The Online Math Open Fall Contest
Official Solutions
October 25 – November 5, 2019

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1. Compute the sum of all positive integers n such that the median of the n smallest prime numbers is n.

   Proposed by Luke Robitaille.

   Answer. $\boxed{25}$.

   Solution. We claim that all the numbers are 3, 4, 5, 6, and 7, for a sum of $3+4+5+6+7=25$. It is straightforward to check that these work by just listing the n smallest prime numbers for these n. Note that $n=1$ and $n=2$ do not work because the n smallest prime numbers are $\{2\}$ and $\{2,3\}$, respectively. Also, note that $n=8$ does not work. Now, we show that $n \geq 9$ does not work. Let $p_1, p_2, \ldots$ be the prime numbers.

   If $n \geq 9$ is even, $n = 2k$ for some $k \geq 5$. If n is the median of $p_1, p_2, \ldots, p_n$, so

   $$n = \frac{p_k + p_{k+1}}{2} \Rightarrow p_k < 2k.$$

   Since no even numbers larger than 2 are prime, any prime below $2k$ must be one of $\{2,3,5,7,9,\ldots 2k-1\}$, which has exactly k members, so if $p_k < 2k$, $p_j = 2j-1$ for $j = 2,3,\ldots k$. However, this is false for $j=5$, so it cannot hold for $k \geq 5$, as desired.

   Similarly, if $n \geq 9$ is odd, write $n = 2k-1$ for some $k \geq 5$. If n is the median of the smallest n primes, we have $p_k = 2k-1$ so $p_k < 2k$. From the same reasoning as earlier, this cannot hold for $k \geq 5$.

   Thus the answer is $\boxed{25}$, as desired. □

2. Let A, B, C, and P be points in the plane such that no three of them are collinear. Suppose that the areas of triangles BPC, CPA, and APB are 13, 14, and 15, respectively. Compute the sum of all possible values for the area of triangle ABC.

   Proposed by Ankan Bhattacharya.

   Answer. $\boxed{84}$.

   Solution. The possible areas are $\pm 13 \pm 14 \pm 15$, where at most one minus sign is used. The desired sum equals $2(13+14+15) = \boxed{84}$. □

3. Let k be a positive real number. Suppose that the set of real numbers x such that $x^2 + k|x| \leq 2019$ is an interval of length 6. Compute k.

   Proposed by Luke Robitaille.

   Answer. $\boxed{670}$.

   Solution. Note that $x^2 + k|x| = (-x)^2 + k \mid -x \mid$ for all real numbers x and k. Then x satisfies $x^2 + k|x| \leq 2019$ if and only if $-x$ does. Then the interval must be $[-c, c]$ for some c, so since it has length 6 it must be $[-3,3]$; then equality must hold at $x=3$, so $9+3k=2019$, so $k = \boxed{670}$. □

4. Maryssa, Stephen, and Cynthia played a game. Each of them independently privately chose one of Rock, Paper, and Scissors at random, with all three choices being equally likely. Given that at least one of them chose Rock and at most one of them chose Paper, the probability that exactly one of them chose Scissors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m+n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{916}$.

   Solution. There are $3^3 - 2^3 = 19$ ways that at least one of them played Rock, and among them there are 3 ways to have two Papers and one Rock. Among the $19-3=16$ remaining possibilities, 6 of them have the three hands appear once each and 3 of them have two Rocks and one Scissors. Therefore the desired probability is $\frac{9}{16}$. The answer is $\boxed{916}$. □

5. Compute the number of ordered pairs (m, n) of positive integers that satisfy the equation $\operatorname{lcm}(m, n) + \gcd(m, n) = m + n + 30$.

   Proposed by Ankit Bisain.

   Answer. $\boxed{16}$.

   Solution. Let $g = \gcd(m, n)$, $a = \frac{m}{g}$ and $b = \frac{n}{g}$. By definition, $\gcd(a, b) = 1$. Now, the equation becomes

$$gab + g = ga + gb + 30$$

$$g(a-1)(b-1) = 30$$

   If g is even, a and b would both be even, contradicting $\gcd(a, b) = 1$. Thus, g is odd.

   Case 1: $g = 1$; then $(a-1)(b-1) = 2 \cdot 3 \cdot 5$, giving the solutions

$$(a, b) = (2, 31), (3, 16), (4, 11), (6, 7), (7, 6), (11, 4), (16, 3), (31, 2)$$

$$(m, n) = (2, 31), (3, 16), (4, 11), (6, 7), (7, 6), (11, 4), (16, 3), (31, 2)$$

   Case 2: $g = 3$; then $(a-1)(b-1) = 2 \cdot 5$, giving

$$(a, b) = (2, 11), (3, 6), (6, 3), (11, 2)$$

   but since $(3, 6)$ and $(6, 3)$ do not satisfy $\gcd(a, b) = 1$, the only valid solutions are

$$(a, b) = (2, 11), (11, 2)$$

$$(m, n) = (6, 33), (33, 6)$$

   Case 3: $g = 5$; then $(a-1)(b-1) = 2 \cdot 3$, giving

$$(a, b) = (2, 7), (3, 4), (4, 3), (7, 2)$$

$$(m, n) = (10, 35), (15, 20), (20, 15), (35, 10)$$

   Case 4: $g = 15$; then $(a-1)(b-1) = 2$, giving

$$(a, b) = (2, 3), (3, 2)$$

$$(m, n) = (30, 45), (45, 30)$$

   We see that there is a total of $\boxed{16}$ solutions. $\square$

6. An ant starts at the origin of the Cartesian coordinate plane. Each minute it moves randomly one unit in one of the directions up, down, left, or right, with all four directions being equally likely; its direction each minute is independent of its direction in any previous minutes. It stops when it reaches a point (x, y) such that $|x| + |y| = 3$. The expected number of moves it makes before stopping can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{3907}$.

   Solution. After one move the ant is at one of the four points $(1, 0), (0, 1), (-1, 0), (0, -1)$, and at each of the four points, there is $\frac{7}{16}$ probability of reaching a point where it stops *after two moves*, and for $\frac{9}{16}$ probability it goes to one of these four points again. This means that the expected number starting from any of these four points is $2 \cdot \frac{16}{7} = \frac{32}{7}$, so the original expectation is $\frac{32}{7} + 1 = \frac{39}{7}$, and the answer is $\boxed{3907}$. $\square$

7. At a concert 10 singers will perform. For each singer x, either there is a singer y such that x wishes to perform right after y, or x has no preferences at all. Suppose that there are n ways to order the singers such that no singer has an unsatisfied preference, and let p be the product of all possible nonzero values of n. Compute the largest nonnegative integer k such that 2^k divides p.

   Proposed by Gopal Goel.

   Answer. $\boxed{38}$.

   Solution. Suppose n is nonzero. Consider placing the singers into groups such that if for some two people x and y, x wishes to perform right after y, then x and y are in the same group. Note that each of these groups can be ordered in exactly one way, since every person, other than one person, is performing immediately after a fixed person in the group. (Each group can be ordered in at least one way, as $n \neq 0$). Then, if there are g groups, we can see that $n = g!$.

   Thus n is among $10!, 9!, \ldots, 1!$. It is not hard to show that all these can be achieved.

   Now, the product is
$$10!9! \cdots 2!1!.$$

   To compute the answer, Legendre's formula gives an answer of
$$\left(\left\lfloor\frac{10}{2}\right\rfloor + \left\lfloor\frac{10}{4}\right\rfloor \ldots\right) + \left(\left\lfloor\frac{9}{2}\right\rfloor + \left\lfloor\frac{9}{4}\right\rfloor \ldots\right) \ldots + \left(\left\lfloor\frac{1}{2}\right\rfloor + \left\lfloor\frac{1}{4}\right\rfloor \ldots\right)$$
$$= 8 + 7 + 7 + 4 + 4 + 3 + 3 + 1 + 1 + 0 = \boxed{38}.$$
   $\square$

8. There are three eight-digit positive integers which are equal to the sum of the eighth powers of their digits. Given that two of the numbers are 24678051 and 88593477, compute the third number.

   Proposed by Vincent Huang.

   Answer. $\boxed{24678050}$.

   Solution. Since 24678051 has this property, it follows trivially that $\boxed{24678050}$ does as well. $\square$

9. Convex equiangular hexagon $ABCDEF$ has $AB = CD = EF = 1$ and $BC = DE = FA = 4$. Congruent and pairwise externally tangent circles γ_1, γ_2, and γ_3 are drawn such that γ_1 is tangent to side $\overline{AB}$ and side $\overline{BC}$, γ_2 is tangent to side $\overline{CD}$ and side $\overline{DE}$, and γ_3 is tangent to side $\overline{EF}$ and side $\overline{FA}$. Then the area of γ_1 can be expressed as $\frac{m\pi}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

   Proposed by Sean Li.

   Answer. $\boxed{14800}$.

   Solution. Set the radius of γ_1 to be r. Let the center of γ_i be O_i, the foot of O_1 onto $\overline{BC}$ be X, the midpoint of $\overline{BC}$ be M, and the center of $ABCDEF$ be O.

   We now do some length chasing. Note $O_1X = r$ and, because $\triangle O_1XB$ is a $30 - 60 - 90$ triangle, $BX = r/\sqrt{3}$ and so $CX = 2 - r/\sqrt{3}$. Moreover, $\triangle O_1O_2O_3$ is equilateral with center O by symmetry, so $OO_1 = 2r/\sqrt{3}$. Finally, by extending sides BC, DE, FA to form an equilateral triangle, we have $OM = \sqrt{3}$.

   Then by the Pythagorean Theorem on right trapezoid $XMOO_1$, we have
$$\left(2 - \frac{r}{\sqrt{3}}\right)^2 + (\sqrt{3} - r)^2 = \left(\frac{2r}{\sqrt{3}}\right)^2,$$

   which leads to $r = \frac{7\sqrt{3}}{10}$. Thus, the area of γ_1 is $\pi r^2 = \frac{147\pi}{100}$, so the answer is $100 \cdot 147 + 100 = 14800$. $\square$

10. Let k be a positive integer. Marco and Vera play a game on an infinite grid of square cells. At the beginning, only one cell is black and the rest are white.

    A turn in this game consists of the following. Marco moves first, and for every move he must choose a cell which is black and which has more than two white neighbors. (Two cells are neighbors if they share an edge, so every cell has exactly four neighbors.) His move consists of making the chosen black cell white and turning all of its neighbors black if they are not already. Vera then performs the following action exactly k times: she chooses two cells that are neighbors to each other and swaps their colors (she is allowed to swap the colors of two white or of two black cells, though doing so has no effect). This, in totality, is a single turn. If Vera leaves the board so that Marco cannot choose a cell that is black and has more than two white neighbors, then Vera wins; otherwise, another turn occurs.

    Let m be the minimal k value such that Vera can guarantee that she wins no matter what Marco does. For $k = m$, let t be the smallest positive integer such that Vera can guarantee, no matter what Marco does, that she wins after at most t turns. Compute $100m + t$.

    Proposed by Ashwin Sah.

    Answer. $\boxed{103}$.

    Solution. We claim that the values of m and t are $m = 1$ and $t = 3$.

    We show that Vera can win after at most three turns if $k = 1$. Let B represent a black square, and let W represent a white square.

    Originally, the position is this.

    W W W W W

    W W W W W

    W W B W W

    W W W W W

    W W W W W

    Then Marco makes it this.

    W W W W W

    W W B W W

    W B W B W

    W W B W W

    W W W W W

    Then Vera swaps two white squares off in the distance so as to not move, and Marco makes it this or something symmetric.

    W W B W W

    W B W B W

    W B B B W

    W W B W W

    W W W W W

    Vera now makes it this.

    W W W W W

    W B B B W

    W B B B W

    W W B W W

    W W W W W

Marco is forced to make it this.

W W W W W

W B B B W

W B B B W

W B W B W

W W B W W

Vera then makes it this.

W W W W W

W B B B W

W B B B W

W B B B W

W W W W W

In this position, Vera wins. Thus Vera wins after at most three turns if $k = 1$. Thus $m = 1$ (since we cannot have $k < 1$), and $t \leq 3$.

Now, for $k = 1$, it needs to be shown that Vera cannot guarantee that she wins after at most two turns. This is left as an exercise to the reader.

Thus, $m = 1$ and $t = 3$, as claimed. Thus, the answer is $100 \cdot 1 + 3 = \boxed{103}$. □

11. Let ABC be a triangle with incenter I such that $AB = 20$ and $AC = 19$. Point $P \neq A$ lies on line AB and point $Q \neq A$ lies on line AC. Suppose that $IA = IP = IQ$ and that line PQ passes through the midpoint of side BC. Suppose that $BC = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Ankit Bisain.

Answer. $\boxed{3902}$.

Solution. Define I' to be the intersection of AI with the circumcircle of ABC. Then, by Simson lines, the foot of altitude I_B of I onto AB, the foot of altitude I_C of I onto AC, and the midpoint of BC are collinear.

Since $I_B I_C || PQ$ and they both go through the midpoint of BC, they must be the same line. Thus, the reflection of A over I is I'. Then $\frac{AI'}{2} = II' = BI' = CI'$ by the so-called "Fact 5." Now, as I' lies on arc BC not containing A, $AB \cdot CI' + AC \cdot BI' = BC \cdot AI'$ by Ptolemy, which yields that $AB + AC = 2BC$. Then $BC = \frac{39}{2}$, making the answer $\boxed{3902}$. □

12. Let $F(n)$ denote the smallest positive integer greater than n whose sum of digits is equal to the sum of the digits of n. For example, $F(2019) = 2028$. Compute $F(1) + F(2) + \cdots + F(1000)$.

Proposed by Sean Li.

Answer. $\boxed{535501}$.

Solution. Let $a(n)$ denote the least positive integer with digit sum n. Partition the positive integers into chains

$$\begin{gathered}
a(1) = 1 \mapsto F(1) \mapsto F(F(1)) \mapsto F(F(F(1))) \mapsto \dots \\
a(2) = 2 \mapsto F(2) \mapsto F(F(2)) \mapsto F(F(F(2))) \mapsto \dots \\
\vdots \\
a(10) = 19 \mapsto F(19) \mapsto F(F(19)) \mapsto F(F(F(19))) \mapsto \dots \\
\vdots
\end{gathered}$$

so a positive integer with digit sum n appears in the chain starting with $a(n)$. Then $F(\mathbb{N})$ sends each positive integer to the next number in its chain. Thus, if $X(n)$ is the least positive integer greater than 1000 with digit sum n, we have

$$(F(1)+F(2)+\cdots+F(1000))-(1+2+\cdots+1000)=\sum_{n=1}^{27}(X(n)-a(n)).$$

It is straightforward to show

$$a(n)=\overline{m\underbrace{9\dots9}_{k\text{ 9's}}}\quad\text{where } n=9k+m,\ 0\le m<9;\qquad X(n)=\begin{cases}10000 & \text{if } n=1,\\ 1000+a(n-1) & \text{if } 2\le n\le 27.\end{cases}$$

In particular, $X(n)-a(n-1)=1000$, so

$$\sum_{n=1}^{27}(X(n)-a(n))=X(1)-a(27)+\sum_{n=2}^{27}(X(n)-a(n-1))=10000-999+26\cdot 1000=35001,$$

and $F(1)+\cdots+F(1000)=35001+(1+2+\cdots+1000)=535501.$ $\square$

13. Compute the number of subsets S with at least two elements of $\{2^2,3^3,\dots,216^{216}\}$ such that the product of the elements of S has exactly 216 positive divisors.

Proposed by Sean Li.

Answer. $\boxed{8}$.

Solution. Let $X(S)$ denote the product of the elements in S whose prime factorization is $p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$ for $e_1\ge e_2\ge\cdots\ge e_k$. Then notice $p_i\mid e_i$ for all primes p_i, so $p_i\mid X(S)$ only if some divisor of $216=(e_1+1)\dots(e_k+1)$ is 1 (mod p_i). Moreover, a quick check shows that all numbers in S are prime powers, for $6^6\notin S$ and anything more than 10^{10} fails due to size reasons.

Now, we take the time to list all primes which divide one less than some divisor of 216:

$$\begin{array}{rrrrr} 2-1\mapsto\varnothing, & 3-1\mapsto\{2\}, & 4-1\mapsto\{3\}, & 6-1\mapsto\{5\}, & 8-1\mapsto\{7\},\\ 9-1\mapsto\{2\}, & 12-1\mapsto\{11\}, & 18-1\mapsto\{17\}, & 24-1\mapsto\{23\}, & 27-1\mapsto\{2,13\},\\ 36-1\mapsto\{5,7\}, & 54-1\mapsto\{53\}, & 72-1\mapsto\{71\}, & 108-1\mapsto\{107\}, & 216-1\mapsto\{5,43\}.\end{array}$$

Out of all powers of these primes, only $2^2,3^3,5^5,7^7,2^8=4^4,11^{11},17^{17},23^{23},2^{26}=2^2\cdot 8^8,53^{53},71^{71},107^{107}$ are expressible as a product of some prime powers of the form $(p^k)^{(p^k)}$ and could be a factor in the prime factorization of the product of the elements of S. So we are trying to select integers, at most one from the first set, to be our (e_i+1), id est they multiply to 216:

- $\{3,9,27\}$ (from $2^2,2^8,2^{26}$),
- $\{4,6,8,12,18,24,54,72,108\}$.

We perform casework on what we choose from the first set.

- If we pick nothing from the first set, then the pairs $(4,54),(12,18)$ multiply to 216. It is easy to check no triples will work, as $4\cdot 6\cdot 8<216$ and $4\cdot 6\cdot 12>216$, so we get two solutions here: $\{3^3,53^{53}\}$ and $\{11^{11},17^{17}\}$.
- If we pick 3 from the first set, then the number (72) and the pairs $(4,18)$ and $(6,12)$ both multiply to $216/3=72$, corresponding to sets $\{2^2,71^{71}\}$, $\{2^2,3^3,17^{17}\}$, and $\{2^2,5^5,11^{11}\}$.
- If we pick 9 from the first set, then the number (24) and the pair $(4,6)$ both multiply to $216/9=24$, corresponding to sets $\{4^4,23^{23}\}$ and $\{3^3,4^4,5^5\}$.
- If we pick 27 from the first set, then the number (8) works, corresponding to the set $\{2^2,7^7,8^8\}$.

Thus, we have a total of 8 sets. □

14. The sequence of nonnegative integers $F_0, F_1, F_2, \ldots$ is defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let d be the largest positive integer such that, for all integers $n \geq 0$, d divides $F_{n+2020} - F_n$. Compute the remainder when d is divided by 1001.

Proposed by Ankit Bisain.

Answer. $\boxed{638}$.

Solution. Extend the sequence to $F_{-1}, F_{-2}, \ldots$ such that $F_{n+2} = F_{n+1} + F_n$ for all integers n. Note that for all integers n, $F_{-n} = -(-1)^n F_n$.

Now, define the sequence $X_n = F_{n+2020} - F_n$. Since X_n satisfies the recurrence $X_n = X_{n-1} + X_{n-2}$, by the Euclidean Algorithm,

$$\gcd(X_n, X_{n-1}) = \gcd(X_{n-1} + X_{n-2}, X_{n-1}) = \gcd(X_{n-2}, X_{n-1})$$

so for all n,

$$d = \gcd(X_n, X_{n+1}).$$

Taking $n = -1010$,

$$d = \gcd(F_{1010} - F_{-1010}, F_{1011} - F_{1009}) = \gcd(2F_{1010}, F_{1010}) = F_{1010}.$$

To compute F_{1010} in modulo 1001, we must compute it in modulo 7, 11, and 13. We have

$$F_0, F_1, \ldots \equiv 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0$$

so it repeats every 16 in modulo 7. Thus,

$$F_{1010} \equiv F_{1010-63\cdot 16} \equiv F_2 \equiv 1 \pmod{7}.$$

Similarly,

$$F_0, F_1, \ldots = 0, 1, 1, 2, 3, 5, 8, 2, 10, 1, 0, 1, \ldots \pmod{11}$$

, so it repeats every 10 in $\pmod{11}$, giving

$$F_{1010} \equiv F_0 \equiv 0 \pmod{11}.$$

For $\pmod{13}$,

$$F_0, F_1, \ldots \equiv 0, 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1,$$
$$12, 0, 12, 12, 11, 10, 8, 5, 0, 5, 5, 10, 2, 12, 1, 0, 1, \ldots \pmod{13}$$

, meaning it repeats every 28 in $\pmod{13}$, giving

$$F_{1010} \equiv F_{1010-28\cdot 36} \equiv F_2 \equiv 1 \pmod{13}.$$

By the Chinese Remainder Theorem, we can compute the answer as 638. □

15. Let A,B,C, and D be points in the plane with $AB = AC = BC = BD = CD = 36$ and such that $A \neq D$. Point K lies on segment AC such that $AK = 2KC$. Point M lies on segment AB, and point N lies on line AC, such that D, M, and N are collinear. Let lines CM and BN intersect at P. Then the maximum possible length of segment KP can be expressed in the form $m + \sqrt{n}$ for positive integers m and n. Compute $100m + n$.

Proposed by James Lin.

Answer. $\boxed{1632}$.

Solution. We first prove that $\angle BPC = 60^\circ$. Note that $\triangle MBD \sim \triangle DCN$. Now $\frac{MB}{BC} = \frac{MB}{BD} = \frac{DC}{CN} = \frac{BC}{CN}$ and $\angle MBC = 60^\circ = \angle BCN$, so $\triangle MBC \sim \triangle BCN$. Then $\angle BPC = 180^\circ - \angle NBC - \angle BCM = 180^\circ - \angle CMB - \angle BCM = \angle CBM = 60^\circ$, as claimed.

Then, P lies on the circumcircle of ABC. Let O be the circumcenter of ABC. By the Triangle Inequality, $KP \leq KO + OP = 12 + 12\sqrt{3} = 12 + \sqrt{432}$, with equality when P is the intersection of ray $\overrightarrow{KO}$ with the circumcircle of ABC. (Using a phantom point argument, it is not hard to see that this position of P can be achieved.) The answer is then $1200 + 432 = \boxed{1632}$. $\square$

16. Let ABC be a scalene triangle with inradius 1 and exradii r_A, r_B, and r_C such that

$$20\left(r_B^2 r_C^2 + r_C^2 r_A^2 + r_A^2 r_B^2\right) = 19\left(r_A r_B r_C\right)^2.$$

If

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = 2.019,$$

then the area of $\triangle ABC$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Tristan Shin.

Answer. $\boxed{201925}$.

Solution. Let r be the inradius of ABC. Then

$$\sum \frac{r^2}{r_A^2} = \sum \left(\frac{s-a}{s}\right)^2,$$

where s is the semiperimeter of ABC. Let A_1, A_2 be on AB and AC, respectively, such that A_1A_2 is parallel to BC and $\triangle AA_1A_2$ has area equal to $\left(\frac{s-a}{s}\right)^2$ of that of $\triangle ABC$. Similarly define B_1, B_2 on BC and BA, C_1, C_2 on CA and CB, with $B_1B_2 \parallel CA$, $C_1C_2 \parallel AB$, and $\triangle BB_1B_2$ having area equal to $\left(\frac{s-b}{s}\right)^2$ of $\triangle ABC$ and $\triangle CC_1C_2$ having area equal to $\left(\frac{s-c}{s}\right)^2$ of $\triangle ABC$. Then $AA_1 = \frac{c(s-a)}{s}$, $BB_2 = \frac{c(s-b)}{s}$, so $A_1B_2 = \frac{c(s-c)}{s}$. Similarly, $B_1C_2 = \frac{a(s-a)}{s}$ and $C_1A_2 = \frac{b(s-b)}{s}$. But note that $C_1C_2 = \frac{c(s-c)}{s}$, $A_1A_2 = \frac{a(s-a)}{s}$, and $B_1B_2 = \frac{b(s-b)}{s}$, so the semiperimeter of hexagon $A_1A_2C_1C_2B_1B_2$ is $S = \sum a \cdot \frac{s-a}{s} = \sum a \cdot \frac{r}{r_A} = r\sum \frac{a}{r_A} = r\sum \frac{s-c}{r_A} + \frac{s-b}{r_A} = r\sum \cot\left(\frac{\pi}{2} - \frac{B}{2}\right) + \cot\left(\frac{\pi}{2} - \frac{C}{2}\right) = r\sum \tan\frac{B}{2} + \tan\frac{C}{2} = 2r\sum \tan\frac{A}{2}$.

Let ω be the incircle of $\triangle ABC$. Consider the A-exradius of $\triangle AA_1A_2$. It is $\frac{s-a}{s} \cdot r_A = \frac{r}{r_A} \cdot r_A = r$, so the A-excircle of $\triangle AA_1A_2$ is a circle tangent to AB and AC with radius r, so it is ω. Thus, segment A_1A_2 is tangent to ω. Similarly, segments B_1B_2 and C_1C_2 are also tangent to ω. Because of $\frac{c(s-a)}{s} < s - a$ and similar inequalities, it is clear that segments A_1B_2, B_1C_2, and C_1A_2 are all tangent to ω. Thus, hexagon $A_1A_2C_1C_2B_1B_2$ has an incircle, namely ω. Thus, its area is its inradius times its semiperimeter, or

$$2r^2 \sum \tan\frac{A}{2}.$$

However, we can also calculate its area by taking the area of $\triangle ABC$ and subtracting off the area of each of $\triangle AA_1A_2$, $\triangle BB_1B_2$, and $\triangle CC_1C_2$. Let K denote the area of $\triangle ABC$. Then this is

$$K\left(1 - \sum\left(\frac{s-a}{s}\right)^2\right).$$

Putting these results together, we get that

$$K = \frac{2r^2 \sum \tan\frac{A}{2}}{1 - \sum\left(\frac{s-a}{s}\right)^2} = \frac{4.038}{1 - \frac{19}{20}} = \frac{2019}{25},$$

so $m = 2019$ and $n = 25$, and $100m + n = 201925$.

Remark: Here is a sketch of another solution. Note that $\frac{1-\frac{19}{20}}{2} = \frac{(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)}{s^2}$ and $2.019 = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)}{(s-a)(s-b)(s-c)} = \frac{(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)}{s}$. Then $s = \frac{\frac{(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)}{s}}{\frac{(s-a)(s-b)+(s-a)(s-c)+(s-b)(s-c)}{s^2}} = \frac{\frac{2019}{1000}}{\frac{1}{40}} = \frac{2019}{25}$. Then $[ABC] = rs = s$, so the answer is $\boxed{201925}$. □

17. For an ordered pair (m, n) of distinct positive integers, suppose, for some nonempty subset S of $\mathbb{R}$, that a function $f : S \to S$ satisfies the property that $f^m(x) + f^n(y) = x + y$ for all $x, y \in S$. (Here $f^k(z)$ means the result when f is applied k times to z; for example, $f^1(z) = f(z)$ and $f^3(z) = f(f(f(z)))$.) Then f is called (m, n)-*splendid*. Furthermore, f is called (m, n)-*primitive* if f is (m, n)-splendid and there do not exist positive integers $a \le m$ and $b \le n$ with $(a, b) \ne (m, n)$ and $a \ne b$ such that f is also (a, b)-splendid. Compute the number of ordered pairs (m, n) of distinct positive integers less than 10000 such that there exists a nonempty subset S of $\mathbb{R}$ such that there exists an (m, n)-primitive function $f : S \to S$.

Proposed by Vincent Huang.

Answer. $\boxed{9998}$.

Solution. Note that the given equation rewrites as $f^m(x) - x = y - f^n(y) - y$. Fixing y and varying x yields $f^m(x) - x = c$ for a constant c, and fixing x while varying y yields $f^n(y) - y = -c$ for a constant c. Combining these equations yields $f^{m+n}(x) = x$ for all x. But now note $0 = f^{m(m+n)}(x) - x = c(m+n)$, hence $c = 0$, so $f^m(x) = f^n(x) = x$. Thus $f^{\gcd(m,n)}(x) = x$ as well, so for f to be primitive it follows that $\{m, n\} = \{\gcd(m, n), 2\gcd(m, n)\}$, from which we get $(m, n) = (d, 2d), (2d, d)$, and now the answer is obviously $\boxed{9998}$. □

18. Define a *modern artwork* to be a nonempty finite set of rectangles in the Cartesian coordinate plane with positive areas, pairwise disjoint interiors, and sides parallel to the coordinate axes. For a modern artwork S, define its *price* to be the minimum number of colors with which Sean could paint the interiors of rectangles in S such that every rectangle's interior is painted in exactly one color and every two distinct touching rectangles have distinct colors, where two rectangles are *touching* if they share infinitely many points. For a positive integer n, let $g(n)$ denote the maximum price of any modern artwork with exactly n rectangles. Compute $g(1) + g(2) + \cdots + g(2019)$.

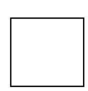
Proposed by Yang Liu and Edward Wan.

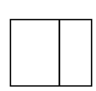
Answer. $\boxed{8068}$.

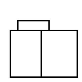
Solution. Define $g'(n)$ as $g'(1) = 1, g'(2) = 2, g'(3) = 3, g'(4) = 3, g'(5) = 3$, and $g'(n) = 4$ for all positive integers $n \ge 6$. We claim $g(n) = g'(n)$ for all positive integers n.

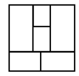
We first show $g(n) \ge g'(n)$ for all positive integers n.

The diagrams below are examples that can be used to prove that $g(1) \ge 1, g(2) \ge 2, g(3) \ge 3$, and $g(6) \ge 4$. (The details are left as an exercise to the reader.)









Also, it can be seen that $g(n+1) \geq g(n)$ for all positive integers n (simply add a rectangle to an n-rectangle example that required $g(n)$ colors).

Combining these facts yields that $g(n) \geq g'(n)$ for all positive integers n.

We now show $g(n) \leq g'(n)$ for all positive integers n.

It is obvious that $g(i) \leq i$ for $i = 1, 2, 3$.

The following lemma is useful.

Lemma. There cannot be four rectangles in any modern artwork so that each pair of them touch each other.

Proof. Let R_1, R_2, R_3, and R_4 be four arbitrary rectangles in a modern artwork. Assume, for contradiction, that each pair of them is touching. If all three of R_2, R_3, R_4 touch the same edge of R_1, it's clear that some two of them do not touch, contradiction. If two of R_2, R_3, R_4 touch opposite edges of R_1, it's clear that they do not touch, contradiction. Hence, by symmetry, it suffices only to analyze the case where R_2, R_3 touch the topmost edge of R_1 and R_4 touches the rightmost edge of R_1. WLOG assume that the center of R_2 is to the left of the center of R_3. Then it's clear that R_2 does not touch R_4, contradiction. We've obtained a contradiction in all cases, and so hence the lemma is proven. ■

By the Lemma, we know that $g(4) \leq 3$, because in any modern artwork of four rectangles, we can simply color a non-touching pair of rectangles red, and the other two rectangles blue and green.

We will now use the Lemma to show that $g(5) \leq 3$. Because of the Lemma, it is sufficient to show that any subgraph of a K_5 which doesn't contain a K_4 is 3-colorable; we shall show this. To do so, let's consider the complement of the graph. If there are two disjoint edges ab, cd in the complement graph, then in the original graph we can color a, b red, c, d green, and the other vertex blue. Otherwise, any two edges share a common vertex, so either they form a triangle or all of them share a vertex. In the former case, color the three vertices of the triangle all red and the other two vertices blue and green. The latter case actually never occurs, because then the original graph has a K_4. As a result of the previous discussion, we've shown that $g(5) \leq 3$.

Notice that $g(n) \leq 4$ for all positive integers n, by the Four Color Theorem.

Thus we have that $g(n) \leq g'(n)$ for all positive integers n.

Thus $g(n) = g'(n)$ for all positive integers n.

Summing from 1 to 2019 gives us a final answer of $1 + 2 + 3 + 3 + 3 + 4 \cdot 2014 = 8068$. □

19. Let ABC be an acute triangle with circumcenter O and orthocenter H. Let E be the intersection of BH and AC and let M and N be the midpoints of HB and HO, respectively. Let I be the incenter of AEM and J be the intersection of ME and AI. If $AO = 20$, $AN = 17$, and $\angle ANM = 90°$, then $\frac{AI}{AJ} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

    Proposed by Tristan Shin.

    Answer. $\boxed{1727}$.

    Solution. Since $\angle AEB = \frac{\pi}{2}$, we have that $AENM$ is cyclic.

    Let ω be the nine point circle of ABC. Note that N is the center of ω and M and E are on ω. Now, the radius of ω is $\frac{R}{2}$, where R is the circumradius of ABC. Now, note that, considering concyclic points A, E, N, M, we have $MN = NE = \frac{R}{2} = 10$ and $AN = 17$; then $AN > MN = NE$, so it can be shown that this yields that A lies on arc ME not containing N on $(AENM)$. Then, applying Ptolemy's Theorem to cyclic quadrilateral $AENM$, $17EM = AN \cdot EM = AE \cdot NM + AM \cdot NE = 10(AE + AM)$.

By the angle bisector theorem on AME, $\frac{MJ}{JE} = \frac{MA}{AE}$, so $MJ = \frac{AM \cdot ME}{AM+AE}$. By the angle bisector theorem on MAJ, $\frac{AI}{IJ} = \frac{AM}{MJ} = \frac{AM+AE}{ME} = \frac{17}{10}$. Thus,

$$\frac{AI}{AJ} = \frac{AM + AE}{AM + AE + ME} = \frac{17}{27},$$

so $100m + n = 1727$.

$\square$

20. Define a *crossword puzzle* to be a 15×15 grid of squares, each of which is either black or white. In a crossword puzzle, define a *word* to be a sequence of one or more consecutive white squares in a row or column such that the squares immediately before and after the sequence both are either black or nonexistent. (The latter case would occur if an end of a word coincides with an end of a row or column of the grid.) A crossword puzzle is *tasty* if every word consists of an even number of white squares. Compute the sum of all nonnegative integers n such that there exists a tasty crossword puzzle with exactly n white squares.

    Proposed by Luke Robitaille.

    Answer. $\boxed{4900}$.

    Solution. I claim the possible values of n are $0, 4, 8, \ldots, 196$.

    Note that if we ignore the rightmost column and the bottom row of the grid, then the rest of the grid can be partitioned into forty-nine 2×2 squares. For any $k = 0, 1, \ldots, 49$, choose any k of those 2×2 squares, and color all the cells in those squares white, and color the cells in the other $49 - k$ of those squares, as well as the cells in the rightmost column and the bottom row of the grid, black. The result is a crossword puzzle with exactly $4k$ white cells. It is not hard to see that this crossword puzzle is tasty. Thus $4k$ is a possible value of n, so $0, 4, 8, \ldots, 196$ all are possible values of n.

    Now the other direction: suppose there is a tasty crossword puzzle with exactly n white squares. Label the rows 1 to 15 in order from top to bottom; label the columns 1 to 15 in order from left to right. Place the letter a in every square at the intersection of an odd-numbered row and an odd-numbered column; place the letter b in every square at the intersection of an odd-numbered row and an even-numbered column; place the letter c in every square at the intersection of an even-numbered row and an odd-numbered column; place the letter d in every square at the intersection of an even-numbered row and an even-numbered column. Let A be the number of white squares that contain the letter a; define B, C, and D analogously. Consider all horizontal words (words that are a subset of some row) in the grid; every such word consists of an even number of white squares (since the crossword puzzle is tasty), and every white square in the grid is in exactly one such word. Considering such words in odd-numbered rows, we get $A = B$. A similar consideration of even-numbered rows yields that $C = D$. Similar consideration of vertical words yields that $A = C$ and $B = D$. Thus $A = B = C = D$. Then $n = A+B+C+D = 4D$. Now D is a nonnegative integer; furthermore, D is at most the total number of cells containing d's in the grid, which is 49. Thus n is among $0, 4, 8, \ldots, 196$, as desired.

    Thus the possible values of n are $0, 4, 8, \ldots, 196$, as claimed. Thus the answer is $0+4+8+\cdots+196 = 4(1+2+\cdots+49) = 4 \cdot \frac{49 \cdot 50}{2} = \boxed{4900}$. $\square$

21. Let p and q be prime numbers such that $(p-1)^{q-1} - 1$ is a positive integer that divides $(2q)^{2p} - 1$. Compute the sum of all possible values of pq.

    Proposed by Ankit Bisain.

    Answer. $\boxed{85}$.

    Solution. First, note that $p > 2$ is obvious.

    We claim $q \mid p - 1$. If it doesn't, $q \mid (p-1)^{q-1} - 1$ by Fermat's Little Theorem, so $q \mid (2q)^{2p} - 1$, meaning $q \mid -1$, contradiction. Now, we take cases on q.

Case 1: $q = 2$ The condition becomes $p-2 \mid 4^{2p}-1 = 16^p - 1$. Note that $15|16^p-1$. For any prime k dividing $p-2$ (which is odd since $p > 2$ is prime), $k \mid 16^p-1$, but since the order of 16 $\pmod{k}$ divides $k-1$ and $k-1 \le p-3 < p$, the order must be 1, so $k \mid 15$. If k divides $\frac{16^p-1}{15} = 16^{p-1}+\cdots+16+1$, then $0 \equiv 16^{p-1}+\cdots+16+1 \equiv 1+\cdots+1+1 = p \equiv 2 \pmod{k}$ (as $16 \equiv 1 \pmod{k}$ and $p \equiv 2 \pmod{k}$), contradicting $k > 2$. Thus, $k \nmid \frac{16^p-1}{15}$. Then $\gcd(p-2, \frac{16^p-1}{15} = 1$, so since $p-2|16^p-1 = \frac{16^p-1}{15}\cdot 15$, we have, $p-2 \mid 15$, so $p \in \{3,5,7,17\}$. All these can be seen to work.

Case 2: $q > 2$ Since $q-1$ is even, we get $p \mid (p-1)^{q-1}-1$, so $p \mid (2q-1)(2q+1)$. Since $q \mid p-1$, $p > q+1$, so $2p > 2q+2 > 2q+1 > 2q-1$, and $p \mid 2q-1$ or $p \mid 2q+1$, so $p \in \{2q-1, 2q+1\}$. Since $q > 2$ and $q \mid p-1$, this forces $p = 2q+1$. Then, the condition becomes

$$(2q)^{q-1}-1 \mid (2q)^{2p}-1 \Rightarrow q-1 \mid 2p \Rightarrow q-1 \mid 2(2q+1) = 4(q-1)+6,$$

so $q-1 \mid 6$, so $q \in \{3,7\}$. Since $p = 2q+1$ is also prime, the only solution from this case is $(p,q) = (7,3)$. This can be seen to work.

For the answer extraction, the sum of possible values of pq is $2\cdot 3+2\cdot 5+2\cdot 7+2\cdot 17+3\cdot 7 = \boxed{85}$. □

22. For finite sets A and B, call a function $f : A \to B$ an *antibijection* if there does not exist a set $S \subseteq A \cap B$ such that S has at least two elements and, for all $s \in S$, there exists exactly one element s' of S such that $f(s') = s$. Let N be the number of antibijections from $\{1,2,3,\dots 2018\}$ to $\{1,2,3,\dots 2019\}$. Suppose N is written as the product of a collection of (not necessarily distinct) prime numbers. Compute the sum of the members of this collection. (For example, if it were true that $N = 12 = 2\times 2\times 3$, then the answer would be $2+2+3 = 7$.)

Proposed by Ankit Bisain.

Answer. $\boxed{1363001}$.

Solution. Note that for all x, if $x, f(x), f^2(x), \dots$ eventually cycles with period > 1, then taking S to be the numbers in this cycle gives a contradiction. However, since $\{1,2,\dots 2018\}$ and $\{1,2,\dots 2019\}$ are finite, that sequence must eventually cycle or terminate at 2019. Thus, that sequence eventually is constant or terminates.

Now, let $g(x)$ be the constant value that $x, f(x), f^2(x), \dots$ eventually becomes (let $g(x) = 2019$ if that sequence ever reaches 2019, when it would terminate). Denote $X = \{g(1), g(2), \dots g(2018)\}$. If $|X \cap \{1,2,3,\dots 2018\}| \ge 2$, then taking $S = |X \cap \{1,2,3,\dots 2018\}|$ gives a contradiction. Thus, $|X \cap \{1,2,3,\dots 2018\}| \in \{0,1\}$. Also, X is nonempty.

Draw a directed graph with vertices labelled 1, 2, $\dots$ 2019, such that there is an edge from x to $f(x)$ for all $x \in \{1,2,3,\dots 2018\}$.

Case 1: $X = \{2019\}$. In this case, if the directions of the graph are removed, there is a path from x to 2019 for all $x \in \{1,2,3,\dots 2019\}$, and there are no cycles, so the graph must be a tree. Also, every tree can be converted to an antibijection by directing each edge 'towards' (on the unique path to) 2019. By Cayley's formula, this gives 2019^{2017} functions.

Case 2: $X = \{c\}$ or $\{c, 2019\}$ for some $c \in \{1,2,\dots 2018\}$. We claim that this can be bijected to the set of ordered pairs (tree on vertices $\{1,2,\dots,2019\}$, choice of neighbor of 2019). We can create a function from each of these by drawing the tree, deleting the edge between 2019 and c, and creating functions from the two resulting trees as done in Case 1 (taking c as the 'final value' of the tree with c in it). Similarly, we can create the trees from a function by drawing the graph then connecting c to 2019, so this is a bijection. To count these, note that each tree generates deg(2019) functions. Counting this over all labellings of a fixed tree shows that this is $\frac{2\cdot 2018}{2019}$ on average (there are 2018 edges, each is counted twice), giving $\frac{2\cdot 2018}{2019}\cdot 2019^{2017} = 2\cdot 2018\cdot 2019^{2016}$ functions by Cayley's.

Thus $N = 2019^{2017} + 2\cdot 2018\cdot 2019^{2016} = 3^{2016}\cdot 673^{2016}\cdot 5\cdot 7\cdot 173$, giving an answer of $2016\cdot 3 + 2016\cdot 673 + 5 + 7 + 173 = \boxed{1363001}$. □

23. Let v and w be real numbers such that, for all real numbers a and b, the inequality

$$(2^{a+b}+8)(3^a+3^b) \le v(12^{a-1}+12^{b-1}-2^{a+b-1})+w$$

holds. Compute the smallest possible value of $128v^2+w^2$.

Proposed by Luke Robitaille.

Answer. $\boxed{62208}$.

Solution. Taking $a=b=2$ gives $432 \le 16v+w$, so $62208 = \frac{432^2}{3} \le \frac{(16v+w)^2}{(\sqrt{2})^2+1^2} \le (8\sqrt{2}v)^2+w^2 = 128v^2+w^2$ by Cauchy-Schwarz.

We claim that 62208 can be achieved. We claim that a construction is $v=18$ and $w=144$. Note that $128\cdot 18^2+144^2=62208$. We now show that this works. The reason this works is because $(2^{a+b}+8)(3^a+3^b) \le 18(12^{a-1}+12^{b-1}-2^{a+b-1})+144$ is equivalent, after some algebra, to $(2^{a+b+1}+16)(3^a+3^b+9) \le 3(12^a+12^b+144)$. Now $2^{a+b+1} \le 4^a+4^b$ by AM-GM, so it suffices to prove $(4^a+4^b+4^2)(3^a+3^b+3^2) \le 3(12^a+12^b+12^2)$, which is true by Chebyshev's Inequality.

Thus we are done; the answer is $\boxed{62208}$. □

24. Let ABC be an acute scalene triangle with orthocenter H and circumcenter O. Let the line through A tangent to the circumcircle of triangle AHO intersect the circumcircle of triangle ABC at A and $P \neq A$. Let the circumcircles of triangles AOP and BHP intersect at P and $Q \neq P$. Let line PQ intersect segment BO at X. Suppose that $BX=2$, $OX=1$, and $BC=5$. Then $AB\cdot AC = \sqrt{k}+m\sqrt{n}$ for positive integers k, m, and n, where neither k nor n is divisible by the square of any integer greater than 1. Compute $100k+10m+n$.

Proposed by Luke Robitaille.

Answer. $\boxed{29941}$.

Solution. Denote by $\measuredangle$ directed angles modulo π. Lemma 1: H,Q,O,X are concyclic. Proof: $\measuredangle HQX = \measuredangle HQP = \measuredangle HBP = \measuredangle BHP + \measuredangle HPB = \measuredangle BHA + \measuredangle AHP + \measuredangle HPA + \measuredangle APB = \measuredangle HAP + \measuredangle BHA + \measuredangle APB = \measuredangle HAP + \measuredangle BHA + \measuredangle ACB = \measuredangle HAP + 2\measuredangle ACB = \measuredangle HAP + \measuredangle AOB = \measuredangle HOA + \measuredangle AOB = \measuredangle HOB = \measuredangle HOX$, so H,Q,O,X are concyclic, as desired.

Lemma 2: A,H,X are collinear. Proof: $\measuredangle XHO = \measuredangle XQO = \measuredangle PQO = \measuredangle PAO = \measuredangle AHO$, so A,H,X are collinear, as desired.

The rest is easy. Let $AH\cap BC = F$ and let M be the midpoint of BC. Then $BM=\frac{5}{2}$. Furthermore, F lies on segment BM with $\frac{BF}{FM}=\frac{BX}{XO}=2$. Thus $FM=\frac{5}{6}$. Now let Y be the foot of the perpendicular from O to AF. Then $YF = OM = \sqrt{BO^2-BM^2} = \sqrt{3^2-(\frac{5}{2})^2} = \frac{\sqrt{11}}{2}$ and $AY = \sqrt{AO^2-OY^2} = \sqrt{AO^2-FM^2} = \sqrt{3^2-(\frac{5}{6})^2} = \frac{\sqrt{299}}{6}$; then $AF = AY+YF = \frac{\sqrt{299}+3\sqrt{11}}{6}$. Now let A' be the reflection of A over O. Then $AA'=6$. Also, triangle ABF is similar to triangle $AA'C$. Thus, $AB\cdot AC = AF\cdot AA' = \sqrt{299}+3\sqrt{11}$. Thus the answer is $29900+30+11=\boxed{29941}$. □

25. The sequence $f_0, f_1, \dots$ of polynomials in $\mathbb{F}_{11}[x]$ is defined by $f_0(x)=x$ and $f_{n+1}(x)=f_n(x)^{11}-f_n(x)$ for all $n\ge 0$. Compute the remainder when the number of nonconstant monic irreducible divisors of $f_{1000}(x)$ is divided by 1000.

Proposed by Ankan Bhattacharya.

Answer. $\boxed{301}$.

Solution. Throughout this solution, "irreducible" will always refer to non-unit polynomials.

It's not hard to show that $f_n(x) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{11^{n-k}}$ for all $n \geq 0$. In particular, $f_{11^n}(x) = x^{11^{11^n}} - x$ for all x, and so

$$x^{11^{11^2}} - x = f_{121} \mid f_{1000} \mid f_{1331} = x^{11^{11^3}} - x.$$

It is well-known that the monic irreducible factors of $x^{11^n} - x$ are exactly the monic irreducible polynomials of degree dividing n, each appearing exactly once in a factorization of $x^{11^n} - x$ into monic irreducible factors. Thus, all monic irreducible divisors of f_{1000}/f_{121} are of degree 11^3. It follows from this and analogous arguments regarding f_{121}, f_{11}, f_1 that the number of monic irreducible factors of f_{1000} equals

$$\begin{aligned} &11^1 + \frac{11^{11} - 11^1}{11^1} + \frac{11^{121} - 11^{11}}{11^2} + \frac{11^{1000} - 11^{121}}{11^3} \\ &= 11^{997} + 10 \cdot 11^{118} + 10 \cdot 11^9 + 10 \\ &\equiv \boxed{301} \pmod{1000}. \end{aligned}$$

□

26. Let $p = 491$ be prime. Let S be the set of ordered k-tuples of nonnegative integers that are less than p. We say that a function $f\colon S \to S$ is *k-murine* if, for all $u, v \in S$, $\langle f(u), f(v)\rangle \equiv \langle u, v\rangle \pmod{p}$, where $\langle (a_1, \ldots, a_k), (b_1, \ldots, b_k)\rangle = a_1b_1 + \cdots + a_kb_k$ for any $(a_1, \ldots a_k), (b_1, \ldots b_k) \in S$.

    Let $m(k)$ be the number of k-murine functions. Compute the remainder when $m(1) + m(2) + m(3) + \cdots + m(p)$ is divided by 488.

    Proposed by Brandon Wang.

    Answer. $\boxed{18}$.

    Solution. Let $\vec{e}_1, \ldots, \vec{e}_k$ be the standard basis. We see that $f(\vec{e}_1) = \vec{u}_1, \ldots, f(\vec{e}_k) = \vec{u}_k$ are an orthonormal basis. Furthermore, for all $\vec{v}, i$,

$$\langle f(\vec{v}), \vec{u}_i\rangle = \langle \vec{v}, \vec{e}_i\rangle,$$

so if $\vec{v} = \sum a_i\vec{e}_i$, $f(\vec{v}) = \sum a_i\vec{u}_i$, thus f is the linear map taking $\vec{e}_i$ to $\vec{u}_i$ for all i. Thus, $m(k)$ counts the number of orthonormal bases $(\vec{u}_1, \ldots, \vec{u}_k)$.

First, we need a lemma: Let $\vec{v}_1, \ldots, \vec{v}_\ell$ be orthonormal. Then, we can extend this to an orthonormal basis $\vec{v}_1, \ldots, \vec{v}_k$.

Proof: Let $V = (\mathbb{F}_p)^k$ and let $W = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_\ell)$. Let $U = W^\perp$ be the orthogonal complement of W. We will show that there exists $\vec{u} \in U$, with $\|\vec{u}\|^2 = 1$.

Let $t = k - \ell = \dim U$. If $t = 1$, then let $U = \operatorname{span}(\vec{u})$. Say M is the matrix with columns $\vec{v}_1, \ldots, \vec{v}_{k-1}, \vec{u}$, then $M^{\mathrm{T}}M$ is a diagonal matrix with diagonal entries $1, 1, \ldots, 1, \|\vec{u}\|^2$, so $\left\|\frac{\vec{u}}{\det M}\right\|^2 = 1$ and so $\vec{v}_k = \frac{\vec{u}}{\det M}$ works.

Else, first take an orthogonal basis of U as follows: Take $\vec{u}_1$ with $||\vec{u}_1||^2 \neq 0$, and replace U with $U \cap \operatorname{span}(\vec{u}_1)^\perp$, and repeat on the new U to get $\vec{u}_2, \ldots, u_t$. Note in general that after we have generated $\vec{u}_1, \ldots, \vec{u}_s$, then $U^\perp$ is currently $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_\ell, \vec{u}_1, \ldots, \vec{u}_s)$. Clearly the span is contained in $U^\perp$. To show that $U^\perp$ does not intersect U, note that if $\vec{x} \in U^\perp \cap U$, then $\langle \vec{x}, \vec{v}\rangle = 0$ for all $\vec{v} \in V$, thus forcing $\vec{x} = 0$.

Now, we claim that U must contain an element with nonzero norm. Otherwise, for all $\vec{u}, \vec{u}' \in U$,

$$\langle \vec{u}, \vec{u}'\rangle = \frac{\langle \vec{u} + \vec{u}', \vec{u} + \vec{u}'\rangle - \langle \vec{u}, \vec{u}\rangle - \langle \vec{u}', \vec{u}'\rangle}{2} = 0,$$

so $U \in U^\perp$, contradiction.

Thus we can find an orthonormal basis. Now suppose $\|\vec{u}_1\|^2 = a_1, \|\vec{u}_2\|^2 = a_2$. Suppose that a_1, a_2 are not quadratic residues (else we can normalize one of the vectors and decrease t).

Note that $\|c\vec{u}_1 + d\vec{u}_2\|^2 = c^2a_1 + d^2a_2$. Note that as c^2 varies, c^2a_1 varies over all $\frac{p+1}{2}$ nonquadratic residues and 0. Similarly so varies d^2. So, by Cauchy-Davenport, there exists c, d so that $c^2a_1+d^2a_2 = 1$, and setting $\vec{u} = c\vec{u}_1 + d\vec{u}_2$ decreases t.

Thus we can always add a vector to our orthonormal set and so we can eventually get a basis. ■

Now, let $\chi(k)$ denote the number of vectors $\vec{v}$ in $(\mathbb{F}_p)^k$ with $\|\vec{v}\|^2 = 1$. We claim that $m(k) = \chi(1)\cdots\chi(k)$. Indeed, we claim that if we pick $\vec{u}_k, \dots, \vec{u}_1$ in that order, then there are $\chi(i)$ choices for u_i. For $i = k$ this is clear. For $i < k$, suppose we have picked $\vec{u}_k, \dots, \vec{u}_{i+1}$. Then, extend this to an orthonormal basis $\vec{u}_k, \dots, \vec{u}_{i+1}, \vec{v}_i, \dots, \vec{v}_1$.

Let T be the transformation taking v_j to e_j for $j \le i$ and u_j to e_j otherwise. Since T and T^{-1} both preserve dot product, it is clear that picking u_i is equivalent to picking e_i so that $\|e_i\|^2 = 1$ and e_i orthogonal to $e_{i+1}, \dots, e_n$, so we clearly have $\chi(i)$ choices for u_i.

Thus it suffices to compute $\chi(k)$ for all k. Note that $k! \mid \chi(k)$, so it suffices to compute $\chi(k)$ for $k \le 60$. Let A_k be the number of vectors $\vec{v}$ so that $\|\vec{v}\|^2 = 0$, $B_k = \chi(k)$, and C_k the number so that $\|\vec{v}\|^2 = r$, where r is a fixed nonresidue. We see that

$$A_k + \frac{p-1}{2}B_k + \frac{p-1}{2}C_k = p^k.$$

Let $D_k = \frac{p-1}{2}B_k$ and $E_k = \frac{p-1}{2}C_k$.

Next, note that if $a_1^2 + \cdots + a_{k+1}^2 \equiv 0$, then either $a_{k+1} = 0$, or $a_1^2 + \cdots + a_k^2$ is a nonresidue with two choices for a_{k+1} since $p \equiv 3 \pmod 4$, so

$$A_{k+1} = A_k + 2E_k.$$

Next, we compute D_{k+1}. If $a_1^2 + \cdots + a_k^2 = 0$, then there are $p-1$ choices for a_{k+1}. Now we do casework on $q = a_1^2 + \cdots + a_k^2$. We compute the number of a so that $q + a^2$ is a square, or $q + a^2 = b^2$ or $q = (b-a)(b+a)$. So, if $b + a = s, b - a = \frac{q}{s}$, $a = s - \frac{q}{s}$. Under $s \to -\frac{q}{s}$, a is fixed.

Thus, if q is a square, then we have $\frac{p-1}{2}$ solutions for a. Similarly, if q is a nonsquare, we have two fixed points. At these fixed points, we see that $s + \frac{q}{s} = 0$, so $b = 0$. Thus we only want non-fixed point a's, and since there are two s's that are fixed, there are $p-3$ non-fixed s's, so we have that there are $\frac{p-3}{2}$ solutions to a where $b \neq 0$. Note that if q is a square, then b is never zero.

So, $D_{k+1} = (p-1)A_k + \frac{p-1}{2}D_k + \frac{p-3}{2}E_k$, and since $A_{k+1} + D_{k+1} + E_{k+1} = p(A_k + D_k + E_k)$, we have $E_{k+1} = \frac{p+1}{2}D_k + \frac{p-1}{2}E_k$.

Scaling back, we have

$$\begin{aligned} A_{k+1} &= A_k + (p-1)C_k, \\ B_{k+1} &= 2A_k + \frac{p-1}{2}B_k + \frac{p-3}{2}C_k, \\ C_{k+1} &= \frac{p+1}{2}B_k + \frac{p-1}{2}C_k. \end{aligned}$$

Taking $p - 3$ and recalling that B_k, C_k are both even, we get

$$A_{k+1} = A_k + 2C_k, B_{k+1} = 2A_k + B_k, C_{k+1} = 2B_k + C_k.$$

Solving the recursion, we see that $A_k = \sum 2^{3m}\binom{k}{3m}$, $B_k = \sum 2^{3m+1}\binom{k}{3m+1}$, $C_k = \sum 2^{3m+2}\binom{k}{3m+2}$. In particular, after a roots of unity filter, we get

$$3B_k = 3^k + (1+2\omega)^k \cdot \omega^2 + (1-2\omega^2)^k \cdot \omega.$$

Here, note that $1 + 2\omega = -i\sqrt{3}$. So, $3B_k = 3^k + (i\sqrt{3})^k(\omega^2 + (-1)^k \cdot \omega)$.

Now we see that

$$B_k = \begin{cases} 3^{4t} + 3^{2t} & k = 4t+1 \\ 3^{4t+1} + 3^{2t} & k = 4t+2 \\ 3^{4t+2} - 3^{2t+1} & k = 4t+3 \\ 3^{4t+3} - 3^{2t+1} & k = 4t+4 \end{cases}.$$

Modulo 8, we have $B_1 = 2, B_2 = 4$, so the sum is 2. Modulo 61, we have $B_1 = 2, B_2 = 4, B_3 = 6, B_4 = 24, B_5 = 29, B_6 = 8, B_7 = 31, B_8 = 25, B_9 = 54$, and $B_{10} = 0$, so we see that the sum is $18 \pmod{61}$ and is thus $\boxed{18} \pmod{488}$.

Remark: This problem asks to count the size of the orthogonal group: `https://en.wikipedia.org/wiki/Orthogonal_group` . A general formula is given at the linked article, which can be proven by computing the B_i's explicitly. □

27. A *complex set*, along with its *complexity*, is defined recursively as the following:

- The set $\mathbb{C}$ of complex numbers is a complex set with complexity 1.
- Given two complex sets C_1, C_2 with complexity c_1, c_2 respectively, the set of all functions $f : C_1 \to C_2$ is a complex set denoted $[C_1, C_2]$ with complexity $c_1 + c_2$.

A *complex expression*, along with its *evaluation* and its *complexity*, is defined recursively as the following:

- A single complex set C with complexity c is a complex expression with complexity c that evaluates to itself.
- Given two complex expressions E_1, E_2 with complexity e_1, e_2 that evaluate to C_1 and C_2 respectively, if $C_1 = [C_2, C]$ for some complex set C, then (E_1, E_2) is a complex expression with complexity $e_1 + e_2$ that evaluates to C.

For a positive integer n, let a_n be the number of complex expressions with complexity n that evaluate to $\mathbb{C}$. Let x be a positive real number. Suppose that

$$a_1 + a_2 x + a_3 x^2 + \cdots = \frac{7}{4}.$$

Then $x = \frac{k\sqrt{m}}{n}$, where k,m, and n are positive integers such that m is not divisible by the square of any integer greater than 1, and k and n are relatively prime. Compute $100k + 10m + n$.

Proposed by Luke Robitaille and Yannick Yao.

Answer. $\boxed{1359}$.

Solution. Lemma 1. If a complex expression E has complexity e and evaluates to a complex set C that has complexity c, then $e \geq c$ and $e \equiv c \pmod 2$; furthermore, if $e = c$, then $E = C$.

Proof. This is not hard and is left to the reader as an exercise. ■

Let $C_k = \frac{\binom{2k}{k}}{k+1}$ be the kth Catalan number for $k \geq 0$.

Lemma 2. For all positive integers i, there are exactly C_{i-1} complex sets of complexity i.

Proof. This follows inductively without difficulty from the well-known fact that $\sum_{k=0}^{n} C_k C_{n-k} = C_{n+1}$ for all nonnegative integers n. ■

Now define a function f from nonnegative integers to integers recursively by $f(0) = 1$ and, for $x > 0$, $f(x) = \sum C_a f(b) f(c)$, where the sum is over all ordered triples (a, b, c) of nonnegative integers such that $a + b + c = x - 1$.

Lemma 3. For any nonnegative integer n, the following is true: for any positive integer k and for any complex set C of complexity k, the number of complex expressions of complexity $k + 2n$ that evaluate to C is $f(n)$.

Proof. Proceed by strong induction on n.

For $n = 0$, any complex expression of complexity k that evaluates to C must be C by Lemma 1; also, C is such a complex expression. Thus, there is exactly $1 = f(0)$ such complex expression, as desired.

For $n > 0$, any complex expression E of complexity $k + 2n$ that evaluates to C is such that $E \neq C$ (since the complexity of the complex expression C is k, and $k \neq k + 2n$ as $n > 0$), so $E = (E_1, E_2)$ for some complex expressions E_1 and E_2 such that, for some complex set C', E_1 evaluates to $[C', C]$, and E_2 evaluates to C'. (Note that C', as the evaluation of E_2, is uniquely determined by E.) The complexity of C' is $a + 1$ for some nonnegative integer a. Then the complexity of $[C', C]$ is $a + 1 + k$. Then, by Lemma 1, the complexity of E_1 is $a+1+k+2b$ and the complexity of E_2 is $a+1+2c$ for some nonnegative integers b and c. Then the complexity of E is $(a+1+k+2b)+(a+1+2c) = k+2(a+b+c+1)$; then $k + 2(a + b + c + 1) = k + 2n$, so $a + b + c = n - 1$. Now a,b, and c are nonnegative integers, and $b, c < n$. (Note that a, b, c are uniquely determined by E.)

Now, for any ordered triple (a, b, c) of nonnegative integers such that $a + b + c = n - 1$, there are C_a complex sets C' of complexity $a + 1$. Then once C' is chosen, as $b, c < n$, by the strong inductive hypothesis there are $f(b)$ complex expressions E_1 of complexity $a + 1 + k + 2b$ that evaluate to $[C', C]$, and there are $f(c)$ complex expressions E_2 of complexity $a + 1 + 2c$ that evaluate to C'. This yields $C_a f(b)f(c)$ complex expressions $E = [E_1, E_2]$ of complexity $k + 2n$ that evaluate to C, for this case of what (a, b, c) is.

Now if we sum over all cases to get $\sum C_a f(b)f(c)$ complex expressions (where the sum is over all ordered triples (a, b, c) of nonnegative integers such that $a+b+c = n-1$), then we count only complex expressions of complexity $k+2n$ that evaluate to C, we count all such complex expressions, and (by the parenthetical notes above that note that C', a, b, c are uniquely determined by E) we don't count any such complex expression more than once. Thus there are exactly $\sum C_a f(b)f(c)$ complex expressions of complexity $k + 2n$ that evaluate to C (where the sum is over all ordered triples (a, b, c) of nonnegative integers such that $a + b + c = n - 1$); by the definition of f, there are thus exactly $f(n)$ complex expressions of complexity $k + 2n$ that evaluate to C, as desired.

Thus Lemma 3 is true. $\blacksquare$

Now, for all positive integers i, $a_i = 0$ if i is even and $a_i = f(\frac{i-1}{2})$ if i is odd. Let $y = x^2$; then $y > 0$ and $f(0)+f(1)y+f(2)y^2+\cdots = \frac{7}{4}$. Now the sequence $f(0), f(1), f(2), \ldots$ is sequence A127632 in the OEIS, so by a result listed at https://oeis.org/A127632 we know that $f(0) + f(1)y + f(2)y^2 + \cdots = \frac{2}{1+\sqrt{2\sqrt{1-4y}-1}}$. Thus $\frac{2}{1+\sqrt{2\sqrt{1-4y}-1}} = \frac{7}{4}$, so $y = \sqrt{444}2401$, so $x = \sqrt{y} = \frac{2\sqrt{111}}{49}$, so the answer is $2 \cdot 100 + 111 \cdot 10 + 49 = \boxed{1359}$.

Remark: Of course, contestants would not be allowed to use the OEIS during the contest. Here is a (somewhat non-rigorous) way they could nevertheless arrive at the correct value of y. Let $F = f(0) + f(1)y + f(2)y^2 \cdots = \frac{7}{4}$ and let $C = C_0 + C_1 y + C_2 y^2 + \ldots$. (Assume that the series for C converges. This is not rigorous.) The recursion for f yields that $1+yCF^2 = F$. (This equation is noted on the OEIS page linked above.) Thus $yC = \frac{F-1}{F^2} = \frac{12}{49}$. The well-known recursion for the Catalan numbers yields the well-known (found on Wikipedia, among other places) equation $1+yC^2 = C$. Then $y = yC - (yC)^2 = \frac{12}{49} - (\frac{12}{49})^2 = \frac{444}{2401}$, so we may finish as before. $\square$

28. Let S be the set of integers modulo 2020. Suppose that $a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c$ are arbitrary elements of S. For any $x_1, x_2, ..., x_{2020} \in S$, define $f(x_1, x_2, ..., x_{2020})$ to be the 2020-tuple whose ith coordinate is $x_{i-2} + a_i x_{2019} + b_i x_{2020} + c x_i$, where we set $x_{-1} = x_0 = 0$. Let m be the smallest positive integer such that, for some values of $a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c$, we have, for all $x_1, x_2, ..., x_{2020} \in S$, that $f^m(x_1, x_2, ..., x_{2020}) = (0, 0, ..., 0)$. For this value of m, there are exactly n choices of the tuple $(a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c)$ such that, for all $x_1, x_2, ..., x_{2020} \in S$, $f^m(x_1, x_2, ..., x_{2020}) = (0, 0, ..., 0)$. Compute $100m + n$.

Proposed by Vincent Huang.

Answer. $\boxed{103020}$.

Solution. Clearly f is equivalent to multiplying the matrix $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2020} \end{bmatrix}$ by $M+cI$, where $M = \begin{bmatrix} 0 & 0 & \dots & 0 & a_1 & b_1 \\ 0 & 0 & \dots & 0 & a_2 & b_2 \\ 1 & 0 & \dots & 0 & a_3 & b_3 \\ 0 & 1 & \dots & 0 & a_4 & b_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{2020} & b_{2020} \end{bmatrix}$.

Note that if solutions exist for some m, then the minimal polynomial of M must have degree at most m.

The main observation is that, for $1 \le i \le 1009$, the matrix M^i has its first 2018 columns equal to the last 2018 columns of M^{i-1}. This is easy to see. Now it follows that $M^0, M^1, \dots, M^{1009}$ are linearly independent, because the first column of each such M^i consists of all 0s and a single 1 in the $2i+1$th position, so no linear combination of them can sumt o zero. From this it follows that $m \ge 1010$.

Next we'll show that solutions exist for $m = 1010$ and find all of them. First we'll characterize all possible solutions. Note by the observation in the previous paragraph that for each $1 \le i \le 1009$, M^i has all 0 entries in the even positions of the first column and the odd positions of the second column. Therefore, for a linear dependence between $M^0, M^1, \dots, M^{1010}$ to exist, M^{1010} must also have all 0 entries in the even positions of the first column and all odd positions of the second column, so $a_2 = a_4 = \cdots = a_{2020} = 0$ and $b_1 = b_3 = \cdots = b_{2019} = 0$. Similarly, each M^i for $1 \le i \le 1009$ has the property that the $2j+1$th position of the first column and the $2j+2$th position of the second column are equal, so M^{1010} must also have this property, meaning that $a_{2j+1} = b_{2j+2} = c_j$ for some constants c_j.

Then it's not hard to verify that M is the square of the 2020×2020 matrix $N = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & c_0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & c_1 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 \end{bmatrix}$.

Now since the minimal polynomial of N has degree ≤ 2020, it follows N's minimal polynomial has degree exactly 2020 as we wanted. Furthermore, the first 2019 columns of N^i are the last 2019 columns of N^{i-1}, so it follows that N^i's first column for $0 \le i \le 2019$ is just all 0s along with a 1 on the $2i+1$th position, while N^{2020}'s first column alternates between the c_is and 0s. Then it's clear that the minimal polynomial of N must be $N^{2020} - c_{1009}N^{2018} - \cdots - c_0$, so the minimal polynomial of M is $M^{1010} - c_{1009}M^{1009} - \cdots - c_0$. Since we want M's minimal polynomial to be $(M+c)^{1010}$ for some residue c modulo 2020, for each c we just expand $(M+c)^{1010}$ and set the c_i to be the appropriate values, and we are done; clearly this shows $m = 1010$ and $n = 2020$, so the answer is $100 \cdot 1010 + 2020 = \boxed{103020}$. $\square$

29. Let ABC be a triangle. The line through A tangent to the circumcircle of ABC intersects line BC at point W. Points $X, Y \neq A$ lie on lines AC and AB, respectively, such that $WA = WX = WY$. Point X_1 lies on line AB such that $\angle AXX_1 = 90^\circ$, and point X_2 lies on line AC such that $\angle AX_1X_2 = 90^\circ$. Point Y_1 lies on line AC such that $\angle AYY_1 = 90^\circ$, and point Y_2 lies on line AB such that $\angle AY_1Y_2 = 90^\circ$. Let lines AW and XY intersect at point Z, and let point P be the foot of the perpendicular from A to line X_2Y_2. Let line ZP intersect line BC at U and the perpendicular bisector of segment BC at V. Suppose that C lies between B and U. Let x be a positive real number. Suppose that $AB = x+1$, $AC = 3$, $AV = x$, and $\frac{BC}{CU} = x$. Then $x = \frac{\sqrt{k}-m}{n}$ for positive integers k,m, and n such that k is not divisible by the square of any integer greater than 1. Compute $100k + 10m + n$.

Proposed by Ankit Bisain, Luke Robitaille, and Brandon Wang.

Answer. $\boxed{264143}$.

Solution. Let O be the circumcenter of ABC. Key Claim: $V = O$ and $(U, AO \cap BC; B, C) = -1$. Proof of Key Claim: Perform a $\sqrt{bc}$-inversion. Denote the image of any point K by K'. Throughout, note that $AB \neq AC$ and $\angle BAC \neq 90°$. Let ℓ be the perpendicular bisector of BC, and let m be the line through A parallel to BC. Now $W' = m \cap (ABC)$. Then the image of (AXY) becomes the perpendicular bisector of AW', which is ℓ. Thus $X' = AB \cap \ell$ and $Y' = AC \cap \ell$. Now X_1' is the foot of the perpendicular from X' to AC, and X_2' is the foot of the perpendicular from X_1' to AB. Similarly Y_1' is the foot of the perpendicular from Y' to AB, and Y_2' is the foot of the perpendicular from Y_1' to AC. Now $Z' = (AX'Y') \cap m$. Also, note that $P = (AX_1X_2) \cap (AY_1Y_2)$. Then $P' = X_1'X_2' \cap Y_1'Y_2'$. Let $S = AO \cap BC$, and let T be on BC such that $(T, S; B, C) = -1$. We want $V = O$ and $U = T$; i.e., we want Z, P, O, T collinear. (We have $T \neq \infty$ as $AB \neq AC$ and $\angle BAC \neq 90°$, so $T' \neq A$.) Thus we need A, Z', P', O', T' concyclic.

Any degenerate cases that might occur are left as a tacit exercise to the reader. Let $R = X'X_1' \cap Y'Y_1'$. Let M be the midpoint of BC. Let R_1 be the reflection of R over M. Let γ be the circle with diameter AR_1. I claim that A, Z', P', O', T' all lie on γ. Obviously A lies on γ.

Note that R is the orthocenter of $AX'Y'$. Also, note that Z' is the intersection of the A-altitude and the circumcircle of $\triangle AX'Y'$. Then R and Z' are reflections of one another over ℓ; then $Z'R_1$ is parallel to ℓ, so $Z'R_1 \perp BC$. Also, R, A, Z' all lie on m. Then $AZ' || BC$. Thus $AZ' \perp Z'R_1$, so Z' lies on γ.

Note that quadrilaterals $CMX'X_1'$ and $BMY'Y_1'$ are cyclic. Then, denoting directed angles modulo π by $\measuredangle$, we have $\measuredangle X'MX_1' = \measuredangle X'CY' = \measuredangle Y'BX' = \measuredangle Y'MY_1'$. Thus M, X_1', Y_1' are collinear. Note that $RX_1'P'Y_1'$ is a parallelogram. Then the midpoints of RP' and RR_1 both lie on line $X_1'Y_1'$. Thus lines $P'R_1$ and $X_1'Y_1'$ are parallel or coincide (assuming for the moment that $P' \neq R_1$). Also, P' is the orthocenter of $\triangle AX_1'Y_1'$, so $AP' \perp X_1'Y_1'$. Thus $AP' \perp P'R_1$, so P' lies on γ. (We assumed $P' \neq R_1$; if $P' = R_1$, then P' lies on γ, so it was okay to assume that.)

Note that O' is the reflection of A over BC. Let n be the reflection of m over BC. Now AO' is perpendicular to BC, while O' and R_1 both lie on n, which is parallel to BC. Then O' lies on γ.

Let H be the orthocenter of ABC. Consider complete quadrilteral $ABCMX'Y'$. Then H, which is the orthocenter of $\triangle ABC$, M, which is the orthocenter of $\triangle X'MB$, as $X'M \perp BM$, and R, which is the orthocenter of $\triangle AX'Y'$, all lie on the Steiner line of this complete quadrilateral. Thus H, M, R are collinear.

Now, in $\triangle ABC$, the A-altitude intersects (ABC) at S'. Now T' lies on (ABC) such that $(T', S'; B, C) = -1$. Let A_1 be the antipode of A on (ABC). Now $S' \neq A_1$. Now $S'A_1 \perp AS' \perp BC$, $S'A_1 || BC$, so $S'A_1 \cap BC = \infty$. Let $L = T'A_1 \cap BC$. Then, projecting through A_1, $-1 = (T', S'; B, C) = (L, \infty; B, C)$, so L is the midpoint of BC. Thus $L = M$. Thus T', A_1, M are collinear. It is well-known that M is the midpoint of HA_1. Thus T', A_1, H, M are collinear.

Now, as $H \neq M$, we have that T', A_1, R all lie on line HM. Note that A does not lie on line HM (as $AB \neq AC$ and $\angle BAC \neq 90°$), so $T' \neq A$. Note that $T' \neq A_1$. If $T' = R$, then T' lies on γ. Otherwise, $\measuredangle AT'R = \measuredangle AT'A_1 = 90°$, so T' lies on γ. Thus, in either case, T' lies on γ.

Thus, A, Z', P', O', T' are concyclic. Thus Z, P, O, T are collinear (as none of them lie at infinity). Thus the Key Claim is true. ■

Now $\frac{BU}{CU} = \frac{BC}{CU} + 1 = x + 1$. Then $(U, S; B, C) = -1$; then S lies on segment BC, as U lies on ray BC beyond C. Then $\frac{[ABO]}{[ACO]} = \frac{BS}{CS} = \frac{BU}{CU} = x + 1$. Now let M_B and M_C be the midpoints of AC and AB, respectively. Then $[ABO] = \frac{AB \cdot OM_C}{2}$ and $[ACO] = \frac{AC \cdot OM_B}{2}$. Thus $\frac{(x+1)OM_C}{3OM_B} = \frac{[ABO]}{[ACO]} = x + 1$. Thus $OM_C = 3OM_B$, so $OM_C^2 = 9OM_B^2$. Now $OM_C^2 = x^2 - (\frac{x+1}{2})^2$ and $OM_B^2 = x^2 - \frac{9}{4}$. Thus $x^2 - (\frac{x+1}{2})^2 = 9(x^2 - \frac{9}{4})$. Thus $3x^2 - 2x - 1 = 36x^2 - 81$, so $33x^2 + 2x - 80 = 0$. Thus, as $x > 0$, we get $x = \frac{\sqrt{2641}-1}{33}$, so the answer is $2641 \cdot 100 + 1 \cdot 10 + 33 = \boxed{264143}$. □

30. For a positive integer n, we say an *n-transposition* is a bijection $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that there exist exactly two elements i of $\{1, 2, \dots, n\}$ such that $\sigma(i) \neq i$.

Fix some four pairwise distinct n-transpositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Let q be any prime, and let $\mathbb{F}_q$ be the integers modulo q. Consider all functions $f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q$ that satisfy, for all integers i with $1 \leq i \leq n$ and all $x_1, \dots x_{i-1}, x_{i+1}, \dots, x_n, y, z \in \mathbb{F}_q^n$,

$$f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) + f(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i-1}, y + z, x_{i+1}, \dots, x_n),$$

and that satisfy, for all $x_1, \dots, x_n \in \mathbb{F}_q^n$ and all $\sigma \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$,

$$f(x_1, \dots, x_n) = -f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

(Note that the equalities in the previous sentence are in $\mathbb{F}_q$. Note that, for any $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{F}_q$, we have $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$, where $a_1 + b_1, \dots, a_n + b_n \in \mathbb{F}_q$.)

For a given tuple $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$, let $g(x_1, \dots, x_n)$ be the number of different values of $f(x_1, \dots, x_n)$ over all possible functions f satisfying the above conditions.

Pick $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$ uniformly at random, and let $\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be the expected value of $g(x_1, \dots, x_n)$. Finally, let

$$\kappa(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = -\lim_{q \to \infty} \log_q \left(-\ln \left(\frac{\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4) - 1}{q - 1} \right) \right).$$

Pick four pairwise distinct n-transpositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ uniformly at random from the set of all n-transpositions. Let $\pi(n)$ denote the expected value of $\kappa(\sigma_1, \dots, \sigma_4)$. Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients such that $q(-3) \neq 0$ and such that $\pi(n) = \frac{p(n)}{q(n)}$ for infinitely many positive integers n. Compute $\frac{p(-3)}{q(-3)}$.

Proposed by Gopal Goel.

Answer. $\boxed{197}$.

Solution. Let I_n be the set of all n-transpositions.

Definition. Fix some subset $T \subseteq I_n$. We say that a multilinear function $f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q$ is T-*good* if

$$f(x_1, \dots, x_n) = -f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $\sigma \in T$ and $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$.

Definition. Fix some subset $T \subseteq I_n$. For any $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$, let $g_T(x_1, \dots, x_n)$ denote the number of different values $f(x_1, \dots, x_n)$ takes over all T-good functions f.

Definition. Fix some subset $T \subseteq I_n$. Let G_T be the graph with vertex set $[n] = \{1, 2, \dots, n\}$ and edge $\{a, b\}$ if and only if the transposition (a, b) is in T. Let $C_1 \sqcup \dots \sqcup C_r$ be partition of the vertex set into connected components of G_T. Call a matrix $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$ T-*invertible* if the sets $\{x_i\}_{i \in C_k}$ are each a set of linearly independent vectors.

Definition. Let $\varepsilon(q, T)$ be the expected value of $g_T(x_1, \dots, x_n)$ if $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$ is chosen uniformly at random. Also define

$$\kappa(T) = -\lim_{q \to \infty} \log_q \left(-\ln \left(\frac{\varepsilon(q, T) - 1}{q - 1} \right) \right).$$

We have the following lemma. Lemma. Let f be a T-good function for some $T \subseteq I_n$. Let $C_1 \sqcup \dots \sqcup C_r$ be the connected components of G_T. Then, if $a, b \in C_k$, then

$$f(x_1, \dots, x_n) = -f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where σ is the transposition swapping a and b.

Sketch. Let $\bar{T}$ be the set of transpositions σ such that

$$f(x_1, \dots, x_n) = -f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $(x_1, \dots, x_n) \in (\mathbb{F}_q^n)^n$. It suffices to show that if $(a, b), (b, c) \in \bar{T}$, then $(a, c) \in \bar{T}$. This follows since $(a, b) \circ (b, c) \circ (a, b) = (a, c)$ and $(-1)^3 = -1$.

We have the following key linear algebra claim.

Claim (Main Linear Algebra Step).Fix some set $T \subseteq I_n$. Then $g_T(x_1, \ldots, x_n) = q$ if $(x_1, \ldots, x_n)$ is T-invertible, and $g_T(x_1, \ldots, x_n) = 1$ otherwise.

Proof. First suppose that $X = (x_1, \ldots, x_n)$ is not T-invertible. As above, let $C_1 \sqcup \cdots \sqcup C_r$ be partition of the vertex set into connected components of G_T. Then, there is some $1 \leq k \leq r$ such that the set of vectors $\{x_i\}_{i \in C_k}$ is linearly dependent. So

$$x_j + \sum_{\substack{i \in C_k \\ i \neq j}} \alpha_i x_i = 0$$

for some $j \in C_k$ and scalars $\alpha_i \in \mathbb{F}_q$.

Note that if $x_a = x_b$ for $a, b \in C_k$ and $a \neq b$, then $f(x_1, \ldots, x_n) = 0$ (this is due to the lemma). Thus, $f(x_1, \ldots, x_n)$ is unchanged if we replace x_j with $x_j + \sum_{\substack{i \in C_k \\ i \neq j}} \alpha_i x_i$, or 0. But $f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) = 0$ by multilinearity, so we have $f(X) = 0$. Thus, if X is not T-invertible, then $f(X) = 0$, so $g_T(X) = 1$.

Now, suppose that $X = (x_1, \ldots, x_n)$ is T-invertible. We'll construct a T-good function f such that $f(X) \neq 0$. By scaling this function, we see that $h(X)$ can take any value in $\mathbb{F}_q$ for T-good functions h, so $g(X) = q$.

Extend each set $\{x_i\}_{i \in C_k}$ to a basis $D_k = \{z_i^{(k)}\}_{i \in [n]}$ of $\mathbb{F}_q^n$. In particular, we have $z_i^{(k)} = x_i$ if $i \in C_k$. Now, for any $(y_1, \ldots, y_n) \in (\mathbb{F}_q^n)^n$, set

$$f(y_1, \ldots, y_n) = \prod_{k=1}^{r} \det(w_1^{(k)}, \ldots, w_n^{(k)}),$$

where $w_i^{(k)} = y_i$ if $i \in C_k$, and $w_i^{(k)} = z_i^{(k)}$ otherwise. It is easy to check that this function is T-good, and that $f(x_1, \ldots, x_n) \neq 0$, so we're done. This prove the Main Linear Algebra Step.

We have the following well known counting step.

Claim (Counting T-Invertible Matrices). Let $T \subseteq [n]$, and let $C_1 \sqcup \cdots \sqcup C_r$ be the connected components of G_T. Then,

$$\frac{\varepsilon(q, T) - 1}{q - 1} = \prod_{k=1}^{r} (1 - q^{-n})(1 - q^{-n+1}) \cdots (1 - q^{-n+|C_k|-1}).$$

Proof. By the previous claim, $\frac{\varepsilon(q,T)-1}{q-1}$ is simply the probability that a randomly chosen element of $(\mathbb{F}_q^n)^n$ is T-good.

Let's count the number of ordered lists of m vectors in $\mathbb{F}_q^n$ that are linearly independent. We have $q^n - 1$ choices for the first vector, $q^n - q$ for the second, $q^n - q^2$ for the third, and so on. Thus the number of lists is

$$(q^n - 1) \cdot (q^n - q) \cdots (q^n - q^{n-m+1}).$$

Now, the number of T-good matrices is just the product of the above quantity where m ranges over all the $|C_k|$, since we just have to choose the vectors such that the sets $\{x_i\}_{i \in C_k}$ are sets of linearly independent vectors. Thus, the number of T-good matrices is

$$\prod_{k=1}^{r} (q^n - 1) \cdot (q^n - q) \cdots (q^n - q^{n-|C_k|+1}).$$

The result follows since the number of total matrices is $q^{n^2} = q^{n|C_1|} \cdots q^{n|C_k|}$. This proves the Claim.

We'll now evaluate the limit. Claim (Evaluating the Limit). Let $T \subseteq I_n$, and let $C_1 \sqcup \cdots \sqcup C_r$ be the connected components of G_T. Let γ be the size of the largest connected components. Then, $\kappa(T) = n + 1 - \gamma$.

Proof. Note that

$$-\ln[((1 - q^{-n})(1 - q^{-n+1}) \cdots (1 - q^{-n+|C_k|-1})] = q^{-n+|C_k|-1} + O(q^{-n+|C_k|-2}).$$

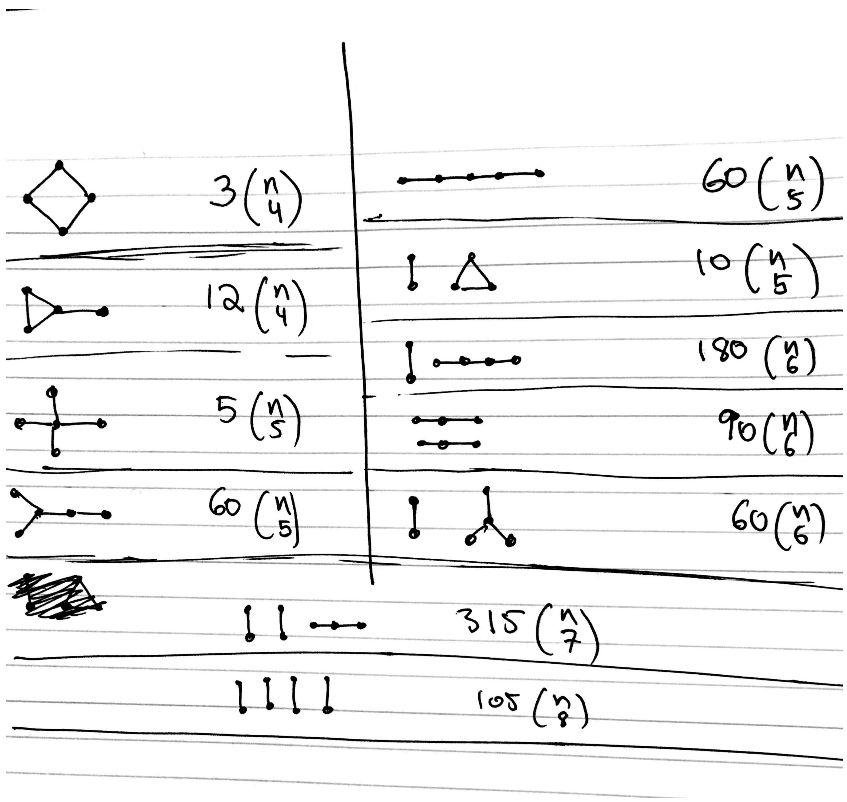
Now, suppose that ℓ of $C_1, \ldots, C_r$ are all the "largest connected component", so they all have size γ. Then, we see that

$$-\ln\left[\prod_{k=1}^{r}(1-q^{-n})(1-q^{-n+1})\cdots(1-q^{-n+|C_k|-1})\right] = \ell\cdot q^{-n+\gamma-1} + O(q^{-n+\gamma-2}).$$

The result then follows by taking $\log_q$ and noting that $\log_q \ell \to 0$ as $q \to \infty$. This proves the Claim.

We see that $\pi(n) = n+1-\mathbb{E}\gamma$, where $\mathbb{E}\gamma$ is the expected size of the largest connected components over all labeled graphs on n vertices and four edges. Let g_γ be the number of graphs with four edges and largest connected components size γ.

Here is a list of all non-isomorphic graphs on four vertices (we're ignoring lone vertices), and the number of labeled graphs on n vertices corresponding to each isomorphism class.



From here, we calculate

$$\begin{aligned}
g_2 &= 105\binom{n}{8}\\
g_3 &= 315\binom{n}{7} + 90\binom{n}{6} + 10\binom{n}{5}\\
g_4 &= 240\binom{n}{6} + 15\binom{n}{4}\\
g_5 &= 125\binom{n}{5},
\end{aligned}$$

yielding a final answer of

$$\pi(n) = n+1-\frac{210\binom{n}{8}+945\binom{n}{7}+1230\binom{n}{6}+655\binom{n}{5}+60\binom{n}{4}}{\binom{\binom{n}{2}}{4}}.$$

A calculation yields $\frac{p(-3)}{q(-3)} = 197$. $\square$