Functional Equations

Unless otherwise specified, all functions are real-valued and defined for all $x \in \mathbb{R}$.

1. Find all solutions of $f(x+y) + f(x-y) = 2f(x)\cos y$.

2. A function f has the following property for a fixed value of a and all values of x:

   $$f(x+a) = \frac{1+f(x)}{1-f(x)}.$$

   Show that f is periodic.

3. Find a function f such that $f(x+y) = f(x) + f(y) + xy$.

4. Find a function f such that $3f(2x+1) = f(x) + 5x$. Are there any others?

5. Find a function f such that $xf(x) + 2xf(-x) = -1$.

6. Find all solutions of $xf(y) + yf(x) = (x+y)f(x)f(y)$.

7. Find a solution f defined on all $x > 0$, such that $f(xy) = xf(y) + yf(x)$.

8. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathbb{Z}$.

9. Find all monotonic solutions to

   $$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}.$$

10. Find all monotonic solutions to $f^2(x) = f(x+y)f(x-y)$.

11. Let f be defined for all $x \neq 0, 1$. Solve:
$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$

12. Find all positive solutions to $f(x+y)f(x-y) = [f(x)f(y)]^2$.

13. **IMO 1977** Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying $f(n+1) > f(f(n))$ for all n. Prove that $f(n) = n$ for all n.

14. **IMO 1990** Find a function $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfies
$$f(xf(y)) = f(x)/y$$
for all $x, y \in \mathbb{Q}^+$.

15. Find a positive function $f : \mathbb{R} \to \mathbb{R}_+$ which transforms three terms of the arithmetic sequence $x, x+y, x+2y$ into corresponding terms $f(x), f(x+y), f(x+2y)$ of a geometric sequence. In other words:
$$[f(x+y)]^2 = f(x) \cdot f(x+2y).$$
What is the general form of this function?

16. Find a function that satisfies $f(x+y) = f(x) + f(y) + f(x)f(y)$.

17. Find all sequences $f(n)$ of positive integers satisfying $f(f(f(n))) + f(f(n)) + f(n) = 3n$.

18. **IMO 1988** The function f is defined on the set of positive integers as follows:
$$f(1) = 1, \qquad f(3) = 3, \qquad f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n), \qquad f(4n+3) = 3f(2n+1) - 2f(n).$$
Find all values of n for which $f(n) = n$ and $1 \leq n \leq 1988$.

19. **IMO 1968** For some positive, real constant a,
$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f^2(x)}.$$
(a) Prove that the function f is periodic.

(b) For $a = 1$, give an example of a non-constant function with the required properties.

20. **IMO 1993** Does there exist a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that
$$f(1) = 2, \qquad f(f(n)) = f(n) + n, \qquad f(n) < f(n+1) \qquad \forall n \in \mathbb{N}?$$