Warmup Problems

Algebra

1. Evaluate the sum
$$\sum_{n=1}^{2018}(-1)^n\frac{n^2+n+1}{n!}.$$

2. The sequence (a_n) is defined by $a_1 = 0$,
$$a_{n+1} = \frac{a_1 + a_2 + \cdots + a_n}{n} + 1.$$
Prove that $a_{2016} > \frac{1}{2} + a_{1000}$.

3. Define a sequence by $a_0 = 1$, $a_{2n+1} = a_n$, and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number occurs in the set
$$\left\{\frac{a_{n-1}}{a_n} : n \geq 1\right\}.$$

4. Find all functions $f : (0, \infty) \to (0, \infty)$ with the following properties: $f(x+1) = f(x)+1$ and $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$.

5. Write the sum
$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{k^3+9k^2+26k+24}$$
in the form $\frac{p(n)}{q(n)}$ where p, q are polynomials with integer coefficients.

Combinatorics

1. The integers $1, 2, \ldots, n$ are placed in order so that each value is either strictly bigger than all the preceding values, or it is strictly smaller than all of the preceding values. How many ways can this be done?

2. Tanya and Serezha have a heap of 2016 candies. They make moves in turn, Tanya moves first. At each move a player can eat either one candy or (if the number of candies is even at the moment) exactly half of all candies. The player that cannot move loses. Which of the players has a winning strategy?

3. One hundred gnomes whose weights are $1, 2, \ldots, 100$ pounds are gathered on the left bank of a river. The gnomes cannot swim, but on the shore there is a rowing boat with capacity 100 pounds. Because of the current, it's hard to swim back, so each gnome has just enough strength to row from the right bank to the left bank at most once. Note that the rower does not change during any trip between the two banks. Can all of the gnomes cross to the right bank of the river?

4. Each cell of 100×100 table is coloured black or white. Every cell sharing an edge with the boundary of the table is coloured black. Suppose that in every 2×2 square there are cells of both colors. Prove that there exists a 2×2 square that is coloured like a 2×2 subgrid of a chessboard.

5. On the board a sequence of positive integers $a_1, a_2, \ldots, a_n$ is written. Max wants to write under each number, a number $b_i \geq a_i$ so that for any two of the numbers $b_1, b_2, \ldots, b_n$ one divides the other. Prove that Max can write out the required numbers so that $b_1 b_2 \cdots b_n \leq 2^{(n-1)/2} a_1 a_2 \cdots a_n$.

Geometry

1. Let ABC be a right angled triangle of area 1. Let $A'B'C'$ be the points obtained by reflecting A, B, C respectively in their opposite sides. Find the area of $\Delta A'B'C'$.

2. Let ABC be an isosceles triangle with $AB = AC$ and let the point L on AC be such that BL bisects $\angle ABC$. The point D is chosen on BC and a point E is chosen on side AB so that $AE = \frac{1}{2}AL = CD$. Prove that $LE = LD$.

3. There is an invisible triangle ABC in the plane. You are given one vertex A, the circumcentre O, and Lemoine's point L (the intersection of the reflection of the medians in the corresponding angle bisectors). Describe how to construct the triangle ABC from A, O, L using only compass and straightedge.

4. Point P is taken in the interior of the triangle ABC, so that
$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C).$$
Let L be the foot of the angle bisector of $\angle B$. The line PL meets the circumcircle of $\triangle APC$ at point Q. Prove that QB is the angle bisector of $\angle AQC$.

5. In acute angled triangle $\triangle ABC$, it holds that $BC < AC < AB$. Let I, O, H be the incentre, circumcentre and orthocentre of $\triangle ABC$ respectively. Let the point D on BC and E on AC satisfy that $AE = BD$, $CD + CE = AB$. If the intersection of BE and AD is K, prove that KH is parallel to IO and $KH = 2IO$.

Number Theory

1. How many pairs of positive integers (x, y) are there with $x \leq y$ such that
$$\gcd(x, y) = 5! \text{ and } \operatorname{lcm}(x, y) = 50!.$$

2. Find all integer solutions (x, y) to $x^2 + 84x + 2008 = y^2$.

3. Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers
$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$
for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

4. For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g $P(6) = 36$). For every positive integer n, define the sequence
$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad \text{for } k = 1, 2, \ldots, 2017$$
Is there a positive integer n such that for every k with $1 \leq k \leq 2018$, the number $a_k(n)$ is a perfect square if and only if k is a perfect square?

5. Find the number of ordered 64-tuples $(x_0, x_1, \ldots, x_{63})$ such that $x_0, x_1, \ldots, x_{63}$ are distinct elements of $\{1, 2, \ldots, 2017\}$ and
$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$
is divisible by 2017.