Combinatorics

C1 The answer is no. James places his rooks on the white squares on the top row. Sarah must place her rook somewhere beneath the rooks, else James will win on the first turn. On each turn, all of James's rooks will descend one row, and will go to all of the black squares if Sarah's rook is on a black square, and all of the white squares otherwise. By doing this, on Sarah's turn she can move horizontally, but cannot move vertically past the line of rooks, as it is blocked by a rook in the same column. In particular, the rooks will slowly march Sarah's rook until they are on the second last row and Sarah's rook is on the last row. On the next turn, James captures Sarah's rook, and claims victory.

C2 We first prove the following lemma.

Lemma. Let G be a bipartite graph between sides A and B where $|A| = |B| = M$ and such that each vertex is adjacent to more than $M/2$ vertices on the other side. Then G has a perfect matching.

Proof. Let H be the largest matching between A and B and assume for contradiction that $|H| < M$. Suppose that $u \in A$ and $v \in B$ are not matched in H. If (u, v) is an edge, then adding (u, v) to A makes it larger, which is impossible. Therefore all of the neighbors of each of u and v are already matched in H. This implies u is neighbors with more than half of the vertices of B matched in H and v is neighbors with more than half of the vertices of A matched in H. Therefore there is an edge $(x, y) \in H$ such that (u, x) and (v, y) are edges. Removing (x, y) from H and adding (u, x) and (v, y) to H again increases the size of A, which yields a contradiction and completes the proof. □

Let C_1, C_2 and C_3 be the sets of students in the three classes. By the lemma, there is a matching H of friends between C_1 and C_2. Now consider a student $u \in C_3$. There are fewer than $M/4$ students in each of C_1 and C_2 friends with u and thus u must friends with both members of more than $M/2$ of the pairs in H. Applying the lemma again matches each student of C_3 with a pair in H, which yields the desired teams.

C3 Let $g = \gcd(n + 1, m + 1)$, and suppose we have a labeling with the given properties.

We will further assign each ferry one of g colours according to its label value modulo g. For each colour i:

- Let n_i be the total number of ferries with colour i.
- Let a_i be the number of (city, ferry) pairs for which the ferry has colour i, the city is on a river bank, and ferry goes to the city, and the city is on a river bank.
- Let b_i be the number of ferries that have colour i and go the island city.

Note that $a_i + b_i$ counts each ferry exactly twice (once for each city it goes to), so $n_i = \frac{a_i+b_i}{2}$ for each i. In particular $a_i + b_i$ is even for all i.

Consider a city on the left bank. Because the ferries going to that city are all labeled with consecutive integers, it must be that there are exactly $\frac{n+1}{g}$ ferries of each colour going to the city. Similarly, there are exactly $\frac{m+1}{g}$ ferries of each colour going to each city on the right bank. Therefore a_i is the same for every colour. Let A denote this common value.

Similarly, the ferries going to the island city are labeled with consecutive integers, so $|b_i - b_j| \leq 1$ for all i, j. But we know $A + b_i$ is even for all i, which is impossible if $|b_i - b_j| = 1$. Therefore, b_i and hence n_i are also constant for all i. Let N denote this common value of n_i.

Finally, consider the set of ferries traveling between the two coasts. There should be exactly nm such ferries. However, the total number of ferries of each colour is exactly $n_i - a_i = N - A$, which is constant. Therefore, $nm \equiv 0 \mod g$. However, we also know $nm \equiv (-1)^2 \equiv 1 \mod g$, which is a contradiction.

Number Theory

N1 There does not exist a solution. Assuming otherwise, by dividing through by powers of 2 we can assume that $\gcd(a, b, c, d) = 1$. Since $x^4 \equiv 0, 1 \pmod{16}$ for all integers x, the only way for $a^4 + 2b^4 + 4c^4 + 8d^4 \equiv 0 \pmod{16}$ is for a, b, c, d to all be even, contradiction. Thus no solutions in the positive integers exists.

N2 There are not. Assume otherwise, and let $p_1 < p_2 < \cdots < p_7$ be the odd primes. Then we have that

$$p_3p_4p_5p_6p_7 \mid p_2^8 - p_1^8 = (p_2^4 + p_1^4)(p_2^2 + p_1^2)(p_2 + p_1)(p_2 - p_1).$$

Since $0 < p_2 - p_1 < p_2 < p_n$ for $n \geq 3$, we have $p_n \nmid p_2 - p_1$ for all $n \geq 3$. Also, the primes are odd, whence $2 \mid p_2^i + p_1^i$ for all i and we can divide by powers of 2 on the right. In particular, we have that

$$p_3p_4p_5p_6p_7 \mid \frac{p_2^4 + p_1^4}{2} \frac{p_2^2 + p_1^2}{2} \frac{p_2 + p_1}{2}.$$

If four of the $p_3, \ldots, p_7$ divide $\frac{p_2^4+p_1^4}{2}$, then we have

$$p_2^4 > \frac{p_2^4 + p_1^4}{2} \geq p_3p_4p_5p_6 > p_3^4,$$

contradiction. Similarly, we cannot have two of the primes dividing $\frac{p_2^2+p_1^2}{2}$, and we cannot have one of the primes dividing $\frac{p_2+p_1}{2}$. Therefore the number of primes among $p_3, \ldots, p_7$ dividing the right hand side is at most $3 + 1 + 0 = 4$, which is not enough as we have 5 of them. Thus there is no solution.

N3 Pick a large prime p and set $a_{(p-1)t+k} = (kp + 1)p^{2t+1}$ for each $0 \leq k \leq p - 2$ and $t \geq 0$. Consider a finite set S of positive integers. Let $t = \lfloor \frac{\min S}{p-1} \rfloor$ and let m be the number of elements of S between $t(p - 1)$ and $t(p - 1) - 1$ where $m \leq p - 1$. It follows by construction that

$$\sum_{n \in S} a_n \equiv mp^{2t+1} \pmod{p^{2t+2}}$$

and thus p^{2t+1} is the largest power of p dividing $\sum_{n \in S} a_n$, which implies that it is not a perfect square. Fix some n and let $t = \lfloor \frac{n}{p-1} \rfloor$. Note that

$$a_n < a_{(p-1)(t+1)} < p^{2t+2} < p^{\frac{2n}{p-1}+2} < p^2 \cdot 1.01^n$$

as long as $p^{\frac{2}{p-1}} < 1.01$, which is true for sufficiently large p since $\frac{2\log p}{p-1} \to 0$ as $p \to \infty$.

Algebra

A1 The only solution is $f(x) = c$, a constant. Take $y = z = 0$ to get $f(0) \geq f(x)$ for all x. Next, let $(x, y, z) = (-3z, 0, z)$, and we get

$$3f(0) \geq f(-3z) + f(z) + f(-2z) \geq 3f(0).$$

Thus equality holds in the inequalities used, and in particular $f(0) = f(z)$. But z was arbitrary, whence f is a constant function (which does satisfy the requirements).

A2 We claim that the solutions are $f(x) = 0$ and $f(x) = x^2 + c$ for a constant c (which do satisfy the equation). Define $g(x) = f(x) - x^2$, and then the given equation translates to

$$g(y) = f(y) - y^2 = f(y + f(x)) - f(x)^2 - 2yf(x) - y^2 = g(y + f(x)) = g(y + x^2 + g(x)),$$

for all $x, y \in \mathbb{R}$. If $g(x) = -x^2$ for all x, then we get $f(x) = 0$ for all x. Otherwise, there exists an x such that $k = g(x) + x^2$ is non-zero. Thus we have $g(y) = g(y + k)$ for all y. Therefore, for all x, y,

$$g(y + x^2 + g(x)) = g(y) = g(y + (x + k)^2 + g(x + k)) = g(y + x^2 + g(x) + 2kx + k^2).$$

Let $y = -x^2 - g(x)$, and then $g(0) = g(2kx + k^2)$ for all x. Since $k \neq 0$, as x varies over $\mathbb{R}$, $2kx + k^2$ varies over all of $\mathbb{R}$, whence g is a constant. This translates to $f(x) = x^2 + c$, as desired.

A3 Letting $m = n+1$, we have $|a_n - a_{n+1}| < 100$. Applying this inequality repeatedly, we obtain $|n - m| > \frac{|a_n - a_m|}{100}$ for all positive integers n, m. (*) Also note that no number occurs twice in the sequence, or we would have $\frac{|a_n - a_m|}{|a_n - a_m|} = 0 < \frac{1}{100}$.

First suppose $a_n \geq n + 5000$ for some n. Since every positive integer occurs somewhere in the sequence and since $1, 2, \ldots, n$ cannot all appear in $(a_1, a_2, \ldots, a_{n-1})$, it follows that there exists some $m > n$ for which $a_m \leq n$. Choose u such that $u \leq m$ and a_u is as large as possible. Then $a_u \geq n + 5000$ since we can always choose $u = n$. Since the sequence covers every positive integer, we know $a_u + 1 = a_v$ for some v, and since a_u was chosen to be maximal for $u \leq n$, we must also have $v > m$. Therefore, we have found u, v with the following properties:

(a) $a_v - 1 = a_u \geq a_m + 5000$.

(b) $u \leq m \leq v$.

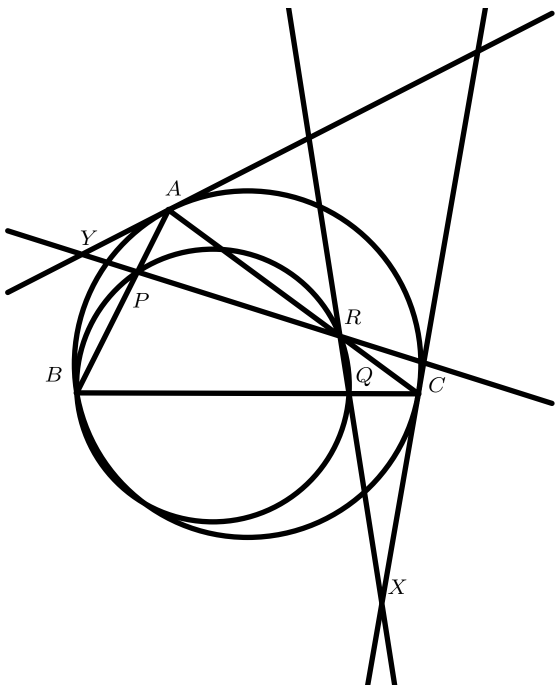
From (a) and (*), we know that $v - m > 50$ and $m - u > 50$, so $v - u > 100$. Therefore, $\frac{|a_u - a_v|}{|u - v|} = \frac{1}{|a_u - a_v|} < \frac{1}{100}$, which is a contradiction.

Next suppose $a_n \leq n - 5000$ for some n. As above, choose u such that $u \leq n$ and a_u is as large as possible. Since $a_1, a_2, \ldots, a_n$ are all distinct, one of them must be at least n, and hence $a_u \geq n$. Again, the sequence covers every positive integer, so we know $a_u + 1 = a_v$ for some v, and since a_u was chosen to be maximal for $u \leq n$, we must also have $v > n$. This leads to the same contradiction as above.

Therefore, we conclude $n - 5000 < a_n < n + 5000$ for all n.

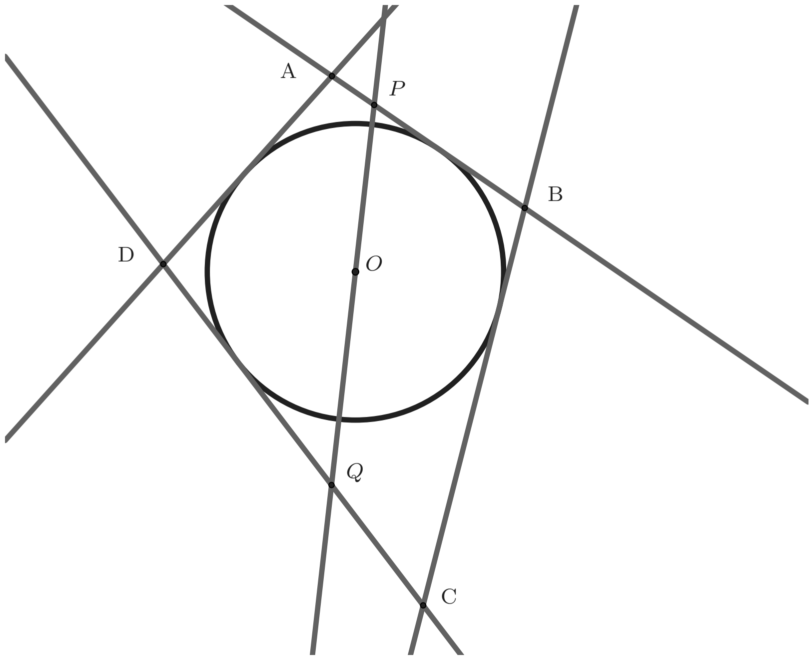
Geometry

G1



We have $\angle YAP = \angle YAB = \angle ACB = \angle RCQ$ as AY is tangent to the circumcircle of ABC. Also, $\angle APY = \angle RPB = 180° - \angle RQB = \angle RQC$, whence $\Delta YAP \sim \Delta RQC$. But $AP = CQ$, whence $\Delta YAP \simeq \Delta RQC$, and so $PY = RQ$. Similarly, we can show that $\Delta PAR \simeq \Delta QCX$, whence $PR = QX$. Adding these together gives $RY = RP + PY = QX + RQ = RX$.

G2

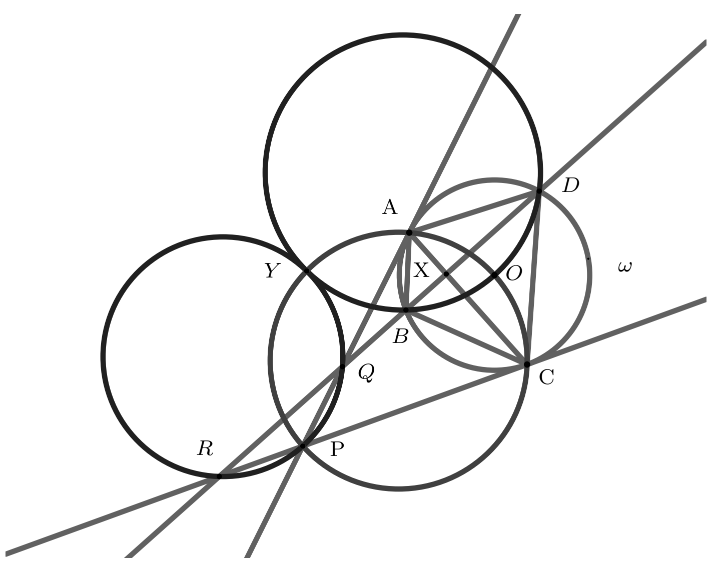


Without loss of generality, assume ℓ intersects AB at P and CD at Q. Let r be the radius of the incircle of $ABCD$. Since $APQD$ and $BPQC$ have the same perimeter, $AP+AD+QD = BP+BC+CQ$ and it follows that

$$[OPADQ] = \frac{r}{2}(AP + AD + QD) = \frac{r}{2}(BP + BC + CQ) = [OPBCQ]$$

However, since $[APQD] = [BPQC]$ it follows that the difference between the areas of $[OPADQ]$ and $[OPBCQ]$ is $2[OPQ]$ and thus $[OPQ] = 0$. This implies that O, P and Q are collinear.

G3



Let the tangents to ω at A and C intersect at P and intersect BD at Q and R. Let AC and BD intersect at X. Now let Y be the second intersection of line OX with the circumcircle of AOC. Since $YAOC$ and $ABCD$ are cyclic, power of a point implies that $OX \cdot XY = AX \cdot XC = BX \cdot XD$ and hence $YBOD$ is cyclic. Now note that P also lies on the circumcircle of $YAOC$ since $\angle OAP = \angle OCP = 90^\circ$. Therefore since the lines QBD and OP are both perpendicular to AC, we have that

$$\angle AQX = \angle APO = \angle AYO = \angle AYX$$

which implies that $QXAY$ is cyclic. Similarly, we have that $\angle XRC = \angle OPC = \angle OYC = \angle XYC$ and hence $RYXC$ is cyclic. Therefore

$$180^\circ - \angle YQP = \angle YQA = \angle YXA = 180^\circ - \angle YXC = \angle YRC = \angle YRP$$

since $RYXC$ and $QXAY$ are cyclic. This implies that $YQPR$ is cyclic. Observe that $\angle PQR = \angle AQB = \angle APO = \angle OPC = \angle PRQ$ and hence PQR is isosceles. Let O_1 and O_2 be the centers of (BOD) and (PQR), respectively. It follows that O_1O and O_2P are both perpendicular to BD. Let O_1O intersect (BOD) again at O'. Since O' is diametrically opposite to O, we have that $\angle O'YO = 90^\circ$. Since Y lies on the circle with diameter OP, we have that $\angle OYP = 90^\circ$. Thus O', Y and P are collinear. Since O_1O' and O_2P are parallel, we have that $\angle O_2YP = \angle O_2PY = \angle O_1O'Y = \angle O_1YO'$ and therefore O_1, Y and O_2 are collinear. This proves that (BOD) and (PQR) are tangent at Y.