Day 2

P4. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR are concurrent.

Common remarks. The introductory steps presented here are used in all solutions below.

Since $AC = BC = AD$, we have $\angle ABC = \angle BAC = \angle ACD = \angle ADC$. Since the quadrilaterals $APRQ$ and $AQCD$ are cyclic, we obtain

$$\angle CRA = 180^\circ - \angle ARP = 180^\circ - \angle AQP = \angle DQA = \angle DCA = \angle CBA,$$

so the points A, B, C, and R lie on some circle γ.

Solution 1. Introduce the point $X = AQ \cap CD$; we need to prove that B, R and X are collinear.

By means of the circle $(APRQ)$ we have

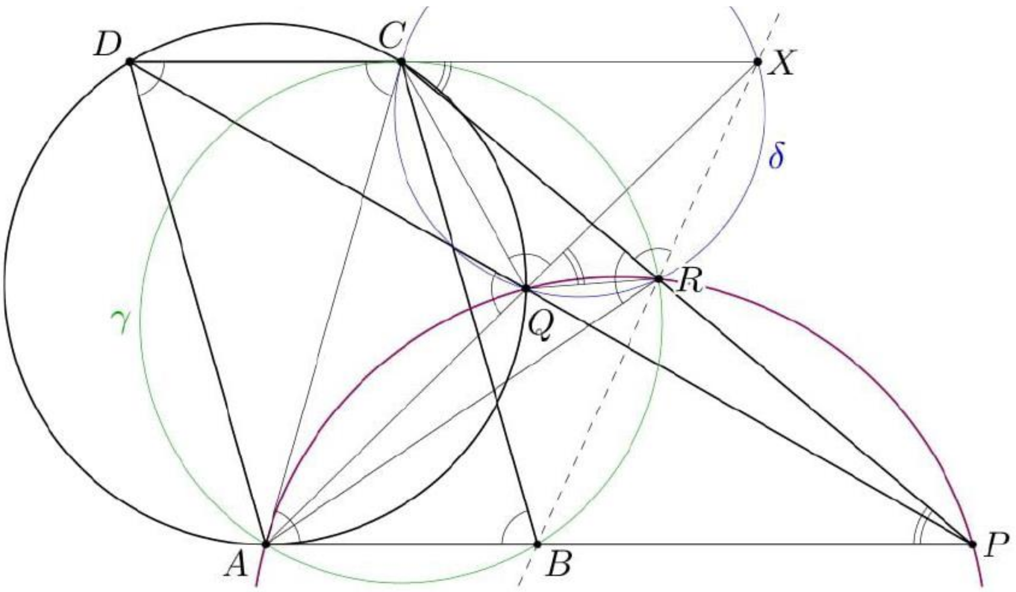
$$\angle RQX = 180^\circ - \angle AQR = \angle RPA = \angle RCX$$

(the last equality holds in view of $AB \parallel CD$), which means that the points C, Q, R, and X also lie on some circle δ.

Using the circles δ and γ we finally obtain

$$\angle XRC = \angle XQC = 180^\circ - \angle CQA = \angle ADC = \angle BAC = 180^\circ - \angle CRB,$$

that proves the desired collinearity.



Solution 2. Let α denote the circle $(APRQ)$. Since

$$\angle CAP = \angle ACD = \angle AQD = 180^\circ - \angle AQP,$$

the line AC is tangent to α.

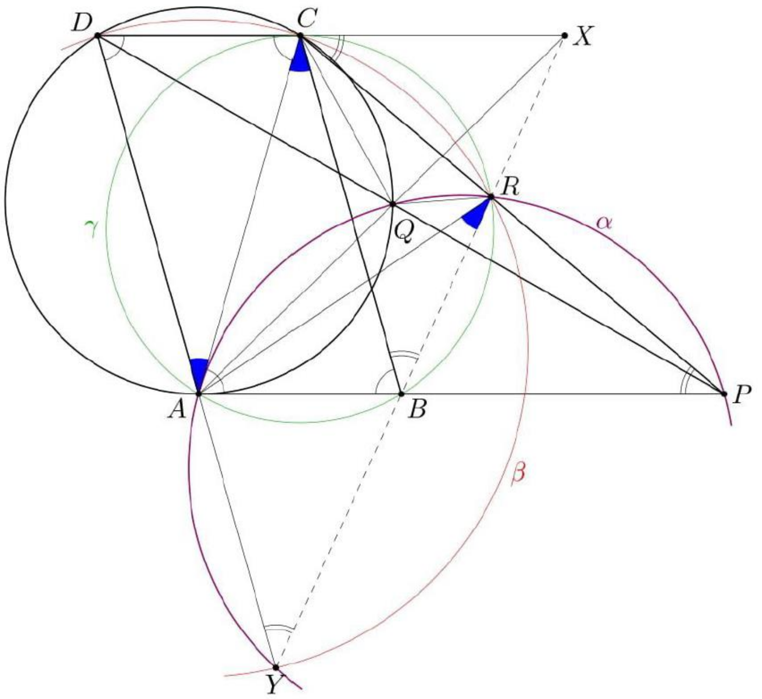
Now, let AD meet α again at a point Y (which necessarily lies on the extension of DA beyond A). Using the circle γ, along with the fact that AC is tangent to α, we have

$$\angle ARY = \angle CAD = \angle ACB = \angle ARB,$$

so the points Y, B, and R are collinear.

Applying Pascal's theorem to the hexagon $AAYRPQ$ (where AA is regarded as the tangent to α at A), we see that the points $AA \cap RP = C$, $AY \cap PQ = D$, and $YR \cap QA$ are collinear. Hence the lines CD, AQ, and BR are concurrent.

Comment 1. Solution 2 consists of two parts: (1) showing that BR and DA meet on α; and (2) showing that this yields the desired concurrency. Solution 3 also splits into those parts, but the proofs are different.



Solution 3. As in Solution 1, we introduce the point $X = AQ \cap CD$ and aim at proving that the points B, R, and X are collinear. As in Solution 2, we denote $\alpha = (APQR)$; but now we define Y to be the second meeting point of RB with α.

Using the circle α and noticing that CD is tangent to γ, we obtain

$$\angle RYA = \angle RPA = \angle RCX = \angle RBC. \tag{1}$$

So $AY \parallel BC$, and hence Y lies on DA.

Now the chain of equalities (1) shows also that $\angle RYD = \angle RCX$, which implies that the points C, D, Y, and R lie on some circle β. Hence, the lines CD, AQ, and YBR are the pairwise radical axes of the circles $(AQCD)$, α, and β, so those lines are concurrent.

Comment 2. The original problem submission contained an additional assumption that $BP = AB$. The Problem Selection Committee removed this assumption as superfluous.

P5 Given a positive integer n, find the smallest value of $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$ over all permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$.

Answer: The minimum of such sums is $\lfloor \log_2 n \rfloor + 1$; so if $2^k \leqslant n < 2^{k+1}$, the minimum is $k+1$.

Solution 1. Suppose that $2^k \leqslant n < 2^{k+1}$ with some nonnegative integer k. First we show a permutation $(a_1, a_2, \ldots, a_n)$ such that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor = k+1$; then we will prove that $\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor \geqslant k+1$ for every permutation. Hence, the minimal possible value will be $k+1$.

I. Consider the permutation

$$(a_1) = (1), \quad (a_2, a_3) = (3, 2), \quad (a_4, a_5, a_6, a_7) = (7, 4, 5, 6), \quad \ldots$$
$$(a_{2^{k-1}}, \ldots, a_{2^k - 1}) = (2^k - 1, 2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 2),$$
$$(a_{2^k}, \ldots, a_n) = (n, 2^k, 2^k + 1, \ldots, n-1).$$

This permutation consists of $k+1$ cycles. In every cycle $(a_p, \ldots, a_q) = (q, p, p+1, \ldots, q-1)$ we have $q < 2p$, so

$$\sum_{i=p}^{q} \left\lfloor \frac{a_i}{i} \right\rfloor = \left\lfloor \frac{q}{p} \right\rfloor + \sum_{i=p+1}^{q} \left\lfloor \frac{i-1}{i} \right\rfloor = 1;$$

The total sum over all cycles is precisely $k+1$.

II. In order to establish the lower bound, we prove a more general statement.

Claim. If $b_1, \ldots, b_{2^k}$ are distinct positive integers then

$$\sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant k+1.$$

From the Claim it follows immediately that $\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant k+1$.

Proof of the Claim. Apply induction on k. For $k=1$ the claim is trivial, $\left\lfloor \frac{b_1}{1} \right\rfloor \geqslant 1$. Suppose the Claim holds true for some positive integer k, and consider $k+1$.

If there exists an index j such that $2^k < j \leqslant 2^{k+1}$ and $b_j \geqslant j$ then

$$\sum_{i=1}^{2^{k+1}} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor + \left\lfloor \frac{b_j}{j} \right\rfloor \geqslant (k+1)+1$$

by the induction hypothesis, so the Claim is satisfied.

Otherwise we have $b_j < j \leqslant 2^{k+1}$ for every $2^k < j \leqslant 2^{k+1}$. Among the 2^{k+1} distinct numbers $b_1, \ldots, b_{2^{k+1}}$ there is some b_m which is at least 2^{k+1}; that number must be among $b_1 \ldots, b_{2^k}$. Hence, $1 \leqslant m \leqslant 2^k$ and $b_m \geqslant 2^{k+1}$.

We will apply the induction hypothesis to the numbers

$$c_1 = b_1, \ldots, c_{m-1} = b_{m-1}, \quad c_m = b_{2^k+1}, \quad c_{m+1} = b_{m+1}, \ldots, c_{2^k} = b_{2^k},$$

so take the first 2^k numbers but replace b_m with b_{2^k+1}. Notice that

$$\left\lfloor \frac{b_m}{m} \right\rfloor \geqslant \left\lfloor \frac{2^{k+1}}{m} \right\rfloor = \left\lfloor \frac{2^k + 2^k}{m} \right\rfloor \geqslant \left\lfloor \frac{b_{2^k+1} + m}{m} \right\rfloor = \left\lfloor \frac{c_m}{m} \right\rfloor + 1.$$

For the other indices i with $1 \leqslant i \leqslant 2^k$, $i \neq m$ we have $\left\lfloor \frac{b_i}{i} \right\rfloor = \left\lfloor \frac{c_i}{i} \right\rfloor$, so

$$\sum_{i=1}^{2^{k+1}} \left\lfloor \frac{b_i}{i} \right\rfloor = \sum_{i=1}^{2^k} \left\lfloor \frac{b_i}{i} \right\rfloor \geqslant \sum_{i=1}^{2^k} \left\lfloor \frac{c_i}{i} \right\rfloor + 1 \geqslant (k+1)+1.$$

That proves the Claim and hence completes the solution. □

Solution 2. We present a different proof for the lower bound.

Assume again $2^k \leqslant n < 2^{k+1}$, and let $P = \{2^0, 2^1, \ldots, 2^k\}$ be the set of powers of 2 among $1, 2, \ldots, n$. Call an integer $i \in \{1, 2, \ldots, n\}$ and the interval $[i, a_i]$ *good* if $a_i \geqslant i$.

Lemma 1. The good intervals cover the integers $1, 2, \ldots, n$.

Proof. Consider an arbitrary $x \in \{1, 2 \ldots, n\}$; we want to find a good interval $[i, a_i]$ that covers x; i.e., $i \leqslant x \leqslant a_i$. Take the cycle of the permutation that contains x, that is $(x, a_x, a_{a_x}, \ldots)$. In this cycle, let i be the first element with $a_i \geqslant x$; then $i \leqslant x \leqslant a_i$. □

Lemma 2. If a good interval $[i, a_i]$ covers p distinct powers of 2 then $\left\lfloor \frac{a_i}{i} \right\rfloor \geqslant p$; more formally, $\left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \left| [i, a_i] \cap P \right|$.
Proof. The ratio of the smallest and largest powers of 2 in the interval is at least 2^{p-1}. By Bernoulli's inequality, $\frac{a_i}{i} \geqslant 2^{p-1} \geqslant p$; that proves the lemma. □

Now, by Lemma 1, the good intervals cover P. By applying Lemma 2 as well, we obtain that

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor = \sum_{i \text{ is good}}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i \text{ is good}}^{n} \left| [i, a_i] \cap P \right| \geqslant |P| = k + 1.$$

Solution 3. We show yet another proof for the lower bound, based on the following inequality.

Lemma 3.

$$\left\lfloor \frac{a}{b} \right\rfloor \geqslant \log_2 \frac{a+1}{b}$$

for every pair a, b of positive integers.
Proof. Let $t = \left\lfloor \frac{a}{b} \right\rfloor$, so $t \leqslant \frac{a}{b}$ and $\frac{a+1}{b} \leqslant t + 1$. By applying the inequality $2^t \geqslant t + 1$, we obtain

$$\left\lfloor \frac{a}{b} \right\rfloor = t \geqslant \log_2(t+1) \geqslant \log_2 \frac{a+1}{b}. \qquad \square$$

By applying the lemma to each term, we get

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \sum_{i=1}^{n} \log_2 \frac{a_i+1}{i} = \sum_{i=1}^{n} \log_2(a_i+1) - \sum_{i=1}^{n} \log_2 i.$$

Notice that the numbers $a_1+1, a_2+1, \ldots, a_n+1$ form a permutation of $2, 3, \ldots, n+1$. Hence, in the last two sums all terms cancel out, except for $\log_2(n+1)$ in the first sum and $\log_2 1 = 0$ in the second sum. Therefore,

$$\sum_{i=1}^{n} \left\lfloor \frac{a_i}{i} \right\rfloor \geqslant \log_2(n+1) > k.$$

As the left-hand side is an integer, it must be at least $k+1$.

P6 The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a *path from X to Y* is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Solution 1. We write $X \to Y$ (or $Y \leftarrow X$) if the road between X and Y goes from X to Y. Notice that, if there is any route moving from X to Y (possibly passing through some cities more than once), then there is a path from X to Y consisting of some roads in the route. Indeed, any cycle in the route may be removed harmlessly; after some removals one obtains a path.

Say that a path is *short* if it consists of one or two roads.

Partition all cities different from A and B into four groups, $\mathcal{I}$, $\mathcal{O}$, $\mathcal{A}$, and $\mathcal{B}$ according to the following rules: for each city C,

$$C \in \mathcal{I} \iff A \to C \leftarrow B; \qquad C \in \mathcal{O} \iff A \leftarrow C \to B;$$
$$C \in \mathcal{A} \iff A \to C \to B; \qquad C \in \mathcal{B} \iff A \leftarrow C \leftarrow B.$$

Lemma. Let $\mathcal{P}$ be a diverse collection consisting of p paths from A to B. Then there exists a diverse collection consisting of at least p paths from A to B and containing all short paths from A to B.

Proof. In order to obtain the desired collection, modify $\mathcal{P}$ as follows.

If there is a direct road $A \to B$ and the path consisting of this single road is not in $\mathcal{P}$, merely add it to $\mathcal{P}$.

Now consider any city $C \in \mathcal{A}$ such that the path $A \to C \to B$ is not in $\mathcal{P}$. If $\mathcal{P}$ contains at most one path containing a road $A \to C$ or $C \to B$, remove that path (if it exists), and add the path $A \to C \to B$ to $\mathcal{P}$ instead. Otherwise, $\mathcal{P}$ contains two paths of the forms $A \to C \dashrightarrow B$ and $A \dashrightarrow C \to B$, where $C \dashrightarrow B$ and $A \dashrightarrow C$ are some paths. In this case, we recombine the edges to form two new paths $A \to C \to B$ and $A \dashrightarrow C \dashrightarrow B$ (removing cycles from the latter if needed). Now we replace the old two paths in $\mathcal{P}$ with the two new ones.

After any operation described above, the number of paths in the collection does not decrease, and the collection remains diverse. Applying such operation to each $C \in \mathcal{A}$, we obtain the desired collection. □

Back to the problem, assume, without loss of generality, that there is a road $A \to B$, and let a and b denote the numbers of roads going out from A and B, respectively. Choose a diverse collection $\mathcal{P}$ consisting of N_{AB} paths from A to B. We will transform it into a diverse collection $\mathcal{Q}$ consisting of at least $N_{AB} + (b - a)$ paths from B to A. This construction yields

$$N_{BA} \geqslant N_{AB} + (b - a); \qquad \text{similarly, we get} \quad N_{AB} \geqslant N_{BA} + (a - b),$$

whence $N_{BA} - N_{AB} = b - a$. This yields the desired equivalence.

Apply the lemma to get a diverse collection $\mathcal{P}'$ of at least N_{AB} paths containing all $|\mathcal{A}| + 1$ short paths from A to B. Notice that the paths in $\mathcal{P}'$ contain no edge of a short path from B to A. Each non-short path in $\mathcal{P}'$ has the form $A \to C \dashrightarrow D \to B$, where $C \dashrightarrow D$ is a path from some city $C \in \mathcal{I}$ to some city $D \in \mathcal{O}$. For each such path, put into $\mathcal{Q}$ the path $B \to C \dashrightarrow D \to A$; also put into $\mathcal{Q}$ all short paths from B to A. Clearly, the collection $\mathcal{Q}$ is diverse.

Now, all roads going out from A end in the cities from $\mathcal{I} \cup \mathcal{A} \cup \{B\}$, while all roads going out from B end in the cities from $\mathcal{I} \cup \mathcal{B}$. Therefore,

$$a = |\mathcal{I}| + |\mathcal{A}| + 1, \quad b = |\mathcal{I}| + |\mathcal{B}|, \qquad \text{and hence} \quad a - b = |\mathcal{A}| - |\mathcal{B}| + 1.$$

On the other hand, since there are $|\mathcal{A}| + 1$ short paths from A to B (including $A \to B$) and $|\mathcal{B}|$ short paths from B to A, we infer

$$|\mathcal{Q}| = |\mathcal{P}'| - (|\mathcal{A}| + 1) + |\mathcal{B}| \geqslant N_{AB} + (b - a),$$

as desired.

Solution 2. We recall some graph-theoretical notions. Let G be a finite graph, and let V be the set of its vertices; fix two distinct vertices $s, t \in V$. An *(s,t)-cut* is a partition of V into two parts $V = S \sqcup T$ such that $s \in S$ and $t \in T$. The *cut-edges* in the cut (S,T) are the edges going from S to T, and the *size* $e(S,T)$ of the cut is the number of cut-edges.

We will make use of the following theorem (which is a partial case of the Ford–Fulkerson "min-cut max-flow" theorem).

Theorem (Menger). Let G be a directed graph, and let s and t be its distinct vertices. Then the maximal number of edge-disjoint paths from s to t is equal to the minimal size of an (s,t)-cut.

Back to the problem. Consider a directed graph G whose vertices are the cities, and edges correspond to the roads. Then N_{AB} is the maximal number of edge-disjoint paths from A to B in this graph; the number N_{BA} is interpreted similarly.

As in the previous solution, denote by a and b the out-degrees of vertices A and B, respectively. To solve the problem, we show that for any (A,B)-cut (S_A, T_A) in our graph there exists a (B,A)-cut (S_B, T_B) satisfying

$$e(S_B, T_B) = e(S_A, T_A) + (b - a).$$

This yields

$$N_{BA} \leqslant N_{AB} + (b - a); \qquad \text{similarly, we get} \quad N_{AB} \leqslant N_{BA} + (a - b),$$

whence again $N_{BA} - N_{AB} = b - a$.

The construction is simple: we put $S_B = S_A \cup \{B\} \setminus \{A\}$ and hence $T_B = T_A \cup \{A\} \setminus \{B\}$. To show that it works, let A and B denote the sets of cut-edges in (S_A, T_A) and (S_B, T_B), respectively. Let a_s and $a_t = a - a_s$ denote the numbers of edges going from A to S_A and T_A, respectively. Similarly, denote by b_s and $b_t = b - b_s$ the numbers of edges going from B to S_B and T_B, respectively.

Notice that any edge incident to none of A and B either belongs to both A and B, or belongs to none of them. Denote the number of such edges belonging to A by c. The edges in A which are not yet accounted for split into two categories: those going out from A to T_A (including $A \to B$ if it exists), and those going from $S_A \setminus \{A\}$ to B — in other words, going from S_B to B. The numbers of edges in the two categories are a_t and $|S_B| - 1 - b_s$, respectively. Therefore,

$$|\mathsf{A}| = c + a_t + (|S_B| - b_s - 1). \qquad \text{Similarly, we get} \quad |\mathsf{B}| = c + b_t + (|S_A| - a_s - 1),$$

and hence

$$|\mathsf{B}| - |\mathsf{A}| = (b_t + b_s) - (a_t + a_s) = b - a,$$

since $|S_A| = |S_B|$. This finishes the solution.