# Three Lemmas in Geometry

Yufei Zhao
Massachusetts Institute of Technology

yufei.zhao@gmail.com

## 1 Diameter of incircle

1. (IMO 1992) In the plane let $\mathcal{C}$ be a circle, $\ell$ a line tangent to the circle $\mathcal{C}$, and $M$ a point on $\ell$. Find the locus of all points $P$ with the following property: there exists two points $Q, R$ on $\ell$ such that $M$ is the midpoint of $QR$ and $\mathcal{C}$ is the inscribed circle of triangle $PQR$.

   *Solution:* Let $\mathcal{C}$ touch $\ell$ at $D$, and $DT$ be a diameter of $\mathcal{C}$. For any such $P, Q, R$ described in the problem, the line $PT$ must intersect $\ell$ at a point $F$ such that $MD = MF$, by the lemma. The point $F$ depends only on $M$, $\ell$, and $\mathcal{C}$. It follows that $P$ must lie on the ray $FE$ beyond $E$.

   Conversely, given a point $P$ lying on the ray $FE$ beyond $E$, let the tangents from $P$ to $\mathcal{C}$ meet $\ell$ at $Q$ and $R$. By the lemma we must have $QF = RD$, from which it follows that $M$ is the midpoint of $QR$. Therefore, the locus is the ray $FE$ beyond $E$.

2. (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle $\omega$ of triangle $BCD$ meets $CD$ at $E$. Let $F$ be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle $ACF$ meet line $CD$ at $C$ and $G$. Prove that the triangle $AFG$ is isosceles.

   *Solution:* Observe that $F$ is the center of the excircle of $ADC$ opposite to $A$ (since the center satisfies the two defining properties of $F$). Let line $AC$ touch this excircle at $X$. Then, using fact that $GACF$ is cyclic, we have

   $$\angle GAF = \angle GCF = \angle XCF = \angle AGF,$$

   and therefore $AFG$ is isosceles.

3. (IMO Shortlist 2005) In a triangle $ABC$ satisfying $AB + BC = 3AC$ the incircle has centre $I$ and touches the sides $AB$ and $BC$ at $D$ and $E$, respectively. Let $K$ and $L$ be the symmetric points of $D$ and $E$ with respect to $I$. Prove that the quadrilateral $ACKL$ is cyclic.

   *Solution:* Let $x = AD$, $y = BD = BE$, $z = CE$. Then $AB = x + y$, $BC = y + z$, $AC = x + z$, so the condition that $AB + BC = 3AC$ is equivalent to $x + y + y + z = 3x + 3z$, or equivalently $y = x + z$.

   Let line $CK$ meet $AB$ at $M$ and line $AL$ meet $BC$ at $N$. Then by the lemma, $BM = AD = x$, so $MD = BD - BM = y - x = z$. Similarly $NE = x$. By comparing the lengths of the legs, we see that the right triangles $MDK$ and $CEL$ are congruent, and so are $ADK$ and $NEL$. Therefore $\angle MKD = \angle CLE$ and $\angle AKD = \angle NLE$. Adding gives $\angle MKA = \angle CLN$, so $\angle AKC = \angle ALC$, and hence $ACKL$ is cyclic.

4. (Nagel line) Let $ABC$ be a triangle. Let the excircle of $ABC$ opposite to $A$ touch side $BC$ at $D$. Similarly define $E$ on $AC$ and $F$ on $AB$. Then $AD$, $BE$, $CF$ concur (why?) at a point $N$ known as the *Nagel point*.

   Let $G$ be the centroid of $ABC$ and $I$ the incenter of $ABC$. Show that $I, G, N$ lie in that order on a line (known as the *Nagel line*, and $GN = 2IG$.

*Solution:* Let the incircle of $ABC$ touch $BC$ at $X$, and let $XY$ be a diameter of the incircle. By the lemma, $A, Y, D$ are collinear. Let $M$ be the midpoint of $BC$. Then $MI$ is a midline of triangle $XYD$, so $IM$ and $YD$ are parallel. The dilation centered at $G$ with ratio $-2$ takes $M$ to $A$, and thus it takes line $IM$ to the line through $A$ parallel to $IM$, namely the line $AD$. Hence the image of $I$ under the dilation lies on the line $AD$. Analogously, it must also lie on $BE$ and $CF$, and therefore the image of $I$ is precise $N$. This proves that $I, G, N$ are collinear in that order with $GN = 2IG$.

5. (USAMO 2001) Let $ABC$ be a triangle and let $\omega$ be its incircle. Denote by $D_1$ and $E_1$ the points where $\omega$ is tangent to sides $BC$ and $AC$, respectively. Denote by $D_2$ and $E_2$ the points on sides $BC$ and $AC$, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by $P$ the point of intersection of segments $AD_2$ and $BE_2$. Circle $\omega$ intersects segment $AD_2$ at two points, the closer of which to the vertex $A$ is denoted by $Q$. Prove that $AQ = D_2P$.

   *Solution:* From the lemma we know that $D_1Q$ is a diameter of the incircle. Let the incenter of $ABC$ be $I$, its centroid be $G$, and the midpoint of $BC$ be $M$. Note that $P$ is the Nagel point of $ABC$. From the previous problem, we know that the dilation centered at $G$ with ratio $-2$ sends $M$ to $A$ and $I$ to $P$, and hence sends segment $IM$ to $PA$, thus $PA = 2IM$. On the other hand, a dilation centered at $D_1$ with ratio 2 sends $IM$ to $QD_2$, so $QD_2 = 2IM = PA$. Therefore, $AQ = PA - QP = QD_2 - QP = D_2P$.

6. (Tournament of Towns 2003 Fall) Triangle $ABC$ has orthocenter $H$, incenter $I$ and circumcenter $O$. Let $K$ be the point where the incircle touches $BC$. If $IO$ is parallel to $BC$, then prove that $AO$ is parallel to $HK$.

   *Solution:* Let $KE$ be a diameter of the incircle, and let line $AE$ meet $BC$ at $D$. Let $M$ be the midpoint of $BC$. By the lemma, $M$ is also the midpoint of $KD$. Since $IO$ is parallel to $BC$, $KMOI$ is a rectangle. Since $I$ is the midpoint of $KE$ and $M$ is the midpoint of $KD$, we see that $O$ must be the midpoint of $ED$. Thus lines $AE$ and $AO$ coincide.

   Let $G$ be the centroid of $ABC$. A dilation centered at $G$ with ratio $-2$ takes $M$ to $A$ and $O$ to $H$ (by Euler line). So it takes segment $MO$ to $AH$, and hence $AH = 2MO = EK$. Since $AH$ and $EK$ are both perpendicular to $BC$, it follows that $AHKE$ is a parallelogram, and hence $HK$ is parallel to $AE$, which coincides with line $AO$.

7. (IMO 2008) Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles $ABC$ and $ADC$ by $\omega_1$ and $\omega_2$ respectively. Suppose that there exists a circle $\omega$ tangent to the ray $BA$ beyond $A$ and to the ray $BC$ beyond $C$, which is also tangent to the lines $AD$ and $CD$. Prove that the common external tangents of $\omega_1$ and $\omega_2$ intersect on $\omega$.

   (Hint: show that $AB + AD = CB + CD$. What does this say about the lengths along $AC$?)

   *Solution:* Chasing some lengths using equal tangents yields $AB + AD = CB + CD$ (details omitted). Let $\omega_1$ and $\omega_2$ touch $AC$ at $P$ and $Q$ respectively. Then $AP = \frac{1}{2}(AB + AC - BC) = \frac{1}{2}(CD + AC - AD) = CQ$.

   Let $PP'$ be a diameter of $\omega_1$, and let $QQ'$ be a diameter of $\omega_2$. By the lemma, $B, P', Q$ are collinear, and $P, Q', D$ are collinear.

   Choose point $T$ on $\omega$ so that the tangent to $\omega$ at $T$ is parallel to $AC$ and furthermore puts $\omega$ and $B$ on different sides. Then the dilation centered at $B$ that sends $\omega_1$ to $\omega$ must send $P'$ to $T$, so $B, P', Q, T$ are collinear. Analogously, the dilation centered at $D$ (with negative ratio) that sends $\omega_2$ to $\omega$ must take $Q'$ to $T$, so $P, Q', D, T$ are collinear.

   Now, $PP'$ and $Q'Q$ are parallel diameters of $\omega_1$ and $\omega_2$, and lines $P'Q$ and $PQ'$ meet at $T$. It follows that there is a dilation with positive ratio centered at $T$ that takes $\omega_1$ to $\omega_2$, and hence $T$ is the intersection of the common external tangents of $\omega_1$ and $\omega_2$. Since $T$ lies on $\omega$, we are done.

## 2 Center of spiral similarity

1. (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle CBA = \angle DCA = \angle EDA.$$

   Diagonals $BD$ and $CE$ meet at $P$. Prove that line $AP$ bisects side $CD$.

   *Solution:* Since $A$ is the center of the spiral similarity sending $BC$ to $DE$, by the lemma we know that $ABCP$ and $APDE$ are both cyclic. Furthermore, since $\angle ACD = \angle ABC$, the circumcircle of $ABCP$ is tangent to $CD$. Since $\angle ADC = \angle AEC$, the circumcircle of $APDE$ is also tangent to $CD$. Let $AP$ meet $CD$ at $M$. Then by power of a point, $MC^2 = MP \cdot MA = MD^2$, so $MC = MD$, as desired.

2. (USAMO 2006) Let $ABCD$ be a quadrilateral, and let $E$ and $F$ be points on sides $AD$ and $BC$, respectively, such that $AE/ED = BF/FC$. Ray $FE$ meets rays $BA$ and $CD$ at $S$ and $T$, respectively. Prove that the circumcircles of triangles $SAE$, $SBF$, $TCF$, and $TDE$ pass through a common point.

   *Solution:* Note that $ABCD$ cannot be a parallelogram, for otherwise $FE$ and $BA$ would be parallel. Let $P$ be the center of spiral similarity that carries $AD$ to $BC$. It must also carry $E$ to $F$ since $AE/ED = BF/FC$. Since the spiral similarity carries $AE$ to $BF$, it follows by the lemma that $PAES$ and $PBFS$ are concyclic. Similarly, since the spiral similarity carries $DE$ to $CF$, $PEDT$ and $PFCT$ are concyclic. Therefore the circumcircles of $SAE, SBF, TCF, TDE$ all pass through $P$.

3. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle $\omega$ with center $O$. Diagonals $AC$ and $BD$ meet at $P$. The circumcircles of triangles $ABP$ and $CDP$ meet at $P$ and $Q$. Assume that points $O, P$, and $Q$ are distinct. Prove that $\angle OQP = 90°$.

   *Solution:* Let $M$ and $N$ be the midpoints of $AC$ and $BD$, respectively. Let $\mathbf{T}$ be the spiral similarity that carries $A$ to $B$ and $C$ to $D$. By the lemma (and the fact after it), $Q$ is the center of the spiral similarity. Since $\mathbf{T}$ carries $AC$ to $BD$, it preserves midpoints, so $\mathbf{T}$ brings $M$ to $N$. Using the lemma again, we see that $M, N, P, Q$ are concyclic. Since $\angle OMP = \angle ONP = 90°$, points $O, P, M, N$ are concyclic with diameter $OP$. Therefore, $M, N, P, Q, O$ are concyclic with diameter $OP$, and therefore $\angle OQP = 90°$.

4. Let $ABCD$ be a quadrilateral. Let diagonals $AC$ and $BD$ meet at $P$. Let $O_1$ and $O_2$ be the circumcenters of $APD$ and $BPC$. Let $M$, $N$ and $O$ be the midpoints of $AC$, $BD$ and $O_1O_2$. Show that $O$ is the circumcenter of $MPN$.

   *Solution:* Let the circumcircles of $APD$ and $BPC$ meet at $P$ and $Q$. Let $\mathbf{T}$ denote the spiral similarity that sends $AD$ to $CB$. Then $\mathbf{T}$ is centered at $Q$ by the lemma. Let $\mathbf{id}$ denote the identity transformation, and consider the transformation $\mathbf{R} = \frac{1}{2}(\mathbf{id} + \mathbf{T})$. This is another spiral similarity centered at $Q$ (if you're not convinced, think about it in terms of multiplication by complex numbers). Then $\mathbf{R}(A) = M$, $\mathbf{R}(D) = N$, $\mathbf{R}(O_1) = O$. Since $O_1$ is the circumcenter of $QAD$, the transformation yields that $O$ is the circumcenter of $QMN$, whose circumcircle must again pass through $P$ by the lemma. This proves the desired fact.

5. (Miquel point of a quadrilateral) Let $\ell_1, \ell_2, \ell_3, \ell_4$ be four lines in the plane, no two parallel. Let $\mathcal{C}_{ijk}$ denote the circumcircle of the triangle formed by the lines $\ell_i, \ell_j, \ell_k$ (these circles are called *Miquel circles*). Then $\mathcal{C}_{123}, \mathcal{C}_{124}, \mathcal{C}_{134}, \mathcal{C}_{234}$ pass through a common point (called the *Miquel point*).

   (It's not too hard to prove this result using angle chasing, but can you see why it's almost an immediate consequence of the lemma?)

*Solution:* Let $P_{ij}$ denote the intersection of $\ell_i$ and $\ell_j$. Let $\mathcal{C}_{134}$ and $\mathcal{C}_{234}$ meet at $P$. Then by the lemma, $P$ is the center of the spiral similarity that sends $P_{13}$ to $P_{23}$ and $P_{14}$ to $P_{24}$. It follows that $P$ is also the center of the spiral similarity that sends $P_{13}$ to $P_{14}$ and $P_{23}$ to $P_{24}$. Applying the lemma again, we find that $\mathcal{C}_{123}$ and $\mathcal{C}_{124}$ also pass through $P$, as desired.

6. (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides $BC$ and $AD$ equal in length and not parallel. Let $E$ and $F$ be interior points of the sides $BC$ and $AD$ respectively such that $BE = DF$. The lines $AC$ and $BD$ meet at $P$, the lines $BD$ and $EF$ meet at $Q$, the lines $EF$ and $AC$ meet at $R$. Consider all the triangles $PQR$ as $E$ and $F$ vary. Show that the circumcircles of these triangles have a common point other than $P$.

   *Solution:* Let $S$ be the center of the spiral similarity that carries $AD$ to $CB$, then it must also carry $F$ to $E$. Using the lemma, we see that $SPAD$, $SRAF$, $SQFD$ are all cyclic. Note that they are the Miquel circles of the quadrilateral formed by the lines $AD, AP, PD, QF$, and thus $S$ is the Miquel point of these circles. The remaining Miquel circle passes through $P, Q, R, S$, and hence $S$ lies on the circumcircle of $PQR$. Note that $S$ is the desired point, as it does not depend on the choice of $E$ and $F$.

7. (IMO Shortlist 2006) Points $A_1, B_1$ and $C_1$ are chosen on sides $BC, CA$, and $AB$ of a triangle $ABC$, respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1$, and $CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2, B_2$, and $C_2$, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points $A_3, B_3$, and $C_3$ are symmetric to $A_1, B_1, C_1$ with respect to the midpoints of sides $BC, CA$, and $AB$, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

   *Solution:* By the lemma, $C_2$ is the center of the spiral similarity that takes $A_1$ to $B$ and $B_1$ to $A$. So triangles $C_2A_1B_1$ and $C_2BA$ are similar. But $C_2$ is also the center of the spiral similarity that takes $A_1$ to $B_1$ and $B$ to $A$. Then because $BA_1 = CA_3$ and $AB_1 = CB_3$,

$$\frac{C_2A_1}{C_2B_1} = \frac{BA_1}{AB_1} = \frac{CA_3}{CB_3}.$$

   Since $\angle A_1C_2B_1 = \angle A_3CB_3$, triangles $CA_3B_3$, $C_2A_1B_1$, and $C_2BA$ are similar. So $\angle CA_3B_3 = \angle C_2BA$. Similarly, $\angle BA_3C_3 = \angle B_2CA$. Then

$$\begin{aligned}\angle B_2A_2C_2 = \angle B_2AC_2 = 180^\circ - \angle B_2C_2A - \angle C_2B_2A &= 180^\circ - \angle B_2CA - \angle C_2BA \\ &= 180^\circ - \angle BA_3C_3 - \angle CA_3B_3 = \angle B_3A_3C_3.\end{aligned}$$

   Similarly $\angle A_2B_2C_2 = \angle A_3B_3C_3$, hence triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

## 3 Symmedian

1. (Poland 2000) Let $ABC$ be a triangle with $AC = BC$, and $P$ a point inside the triangle such that $\angle PAB = \angle PBC$. If $M$ is the midpoint of $AB$, then show that $\angle APM + \angle BPC = 180^\circ$.

   *Solution:* Since $\angle CAB = \angle CBA$, $\angle PAB = \angle PBC$ implies that $\angle PAC = \angle PBA$, and thus the circumcircle of $ABP$ is tangent to $CA$ and $CB$. It follows by the lemma that line $CP$ is a symmedian of $APB$, and therefore $\angle APM = 180^\circ - \angle BPC$.

2. (IMO Shortlist 2003) Three distinct points $A, B, C$ are fixed on a line in this order. Let $\Gamma$ be a circle passing through $A$ and $C$ whose center does not lie on the line $AC$. Denote by $P$ the intersection of the tangents to $\Gamma$ at $A$ and $C$. Suppose $\Gamma$ meets the segment $PB$ at $Q$. Prove that the intersection of the bisector of $\angle AQC$ and the line $AC$ does not depend on the choice of $\Gamma$.

   *Solution:* We follow the method of the first proof of the lemma. Let the bisector of $\angle AQC$ meet $A$ at $R$, then $AR/RC = AQ/QC$, so it suffices to show that $AQ/QC$ does not depend on $\Gamma$.

Applying Sine law repeatedly, we find that

$$\frac{AQ}{QC} = \frac{\sin\angle ACQ}{\sin\angle CAQ} = \frac{\sin\angle PAQ}{\sin\angle PCQ} = \frac{\frac{PQ}{AP}\sin\angle AQP}{\frac{PQ}{CP}\sin\angle CQP} = \frac{\sin\angle AQB}{\sin\angle CQB} = \frac{\frac{AB}{AQ}\sin\angle ABQ}{\frac{CB}{CQ}\sin\angle CBQ} = \frac{AB}{CB}\frac{QC}{AQ}.$$

Thus $AQ/QC = \sqrt{AB/CB}$, which is independent of $\Gamma$, as desired.

3. (Vietnam TST 2001) In the plane, two circles intersect at $A$ and $B$, and a common tangent intersects the circles at $P$ and $Q$. Let the tangents at $P$ and $Q$ to the circumcircle of triangle $APQ$ intersect at $S$, and let $H$ be the reflection of $B$ across the line $PQ$. Prove that the points $A$, $S$, and $H$ are collinear.

   *Solution:* We will only do the configuration where $B$ is closer to line $PQ$ than $A$. You should think about what happens in the other configuration, which is analogous.

   Since $AS$ coincides with the symmedian of $APQ$, it suffices to show that $H$ lies on this symmedian. Note that $AB$ coincides with a median of $APQ$. Indeed, let line $AB$ meet $PQ$ at $M$, then by Power of a Point, $MP^2 = MB \cdot MA = MQ^2$, so $MP = MQ$.

   Since $\angle PHQ = \angle PBQ = 180° - \angle BPQ - \angle BQP = 180° - \angle BAP - \angle BAQ = 180° - \angle PAQ$, we see that $APHQ$ is cyclic. Then $\angle HAQ = \angle HPQ = \angle BPQ = \angle BAP$. Since $AB$ coincides with a median of $APQ$, it follows that $AH$ coincides with a symmedian of $APQ$, and hence $A, H, S$ are collinear.

4. (USA TST 2007) Triangle $ABC$ is inscribed in circle $\omega$. The tangent lines to $\omega$ at $B$ and $C$ meet at $T$. Point $S$ lies on ray $BC$ such that $AS \perp AT$. Points $B_1$ and $C_1$ lies on ray $ST$ (with $C_1$ in between $B_1$ and $S$) such that $B_1T = BT = C_1T$. Prove that triangles $ABC$ and $AB_1C_1$ are similar to each other.

   *Solution:* Let $M$ be the midpoint of $BC$. Since $BT$ is tangent to $\omega$, we have $\angle TBA = 180° - \angle BCA$. By the lemma, we have $\angle BAT = \angle CAM$. Applying Sine law to triangles $BAT$ and $CAM$, we get

$$\frac{BT}{AT} = \frac{\sin\angle BAT}{\sin\angle TBA} = \frac{\sin\angle CAM}{\sin\angle BCA} = \frac{MC}{AM}.$$

   Since $TB = TC_1$, we have $TC_1/TA = MC/MA$. Note that $\angle TMS = \angle TAS = 90°$, so $TMAS$ is cyclic, and hence $\angle AMS = \angle ATS$. Therefore, triangles $AMC$ and $ATC_1$ are similar. Analogously, triangles $AMB$ and $ATB_1$ are similar. Combine the two results, and we see that $ABC$ and $AB_1C_1$ are similar.

5. Let $ABC$ be a triangle. Let $X$ be the center of spiral similarity that takes $B$ to $A$ and $A$ to $C$. Show that $AX$ coincides with a symmedian of $ABC$.

   *Solution:* Let the tangents to the circumcircle of $ABC$ at $B$ and $C$ meet at $D$. Let $AD$ meet the circumcircle of $BCD$ again at $X'$. Then

$$\angle ABX' = \angle BX'D - \angle BAX' = \angle BCD - \angle BAD = \angle BAC - \angle BAD = \angle X'AC,$$

   and analogously we have $\angle ACX' = \angle X'AB$. Therefore $X = X'$, and it lies on the symmedian $AD$.

   *Second solution, not using the lemma:* Let $Y$ be a point on the circumcircle of $ABC$ so that $AY$ coincides with a symmedian of $ABC$. Let $X'$ be the midpoint of $AY$. Let $M$ be the midpoint of $BC$ and $N$ the midpoint of $AC$. Since $AY$ is a symmedian, $\angle BAY = \angle MAC$. Additionally we have $\angle BYA = \angle MCA$, so triangles $ABY$ and $AMC$ are similar. Since $X'$ is the midpoint of $Y$ and $N$ is the midpoint of $AC$, we see that $ABX'$ and $AMN$ are similar. Hence

$$\angle ABX' = \angle AMN = \angle MAB = \angle CAX'.$$

   Analogously $\angle ACX' = \angle BAX'$. Therefore $X = X'$, and it lies on a symmedian.

6. (USA TST 2008) Let $ABC$ be a triangle with $G$ as its centroid. Let $P$ be a variable point on segment $BC$. Points $Q$ and $R$ lie on sides $AC$ and $AB$ respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as $P$ varies along segment $BC$, the circumcircle of triangle $AQR$ passes through a fixed point $X$ such that $\angle BAG = \angle CAX$.

   *Solution:* Let $X$ be the center of spiral similarity $\mathbf{T}$ that carries $B$ to $A$ and $A$ to $C$, as in the previous problem. Triangles $BRP$ and $PQC$ are similar as the corresponding sides are parallel. Since $AR/RB = QP/RB = QC/RP = CQ/QA$, we see that $\mathbf{T}$ must carry $R$ to $Q$. Thus triangles $ARX$ and $XQC$ are similar, so $\angle ARX = \angle XQC$, and hence $ARXQ$ is cyclic. Note that $\angle BAG = \angle CAX$ since $X$ lies on the symmedian. Therefore $X$ has the required properties.

7. (USA 2008) Let $ABC$ be an acute, scalene triangle, and let $M$, $N$, and $P$ be the midpoints of $BC$, $CA$, and $AB$, respectively. Let the perpendicular bisectors of $AB$ and $AC$ intersect ray $AM$ in points $D$ and $E$ respectively, and let lines $BD$ and $CE$ intersect in point $F$, inside of triangle $ABC$. Prove that points $A$, $N$, $F$, and $P$ all lie on one circle.

   *Solution:* Let $X$ be as in Problem 5. Then $\angle ABX = \angle XAC = \angle BAM = \angle ABF$, and analogously $\angle ACX = \angle ACF$, so $X = F$. Then this problem follows as a special case of the previous problem.

8. Let $A$ be one of the intersection points of circles $\omega_1, \omega_2$ with centers $O_1, O_2$. The line $\ell$ is tangent to $\omega_1, \omega_2$ at $B, C$ respectively. Let $O_3$ be the circumcenter of triangle $ABC$. Let $D$ be a point such that $A$ is the midpoint of $O_3D$. Let $M$ be the midpoint of $O_1O_2$. Prove that $\angle O_1DM = \angle O_2DA$.

   (Hint: use Problem 5.)

   *Solution:* We have $\angle AO_3C = 2\angle ABC = \angle AO_1B$, and triangles $AO_3C$ and $AO_1B$ are both isosceles, hence they are similar. So the spiral similarity at $A$ that carries $O_1$ to $B$ also carries $O_3$ to $C$, and it follows that $AO_1O_3$ and $ABC$ are similar. Analogously, triangles $ABC$ and $AO_3O_2$ are similar. So triangles $AO_1O_3$ and $AO_3O_2$ are similar.

   Compare triangles $AO_1D$ and $ADO_2$. We have $\angle O_1AD = 180^\circ - \angle O_1AO_3 = 180^\circ - \angle O_3AO_2 = \angle DAO_2$. Also $AO_1/AD = AO_1/AO_3 = AO_3/AO_2 = AD/AO_2$. It follows that triangls $AO_1D$ and $ADO_2$ are similar. It follows by Problem 5 that $DA$ is a symmedian of $DO_1O_2$, and thus $\angle O_1DM = \angle O_2DA$.