# The Online Math Open Spring Contest
## Official Solutions
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Online Math Open

# Acknowledgements

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1. Let $A_n$ denote the answer to the $n$th problem on this contest ($n = 1, \ldots, 30$); in particular, the answer to this problem is $A_1$. Compute $2A_1(A_1 + A_2 + \cdots + A_{30})$.

   *Proposed by Yang Liu.*

   **Answer.** $\boxed{0}$.

   **Solution.** Since $A_1$ is the answer to this problem, we know that $A_1 = 2A_1(A_1 + A_2 + \cdots + A_{30})$. This means that either $A_1 = 0$ or $A_1 + A_2 + \cdots + A_{30} = \frac{1}{2}$. The latter is impossible because all answers are nonnegative integers.

   Therefore, $A_1 = 0$. $\qquad\square$

2. Let $x$, $y$, and $z$ be real numbers such that $x + y + z = 20$ and $x + 2y + 3z = 16$. What is the value of $x + 3y + 5z$?

   *Proposed by James Lin.*

   **Answer.** $\boxed{12}$.

   **Solution.** We present three different solutions.

   **Solution 1.** Note that $x+y+z, x+2y+3z, x+3y+5z$ form an arithmetic sequence, giving us answer of 12.

   **Solution 2.** Subtracting the first equation from twice the second gives that $x + 3y + 5z = 2(x + 2y + 3z) - (x + y + z) = 2(16) - 20 = 12$.

   **Solution 3.** Note that we are given three variables but only two equations, so assuming that the answer is constant, we can assume $x = 0$. Then, $y + z = 20$ and $2y + 3z = 16$, and solving gives $y = 44$ and $z = -24$. Hence, $3y + 5z = 132 - 120 = 12$.

   $\qquad\square$

3. A store offers packages of 12 pens for \$10 and packages of 20 pens for \$15. Using only these two types of packages of pens, find the greatest number of pens \$173 can buy at this store.

   *Proposed by James Lin.*

   **Answer.** $\boxed{224}$.

   **Solution.** For every \$30, it's clear that our best option is to buy 40 pens through the latter option. After we do this 5 times, we are left with \$23, which we can use to either buy two packages of 12 pens or a package of 20 pens. The first option is better, giving $5 \cdot 40 + 2 \cdot 12 = 224$ pens. $\qquad\square$

4. Given that $x$ is a real number, find the minimum value of $f(x) = |x+1|+3|x+3|+6|x+6|+10|x+10|$.

   *Proposed by Yannick Yao.*

   **Answer.** $\boxed{54}$.

   **Solution.** Notice that it suffices to minimize the last term because its coefficients is as large as the sum of the other three (in other words, the slope of $f(x)$ will be nonpositive when $x + 10 < 0$, and will be nonnegative when $x + 10 > 0$). Therefore the minimum is achieved when $x + 10 = 0$, or $x = -10$, and this minimum is $f(-10) = 9 + 3 \cdot 7 + 6 \cdot 4 = 54$. $\qquad\square$

5. Let $\ell$ be a line with negative slope passing through the point $(20, 16)$. What is the minimum possible area of a triangle that is bounded by the $x$-axis, $y$-axis, and $\ell$?

   *Proposed by James Lin.*

**Answer.** $\boxed{640}$.

**Solution.** Let $l$ have a slope of $-k$ for a positive real number $k$, so that $l$ intersects the $x$-axis at $(20+\frac{16}{k}, 0)$ and the $y$-axis at $(0, 16+20k)$. Then, the area is $320+200k+\frac{128}{k} = \frac{(10k\sqrt{2}-8\sqrt{2})^2}{k}+640 \geq 640$, giving a minimal area of $640$ at $k = \frac{4}{5}$. $\qquad\square$

6. In a round-robin basketball tournament, each basketball team plays every other basketball team exactly once. If there are 20 basketball teams, what is the greatest number of basketball teams that could have at least 16 wins after the tournament is completed?

   *Proposed by James Lin.*

   **Answer.** $\boxed{7}$.

   **Solution.** We will show that the answer is 7. It's clear that each team with at least 16 wins must have at most 3 losses. Assume for the sake of contradiction that there are 8 such teams with at most 3 losses. Then, consider the $\binom{8}{2} = 28$ games among these 8 teams, which must consist of 28 losses. By the Pigeonhole Principle, some team must have at least $\frac{28}{8} = 3.5$ losses, which is a contradiction. Hence, our answer is at most 7. For the construction, let the 7 teams with at least 16 wins be labeled $0, 1, \cdots, 6$. Say that team $i$ beats team $j$ for $i \neq j$ if and only if $i - j \pmod 7 \in \{1, 2, 3\}$. Then, each team will have 3 wins and 3 losses among these 7 teams, and let these teams beat all of the 13 other teams. In this scenario, it is clear that each team has 16 wins. $\qquad\square$

7. Compute the number of ordered quadruples of positive integers $(a, b, c, d)$ such that
   $$a! \cdot b! \cdot c! \cdot d! = 24!.$$

   *Proposed by Michael Kural.*

   **Answer.** $\boxed{28}$.

   **Solution.** Without loss of generality assume $a \leq b \leq c \leq d$. Of course, $d \leq 24$. If $d = 24$, then $a = b = c = 1$, so we get the solution $(1, 1, 1, 24)$.

   Otherwise, we must have $23 \mid a! \cdot b! \cdot c! \cdot d!$, so $23 \mid d!$. But as 23 is prime, we must have $d = 23$. So $a! \cdot b! \cdot c! = 24$. Now $c \leq 4$. If $c = 4$, then $a = b = 1$, so we get the solution $(1, 1, 4, 23)$.

   Otherwise, $3 \mid a! \cdot b! \cdot c!$, so $3 \mid c!$, and $c = 3$. Thus $a! \cdot b! = 4$, from which it is clear that $a = b = 2$. Thus we get the final solution $(2, 2, 3, 23)$.

   Finally, the number of ordered quadruples is the number of nonequivalent permutations of $(1, 1, 1, 24)$, $(1, 1, 4, 23)$, and $(2, 2, 3, 23)$, which is $\frac{4!}{3!} + \frac{4!}{2!} + \frac{4!}{2!} = 4 + 12 + 12 = 28$. $\qquad\square$

8. Let $ABCDEF$ be a regular hexagon of side length 3. Let $X, Y,$ and $Z$ be points on segments $AB, CD,$ and $EF$ such that $AX = CY = EZ = 1$. The area of triangle $XYZ$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where $a, b, c$ are positive integers such that $b$ is not divisible by the square of any prime and $\gcd(a, c) = 1$. Find $100a + 10b + c$.

   *Proposed by James Lin.*

   **Answer.** $\boxed{2134}$.

**Solution.** We present three solutions to this problem.

**Solution 1.** Extend lines $AB, CD, EF$ to intersect at $AB \cap CD = G, CD \cap EF = H, EF \cap AB = I$. Then, $GH = HI = IG = 9$. Note that $IX = 4, IZ = 5$, and $\angle XIZ = 60°$, so by the Law of Cosines, $XZ = \sqrt{21}$. Then, since $XY = YZ = ZX$, the area of $XYZ$ is $\dfrac{21\sqrt{3}}{4}$, so the answer is 2134.

**Solution 2.** Notice that $\dfrac{[XIZ]}{[GHI]} = \dfrac{4}{9} \cdot \dfrac{5}{9} = \dfrac{20}{81}$, so since $[XIZ] = [YGX] = [ZHY]$, we get that $\dfrac{[XYZ]}{[GHI]} = \dfrac{21}{81}$. Because $[GHI] = \dfrac{81\sqrt{3}}{4}$, it follows that $[XYZ] = \dfrac{21\sqrt{3}}{4}$ and the answer is 2134.

**Solution 3.** Let $O$ be the center of equilateral triangle $XYZ$, so it's the center of $ABCDEF$ as well. Let the foot of $O$ to $AB$ be $K$. Then $XK = \dfrac{1}{2}$ and $KO = \dfrac{3\sqrt{3}}{2}$, so $XO = \sqrt{7}$ giving $[XYZ] = \dfrac{21\sqrt{3}}{4}$. Hence the answer is 2134. $\qquad\square$

9. Let $f(n) = 1 \times 3 \times 5 \times \cdots \times (2n - 1)$. Compute the remainder when $f(1) + f(2) + f(3) + \cdots + f(2016)$ is divided by 100.

   *Proposed by James Lin.*

   **Answer.** $\boxed{24}$.

   **Solution.** We evaluate modulo 4 and modulo 25. $f(1), f(2), f(3), f(4)$ are $1, 3, 3, 1 \pmod 4$, respectively, and repeat every four integers, so hence our answer is $504 \cdot (1 + 3 + 3 + 1) \equiv 0 \pmod 4$. Notice that $f(n)$ is divisible by 25 for $n \geq 8$. $f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) \equiv 1 + 3 + 5 \cdot (3 + 1 + 4 + 4 + 2) \equiv 24 \pmod{25}$, so the last two digits are 24. $\qquad\square$

10. Lazy Linus wants to minimize his amount of laundry over the course of a week (seven days), so he decides to wear only three different T-shirts and three different pairs of pants for the week. However, he doesn't want to look dirty or boring, so he decides to wear each piece of clothing for either two or three (possibly nonconsecutive) days total, and he cannot wear the same outfit (which consists of one T-shirt and one pair of pants) on two different (not necessarily consecutive) days. How many ways can he choose the outfits for these seven days?

    *Proposed by Yannick Yao.*

    **Answer.** $\boxed{90720}$.

    **Solution.** The problem is equivalent to the number of ways to choose 7 out of 9 squares in a 3 by 3 grid and label them from 1 to 7 inclusive such that the two blank squares don't lie on the same row or column. Once this configuration is fixed, we can map each column to a T-shirt and each row to a pair of pants (and therefore each square correspond to a possible outfit), and the number in each square (or lack thereof) signifies which day to where this outfit, if at all. There are $\frac{3^2 \cdot 2^2}{2} = 18$ ways to choose the two blanks and $7! = 5040$ ways to label the 7 other squares, for $18 \cdot 5040 = 90720$ ways in total. $\qquad\square$

11. For how many positive integers $x$ less than 4032 is $x^2 - 20$ divisible by 16 and $x^2 - 16$ divisible by 20?

    *Proposed by Tristan Shin.*

    **Answer.** $\boxed{403}$.

    **Solution.** We just $a|b$ to denote that $b/a$ is an integer.

    We must solve the system of quadratic congruences of $x^2 \equiv 4 \pmod{16}$ and $x^2 \equiv 16 \pmod{20}$. The first is equivalent to $16 \mid (x-2)(x+2)$, while the second is $5, 4 \mid (x-4)(x+4)$. $5 \mid (x-4)(x+4)$ is

equivalent to $x \equiv \pm 1 \pmod 5$, while $4 \mid x^2 - 16$ is equivalent to $x \equiv 0 \pmod 2$. These two give that $x \equiv 4, 6 \pmod{10}$.

Now, I claim that $x \equiv 2 \pmod 4$. Assume not, then neither $x - 2$ nor $x + 2$ is divisible by 4, so then $(x - 2)(x + 2)$ is not divisible by 16, contradiction. Thus, $x \equiv 2 \pmod 4$ and $x \equiv 4, 6 \pmod{10}$, so $x \equiv 6, 14 \pmod{20}$. It suffices to confirm that every number of this type works. Let $x = 20k + 10 \pm 4$, then $x^2 = (20k + 10)^2 \pm 8(20k + 10) + 16 = 400k^2 + 400k + 116 \pm 160k \pm 80$. Now, $x^2 \equiv 0k^2 + 0k + 4 \pm 0k \pm 0 \pmod{16}$, so $x^2 - 20$ is divisible by 16, and $x^2 \equiv 0k^2 + 0k + 16 \pm 0k \pm 0 \pmod{20}$, so $x^2 - 16$ is divisible by 20.

Thus, every 20 integers, there will be 2, for 402 up until 4020. But between 4021 and 4032, there is only one: 4026, as 4034 is too large. Therefore, there are 403 such positive integers.

$\square$

12. A *9-cube* is a nine-dimensional hypercube (and hence has $2^9$ vertices, for example). How many five-dimensional faces does it have?

(An $n$ dimensional hypercube is defined to have vertices at each of the points $(a_1, a_2, \cdots, a_n)$ with $a_i \in \{0, 1\}$ for $1 \le i \le n$.)

*Proposed by Evan Chen.*

**Answer.** $\boxed{2016}$.

**Solution.** Without loss of generality let's consider the 9-cube as $[0, 1]^9$ in the 9-dimension Euclidean space. On each 5-dimensional face, there are $9 - 5 = 4$ coordinates that are fixed, and each of them can be 0 or 1. Therefore, there are $\binom{9}{4} \cdot 2^4 = 2016$ 5-dimensional faces in total. $\square$

13. For a positive integer $n$, let $f(n)$ be the integer formed by reversing the digits of $n$ (and removing any leading zeroes). For example $f(14172) = 27141$. Define a sequence of numbers $\{a_n\}_{n \ge 0}$ by $a_0 = 1$ and for all $i \ge 0$, $a_{i+1} = 11a_i$ or $a_{i+1} = f(a_i)$. How many possible values are there for $a_8$?

*Proposed by James Lin.*

**Answer.** $\boxed{13}$.

**Solution.** Note that we can have $a_i = a_{i+1}$ whenever $a_i \le 14641$, so it's clear that we can assume $a_0 = 1$, $a_1 = 11$, $a_2 = 121$, $a_3 = 1331$, $a_4 = 14641$, $a_5 = 161051$, and find all possible values among $a_0, a_1, \ldots, a_8$. Notice that $f(11^6) = 11f(11^5)$ because $11^5$ does not have consecutive digits adding up to at least 10, so no digits carry. Now, we see that the other numbers we can have are $f(11^5), 11^6, f(11^6), 11^7, f(11^7), 11f(11^6), 11^8$, which we can easily check are distinct, giving a total of $6 + 7 = 13$ possible values for $a_8$. $\square$

14. Let $ABC$ be a triangle with $BC = 20$ and $CA = 16$, and let $I$ be its incenter. If the altitude from $A$ to $BC$, the perpendicular bisector of $AC$, and the line through $I$ perpendicular to $AB$ intersect at a common point, then the length $AB$ can be written as $m + \sqrt{n}$ for positive integers $m$ and $n$. What is $100m + n$?

*Proposed by Tristan Shin.*

**Answer.** $\boxed{460}$.

**Solution.** First, assume that we have an arbitrary triangle with side lengths $a$, $b$, and $c$ that satisfy this concurrency. Let $D$ be the foot of the altitude from $A$ to $BC$. We will prove a lemma that is known as Carnot's Theorem.

**Lemma.** Let $ABC$ be a triangle, and let $D, E, F$ be on $BC, AC, AB$ respectively. If the perpendiculars through $D, E, F$ to their respective sides concur, then

$$BD^2 + CE^2 + AF^2 = CD^2 + AE^2 + BF^2.$$

*Proof.* Suppose the perpendiculars concur at a point $P$. Then

$$PB^2 - PC^2 = (BD^2 + PD^2) - (CD^2 + PD^2) = BD^2 - CD^2.$$

Similarly,

$$PC^2 - PA^2 = CE^2 - AE^2$$

and

$$PA^2 - PB^2 = AF^2 - BF^2$$

so summing the three equations yields the desired result.

$\square$

Now by Carnot's Theorem, we need $BD^2 - DC^2 + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + (s-a)^2 - (s-b)^2 = 0$. Simplifying this gives $BD^2 - DC^2 + c\,(b-a) = 0$. By the Perpendicularity Lemma, $BD^2 - DC^2 = BA^2 - AC^2 = c^2 - b^2$, so we have $c^2 + (b-a)\,c - b^2 = 0$. With $a = 20$ and $b = 16$, we have that $c^2 - 4c - 256 = 0$, so $c = 2 \pm \sqrt{260}$. But $c$ must be positive, so $c = 2 + \sqrt{260}$ and $100m + n = 460$.

$\square$

15. Let $a, b, c, d$ be four real numbers such that $a + b + c + d = 20$ and $ab + bc + cd + da = 16$. Find the maximum possible value of $abc + bcd + cda + dab$.

    *Proposed by Yannick Yao.*

    **Answer.** $\boxed{80}$.

    **Solution.** Note that $ab + bc + cd + da = (a+c)(b+d) = 16$, which implies along with $(a+c) + (b+d) = 20$ and Vieta's Theorem that $a + c, b + d$ are roots of the equation $x^2 - 20x + 16x$. Solving the quadratic, we get without loss of generality $a + c = 10 - 2\sqrt{21}$ and $b + d = 10 + 2\sqrt{21}$. Now

    $$abc + bcd + cda + dab = (a+c)bd + (b+d)ac$$

    so it suffices to maximize $bd$ and $ac$ subject to the constraints $a + c = 10 - 2\sqrt{21}$ and $b + d = 10 + 2\sqrt{21}$ (noting that $10 - 2\sqrt{21} > 0$). By AM-GM, $ac$ is maximized when $a = c = 5 - \sqrt{21}$ and $bd$ is maxmized when $b = d = 5 + \sqrt{21}$. Thus the answer is

    $$(5 + \sqrt{21})^2(10 - 2\sqrt{21}) + (5 - \sqrt{21})^2(10 + 2\sqrt{21}) = 80$$

    $\square$

16. Jay is given a permutation $\{p_1, p_2, \ldots, p_8\}$ of $\{1, 2, \ldots, 8\}$. He may take two dividers and split the permutation into three non-empty sets, and he concatenates each set into a single integer. In other words, if Jay chooses $a, b$ with $1 \le a < b < 8$, he will get the three integers $\overline{p_1 p_2 \ldots p_a}$, $\overline{p_{a+1} p_{a+2} \ldots p_b}$, and $\overline{p_{b+1} p_{b+2} \ldots p_8}$. Jay then sums the three integers into a sum $N = \overline{p_1 p_2 \ldots p_a} + \overline{p_{a+1} p_{a+2} \ldots p_b} + \overline{p_{b+1} p_{b+2} \ldots p_8}$. Find the smallest positive integer $M$ such that no matter what permutation Jay is given, he may choose two dividers such that $N \le M$.

    *Proposed by James Lin.*

    **Answer.** $\boxed{1404}$.

**Solution.** We want to use the divisors to split the permutation into sets of size $3, 3, 2$ in some order. We consider all three possible uses of these divisors, let these three uses be $U_1 = 3 - 3 - 2$, $U_2 = 3 - 2 - 3$, $U_3 = 2 - 3 - 3$. $p_1$ appears as the hundreds digit in $U_1$ and $U_2$, $p_6$ in $U_2$ and $U_3$, $p_3$ in $U_3$, and $p_4$ in $U_1$. Note that $2p_1 + 2p_6 + p_3 + p_4 \leq 2(8+7) + 6 + 5 = 41$, and we wish to maximize $m = \min(p_1 + p_4, p_1 + p_6, p_3 + p_6)$. We can achieve the clear maximum of $m = 13$ by $\{p_1, p_6\} = \{7, 8\}, p_4 = 13 - p_1, p_3 = 13 - p_6$, or $\{p_1, p_6\} = \{6, 8\}, p_4 = 13 - p_1, p_3 = 13 - p_6$, and any other possibilities will match up $k \leq 5$ with a number less than 8. For $U_2$, $m < p_1 + p_6$, so we do not worry about this case. But since $p_3$ and $p_6$ happen to both be units digits in $U_1$, and $p_1$ and $p_4$ happen to both be tens digits in $U_3$, choosing any of our four possibilities for $(p_1, p_3, p_4, p_6)$ does not affect our minimum value for $N$. Since $p_7$ is a tens digit of both $U_1$ and $U_3$, and $p_8$ is a units digit of both $U_1$ and $U_3$, we can set $p_7 = 4$ and $p_8 = 1$. Then we set $\{p_2, p_5\} = \{2, 3\}$ in some order since $p_2, p_5$ are both tens digits in $U_1$ and both units digits in $U_3$. Now, we get $U_1 = 1404$ and $U_3 = 1476$, so $M = 1404$   $\square$

17. A set $S \subseteq \mathbb{N}$ satisfies the following conditions:

    (a) If $x, y \in S$ (not necessarily distinct), then $x + y \in S$.
    (b) If $x$ is an integer and $2x \in S$, then $x \in S$.

    Find the number of pairs of integers $(a, b)$ with $1 \leq a, b \leq 50$ such that if $a, b \in S$ then $S = \mathbb{N}$.

    *Proposed by Yang Liu.*

**Answer.**   $\boxed{2068}$ .

**Solution.** Call a pair $(a, b)$ *forcing* if $a, b \in S$ forces $S = \mathbb{N}$.

**Lemma.** $(a, b)$ is forcing if and only if $a, b$ share no odd factor that is greater than 1.

*Proof.* If $a, b$ share an odd factor greater than 1, then neither operation can change this. Therefore, $1 \notin S$, so $S$ is not forced to be $\mathbb{N}$.

We proceed by induction on the pairs, where pairs are sorted by first coordinate, then by second coordinate. The base cases are clear, since if $1 \in S$, then using condition (a) repeatedly shows that $S = \mathbb{N}$.

Assume that $a < b$. If either is even, we can halve it and finish by induction. This doesn't change whether they have a common odd factor greater than 1. Now assume that $a, b$ are both odd.

Now because $a, b \in S$, and $a, b$ are both odd, $\frac{a+b}{2}$ is in the set (use condition (a), then (b)). It's easy to check that $a, \frac{a+b}{2}$ share no common odd factor if $a, b$ do not. More explicitly, if an odd number $p | \frac{a+b}{2}$ and $p | a$, then $p | a + b$ and therefore, $p | (a+b) - a = b$. Now because $\frac{a+b}{2} < b$, we can finish by induction.

$\square$

To finish, one could use the principle of inclusion and exclusion to eliminate pairs that share common odd prime factors. The odd primes less than 50 are $3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$. Therefore, the final answer is

$$50^2 - \left\lfloor \frac{50}{3} \right\rfloor^2 - \left\lfloor \frac{50}{5} \right\rfloor^2 - \left\lfloor \frac{50}{7} \right\rfloor^2 - \left\lfloor \frac{50}{11} \right\rfloor^2 - \left\lfloor \frac{50}{13} \right\rfloor^2 - \left\lfloor \frac{50}{17} \right\rfloor^2 - \left\lfloor \frac{50}{19} \right\rfloor^2 - \left\lfloor \frac{50}{23} \right\rfloor^2 - \left\lfloor \frac{50}{29} \right\rfloor^2 - \left\lfloor \frac{50}{31} \right\rfloor^2 - \left\lfloor \frac{50}{37} \right\rfloor^2$$

$$- \left\lfloor \frac{50}{41} \right\rfloor^2 - \left\lfloor \frac{50}{43} \right\rfloor^2 - \left\lfloor \frac{50}{47} \right\rfloor^2 + \left\lfloor \frac{50}{15} \right\rfloor^2 + \left\lfloor \frac{50}{21} \right\rfloor^2 + \left\lfloor \frac{50}{33} \right\rfloor^2 + \left\lfloor \frac{50}{35} \right\rfloor^2 + \left\lfloor \frac{50}{39} \right\rfloor^2 = 2068.$$

$\square$

**Remark.** I apologize for the inclusion-exclusion part of the problem. I could not find a cleaner answer extraction. If people have better answer extraction ideas, please post them in the corresponding forum on AoPS.

Also, it might be nicer to think about the induction described as an algorithm. It was worded as induction in this solution as it is easier to word solutions that way.

18. Kevin is in kindergarten, so his teacher puts a $100 \times 200$ addition table on the board during class. The teacher first randomly generates distinct positive integers $a_1, a_2, \ldots, a_{100}$ in the range $[1, 2016]$ corresponding to the rows, and then she randomly generates distinct positive integers $b_1, b_2, \ldots, b_{200}$ in the range $[1, 2016]$ corresponding to the columns. She then fills in the addition table by writing the number $a_i + b_j$ in the square $(i, j)$ for each $1 \leq i \leq 100$, $1 \leq j \leq 200$.

During recess, Kevin takes the addition table and draws it on the playground using chalk. Now he can play hopscotch on it! He wants to hop from $(1, 1)$ to $(100, 200)$. At each step, he can jump in one of 8 directions to a new square bordering the square he stands on a side or at a corner. Let $M$ be the minimum possible sum of the numbers on the squares he jumps on during his path to $(100, 200)$ (including both the starting and ending squares). The expected value of $M$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers $p, q$. Find $p + q$.

*Proposed by Yang Liu.*

**Answer.** $\boxed{30759351}$.

**Solution.** Say $(1, 1)$ is upper left, and $(100, 200)$ is bottom right. Note that Kevin must hop in at least 200 squares to get from $(1, 1)$ to $(100, 200)$. He also must hop in each row and column at least once.

Therefore, we can see that

$$M \geq \sum_{i=1}^{200} b_i + \sum_{i=1}^{100} a_i + 100 \min(a_1, a_2, \ldots, a_{100}).$$

The extra term with the minimum at the end comes from the fact that the numbers written on the rows must be added at least 200 times in total, as Kevin hops on at least 200 squares. To see that this minimum is attainable, Kevin could just hop diagonally until he gets to the row with minimum sum, stay in that row for 100 steps while increasing his column number one by one, and then hopping diagonally until the finish.

By linearity of expectation,

$$\mathbb{E}[M] = \sum_{i=1}^{200} \mathbb{E}[b_i] + \sum_{i=1}^{100} \mathbb{E}[a_i] + 100 \cdot \mathbb{E}[\min(a_1, a_2, \ldots, a_{100})] = 300 \cdot \frac{2017}{2} + 100 \mathbb{E}[\min(a_1, a_2, \ldots, a_{100})],$$

as $\mathbb{E}[a_i] = \mathbb{E}[b_i] = \frac{2017}{2}$. To finish, we need to compute the expected value of the minimum. This is a classical problem with many ways to do it, one of which is a straightforward computation using the Hockey-Stick Identity. Instead, I will present an alternate proof, one that I like better.

Say $a_1 < a_2 < \cdots < a_{100}$. Consider the ranges of integers (some of which may be empty) $[1, a_1 - 1], [a_1 + 1, a_2 - 1], \ldots, [a_{99} + 1, a_{100} - 1], [a_{100} + 1, 2016]$. There are 101 of these ranges, with 1916 total integers among them. We can compute the min by taking the first of these ranges and adding 1 to its length. By symmetry, the average length of the first range will be $\frac{1916}{101}$, so the expected value of the minimum number is $\frac{1916}{101} + 1 = \frac{2017}{101}$.

Plugging this in, our final answer is $\frac{2017}{2} \cdot 300 + 100 \cdot \frac{2017}{101} = \frac{30759250}{101} \implies p + q = 30759351$. $\qquad \square$

19. Let $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers.

    Define a function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ with $f(0) = 1$ and

    $$f(n) = 512^{\lfloor n/10 \rfloor} f(\lfloor n/10 \rfloor)$$

    for all $n \geq 1$. Determine the number of nonnegative integers $n$ such that the hexadecimal (base 16) representation of $f(n)$ contains no more than 2500 digits.

    *Proposed by Tristan Shin.*

    **Answer.** $\boxed{10010}$.

    **Solution.** I claim that $f(n) = 2^{n-s(n)}$, where $s(n)$ is the sum of the digits of $n$ in base 10. We proceed by induction on $d$, the number of digits of $n$ in base 10. For $d = 1$, we have $f(\lfloor n/10 \rfloor) = f(0) = 1$, and the 512 part becomes just 1 also, so $f(n) = 1 = 2^{n-n}$. Now, assume for some $d = k \geq 1$, $f(n) = 2^{n-s(n)}$ for all $n$ with $k$ digits. The base case of $k = 1$ has just been proven. Then, let $n = 10a + b$ for a $k$ digit number $a$ and a single digit number $b$. Then $f(n) = 512^a f(a)$. But $a$ is a $k$ digit number, so $f(a) = 2^{a-s(a)}$. Thus, $f(n) = 2^{9a+a-s(a)} = 2^{10a-s(a)}$. But note that $s(a) = s(a) + s(b) - b$ as $b = s(b)$, and $s(a) + s(b) = s(10a) + s(b) = s(10a + b) = s(n)$, so $f(n) = 2^{10a+b-s(n)} = 2^{n-s(n)}$. Thus, by induction on $d$, we have $f(n) = 2^{n-s(n)}$ for all nonnegative integers $n$.

    Then the condition about $50^2 = 2500$ digits means that $\lfloor \log_{16} f(n) \rfloor + 1 \leq 2500$. This is equivalent to $\left\lfloor \frac{n-s(n)}{4} \right\rfloor \leq 2499$, which in turn is equivalent to $\frac{n-s(n)}{9} < 1111 + \frac{1}{9}$. But note that $n - s(n)$ is always divisible by 9, so the LHS must be an integer, implying that $\frac{n-s(n)}{9} \leq 1111$. Now, if we let $n = \sum_{i=0}^{\infty} a_i 10^i$ for integers $a_i \in [0, 9]$, then $\frac{n-s(n)}{9} = a_1 + 11a_2 + 111a_3 + \dots$. This shows that $n - s(n)$ is always nonnegative. We can then easily determine that for every integer $k \in [0, 1111]$ except for 1110 and those in $[0, 1109]$ that are $10, 21, 32, 43, 54, 65, 76, 87, 98, 109, 110 \pmod{111}$, there exist exactly 10 solutions to $\frac{n-s(n)}{9} = k$. This is a total of 1001 numbers, so 10010 solutions. Alternatively, one can see there are 1001 numbers by noting that this is essentially a worse version of base 10, but we can still plug in digits 0 to 9. $\qquad \square$

20. Define $A(n)$ as the average of all positive divisors of the positive integer $n$. Find the sum of all solutions to $A(n) = 42$.

    *Proposed by Yannick Yao.*

    **Answer.** $\boxed{1374}$.

    **Solution.** Notice that the function $A$ is multiplicative, which means whenever $m, n$ are relatively prime, we have $A(m)A(n) = A(mn)$. So we can focus our attention to $A(p^t) = \frac{(p^{t+1}-1)/(p-1)}{(t+1)} \leq 42$ for prime $p$ and positive integer $t$. Then since we don't want prime factors other than 2,3,7 in either numerator or denominator (which is because any prime factor greater or equal to 11 on the numerator are too big since they requires something to the 10th power or higher to cancel this factor out on the denominator, and $A(2^{10}) = \frac{2047}{11} > 42$ already. Cancelling a factor of 5 also requires a fourth power, and $A(5^4) = \frac{781}{5} > 42$ is already too large and $A(3^4) = 121/5, A(2^4) = 31/5$, both creating factors that are too large to cancel out), the candidate components are further narrowed down to $A(2^t)$ for $t = 0, 1, 2, 5$ and $A(p) = \frac{p+1}{2}$ for primes $p$ between 3 and 83 inclusive.

    Here is a complete list of all candidates at this point:

    $$A(1) = 1, A(2) = 3/2, A(4) = 7/3, A(32) = 21/2,$$

    $$A(3) = 2, A(5) = 3, A(7) = 4, A(11) = 6, A(13) = 7, A(17) = 9, A(23) = 12,$$

    $$A(31) = 16, A(41) = 21, A(47) = 24, A(53) = 27, A(71) = 36, A(83) = 42.$$

We can now case work on the maximum power of 2 dividing $n$, let this power be $2^t$.

If $t = 0$ then we look for *distinct* primes whose $A$-values multiply to 42, so we have $83, 3 \cdot 41 = 123, 11 \cdot 13 = 143, 3 \cdot 5 \cdot 13 = 195$.

If $t = 1$ then the $A$-values multiply to $\frac{42}{3/2} = 28$, so we have $2 \cdot 7 \cdot 13 = 182$.

If $t = 2$ then the $A$-values multiply to $\frac{42}{7/3} = 18$, so we have $2^2 \cdot 3 \cdot 17 = 204, 2^2 \cdot 5 \cdot 11 = 220$.

If $t = 5$ then the $A$-values multiply to $\frac{42}{21/2} = 4$, so we have $2^5 \cdot 7 = 224$.

Summing up all cases, we get that there are 8 possibilities: $83, 123, 143, 182, 195, 204, 220, 224$, and their sum is 1374. $\qquad\square$

21. Say a real number $r$ is *repetitive* if there exist two distinct complex numbers $z_1, z_2$ with $|z_1| = |z_2| = 1$ and $\{z_1, z_2\} \neq \{-i, i\}$ such that

$$z_1(z_1^3 + z_1^2 + rz_1 + 1) = z_2(z_2^3 + z_2^2 + rz_2 + 1).$$

There exist real numbers $a, b$ such that a real number $r$ is *repetitive* if and only if $a < r \leq b$. If the value of $|a| + |b|$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers $p$ and $q$, find $100p + q$.

*Proposed by James Lin.*

**Answer.** $\boxed{2504}$.

**Solution.** Let the argument of $z$, with $|z| = 1$, be $\theta$. Note that $f(z) = z^4 + z^3 + rz^2 + z + 1$ is in the direction of argument $2\theta$ with a signed magnitude of $r + 2\cos\theta + 2\cos(2\theta) = 4\cos^2\theta + 2\cos\theta - 2 + r = 4(\cos\theta + \frac{1}{4})^2 - \frac{9}{4} + r$.

For $r > \frac{9}{4}$, then note that this signed magnitude is always positive. Then, $f(z_1) = f(z_2)$ can only happen for $z_1 \neq z_2$ if their arguments $\theta_1, \theta_2$ satisfy $2\theta_1 = 2\theta_2 \implies \theta_1 = \theta_2 + \pi$. But then their signed magnitudes can only be equal if $\cos\theta_1 = \cos\theta_2$, but that implies $\{z_1, z_2\} = \{i, -i\}$. Hence $f(z_1) \neq f(z_2)$ for all distinct $z_1, z_2$. For $r < -4$, the signed magnitude is always negative since $|\cos\theta + \frac{1}{4}| \leq \frac{5}{4}$ over all $\theta$. Once again, this implies that $f(z_1) \neq f(z_2)$.

However, if $-4 \leq r \leq \frac{9}{4}$, then there exists a unique value for $\cos\theta$ with $-\frac{1}{4} \leq \cos\theta \leq 1$ such that the signed magnitude is equal to 0. For all $\cos\theta \neq 1$, there are two values of $\theta$ giving the desired value of $\cos\theta$, showing there exist distinct $z_1, z_2$ giving $f(z_1) = f(z_2) = 0$. However, $\cos\theta = 1$ only when $\theta = 0$, meaning for $r = -4$ we see that the signed magnitude is always negative except for $\theta = 0$, when the signed magnitude is exactly 0. Hence $r = -4$ is also *repetitive*. Hence $a = -4$ and $b = \frac{9}{4}$ so $100p + q = 2504$. $\qquad\square$

22. Let $ABC$ be a triangle with $AB = 5$, $BC = 7$, $CA = 8$, and circumcircle $\omega$. Let $P$ be a point inside $ABC$ such that $PA : PB : PC = 2 : 3 : 6$. Let rays $\overrightarrow{AP}$, $\overrightarrow{BP}$, and $\overrightarrow{CP}$ intersect $\omega$ again at $X$, $Y$, and $Z$, respectively. The area of $XYZ$ can be expressed in the form $\frac{p\sqrt{q}}{r}$ where $p$ and $r$ are relatively prime positive integers and $q$ is a positive integer not divisible by the square of any prime. What is $p + q + r$?

*Proposed by James Lin.*

**Answer.** $\boxed{940}$.

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**Solution.** Let the pedal triangle of $P$ with respect to $ABC$ be $DEF$ such that $D$ is on $BC$, $E$ is on $CA$, and $F$ is on $AB$. Note that $\angle PYX = \angle BYX = \angle BAX = \angle FAP = \angle FEP$. Similarly, $\angle PYZ = \angle DEP$, so then $\angle XYZ = \angle DEF$. Similarly $\angle YZX = \angle EFD$, so $\triangle DEF \sim \triangle XYZ$. Then by the Law of Sines on triangles $DEP$ and $FEP$,

$$
\begin{aligned}
\frac{YX}{YZ} &= \frac{ED}{EF} \\
&= \frac{\left(\dfrac{EP \sin EPD}{\sin EDP}\right)}{\left(\dfrac{EP \sin EPF}{\sin EFP}\right)} \\
&= \frac{\sin EPD}{\sin EPF} \cdot \frac{\sin EFP}{\sin EDP} \\
&= \frac{\sin C}{\sin A} \cdot \frac{\sin EAP}{\sin ECP} \\
&= \frac{BA}{BC} \cdot \frac{PC}{PA}.
\end{aligned}
$$

Symmetry shows that $YZ : ZX : XY = PA \cdot BC : PB \cdot CA : PC \cdot AB = 7 : 12 : 15$.

Note that $\cos BAC = \dfrac{5^2 + 8^2 - 7^2}{2 \cdot 5 \cdot 8} = \dfrac{1}{2}$, so $\angle BAC = 60°$, so then the circumradius $R$ of $\omega$ is $\dfrac{BC}{2 \sin A} = \dfrac{7}{\sqrt{3}}$. By Heron's formula, a triangles with side lengths $7, 12, 15$ has area $\sqrt{17 \cdot 10 \cdot 5 \cdot 2} = 10\sqrt{17}$ and circumradius $\dfrac{7 \cdot 12 \cdot 15}{4 \cdot 10\sqrt{17}} = \dfrac{63}{2\sqrt{17}}$. Since $XYZ$ also has circumcircle $\omega$, we can scale the $7 - 12 - 15$ triangle to find the area of $XYZ$ is $10\sqrt{17} \cdot \left(\dfrac{\dfrac{7}{\sqrt{3}}}{\dfrac{63}{2\sqrt{17}}}\right)^2 = \dfrac{680\sqrt{17}}{243}$, so the answer is $940$. $\qquad\square$

23. Let $S$ be the set of all $2017^2$ lattice points $(x, y)$ with $x, y \in \{0\} \cup \{2^0, 2^1, \cdots, 2^{2015}\}$. A subset $X \subseteq S$ is called BQ if it has the following properties:

(a) $X$ contains at least three points, no three of which are collinear.

(b) One of the points in $X$ is $(0, 0)$.

(c) For any three distinct points $A, B, C \in X$, the orthocenter of $\triangle ABC$ is in $X$.

(d) The convex hull of $X$ contains at least one horizontal line segment.

Determine the number of BQ subsets of $S$.

*Proposed by Vincent Huang.*

**Answer.** $\boxed{17274095}$.

**Solution.** First we will determine the possible kinds of BQ subsets $X$.

Consider the convex hull of $X$. Clearly it cannot have any obtuse angles or else we take $A, B, C$ with $\angle ABC$ obtuse to get that the orthocenter of $ABC$ is outside the convex hull, a clear contradiction. Thus we can conclude that the convex hull has either 3 or 4 sides, since it is well-known that any convex polygon of more than 4 sides has an obtuse angle.

If the convex hull of $X$ has four sides, then for it to not have obtuse angles, it must be a rectangle. If the rectangle is $ABCD$, we can't have a point $P$ inside or else WLOG assume $P$ is strictly inside

triangle $ABC$ and then the orthocenter of $PAC$ is outside the rectangle, a contradiction. The rectangle contains $(0,0)$ as its bottom left vertex, so it must be of the form $(0,0),(2^x,0),(0,2^y),(2^x,2^y)$ with $0 \le x,y \le 2015$. This yields $2016^2$ possibilities for $X$.

If the convex hull has three sides, it is a non-obtuse triangle $ABC$. Then its orthocenter $H$ is also obviously in the set. I claim that no other points can be in the set. Suppose another point $P$ is in the set, and WLOG assume $P$ is strictly inside triangle $BCH$. Now consider the orthocenter of $PBC$, and it is easy to see from $\angle BPC > \angle BHC \ge 90°$ that the orthocenter of $PBC$ is outside the convex hull of $X$, a contradiction.

Now we must count the number of triangles whose vertices are in the set and whose orthocenters are also in the set $S$. We split into two cases:

**Case 1.** The orthocenter is $(0,0)$.

Then the other two vertices of the triangle are $(2^x,0),(0,2^y)$ which yields $2016^2$ possibilities as before.

**Case 2.** The orthocenter is not $(0,0)$.

**Case 2.1.** Suppose there is no point in $X$ of the form $(a,0)$ with $a \ne 0$. We need a horizontal segment somewhere in the convex hull, and since the convex hull is non-obtuse this implies the horizontal side is somewhere above $(0,0)$, so the set $X$ is $(0,0),(0,a),(b,a)$ for some $a,b$. This yields $2016^2$ more possibilities.

**Case 2.2.** There is another point in the set which is of the form $(a,0)$, since the convex hull of $X$ contains at least one horizontal segment. Let the third point of the triangle be $(b,c)$ and from coordinates we can easily solve for the orthocenter which is $\left(b, \dfrac{b(a-b)}{c}\right)$. Notice that since the orthocenter is inside the triangle that $b(a-b) \le c^2$.

Since $a,b,c,\dfrac{b(a-b)}{c}$ are powers of two, we conclude $a-b$ is a power of two and thus either $a=b$ or $a=2b$.

**Case 2.2.1.** $a=b$. Then our set is just $(0,0),(a,0),(a,c)$ and the orthocenter is $(a,0)$,m which is already inside $X$. So we need $1 \le a \le 2^{2015}, 1 \le c \le 2^{2015}$ which yields another $2016^2$ possibilities.

**Case 2.2.2.** $a=2b$. So our points are $(0,0),(2b,0),(b,c),\left(b,\dfrac{b^2}{c}\right)$. Recall from earlier that $b \le c$ but also every coordinate must be $\le 2^{2015}$. So let $b = 2^x$ with $0 \le x \le 2014$ and let $c = 2^y$. We know $c|b^2, c \ge b$ which translates into $x \le y \le \min(2x, 2015)$. Counting the number of pairs $(x,y)$ satisfying this is easy to do by splitting the sum at $x = 1007$: When $x \le 1007$ we have $x + 1$ choices for $y$ and when $1008 \le x \le 2014$ we have $2016 - x$ choices for $y$, yielding a total of $(1 + 2 + ... + 1008) + (1008 + 1007 + ... + 2) = 1008 \cdot 1009 - 1$ pairs $(x,y)$.

Thus our final answer is the sum $4 \cdot 2016^2 + 1008 \cdot 1009 - 1 = 17274095$

$\square$

24. Bessie and her 2015 bovine buddies work at the Organic Milk Organization, for a total of 2016 workers. They have a hierarchy of bosses, where obviously no cow is its own boss. In other words, for some pairs of employees $(A, B)$, $B$ is the boss of $A$. This relationship satisfies an obvious condition: if $B$ is the boss of $A$ and $C$ is the boss of $B$, then $C$ is also a boss of $A$. Business has been slow, so Bessie hires an outside organizational company to partition the company into some number of groups. To promote growth, every group is one of two forms. Either no one in the group is the boss of another in the group, or for every pair of cows in the group, one is the boss of the other. Let $G$ be the minimum number of groups needed in such a partition. Find the maximum value of $G$ over all possible company structures.

*Proposed by Yang Liu.*

**Answer.** $\boxed{63}$.

**Solution.** This solution will use the language of *posets*, *chains*, and *antichains*. A poset is exactly the structure defined in the problem. A chain is a subset of elements of the poset such that all pairs are comparable, and an antichain is a subset of elements of the poset such that no two are comparable. This problem is then asking to cover $G$ with chains and antichains.

Note that $2016 = \frac{63 \cdot 64}{2}$. I claim that the answer is 63. The upper bound will be shown in the following lemma.

**Lemma.** In a poset with $\frac{k(k+1)}{2}$ elements, it can be covered using at most $k$ chains or antichains.

*Proof.* This can be done easily with Dilworth's Theorem, but I will present a proof here that doesn't appeal to Dilworth's Theorem. For each $v \in G$, label $v$ with the longest path in the poset that starts at $v$. Let $f(v)$ denote this label. More explicitly, if we let $<$ denote the binary comparator on $G$, $f(v)$ is the largest $m$ such that there exists a sequence $v = v_1 < v_2 < \cdots < v_m$, where $v_i \in G \forall i$. You can think of this as grouping the poset into "layers". If $f(v) \geq k$ for some $v$, then there exists a chain of length $k$ starting at $v$. Now delete this chain. Afterwards, we are left with a poset with $\frac{(k-1)k}{2}$ elements, which can be covered in $k-1$ chains or antichains by induction. So in this case we use $k$ in total.

Otherwise, $f(v) < k$ for all $v$. This now admits a decomposition into $k-1$ antichains, where the $i$-th antichain is simply the set of all $v$ such that $f(v) = i$. To see that each of these sets is an antichain, assume that there exist $u, v$ within the same set such that $u < v$. But this obviously means that $f(u) \geq f(v) + 1$, contradiction. So in this case, there exists a covering using only $k-1$ antichains, as desired.

$\square$

Now we prove the lower bound by providing a construction. For the construction, let the elements be grouped into $k$ groups $G_1, G_2, \ldots, G_k$ such that $|G_i| = i$ and such that for all $i < j$, if $v_i \in G_i, v_j \in G_j$, then $v_i > v_j$. These are also the only relations, which means that each $G_i$ is an antichain. This has a total of $\frac{k(k+1)}{2}$ elements. I claim that it needs at least $k$ chains or antichains to cover it. We proceed by induction.

If a chain is used, it might as well be of length $k$, since deleting more vertices doesn't hurt us later on. After deleting this chain of length $k$, we have reduced the poset to the case with $\frac{(k-1)k}{2}$ elements of the same construction, which requires $k-1$ more chains or antichains. If no chain is used, then obviously we need at least $k$ antichains to cover everything. Therefore, we are done.

$\square$

25. Given a prime $p$ and positive integer $k$, an integer $n$ with $0 \leq n < p$ is called a $(p, k)$-Hofstadterian residue if there exists an infinite sequence of integers $n_0, n_1, n_2, \ldots$ such that $n_0 \equiv n$ and $n_{i+1}^k \equiv n_i$ (mod $p$) for all integers $i \geq 0$. If $f(p, k)$ is the number of $(p, k)$-Hofstadterian residues, then compute
$$\sum_{k=1}^{2016} f(2017, k).$$
*Proposed by Ashwin Sah.*

**Answer.** $\boxed{1296144}$.

**Solution.** Let $p = 2017$ throughout this solution. Also, let $p$ prime factorize as $\prod_{i=1}^{m} q_i^{e_i}$.

For an integer $k$, let $d(k, p-1)$ denote the largest divisor of $p-1$ that is relatively prime to $k$. Using primitive roots, it is not hard to see that there are exactly $d(k, p-1) + 1$ $(p, k)$-Hofstaderian residues. The plus 1 comes from including 0. Now we must compute $p - 1 + \sum_{k=1}^{p-1} d(k, p-1)$.

To compute this sum, say that exactly the primes $q_{a_1}, q_{a_2}, \ldots, q_{a_j}$ divide $\gcd(k, p-1)$. It's easy to see that $(p-1)\prod_{i=1}^{j}\left(1 - \frac{1}{q_{a_i}}\right)$ values of $k$ satisfy this. For these $k$, $d(k, p-1) = \frac{\phi(p-1)}{\prod_{i=1}^{j}(q_{a_i}-1)}$. Therefore, our sum can be rewritten as (where the sum is over all subsets of the prime factors of $p-1$)

$$\sum_{q_{a_1}, q_{a_2}, \ldots, q_{a_j}} \frac{\phi(p-1)}{\prod_{i=1}^{j}(q_{a_i}-1)} \cdot \frac{p-1}{\prod q_{a_i}^{e_{a_i}}} = (p-1)\phi(p-1)\prod\left(1 + \frac{1}{q_i^{e_i}(q_i-1)}\right) = 1296144,$$

as desired.

$\square$

26. Let $S$ be the set of all pairs $(a, b)$ of integers satisfying $0 \le a, b \le 2014$. For any pairs $s_1 = (a_1, b_1), s_2 = (a_2, b_2) \in S$, define

$$s_1 + s_2 = ((a_1 + a_2)_{2015}, (b_1 + b_2)_{2015}) \text{ and } s_1 \times s_2 = ((a_1 a_2 + 2b_1 b_2)_{2015}, (a_1 b_2 + a_2 b_1)_{2015}),$$

where $n_{2015}$ denotes the remainder when an integer $n$ is divided by 2015.

Compute the number of functions $f : S \to S$ satisfying

$$f(s_1 + s_2) = f(s_1) + f(s_2) \text{ and } f(s_1 \times s_2) = f(s_1) \times f(s_2)$$

for all $s_1, s_2 \in S$.

*Proposed by Yang Liu.*

**Answer.** $\boxed{81}$.

**Solution 1.** We can think of $S$ as the set of all $a + b\sqrt{2}$, where $a, b$ are taken $\pmod{2015}$. Then the sum and product work how we expect:

$$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_1) + (b_1 + b_2)\sqrt{2}$$

and

$$(a_1 + b_1\sqrt{2}) \times (a_2 + b_2\sqrt{2}) = (a_1 a_2 + 2b_1 b_2) + (b_1 a_2 + a_1 b_2)\sqrt{2}$$

We want to find all functions $f : S \to S$ that preserve both addition and multiplication. First, suppose such a function $f$ exists. Note that for any $a \in S$, $f(a + a) = f(a) + f(a) = 2f(a)$, $f(3a) = f(a) + f(2a) = 3f(a)$, and so on, showing that for any positive integer $n$, $f(na) = nf(a)$. Since all integers are taken $\pmod{2015}$, we equivalently have $f(na) = nf(a)$ for any remainder $n \pmod{2015}$.

Let $f(1) = x$ and $f(\sqrt{2}) = y$. From $f(na) = nf(a)$ and preservation of addition, we get

$$f(a + b\sqrt{2}) = af(1) + bf(\sqrt{2}) = ax + by.$$

Additionally, note that

$$2x = 2f(1) = f(2) = f(\sqrt{2})^2 = y^2,$$

so $x = \frac{y^2}{2}$. Also,

$$\frac{y^3}{2} = xy = f(1)f(\sqrt{2}) = f(\sqrt{2}) = y$$

so $y^3 = 2y$. Thus any valid $f$ must be in the form

$$f(a + b\sqrt{2}) = a\left(\frac{y^2}{2}\right) + by$$

for some $y \in S$ satisfying $y^3 = 2y$. We claim that this is also sufficient for $f$ to preserve both addition and multiplication. It is clear that such an $f$ preserves addition, and

$$
\begin{aligned}
f(a + b\sqrt{2})f(c + d\sqrt{2}) &= \left(\frac{ay^2}{2} + by\right)\left(\frac{cy^2}{2} + dy\right) \\
&= \frac{acy^4}{4} + \frac{bcy^3}{2} + \frac{ady^3}{2} + bdy^2 \\
&= \frac{acy^2}{2} + bcy + ady + bdy^2 \\
&= (ac + 2bd)\left(\frac{y^2}{2}\right) + (bd + ad)(y) \\
&= f((ac + 2bd) + (bd + ad)\sqrt{2}) \\
&= f((a + b\sqrt{2})(c + d\sqrt{2}))
\end{aligned}
$$

so it preserves multiplication as well. Thus it suffices to find the number of solutions to $y^3 = 2y$ in $S$. Now for some $a, b \pmod{2015}$, $(a + b\sqrt{2})^3 = 2(a + b\sqrt{2})$ is a set of polynomial equations in $a, b$. So by the Chinese Remainder Theorem, the answer is the product of the number of solutions to $y^3 = 2y$ when the coefficients are taken $\pmod 5$, $\pmod{13}$, and $\pmod{31}$.

Let $\mathbb{F}_p$ denote integers $\pmod p$, and let $S_p$ be the set of all $a + b\sqrt{2}$, where $a, b \in \mathbb{F}_\mathbb{l}$ for an odd prime $p$. Suppose two elements of $S_p$ multiply to 0; i.e.

$$0 = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2}$$

Then $ac + 2bd \equiv 0 \pmod p$ and $bc + ad \equiv 0 \pmod p$.

Now if $a = b = 0$ or $c = d = 0$, we clearly get a solution. Assuming neither of these hold, now suppose $a = 0 \pmod p$; then $bd = 0 \pmod p$ and $bc = 0 \pmod p$. This is only possible if either $a = b = 0$ or $c = d = 0$, so this is a contradiction. Similarly, we can assume that all of $b, c, d$ are nonzero $\pmod p$. Then

$$c \equiv -\frac{2bd}{a} \implies \frac{2b^2 d}{a} = ad$$

which is equivalent to $\left(\frac{a}{b}\right)^2 = 2$. [1] It's well-known that in $\mathbb{F}_p$ for odd $p$, the equation $z^2 = 2$ has no solution if $p \equiv 3, 5 \pmod 8$, and there exists (nonzero) $g \in \mathbb{F}_\mathbb{l}$ such that $z^2 = 2$ if and only $z = \pm g$. So in $S_p$, for $p \equiv 3, 5 \pmod 8$, $(a + b\sqrt{2})(c + d\sqrt{2})$ implies $a + b\sqrt{2} = 0$ or $c + d\sqrt{2} = 0$.

If $(a + b\sqrt{2})(c + d\sqrt{2}) = 0$ in $S_p$ for $p \equiv 1, 7 \pmod 8$ (and neither $a = b = 0$ nor $c = d = 0$ holds), then $a = gb$, which implies $c = -gd$ from $bc + ad = 0$, or $a = -gb$, which implies $c = gd$. Thus

$$\{a + b\sqrt{2}, c + d\sqrt{2}\} = \{k(g - \sqrt{2}), \ell(g + \sqrt{2})\}$$

for some $k, \ell \in \mathbb{F}_p$.

We finally now consider the number of solutions to $y^3 = 2y$ in $S_p$ for $p = 5, 13, 31$.

If $p = 5$ or $p = 13$, then $p \equiv 3, 5 \pmod 8$, so $mn = 0$ implies $m = 0$ or $n = 0$ for $m, n \in S_p$. Note then that if $y^3 = 2y$, then $y(y - \sqrt{2})(y + \sqrt{2}) = 0$. Thus either $y = 0$, $y = \sqrt{2}$, or $y = -\sqrt{2}$. So in both cases, there are 3 solutions to $y^3 = 2y$.

If $p = 31$, then $p \equiv 7 \pmod 8$, so $mn = 0$ implies $m = 0$, $n = 0$, or $\{m, n\} = \{k(8 - \sqrt{2}), \ell(8 + \sqrt{2})\}$ (noting that $8^2 \equiv 2 \pmod{31}$). So if $y(y^2 - 2) = 0$, either $y = 0$, $y^2 = 2$, or $y = k(8 \pm \sqrt{2})$.

If $y = 0$, of course we have 1 solution for $y$.

If $y^2 = 2$, then either $y = \pm\sqrt{2}$ or $y - \sqrt{2} = k(8 \pm \sqrt{2})$. If the second case holds, then $k(8 \pm \sqrt{2}) + 2\sqrt{2} = \ell(8 \mp \sqrt{2})$, implying $8k = \mp 8(2 \pm k)$. Thus either $8k = 16 - 8k$ or $8k = -16 - 8k$, yielding $k = 1$ and

_____
[1] Note that $\frac{a}{b}$ is in $\mathbb{F}_p$, **not** $S_p$. We must be careful to distinguish $\sqrt{2} \in S_p$ and an element of $\mathbb{F}_p$ which is a root of $z^2 - 2$.

$y - \sqrt{2} = 8 - \sqrt{2}$, or $k = -1$ and $y - \sqrt{2} = -8 + \sqrt{2}$. These two cases give the solutions $y = 8$ and $y = -8$ additionally, so there are 4 solutions to $y^2 = 2$.

If $y = k(8 \pm \sqrt{2})$, then $y^2 - 2 = \ell(8 \mp \sqrt{2})$. Note that

$$y^2 - 2 = -2 + k^2(4 \pm 16\sqrt{2})$$

so

$$(4k^2 - 2) = \mp 8(\pm 16k^2) = -4k^2$$

Implying $4k^2 = 1$ and so $k = 15$ or $k = 16$. So in this case we get the 4 solutions $k = 15(8 + \sqrt{2})$, $k = 16(8 + \sqrt{2})$, $k = 15(8 - \sqrt{2})$, and $k = 16(8 - \sqrt{2})$.

In total, the $p = 31$ case gives $1 + 4 + 4 = 9$ solutions for $y^3 = 2y$ in $S_{31}$. The other two primes, $p = 5$ and $p = 13$, each yield 3 solutions, so the number of solutions to $y^3 = y$ in $S$ is $3 \cdot 3 \cdot 9 = 81$ (by the Chinese Remainder Theorem). So our final answer is 81. □

**Solution 2.** There is an alternate solution which avoids much of the casework of the $p = 31$ case, but requires a little more tricky theory. The key is to note that in the ring $\mathbb{Z}[\sqrt{2}]$, 2015's full factorization into primes is not $5 \times 13 \times 31$, but rather $5 \times 13 \times (-1 + 4\sqrt{2}) \times (1 + 4\sqrt{2})$. (We use the fact that $\mathbb{Z}[\sqrt{2}]$ is a unique factorization domain, which allows us to identify primes and irreducibles.)

To prove that each of these factors are each indeed prime, we consider the multiplicative norm $N(a + b\sqrt{2}) = a^2 - 2b^2$. $-1 + 4\sqrt{2}$ and $1 + 4\sqrt{2}$ each have norm $-31$, which is prime. Thus if two elements of $\mathbb{Z}[\sqrt{2}]$ multiply to $\pm 1 + 4\sqrt{2}$, then one must have norm $\pm 1$, which would make it a unit. This implies that both are irreducible, and therefore prime. Now note that if $a + b\sqrt{2} \mid 5$, then by taking norms we get $a^2 - 2b^2 \mid 5^2$, so either $a + b\sqrt{2}$ is a unit or $5 \mid a^2 - 2b^2$. But as 2 is not a quadratic residue (mod 5), this is impossible unless $5 \mid a$ and $5 \mid b$, which would make $\frac{5}{a+b\sqrt{2}}$ a unit. Therefore 5, and by a similar argument 13, are each prime in $\mathbb{Z}[\sqrt{2}]$.

Let $5, 13, -1 + 4\sqrt{2}$, and $1 + 4\sqrt{2}$ be denoted by $p_1, p_2, p_3$, and $p_4$, respectively. Note that

$$S = \mathbb{Z}[\sqrt{2}]/(2015) = \mathbb{Z}[\sqrt{2}]/(p_1 p_2 p_3 p_4)$$

Now by the Chinese Remainder Theorem, there is an isomorphism between $S$ and the product of the quotient rings formed by each of the ideals $(p_i)$, assuming these ideals pairwise coprime (which is easy to check):

$$R/(p_1 p_2 p_3 p_4) \cong R/(p_1) \oplus R/(p_2) \oplus R/(p_3) \oplus R/(p_4)$$

where $R = \mathbb{Z}[\sqrt{2}]$ is a commutative ring. Thus the number of solutions to $y^3 = 2y$ in $S = R/(p_1 p_2 p_3 p_4)$ is the product of the number of solutions in each $R/(p_i)$.

Now the quotient $R/(p_i)$ of a commutative ring by a prime ideal is an integral domain. (In fact, since it is finite, it is a field as well; explicitly we have the decomposition

$$S \cong \mathbb{F}_{5^2} \oplus \mathbb{F}_{13^2} \oplus \mathbb{F}_{31}^2,$$

but we won't need this for our purposes.) So within each integral domain, $y(y - \sqrt{2})(y + \sqrt{2}) = 0$ only holds when one of the terms vanishes, i.e. when $y = 0$ or $y = \pm\sqrt{2}$. So the total number of solutions is $3^4 = 81$. □

27. Let $ABC$ be a triangle with circumradius 2 and $\angle B - \angle C = 15°$. Denote its circumcenter as $O$, orthocenter as $H$, and centroid as $G$. Let the reflection of $H$ over $O$ be $L$, and let lines $AG$ and $AL$ intersect the circumcircle again at $X$ and $Y$, respectively. Define $B_1$ and $C_1$ as the points on the circumcircle of $ABC$ such that $BB_1 \parallel AC$ and $CC_1 \parallel AB$, and let lines $XY$ and $B_1C_1$ intersect at $Z$. Given that $OZ = 2\sqrt{5}$, then $AZ^2$ can be expressed in the form $m - \sqrt{n}$ for positive integers $m$ and $n$. Find $100m + n$.

*Proposed by Michael Ren.*

**Answer.** $\boxed{3248}$.

**Solution.** Let $\Omega$ be the circumcircle of $ABC$, and $A_1$ is on $\Omega$ such that $AA_1 \parallel BC$. Let lines $BB_1$ and $CC_1$ intersect at $P$, lines $CC_1$ and $AA_1$ intersect at $Q$, and lines $AA_1$ and $BB_1$ intersect at $R$. Note that $A$, $G$, and $P$ are collinear. Also, note that $H$ is the circumcenter and $O$ is the nine-point center of $PQR$, so hence $L$ is the orthocenter of $PQR$. Hence, $L$ is the incenter of $A_1B_1C_1$. Let $\omega_1$ be the incircle of $A_1B_1C_1$, so $\omega_1$ has center $L$ and let it touch side $B_1C_1$ at $D$. Similarly, let $\omega_2$ be the $A_1$-excircle of $A_1B_1C_1$, so $\omega_2$ has center $P$ and let it touch side $B_1C_1$ at $E$. Note that $A$ is the midpoint of arc $B_1A_1C_1$, so let $AA_1$ (the external angle bisector of $\angle B_1A_1C_1$) intersect $B_1C_1$ at $K$.

Let $\mathcal{F}$ be the inversion centered at $A_1$ with radius $\sqrt{A_1B_1 \cdot A_1C_1}$, and let $\mathcal{G}$ be the reflection through the internal angle bisector of $\angle B_1A_1C_1$. Let $\mathcal{H} = \mathcal{F} \circ \mathcal{G}$. Note that $\angle LA_1K = \angle LDK = 90°$, so $A_1LDK$ is cyclic. Note that $\mathcal{H}(\omega_1)$ is the $A_1$-mixtillinear excircle of $A_1B_1C_1$. Hence, the line through $\mathcal{H}(L) = P$ and $\mathcal{H}(K) = A_1$ intersects $\Omega$ at $\mathcal{H}(D)$, so then $X$ is the tangency point of the $A_1$-mixtillinear excircle of $A_1B_1C_1$ and $\Omega$. Similarly, since $\angle PA_1K = \angle PEK = 90°$, $A_1PEK$ is cyclic. Note that $\mathcal{H}(\omega_2)$ is the $A_1$-mixtillinear incircle of $A_1B_1C_1$, so the line through $\mathcal{H}(P) = L$ and $\mathcal{H}(K) = A_1$ intersects $\Omega$ at $\mathcal{H}(E)$, so then $Y$ is the tangency point of the $A_1$-mixtillinear incircle of $A_1B_1C_1$ and $\Omega$.

Let the tangent of $X$ to $\Omega$ be $\ell_1$, and the tangent of $Y$ to $\Omega$ be $\ell_2$. Let $\ell_1 \cap \ell_2 = T$. Note that $\mathcal{H}(\ell_1)$ is the circle tangent to $BC$ at $D$ passing through $A_1$, and $\mathcal{H}(\ell_2)$ is the circle tangent to $BC$ at $E$ passing through $A_1$. Let $\mathcal{H}(T) = S$ be the second intersection of the two circles, so that $A_1T$ is the radical axis of these two circles. But the midpoint $M$ of side $B_1C_1$ is the midpoint of $DE$, so $A_1, S, M$ are collinear so hence $A_1T$ is the $A_1$-symmedian of triangle $A_1B_1C_1$. Let the tangents of $B_1$ and $C_1$ to $\Omega$ be $F$, so then note that $Z$ lies on the polar of both $T$ and $F$ with respect to $\Omega$. Hence, $Z$ is the pole of line $TF$, but $T$ and $F$ both lie on the $A_1$-symmedian of $A_1B_1C_1$, so $Z$ lies on the polar of $A_1$ and hence $ZA_1$ is tangent to $\Omega$.

Now, let the foot of the perpendicular from $Z$ to line $AA_1$ be $Z_1$. Note that $ABA_1 = \angle B - \angle C = 15°$ so $AA_1 = \sqrt{6} - \sqrt{2}$, and $\angle AA_1Z = 180 - (\angle B - \angle C) = 165°$ so $\angle ZA_1Z_1 = \angle 15°$. Note that $ZA_1 = 4$, so $A_1Z_1 = \sqrt{6} + \sqrt{2}$ and $ZZ_1 = \sqrt{6} - \sqrt{2}$ so by the Pythagorean Theorem, $AZ^2 = 32 - \sqrt{48}$, so our answer is 3248. $\qquad\square$

28. Let $N$ be the number of polynomials $P(x_1, x_2, \ldots, x_{2016})$ of degree at most 2015 with coefficients in the set $\{0, 1, 2\}$ such that $P(a_1, a_2, \cdots, a_{2016}) \equiv 1 \pmod{3}$ for all $(a_1, a_2, \cdots, a_{2016}) \in \{0, 1\}^{2016}$.

Compute the remainder when $v_3(N)$ is divided by 2011, where $v_3(N)$ denotes the largest integer $k$ such that $3^k | N$.

*Proposed by Yang Liu.*

**Answer.** $\boxed{189}$.

**Solution.** Let $n = 2016$ throughout this solution. We therefore need to compute the number of $n$-variable polynomials of degree at most $n - 1$ such that it is identically 1 on the set $\{0, 1\}^n$. We first prove a useful lemma. Say a multivariate polynomial is *simple* if the exponent of every variable is at most 1 in all terms. Therefore, $x_1x_2x_3$ is simple, while $x_1x_2 + x_3^2x_4$ isn't.

**Lemma.** If $Q(x_1, x_2, \ldots, x_n)$ is simple and $Q(x_1, x_2, \ldots, x_n) = 0 \,\forall\, (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$, then $Q \equiv 0$.

*Proof.* We go by induction on $n$, the number of variables. If $x_1$ doesn't appear then we are done by induction. Otherwise, we can find simple polynomials $P_0(x_2, x_3, \ldots, x_n), P_1(x_2, x_3, \ldots, x_n)$ such $Q = x_1P_1 + P_2$. Since $x_1$ appears, $P_1$ is not identically 0 as a polynomial. By induction, there exists a $(n-1)$-tuple $a = (a_2, a_3, \ldots, a_n) \in \{0, 1\}^{n-1}$ such that $P_1(a_2, a_3, \ldots, a_n) \neq 0$. Then the single variable polynomial $x_1 \cdot P_1(a) + P_2(a) = 0$ for $x_1 \in \{0, 1\}$, a contradiction because the degree of the previous term is 1 but has 2 roots. $\qquad\square$

Let $Q(x) = P(x) - 1$. If $P(x) = 1$ for all $x \in \{0,1\}^n$, then $Q(x) = 0$ for all $x \in \{0,1\}^n$, and the degree of $Q$ is at most $n-1$ (where the degree of the 0 polynomial is defined to be $-1$). Consider all terms of the form $\prod_{i=1}^n x_i^{e_i}$ where $e_i \geq 2$ for some $i$, and $\sum e_i \leq n-1$. One can easily count that there are $\binom{2n-1}{n-1} - 2^n + 1$ of these terms. Therefore, there are $3^{\binom{2n-1}{n-1} - 2^n + 1}$ ways to choose the coefficients of these terms. I now claim that the coefficients of the remaining terms (where $e_i \leq 1 \, \forall \, i$) must be fixed.

To prove this, note that for $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$, $\prod_{i=1}^n x_i^{e_i} = \prod_{i=1}^n x_i^{\min(e_i,1)}$, as $x_i^2 = x_i$ if $x_i \in \{0,1\}$. Now transform $Q$ into a new polynomial $Q'$ by replacing every term $\prod_{i=1}^n x_i^{e_i}$ with $\prod_{i=1}^n x_i^{\min(e_i,1)}$. Now by the above lemma, $Q'$ must be the zero polynomial. Therefore, there was exactly one way to choose the coefficients of each simple monomial in $Q$ that makes the coefficient vanish in $Q'$. This proves the claim.

Therefore, the number of different $Q$ is $3^{\binom{2n-1}{n-1} - 2^n + 1}$. Extracting the answer is easy from here.

$\square$

**Remark.** Simulating the above proof in the correct generality leads to a proof of the *Combinatorial Nullstellensatz*. In fact, the above lemma follows directly from an application of this theorem. For a proof and applications, see *Problems From the Book* Chapter 23, or this abbreviated handout.

29. Yang the Spinning Square Sheep is a square in the plane such that his four legs are his four vertices. Yang can do two different types of *tricks*:

    (a) Yang can choose one of his sides, then reflect himself over the side.
    (b) Yang can choose one of his legs, then rotate $90°$ counterclockwise around the leg.

    Yang notices that after 2016 tricks, each leg ends up in exactly the same place the leg started out in! Let there be $N$ ways for Yang to perform his 2016 tricks. What is the remainder when $N$ is divided by 100000?

    *Proposed by James Lin.*

    **Answer.** $\boxed{20000}$.

    **Solution.** Assume Yang has his sides parallel to the coordinate axes. Denote the reflection tricks as $R_1, U_1, L_1, D_1$ depending on whether the reflection takes his body to the right, up, left, or down. Similarly define $R_2, U_2, L_2, D_2$ for the rotations. First we count the number of ways for Yang to arrive back at his original square after 2016 tricks, regardless of his orientation or direction. Assume his only moves are $R, U, L, D$. Then it's clear that the number of ways is $\sum_{k=0}^{1008} \binom{2016}{2k}\binom{2k}{k}\binom{2016-2k}{1008-k} = \binom{2016}{1008}\binom{1008}{k}\binom{1008}{1008-k}$. By Vandermonde's Identity, this is equal to $\binom{2016}{1008}^2$. Multiplying by $2^{2016}$ to take into account that each move may be a rotation or a reflection gives a total of $\binom{2016}{1008}^2 \cdot 2^{2016}$.

    Now, we consider the orientations and directions. Take the right side of Yang and denote it as his head. We give Yang two orientations: $S$ for when his legs are in the same counterclockwise order as originally and $O$ for when his legs are in the opposite counterclockwise order as originally. Let the subscripts $0, 1, 2, 3$ denote whether Yang's head is facing east, north, west, or south, respectively. Now, split the orientations/directions of Yang into two groups: Group $A$ consists of $S_0, S_2, O_1, O_3$ and Group $B$ consists of $S_1, S_3, O_0, O_2$. Note that each rotation keeps the same orientation but increases the index by 1, and each reflection flips the orientation but keeps the index the same modulo 2. Hence, each move switches the group the sequence is in, so Yang must be in Group $A$ after 2016 tricks.

Now, fix a sequence of moves that results in Yang at his original square. Note that a rotation with $R$ and $L$, and similarly for a reflection with $R$ and $L$, are indistinguishable when it comes to orientation and direction. Thus, we denote them both by $H$. Similarly, we denote $U$ and $D$ by the single movement $V$. We split into two cases: a case where there are two consecutive $H$'s or two consecutive $V$'s, or another case where the sequence is either $HVHV\cdots HV$, or $VHVH\cdots VH$.

**Case 1.** Assume that there are two consecutive $H$'s. We will show that the sequence is equally likely to be any of the four orientations/directions in Group $A$ after 2016 tricks. Let $\mathcal{M}$ be the sequence before the two $H$'s. We perform a bijection by varying over all rotation/reflection possibilities of the two $H$'s. Let $H_1$ be a horizontal reflection, and $H_2$ be a horizontal rotation. $H_1H_1$ gives the same orientation as $\mathcal{M}$ and an index increase of 2, $H_2H_2$ gives the same orientation as $\mathcal{M}$ and an index increase of 0 or 4, and both of $H_1H_2$ and $H_2H_1$ flip the orientation and one gives an increase of 1 while the other gives an increase of 3, depending on the direction of $\mathcal{M}$. Thus, Yang is equally like to take any orientation/direction in the group he must be in after the two $H$'s, and no different choices for the rotation/reflection of the $H$'s will result in the same orientation/direction of Yang ever again. Hence, exactly one of the four choices for the rotation/reflection of the two $H$'s will allow Yang to go back to $S_0$ after 2016 moves.

**Case 2.** Assume the sequence is $HVHV\cdots HV$. $VHVH\cdots VH$ follows analogously. The number of ways this can result in Yang being in his original square is $\binom{1008}{504}^2 \cdot 2^{2016}$, since we must choose the $H$ and $V$'s to be $L$ and $R$, or $U$ and $D$, respectively. Now, we show that after $2n$ moves, Yang is equally likely to be in either $S_{2n}$ or $O_{2n+1}$, for $n \geq 1$. Define $V_1$ and $V_2$ similarly to how $H_1$ and $H_2$ were defined in the previous case. After one move, note that both $H_1V_1$ and $H_2V_2$ give $S_2$, and $H_1V_2$ and $H_2V_1$ both give $O_3$. Now, assume that $2n = k$ holds, we will show it for $2n = k+2$. Assume that For each way Yang is in $S_{2k}$, $H_1V_1$ and $H_2V_2$ both give $S_{2k+2}$, $H_1V_2$ and $H_2V_1$ are both in $O_{2k+3}$; and for each way Yang is in $O_{2k+1}$, $H_1V_1$ and $H_2V_2$ both give $O_{2k+3}$ and $H_1V_2$ and $H_2V_1$ both give $S_{2k+2}$. Hence, since $2016 \equiv 0 \pmod{4}$, Yang has a $\frac{1}{2}$ chance of being in his original position. Hence, our final answer is $\binom{2016}{1008}^2 \cdot 2^{2014} + \binom{1008}{504}^2 \cdot 2^{2015}$ after doubling to account for $VHVH\cdots VH$. Now, we wish to evaluate this modulo 100000, which by the Chinese Remainder Theorem we simply have to do for 32 and $5^5 = 3125$. It's clear that our answer is divisible by 32.

Note that to compute $\binom{2016}{1008}$ and $\binom{1008}{504}^2 \pmod{3125}$, we only need to compute $\binom{2016}{1008}$ and $\binom{1008}{504} \pmod{125}$ since it's easy to check that $v_5\binom{2016}{1008} = \binom{1008}{504} = 2$. Here, we use the fact that $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^3}$ for all integers $a, b$ and $p > 3$. Using this fact, we get that $\binom{2016}{1008} \equiv \binom{2000}{1000}\binom{16}{8} \equiv \binom{16}{8}^2 \pmod{125}$. Then, $\binom{2016}{1008}^2 \equiv 5^4 \cdot 2574^4 \equiv 625 \pmod{3125}$. Similarly, $\binom{1008}{504}^2 \equiv 5^4 \cdot 14^4 \equiv 625 \pmod{3125}$. Also, note that $2^{2014} \equiv 4 \pmod 5$ and $2^{2015} \equiv 3 \pmod 5$, so $\binom{2016}{1008}^2 \cdot 2^{2014} + \binom{1008}{504}^2 \cdot 2^{2015} \equiv 2500 + 1875 \equiv 1250 \pmod{3125}$. Then, since our answer is 1250 (mod 3125) and 0 (mod 32), our final answer modulo 100000 is 20000. $\qquad\square$

30. In triangle $ABC$, $AB = 3\sqrt{30} - \sqrt{10}$, $BC = 12$, and $CA = 3\sqrt{30} + \sqrt{10}$. Let $M$ be the midpoint of $AB$ and $N$ be the midpoint of $AC$. Denote $l$ as the line passing through the circumcenter $O$ and orthocenter $H$ of $ABC$, and let $E$ and $F$ be the feet of the perpendiculars from $B$ and $C$ to $l$, respectively. Let $l'$ be the reflection of $l$ in $BC$ such that $l'$ intersects lines $AE$ and $AF$ at $P$ and $Q$, respectively. Let lines $BP$ and $CQ$ intersect at $K$. $X$, $Y$, and $Z$ are the reflections of $K$ over the perpendicular bisectors of

sides $BC$, $CA$, and $AB$, respectively, and $R$ and $S$ are the midpoints of $XY$ and $XZ$, respectively. If lines $MR$ and $NS$ intersect at $T$, then the length of $OT$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers $p$ and $q$. Find $100p + q$.

*Proposed by Vincent Huang and James Lin.*

**Answer.** $\boxed{11271}$.

**Solution.** Let $O_1, O_2$, and $O_3$ be the circumcenters of $AHO$, $BHO$ and $CHO$. Remark that $BE$, $CF$, and the line through $A$ perpendicular to $OH$ are mutually parallel and thus concur at $\infty_{\perp OH}$. Then since $AH, AO$ are isogonal w.r.t. $\angle BAC$, we know that $AO_1, BO_2, CO_3$ concur at the isogonal conjugate of $\infty_{\perp OH}$, and the isogonal conjugate of a point at infinity is a point on $\omega$. Let this point of concurrence be $W$.

**Lemma 1.** $W$ lies on $l'$.

*Proof.* Let $L$ be the midpoint of side $BC$, and let $G'$ be the reflection of centroid $G$ over $M$. Let $\mathcal{F}$ be the reflection over line $BC$, $\mathcal{G}$ be the reflection over the perpendicular bisector, and $\mathcal{H}$ be the reflection over the point $L$. Note that $\mathcal{F}(l) = \mathcal{G}(\mathcal{H}(l))$. Now, note that $\mathcal{H}(l)$ is a homothety of factor 2 from $A$. $\square$

Now, let $U$ be the foot of the altitude from $A$ to $AOH$, and let $V$ be where line $AU$ intersect the circumcircle of $ABC$ again. Note that $\mathcal{H}(U) = V$ and $\mathcal{G}(V) = W$, so $W$ is on $l'$.

**Lemma 2.** Let $AW$ intersect line $OH$ at $D$. Then $D$ and $K$ are isogonal conjugates.

*Proof.* Let $J$ be the intersection of $l$ and line $BC$. Note that $AEWJPD$ is a complete quadrilateral formed by the four lines $AE$, $AW$, $JE$ and $JW$. By the dual of Desargues' Involution Theorem, there must be an involution mapping $BE \to BW$, $BA \to BJ$, and $BP \to BD$. But since the reflection across the angle bisector of $\angle ABC$ is an involution mapping $BE \to BW$ and $BA \to BJ$, it must be the same as the prior involution. Thus it also maps $BP \to BD$, implying that $BP$ and $BD$ are isogonal with respect to $\angle ABC$. Similarly, we can show that the lines $CQ$ and $CD$ are isogonal with respect to $\angle BCA$, so $D$ and $K$ are isogonal conjugates with respect to triangle $ABC$. $\square$

**Lemma 3.** Let $\mathcal{I}$ be the homothety of factor $-\dfrac{1}{2}$ centered at $G$. Then $\mathcal{I}(D) = T$.

*Proof.* We will show that $N$, $S$, and $\mathcal{I}(D)$ are collinear. Let the foot of $K$ to sides $BC$ and $BA$ be $K_1$ and $K_3$, respectively. Let the reflection of $K$ over $K_1$ and $K_3$ be $K_A$ and $K_C$, respectively. Note that $AK_CBZ$ is a parallelogram because $M$ is the midpoint of $K_CZ$. Similarly, $CK_ABX$ is a parallelogram. $BK$ is the diameter of the circumcircle of $BK_1KK_3$, so it follows that $BD$ is perpendicular to $K_1K_3$ from Lemma 2, and hence $K_AK_C$. But since $BK_A = BK_C$, $BD$ is the perpendicular bisector of $K_AK_C$. Now, note that $\overrightarrow{SN} = \overrightarrow{N} - \overrightarrow{S} = \dfrac{\overrightarrow{A} + \overrightarrow{C}}{2} - \dfrac{\overrightarrow{X} + \overrightarrow{Z}}{2} = \dfrac{\overrightarrow{A} - \overrightarrow{Z}}{2} + \dfrac{\overrightarrow{C} - \overrightarrow{X}}{2} = \dfrac{\overrightarrow{ZA} + \overrightarrow{XC}}{2} = \dfrac{\overrightarrow{BK_C} + \overrightarrow{BK_A}}{2} = \overrightarrow{BM_B}$, where $M_B$ is the midpoint of $K_AK_C$, so hence $BD \parallel SN$. But since $\mathcal{I}(B) = N$, it follows that $BD \parallel N\mathcal{I}(D)$ and hence $N, S$, and $\mathcal{I}(D)$ are collinear. Similarly, it follows that $M, R$, and $\mathcal{I}(D)$ are collinear, so $\mathcal{I}(D) = T$, as desired.

$\square$

Now, note that the conditions give us that the circumradius $R = 10$, $\cos A = \dfrac{4}{5}$, and $\cos(B - C) = \dfrac{1}{2}$. This gives $AO = 10, OH = 14$, and $HA = 16$. Then, $\sin DAH = \cos AOH = \dfrac{1}{7}$ and $\sin DAO =$

$\cos AHO = \dfrac{11}{14}$, so by the Ratio Lemma we get that $HD : DO = 16 : 55$. Then, since $HG : GO = 2 : 1$, we get that $HT : TO = 63 : 8$ from Lemma 3. Then, $OT = 14 \times \dfrac{8}{71} = \dfrac{112}{71}$, so the answer is 11271. $\quad \square$