0.1 Good Algebra Problems

Problem 0.1.1. *Let A be a set of $n \geq 2$ positive integers, and let $f(x) = \sum_{a \in A} x^a$. Prove that there exists a complex number z with $|z| = 1$ and $|f(z)| = \sqrt{n-2}$.*

Problem 0.1.2. *Does there exist positive reals $a_0, a_1, \ldots, a_{19}$, such that the polynomial $P(x) = x^{20} + a_{19}x^{19} + \ldots + a_1 x + a_0$ does not have any real roots, yet all polynomials formed from swapping any two coefficients a_i, a_j has at least one real root?*

Problem 0.1.3. *Determine if there exists a (three-variable) polynomial $P(x, y, z)$ with integer coefficients satisfying the following property: a positive integer n is not a perfect square if and only if there is a triple (x, y, z) of positive integers such that $P(x, y, z) = n$.*

Problem 0.1.4. *Carl chooses a functional expression E which is a finite nonempty string formed from a set $x_1, x_2, \ldots$ of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets S denote the set of functions $f \colon \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers $x_1, x_2, \ldots$. Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Can Carl choose his functional equation such that $|S| = 1$ and $S \subseteq X$?*

Problem 0.1.5. *Given a set S of n variables, a binary operation $\times$ on S is called simple if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called full if it contains each variable in S at least once, and two strings are equivalent if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.*

Problem 0.1.6. *Consider solutions to the equation*

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

Problem 0.1.7. *Determine whether or not there exist two different sets A, B, each consisting of at most 2011^2 positive integers, such that every x with $0 < x < 1$ satisfies the following inequality:*

$$\left| \sum_{a \in A} x^a - \sum_{b \in B} x^b \right| < (1-x)^{2011}.$$

Problem 0.1.8. *We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form*

$$f(x_1, \cdots, x_k) = \max_{i=1,\cdots,m} \min_{j=1,\cdots,n} P_{i,j}(x_1, \cdots, x_k),$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Problem 0.1.9. *Let n be a fixed integer with $n \geq 2$. We say that two polynomials P and Q with real coefficients are block-similar if for each $i \in \{1, 2, \ldots, n\}$ the sequences*

$$\begin{aligned} &P(2015i), P(2015i-1), \ldots, P(2015i-2014) \quad \textit{and} \\ &Q(2015i), Q(2015i-1), \ldots, Q(2015i-2014) \end{aligned}$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n+1$. (b) Prove that there do not exist distinct block-similar polynomials of degree n.