Mock Olympiad #5

July 10, 2009

1. An eccentric mathematician has a ladder with n rungs that he always ascends and descends in the following way: When he ascends, each step he takes covers a rungs of the ladder, and when he descends, each step he takes covers b rungs of the ladder, where a and b are fixed positive integers. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and come back to ground level again. Find, with proof, the minimum value of n, expressed in terms of a and b.

2. Let A', B', C', D', E', F' be midpoints of the sides AB, BC, CD, DE, EF, FA of convex hexagon $ABCDEF$. Also let p denote the perimeter of $ABCDEF$ and p' denote the perimeter of $A'B'C'D'E'F'$. If all inner angles of hexagon $A'B'C'D'E'F'$ are equal, prove that

$$p \geq \frac{2 \cdot \sqrt{3}}{3} \cdot p_1.$$

When does equality hold?

3. Let X be a set of $2k$ elements and $\mathcal{F}$ a family of subsets of X each of cardinality k such that each subset of X of cardinality $k-1$ is contained in precisely one member of $\mathcal{F}$. Show that $k+1$ is prime.