Mock Olympiad #1

1. Given trapezoid $AYCD$ with parallel sides AY and CD, assume that there exist points E on line AD outside segment AD, and F inside segment YC such that $\angle YCE = \angle ADF$. Denote by I the point of intersection of AY and EF, and by J the point of intersection of CD and EF. Let K be the midpoint of segment EF, assume it does not lie on line CD. Prove that I belongs to the circumcircle of CDK if and only if K belongs to the circumcircle of AYJ.

2. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

3. Prove that for any four positive real numbers a, b, c, d the inequality

$$\frac{(a-b)(a-c)}{a+b+c} + \frac{(b-c)(b-d)}{b+c+d} + \frac{(c-d)(c-a)}{c+d+a} + \frac{(d-a)(d-b)}{d+a+b} \geq 0$$

holds. Determine all cases of equality.