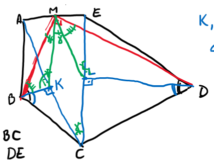
1.



$AB = BC$
$CD = DE$
$\measuredangle ABC + \measuredangle CDE = 180$

K, L = midpoint of AC, CE

$\triangle BKC \sim \triangle CLD$ ← here's why

$CKML$ = parallelogram

$\measuredangle BKM = \measuredangle MLD = 90° + \gamma$

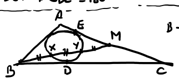
It remains to prove that $\measuredangle MBK = \measuredangle DML$

i.e. that $\triangle MBK \sim \triangle DML$.

$\measuredangle BKM = \measuredangle MLD$ ✓

$\frac{BK}{KM} = \frac{ML}{LD}$ ✓ because $\frac{BK}{LC} = \frac{KC}{LD}$

2.



$B - X - Y - M$
$BC = a$
$CA = b$
$AB = c$

$\begin{cases} BX \cdot BY = BD^2 \\ MX \cdot MY = ME^2 \end{cases} \Rightarrow BD = ME$

$BD = \frac{a-b+c}{2}$, $ME = \frac{b}{2} - \frac{b+c-a}{2} = \frac{a-c}{2}$

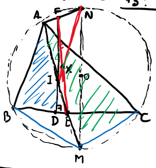
$BX = \frac{1}{3}BM$, $BY = \frac{2}{3}BM$ $\quad \cancel{a} - c = \cancel{a} - b + c \Rightarrow \boxed{b = 2c}$.

$\left(\frac{a-c}{2}\right)^2 = BD^2 = \frac{2}{9}BM^2 = \frac{2a^2+2c^2-b^2}{18} = \frac{2a^2-2c^2}{18}$ $\quad BM^2 = \frac{2a^2+2c^2-b^2}{4}$

$\frac{1}{4}\cancel{(a-c)}(a-c) = \frac{1}{9}\cancel{(a-c)}(a+c) \Rightarrow 4(a+c) = 9(a-c) \Rightarrow \boxed{5a = 13c}$

$a : b : c = \underline{13 : 10 : 5}$.

3.



Prove that NI bisects EF, i.e. $XE = XF$

$FADE$ cyclic.

Let's „compute" $\frac{EX}{XF}$ (and prove it equals 1)

Apply Menelaus' thm. on $\triangle AEF$ and line NXI

$\frac{EX}{XF} = \frac{EI}{IA} \cdot \frac{AN}{NF}$, where $\frac{EI}{IA} = \frac{EB}{BA}$

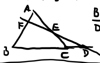
$\frac{AN}{NF} = \frac{AM}{MI}$ $(FI \parallel MN)$

So, $\frac{EX}{XF} = \frac{EB}{BA} \cdot \frac{AM}{MI} \overset{?}{=} 1$

It remains to show that $\frac{AM}{MI} = \frac{AB}{BE}$.

But $\frac{AM}{MI} = \frac{AM}{MC} = \frac{AB}{BE}$ because $\triangle AMC \sim \triangle ABE$

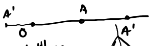
Menelaus' theorem



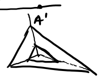
$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$

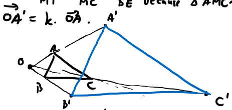
$\frac{BD}{DC} = \frac{BF}{FA} \cdot \frac{AE}{EC}$

$H_{O,k}$: $A \to A'$ such that $\vec{OA'} = k \cdot \vec{OA}$.



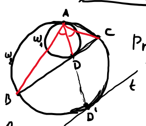
$A'B' \parallel AB$





Whenever $A'B' \parallel AB$, $A'C' \parallel AC$, $B'C' \parallel BC$, AA', BB', CC' are concurrent or parallel.

4.



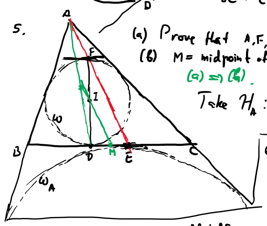
Prove that AD bisects $\measuredangle BAC$.

$H_A : \omega_1 \to \omega_2$

$D \to D'$, $t \parallel BC \Rightarrow D'$ is the midpoint of $\widehat{BC}$

$BC \to t$

5.



(a) Prove that A, F, E are collinear

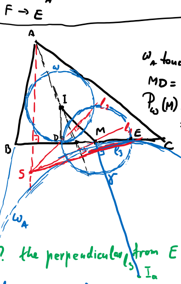
(b) M = midpoint of $BC \Rightarrow$ prove that MI bisects AD

(a) ⇒ (b). But why $F \in AE$?

Take H_A : $\omega \to \omega_A$

$F \to E$

6.



$AB \neq AC$

M = midpoint of BC

Prove that the perpendiculars from M to AI and from D to MI meet on the altitude from A.

ω_A touches BC at E

$MD = ME$

$P_\omega(M) = MD^2 = ME^2 = P_{\omega_A}(M)$

rad. axis of ω, γ is ℓ_2

rad. axis of ω, ω_A is ℓ_1

What is the rad. axis of γ and ω_A? the perpendicular ℓ_3 from E to MI_a

So, ℓ_1, ℓ_2 and ℓ_3 meet at S.

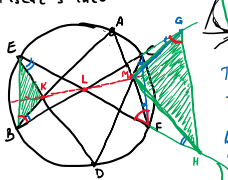
$\ell_2 \perp MI \parallel AE$: ℓ_2 is the altitude in $\triangle ADE$ from D.

$\to \ell_3 \perp MI_a \parallel AD$: ℓ_3 is the altitude in $\triangle ADE$ from E.

$\Rightarrow S$ is the orthocenter of $\triangle ADE$, so it lies on the altitude from A as well.

Why is $MI_a \parallel AD$? Why is $MI \parallel AE$!

7. Pascal's theorem



Take $G \in BC$ with $MG \parallel BK$

Then G is on the circle CMF.

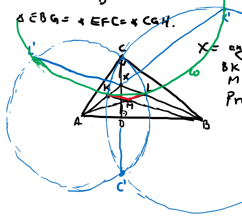
Take $H \in EF$ with $MH \parallel EK$

Then H is also on the circle CMF.

If we can show that $GH \parallel BE$, we would deduce that $\triangle BEK$, $\triangle GHM$ are homothetic and finish.

$\measuredangle EBG = \measuredangle EFC = \measuredangle CGH$.

8.



X = any point on the altitude CD.

$BK = BC$, $AL = AC$

$M = AL \cap BK$.

Prove that $MK = ML$.

$XL \cdot XL' = XC \cdot XC' = XK \cdot XK'$

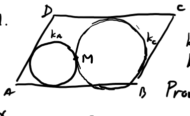
$AL^2 = AC^2 = AK \cdot AK'$

$\Rightarrow AL$ is tangent to ω

BK is tangent to ω

Homework:

1.



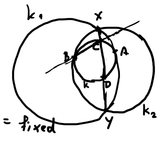
$ABCD$ = parallelogram

k_A is tangent to AB, AD

k_C is tangent to CB, CD, k_A

Prove that M is on AC.

2.



k_1, k_2 = fixed

Prove that, as k varies, each of the lines AC, AD, BC, BD passes through a fixed point.

Hint. These fixed points are on k_1, k_2.