IMO Winter Camp Mock Olympiad 2010
Time: 4 Hours

1. Let $ABCD$ be a parallelogram with $AC/BD = k$. The bisectors of the angles formed by AC and BD intersect the sides of $ABCD$ at K, L, M, N. Prove that the ratio of the areas of $KLMN$ and $ABCD$ is $2k/(k+1)^2$.

2. Each edge of an $m \times n$ rectangular grid is oriented with an arrow such that

   (a) the border is oriented clockwise, and

   (b) each interior vertex has two arrows coming out of it, and two arrows going into it.

   Prove that there is at least one square whose edges are oriented clockwise.

3. Let $\mathbb{Z}^*$ denote the set of non-zero integers. A function $f : \mathbb{Z}^* \to \mathbb{Z}^{\geq 0}$ satisfies the following properties:

   (a) $f(m+n) \geq \min\{f(m), f(n)\}$ for all $m, n \in \mathbb{Z}^*, m+n \neq 0$.

   (b) $f(mn) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}^*$.

   (c) $f(2010) = 1$.

   Determine the minimum and maximum possible value of $f(2010!)$.

4. If a, b, c are positive real numbers such that $a + b + c = 3$, show that

$$\frac{1}{2+a^2+b^2} + \frac{1}{2+b^2+c^2} + \frac{1}{2+c^2+a^2} \leq \frac{3}{4}$$