Number theory 2

September 2, 2020

Problem 1. *Let n points be given in the interior of the rectangle R, such that no two points lie on a line parallel to the sides of R. The rectangle is partitioned into several rectangles, so that none of n points is in the interior of a small rectangle. Prove that R is partitioned into at least $n+1$ parts.*

Problem 2. *Prove that if $n \geq 2$, then it is impossible to partition the set $\{1, 2, ..., n\}$ into two sets, so that the product of the elements in the sets are equal.*

Problem 3. *Let $n > 1$ be an integer. Let $\sigma(n)$ be the sum of the positive divisors of n. Prove that $\sigma(n-1)\sigma(n)\sigma(n+1)$ is even.*

Problem 4. *Denote by $\tau(n)$ the number of positive divisors of n. Prove that if a, b are positive integers, then $\tau(ab) \geq \tau(a) + \tau(b) - 1$.*

Problem 5. *Find all positive integers n, which have at least 4 distinct positive divisors and $n = {d_1}^2 + {d_2}^2 + {d_3}^2 + {d_4}^2$, where d_1, d_2, d_3, d_4 are the four smallest divisors of n.*

Problem 6. *$2n$ cells of a $n \times n$ board are colored red. Prove that there exists $k > 1$ and $2k$ distinct red cells $a_1, ..., a_{2k}$ so that a_{2i} and a_{2i+1} are in the same row, and a_{2i-1} and a_{2i} in the same column (we assume that $a_{2k+1} = a_1$).*