The Online Math Open Fall Contest
October 26 – November 6, 2018

Online Math Open

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student may belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.

Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2018 Fall Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs, are prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams may not communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com` (include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Leonhard has five cards. Each card has a nonnegative integer written on it, and any two cards show relatively prime numbers. Compute the smallest possible value of the sum of the numbers on Leonhard's cards.

   Note: Two integers are relatively prime if no positive integer other than 1 divides both numbers.

2. Let $(p_1, p_2, \dots) = (2, 3, \dots)$ be the list of all prime numbers, and $(c_1, c_2, \dots) = (4, 6, \dots)$ be the list of all composite numbers, both in increasing order. Compute the sum of all positive integers n such that $|p_n - c_n| < 3$.

3. Katie has a list of real numbers such that the sum of the numbers on her list is equal to the sum of the squares of the numbers on her list. Compute the largest possible value of the arithmetic mean of her numbers.

4. Compute the largest integer that can be expressed in the form $3^{x(3-x)}$ for some real number x.

5. In triangle ABC, $AB = 8, AC = 9$, and $BC = 10$. Let M be the midpoint of BC. Circle ω_1 with area A_1 passes through A, B, and C. Circle ω_2 with area A_2 passes through A, B, and M. Then $\frac{A_1}{A_2} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

6. Patchouli is taking an exam with $k > 1$ parts, numbered Part $1, 2, \dots, k$. It is known that for $i = 1, 2, \dots, k$, Part i contains i multiple choice questions, each of which has $(i + 1)$ answer choices. It is known that if she guesses randomly on every single question, the probability that she gets exactly one question correct is equal to 2018 times the probability that she gets no questions correct. Compute the number of questions that are on the exam.

7. Compute the number of ways to erase 24 letters from the string "OMOMO$\cdots$OMO" (with length 27), such that the three remaining letters are O, M and O in that order. Note that the order in which they are erased does not matter.

8. Let ABC be the triangle with vertices located at the center of masses of Vincent Huang's house, Tristan Shin's house, and Edward Wan's house; here, assume the three are not collinear. Let $N = 2017$, and define the A-*ntipodes* to be the points $A_1, \dots, A_N$ to be the points on segment BC such that $BA_1 = A_1A_2 = \cdots = A_{N-1}A_N = A_NC$, and similarly define the B, C-ntipodes. A line ℓ_A through A is called a *qevian* if it passes through an A-ntipode, and similarly we define qevians through B and C. Compute the number of ordered triples (ℓ_A, ℓ_B, ℓ_C) of concurrent qevians through A, B, C, respectively.

9. Ann and Drew have purchased a mysterious slot machine; each time it is spun, it chooses a random positive integer such that k is chosen with probability 2^{-k} for every positive integer k, and then it outputs k tokens. Let N be a fixed integer. Ann and Drew alternate turns spinning the machine, with Ann going first. Ann wins if she receives at least N total tokens from the slot machine before Drew receives at least $M = 2^{2018}$ total tokens, and Drew wins if he receives M tokens before Ann receives N tokens. If each person has the same probability of winning, compute the remainder when N is divided by 2018.

10. Compute the largest prime factor of $357! + 358! + 359! + 360!$.

11. Let an ordered pair of positive integers (m, n) be called *regimented* if for all nonnegative integers k, the numbers m^k and n^k have the same number of positive integer divisors. Let N be the smallest positive integer such that $\left(2016^{2016}, N\right)$ is regimented. Compute the largest positive integer v such that 2^v divides the difference $2016^{2016} - N$.

12. Three non-collinear lattice points A, B, C lie on the plane $1 + 3x + 5y + 7z = 0$. The minimal possible area of triangle ABC can be expressed as $\frac{\sqrt{m}}{n}$ where m, n are positive integers such that there does not exists a prime p dividing n with p^2 dividing m. Compute $100m + n$.

13. Compute the largest possible number of distinct real solutions for x to the equation
$$x^6 + ax^5 + 60x^4 - 159x^3 + 240x^2 + bx + c = 0,$$
where a, b, and c are real numbers.

14. In triangle ABC, $AB = 13, BC = 14, CA = 15$. Let Ω and ω be the circumcircle and incircle of ABC respectively. Among all circles that are tangent to both Ω and ω, call those that contain ω *inclusive* and those that do not contain ω *exclusive*. Let $\mathcal{I}$ and $\mathcal{E}$ denote the set of centers of inclusive circles and exclusive circles respectively, and let I and E be the area of the regions enclosed by $\mathcal{I}$ and $\mathcal{E}$ respectively. The ratio $\frac{I}{E}$ can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Compute $100m + n$.

15. Iris does not know what to do with her 1-kilogram pie, so she decides to share it with her friend Rosabel. Starting with Iris, they take turns to give exactly half of total amount of pie (by mass) they possess to the other person. Since both of them prefer to have as few number of pieces of pie as possible, they use the following strategy: During each person's turn, she orders the pieces of pie that she has in a line from left to right in increasing order by mass, and starts giving the pieces of pie to the other person beginning from the left. If she encounters a piece that exceeds the remaining mass to give, she cuts it up into two pieces with her sword and gives the appropriately sized piece to the other person.

    When the pie has been cut into a total of 2017 pieces, the largest piece that Iris has is $\frac{m}{n}$ kilograms, and the largest piece that Rosabel has is $\frac{p}{q}$ kilograms, where m, n, p, q are positive integers satisfying $\gcd(m, n) = \gcd(p, q) = 1$. Compute the remainder when $m + n + p + q$ is divided by 2017.

16. Jay has a 24×24 grid of lights, all of which are initially off. Each of the 48 rows and columns has a switch that toggles all the lights in that row and column, respectively, i.e. it switches lights that are on to off and lights that are off to on. Jay toggles each of the 48 rows and columns exactly once, such that after each toggle he waits for one minute before the next toggle. Each light uses no energy while off and 1 kiloJoule of energy per minute while on. To express his creativity, Jay chooses to toggle the rows and columns in a random order. Compute the expected value of the total amount of energy in kiloJoules which has been expended by all the lights after all 48 toggles.

17. A hyperbola in the coordinate plane passing through the points $(2, 5)$, $(7, 3)$, $(1, 1)$, and $(10, 10)$ has an asymptote of slope $\frac{20}{17}$. The slope of its other asymptote can be expressed in the form $-\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m + n$.

18. On Lineland there are 2018 bus stations numbered 1 through 2018 from left to right. A self-driving bus that can carry at most N passengers starts from station 1 and drives all the way to station 2018, while making a stop at each bus station. Each passenger that gets on the bus at station i will get off at station j for some $j > i$ (the value of j may vary over different passengers). Call any group of four distinct stations i_1, i_2, j_1, j_2 with $i_u < j_v$ for all $u, v \in \{1, 2\}$ a *good* group. Suppose that in any good group i_1, i_2, j_1, j_2, there is a passenger who boards at station i_1 and de-boards at station j_1, or there is a passenger who boards at station i_2 and de-boards at station j_2, or both scenarios occur. Compute the minimum possible value of N.

19. Players $1, 2, \ldots, 10$ are playing a game on Christmas. Santa visits each player's house according to a set of rules:

    - Santa first visits player 1. After visiting player i, Santa visits player $i + 1$, where player 11 is the same as player 1.
    - Every time Santa visits someone, he gives them either a present or a piece of coal (but not both).
    - The absolute difference between the number of presents and pieces of coal that Santa has given out is at most 3 at every point in time.
    - If Santa has a choice between giving out a present and a piece of coal, he chooses with equal probability.

Let p be the probability that player 1 gets a present before player 2 does. If $p = \frac{m}{n}$ for relatively prime positive integers m and n, then compute $100m + n$.

20. For positive integers k, n with $k \le n$, we say that a k-tuple $(a_1, a_2, \ldots, a_k)$ of positive integers is *tasty* if

    - there exists a k-element subset S of $[n]$ and a bijection $f : [k] \to S$ with $a_x \le f(x)$ for each $x \in [k]$,
    - $a_x = a_y$ for some distinct $x, y \in [k]$, and
    - $a_i \le a_j$ for any $i < j$.

    For some positive integer n, there are more than 2018 tasty tuples as k ranges through $2, 3, \ldots, n$. Compute the least possible number of tasty tuples there can be.

    Note: For a positive integer m, $[m]$ is taken to denote the set $\{1, 2, \ldots, m\}$.

21. Suppose that a sequence $a_0, a_1, \ldots$ of real numbers is defined by $a_0 = 1$ and

$$a_n = \begin{cases} a_{n-1}a_0 + a_{n-3}a_2 + \cdots + a_0 a_{n-1} & \text{if } n \text{ odd} \\ a_{n-1}a_1 + a_{n-3}a_3 + \cdots + a_1 a_{n-1} & \text{if } n \text{ even} \end{cases}$$

    for $n \ge 1$. There is a positive real number r such that

$$a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots = \frac{5}{4}.$$

    If r can be written in the form $\frac{a\sqrt{b} - c}{d}$ for positive integers a, b, c, d such that b is not divisible by the square of any prime and $\gcd(a, c, d) = 1$, then compute $a + b + c + d$.

22. Let ABC be a triangle with $AB = 2$ and $AC = 3$. Let H be the orthocenter, and let M be the midpoint of BC. Let the line through H perpendicular to line AM intersect line AB at X and line AC at Y. Suppose that lines BY and CX are parallel. Then $[ABC]^2 = \frac{a + b\sqrt{c}}{d}$ for positive integers a, b, c and d, where $\gcd(a, b, d) = 1$ and c is not divisible by the square of any prime. Compute $1000a + 100b + 10c + d$.

23. Consider all ordered pairs (a, b) of positive integers such that $\frac{a^2 + b^2 + 2}{ab}$ is an integer and $a \le b$. We label all such pairs in increasing order by their distance from the origin. (It is guaranteed that no ties exist.) Thus $P_1 = (1, 1), P_2 = (1, 3)$, and so on. If $P_{2020} = (x, y)$, then compute the remainder when $x + y$ is divided by 2017.

24. Let $p = 101$ and let S be the set of p-tuples $(a_1, a_2, \ldots, a_p) \in \mathbb{Z}^p$ of integers. Let N denote the number of functions $f : S \to \{0, 1, \ldots, p-1\}$ such that

    - $f(a + b) + f(a - b) \equiv 2\big(f(a) + f(b)\big) \pmod{p}$ for all $a, b \in S$, and
    - $f(a) = f(b)$ whenever all components of $a - b$ are divisible by p.

    Compute the number of positive integer divisors of N. (Here addition and subtraction in $\mathbb{Z}^p$ are done component-wise.)

25. Given two positive integers x, y, we define $z = x \oplus y$ to be the bitwise XOR of x and y; that is, z has a 1 in its binary representation at exactly the place values where x, y have differing binary representations. It is known that $\oplus$ is both associative and commutative. For example, $20 \oplus 18 = 10100_2 \oplus 10010_2 = 110_2 = 6$. Given a set $S = \{a_1, a_2, \ldots, a_n\}$ of positive integers, we let $f(S) = a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n$. We also let $g(S)$ be the number of divisors of $f(S)$ which are at most 2018 but greater than or equal to the largest element in S (if S is empty then let $g(S) = 2018$). Compute the number of 1s in the binary representation of $\displaystyle\sum_{S \subseteq \{1, 2, \ldots, 2018\}} g(S)$.

26. Let $p = 2027$ be the smallest prime greater than 2018, and let $P(X) = X^{2031} + X^{2030} + X^{2029} - X^5 - 10X^4 - 10X^3 + 2018X^2$. Let $\mathrm{GF}(p)$ be the integers modulo p, and let $\mathrm{GF}(p)(X)$ be the set of rational functions with coefficients in $\mathrm{GF}(p)$ (so that all coefficients are taken modulo p). That is, $\mathrm{GF}(p)(X)$ is the set of fractions $\frac{P(X)}{Q(X)}$ of polynomials with coefficients in $\mathrm{GF}(p)$, where $Q(X)$ is not the zero polynomial. Let $D\colon \mathrm{GF}(p)(X) \to \mathrm{GF}(p)(X)$ be a function satisfying

$$D\left(\frac{f}{g}\right) = \frac{D(f)\cdot g - f\cdot D(g)}{g^2}$$

for any $f, g \in \mathrm{GF}(p)(X)$ with $g \neq 0$, and such that for any nonconstant polynomial f, $D(f)$ is a polynomial with degree less than that of f. If the number of possible values of $D(P(X))$ can be written as a^b, where a, b are positive integers with a minimized, compute ab.

27. Let $p = 2^{16} + 1$ be a prime. Let N be the number of ordered tuples (A, B, C, D, E, F) of integers between 0 and $p-1$, inclusive, such that there exist integers x, y, z not all divisible by p with p dividing all three of $Ax + Ez + Fy, By + Dz + Fx, Cz + Dy + Ex$. Compute the remainder when N is divided by 10^6.

28. Let ω be a circle centered at O with radius $R = 2018$. For any $0 < r < 1009$, let γ be a circle of radius r centered at a point I satisfying $OI = \sqrt{R(R-2r)}$. Choose any $A, B, C \in \omega$ with AC, AB tangent to γ at E, F, respectively. Suppose a circle of radius r_A is tangent to AB, AC, and internally tangent to ω at a point D with $r_A = 5r$. Let line EF meet ω at P_1, Q_1. Suppose P_2, P_3, Q_2, Q_3 lie on ω such that $P_1P_2, P_1P_3, Q_1Q_2, Q_1Q_3$ are tangent to γ. Let P_2P_3, Q_2Q_3 meet at K, and suppose KI meets AD at a point X. Then as r varies from 0 to 1009, the maximum possible value of OX can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that b is not divisible by the square of any prime and $\gcd(a, c) = 1$. Compute $10a + b + c$.

29. For integers $0 \le m, n \le 2^{2017} - 1$, let $\alpha(m, n)$ be the number of nonnegative integers k for which $\lfloor m/2^k \rfloor$ and $\lfloor n/2^k \rfloor$ are both odd integers. Consider a $2^{2017} \times 2^{2017}$ matrix M whose (i, j)th entry (for $1 \le i, j \le 2^{2017}$) is

$$(-1)^{\alpha(i-1, j-1)}.$$

For $1 \le i, j \le 2^{2017}$, let $M_{i,j}$ be the matrix with the same entries as M except for the (i, j)th entry, denoted by $a_{i,j}$, and such that $\det M_{i,j} = 0$. Suppose that A is the $2^{2017} \times 2^{2017}$ matrix whose (i, j)th entry is $a_{i,j}$ for all $1 \le i, j \le 2^{2017}$. Compute the remainder when $\det A$ is divided by 2017.

30. Let ABC be an acute triangle with $\cos B = \frac{1}{3}, \cos C = \frac{1}{4}$, and circumradius 72. Let ABC have circumcenter O, symmedian point K, and nine-point center N. Consider all non-degenerate hyperbolas $\mathcal{H}$ with perpendicular asymptotes passing through A, B, C. Of these $\mathcal{H}$, exactly one has the property that there exists a point $P \in \mathcal{H}$ such that NP is tangent to $\mathcal{H}$ and $P \in OK$. Let N' be the reflection of N over BC. If AK meets PN' at Q, then the length of PQ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are positive integers such that c is not divisible by the square of any prime. Compute $100a + b + c$.