2019 Winter Camp Mock Olympiad

1. A circle passing through the vertices A and B of a cyclic quadrilateral $ABCD$ intersects diagonals AC and BD at E and F, respectively. The lines AF and BC meet at a point P, and the lines BE and AD meet at a point Q. Prove that PQ is parallel to CD.

2. Let $1 = d_0 < d_1 < \cdots < d_m = 4k$ be all positive divisors of $4k$, where k is a positive integer. Prove that there exists $i \in \{1, \ldots, m\}$ such that $d_i - d_{i-1} = 2$.

3. Let $x_1, x_2, \ldots, x_n$ be a sequence of positive integers satisfying that the decimal representation of x_i is not the beginning of the decimal representation of x_j for any $i \neq j$. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 3$$

4. In a country, some pairs of cities are connected by two-way roads such that it is possible to travel between any two cities in the country by a sequence of roads. James wants to put a flag in each city so that no two cities connected by a single road have the same flag. If James cannot use fewer than $n+1$ distinct flags to do this, prove that he can close $\frac{n(n-1)}{2}$ roads so that it is still possible to travel between any two cities by a sequence of roads.

5. Let $a_1, a_2, \ldots, a_{2^k+1}$ be distinct positive integers where $k \geq 2$. Prove that the number

$$\prod_{1 \leq i < j \leq 2^k+1} (a_i + a_j)$$

has at least $k+1$ distinct prime divisors.