# Combinatorial Ideas in Inequalities

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## 1 2019 ISL C2 – Construction

You are given a set of $n$ blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number $r$ with $0 \leq r \leq 2n-2$ you can choose a subset of the blocks whose total weight is at least $r$ but at most $r+2$.

## 2 2015 ISL A1 – Induction

Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer $k$. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

## 3 2019 TOT Fall Advanced P5 – sums and bounding

An increasing sequence of positive numbers $\ldots, a_{-1}, a_0, a_1, \ldots$ is given. For each positive integer $k$, let

$$b_k = \left\lceil \max_{i \in \mathbb{Z}} \frac{a_i + a_{i+1} + \cdots + a_{i+k-1}}{a_{i+k-1}} \right\rceil.$$

Prove that either $b_k = k$ for all $k$, or the sequence $b_k$ is eventually constant.

**Solution sketch:** Assume $b_k \neq k$ for some $k$. Then $b_k \leq k-1$, hence for all $i$, $a_i + a_{i+1} + \cdots + a_{i+k-1} \leq (k-1)a_{i+k-1}$. Hence $(k-1)a_i \leq (k-2)a_{i+k-1}$ for all $i$.

Now we have, for all $i$,

$$\sum_{j=-\infty}^{i} a_j \leq (k-1) \sum_{j=-\infty}^{0} a_{i-j(k-1)} \leq (k-1) \sum_{j=-\infty}^{0} \left(\frac{k-1}{k-2}\right)^j a_i < (k-1)(k-2)a_i,$$

hence the sequence $b$ is bounded above by $(k-1)(k-2)$, and since it is discrete and non-decreasing, it is eventually constant.

## 4 Recommended Problems

These problems are roughly in order of difficulty and equally distributed in difficulty relative to the later sections.

1. Let the sequence $\{a_n\}$ be defined recursively as follows:

$$a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + \frac{1}{a_n}, n = 1, 2, \dots$$

   Prove that $a_{180} > 19$.

2. Define the sequence of real numbers $x_1, x_2, \dots, x_{2021}$, such that $x_1$ is any real number and $x_n = 1 - x_1 x_2 \cdots x_{n-1}$ for all $n > 1$. Show that $x_{2021} > \frac{2021}{2022}$.

3. Fix a positive integer $k$, and let $a_1, a_2, \dots, a_k$ be a sequence of nonnegative integers such that for any $i, j$ with $i + j \leq k$, we have that $a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$. Classify all such sequences.

4. Let $a_0, a_1, \dots$ be an arbitary infinite sequence of positive numbers. Show that the inequality $1 + a_n > \frac{n+1}{n} a_{n-1}$ holds for infinitely many positive integers $n$.

5. A sequence of real numbers $a_1, a_2, \dots$ satisfies the relation

$$a_n = -\max_{i_1+i_2+\ldots+i_k=n} (a_{i_1} + a_{i_2} + \ldots + a_{i_k}) \qquad \text{for all} \quad n > 2017.$$

   Then, prove that this sequence is bounded.

6. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and $s$ be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \leq k \leq n-1\} \text{ for all } n > s.$$

   Prove there exist positive integers $\ell \leq s$ and $N$, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \geq N.$$

## 5 Easier Problems (IMO 0-1,4 level)

1. Let $a_0 < a_1 < a_2 \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

2. Let $n$ be an positive integer. Find the smallest integer $k$ with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into $k$ groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

3. Let $a_1, \dots, a_{2020}$ be a sequence of real numbers such that $a_1 = 2^{-2019}$, and $a_{n-1}^2 a_n = a_n - a_{n-1}$. Prove that $a_{2020} < \frac{1}{2^{2019}-1}$.

4. Let $a_1, a_2, \dots, a_n$ be a sequence of real numbers, and let $m$ be a fixed positive integer less than $n$. We say an index $k$ with $1 \leq k \leq n$ is good if there exists some $\ell$ with $1 \leq \ell \leq m$ such that $a_k + a_{k+1} + \ldots + a_{k+\ell-1} \geq 0$, where the indices are taken modulo $n$. Let $T$ be the set of all good indices. Prove that $\sum\limits_{k \in T} a_k \geq 0$.

5. Let $a_0$ be an irrational number such that $0 < a_0 < \frac{1}{2}$. Define $a_n = \min\{2a_{n-1}, 1 - 2a_{n-1}\}$ for $n \geq 1$. Can it happen that $a_n > \frac{7}{40}$ for all $n$? Let $n$ be a positive integer, and consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \cdots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

6. Let $n$ be a positive integer. Find the number of permutations $a_1$, $a_2$, $\ldots a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n$$

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# 6 Medium Problems (IMO 1,4-2,5 level)

1. Let $n \geq 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of $n$ positive real numbers with sum equal to 2. Let $X$ be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left| 1 - \sum_{i \in X} a_i \right|$$

is minimised. Prove that there exists a strictly increasing sequence of $n$ positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

2. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leq i \leq n} |x_1 + \cdots + x_i|.$$

Given $n$ real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price $D$. Greedy George, on the other hand, chooses $x_1$ such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses $x_2$ such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the $i$-th step he chooses $x_i$ among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price $G$.

Find the least possible constant $c$ such that for every positive integer $n$, for every collection of $n$ real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leq cD$.

3. We have $2^m$ sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are $a$ and $b$, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least $4^m$.

4. A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k, x_{2k-1} = (-1)^{k+1} x_k$ for all $k \geq 1$. Prove that $\forall n \geq 1$, $x_1 + x_2 + \ldots + x_n \geq 0$.

5. For each positive integer $n$, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

6. Given an integer $n \geq 2$ and real numbers $x_1, x_2, \ldots, x_n$ in the interval $[0, 1]$, prove that there exist real numbers $a_0, a_1, \ldots, a_n$ satisfying the following conditions:
   - $a_0 + a_n = 0$
   - $|a_i| \leq 1$, for $i = 0, 1, \ldots, n$
   - $|a_i - a_{i-1}| = x_i$, for $i = 1, 2, \ldots, n$.
7. Suppose that a sequence $\{a_n\}$ of integers has the following property: for all sufficiently large $n$, $a_n$ equals the number of indices $i$, $1 \leq i < n$, such that $a_i + i \geq n$. Find the maximum possible number of integers which occur infinitely many times in this sequence.
8. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0, a_1 = 1$, and for every $n \geq 2$ there exists $1 \geq k \geq n$ satisfying
$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$
Find the maximum possible value of $a_{2018} - a_{2017}$.
9. Find all positive integers $n$ such that the following statement holds: Suppose real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, each of which is either $-1$ or 1, such that
$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \leq 1.$$
10. Prove that for $n > 1$ and real numbers $a_0, a_1, \ldots, a_n, k$ with $a_1 = a_{n-1} = 0$,
$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

# 7 Hard Problems (IMO 2,5-3,6)

1. Let $n \geq 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that
$$a_1 + a_2 + \cdots + a_n = 0.$$
Define the set $A$ by
$$A = \{(i, j) \,|\, 1 \leq i < j \leq n, |a_i - a_j| \geq 1\}$$
Prove that, if $A$ is not empty, then
$$\sum_{(i,j)\in A} a_i a_j < 0.$$
2. The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:
   - $1 \leq a_j \leq 2015$ for all $j \geq 1$,
   - $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

   Prove that there exist two positive integers $b$ and $N$ for which
$$\left|\sum_{j=m+1}^{n} (a_j - b)\right| \leq 1007^2$$
for all integers $m$ and $n$ such that $n > m \geq N$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let $f^m$ be $f$ applied $m$ times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let $k_n$ be the smallest such $k$. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

4. Let $x_0, x_1, \ldots, x_{n_0-1}$ be integers, and let $d_1, d_2, \ldots, d_k$ be positive integers with $n_0 = d_1 > d_2 > \cdots > d_k$ and $\gcd(d_1, d_2, \ldots, d_k) = 1$. For every integer $n \geq n_0$, define
$$x_n = \left\lfloor \frac{x_{n-d_1} + x_{n-d_2} + \cdots + x_{n-d_k}}{k} \right\rfloor.$$
Show that the sequence $\{x_n\}$ is eventually constant.

5. Find all functions $f \colon \mathbb{Z}^2 \to [0, 1]$ such that for any integers $x$ and $y$,
$$f(x, y) = \frac{f(x-1, y) + f(x, y-1)}{2}.$$

# 8 Very Hard Problems (IMO 3-6+)

1. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences
$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$
are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

2. For any finite sets $X$ and $Y$ of positive integers, denote by $f_X(k)$ the $k^{\text{th}}$ smallest positive integer not in $X$, and let
$$X * Y = X \cup \{f_X(y) : y \in Y\}.$$
Let $A$ be a set of $a > 0$ positive integers and let $B$ be a set of $b > 0$ positive integers. Prove that if $A * B = B * A$, then
$$\underbrace{A * (A * \cdots (A * (A * A)) \cdots )}_{\text{A appears } b \text{ times}} = \underbrace{B * (B * \cdots (B * (B * B)) \cdots )}_{\text{B appears } a \text{ times}}.$$

3. For any two different real numbers $x$ and $y$, we define $D(x, y)$ to be the unique integer $d$ satisfying $2^d \leq |x - y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the scales of $x$ in $\mathcal{F}$ are the values of $D(x, y)$ for $y \in \mathcal{F}$ with $x \neq y$. Let $k$ be a given positive integer. Suppose that each member $x$ of $\mathcal{F}$ has at most $k$ different scales in $\mathcal{F}$ (note that these scales may depend on $x$). What is the maximum possible size of $\mathcal{F}$?