2021 Winter Camp Mock Olympiad

Each of the following eight problems has a maximum point value. Your score will be the sum of the three highest scores you obtain on the problems. You have 4.5 hours to complete this test.

1. [**5**] Prove or disprove: There exists an infinite arithmetic sequence with a positive integer first term and positive integer common difference for which no two terms have digits that are permutations of each other.

2. [**6**] Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 be the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.

3. [**7**] Determine if it is possible to construct 100 lines in the plane so they produce exactly 2021 intersection points.

4. [**9**] Find the minimum c such that the following inequality is true for all positive numbers x, y, z:
$$\frac{x^3}{x^3+y^2z}+\frac{y^3}{y^3+z^2x}+\frac{z^3}{z^3+x^2y}\le c.$$

5. [**11**] Prove that for any prime p of the form $4k+1$ ($k \in \mathbb{N}$), the following equality holds:
$$\sum_{j=1}^{p-1}\lfloor\sqrt{jp}\rfloor=\frac{(p-1)(2p-1)}{3}.$$

6. [**12**] Given a cyclic quadrilateral $ABCD$, let E be the intersection of the diagonals and M be the midpoint of AB. Let P, Q, and R be the feet of the perpendiculars from E to DA, AB, and BC respectively. Show that M lies on the circumcircle of $\triangle PQR$.

7. [**13**] The integers $a_1, \ldots, a_n$ give at least $k+1$ different remainders modulo $n+k$. Prove that there is a non-empty subset of these n integers which sums to 0 modulo $n+k$.

8. [**16**] Let G be a connected graph with $n > 1$ vertices. The maximal independent set of G is defined as the largest set of vertices so that no two are neighbours in G, and its size is denoted as $\alpha(G)$. Prove that there is an induced subgraph H of G with size at least $\alpha(G)/2$ where all degrees are odd.

9. [**2021**] Dissect the unit square into 2021 triangles each with area $\frac{1}{2021}$.