Combinatorics

C1 James and Sarah play a game on a 100×100 chessboard. First, James places 50 kings on the board and then Sarah places a rook. They then move in turns beginning with James as follows. On his turn, James moves each of the kings one square in any direction. On her turn, Sarah can move the rook horizontally or vertically by any number of squares without passing over a king. James' goal is to capture the rook, and Sarah's is to avoid capture. Is there a strategy allowing Sarah to avoid capture indefinitely?

C2 In a school, there are three classes each containing M students. Every student is friends with more than $\frac{3}{4}M$ people in each of the other two classes. Prove that the students can be split into M teams of three friends from different classes.

C3 There are m villages on the left bank of a river, n villages on the right bank and one village on an island, where $\gcd(m+1, n+1) > 1$. Every two villages separated by water are connected by ferry labelled with a positive integer. Prove that it is impossible for the labels on the ferries going to each village to be distinct consecutive integers when arranged in some order.

Number Theory

N1 Are there positive integers a, b, c, d satisfying $a^4 + 2b^4 + 4c^4 + 8d^4 = 16abcd$?

N2 Are there seven distinct odd prime numbers satisfying that the difference of the eighth powers of any two of them is divisible by the remaining five primes?

N3 Prove that there is a strictly increasing positive integer sequence (a_n) such that $a_n \leq 1.01^n K$ for some $K > 0$ and the sum of any finite number of distinct terms in (a_n) is not a perfect square.

Algebra

A1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$.

A2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$

for all $x, y \in \mathbb{R}$.

A3 In a sequence of natural numbers (a_n), every natural number occurs and

$$\frac{1}{100} < \frac{|a_n - a_m|}{|n-m|} < 100$$

for each $m \neq n$. Prove that $|a_n - n| < 5000$ for all n.

Geometry

G1 Let P, Q, R be points on the sides AB, BC, CA of a triangle ABC such that $AP = CQ$ and the quadrilateral $RPBQ$ is cyclic. The tangents to the circumcircle of triangle ABC at the points C and A intersect the lines RQ and RP at the points X and Y, respectively. Prove that $RX = RY$.

G2 A convex quadrilateral $ABCD$ has an inscribed circle with centre O. A line ℓ divides $ABCD$ into two quadrilaterals with equal areas and perimeters. Prove that ℓ passes through O.

G3 The quadrilateral $ABCD$ has perpendicular diagonals and is inscribed in a circle ω with centre O. The tangents to ω at A and C together with line BD form the triangle Δ. Prove that the circumcircles of BOD and Δ are tangent.