2010 IMO Summer Training Warm Up Problems

Algebra

A1 (Romania TST 2005) Let r, s be fixed rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = f(x + r) + y + s$$

for all $x, y \in \mathbb{Q}$, where $\mathbb{Q}$ denotes the rational numbers.

A2 Let x, y, z be any positive real numbers such that $x + y + z = 3$. Prove that

$$\frac{x^3}{(y+2z)^2} + \frac{y^3}{(z+2x)^2} + \frac{z^3}{(x+2y)^2} \geq \frac{1}{3}.$$

A3 (IMO Shortlist 2001) Let $x_1, x_2, \cdots, x_n$ be any real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

A4 (IMO 2007) Let $a_1, \cdots, a_n$ be n real numbers and $d = \max\limits_{1\leq i\leq j\leq n} \{|a_i - a_j|\}$.

(a) If $x_1 \leq x_2 \leq \ldots \leq x_n$ are real numbers, prove that

$$\max_{i=1,\ldots,n} \{|x_i - a_i|\} \geq \frac{d}{2}.$$

(b) Prove that there exist $x_1 \leq x_2 \leq \ldots \leq x_n$ such that equality holds in (a).

A5 (Russia 2005, Grade 11) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be $2n$ pairwise distinct real numbers. Find the maximum possible finite number of real solutions to the following equation.

$$|x - a_1| + \ldots + |x - a_n| = |x - b_1| + \ldots + |x - b_n|.$$

A6 (Brazil 2003) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a strictly increasing function[1] on the *positive* reals such that

$$f\left(\frac{2xy}{x+y}\right) = \frac{f(x)+f(y)}{2} \text{ for all } x, y \in \mathbb{R}^+ .$$

Show that $f(x) < 0$ for some value of x.

A7 (Romania 1999) Let $x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$

[1] To avoid confusion, a function is strictly increasing if $f(x) < f(y)$ for $x < y$.

Combinatorics

C1 (Romania TST 2005) Let S be a set of 2009 elements and $P(S)$ be the set of all subsets of S. Let $f : P(S) \to \mathbb{R}$ be a function such that

$$f(X \cap Y) = \min\{f(X), f(Y)\}$$

for all $X, Y \in P(S)$. Find the maximum possible number of elements in the range of f.

C2 (Australia 1990) Let n, k be positive integers. In a certain library, there are n shelves, each holding at least one book. k new shelves are acquired and the books are arranged on the $n + k$ shelves, again with at least one book on each shelf. A book is said to be *special* if it is in a shelf with fewer books in the new arrangement than it was in the original arrangement. Prove that there are at least $k + 1$ special books in the rearranged library.

The scenarios in Problems C3-C5 can all be represented as a graph. It is easier to see this in some problems than in others.

C3 There are n cities, where each pair of cities is joined by a one-way road. Prove that there exists a city that can reach any other city via at most two roads.

C4 (Iran 1998) Let $n \geq 3$ be an integer. Let G be a grid whose entries are all $0, 1$ or -1 such that each row and each column contains exactly one 1 and one -1. Prove that the rows and the columns of the grid can be re-ordered such that the resulting grid is the negative of G.

C5 (Iran 2005) A simple polygon is one where the perimeter of the polygon does not intersect itself (but is not necessarily convex). Prove that a simple polygon $\mathcal{P}$ contains a diagonal which is completely inside $\mathcal{P}$ such that the diagonal divides the perimeter into two parts both containing at least $n/3 - 1$ vertices. (Do not count the vertices which are endpoints of the diagonal.)

C6 (IMO Shortlist 2006) Let S be a finite set of points in the plane such that no three points are collinear. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P and $b(P)$ be the number of points of S outside of P. Prove that for all real numbers x, the polynomial

$$\sum_P x^{a(P)}(1 - x)^{b(P)} = 1,$$

where sum is over all convex polygons P with vertices in S. (Note: A line segment, a point and the empty-set are considered convex polygons of 2, 1, 0 vertices respectively.)

C7 (Iran 2005) Given n points on the plane with no three collinear, a set of k points is said to be *polite* if they determine a convex k-gon that contains no other given points in its interior. Let c_k denote the number of k-polite subsets of the given points. Show that the series

$$\sum_{k=3}^{n} (-1)^k c_k$$

is independent of the configuration of the points and depends only on n.

Geometry

G1 (Britain 2006) Given triangle ABC with $|AB| < |AC|$, let P be on side AC such that $|CP| = |AB|$ and Q be on ray BA such that $|BQ| = |AC|$. Let R be the intersection of PQ and the perpendicular bisector of BC. Prove that

$$\angle BAC + \angle BRC = 180^o.$$

G2 Let ABC be a triangle with altitudes BE and CF, orthocentre H and circumcentre O and Y, Z the midpoints of AC, AB, respectively. Let $C' = EF \cap YZ$. Prove that $CC' \perp OH$.

G3 (Romania 2006) Let ΔABC be a triangle with incircle γ. Consider the circle passing through BC and internally tangent to γ; let ℓ be the common tangent line. Let A_1 be the intersection of ℓ and BC. Define B_1, C_1 similarly. Prove that A_1, B_1, C_1 are collinear.

G4 (Japan 2007) Let ΔABC be a triangle with circumcircle ω. Let the circle tangent to sides AB and AC and internally tangent to ω touch ω at A_1. Define B_1, C_1, similarly. Prove that AA_1, BB_1, CC_1 are concurrent.

G5 (IMO Shortlist 1999) Let ABC be a triangle with interior point P. Prove that

$$\min\{|PA|, |PB|, |PC|\} + |PA| + |PB| + |PC| < |AB| + |BC| + |CA|.$$

G6 (China TST 2008) Given a triangle ABC, a line l intersect the lines BC, CA, AB at D, E, F respectively. Let O_1, O_2, O_3 be the circumcentres of $\Delta AEF, \Delta BFD, \Delta CDE$, respectively. Prove that the orthocentre of $\Delta O_1O_2O_3$ lies on l.

G7 (Romania TST 2005 Special Case) Given a convex hexagon $ABCDEF$ of area 1, prove that there exist a triangle formed by three consecutive vertices of the hexagon, whose area is at most 1/6.

Number Theory

N1 (Romanian Masters in Mathematics 2009) Let k be a positive integer. Let $a_1, a_2, \cdots, a_k$ be positive integers and $d = gcd(a_1, a_2, \cdots, a_k)$ and $n = a_1 + a_2 + \cdots + a_k$. Prove that

$$\frac{d \cdot (n-1)!}{a_1! a_2! \cdots a_k!}$$

is an integer.

N2 (IberoAmerican 2004) Find all pairs of positive integers (n, k) such that the following statement is true: there exist integers a, b such that $gcd(a, n) = 1, gcd(b, n) = 1$ and $a + b = k$.

N3 (Olymon) Prove that there are no integers a, b such that $a^3 + b^4 = 2^{2003}$.

N4 Solve (a). If (a) is too easy, then solve (b).

(a) (Romania TST 1999) Prove that for any non-negative integer n, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k} 4^{n-k} 3^k$$

is the sum of two consecutive perfect squares.

(b) (Romania TST 2004) Prove that for all positive integers m, n with m odd, the following number is an integer:

$$\frac{1}{3^m n} \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

N5 (Iran 2007) Find all polynomials with integer coefficients such that for all positive integers a, b, c, $f(a) + f(b) + f(c)$ is divisible by $a + b + c$.

N6 (IMO Shortlist 2004) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $(m^2+n)^2$ is divisible by $f(m)^2+f(n)$ for all $m, n \in \mathbb{N}$.

N7 (Turkey 2005) Find all primes p such that the number of ordered pairs of integers (x, y) with $0 \le x, y < p$ to the following equation is p.

$$y^2 \equiv x^3 - x (\text{mod } p).$$

Hints

A1 Use a substitution to turn this equation into the Cauchy equation $g(x+y) = g(x) + g(y)$.

A2 Use AM-GM to clear the denominators.

A3 Two words, Cauchy-Schwarz. Then try to apply an inequality that results in a telescoping sum.

A4 Suppose $d = |a_i - a_j|$ for some $i \leq j$. Then we know that $x_i, \cdots, x_j$ have to be. What about the remaining parts of the sequence?

A5 Let $\{c_1, \cdots, c_{100}\} = \{a_1, \cdots, a_{50}\} \cup \{b_1, \cdots, b_{50}\}$ such that $c_1 < c_2 < \cdots < c_{100}$. As you go from $x = -\infty$ to $x = \infty$, what does the graph

$$|x - a_1| + \cdots + |x - a_{50}| - |x - b_1| - \cdots - |x - b_{50}|$$

look like? As x passes through a value c_i, what happens to the slope?

A6 Think harmonic (and not in the geometric or musical sense).

A7 This is the only inequality I know of, that can be solved using contradiction.

C1 Focus on the sets with 2008 elements.

C2 Try to use a nice invariant.

C3 Use induction.

C4 Let each row and each column be vertices of a graph. Join two vertices by an red edge the entry in the corresponding row and column is 1 and a blue edge if the entry is -1. The graph consists of a union of cycle. (Why?) Now what does switching rows and columns correspond to in the graph?

C5 Triangulate. Then create a graph.

C6 Try a method that *probably* works. :)

C7 Let S be the set of subsets of the points. Partition S so that each part consists of sets with the same convex hull.

G1 There are many ways to solve this problem. Try to *reflect* on how you can solve this problem nicely.

G2 Points have power. Check out their power. And what does the nine-point circle pass through?

G3 Either use Menelaos' Theorem and a bit of bash, or prove that A_1, B_1, C_1 lie on the radical axis of two certain circles.

G4 There is a known solution using inversion and one using Monge's Theorem. (Do the former if you do not know what Monge's Theorem is.)

G5 Consider the midpoints of the sides AB, BC, CA.

G6 Miquel should tell you something about the three circles and the circumcircle of ΔABC. The second part of the problem then shouldn't be too difficult.

G7 Join opposing vertices by a line. Partition the hexagon into four parts; three quadrilaterals and the triangle formed by the three lines joining opposing vertices.

N1 Note that $d = c_1a_1 + \cdots + c_ka_k$ for some $c_1, \cdots, c_k \in \mathbb{Z}$.

N2 The Chinese team should definitely get this. :)

N3 Modular magic! :)

N4 (a) The first step is clearly to apply Binomial Theorem. Note that a number n is the product of two consecutive positive integers if and only if $4n + 1$ is a perfect square. In (b), use a similar idea in (a), but requires something more *complex*.

N5 As in most problems involving polynomials with integer coefficients, use the fact that $a-b|f(a)-f(b)$ for all distinct $a, b \in \mathbb{Z}$.

N6 Try to prove that $f(1) = 1$ and $f(p-1) = p-1$ for all primes. Then try to prove that $(p-1)^2 + f(n) \mid (n - f(n))^2$ for all $n \in \mathbb{N}$ and primes p.

N7 Solve the problem for small primes. Conjecture an answer, and then use the main difference between primes that work and the primes that do not work.