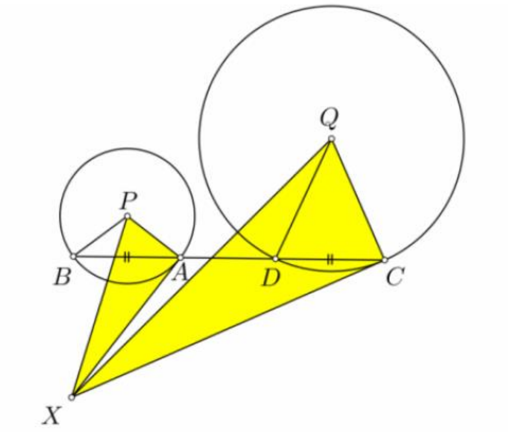
Denote $R, S$ as intersection of the pairs of lines $PB, QC$ and $PA, QD$. Note that $BMCR$ and $ALDS$ are cyclic quadrilateral. Thus

$$\begin{aligned}\angle KMN &= \angle BRC = 180^\circ - (\angle BCR + \angle CBR) \\ &= 180^\circ - (\angle PBA + \angle QDC) \\ &= 180^\circ - (\angle PAD + \angle QDA) = \angle ASD = \angle KLN.\end{aligned}$$

Hence, $K, L, M, N$ belong to the same circle.



Now, consider the following claim: Let be given two circles $(P, R), (Q, R')$ and some line cuts them at $B, A, D, C$ such that $AB = CD$ (see the figure). The tangent lines at $A, C$ respectively of $(P), (Q)$ meet at $X$ then $\frac{XP}{XQ} = \frac{R}{R'}$.




Indeed, by applying the sine law for triangle $XAC$, we get

$$\frac{XA}{XC} = \frac{\sin XCA}{\sin XAC} = \frac{\sin \frac{CQD}{2}}{\sin \frac{APB}{2}} = \frac{CD}{AB} \cdot \frac{R}{R'} = \frac{R}{R'} = \frac{PA}{QC}.$$

Thus two triangles $XPA$ and $XQC$ are similar, which implies that $\frac{XP}{XQ} = \frac{R}{R'}$. The claim is proved.

Back to the problem, denote $X, Y$ as the external and the internal homothety centers of $(P), (Q)$ then $\frac{XP}{XQ} = \frac{YP}{YQ} = k$ with $k$ is the ratio of radius of $(P), (Q)$. It is easy to check that these radiuses are different, otherwise the the tangent lines of $(P), (Q)$

will be parallel and points $K, L, M, N$ will not exist, thus $k \neq 1$. In the other hand, by applying the above claim, we get

$$\frac{MP}{MQ} = \frac{NP}{NQ} = \frac{KP}{KQ} = \frac{LP}{LQ} = k.$$

Hence, six points $X, Y, M, N, K, L$ are all belong to the Apollonius circle with ratio $k$ constructing on the segment $PQ$. Thus, the point $X$, which also is the intersection of two common external tangent lines of $(P), (Q)$, is on $(\omega)$. $\square$