Problems From ISL I thought were pretty cool

Please note, this problem set is intended to be even more difficult than my others, so I recommend working together.

Warm Warmup Problems:

1. (2015 ISL C1) In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes. Let A and B be two towns, with B being to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of all towns on its way. Prove that there is exactly one town which cannot be swept away by any other one.

2. (2012 ISL C1) Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

3. (2017 ISL C1) A rectangle R with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of R are either all odd or all even.

4. (2011 ISL C2) Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

5. (2013 ISL C3) A crazy physicist discovered a new kind of particle which he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

   (a) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

   (b) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

   Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

Hot Problems:

1. (2018 ISL C6) Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

   (a) If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.

   (b) If no such pair exists, we write down two times the number 0.

   Prove that, no matter how we make the choices in (a), operation (b) will be performed only finitely many times.

2. (2010 ISL C6) Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The length of a string is the number of pearls on it. One cuts these strings in some steps by the following rules. In each step:

   (a) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

   (b) Next, one cuts each chosen string into two parts differing in length by at most one. (For instance, if there are strings of $5, 4, 4, 2$ black pearls, strings of $8, 4, 3$ white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4), (3, 2), (2, 2)$ and $(2, 2)$ respectively.) The process stops immediately after the step when a first isolated white pearl appears.

   Prove that at this stage, there will still exist a string of at least two black pearls.

3. (2011 ISL C5) Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn 90 degrees clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear. Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard.

4. (2015 ISL C6) Let S be a nonempty set of positive integers. We say that a positive integer n is clean if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

5. (2013 ISL C6) In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

6. (2016 ISL C8) Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

7. (2015 ISL C7) In a company of people some pairs are enemies. A group of people is called unsociable if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

8. (2017 ISL C8) Let n be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The neighborhood of a lattice point c consists of all lattice points within the axis-aligned $(2n+1) \times (2n+1)$ square entered at c, apart from c itself. We call a butterfly lonely, crowded, or comfortable, depending on whether the number of butterflies in its neighborhood N is respectively less than, greater than, or equal to half of the number of lattice points in N. Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

9. Alice has a map of Wonderland, a country consisting of $n \geq 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns. Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.