Warmup Problems

Algebra

1. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$.

2. Find all triples (a, b, c) of real numbers such that the following system holds:
$$\begin{cases} a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ a^2+b^2+c^2 = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \end{cases}$$

3. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f^{f^{f(x)}(y)}(z) = x+y+z+1$$
for all $x, y, z \in \mathbb{N}$.

4. Let n be a positive integer and $a_1, a_2, \ldots, a_n$ non-zero real numbers. What is the least number of non-zero coefficients that the polynomial $P(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ can have?

Combinatorics

1. Let n be a positive integer. Compute the number of words w that satisfy the following three properties.
   1. w consists of n letters from the alphabet $\{a, b, c, d\}$.
   2. w contains an even number of a's
   3. w contains an even number of b's.

   For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd, dc.

2. Let $\mathcal{P}$ be a finite set of squares on an infinite chessboard. Kelvin the Frog notes that $\mathcal{P}$ may be tiled with only 1×2 dominoes, while Alex the Kat notes that $\mathcal{P}$ may be tiled with only 2×1 dominoes. The dominoes cannot be rotated in each tiling. Prove that the area of $\mathcal{P}$ is a multiple of 4.

3. Alex the Kat and Kelvin the Frog play a game on a complete graph with n vertices. Kelvin goes first, and the players take turns selecting either a single edge to remove from the graph, or a single vertex to remove from the graph. Removing a vertex also removes all edges incident to that vertex. The player who removes the final vertex wins the game. Assuming both players play perfectly, for which positive integers n does Kelvin have a winning strategy?

4. A cube consisting of $(2N)^3$ unit cubes is pierced by several needles parallel to the edges of the cube (each needle pierces exactly $2N$ unit cubes). Each unit cube is pierced by at least one needle. Let us call any subset of these needles *regular* if there are no two needles in this subset that pierce the same unit cube. What is the maximum size of a regular subset that does exist for sure? (That is, for every piercing of the cube such that each unit cube is pierced by one needle, there is a regular subset of that size.)

Geometry

1. Convex pentagon $ABCDE$ has side lengths $AB = 5$, $BC = CD = DE = 6$, and $EA = 7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

2. Let ABC be an acute triangle with circumcircle Γ and let D be the midpoint of minor arc BC. Let E, F be on Γ such that $DE \perp AC$ and $DF \perp AB$. Lines BE and DF meet at G, and lines CF and DE meet at H. Show that $BCHG$ is a parallelogram.

3. Janabel has a device that, when given two distinct points U and V in the plane, draws the perpendicular bisector of UV. Show that if three lines forming a triangle are drawn, Janabel can mark the orthocenter of the triangle using this device, a pencil, and no other tools.

4. Consider horizontal and vertical segments in the plane that may intersect each other. Let n denote their total number. Suppose that we have m curves starting from the origin that are pairwise disjoint except for their endpoints. Assume that each curve intersects exactly two of the segments, a different pair for each curve. Prove that $m = O(n)$.

Number Theory

1. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive divisors of n.

2. Prove or disprove: If $n > 1$ is an odd integer satisfying $n \mid 2^{\frac{n-1}{2}} + 1$, then n is prime.

3. Let us say that the pair (m, n) of two positive different integers m and n is nice if mn and $(m+1)(n+1)$ are perfect squares. Prove that for each positive integer m there exists at least one $n > m$ such that the pair (m, n) is nice.

4. For any positive integer n, let $\tau(n)$ denote the number of positive integer divisors of n, $\sigma(n)$ denote the sum of the positive integer divisors of n, and $\varphi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n. Let $a, b > 1$ be integers. Brandon has a calculator with three buttons that replace the integer n currently displayed with $\tau(n)$, $\sigma(n)$, or $\varphi(n)$, respectively. Prove that if the calculator currently displays a, then Brandon can make the calculator display b after a finite (possibly empty) sequence of button presses.