1. **[5]** Prove or disprove: There exists an infinite arithmetic sequence with a positive integer first term and positive integer common difference for which no two terms have digits that are permutations of each other.

   Source: Original problem by Alex

   No such sequence exists. In other words, for any arithmetic sequence

   $$a, a+d, a+2d, \ldots$$

   we can find two terms whose digits are permutations of each other. In particular, consider $a + (10^j + 10^k)d$ and $a + (10^{j+1} + 10^k)d$ where $k \gg j \gg \log a, \log d$.

   For example, for $d = 123$ and $a = 4567890$, this would look like

   $$1230001230004567890$$

   and

   $$1230012300004567890.$$

2. **[6]** Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 be the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.

   Source: https://artofproblemsolving.com/community/c6h1754412p11448685

   Observe that A_0 is the circumcenter of cyclic quadrilateral BCB_1C_1. So $\triangle B_1A_0C_1$, $\triangle BA_0C_1$, and $\triangle CA_0B_1$ are isosceles. Since $PQ||BC$ and $\triangle BA_0C_1$ isosceles, $\triangle APB_1$ isosceles as well. Similarly, $\triangle AQC_1$ is isosceles.

   In summary,
$$B_1M = MC_1,\ C_1Q = QA \text{ and } AP = PB_1.$$

   Therefore the contact triangle of $\triangle PA_0Q$ is just $\triangle AB_1C_1$. So the incenter of $\triangle PA_0Q$ lies on the line through A perpendicular to PQ. That is, the altitude of $\triangle ABC$ from A.

3. [**7**] Determine if it is possible to construct 100 lines in the plane so they produce exactly 2021 intersection points.

Source: Original problem by Alex

It is indeed possible. Consider bundles of pairwise parallel lines. Assume there are a_i in each bundle. So lines within the same bundle have 0 intersections between them, but any two lines in different bundles intersect once. Also, we constrain any intersection point to only be between two lines. The total number of intersections is then $\sum_{i<j} a_i a_j$.

It is enough to find $a_1, a_2, \ldots, a_n$ such that

$$\sum_{i=1}^{n} a_i = 100$$

and

$$\sum_{1 \leq i < j \leq n} a_i a_j = 2021.$$

There are many examples, one such is

$$n = 22,\ a_1 = 77,\ a_2 = 3, \text{ and } a_3 = a_4 = \cdots = a_{22} = 1.$$

4. **[9]** Find the minimum c such that the following inequality is true for all positive numbers x, y, z:

$$\frac{x^3}{x^3+y^2z}+\frac{y^3}{y^3+z^2x}+\frac{z^3}{z^3+x^2y}\le c.$$

Source: Original problem by Alex

We claim that the minimum c is 2. We can get arbitrarily close by setting $(x, y, z) = (1, 1, \epsilon)$ for $\epsilon > 0$.

To prove this, WLOG let x be the largest among x, y, z. Then we have

$$\begin{aligned} \frac{x^3}{x^3+y^2z} &< 1, \\ \frac{y^3}{y^3+z^2x} &\le \frac{y^3}{y^3+z^3}, \\ \text{and}\quad \frac{z^3}{z^3+x^2y} &\le \frac{z^3}{z^3+y^3}. \end{aligned}$$

Summing the three gives the desired inequality.

5. **[11]** Prove that for any prime p of the form $4k+1$ ($k \in \mathbb{N}$), the following equality holds:

$$\sum_{j=1}^{p-1} \lfloor \sqrt{jp} \rfloor = \frac{(p-1)(2p-1)}{3}.$$

Source: https://www.komal.hu/verseny/2004-03/mat.e.shtml

We count the sum as follows:

$$\sum_{j=1}^{p-1} \lfloor \sqrt{jp} \rfloor = \sum_{j=1}^{p-1} \sum_{\substack{1 \le i \le p \\ i^2 \le jp}} 1.$$

Swapping the order of summation, this is equal to

$$\begin{aligned}\sum_{i=1}^{p-1} \sum_{\substack{1 \le j \le p-1 \\ \frac{i^2}{p} \le j}} &= \sum_{i=1}^{p-1} p - 1 - \lfloor \frac{i^2}{p} \rfloor \\ &= \sum_{i=1}^{p-1} p - 1 - \frac{i^2 - (i^2 \pmod p)}{p}.\end{aligned}$$

Since p is of the form $4k+1$, it is known that -1 is a quadratic residue, and so we can pair up quadratic residues with their additive inverse. Thus,

$$\sum_{i=1}^{p-1} (i^2 \pmod p) = \frac{p(p-1)}{2}.$$

So

$$\begin{aligned}\sum_{i=1}^{p-1} p - 1 - \frac{i^2 - (i^2 \pmod p)}{p} &= (p-1)^2 - \frac{(p-1)(2p-1)}{6} + \frac{(p-1)}{2} \\ &= \frac{(p-1)(2p-1)}{3}.\end{aligned}$$

6. **[12]** Given a cyclic quadrilateral $ABCD$, let E be the intersection of the diagonals and M be the midpoint of AB. Let P, Q, and R be the feet of the perpendiculars from E to DA, AB, and BC respectively. Show that M lies on the circumcircle of $\triangle PQR$.

   Source: https://www.komal.hu/feladat?a=feladat&f=A729&l=en

   Let X and Y be the midpoints of AE and BE respectively. We claim that $\triangle PXM \simeq \triangle MYR$. First note that $PX = \frac{AE}{2} = MY$ and $RY = \frac{BE}{2} = MX$. Now we can also see that

$$\angle PXM = \angle PXE + \angle EXM = 2\angle EAD + \pi - \angle AEB.$$

   Similarly,

$$\angle RYM = \angle RYE + \angle EYM = 2\angle EBC + \pi - \angle AEB.$$

   Since $ABCD$ is cyclic, we can conclude that $\angle PXM = \angle RYM$. So by SAS, $\triangle PXM \simeq \triangle MYR$.

   To finish, it suffices to show that $\angle PQR = \angle PMR$. Note that $BQER$ and $AQEP$ are cyclic, and so

$$\angle PQR = \angle PQE + \angle EQR = 2\angle CAD.$$

   Now

$$\begin{aligned}
\angle PMR &= \angle XMY - \angle PMX - \angle YMR \\
&= \angle AEB - (\pi - \angle PXM) \\
&= 2\angle EAD \\
&= 2\angle CAD.
\end{aligned}$$

   So we are done.

7. [**13**] The integers $a_1, \dots, a_n$ give at least $k+1$ different remainders modulo $n+k$. Prove that there is a non-empty subset of these n integers which sums to 0 modulo $n+k$.

Source: https://www.komal.hu/verseny/2000-12/A.h.shtml

Without loss of generality, say that $a_1, a_2, \dots, a_{k+1}$ are $k+1$ of our different remainders.

Consider the following three types of subsets:

- $x_i = \{a_i\}$ for $1 \le i \le k+1$
- $y_i = \{a_j \mid 1 \le j \le k+1\} \setminus \{a_i\}$ for $1 \le i \le k+1$
- $z_i = \{a_j \mid 1 \le j \le i\}$ for $k+1 \le i \le n$

There are $n+k+2$ subsets in total. Let $\Sigma(S)$ denote the sum of the elements in a set S, modulo $n+k$. Assume for the sake of contradiction that no subset S (possibly not of the three types) has $\Sigma(S) = 0$.

Observe that the Σs for the xs are all distinct and the Σs for the ys are all distinct. Furthermore, note that for any subset s of the three types and any z_i, either $s \subsetneq z_i$ or $z_i \subsetneq s$. So if $\Sigma(s) = \Sigma(z_i)$, we could consider $\Sigma(s \setminus z_i)$ or $\Sigma(z_i \setminus s)$, giving a contradiction.

The last possibility is that we have $S(x_i) = S(y_j)$. However, if $i \neq j$, $x_i \subsetneq y_j$ and as we saw above, this gives a contradiction. But if $i = j$, then this implies that

$$2a_j \equiv \sum_{i=1}^{k+1} a_i \pmod{n+k}.$$

This has at most two solutions, and so this can happen at most twice. Thus, the Σs of these $n+k+2$ subsets give at least $n+k$ distinct remainders and so one of them must be 0, giving a final contradiction.

8. **[16]** Let G be a connected graph with $n > 1$ vertices. The maximal independent set of G is defined as the largest set of vertices so that no two are neighbours in G, and its size is denoted as $\alpha(G)$. Prove that there is an induced subgraph H of G with size at least $\alpha(G)/2$ where all degrees are odd.

   Source: https://yufeizhao.com/pm/ps.pdf

   We will use the following algorithm:

   1. For any graph G, consider the largest independent set A (if there are multiple, pick a single one arbitrarily). Color vertices in A and with degree > 1 red. Color vertices in A and with degree $= 1$ green. Color vertices not in A blue.
   2. If there is some blue vertex which has no green neighbor, skip to step 7. Otherwise, move on to step 3.
   3. For each blue vertex, put it into H with probability $\frac{1}{2}$.
   4. For each red vertex, if it has an odd number of neighbors in H, put it in H.
   5. For each blue vertex in H, consider its green neighbors (based on step 2, every blue neighbor has at least one). Depending on the parity of its degree in H, either put all or all but one of the green neighbors into H.
   6. We are now done.
   7. This is the situation where there is some blue vertex with no green neighbor. Remove this blue vertex and recurse.

   It's a little unclear, but this is guaranteed to return a subgraph with all odd degrees. Let's first discuss step 7. Call G' the graph with the removed blue vertex. We will appeal to induction, with our hypothesis being that we can find H with size $\alpha(G)/2$ for graphs G of a maximum size and without isolated vertices. Our base cases are the situations where we go through steps 3 to 6 and will be proved later. Then note that $\alpha(G') = \alpha(G)$, $|G'| < |G|$, and G' has no isolated vertices. So H' found from G' still works with G and the induction step is done.

   Now let's analyze our base cases. For each blue vertex b, let J_b denote the set of its green neighbors. Note that if b is chosen in step 3, then it contributes either $1 + |J_b|$ or $|J_b|$ to H through step 5. So the expected number of vertices added through either step 3 or 5 is at least

   $$\sum_{b \in V_{\text{blue}}} \frac{1}{2}|J_b| = \frac{1}{2}|V_{\text{green}}|.$$

   The expected number of vertices added through step 4 is $\frac{1}{2}|V_{\text{red}}|$. So in total,

   $$\mathbb{E}[|H|] \geq \frac{1}{2}|V_{\text{green}}| + \frac{1}{2}|V_{\text{red}}| = \frac{\alpha(G)}{2}$$

   and so there is a subgraph with all odd degrees and size at least $\alpha(G)/2$.

9. **[2021]** Dissect the unit square into 2021 triangles each with area $\frac{1}{2021}$.

This is a reference to Monsky's theorem, which states that a square cannot be dissected into an odd number of triangles with equal area.