The Online Math Open Fall Contest
October 27 – November 7, 2017

Online Math Open

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.

Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2017 Fall Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

October 27 – November 7, 2017

from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams cannot communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com`. (Include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

OMO Fall 2017
October 27 – November 7, 2017

1. Jingoistic James wants to teach his kindergarten class how to add in Chimese. The Chimese also use base-10 number system, but have replaced the digits 0-9 with ten of its own characters. For example, the two-digit number 十六 represents 16. What is the sum of the two-digit numbers 十六 and 六十?

2. The numbers a, b, c, d are $1, 2, 2, 3$ in some order. What is the greatest possible value of $a^{b^{c^d}}$?

3. The USAMO is a 6 question test. For each question, you submit a positive integer number p of pages on which your solution is written. On the ith page of this question, you write the fraction i/p to denote that this is the ith page out of p for this question. When you turned in your submissions for the 2017 USAMO, the bored proctor computed the sum of the fractions for all of the pages which you turned in. Surprisingly, this number turned out to be 2017. How many pages did you turn in?

4. Steven draws a line segment between every two of the points
$$A(2,2), B(-2,2), C(-2,-2), D(2,-2), E(1,0), F(0,1), G(-1,0), H(0,-1).$$
How many regions does he divide the square $ABCD$ into?

5. Henry starts with a list of the first 1000 positive integers, and performs a series of steps on the list. At each step, he erases any nonpositive integers or any integers that have a repeated digit, and then decreases everything in the list by 1. How many steps does it take for Henry's list to be empty?

6. A convex equilateral pentagon with side length 2 has two right angles. The greatest possible area of the pentagon is $m + \sqrt{n}$, where m and n are positive integers. Find $100m + n$.

7. Let S be a set of 13 distinct, pairwise relatively prime, positive integers. What is the smallest possible value of $\max_{s\in S} - \min_{s\in S}$?

8. A permutation of $\{1, 2, 3, \ldots, 16\}$ is called *blocksum-simple* if there exists an integer n such that the sum of any 4 consecutive numbers in the permutation is either n or $n+1$. How many blocksum-simple permutations are there?

9. Let a and b be positive integers such that $(2a + b)(2b + a) = 4752$. Find the value of ab.

10. Determine the value of $-1 + 2 + 3 + 4 - 5 - 6 - 7 - 8 - 9 + ... + 10000$, where the signs change after each perfect square.

11. Let $\{a, b, c, d, e, f, g, h, i\}$ be a permutation of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\gcd(c, d) = \gcd(f, g) = 1$ and
$$(10a + b)^{c/d} = e^{f/g}.$$
Given that $h > i$, evaluate $10h + i$.

12. Bill draws two circles which intersect at X, Y. Let P be the intersection of the common tangents to the two circles and let Q be a point on the line segment connecting the centers of the two circles such that lines PX and QX are perpendicular. Given that the radii of the two circles are $3, 4$ and the distance between the centers of these two circles is 5, then the largest distance from Q to any point on either of the circles can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

13. We define the sets of lattice points $S_0, S_1, \ldots$ as $S_0 = \{(0,0)\}$ and S_k consisting of all lattice points that are exactly one unit away from exactly one point in S_{k-1}. Determine the number of points in S_{2017}.

14. Let S be the set of all points $(x_1, x_2, x_3, \ldots, x_{2017})$ in $\mathbb{R}^{2017}$ satisfying $|x_i| + |x_j| \leq 1$ for any $1 \leq i < j \leq 2017$. The volume of S can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

15. Find the number of integers $1 \le k \le 1336$ such that $\binom{1337}{k}$ divides $\binom{1337}{k-1}\binom{1337}{k+1}$.

16. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two parabolas with distinct directrices ℓ_1 and ℓ_2 and distinct foci F_1 and F_2 respectively. It is known that $F_1F_2||\ell_1||\ell_2$, F_1 lies on $\mathcal{P}_2$, and F_2 lies on $\mathcal{P}_1$. The two parabolas intersect at distinct points A and B. Given that $F_1F_2 = 1$, the value of AB^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

17. For a positive integer n, define $f(n) = \sum_{i=0}^{\infty} \frac{\gcd(i,n)}{2^i}$ and let $g : \mathbb{N} \to \mathbb{Q}$ be a function such that $\sum_{d|n} g(d) = f(n)$ for all positive integers n. Given that $g(12321) = \frac{p}{q}$ for relatively prime integers p and q, find $v_2(p)$.

18. Let a, b, c be real nonzero numbers such that $a + b + c = 12$ and
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = 1.$$
Compute the largest possible value of $abc - (a + 2b - 3c)$.

19. Tessa the hyper-ant is at the origin of the four-dimensional Euclidean space $\mathbb{R}^4$. For each step she moves to another lattice point that is 2 units away from the point she is currently on. How many ways can she return to the origin for the first time after exactly 6 steps?

20. Let $p = 2017$ be a prime. Suppose that the number of ways to place p indistinguishable red marbles, p indistinguishable green marbles, and p indistinguishable blue marbles around a circle such that no red marble is next to a green marble and no blue marble is next to a blue marble is N. (Rotations and reflections of the same configuration are considered distinct.) Given that $N = p^m \cdot n$, where m is a nonnegative integer and n is not divisible by p, and r is the remainder of n when divided by p, compute $pm + r$.

21. Iris has an infinite chessboard, in which an 8×8 subboard is marked as Sacred. In order to preserve the Sanctity of this chessboard, her friend Rosabel wishes to place some indistinguishable Holy Knights on the chessboard (not necessarily within the Sacred subboard) such that:

   - No two Holy Knights occupy the same square;
   - Each Holy Knight attacks at least one Sacred square;
   - Each Sacred square is attacked by exactly one Holy Knight.

   In how many ways can Rosabel protect the Sanctity of Iris' chessboard? (A Holy Knight works in the same way as a knight piece in chess, that is, it attacks any square that is two squares away in one direction and one square away in a perpendicular direction. Note that a Holy Knight does *not* attack the square it is on.)

22. Given a sequence of positive integers $a_1, a_2, a_3, \ldots, a_n$, define the *power tower function*
$$f(a_1, a_2, a_3, \ldots, a_n) = a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_n}}}}}}.$$
Let $b_1, b_2, b_3, \ldots, b_{2017}$ be positive integers such that for any i between 1 and 2017 inclusive,
$$f(a_1, a_2, a_3, \ldots, a_i, \ldots, a_{2017}) \equiv f(a_1, a_2, a_3, \ldots, a_i + b_i, \ldots, a_{2017}) \pmod{2017}$$
for all sequences $a_1, a_2, a_3, \ldots, a_{2017}$ of positive integers greater than 2017. Find the smallest possible value of $b_1 + b_2 + b_3 + \cdots + b_{2017}$.

23. Call a nonempty set V of nonzero integers *victorious* if there exists a polynomial $P(x)$ with integer coefficients such that $P(0) = 330$ and that $P(v) = 2|v|$ holds for all elements $v \in V$. Find the number of victorious sets.

24. Senators Sernie Banders and Cedric "Ced" Truz of OMOrica are running for the office of Price Dent. The election works as follows: There are 66 states, each composed of many adults and 2017 children, with only the latter eligible to vote. On election day, the children each cast their vote with equal probability to Banders or Truz. A majority of votes in the state towards a candidate means they "win" the state, and the candidate with the majority of won states becomes the new Price Dent. Should both candidates win an equal number of states, then whoever had the most votes cast for him wins.

    Let the probability that Banders and Truz have an unresolvable election, i.e., that they tie on both the state count and the popular vote, be $\frac{p}{q}$ in lowest terms, and let m, n be the remainders when p, q, respectively, are divided by 1009. Find $m + n$.

25. For an integer k let T_k denote the number of k-tuples of integers $(x_1, x_2, ...x_k)$ with $0 \le x_i < 73$ for each i, such that $73 | x_1^2 + x_2^2 + ... + x_k^2 - 1$. Compute the remainder when $T_1 + T_2 + ... + T_{2017}$ is divided by 2017.

26. Define a sequence of polynomials $P_0, P_1, ...$ by the recurrence $P_0(x) = 1, P_1(x) = x, P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$. Let $S = \left|P'_{2017}\left(\frac{i}{2}\right)\right|$ and $T = \left|P'_{17}\left(\frac{i}{2}\right)\right|$, where i is the imaginary unit. Then $\frac{S}{T}$ is a rational number with fractional part $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

27. For a graph G on n vertices, let $P_G(x)$ be the unique polynomial of degree at most n such that for each $i = 0, 1, 2, \ldots, n$, $P_G(i)$ equals the number of ways to color the vertices of the graph G with i distinct colors such that no two vertices connected by an edge have the same color. For each integer $3 \le k \le 2017$, define a k-*tasty* graph to be a connected graph on 2017 vertices with 2017 edges and a cycle of length k. Let the *tastiness* of a k-tasty graph G be the number of coefficients in $P_G(x)$ that are odd integers, and let t be the minimal tastiness over all k-tasty graphs with $3 \le k \le 2017$. Determine the sum of all integers b between 3 and 2017 inclusive for which there exists a b-tasty graph with tastiness t.

28. Let ABC be a triangle with $AB = 7, AC = 9, BC = 10$, circumcenter O, circumradius R, and circumcircle ω. Let the tangents to ω at B, C meet at X. A variable line ℓ passes through O. Let A_1 be the projection of X onto ℓ and A_2 be the reflection of A_1 over O. Suppose that there exist two points Y, Z on ℓ such that $\angle YAB + \angle YBC + \angle YCA = \angle ZAB + \angle ZBC + \angle ZCA = 90^\circ$, where all angles are directed, and furthermore that O lies inside segment YZ with $OY * OZ = R^2$. Then there are several possible values for the sine of the angle at which the angle bisector of $\angle AA_2O$ meets BC. If the product of these values can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with b squarefree and a, c coprime, determine $a + b + c$.

29. Let $p = 2017$. If A is an $n \times n$ matrix composed of residues $\pmod p$ such that $\det A \not\equiv 0 \pmod p$ then let $\mathrm{ord}(A)$ be the minimum integer $d > 0$ such that $A^d \equiv I \pmod p$, where I is the $n \times n$ identity matrix. Let the maximum such order be a_n for every positive integer n. Compute the sum of the digits when $\sum_{k=1}^{p+1} a_k$ is expressed in base p.

30. We define the bulldozer of triangle ABC as the segment between points P and Q, distinct points in the plane of ABC such that $PA \cdot BC = PB \cdot CA = PC \cdot AB$ and $QA \cdot BC = QB \cdot CA = QC \cdot AB$. Let XY be a segment of unit length in a plane $\mathcal{P}$, and let $\mathcal{S}$ be the region of $\mathcal{P}$ that the bulldozer of XYZ sweeps through as Z varies across the points in $\mathcal{P}$ satisfying $XZ = 2YZ$. Find the greatest integer that is less than 100 times the area of $\mathcal{S}$.