Games

Alex Song

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1 Introduction

Problems about games on the Math Olympiads can be found in all four of the main subjects, algebra, combinatorics, geometry, and number theory. Game theory problems are unique in that any strategy for any player must deal with all possible plays of the other player(s). Thus, in a game theory problem, say with a winner and a loser, proving that the winner indeed is the winner requires a proposal for a sequence of moves of the winner against any possible strategy of the loser. In other problems where, perhaps, one player P attempts to maximize a quantity F against another player Q, we must compute

$$\max_{P\text{'s moves}} \min_{Q\text{'s moves}} \max_{P\text{'s moves}} \min_{Q\text{'s moves}} \dots F,$$

where the $\dots$ goes until the end of the game. Fortunately, in most game theory problems, the inequalities implied by the above expression are themselves quite loose.

One central theme in two-player games with a winner and a loser (or more generally any other zero-sum game), where the players are, as above, named P and Q, is that the optimal outcome for P is also the optimal outcome for Q. This is seen because, for instance, if P is trying to maximize a value F, then by the zero-sum assumption, Q will be trying to maximize $-F$. But we have

$$\max_{P\text{'s moves}} \min_{Q\text{'s moves}} \max_{P\text{'s moves}} \min_{Q\text{'s moves}} \dots F = \min\left(\min_{P\text{'s moves}} \max_{Q\text{'s moves}} \min_{P\text{'s moves}} \max_{Q\text{'s moves}} \dots - F\right).$$

Specifically, given any two-player game where there is a winner and a loser, exactly one player can guarantee a victory.

2 Position Analysis

Often, if the game in question progresses in a downward manner (for example, if players are taking stones from a pile of stones), it is good to analyze at least the winner in appropriately small positions for intuition. Sometimes, this analysis can lead to a general solution for all positions.

Problem 1 (COMC 2013)**.** *Alice and Bob play a game. Initially, the nonnegative integers m and n are written on the blackboard. Alice and Bob take turns erasing a nonnegative integer from the board, replacing it with a lesser nonnegative integer that is different from all previously written nonnegative integers. Alice goes first. For which pairs (m, n) does she win?*

Proof. We see that the game is symmetric, so we only need to consider positions as pairs of numbers (m, n) (with possibly some prohibited numbers, which don't matter. (why?)). When $(m, n) = (0, 1)$ or $(1, 0)$, Bob wins because Alice has no move – this is indeed the simplest position. Thus, in the position $(1, n)$ or $(0, n)$, where $n \neq 0, 1$, Alice must win. Now we consider the next simplest position $(2,3)$, where Bob again wins. We can induct forward to see that Bob wins in the positions $(2i,2i+1)$ and $(2i+1,2i)$and Alice wins in all other positions.

□

3 Symmetric Games and Copying

In some problems, the situation is symmetric – that is, the choices in the possible moves do not depend on the player making the move. In this case, one good thing to think about is whether one player can exploit the symmetry of the situation by making a move based on the other players' strategies.

Here is a classic example:

Problem 2. *A natural number n is written on the blackboard. Alice and Bob play a game, with Alice going first. On their turn, they erase the number m on the blackboard and replace it with exactly one of $m-1$, $m-2$, and $m-3$. The first player to write a negative number loses. Who wins?*

Proof. Bob wins when n is divisble by 4, because he may subtract $4-k$ from the number the turn after Alice subtracts k from the number. On the other hand, Alice wins when n is not divisble by 4, because she may replace n with a number divisble by 4 on her first move. □

However, we can show the same result without describing the strategy in full:

Proof. Alice clearly wins for $n = 1,2,3$. Let the set of natural numbers n for which Alice wins be S. Now, if $n-1, n-2, n-3 \in S$, then we have that $n \notin S$ because Bob can use Alice's winning strategy after any of Alice's possible first moves. On the other hand, if $n \notin S$, then we must have that $n-1, n-2, n-3 \in S$, as Bob must be able to win against any of Alice's first moves. (We are using the symmetry of the game here.) By induction, we can find that S is precisely the set of numbers not divisible by 4.

□

Here's a problem where the move-by-move winning strategy is much harder to state.

Problem 3 (TOT 2005)**.** *Calvin and James wish to divide 25 coins, of denominations $1,2,3,\dots,25$ kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. Calvin makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kopeks wins. Which player has a winning strategy?*

Proof. Note first that there is indeed a winner because $1+\cdots+25 = 425$ is odd.

We claim that James wins. For every initial coin Calvin may choose, James can decide to either take it or give it to Calvin. These two scenarios are exactly mirror opposites, so James wins in exactly one of them, and he chooses precisely that one.

□

4 How does the game end?

Often, examining how a finite game ends can give insight to how the players can approach the game. This is especially true when the players are required to make moves that satisfy some property X, and then the game ends at a point where a player can no longer make such a move. Here is an example:

Problem 4 (USAMO 1999)**.** *Alice and Bob play a game. Initially, there is a row of n unfilled boxes. Alice and Bob take turns filling an unfilled box with either a B or an O (either player can fill either letter). The player who first creates three consecutive boxes that spell BOB wins. For which n does Alice win? For which n does Bob win?*

Proof. Note that it is not clear that there is necessarily a winner. For example, the row can be eventually filled $BBBBB$.... However, one way to guarantee that the game ends is if there is a cluster of four consecutive boxes, where the middle two are unfilled and the first and last are filled with B. Then, if a player places either a B or an O in the middle two box, the other player may place the opposite letter in the other box to win.

Consider a scenario where every move by Alice (or symmetrically, Bob) has no moves for which Bob does not immediately win in the next move. Then, the only gaps left in the grid must be of size 1 or 2 (for example, on a gap of size 3 or more, say, B____O, Alice may put a B on the left of the gap safely to make BB___O). Furthermore, on gaps of size 1, the two letters on either side of the gap must be of different letters, or otherwise Alice can win immediately. Now Alice may safely place an O in the middle of the gap.

Thus, right before the end of the game (if it ends), all gaps of unfilled boxes must be of size 2, and so if the game ends, then Alice must win when n is odd, and Bob must win when n is even.

It remains to show that either player can make the game end by placing the B__B cluster. Some careful casework is needed and some small cases are needed, but for n sufficiently large, both Alice and Bob can do it.

□

5 Subgames

In games where the players split a pile into multiple piles (or similar variants), it is often useful to consider subgames. Often, piles of the same size can be paired together, for instance.

Problem 5 (Rioplatense 2013)**.** *Two players A and B play alternatively in a convex polygon with $n \geq 5$ sides. In each turn, the corresponding player has to draw a diagonal that does not cut inside the polygon previously drawn diagonals. A player loses if after his turn, one quadrilateral is formed such that its two diagonals are not drawn. A starts the game. For each positive integer n, find a winning strategy for one of the players.*

Proof. We claim that when n is even, Alice wins, and when n is odd, Bob wins. When n is even and at least 8, she can split the n-gon into two $(n+2)/2$-gons and then copy Bob's move on the opposite game. When n is odd, Alice must split the n-gon into an odd polygon and an even polygon. Then, Bob splits the larger polygon into a polygon with the same size as the smaller polygon and a third polygon, which also has odd size (why?). (For example, if $n = 13$ and Alice splits it into 5 and 10, then Bob splits it further into 5,5,7.) Now, whenever Alice plays in one of the equal polygons, Bob copies; and if Alice plays in the new odd polygon, Bob wins via induction.

□

6 Practice Problems

6.1 List 1

1. (COMC 2007) A and B are playing a two person game with the following rules:
   - Initially there is a pile of N stones, with $N \geq 2$.
   - The players alternate turns, with A going first. On his first turn, A must remove at least 1 and at most $N1$ stones from the pile.
   - If a player removes k stones on their turn, then the other player must remove at least 1 and at most $2k - 1$ stones on their next turn.

- The player who removes the last stone wins the game

Who wins?

2. (TOT 2002) A game is played on a 23×23 board. The first player controls two white chips which start in the bottom-left and the top-right corners. The second player controls two black ones which start in the bottom-right and the top-left corners. The players move alternately. In each move, a player can move one of the chips under control to a vacant square which shares a common side with its current location. The first player wins if the two white chips are located on two squares sharing a common side. Can the second player prevents the first player from winning?

3. (JBMO 2014; Russia 2011)

   For a positive integer n, two players A and B play the following game: Given a pile of s stones, the players take turn alternatively with A going first. On each turn the player is allowed to take either one stone, or a prime number of stones, or a positive multiple of n stones. The winner is the one who takes the last stone. Assuming both A and B play perfectly, for how many values of s the player A cannot win?

   Can you find a bound N in terms of n for which A wins for all $s \geq N$?

4. (Benelux 2017)

   Let $n \geq 2$ be an integer. Alice and Bob play a game concerning a country made of n islands. Exactly two of those n islands have a factory. Initially there is no bridge in the country. Alice and Bob take turns in the following way. In each turn, the player must build a bridge between two different islands I_1 and I_2 such that:

   - I_1 and I_2 are not already connected by a bridge.
   - at least one of the two islands I_1 and I_2 is connected by a series of bridges to an island with a factory (or has a factory itself). (Indeed, access to a factory is needed for the construction.)

   As soon as a player builds a bridge that makes it possible to go from one factory to the other, this player loses the game. (Indeed, it triggers an industrial battle between both factories.) If Alice starts, then determine (for each $n \geq 2$) who has a winning strategy. (Note: It is allowed to construct a bridge passing above another bridge.)

5. (Russia 2014)

   Peter and Bob play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. Peter goes first. Who has a winning strategy?

6. (Russia 1993)

   On a board, there are n equations in the form $*x^2 + *x + *$. Two people play a game where they take turns. During a turn, you are allowed to change a star into a number not equal to zero. After $3n$ moves, there will be n quadratic equations. The first player is trying to make more of the equations not have real roots, while the second player is trying to do the opposite. What is the maximum number of equations that the first player can create without real roots no matter how the second player acts?

7. (TOT 2003)

   A chocolate bar in the shape of an equilateral triangle with side of the length n, consists of triangular chips with sides of the length 1, parallel to sides of the bar. Two players take turns eating up the chocolate. Each player breaks off a triangular piece (along one of the lines), eats it up and passes leftovers to the other player (as long as bar contains more than one chip, the player is not allowed to eat it completely). A player who has no move or leaves exactly one chip to the opponent, loses. For each n, find who has a winning strategy.

8. (TOT 2003)

   Two players in turns play a game. Each player has 1000 cards with numbers written on them; namely, First Player has cards with numbers $2, 4, \dots, 2000$ while Second Player has cards with numbers $1, 3, \dots, 2001$. In each his turn, a player chooses one of his cards and puts it on a table; the opponent sees it and puts his card next to the first one. Player, who put the card with a larger number, scores 1 point. Then both cards are discarded. First Player starts. After 1000 turns the game is over; First Player has used all his cards and Second Player used all but one. What are the maximal scores, that players could guarantee for themselves, no matter how the opponent would play?

6.2 List 2

1. (TOT 2008)

   Alice and Brian are playing a game on a $1 \times (N + 2)$ board. To start the game, Alice places a checker on any of the N interior squares. In each move, Brian chooses a positive integer n. Alice must move the checker to the n-th square on the left or the right of its current position. If the checker moves off the board, Alice wins. If it lands on either of the end squares, Brian wins. If it lands on another interior square, the game proceeds to the next move. For which values of N does Brian have a strategy which allows him to win the game in a finite number of moves?

2. (IMO Shortlist 2015)

   Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

   (i) A player cannot choose a number that has been chosen by either player on any previous turn.

   (ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

   (iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

   The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

3. (IMO Shortlist 2012)

   Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \dots$ by the following rules: (a) On every move of his B passes 1 coin from every box to an adjacent box. (b) On every move of hers A chooses several coins that were not involved in B's previous move and are in different boxes. She passes every coin to and adjacent box. Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

4. (IMO 1990)

   Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, $\dots$ alternately according to the following rules:

I.) Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

II.) Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does:

a.) $\mathcal{A}$ have a winning strategy? b.) $\mathcal{B}$ have a winning strategy? c.) Neither player have a winning strategy?

5. (Rioplatense 2010)

   Alice and Bob play the following game. To start, Alice arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's turn consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number k at most k times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer n, determine who has a winning strategy.

6. (Russia 2012)

   On a circle there are $2n + 1$ points, dividing it into equal arcs ($n \geq 2$). Two players take turns to erase one point. If after one player's turn, it turned out that all the triangles formed by the remaining points on the circle were obtuse, then the player wins and the game ends. Who has a winning strategy: the starting player or his opponent?

7. (Russia 2009)

   There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts two or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

8. (Cono Sur 2012)

   A and B play alternating turns on a 2012×2013 board with enough pieces of the following types:

   Type 1: Piece like Type 2 but with one square at the right of the bottom square. Type 2: Piece of 2 consecutive squares, one over another. Type 3: Piece of 1 square.

   At his turn, A must put a piece of the type 1 on available squares of the board. B, at his turn, must put exactly one piece of each type on available squares of the board. The player that cannot do more movements loses. If A starts playing, decide who has a winning strategy.

   Note: The pieces can be rotated but cannot overlap; they cannot be out of the board. The pieces of the types 1, 2 and 3 can be put on exactly 3, 2 and 1 squares of the board respectively.

6.3 List 3

1. (RMM 2015)

For an integer $n \geq 5$, two players play the following game on a regular n-gon. Initially, three consecutive vertices are chosen, and one counter is placed on each. A move consists of one player sliding one counter along any number of edges to another vertex of the n-gon without jumping over another counter. A move is legal if the area of the triangle formed by the counters is strictly greater after the move than before. The players take turns to make legal moves, and if a player cannot make a legal move, that player loses. For which values of n does the player making the first move have a winning strategy?

2. (IMO 2017)

   A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order:

   - The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.
   - A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.
   - The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

   Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

3. (IMO 2012)

   The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

   At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   (a) If $n \geq 2^k$, then B can guarantee a win.

   (b) For all sufficiently large k, there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.

4. (IMO Shortlist 2014)

   A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared. Determine all possible first moves of the first player after which he has a winning strategy.

7 Hints

7.1 List 1

1. What happens when N is odd?

2. The answer is yes. Can you simplify the situation? Alternatively, what subset of positions can the second player conserve?

3. If player B wins for a value s, what can you say about the values $s + kn$ for $k \geq 1$?

4. How does the game end? When is the end position of the game decided?

5. There is a lot of freedom. Can we reduce it in some way? Also – who wins?

6. Under what conditions does the last player to play on a quadratic "win" that quadratic?

7. What happens for small n?

8. It should be obvious what the second player to put a card on the table should put (there are two choices). Can one of these preserve some property of the remaining cards held by the two players?

7.2 List 2

1. How can Alice lose? For each n, which starting positions result in a win or loss for Alice?

2. What is the answer?

3. The answer is $N = 2 \cdot 2012 - 1$. When N is this value, what is A's strategy? When N is less, think about a simple strategy B can use.

4. Who wins if n_0 is really big?

5. What would a typical strategy for B look like?

6. For $n = 3$ (7 points), who wins and how?

7. It should be clear who should win. Can you write an explicit strategy?

8. The second player wins. How?

7.3 List 3

1. What is the answer?

2. The rabbit wins.

3. For the first part, can B get some information with a sequence of $k + 1$ consecutive guesses? For the second part, what strategy can A use to answer B's questions?

4. Does there always exist the winner? How can we determine it?