2010 Mock 1
Time: 4.5 Hours

1. Let AD be the angle bisector of ΔABC. A line ℓ is tangent to the circumcircles of ΔADB and ΔADC at points M, N, respectively. Prove that ℓ is also tangent to the circle passing through the midpoints of BD, DC, MN.

   Solution 1: (Using homothety:) Let ω_1, ω_2 be circumcircles of $\Delta BAD, \Delta CAD$, respectively with circumcentres O_1, O_2, respectively.

   Let h be the homothety with positive ratio that maps ω_1 to ω_2. Then $h(M) = N$ and $h(O_1) = O_2$. We claim that $h(B) = D$ and $h(D) = C$. Note that $\angle O_1BC = 90 - \angle BAD$ and $\angle O_2DC = 90 - \angle DAC$. Since $\angle BAD = \angle DAC$, $\angle O_1BC = \angle O_2DC$. Therefore, $BO_1||DO_2$. Consequently, $h(B) = D$. Therefore, the centre of h lies on BD. Hence, $h(D)$ is a point (different from D) on ω_2 that lies on BC. Therefore, $h(D) = C$.

   Let E, X, Y be the midpoints of MN, BD, DC, respectively and ω the circumcircle of ΔEXY. Since $N = h(M)$ and $D = h(B)$, $MB||ND$. Therefore, $EX||MB||ND$. Similarly, $EY||MD||NC$. Hence, the homothety h_1 (with positive ratio) that maps MB to EX has the same centre as h and maps D to Y. Since the circumcircle of ΔMBD is tangent to ℓ and ℓ passes through the centre of h_1 and h_1 maps ΔMBD to ΔEXY, the image of this circumcircle is also tangent to ℓ, i.e. the circumcircle of ΔEXY is tangent to h. This completes the problem. $\square$

   Solution 2: (Using angle chasing:) Let ω_1, ω_2 be circumcircles of $\Delta BAD, \Delta CAD$, respectively. We will consider the two tangent lines of ω_1, ω_2 separately.

   We first consider the case when ℓ intersects DA on ray DA. Using the same argument as in Solution 1, AD passes through the midpoint of MN; we will call this point E. I claim that $MD||NC$ and $MB||ND$. Let $\alpha = \angle BAD = \angle DAC, \theta_1 = \angle NMD, \theta_2 = \angle MND$. Note that $\angle BMD = \angle BAD = \alpha$ and $\angle DNC = \angle DAC = \alpha$. Then by properties of tangents, $\angle MBD = \theta_1, \angle NCD = \theta_2$. Looking at the quadrilateral $MNCB$, we have $2(\theta_1 + \theta_2 + \alpha) = 360°$. Hence, $\theta_1 + \theta_2 + \alpha = 180°$. Since $\angle NMD = \theta_1$ and $\angle MNC = \theta_2 + \alpha$ and $\theta_1 + \theta_2 + \alpha = 180°$, $MD||NC$. Since $\angle BMD = \angle DNC = \alpha$, $MB||ND$.

   Let X, Y be the midpoints of BD and DC. Since E is the midpoint of MN, $MB||EX||ND$ and $MD||EY||NC$. Therefore, $\angle MEX = \angle MND = \angle NCD = \angle EYX$. Hence, the circumcircle of ΔMXY is tangent to ℓ, as desired.

   The case when ℓ intersects DA on ray AD is handled similarly. Using the same argument as in Solution 1, AD passes through the midpoint of MN; we will call this point E. I claim that $MD||NC$ and $MB||ND$. Let $\alpha = \angle BAD = \angle DAC, \theta_1 = \angle NMD, \theta_2 = \angle MND$. Note that $\angle BMD = 180 - \angle BAD = 180 - \alpha$ and $\angle DNC = 180° - \angle DAC = 180° - \alpha$. Then by properties of tangents, $\angle MBD = \theta_1, \angle NCD = \theta_2$. Looking at the quadrilateral

$MNCB$, we have $2(\theta_1 + \theta_2 + 180° - \alpha) = 360°$. Hence, $\theta_1 + \theta_2 + 180° - \alpha = 180°$. Since $\angle NMD = \theta_1$ and $\angle MNC = 180 - \alpha + \theta_2$ and $\theta_1 + \theta_2 + 180° - \alpha = 180°$, $MD||NC$. Since $\angle BMD = \angle DNC = 180° - \alpha$, $MB||ND$.

Let X, Y be the midpoints of BD and DC. Since E is the midpoint of MN, $MB||EX||ND$ and $MD||EY||NC$. Therefore, $\angle MEX = \angle MND = \angle NCD = \angle EYX$. Hence, the circumcircle of ΔMXY is tangent to ℓ, as desired. $\square$

2. Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function $T : \mathbb{Z} \to \mathbb{Z}$ such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

   Solution: Suppose such a function T exists. For each positive integer n, let

   $$A(n) = \{x \in \mathbb{Z} \mid T^n(x) = x\}$$

   and

   $$B(n) = \{x \in \mathbb{Z} \mid T^n(x) = x, \text{ and } n \text{ is the smallest positive integer with this property.}\}$$

   Then $P(n) = |A(n)|$ for all positive integers n. This implies that $A(n)$ is finite for all $n \in \mathbb{N}$. Note that

   $$A(n) = \bigcup_{d|n} B(d)$$

   and the sets $B(d), d|n$ are pairwise disjoint. Therefore,

   $$|A(n)| = \sum_{d|n} |B(d)|.$$

   We claim that d divides $|B(d)|$. We represent this as a direct graph: let the set of integers be represented by vertices and there is a direct edge from a to b if and only if $b = T(a)$. Then an element $x \in B(d)$ if and only if x is in a cycle of length d. Since the number of elements in a given cycle of length d is a multiple of d, the number of elements in a cycle of length d is a multiple of d. Therefore, d divides $|B(d)|$.

   Let $m = \deg P$ with $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$. Since P is not a constant, $m \geq 1$. Let p be any prime number. Note that $P(1) = |A(1)| = |B(1)|$ and $P(p) = |A(p)| = |B(p)| + |B(1)| = |B(p)| + |A(1)| = |B(p)| + P(1)$. Since p divides $|B(p)|$, p divides $P(p) - P(1)$. Hence,

   $$p \text{ divides } a_m(p^m - 1) + \cdots + a_1(p - 1).$$

   Therefore, $p | a_m + a_{m-1} + \cdots + a_1$. Since this is true for all primes p,

   $$a_m + a_{m-1} + \cdots + a_1 = 0 \tag{1}$$

Finally, let p, q be any two distinct primes. Then $P(pq) = |A(pq)| = |B(pq)|+|B(p)|+|B(q)|+|B(1)| = |B(pq)|+(|A(p)|-B(1))+(|A(q)|-B(1))+|A(1)| = |B(pq)|+|A(p)|+|A(q)|-|A(1)| = |B(pq)| + P(p) + P(q) - P(1)$. Therefore, $|B(pq)| = P(pq) - P(p) - P(q) + P(1)$. Since pq divides $|B(pq)|$, pq divides $P(pq) - P(p) - P(q) + P(1)$. Therefore,

$$\begin{array}{ccl} pq & \text{divides} & a_m((pq)^m - p^m - q^m + 1) + \cdots + a_1(pq - p - q + 1) \\ & = & a_m(p^m - 1)(q^m - 1) + \cdots + a_1(p-1)(q-1). \end{array}$$

In particular, p divides this expression. Taking this equation modulo p and by (1), we have

$$-a_m(q^m - 1) - \cdots - a_1(q-1) \equiv -(a_m q^m + \cdots + a_1 q) \equiv 0 \pmod{p}.$$

Therefore, every prime q is a root of the equation $a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x \pmod{p}$. We will show that this is impossible. Choose p such that $p > m+1$. By Dirchlet's theorem for prime, for each $i \in \{1, \cdots m+1\}$, there exists a prime q_i such that $q_i \equiv i \pmod{p}$. Since $p > m+1$, $q_1, \cdots, q_{m+1}$ are pairwise distinct modulo p. Then $q_1, \cdots, q_{m+1}$ are pairwise distinct roots of the polynomial $a_m x^m + a_{m-1}x^{m-1} + \cdots + a_x \pmod{p}$. But this polynomial contains at most m roots. This is a contradiction.

Therefore, no such function T exists. $\square$

3. On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

   How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

   Solution: The answer is $4 \cdot ((499)^2 - 1)$.

   Colour the board with four colours A, B, C, D according to the following rules: Given a square in the i^{th} row and j^{th} column, colour the square A if i, j are both odd, B if i is odd and j is even, C if i is even and j is odd and D if i, j are both even. Then a limp rook encounters the same colour exactly every four steps. More specifically, D appears once every four steps.

   Since there are only 499^2 squares coloured D, a limp rook cycle contains at most $4(499^2)$. We claim that a limp rook cycle cannot go through every square coloured D. This will prove that a limp rook cycle passes through at most $4(499^2 - 1)$ squares. We will then construct such a cycle to prove that this length is indeed the maximum.

Colour the squares marked D in a "chessboard" manner in black and white. Suppose that a limp rook cycle passes through every square marked D. Note that two consecutive squares of a limp rook cycle that are coloured D, are "neighbours" of each other (i.e. distance 2 horizontally and/or vertically of each other.) Since there are an odd number of squares marked D, there are two consecutive squares in the limp rook cycle coloured the same colour, marked D. By symmetry, we may assume that these two squares are $(a, b), (a+2, b+2)$, where (i, j) is the square in the i^{th} row and the j^{th} column, for some positive integers a, b. Without loss of generality, suppose (a, b) is going toward $(a+2, b+2)$. Also, assume that the path goes

$$(a, b) \to (a, b+1) \to (a+1, b+1) \to (a+1, b+2) \to (a+2, b+2).$$

It is easy to show that by parity, the limp rook cycle enters any square coloured D horizontally and exits any square coloured D vertically. Consider the square $(a, b+2)$. This square is coloured D. Then the limp rook cycle enters this square horizontally and exits this square vertically. Therefore, the limp rook cycle goes from $(a-1, b+2) \to (a, b+2) \to (a, b+3)$. By the direction of the limp rook cycle, the limp rook goes from $(a, b+3)$ to (a, b) and from $(a+2, b+2)$ to $(a-1, b+2)$. Let P_1, P_2 be these two paths of the limp-rook cycle, respectively. Note that P_1, P_2 does not go through the interior of the quadrilateral whose vertices are $(a, b), (a+2, b+2), (a, b+3), (a-1, b+2)$. Therefore, P_1 and P_2 must intersect. Hence, the limp rook cycle must intersect. This is a contradiction.

Hence, a limp rook cycle contains at most $4(499^2 - 1)$ squares. And now the construction.

2010 Mock 2
Time: 4.5 Hours

1. A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

   (a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), \cdots, P(50)$ are balanced.

   Solution: For each $n \in \mathbb{N}$, let $f : \mathbb{N} \to \mathbb{Z}$ be a function such that $f(n) = -1$ if n has an odd number of prime divisors and $f(n) = 1$ if n has an even number of prime divisors. Then n is balanced if and only if $f(n) = 1$. Note that f is multiplicative. Consider the $2^{50} + 1$ sequences $(km + 1, \cdots, km + 50)$ for $k = 1, 2, \cdots, 2^{50} + 1$. Since there are 2^{50} possible sequences containing ± 1, two of the $2^{50} + 1$ sequences are the same. Hence $(f(um + 1), \cdots, f(um + 50)) = (f(vm + 1), \cdots, f(vm + 50))$ for some distinct $u, v \in \{1, \cdots, 2^{50} + 1\}$. Hence, $f((um + i)(vm + i)) = 1$, which implies that $(um + i)(vm + i)$ is balanced for each $i \in \{1, \cdots, 50\}$. Setting $a = um, b = vm$ yields $P(1), \cdots, P(50)$ each having an even number of prime divisors. $\square$

   (b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

   Solution: Suppose $a \neq b$. WLOG, suppose $a < b$. Then for any $n > b$, $P(n - a) = n(n + b - a)$ is balanced. Therefore, $f(n) = f(n + (b - a))$. Then the sequence $f(1), f(2), \cdots$ is eventually periodic with period d for some $d \mid b - a$. Then for any prime p, $f(dp) = -f(dp^2)$, since dp^2 has exactly one more prime divisor (namely p) than dp. Choose a sufficiently large p such that $f(dp)$ is part of the sequence $f(1), f(2), \cdots$ for which the sequence is periodic. Since dp^2 and dp differs by a multiple of d, $f(dp) = f(dp^2)$. This is a contradiction. Therefore, $a = b$. $\square$

2. In a triangle ABC with $AB \neq AC$, the incircle touches the sides BC, CA, AB at D, E, F, respectively. Line AD meets the incircle again at P. The line EF and the line through P perpendicular to AD meet at Q. Line AQ intersects DE at X and DF at Y. Prove that A is the midpoint of XY.

   Solution 1: Let I be the incentre of $\triangle ABC$. Let AI intersect EF at M. Then clearly M is the midpoint of EF and $\angle AMQ = 90^o$. Since $\angle APQ = 90^o$, $APMQ$ is cyclic. Furthermore, by Power of a Point, note that $AP \cdot AD = AF^2 = AM \cdot AI$ (since $\triangle AFM \sim \triangle AIF$.) Hence, $PMID$ is also cyclic.

   Then, $\angle AQP = \angle AMP = \angle ADI$, and so $\angle QAD = 90^o - \angle AQP = \angle BDI - \angle ADI = \angle BDA$, and so $BC || AQ$. Therefore, $\angle AXE = \angle EDC = \angle CED = \angle AEX$. Therefore, $AE = AX$. Similarly, $AF = AY$. Since $AE = AF$, $AX = AY$, as desired. $\square$

Solution 2: We do the same as Solution 1 to prove that $AQ||BC$.

Since $FPED$ is cyclic, $FPED$ is harmonic. Therefore, the pencil $D(D,F,P,E)$ is harmonic. Intersecting it with line AQ we get a harmonic bundle $(S,A;Y,X)$, where S is some point on XY. Since we want to show A is the midpoint of XY, it suffices to show S is a point at infinity, i.e. that $AQ||BC$. This completes the solution. $\square$

3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(x+y)) = f(yf(x)) + x^2,$$

for all $x,y\in\mathbb{R}$.

Solution: The answers are $f(x)=x, \forall x\in\mathbb{R}$ and $f(x)=-x, \forall x\in\mathbb{R}$.

Substituting $x=0$ into the original equation yields $f(0)=f(yf(0)), \forall y\in\mathbb{R}$. Suppose $f(0)\neq 0$. Then $yf(0)$ ranges over all reals as y varies over all reals. Hence, $f(0)=f(z)$ for all $z\in\mathbb{R}$. Consequently, f is a constant. If $f(x)=c$, a constant, then substituting this into the original equation yields $c=c+x^2$, or $x^2=0$ for all $x\in\mathbb{R}$. This is clearly absurd since $x=1$ violates this condition. Therefore,

$$f(0)=0. \tag{2}$$

Suppose $f(z)=0$. We will prove that $z=0$. Substituting $x=z, y=-z$ into the original equation yields $f(zf(0))=f(-zf(0))+z^2=z^2$. Since $f(0)=0$ by (2), $0=0+z^2$. Therefore, $z=0$. We have

$$f(z)=0 \Leftrightarrow z=0, \forall z\in\mathbb{R} \tag{3}$$

Since $f(0)=0$, substituting $y=0$ into the original equation yields

$$f(xf(x))=x^2, \forall x\in\mathbb{R} \tag{4}$$

Substituting $y=-x$ yields

$$f(-xf(x))=-x^2, \forall x\in\mathbb{R} \tag{5}$$

Lemma 1: f is surjective.

Proof of Lemma 1: This follows from both (4) and (5). *End Proof of Lemma 1*

Lemma 2: f is injective.

Proof of Lemma 2: Suppose $f(z) = f(z+r)$ for some $z, r \in \mathbb{R}$. Substituting $x = z, y = r$ into the original equation yields $f(zf(z+r)) = f(rf(z)) + z^2$. Since $f(z) = f(z+r)$, $f(zf(z)) = f(rf(z)) + z^2$. By (4), $f(zf(z)) = z^2$. Therefore, $f(rf(z)) = 0$. By (3), $rf(z) = 0$. Therefore, $f(z) = 0$ or $r = 0$. If $f(z) = 0$, then $z = 0$, Hence, $f(0) = f(0+r)$. By (3), $r = 0$. Hence, in either case, $r = 0$. Therefore, f is injective. *End Proof of Lemma 2*

Lemma 3: $f(-x) = -f(x), \forall x \in \mathbb{R}$.

Proof of Lemma 3: Let $z \in \mathbb{R}$. If $z = 0$, then clearly, $f(-z) = -zf(z)$, since both are equal to 0. Otherwise, suppose $z \neq 0$. Substituting $x = -z$ yields $f(-zf(-z+y)) = f(yf(-z)) + z^2$. Since $z \neq 0$, $f(-z) \neq 0$. Therefore, $yf(-z)$ ranges over all reals as y varies over all reals. Since f is surjective, we can let $w \in \mathbb{R}$ such that $f(wf(-z)) = -z^2$. Then $f(-zf(-z+w)) = 0$. Hence, $-zf(-z+w) = 0$. Since $z \neq 0$, $f(-z+w) = 0$. Hence, $w = z$. Therefore, by the definition of w, $f(zf(-z)) = -z^2$. But we also know that $f(-zf(z)) = -z^2$ by (5). Since f is injective, $zf(-z) = -zf(z)$. Since $z \neq 0$, $f(-z) = -f(z)$, as desired. $\square$

Since f is surjective, there exist $z \in \mathbb{R}$ such that $f(z) = 1$. Then substituting $x = z$ into (4) yields $f(z) = z^2$. Since $f(z) = 1$, $z = \pm 1$. Therefore, by Lemma 3, either $(f(1), f(-1)) = (1, -1)$ or $(f(-1), f(1)) = (-1, 1)$.

If $f(1) = 1$ and $f(-1) = -1$, then substituting $x = 1$ yields

$$f(f(y+1)) = f(y) + 1, \forall y \in \mathbb{R} \tag{6}$$

and substituting $x = -1$ yields $f(-f(y-1)) = f(-y) + 1$, or by Lemma 3,

$$-f(f(y-1)) = -f(y) + 1, \forall y \in \mathbb{R}. \tag{7}$$

Therefore, $-f(f(y+1)) = -f(y+2) + 1, \forall y \in \mathbb{R}$. Adding this equation with (6) yields $0 = f(y) - f(y+2) + 2$. Hence, $f(y+2) - f(y) = 2$. Similarly, $f(y+4) - f(y+2) = 2$. Therefore,

$$f(y+4) - f(y) = 4, \forall y \in \mathbb{R}. \tag{8}$$

Substituting $x = 2$ into the original equation fields $f(2f(y+2)) = f(yf(2)) + 4$. By (8), we have $f(yf(2)+4) = f(yf(2)) + 4$. Therefore, $f(2f(y+2)) = f(yf(2)+4)$. Since f is injective, $2f(y+2) = yf(2) + 4$. Hence, $f(y) = \frac{f(2)}{2}(y-2) + 2$. Let $c = \frac{f(2)}{2}$. Then $f(y) = cy + (2-2c)$. Since $y = 0$ and $f(0) = 0$, $c = 1$. Then $f(y) = y, \forall y \in \mathbb{R}$, which can be verified as a solution to the functional equation.

If $f(1) = -1$ and $f(-1) = 1$, then substituting $x = 1$ yields

$$f(f(y+1)) = f(-y) + 1 = -f(y) + 1, \forall y \in \mathbb{R} \tag{9}$$

and substituting $x = -1$ yields $f(-f(y-1)) = f(y) + 1$, or by Lemma 3,

$$-f(f(y-1)) = f(y) + 1, \forall y \in \mathbb{R}. \tag{10}$$

Therefore, $-f(f(y+1)) = f(y+2) + 1, \forall y \in \mathbb{R}$. Adding this equation with (8) yields $0 = -f(y) + f(y+2) + 2$. Hence, $f(y) - f(y+2) = 2$. Similarly, $f(y+2) - f(y+4) = 2$. Therefore,

$$f(y) - f(y+4) = 4, \forall y \in \mathbb{R}. \tag{11}$$

Substituting $x = 2$ into the original equation fields $f(2f(y+2)) = f(yf(2)) + 4$. By (8), we have $f(yf(2)-4) = f(yf(2))+4$. Therefore, $f(2f(y+2)) = f(yf(2)-4)$. Since f is injective, $2f(y+2) = yf(2) - 4$. Hence, $f(y) = \frac{f(2)}{2}(y-2) - 2$. Let $c = \frac{f(2)}{2}$. Then $f(y) = cy - (2+2c)$. Since $y = 0$ and $f(0) = 0$, $c = -1$. Then $f(y) = -y, \forall y \in \mathbb{R}$, which can be verified as a solution to the functional equation. This completes the problem. $\square$

2010 Mock 3
Time: 4.5 Hours

1. For any $n \geq 2$, let $N(n)$ be the maximal number of triples $(a_i, b_i, c_i), i = 1, \cdots, N(n)$ consisting of nonnegative integers a_i, b_i, c_i such that the following two conditions are satisfied:

   - $a_i + b_i + c_i = n$ for all $i = 1, \cdots, N(n)$
   - If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j, c_i \neq c_j$.

   Determine $N(n)$ for $n \geq 2$.

   Solution: The answer is $N(n) = \lfloor 2n/3 \rfloor + 1$. For brevity, let $N = N(n)$.

   Note that $\sum_{i=1}^{N} a_i, \sum_{i=1}^{N} b_i, \sum_{i=1}^{N} c_i \geq N(N-1)/2$. Therefore,

$$\frac{3N(N-1)}{2} \leq \sum_{i=1}^{n}(a_i + b_i + c_i) = Nn.$$

   Therefore, $N \leq \lfloor 2n/3 \rfloor + 1$.

   It remains to construct such triples for each n. We will split this into 3 cases; $n \equiv 0 \pmod 3, n \equiv 1 \pmod 3, n \equiv 2 \pmod 3$.

   If $n \equiv 0 \pmod 3$, let $n = 3k$ for some positive integer k. Then $N(n) = 2k+1$. The following is a construction of triples satisfying the conditions given in the problem.

a_i	b_i	c_i
$2k$	0	k
$2k-1$	2	$k-1$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	0
$k-1$	1	$2k$
$k-2$	3	$2k-1$
$\vdots$	$\vdots$	$\vdots$
0	$2k-1$	$k+1$

   If $n \equiv 1 \pmod 3$, let $n = 3k+1$ for some positive integer k. Then $N(n) = 2k+1$. The following is a construction of triples satisfying the conditions given in the problem.

a_i	b_i	c_i
$2k$	0	$k+1$
$2k-1$	2	k
$\vdots$	$\vdots$	$\vdots$
k	$2k$	1
$k-1$	1	$2k+1$
$k-2$	3	$2k$
$\vdots$	$\vdots$	$\vdots$
0	$2k-1$	$k+2$

If $n \equiv 2 \pmod 3$, let $n = 3k+2$ for some positive integer k. Then $N(n) = 2k+2$. The following is a construction of triples satisfying the conditions given in the problem.

a_i	b_i	c_i
$2k+1$	0	$k+1$
$2k$	2	k
$\vdots$	$\vdots$	$\vdots$
k	$2k+2$	0
$k-1$	1	$2k+2$
$k-2$	3	$2k+1$
$\vdots$	$\vdots$	$\vdots$
0	$2k-1$	$k+3$

This completes the problem. $\square$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that there exist $x, y \in \mathbb{R}$ such that

$$f(x - f(y)) > yf(x) + x.$$

Solution: Suppose no such $x, y \in \mathbb{R}$ exist. Then

$$f(x - f(y)) \leq yf(x) + x, \forall x, y \in \mathbb{R}. \tag{12}$$

Substituting $x \leftarrow x + f(0)$ and $y = 0$ into (12) yields

$$f(x) \leq x + f(0). \tag{13}$$

Substituting $x = f(y)$ into (12) and using (13) yields $f(0) \leq yf(f(y)) + f(y) \leq yf(f(y)) + y + f(0)$. Therefore,

$$f(f(y)) \geq -1, \forall y > 0. \tag{14}$$

Therefore, for all $x, y \in \mathbb{R}$ such that $x > f(y)$, we have by (14), (13) and (12),

$$-1 \leq f(f(x - f(y))) \leq f(x - f(y)) + f(0) \leq yf(x) + x + f(0). \tag{15}$$

Suppose there exists $x \in \mathbb{R}$ such that $f(x) > 0$. By (13), we have that $f(y)$ approaches $-\infty$ as y approaches $-\infty$. Then we can choose a sufficiently small y such that $f(y) < x$ and $yf(x) + x + f(0) < -1$. This contradicts (15).

Therefore, $f(x) \leq 0$ for all $x \in \mathbb{R}$. Suppose there exists a positive real x such that $f(x) < 0$. Then for all $y \in \mathbb{R}$, $f(y) \leq 0 < x$. Then by (15), we have $-1 \leq yf(x) + x + f(0)$. This is not possible if y is sufficiently large. Therefore, $f(x) = 0$ for all positive real x.

Finally, choose $y = 1$ and any $x < 0$. Since $f(y) = f(1) = 0$, by (12) $f(x) \leq f(x) + x$. Then $x \geq 0$. This contradicts the choice of x. This solves the problem. $\square$

3. Let P be a polygon that is convex and symmetric about some point O (i.e. if a point is on the polygon, then its reflection in O is also on the polygon). Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the areas of R and P, respectively.

Solution: Let A, B be two vertices of P such that $[OAB]$ is maximum, where $[XYZ]$ denotes the area of XYZ. Let A', B' be the reflections of A, B across O, respectively. Then A', B' are vertices of P, since P is symmetric about O. Let l_1, l_3 be the lines passing through A, A' respectively parallel to OB and l_2, l_4 the lines passing through B, B' respectively parallel to OA. Let P_1 be the parallelogram formed by l_1, l_2, l_3, l_4. Clearly, $P \subseteq P_1$, since $[OAB]$ has maximum area. Note that A, B, A', B' are midpoints of the sides of P_1. Therefore, $ABA'B'$ is a parallelogram. Let P_2 be this parallelogram. Then

$$|P_1| = 2|P_2|.$$

Note that $P_2 \subseteq P$. Let P_3 be the smallest parallelogram whose sides are parallel to that of P_2 such that $P \subseteq P_3$. Note that each side of P_3 contains a vertex of P. Let X be a point on the side of P_3 "closest" to AB which is on P. Let X' be the image of the reflection of X about O. Note that X' is on the side of P_3 closest to $A'B'$ which is on P. Let Y be a point on the side of P_3 closest to BA' which is on P. Let Y' be the image of the reflection of Y about O.

Let x be the ratio of the distance from X to AB to the distance from O to AB. Let y be the ratio of the distance from Y to BA' to the distance from O to BA'. Note that the polygon $AXBYA'X'B'Y'$ is contained in P, since P is convex. By symmetry, we have

$$[AXBYA'X'B'Y'] = 2[OAXB] + 2[OBYA'] = 2(1 + x)\frac{|P_2|}{4} + 2(1 + y)\frac{|P_2|}{4}$$

$$= (1+x)\frac{|P_2|}{2} + (1+y)\frac{|P_2|}{2} = \frac{|P_2|}{2}(2+x+y).$$

Hence,

$$\frac{|P_2|}{2}(2+x+y) \le |P|.$$

Also, note that

$$|P_3| = (1+x)(1+y)|P_2| \le \frac{2(1+x)(1+y)}{2+x+y}|P|$$

Therefore,

$$\frac{|P_2|}{|P|} \le \frac{2}{2+x+y} \Rightarrow \frac{|P_1|}{|P|} \le \frac{4}{2+x+y}$$

If $2+x+y \ge 2\sqrt{2}$, then $\frac{|P_1|}{|P|} \le \frac{4}{2+x+y} \le \sqrt{2}$. Therefore, P_1 satisfies the desired conditions. Otherwise, we may assume that $2+x+y < 2\sqrt{2}$. Let $a = 1+x, b = 1+y$. Therefore, $\frac{|P_3|}{|P|} \le \frac{2ab}{a+b} \le \frac{(a+b)^2}{2(a+b)} = \frac{a+b}{2} < \sqrt{2}$. Thus, P_3 satisfies the desired conditions. We are done. $\square$

2010 Mock 4
Time: 4.5 Hours

1. Find the largest possible integer k, such that the following statement is true:

   Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

   $$\begin{array}{rll} & b_1 \leq b_2 \leq \cdots \leq b_{2009} & \text{the lengths of the blue sides} \\ & r_1 \leq r_2 \leq \cdots \leq r_{2009} & \text{the lengths of the red sides} \\ \textit{and} & w_1 \leq w_2 \leq \cdots \leq w_{2009} & \text{the lengths of the white sides.} \end{array}$$

   Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

   Solution: The answer is $k = 1$.

   We first prove that $k > 0$. Suppose $k = 0$. Then WLOG, suppose $b_{2009} \leq r_{2009} \leq w_{2009}$. Since $k = 0$, these three numbers do not form the sides of a non-dengerate triangle. Therefore, $b_{2009} + r_{2009} \leq w_{2009}$. Since $b_i \leq b_{2009}$ and $r_j \leq r_{2009}$ for all $i, j \in \{1, \cdots, 2009\}$, $b_i + r_j \leq w_{2009}$ for all $i, j \in \{1, \cdots, 2009\}$. Therefore, w_{2009} cannot be the side length of any of the 2009 triangles. This is a contradiction.

   Therefore, $k \geq 1$. The following is an example for which $k = 1$. For $i \in \{1, \cdots, 2008\}$, let

   $$b_i = i, \quad r_i = 2i, \quad w_i = 3i.$$

   Let

   $$b_{2009} = 2009, \quad r_{2009} = 2008 \cdot 3, \quad w_{2009} = 2008 \cdot 3.$$

   Then for all $i \in \{2, \cdots, 2009\}$, it is easy to verify that (b_i, r_{i-1}, w_{i-1}) form the sides of a triangle, since $b_i = i, r_{i-1} = 2i - 2$ and $w_{i-1} = 3i - 3$ and $i + (2i - 2) > 3i - 3$, and $(b_1, r_{2009}, w_{2009})$ form the sides of a triangle, since $1, a, a$ are the sides of a triangle for any positive integer a. $\square$

2. Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

   Solution: WLOG, suppose A, B are closer to F than D, C, respectively. By tangency properties, it suffices to show that $\angle FEG = \angle GHE$.

   Note that $\Delta FCD \sim \Delta FAB$. Let r be the ratio of similarity of ΔFCD to ΔFAB. Let φ map the plane to itself by reflecting a point about the angle bisector of $\angle AFB$ and dilating the result by a ratio of r. We note the following properties of φ:

• Since reflection and dilation preserves the collinearity of any three points, φ preserves the collinearity of any three collinear points.

• $\varphi(A) = C, \varphi(B) = D$ and $\varphi(G) = H$ by the similarity of ΔFAB and ΔFCD.

Let $C' = \varphi(C)$ and $D' = \varphi(D)$. Then by the similarity, C' lies on line AD and $C'C||BD$. Similarly, D' lies on BC and $DD'||AC$. Let $E' = CC' \cap DD'$. Since φ preserves collinearity and (A, E, C) is a triple of collinear points, $(\varphi(A), \varphi(E), \varphi(C)) = (C, \varphi(E), C')$ is a triple of collinear points. Similarly, $(D, \varphi(E), D')$ is a triple of collinear points. Hence, $\varphi(E) = CC' \cap DD' = E'$. Note that $EDE'C$ is a parallelogram. Since H is the midpoint of CD, $EH = HE'$ and E, H, E' are collinear. Since φ preserves angles, $\angle FEG = \angle FE'H$. We want to show this angle is equal to $\angle GHE$ to solve the problem; i.e. $GH||FE'$.

Let $E'' = \varphi(E')$ and $H' = \varphi(H)$. Then by similarity, H' is the midpoint of $C'D'$. Using the same argument as before, we have $E'H' = H'E''$ and E', H', E'' are collinear. Therefore, $EE''||HH'$. Note that $\varphi(\varphi())$ is a dilation about F. Therefore, F, E, E'' are collinear. Therefore, $FE||HH'$. Since φ preserves parallel lines, $FE'||H'\varphi(H')$. But since $\varphi(\varphi())$ is a dilation, $GH||\varphi(\varphi(G))\varphi(\varphi(H)) = H'\varphi(H')$. Hence, $FE'||GH$, as desired. $\square$

3. There exists a sequence of positive integers $a_1, a_2, \cdots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$. Prove that $n \leq 4$.

Solution: Rewrite the equation as

$$(a_{k+1} + 1)(a_{k-1} + 1) = a_k^2 + 1.$$

Since $n \geq 5$,
(1) $(a_3 + 1)(a_1 + 1) = a_2^2 + 1$.
(2) $(a_4 + 1)(a_2 + 1) = a_3^2 + 1$.
(3) $(a_5 + 1)(a_3 + 1) = a_4^2 + 1$.

We claim that each of a_1, a_2, a_3, a_4, a_5 are even. If a_1 is odd, then by (1), a_2 is odd. Then by (2), a_3 is odd. Since a_1 and a_3 are both odd, in (1), $a_2^2 + 1$ is divisible by 4. This is impossible since no integer of the form $a^2 + 1$ is divisible by 4. Therefore, a_1 is even. If a_2 is odd, then similarly, a_3 and a_4 are odd. Sine a_2 and a_4 are both odd, in (2), $a_3^2 + 1$ is divisible by 4. This is again impossible. Therefore, a_2 is even. If a_3 is odd, then by (1), a_2 is odd and by (3), a_4 is odd. Again, $a_3^2 + 1$ is divisible by 4, which is a contradiction. If a_4 is odd, then by (2), a_3 is odd. By (1), a_2 is odd. Again, a_2, a_4 cannot both be odd. If a_5 is odd, then a_4 is odd, whose case has already been handled. Therefore, each of $a_1, \cdots, a_5$ are even.

By (1) and (2), $a_3 + 1 | a_2^2 + 1$ and $a_2 + 1 | a_3^2 + 1$. It suffices to show that there are no pairs of even positive integers x, y such that $x + 1 | y^2 + 1, y + 1 | x^2 + 1$. Suppoe on the contrary that there exist a pair of even numbers (x, y) such that $x + 1 | y^2 + 1$ and $y + 1 | x^2 + 1$. Without loss of generality, suppose x, y are chosen minimally and $x \geq y$. If $x = y$, then $x + 1 | x^2 + 1$, which implies that $x + 1 | 2 \Rightarrow x = 1$. This contradicts x being even. Therefore, $x > y$. The conditions on x, y implies that $x + 1 | (y^2 + 1) + (x^2 - 1) = x^2 + y^2$. Similarly, $y + 1 | x^2 + y^2$. I claim that $\gcd(x + 1, y + 1) = 1$. Let $d = \gcd(x + 1, y + 1)$. Then $x, y \equiv -1 \pmod{d}$. Hence, $x^2 + y^2 \equiv 2 \pmod{d}$. Since $d | x + 1$, $d | x^2 + y^2$. Hence, $x^2 + y^2 \equiv 0 \pmod{d}$. This implies that $0 \equiv 2 \pmod{d}$. Therefore, $d = 1$ or $d = 2$. But $d \neq 2$ since x is even and $d | x + 1$. Therefore, $d = 1$. Since $x + 1 | x^2 + y^2$ and $y + 1 | x^2 + y^2$ and $\gcd(x + 1, y + 1) = 1$, $(x + 1)(y + 1) | x^2 + y^2$. Let

$$\frac{x^2 + y^2}{(x + 1)(y + 1)} = k. \tag{16}$$

for some positive integer k. This can be rewritten as $x^2 - k(y + 1)x + (y^2 - k(y + 1)) = 0$. Let x' be the second root of $t^2 - k(y + 1)t + (y^2 - k(y + 1)) = 0$, other than x. Note that x' is an integer, since x is an integer and $x + x' = k(y + 1)$. Note that

$$k(y + 1) = \frac{x^2 + y^2}{x + 1} = x + \frac{y^2 - x}{x + 1}.$$

If $x < y^2$, then since $x + x' = k(y + 1) = x + \frac{y^2 - x}{x + 1}, x' > 0$. Since $xx' = (y^2 - k(y + 1))$ and $x > y$, $x' < y < x$. Hence, $x' < x$. We conclude that (x', y) is a smaller solution to (16), contradicting the minimality of (x, y). If $x > y^2$, then since $x + 1 | y^2 + 1$, $x + 1 \leq y^2 + 1$. Hence, $y^2 + 1 < x + 1 \leq y^2 + 1$. This is a contradiction. Finally, if $x = y^2$, then since $y + 1 | x^2 + 1$, $y + 1 | y^4 + 1$. Since $y + 1 | y^4 - 1$, $y + 1 | 2$. This implies that $y = 1$. This contradicts that y is even. Therefore, there are no even numbers x, y such that $x + 1 | y^2 + 1$ and $y + 1 | x^2 + 1$. Hence, $n \ngeq 5$. $\square$

2010 Mock 5
Time: 4.5 Hours

1. Let O be the centre of the excircle of ΔABC opposite A. Let M be the midpoint of AC, and P the intersection of MO and BC. Prove that if $\angle BAC = 2\angle ACB$, then $AB = BP$.

Solution 1: Let $\angle ACB = \theta$. Then $\angle BAC = 2\theta$ and $\angle ABC = 180 - 3\theta$. Note that O lies on the angle bisector of $\angle BAC$. Therefore, $\angle BAO = \theta$. Let D be the intersection of AO and BC. To prove that $BA = BP$, it suffices to show that $\angle BAP = 3\theta/2$, i.e. $\angle DAP = \angle PAC = \theta/2 \Leftrightarrow AP$ bisects $\angle DAC \Leftrightarrow AD/AC = DP/PC$. By Sine Law, we have

$$\frac{AD}{AC} = \frac{\sin\theta}{\sin 2\theta}.$$

Since M, P, O are collinear, by Menelaos Theorem on ΔADC,

$$\frac{AO}{OD} \cdot \frac{DP}{PC} \cdot \frac{CM}{MA} = 1.$$

Since $CM = MA$,

$$\frac{DP}{PC} = \frac{OD}{OA}.$$

Let r be the radius of the excircle of ΔABC opposite A. Note $\angle CDO = \angle BDA = 180 - \angle ABC - \angle BAO = 2\theta$. Let X, Y be the foot of the perpendicular from O on BC, AC, respectively. Since $OX = r$, $OD = \frac{r}{\sin\angle CDO} = \frac{r}{\sin 2\theta}$. Furthermore, $OA = \frac{r}{\sin\angle CAO} = \frac{r}{\sin\theta}$. Hence,

$$\frac{DP}{PC} = \frac{OA}{OD} = \frac{\sin\theta}{\sin 2\theta}.$$

By looking at triangle ADC, by Sine Law,

$$\frac{AD}{AC} = \frac{\sin\angle ACD}{\sin\angle ADC} = \frac{\sin\angle ACD}{\sin(180 - ADC)} = \frac{\sin\theta}{\sin 2\theta} = \frac{DP}{PC},$$

as desired. $\square$

Solution 2: Let AO and BC intersect at D. Since AO bisects $\angle BAC$ and $\angle BAC = 2\angle ACB$, $\angle BAD = \angle CAD = \angle ACD$. Therefore, $AD = DC$. Consider the triangles OAC and ODC. Since they have equal altitudes from O and the same altitude from C,

$$\frac{[OAC]}{[ODC]} = \frac{AC}{DC} = \frac{AO}{DO}.$$

Since M is the midpoint of AC, $[OAM] = [OCM]$ and $[PAM] = [PCM]$. Therefore, $[OAP] = [OCP]$. Then,

$$\frac{AC}{AD} = \frac{AC}{DC} = \frac{AO}{DO} = \frac{[OAP]}{[ODP]} = \frac{[OCP]}{[ODP]} = \frac{CP}{DP}.$$

Therefore, AP is the bisector of $\angle DAC$. It follows that $\angle BAP = \angle BAD + \angle DAP = \angle ACP + \angle PAC = \angle APB$. Therefore, $AB = BP$, as desired. $\square$

2. Let f be a non-constant function from the set of positive integers into the set of positive integers such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Solution: Suppose the conclusion of the problem statement is false. Then there exist a finitely number of primes $p_1, \cdots, p_t$ such that the prime divisors of $f(n)$ is a subset of $\{p_1, \cdots, p_t\}$ for every $n \in \mathbb{N}$

First, I claim that there exists a positive integer N such that $m \geq N$ implies $f(m) \neq f(1)$. Suppose the contrary; i.e. there are infinitely many positive integers m such that $f(m) = f(1)$. Let k be the smallest positive integer such that $f(k) \neq f(1)$. Such a positive integer exist since f is a non-constant function. Let $d = |f(k) - f(1)|$. If $f(m) = f(1)$ for some $m > k + d$. Then $m - k \mid |f(m) - f(k)| = |f(1) - f(k)| = d \neq 0$ and $m - k > d$. This is a contradiction.

Let $f(1) = p_1^{a_1} \cdots p_t^{a_t}$ for some non-negative integers $a_1, \cdots, a_t$. Note that $p_1^m p_2^m \cdots p_t^m$ divides $f(p_1^m \cdots p_t^m + 1) - f(1)$. Consider this statement when $m > \max\{a_1, \cdots, a_t\}$. Then $p_i^{a_i} || f(p_1^m \cdots p_t^m + 1)$ for all such m. This implies that $f(p_1^m \cdots p_t^m + 1) = p_1^{a_1} \cdots p_t^{a_t} = f(1)$, since the only possible prime divisors of $f(p_1^m \cdots p_t^m + 1)$. But by our claim, $f(p_1^m \cdots p_t^m + 1) > f(1)$ when m gets sufficiently large. This is a contradiction.

Hence, there are indeed infinitely many primes p such that p divides $f(c)$ for some positive integer c. $\square$

3. A multiple choice contest with n questions was written by K students. The jury assigns the difficulty to each question - a positive integer which is awarded to each student who solves the question. If a student does not solve the question, the student gets 0 points. The student's score is the sum of the scores received for each question. It turns out that when the answer sheets are submitted, the jury can assign the difficulty of each question in a way so that the ranks of the students are in any pre-determined order. What is the maximum possible value of K?

Solution: The maximum value of K is n.

Label the students $S_1, \cdots, S_n$. An example of n students where the rank of the students can be pre-determined is by having each student solve a unique question Q_i. If the ranking order is $S_1, \cdots, S_n$, then assign problem Q_i a score of $n + 1 - i$ for each $i \in \{1, \cdots, n\}$.

Assume the result holds for $K \geq n + 1$. Take $n + 1$ different students, and clone each of them infinitely many times so that we get $n + 1$ types of students. Let us show that if we can find two non-identical teams with finitely many students but with the same results, i.e. for every question, the number of students on each team who solved the question is the same, we arrive at a contradiction.

We can assume every type of students is present on at most one team (otherwise keep removing 1 student of the same type from both teams until this is no longer possible). Without loss of generality, the first team has at least as many students as the second. The sum of the scores of each of the two teams are the same. Then it is impossible to assign the difficulty of questions so that each student on the first team is ranked higher than each student on the second team.

It remains to find the two teams. We will suppose that it is possible and construct a system of n linear equations on $n+1$ variables. The non-zero solution of this system will correspond to the assignment of the teams.

For $j = 1, 2, \cdots, n+1$, if the contestants that solved problem j are on team 1, then let x_j be the number of such contestants. If the contestants that solved problem j are on team 2, then let x_j be the negative number of such contestants.

For $i = 1, 2, \cdots, n$ let equation i state that the difference between the number of contestants on the two teams who solved question i is 0; this is an equation involving only variables $x_1, x_2, \cdots, x_{n+1}$. We get a homogenous system of n equations and $n+1$ unknowns. It has a non-zero solution; it must be rational since all the coefficients in the equation are rational numbers (namely, 0 or 1). Multiplying it by an appropriate constant still gives a solution which contains only integers. This gives us the two teams.