2020 Winter Camp Mock Olympiad

Each of the following eight problems has a maximum point value. Your score will be the sum of the three highest scores you obtain on the problems. You have 4 hours to complete this test.

1. **[4]** Let ABC be an acute triangle with $\angle BAC = 60^{\circ}$. Let D and E be the feet of the perpendiculars from A to the internal angle bisectors of $\angle ABC$ and $\angle ACB$, respectively. Let O be the circumcenter of the triangle ABC. Prove that the circumcircles of the triangles ADE and BOC are tangent to each other.

2. **[5]** Let n be a positive integer. Prove that there exists a positive integer m such that
$$2020m^2 + 2019m + 1$$
is divisible by 2^n.

3. **[6]** Prove that for every $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the interval $(0, \pi/2)$
$$\left(\frac{1}{\sin\alpha_1} + \frac{1}{\sin\alpha_2} + \ldots + \frac{1}{\sin\alpha_n}\right)\left(\frac{1}{\cos\alpha_1} + \frac{1}{\cos\alpha_2} + \ldots + \frac{1}{\cos\alpha_n}\right)$$
$$\leq 2\left(\frac{1}{\sin 2\alpha_1} + \frac{1}{\sin 2\alpha_2} + \ldots + \frac{1}{\sin 2\alpha_n}\right)^2.$$

4. **[8]** The cells of a 8×8 table are initially white. Alice and Bob play a game. First Alice paints n of the fields in red. Then Bob chooses 4 rows and 4 columns from the table and paints all fields in them in black. Alice wins if there is at least one red field left. Find the least value of n such that Alice can win the game no matter how Bob plays.

5. **[12]** Let $a_1, a_2, \ldots, a_{2019}$ be positive integers and P a polynomial with integer coefficients such that, for every positive integer n,
$$P(n) \text{ divides } a_1^n + a_2^n + \cdots + a_{2019}^n.$$
Prove that P is a constant polynomial.

6. **[13]** Let ABC be an acute triangle with $AC > AB > BC$. The perpendicular bisectors of AC and AB cut line BC at D and E respectively. Let P and Q be points on lines AC and AB respectively, both different from A, such that $AB = BP$ and $AC = CQ$, and let K be the intersection of lines EP and DQ. Let M be the midpoint of BC. Show that $\angle DKA = \angle EKM$.

7. **[13]** Let $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that:
$$\sum_{cyc} \frac{1}{x_i^3 + x_{i-1}x_ix_{i+1}} \leq \sum_{cyc} \frac{1}{x_ix_{i+1}(x_i + x_{i+1})}$$

8. **[16]** A regular icosahedron is a regular solid of 20 faces, each in the form of an equilateral triangle, with 12 vertices, so that each vertex is in 5 edges. Twelve indistinguishable candies are glued to the vertices of a regular icosahedron (one at each vertex), and four of these twelve candies are special. André and Lucas want to together create a strategy for the following game:

   - First, André is told which are the four special sweets and he must remove exactly four sweets that are not special from the icosahedron and leave the solid on a table, leaving afterwards without communicating with Lucas.
   - Later, Sponchi, who wants to prevent Lucas from discovering the special sweets, can pick up the icosahedron from the table and rotate it however he wants.
   - After Sponchi makes his move, he leaves the room, Lucas enters and he must determine the four special candies out of the eight that remain in the icosahedron.

   Determine if there is a strategy for which Lucas can always properly discover the four special sweets.