Combinatorics

C1 Suppose a grid with $2m$ rows and $2n$ columns is given, where m and n are positive integers. You may place one pawn on any square of this grid, except the bottom left one or the top right one. After placing the pawn, a snail wants to undertake a journey on the grid. Starting from the bottom left square, it wants to visit every square exactly once, except the one with the pawn on it, which the snail wants to avoid. Moreover, it wants to finish in the top right square. It can only move horizontally or vertically on the grid. On which squares can you put the pawn for the snail to be able to finish its journey?

C2 In a certain club, some pairs of members are friends. Given $k \geq 3$, we say that a club is $k-$good if every group of k members can be seated around a round table such that every two neighbors are friends. Prove that if a club is $6-$good then it is $7-$good.

C3 Two players Arnaldo and Betania play alternately, with Arnaldo being the first to play. Initially there are two piles of stones containing x and y stones respectively. In each play, it is possible to perform one of the following operations:

- Choose two non-empty piles and take one stone from each pile.
- Choose a pile with an odd amount of stones, take one of their stones and, if possible, split into two piles with the same amount of stones.

The player who cannot perform either of operations 1 and 2 loses. Determine who has the winning strategy based on x and y.

C4 Let $n \geq 2$ be an integer. Show that the number of ways to completely cover a $2n \times 2n$ chessboard with 1×2 dominoes is less than the number of ways to completely cover a $3n \times 3n$ chessboard with 1×3 dominoes.

Number Theory

N1 Let a, b, c be integers with $a^3 + b^3 + c^3$ divisible by 18. Prove that abc is divisible by 6.

N2 Determine all ordered pairs (x, y) of nonnegative integers that satisfy the equation

$$3x^2 + 2 \cdot 9^y = x(4^{y+1} - 1).$$

N3 The *triangular numbers* are the integers of the form $\frac{k(k+1)}{2}$ for $k \in \mathbb{Z}$. Determine whether there exists an integer n for which there are at least 6 unordered pairs of triangular numbers which sum to n.

N4 The coefficients of the polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_3x^3 + a_2x^2 + a_0$, $d \geq 2$ are positive integers. Consider the sequence defined by

$$b_1 = a_0, \quad b_{n+1} = P(b_n) \quad \forall n \geq 1.$$

Prove that for any $n \geq 2$ there exists a prime number p, such that p divides b_n and is relatively prime to $b_1 b_2 \ldots b_{n-1}$.

Algebra

A1 Prove that there is a polynomial P such that for any positive integer n,

$$\left\lfloor 2\sqrt{1} \right\rfloor + \left\lfloor 2\sqrt{2} \right\rfloor + \left\lfloor 2\sqrt{3} \right\rfloor + \cdots + \left\lfloor 2\sqrt{n^2} \right\rfloor = P(n).$$

A2 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(-f(x) - f(y)) = 1 - x - y \quad \forall x, y \in \mathbb{Z}.$$

A3 Let $P_0(x) = x^3 - 4x$. The following sequence of polynomials is defined by the recurrence:

$$P_{n+1}(x) = P_n(1 + x)P_n(1 - x) - 1$$

for all integers $n \geq 0$. Prove that $x^{2016} \mid P_{2016}(x)$.

A4 Consider a sequence of positive real numbers $a_1, a_2, a_3, \ldots$, such that $a_1 = 1$, $a_2 = 2$, and

$$a_{mn} = a_m a_n,$$

$$a_{m+n} \leq 2020(a_m + a_n),$$

for all natural numbers m, n. Prove that $a_n = n$ for all natural numbers n.

Geometry

G1 Let ω be a unit circle and AB be a chord of ω such that $AB > 1$. Let S be the point on chord AB satisfying $AS = 1$. The perpendicular bisector of BS intersects ω at C and D. Line DS intersects ω at $E \neq D$. Prove that triangle CSE is equilateral.

G2 Let $\triangle ABC$ be an acute-angled triangle with $AB > AC$, O be its circumcenter and D the midpoint of side BC. The circle with diameter AD meets sides AB, AC again at points E, F respectively. The line passing through D parallel to AO meets EF at M. Show that $EM = MF$.

G3 Triangle ABC has circumcircle ω and centroid G. Let AD, BE, CF be the three altitudes of ABC. Rays AG and GD intersect ω at M and N, respectively. Prove that E, F, M, N are concyclic.

G4 Let I be an incenter of $\triangle ABC$. Denote $D,\ S \neq A$ intersections of AI with BC, $O(ABC)$ respectively. Let $K,\ L$ be incenters of $\triangle DSB,\ \triangle DCS$. Let P be a reflection of I with the respect to KL. Prove that $BP \perp CP$.