# WINTER CAMP 2020: GAME THEORY

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## 1. Games

Two player games are very common. A *winning strategy* for one of the two players (Alice) is a set of rules to follow, such that no matter what the other player does, if Alice follows the rules she will win the game. It is a fact that if Alice and Bob play a game which ends in a finite amount of time, and one of the two players always wins, then there is a winning strategy for either Alice or Bob. This is by no means a trivial remark!

Here are a a few common tricks and strategies one can use when analyzing games.

### 1.1. **Symmetry.**

**Question 1.1.** *There is a table with a square top of radius 10. Two players take turn putting a dollar coin of radius 1 on the table. The player who cannot do so loses the game. Show that the first player can always win.*

*Solution*: The first player places her first coin in the middle of the table. On every subsequent move, the first player places their coin $C_2$ symmetrically to the second players previous coin $C_1$. Since $C_1$ does not pass through the center, it follows that $C_1$ and $C_2$ cannot intersect (why?) And so if $C_1$ fit on the table, so does $C_2$, as the board was symmetric up to that point.

### 1.2. **Pairing.**

**Question 1.2** (USAMO 2004/4)**.** *Alice and Bob play a game on a 6  6 grid. They take turns writing a number in an empty square of the grid (distinct from all previous numbers thus far), and Alice goes first. When all squares are filled, the square in each row with the largest number is colored black. Alice wins if she can then draw a straight line (possibly diagonal) connecting two opposite sides of the grid that stays entirely in black squares. Find, with proof, a winning strategy for one of the players.*

*solution* Consider the squares lying on the main diagonal, the diagonals immediately above and below it, and the 2 corners. This consists of 3 squares in each row. Mark all these squares. Bob can ensure that none of these are ever colored black by always acting in the same row in Alice: if she picks a marked square, he writes a higher number in an unmarked square. And if she picks an unmarked square, he writes a lower number in a marked square.

### 1.3. **Strategy Stealing.**

**Question 1.3.** *The game of Chomp is played on an $m \times n$ board by Alice and Bob as follows. Alice moves first. On a players move, they must place an $X$ on any square $(i, j)$ which does not yet have an $X$ on it, and they also place an $X$ on any square above and/or to the right of that square which does not yet have an $X$ on it. That is, any square $(s, t)$ with $s \geq i$ and $t \geq j$ which does not yet have an $X$ also gets an $X$ put in it. The person who places the last $X$ loses. Determine who has a winning strategy.*

*Solution* The key is that the top right is a 'throwaway' move. In other words, consider the highly related game of 'champ', which is the same except its played on an $m \times n$ board minus the top right piece. Now if first player wins in champ, then Alice can just play the winning champ strategy for chomp, since the top right piece will immediately get an $X$. If first player loses in champ, Alice simply plays her first move for chomp in the top right and then plays the winning strategy for champ as the second player. Note that champ is a difficult game to analyze, and we do not know who wins in general!

## 2. Problems

### 2.1. **Warmup Problems.**

(1) (The matchstick game) Alice and Bob play a game with a pile of 10 matches, with Alice moving first. On each players turn, they must remove between 1 and 3 matches from the pile. The person who empties the pile wins.The initial pile has 10 matches. Figure out a winning strategy for one of the players.

What if instead, the player who empties the pile *loses*?

(2) Alice and Bob play a game in which they take turns removing stones from a heap that initially has $n$ stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many $n$ such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

(3) Alice and Bob play a game in which the first player places a king on an empty 88 chessboard, and then, starting with the second player, they alternate moving the king (in accord with the rules of chess) to a square that has not been previously occupied. The player who cannot move loses. Which player has a winning strategy? What about on a $5 \times 5$ board?

(4) Alice and Bob alternate writing in the entries of a $3 \times 3$ matrix. Alice moves first, and only writes a 1, and Bob only writes a 0, into some empty cell of the matrix. After 9 moves the game is over, and Alice

wins if the determinant is non-zero. Determine whether Alice has a winning strategy.

(5) Alice and Bob play a game as follows. They start with a row of 50 coins, of various values. The players alternate, and at each step they pick either the first or last coin and take it. If Alice plays first, prove that she can guarantee that she will end up with at least as much money as Bob. Find an example where Bob can make more money then Alice if there are 51 coins.

(6) Let $n$ be a positive integer. Alice and Bob play a game with a set of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and $n$ cards are dealt to each of the players. Beginning with Alice, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$ and the last player to discard wins the game. Prove that Bob has a winning strategy.

(7) Two players play a game by starting with the integer 2020, and taking turns replacing the current integer $N$ with either $\lfloor N/2 \rfloor$ or $N-1$. The player who writes down 0 wins. Who has a winning strategy?

### 2.2. **Medium.**

(1) Two players, Jacob and David, play a game in a convex polygon with $n \geq 5$ sides. On each turn, they draw a diagonal which does not intersect any previously drawn diagonal. The player that creates a quadrilateral (with no diagonal yet drawn) loses. If Jacob plays first, for which $n$ can he win?

(2) For a positive integer $n$, two players A and B play the following game: Given a pile of $N$ stones, the players alternate with A going first. On a given turn, a plater is allowed to take either one stone, or a prime number of stones, or $kn$ stones for some positive integer $k$. The winner is the one who takes the last stone. Assuming both A and B play perfectly, for how many integers $N$ does $B$ win?

Can you find a bound N in terms of n for which A wins for all $s \geq N$?

(3) (IMO Shortlist 2017) Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index $i$ in the set $\{1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element $a_i$ from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1} 10^{p-1} = \sum_{i=0}^{p-1} a_i . 10^i .$$

The goal of Eduardo is to make $M$ divisible by $p$, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

(4) There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts two or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

(5) A and B play a version of Tic-Tac-Toe on an infinite grid, where someone wins if they get 5 consecutive $X$'s or $O$'s in a row. Prove that no-one has a winning strategy.

(6) $A$ and $B$ play a game, given an integer $N$, $A$ writes down 1 first, then every player sees the last number written and if it is $n$ then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than $N$. The player who writes $N$ wins. For which values of $N$ does $B$ win?

### 2.3. **Hard.**

(1) (USAMO 1999) Alice and Bob play a game. Initially, there is a row of $n$ unfilled boxes. Starting with Alice, they take turns filling in either S or O in an unfilled box. The player who first creates three consecutive boxes spelling SOS wins (if there is no such player, they tie). For which $n$ can Alice guarantee a win? For which $n$ does Bob guarantee a win?

(2) Fix an integer $k > 2$. Two players, called Ana and Banana, play the following game of numbers. Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number $m$ just written on the blackboard and replaces it by some number $m'$ with $k \leq m' < m$ that is coprime to $m$. The first player who cannot move anymore loses.

An integer $n \geq k$ is called good if Banana has a winning strategy when the initial number is $n$, and bad otherwise.

Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides $n$ if and only if it divides $n'$. Prove that either both $n$ and $n'$ are good or both are bad.

(3) (IMO Shortlist 2014) Alice and Bob play the following game. To start, Alice arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's turn consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number $k$ at most $k$ times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer $n$, determine who has a winning strategy.

(4) (RMM 2019) Alice and Bob play the following game. To start, Alice arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Bob chooses one of the numbers and places a pebble on it. A player's turn consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number $k$ at most $k$ times. The two players alternate taking turns beginning with Alice. The first player who cannot make a move loses. For each positive integer $n$, determine who has a winning strategy.

(5) (RMM 2018) Ann and Bob play a game on the edges of an infinite square grid, playing in turns. Ann plays the first move. A move consists of orienting any edge that has not yet been given an orientation. Bob wins if at any point a cycle has been created. Does Bob have a winning strategy?

(6) Tim and Allen play a game. They start with an empty set $S$. With Tim going first and alternating, on each turn they pick a positive integer greater than 1 which cannot be written as a sum of elements in $S$ (possibly with repetition), and add it to $S$. The player who cannot move loses. Prove the game eventually ends, and determine who has a winning strategy.