Miscellaneous Geometry Facts

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Cyclic Quadrilaterals

1. A convex quadrilateral $ABCD$ is cyclic if and only if either:
   (a) $\angle ADB = \angle ACB$
   (b) $\angle DAB + \angle BCD = 180°$

2. The above two conditions can be restated as a single condition in terms of directed angles: Four points A, B, C and D are concyclic if and only if $\measuredangle ABC = \measuredangle ADC$.

3. (Power of a Point) Let $ABCD$ be a convex quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. $ABCD$ is cyclic if and only if either:
   (a) $AQ \cdot QC = BQ \cdot QD$ or equivalently QAD and QBC are similar
   (b) $PA \cdot PB = PC \cdot PD$ or equivalently PAD and PCB are similar

4. Given a triangle ABC, the intersections of the internal and external bisectors of angle $\angle BAC$ with the perpendicular bisector of BC both lie on the circumcircle of ABC.

5. (Ptolemy's Theorem) A quadrilateral $ABCD$ is cyclic if and only if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

6. Let $ABCD$ be a cyclic quadrilateral such that AB and CD intersect at P and diagonals AC and BD intersect at Q. Then:

$$\frac{BQ}{QD} = \frac{AB \cdot BC}{AD \cdot DC} \quad \text{and} \quad \frac{PB}{PA} = \frac{BC \cdot BD}{AC \cdot AD}$$

7. (Polars) Let $ABCD$ be a cyclic quadrilateral inscribed in circle Γ such that AB and CD intersect at P and diagonals AC and BD intersect at Q. If the tangents drawn from P to Γ touch Γ at R and S, then R, Q and S are collinear.

Circles

1. (Power of a Point) Given a circle Γ with center O and a point P then for any line ℓ through P that intersects Γ at A and B, the value $PA \cdot PB$ is constant as ℓ varies and is equal to the power of the point P with respect to Γ.

(a) The power of P is equal to $r^2 - PO^2$ if P is inside Γ and $PO^2 - r^2$ otherwise.

(b) If PA is tangent to Γ, then the power of P is equal to PA^2.

2. (Radical Axis) Given two circles Γ_1 and Γ_2, the set of all points P with equal powers with respect to Γ_1 and Γ_2 is a line which is the radical axis of the two circles.

   (a) The radical axis is perpendicular to the line through the centers of Γ_1 and Γ_2.

   (b) If Γ_1 and Γ_2 intersect at A and B, then the radical axis passes through A and B.

   (c) If AB is a common tangent with A on Γ_1 and B on Γ_2, then the radical axis passes through the midpoint of AB.

3. (Radical Center) Given three circles Γ_1, Γ_2 and Γ_3, the three radical axes between pairs of the three circles meet at a common point P which is the radical center of the circles.

4. A point P is a circle of radius zero and the radical axis of P and a circle Γ is the line through the midpoints of PA and PB where A and B are points on Γ such that PA and PB are tangent to Γ.

5. (Monge's Theorem) Given three circles Γ_1, Γ_2 and Γ_3. If P, Q and R are the external centers of homothety between pairs of the three circles, then P, Q and R are collinear. If P and Q are internal centers of homothety, then P, Q and R are also collinear.

6. Two circles Γ_1 and Γ_2 intersect at R and have centers O_1 and O_2. If P and Q are the internal and external centers of homothety between the two circles, then $\angle PRQ = 90^\circ$. The lines RP and RQ are the internal and external bisectors of $\angle O_1RO_2$.

Triangle Geometry

1. (Angle Bisector Theorem) Let ABC be a given triangle and let P and Q be the intersections of the internal and external bisectors of angle $\angle ABC$ with line AC. Then

$$\frac{AB}{BC} = \frac{AP}{PC} = \frac{AQ}{QC}$$

2. Angles around the centers of a triangle ABC:

   (a) If I is the incenter of ABC then $\angle BIC = 90^\circ + \frac{a}{2}$, $\angle IBC = \frac{b}{2}$ and $\angle ICB = \frac{c}{2}$.

   (b) If H is the orthocenter of ABC then $\angle BHC = 180^\circ - a$, $\angle HBC = 90^\circ - c$ and $\angle HCB = 90^\circ - b$.

   (c) If O is the circumcenter of ABC then $\angle BOC = 2a$ and $\angle OBC = \angle OCB = 90^\circ - a$.

   (d) If I_a is the A-excenter of ABC then $\angle AI_aB = \frac{c}{2}$, $\angle AI_aC = \frac{b}{2}$ and $\angle BI_aC = 90^\circ - \frac{a}{2}$.

3. Pedal triangles of the centers of a triangle ABC:

   (a) If DEF is the triangle formed by projecting the incenter I onto sides BC, AC and AB, then I is the circumcenter of DEF and $\angle EDF = 90^\circ - \frac{a}{2}$.

   (b) If DEF is the triangle formed by projecting the orthocenter H onto sides BC, AC and AB, then H is the incenter of DEF and $\angle EDF = 180^\circ - 2a$.
   (c) The medial triangle of ABC is the pedal triangle of the circumcenter O of ABC and O is its orthocenter.

4. Alternate methods of defining the orthocenter and circumcenter:
   (a) O is the circumcenter of ABC if and only if $\measuredangle AOB = 2\measuredangle ACB$ and $OA = OB$.
   (b) H is the orthocenter of ABC if and only if H lies on the altitude from A and satisfies that $\measuredangle BHC = 180^\circ - \measuredangle BAC$.

5. Facts related to the orthocenter H of a triangle ABC with circumcircle Γ:
   (a) If O is the circumcenter of ABC, then $\angle BAH = \angle CAO$.
   (b) If D is the point diametrically opposite to A on Γ and M is the midpoint of BC, then M is also the midpoint of HD.
   (c) If AH, BH and CH intersect Γ again at D, E and F, then there is a homothety centered at H sending the pedal triangle of H to DEF with ratio 2.
   (d) If D and E are the intersections of AH with BC and Γ, respectively, then D is the midpoint of HE.
   (e) H lies on the three circles formed by reflecting Γ about AB, BC and AC.
   (f) If M is the midpoint of BC then $AH = 2 \cdot OM$.
   (g) If BH and CH intersect AC and AB at D and E, and M is the midpoint of BC, then M is the center of the circle through B, D, E and C, and MD and ME are tangent to the circumcircle of ADE.

6. Facts related to the incenter I and excenters I_a, I_b, I_c of ABC with circumcircle Γ:
   (a) If the incircle of ABC is tangent to AB and AC at points D and E and s is the semiperimeter of ABC then
   $$AD = AE = \frac{AB + AC - BC}{2} = s - BC$$
   (b) If AI intersects Γ at D then $DB = DI = DC$, D is the midpoint of II_a, and II_a is a diameter of the circle with center D which passes through B and C.
   (c) If AI, BI and CI intersect Γ at D, E and F, then $I_aI_bI_c$, DEF and the pedal triangle of I are similar and have parallel sides.
   (d) I is the orthocenter of $I_aI_bI_c$ and Γ is the nine-point circle of $I_aI_bI_c$.
   (e) If BI and CI intersect Γ again at D and E, then I is the reflection of A in line DE and if M is the intersection of the external bisector of $\angle BAC$ with Γ, then $DMEI$ is a parallelogram.
   (f) If the incircle and A-excircle of ABC are tangent to BC at D and E, $BD = CE$.
   (g) If the A-excircle of ABC is tangent to AB, AC and BC at D, E and F then $AB + BF = AC + CF = AD = AE = s$ where s is the semi-perimeter of ABC.

   (h) If M is the midpoint of arc BAC of Γ, then M is the midpoint of I_bI_c and the center of the circle through I_b, I_c, B and C.

7. (Nine-Point Circle) Given a triangle ABC, let Γ denote the circle passing through the midpoints of the sides of ABC. If H is the orthocenter of ABC, then Γ passes through the midpoints of AH, BH and CH and the projections of H onto the sides of ABC.

8. (Feuerbach's Theorem) The nine-point circle is tangent to the incircle and excircles.

9. (Euler Line) If O, H and G are the circumcenter, orthocenter and centroid of a triangle ABC, then G lies on segment OH with $HG = 2 \cdot OG$.

10. (Symmedian) Given a triangle ABC such that M is the midpoint of BC, the symmedian from A is the line that is the reflection of AM in the bisector of angle $\angle BAC$.

    (a) If the tangents to the circumcircle Γ of ABC at B and C intersect at N, then N lies on the symmedian from A and $\angle BAM = \angle CAN$.
    (b) If the symmedian from A intersects Γ at D, then $AB/BD = AC/CD$.

11. If the median from A in a triangle ABC intersects the circumcircle Γ of ABC at D, then $AB \cdot BD = AC \cdot CD$.

12. (Euler's Formula) Let O, I and I_a be the circumcenter, incenter and A-excenter of a triangle ABC with circumradius R, inradius r and A-exradius r_a. Then:

    (a) $OI = \sqrt{R(R-2r)}$.
    (b) $OI_a = \sqrt{R(R+2r_a)}$.

13. (Poncelet's Porism) Let Γ and ω be two circles with centers O and I and radii R and r, respectively, such that $OI = \sqrt{R(R-2r)}$. Let A, B and C be any three points on Γ such that lines AB and AC are tangent to ω. Then line BC is also tangent to ω.

14. (Apollonius Circle) Let ABC be a given triangle and let P be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line AC at Q and R, then P lies on the circle with diameter QR.

15. Let ABC be a given triangle with incircle ω and A-excircle ω_a. If ω and ω_a are tangent to BC at M and N, then AN passes through the point diametrically opposite to M on ω and AM passes through the point diametrically opposite to N on ω_a.

16. Let ABC be a triangle with incircle ω which is tangent to BC, AC and AB at D, E and F. Let M be the midpoint of BC. The perpendicular to BC at D, the median AM and the line EF are concurrent.

17. Let ABC be a triangle with incenter I and incircle ω which is tangent to BC, AC and AB at D, E and F. The angle bisector CI intersects FE at a point T on the line adjoining the midpoints of AB and BC. It also holds that $BFTID$ is cyclic and $\angle BTC = 90°$.

18. Let ABC be a triangle with incircle ω and let D and E be the points at which ω is tangent to BC and the A-excircle is tangent to BC. Then AE passes through the point diametrically opposite to D on ω.

19. Let ABC be a triangle with A-excenter I_A and altitutde AD. Let M be the midpoint of AD and let K be the point of tangency between the incircle of ABC and BC. Then I_A, K and M are collinear.

Collinearity and Concurrency

1. (Ceva's Theorem) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an even number are on the extensions of the sides (zero or two). Then AD, BE and CF are concurrent if and only if
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$
2. (Menelaus' Theorem) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an odd number are on the extensions of the sides (one or three). Then D, E and F are collinear if and only if
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$
3. (Trig Ceva) Let ABC be a triangle and D, E and F be on the lines BC, AC and AB such that an even number are on the extensions of the sides (zero or two). Then AD, BE and CF are concurrent if and only if
$$\frac{\sin(\angle ABE)}{\sin(\angle CBE)} \cdot \frac{\sin(\angle BCF)}{\sin(\angle ACF)} \cdot \frac{\sin(\angle CAD)}{\sin(\angle BAD)} = 1$$
4. (Casey's Theorem) If A_1, B_1 and C_1 are points on the sides BC, AC and AB of a triangle ABC, then the perpendiculars to their respective sides at these three points are concurrent if and only if $BA_1^2 - CA_1^2 + CB_1^2 - AB_1^2 + AC_1^2 - BC_1^2 = 0$.
5. (Pascal's Theorem) If A, B, C, D, E, F are points on a circle then the intersections of the pairs of lines AB and DE, BC and EF, CD and FA lie on a line.
6. (Pappus' Theorem) If A, C and E lie on one line ℓ_1 and B, D and F lie on a line ℓ_2, then the intersections of the pairs of lines AB and DE, BC and EF, CD and FA lie on a line.
7. (Brianchon's Theorem) If $ABCDEF$ is a hexagon with an inscribed circle then AD, BE and CF are concurrent.
8. (Desargues Theorem) Let ABC and XYZ be triangles. Let D, E, F be the intersections of the pairs of lines AB and XY, BC and YZ, AC and XZ. Then D, E and F are collinear if and only if AX, BY and CZ are concurrent.
9. Pascal's theorem is true when points are not necessarily distinct and many of its applications concern tangent lines when some of the six points are equal.

Trigonometry

1. (Sine Law) Given a triangle ABC with circumradius R
$$\frac{BC}{\sin\angle A}=\frac{AC}{\sin\angle B}=\frac{AB}{\sin\angle C}=2R$$

2. (Cosine Law) Given a triangle ABC
$$BC^2=AB^2+AC^2-2\cdot AB\cdot AC\cdot\cos\angle A$$

3. (Pythagorean Theorem) If ABC is a triangle, then $\angle ABC=90^\circ$ if and only if
$$AB^2+BC^2=AC^2$$

4. Given a triangle ABC and a point D on line BC, then
$$\frac{\sin\angle BAD}{\sin\angle CAD}=\frac{BD\cdot AC}{CD\cdot AB}$$

5. (Stewart's Theorem) Let a,b,c be the side lengths of a triangle ABC and let d be the length of a cevian from A to BC that divides BC into segments of lengths m and n with m closer to B. Then
$$b^2m+c^2n=a(d^2+mn)$$

Miscellaneous Synthetic Facts

1. (Spiral Similarity) Let OAB and OCD be directly similar triangles. Then OAC and OBD are also directly similar triangles.

2. The unique center of spiral similarity sending AB to CD is the second intersection of the circumcircles of QAB and QCD where AC and BD intersect at Q.

3. Lines AB and CD are perpendicular if and only if $AC^2-AD^2=BC^2-BD^2$.

4. (Apollonius Circle) Given two points A and B and a fixed $r>0$, then the locus of points Q such that $AQ/BQ=r$ is a circle Γ with center at the midpoint of Q_1Q_2 where Q_1 and Q_2 are the two points on line AB satisfying $AQ_i/BQ_i=r$ for $i=1,2$.

5. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle Γ tangent to the four sides of $ABCD$ if and only if $AB+CD=AD+BC$.

6. (Simson Line) Let M, N and P be the projections of a point Q onto the sides of a triangle ABC. Then Q lies on the circumcircle of ABC if and only if M, N and P are collinear. If Q lies on the circumcircle of ABC, then the reflections of Q in the sides of ABC are collinear and pass through the orthocenter of the triangle.

7. (Broken Chord Theorem) Let E is the midpoint of major arc $\widehat{ABC}$ of the circumcircle of a triangle ABC where $AB<BC$. If D is the projection of E onto BC, then $AB+BD=DC$.

8. (Butterfly Theorem) Let M be the midpoint of a chord XY of a circle Γ. The chords AB and CD pass through M. If AD and BC intersect chord XY at P and Q, then M is also the midpoint of PQ.

9. (Miquel Point) Let D, E and F be points on sides BC, AC and AB of a triangle ABC. Then the circumcircles of AEF, BDF and CDE are concurrent.

10. (Isogonal Conjugates) Let ABC be a triangle and P be a point. If the reflection of BP in the angle bisector of $\angle ABC$ and the reflection of CP in the angle bisector $\angle ACB$ intersect at Q, then Q lies on the reflection of CP in the angle bisector of $\angle ACB$.

11. (Casey's Theorem) Let O_1, O_2, O_3, O_4 be four circles tangent to a circle O. Let t_{ij} be the length of the external common tangent between O_iO_j if O_i and O_j are tangent to O from the same side and the length of the internal common tangent otherwise. Then

$$t_{12} \cdot t_{34} + t_{41} \cdot t_{23} = t_{13} \cdot t_{24}$$

The converse is also true: if the above equality holds then O_1, O_2, O_3, O_4 are tangent to O.

12. (Transversal Theorem) If A, B and C are collinear and A', B' and C' are points on AP, BP and CP, then A', B' and C' are collinear if and only if

$$BC \cdot \frac{AP}{A'P} + CA \cdot \frac{BP}{B'P} + AB \cdot \frac{CP}{C'P} = 0$$

where all lengths are directed.

13. (Mixtilinear Incircles) Let ABC be a triangle with circumcircle Γ and let ω be a circle tangent internally to Γ and to AB anc AC at X and Y. Then the incenter of ABC is the midpoint of segment XY.

14. (Curvilinear Incircles) Let ABC be a triangle with circumcircle Γ and let D be a point on segment BC. Let ω be a circle tangent to Γ, DA and DC. If ω is tangent to DA and DC at F and E, then the incenter of ABC lies on FE.