**2022 Winter Camp Mock Olympiad**

Each of the following ten problems has a maximum point value. Your score will be the sum of the three highest scores you obtain on the problems. You have 4.5 hours to complete this test.

1. **[3]** A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

2. **[4]** For each positive integer $n$, let $s(n)$ be the sum of the squares of the digits of $n$. For example, $s(15) = 1^2 + 5^2 = 26$. Determine all integers $n \geq 1$ such that $s(n) = n$.

3. **[4]** Let $ABC$ be a triangle. Suppose that $X, Y$ are points in the plane such that $BX, CY$ are tangent to the circumcircle of $ABC$, $AB = BX, AC = CY$ and $X, Y, A$ are in the same side of $BC$. If $I$ be the incenter of $ABC$, prove that $\angle BAC + \angle XIY = 180°$.

4. **[5]** There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer $n$. You may redistribute the balls by a sequence of moves, each of which consists of choosing an $i$ and moving exactly $i$ balls from box $B_i$ into any one other box. For which values of $n$ is it possible to reach the distribution with exactly $n$ balls in each box, regardless of the initial distribution of balls?

5. **[5]** Denote the set of positive rational numbers by $\mathbb{Q}_+$. Find all functions $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ that satisfy
$$\underbrace{f(f(f(\ldots f(f}_{n}(q))\ldots))) = f(nq)$$
for all integers $n \geq 1$ and rational numbers $q > 0$.

6. **[6]** An infinite increasing arithmetic progression is given. A new sequence is constructed in the following way: its first term is the sum of several first terms of the original sequence, its second term is the sum of several next terms of the original sequence and so on, so that each term of the original sequence becomes used in a sum for the new sequence exactly once. Is it possible that the new sequence is a geometric progression?

7. **[6]** Let $ABC$ be a triangle with $B$-excircle $w_B$ and $C$-excircle $w_C$. The circle $w'_B$ is symmetric to $w_B$ with respect to the midpoint of $AC$, the circle $w'_C$ is symmetric to $w_C$ with respect to the midpoint of $AB$. Prove that the radical axis of $w'_B$ and $w'_C$ halves the perimeter of $ABC$.

8. **[7]** David and Jacob write fractions of the form $1/n$ on the board, where $n$ is a natural number. In the $k$-th round, David writes any such fraction on the board, and then Jacob writes any $k$ such fractions on the board. Jacob wins if at any point, the sum of the fractions on the board is equal to some natural number. Can Jacob win?

9. **[7]** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + yf(x^2)) = f(x) + xf(xy)$$

for all real numbers $x$ and $y$.

10. **[7]** Let $A, B, C, D, T$ be points on a circle $\omega$ so that:

- There exists circle $\omega_1$ touching $AB, AC$ and touching $\omega$ internally at $T$;
- There exists circle $\omega_2$ touching $DB, DC$ and touching $\omega$ externally at $T$.

Prove that the common external tangents of $\omega_1$ and $\omega_2$ intersect on $\omega$.