The Online Math Open Spring Contest
March 27–April 7, 2020

Online Math Open

Acknowledgments

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Contest Information

For any questions about the contest, please email the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Contest Window

The Spring 2020 Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

Instructions, Format, and Rules

Please read the contest information which can be found at

`http://internetolympiad.org/pages/n/omo_info`

This includes registration instructions, contest format, and rules. Here is a quick summary of the rules:

1. Four-function calculators are permitted, but other computational aids are not.
2. Physical drawing aids are permitted, but electronic ones are not.
3. Referencing any outside materials to solve problems is prohibited.
4. Communication between members of different teams about the contest is prohibited.

The rules linked above provide details about these rules and should be referenced as the official rules.

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with “Clarification” in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com` (include “Protest” in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Let ℓ be a line and let points A, B, C lie on ℓ so that $AB = 7$ and $BC = 5$. Let m be the line through A perpendicular to ℓ. Let P lie on m. Compute the smallest possible value of $PB + PC$.

2. Po writes down five consecutive integers and then erases one of them. The four remaining integers sum to 153. Compute the integer that Po erased.

3. Given that the answer to this problem can be expressed as $a \cdot b \cdot c$, where a, b, and c are pairwise relatively prime positive integers with $b = 10$, compute $1000a + 100b + 10c$.

4. Let $ABCD$ be a square with side length 16 and center O. Let $\mathcal{S}$ be the semicircle with diameter AB that lies outside of $ABCD$, and let P be a point on $\mathcal{S}$ so that $OP = 12$. Compute the area of triangle CDP.

5. Compute the smallest positive integer n such that there do not exist integers x and y satisfying $n = x^3 + 3y^3$.

6. Alexis has 2020 paintings, the ith one of which is a $1 \times i$ rectangle for $i = 1, 2, \ldots, 2020$. Compute the smallest integer n for which they can place all of the paintings onto an $n \times n$ mahogany table without overlapping or hanging off the table.

7. On a 5×5 grid we randomly place two *cars*, which each occupy a single cell and randomly face in one of the four cardinal directions. It is given that the two cars do not start in the same cell. In a *move*, one chooses a car and shifts it one cell forward. The probability that there exists a sequence of moves such that, afterward, both cars occupy the same cell is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $100m + n$.

8. Let $a > b$ be positive integers. Compute the smallest possible integer value of $\frac{a!+1}{b!+1}$.

9. A magician has a hat that contains a white rabbits and b black rabbits. The magician repeatedly draws pairs of rabbits chosen at random from the hat, without replacement. Call a pair of rabbits *checkered* if it consists of one white rabbit and one black rabbit. Given that the magician eventually draws out all the rabbits without ever drawing out an unpaired rabbit and that the expected value of the number of checkered pairs that the magician draws is 2020, compute the number of possible pairs (a, b).

10. Compute the number of functions $f : \{1, \ldots, 15\} \to \{1, \ldots, 15\}$ such that, for all $x \in \{1, \ldots, 15\}$,
$$\frac{f(f(x)) - 2f(x) + x}{15}$$
is an integer.

11. A mahogany bookshelf has four identical-looking books which are 200, 400, 600, and 800 pages long. Velma picks a random book off the shelf, flips to a random page to read, and puts the book back on the shelf. Later, Daphne also picks a random book off the shelf and flips to a random page to read. Given that Velma read page 122 of her book and Daphne read page 304 of her book, the probability that they chose the same book is $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

12. Convex pentagon $ABCDE$ is inscribed in circle γ. Suppose that $AB = 14$, $BE = 10$, $BC = CD = DE$, and $[ABCDE] = 3[ACD]$. Then there are two possible values for the radius of γ. The sum of these two values is $\sqrt{n}$ for some positive integer n. Compute n.

13. For nonnegative integers p, q, r, let
$$f(p, q, r) = (p!)^p (q!)^q (r!)^r.$$
Compute the smallest positive integer n such that for any triples (a, b, c) and (x, y, z) of nonnegative integers satisfying $a + b + c = 2020$ and $x + y + z = n$, $f(x, y, z)$ is divisible by $f(a, b, c)$.

14. Let S and T be non-empty, finite sets of positive integers. We say that $a \in \mathbb{N}$ is *good* for $b \in \mathbb{N}$ if $a \geq \frac{b}{2} + 7$. We say that an ordered pair $(a, b) \in S \times T$ is *satisfiable* if a and b are good for each other. A subset R of S is said to be *unacceptable* if there are less than $|R|$ elements b of T with the property that there exists $a \in R$ such that (a, b) is satisfiable. If there are no unacceptable subsets of S, and S contains the elements 14, 20, 16, 32, 23, and 31, compute the smallest possible sum of elements of T given that $|T| \geq 20$.

15. Let ABC be a triangle with $AB = 20$ and $AC = 22$. Suppose its incircle touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively, and P is the foot of the perpendicular from D to $\overline{EF}$. If $\angle BPC = 90^\circ$, then compute BC^2.

16. Compute the number of ordered pairs (m, n) of positive integers such that $(2^m - 1)(2^n - 1) \mid 2^{10!} - 1$.

17. Compute the number of integers $1 \leq n \leq 1024$ such that the sequence $\lceil n \rceil$, $\lceil n/2 \rceil$, $\lceil n/4 \rceil$, $\lceil n/8 \rceil$, $\ldots$ does not contain any multiple of 5.

18. Vincent has a fair die with sides labeled 1 to 6. He first rolls the die and records it on a piece of paper. Then, every second thereafter, he re-rolls the die. If Vincent rolls a different value than his previous roll, he records the value and continues rolling. If Vincent rolls the same value, he stops, does *not* record his final roll, and computes the average of his previously recorded rolls. Given that Vincent first rolled a 1, let E be the expected value of his result. There exist rational numbers $r, s, t > 0$ such that $E = r - s \ln t$ and t is not a perfect power. If $r + s + t = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

19. Let ABC be a scalene triangle. The incircle is tangent to lines BC, AC, and AB at points D, E, and F, respectively, and the A-excircle is tangent to lines BC, AC, and AB at points D_1, E_1, and F_1, respectively. Suppose that lines AD, BE, and CF are concurrent at point G, and suppose that lines AD_1, BE_1, and CF_1 are concurrent at point G_1. Let line GG_1 intersect the internal bisector of angle BAC at point X. Suppose that $AX = 1$, $\cos \angle BAC = \sqrt{3} - 1$, and $BC = 8\sqrt[4]{3}$. Then $AB \cdot AC = \frac{j + k\sqrt{m}}{n}$ for positive integers j, k, m, and n such that $\gcd(j, k, n) = 1$ and m is not divisible by the square of any integer greater than 1. Compute $1000j + 100k + 10m + n$.

20. Reimu invented a new number base system that uses exactly five digits. The number 0 in the decimal system is represented as 00000, and whenever a number is incremented, Reimu finds the leftmost digit (of the five digits) that is equal to the "units" (rightmost) digit, increments this digit, and sets all the digits to its right to 0. (For example, an analogous system that uses three digits would begin with 000, 100, 110, 111, 200, 210, 211, 220, 221, 222, 300, ...) Compute the decimal representation of the number that Reimu would write as 98765.

21. For positive integers $i = 2, 3, \ldots, 2020$, let
$$a_i = \frac{\sqrt{3i^2 + 2i - 1}}{i^3 - i}.$$
Let $x_2, \ldots, x_{2020}$ be positive reals such that $x_2^4 + x_3^4 + \cdots + x_{2020}^4 = 1 - \frac{1}{1010 \cdot 2020 \cdot 2021}$. Let S be the maximum possible value of
$$\sum_{i=2}^{2020} a_i x_i (\sqrt{a_i} - 2^{-2.25} x_i)$$
and let m be the smallest positive integer such that S^m is rational. When S^m is written as a fraction in lowest terms, let its denominator be $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for prime numbers $p_1 < \cdots < p_k$ and positive integers α_i. Compute $p_1\alpha_1 + p_2\alpha_2 + \cdots + p_k\alpha_k$.

22. Let ABC be a scalene triangle with incenter I and symmedian point K. Furthermore, suppose that $BC = 1099$. Let P be a point in the plane of triangle ABC, and let D, E, F be the feet of the perpendiculars from P to lines BC, CA, AB, respectively. Let M and N be the midpoints of segments EF and BC, respectively. Suppose that the triples (M, A, N) and (K, I, D) are collinear, respectively, and that the area of triangle DEF is 2020 times the area of triangle ABC. Compute the largest possible value of $\lceil AB + AC \rceil$.

23. In the Bank of Shower, a bored customer lays n coins in a row. Then, each second, the customer performs "The Process." In The Process, all coins with exactly one neighboring coin heads-up before The Process are placed heads-up (in its initial location), and all other coins are placed tails-up. The customer stops once all coins are tails-up.

    Define the function f as follows: If there exists some initial arrangement of the coins so that the customer never stops, then $f(n) = 0$. Otherwise, $f(n)$ is the average number of seconds until the customer stops over all initial configurations. It is given that whenever $n = 2^k - 1$ for some positive integer k, $f(n) > 0$.

    Let N be the smallest positive integer so that
    $$M = 2^N \cdot \big(f(2^2-1) + f(2^3-1) + f(2^4-1) + \cdots + f(2^{10}-1)\big)$$
    is a positive integer. If $M = \overline{b_k b_{k-1} \cdots b_0}$ in base two, compute $N + b_0 + b_1 + \cdots + b_k$.

24. Let A, B be opposite vertices of a unit square with circumcircle Γ. Let C be a variable point on Γ. If $C \notin \{A, B\}$, then let ω be the incircle of triangle ABC, and let I be the center of ω. Let C_1 be the point at which ω meets $\overline{AB}$, and let D be the reflection of C_1 over line CI. If $C \in \{A, B\}$, let $D = C$. As C varies on Γ, D traces out a curve $\mathfrak{C}$ enclosing a region of area $\mathcal{A}$. Compute $\lfloor 10^4 \mathcal{A} \rfloor$.

25. Let $\mathcal{S}$ denote the set of positive integer sequences (with at least two terms) whose terms sum to 2019. For a sequence of positive integers $a_1, a_2, \ldots, a_k$, its *value* is defined to be
    $$V(a_1, a_2, \ldots, a_k) = \frac{a_1^{a_2} a_2^{a_3} \cdots a_{k-1}^{a_k}}{a_1! a_2! \cdots a_k!}.$$
    Then the sum of the values over all sequences in $\mathcal{S}$ is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute the remainder when $m + n$ is divided by 1000.

26. Let ABC be a triangle with circumcircle ω and circumcenter O. Suppose that $AB = 15$, $AC = 14$, and P is a point in the interior of $\triangle ABC$ such that $AP = \frac{13}{2}$, $BP^2 = \frac{409}{4}$, and P is closer to $\overline{AC}$ than to $\overline{AB}$. Let E, F be the points where $\overline{BP}$, $\overline{CP}$ intersect ω again, and let Q be the intersection of $\overline{EF}$ with the tangent to ω at A. Given that $AQOP$ is cyclic and that CP^2 is expressible in the form $\frac{a}{b} - c\sqrt{d}$ for positive integers a, b, c, d such that $\gcd(a, b) = 1$ and d is not divisible by the square of any prime, compute $1000a + 100b + 10c + d$.

27. The *equatorial algebra* is defined as the real numbers equipped with the three binary operations $\natural$, $\sharp$, $\flat$ such that for all $x, y \in \mathbb{R}$, we have
    $$x \natural y = x + y, \quad x \sharp y = \max\{x, y\}, \quad x \flat y = \min\{x, y\}.$$
    An *equatorial expression* over three real variables x, y, z, along with the *complexity* of such expression, is defined recursively by the following:

    - x, y, and z are equatorial expressions of complexity 0;
    - when P and Q are equatorial expressions with complexity p and q respectively, all of $P \natural Q$, $P \sharp Q$, $P \flat Q$ are equatorial expressions with complexity $1 + p + q$.

    Compute the number of distinct functions $f : \mathbb{R}^3 \to \mathbb{R}$ that can be expressed as equatorial expressions of complexity at most 3.

28. Let A_0BC_0D be a convex quadrilateral inscribed in a circle ω. For all integers $i \geq 0$, let P_i be the intersection of lines A_iB and C_iD, let Q_i be the intersection of lines A_iD and BC_i, let M_i be the midpoint of segment P_iQ_i, and let lines M_iA_i and M_iC_i intersect ω again at A_{i+1} and C_{i+1}, respectively. The circumcircles of $\triangle A_3M_3C_3$ and $\triangle A_4M_4C_4$ intersect at two points U and V.

    If $A_0B = 3$, $BC_0 = 4$, $C_0D = 6$, $DA_0 = 7$, then UV can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$ and b is squarefree. Compute $100a + 10b + c$.

29. Let x_0, x_1, …, x_{1368} be complex numbers. For an integer m, let $d(m)$, $r(m)$ be the unique integers satisfying $0 \le r(m) < 37$ and $m = 37d(m) + r(m)$. Define the 1369×1369 matrix $A = \{a_{i,j}\}_{0 \le i,j \le 1368}$ as follows:

$$a_{i,j} = \begin{cases} x_{37d(j)+d(i)} & r(i) = r(j),\ i \ne j \\ -x_{37r(i)+r(j)} & d(i) = d(j),\ i \ne j \\ x_{38d(i)} - x_{38r(i)} & i = j \\ 0 & \text{otherwise} \end{cases}.$$

We say A is r-*murine* if there exists a 1369×1369 matrix M such that r columns of $MA - I_{1369}$ are filled with zeroes, where I_{1369} is the identity 1369×1369 matrix. Let $\mathrm{rk}(A)$ be the maximum r such that A is r-murine. Let S be the set of possible values of $\mathrm{rk}(A)$ as $\{x_i\}$ varies. Compute the sum of the 15 smallest elements of S.

30. Let c be the smallest positive real number such that for all positive integers n and all positive real numbers x_1, …, x_n, the inequality

$$\sum_{k=0}^{n} \frac{(n^3 + k^3 - k^2 n)^{3/2}}{\sqrt{x_1^2 + \cdots + x_k^2 + x_{k+1} + \cdots + x_n}} \le \sqrt{3} \left(\sum_{i=1}^{n} \frac{i^3(4n - 3i + 100)}{x_i} \right) + cn^5 + 100n^4$$

holds. Compute $\lfloor 2020c \rfloor$.