Generating Functions

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1 Introduction

Combinatorial problems will often ask to determine a certain sequence of numbers $a_0, a_1, a_2, \ldots$. A common technique to solve this type of problems is to encode this sequence as a (possibly infinite) polynomial,

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

These polynomial are called **generating functions**. The goal of this session is to develop the basic tools of generating functions and provide some standard techniques for application.

2 Sicherman Dice

Find all possible combinations of dice, A and B, that bear only positive integers and have the same probability distribution as the sum as normal dice.

Solution. $\boxed{(1,2,3,4,5,6),(1,2,3,4,5,6) \textit{ and } (1,2,2,3,3,4),(1,3,4,5,6,8)}$

Consider a regular fair die with numbers $(1,2,3,4,5,6)$. This can be encoded through generating functions

$$\frac{1}{6}x + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{6}x^5 + \frac{1}{6}x^6$$

The sum of two dice rolls can be expressed as

$$\frac{1}{36}\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2$$

Therefore, the goal is the determine all polynomials $f(x)$ and $g(x)$ such that

(i) $f(x)g(x) = \left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2$

(ii) The coefficients of $f(x)$ and $g(x)$ are non-negative

(iii) The sum of the coefficients of each of $f(x)$ and $g(x)$ is 6, $f(1) = g(1) = 6$

(iv) $f(x)$ and $g(x)$ does not have a constant term

First, factor the original polynomial into irreducible functions

$$\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2 = x^2\left(1 + x + x^2\right)^2(1+x)^2\left(1 - x + x^2\right)^2$$

To satisfy (iv), x^2 must be split into both f and g. Next, evaluating the three polynomials

$$1 + x + x^2, 1 + x, 1 - x + x^2$$

at $x = 1$ yields $3, 2, 1$, respectively. Thus, to satisfy (iii), both the polynomials $\left(1 + x + x^2\right)^2$ and $(1+x)^2$ must be split into both f and g. Finally, there are two ways to handing $\left(1 - x + x^2\right)^2$. The first ways is to split them into both f and g, but this will result in die A and B having the same set of numbers. The second way is to give both terms to $f(x)$. Combining these factors,

$$\begin{aligned} f(x) &= x\left(1 + x + x^2\right)(1+x)\left(1 - x + x^2\right)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8 \\ g(x) &= x\left(1 + x + x^2\right)(1+x) = x + 2x^2 + 2x^3 + x^4 \end{aligned}$$

Note that they both satisfy (ii). Therefore, there is only one possible combination of dice, A and B, such that the set of numbers on die A is different from the set of numbers on die B.

3 Partial Fraction Review

Let $f(x)$ be a rational function then there exist real polynomials $p(x)$ and $q(x)$ with $q(x) \neq 0$ such that

$$f(x) = \frac{p(x)}{q(x)}$$

It suffice to assume that $q(x)$ is monic as it is easy to just factor out the leading coefficients. By Fundamental Theorem of Algebra, it is possible to factor the denominator as

$$q(x) = (x-a_1)^{j_1} \cdots (x-a_m)^{j_m} \left(x^2+b_1x+c_1\right)^{k_1} \cdots \left(x^2+b_nx+c_n\right)^{k_n}$$

where $a_1, \ldots a_m, b_1, \ldots, b_n, c_1, \ldots, c_n$ are real numbers with $b_i^2 - 4c_i < 0$, and $j_1, \ldots, j_m, k_1, \ldots, k_n$ are positive integers. Then the partial fraction decomposition of $f(x)$ is

$$f(x) = P(x) + \sum_{i=1}^{m} \sum_{r=1}^{j_1} \frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n} \sum_{r=1}^{k_i} \frac{B_{ir}x + C_{ir}}{(x^2+b_ix+c_i)^r}$$

where $P(x)$ is a polynomial and A_{ir}, B_{ir}, C_{ir} are real constants. There are numerous way to find the constants. The most common technique is to use coefficient matching.
Remark. Although $q(x)$ can always be written in that form, it is not always easy to do so.
Example 1. Apply partial fraction decomposition to

$$\frac{1}{(x+1)(x+2)^2}$$

Solution. The given expression will decompose into something of the form

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Recombining this expression and equating to 1 yields $A = 1$, $B = -1$, $C = -1$.

4 Some Useful Series

The following are a list of common finite and infinite series. Many more series can be derived from these through operations on functions. (Add, subtract, multiply, divide, composition, derivative, etc.)
Finite Series:

1. $\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$
2. $\sum_{k=0}^{N} \binom{N}{k} x^k = (1+x)^N$

Infinite Series:
These series are called formal power series. It differs from an actual power series in that the variables cannot always be replaced by a number. Therefore, convergence is not an issue.

1. $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ (Infinite geometric series)
2. $\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$
3. $\sum_{k=0}^{\infty} \binom{n+k}{k} x^k = \frac{1}{(1+x)^{n+1}}$ (Negative binomial)
4. $\sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x$
5. $\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$
6. $\sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k = \frac{1-\sqrt{1-4x}}{2x}$
7. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin(x)$
8. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos(x)$

5 Solving Recurrence Relations

The easiest application of generating functions is solving recurrence relations.
Fibonacci sequence. Let $F_0 = 0$ and $F_1 = 1$ and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Find the general term of the sequence.
Solution. Define $f(x)$ to be the generating function of F_n and consider the following manipulations

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} F_k x^k \\
&= F_0 + F_1 x + \sum_{k=2}^{\infty} F_k x^k \\
&= 0 + x + \sum_{k=2}^{\infty} (F_{k-1} + F_{k-2}) x^k \\
&= x + x \sum_{k=2}^{\infty} F_{k-1} x^{k-1} + x^2 \sum_{k=2}^{\infty} F_{k-2} x^{k-2} \\
&= x + x \sum_{k=1}^{\infty} F_k x^k + x^2 \sum_{k=0}^{\infty} F_k x^k \\
&= x + x \sum_{k=0}^{\infty} F_k x^k + x^2 \sum_{k=0}^{\infty} F_k x^k \\
&= x + x f(x) + x^2 f(x) \\
(*) &= \frac{x}{1 - x - x^2} \\
&= \frac{x}{\left(1 - \frac{1+\sqrt{5}}{2} x\right)\left(1 - \frac{1-\sqrt{5}}{2} x\right)} \\
&= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1+\sqrt{5}}{2} x} - \frac{1}{1 - \frac{1-\sqrt{5}}{2} x} \right) \\
&= \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{\infty} \left(\frac{1+\sqrt{5}}{2}\right)^k x^k - \sum_{k=0}^{\infty} \left(\frac{1-\sqrt{5}}{2}\right)^k x^k \right) \\
&= \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{\infty} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right) x^k \right)
\end{aligned}$$

The coefficient of x^k is

$$F_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right)$$

which is exactly the well known closed form formula for the Fibonacci Sequence.
Note. Observe that the denominator at $(*)$ is similar to the characteristic polynomial of the recurrence relation!

6 The Snake Oil Method

The snake oil method is a simple yet powerful way to force a combinatorial summation into a generating function double summation. Here is an overview for the steps of the snake oil method

1. Identify the free variable, say n, that the sum depends on and call the sum $f(n)$.
2. Let $F(x)$ be the ordinary generating function for the sequence $f(0), f(1), \ldots$ Note that $F(x)$ is now a double sum.

3. Move everything inside the inner summation and exchange the order of the summation.
4. Manipulation the exponent of x in order to applied a well known series.
5. Identify the coefficient of the resulting generating function.

Example. Let $m, n \geq 0$. Evaluate

$$\sum_{k=0}^{\infty} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$$

Solution. Let m be the free variable and define the generating function

$$\begin{aligned}
F(x) &= \sum_{m=-\infty}^{\infty} x^m \sum_{k=0}^{\infty} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{k+1} \sum_{m=-\infty}^{\infty} \binom{n+k}{m+2k} x^m \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-2k} \sum_{m=-\infty}^{\infty} \binom{n+k}{m+2k} x^{m+2k} \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-2k} \sum_{r=0}^{n+k} \binom{n+k}{r} x^r \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-2k} (1+x)^{n+k} \\
&= (1+x)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{k+1} \left(\frac{-(1+x)}{x^2} \right)^k \\
&= (1+x)^n \frac{1}{2\left(-\frac{1+x}{x^2}\right)} \left(1 - \sqrt{1 - 4\left(-\frac{1+x}{x^2} \right)} \right) \\
&= x(1+x)^{n+1}
\end{aligned}$$

This is the generating function for $\binom{n-1}{m-1}$. Therefore, the original summation is $\binom{n-1}{m-1}$.
Remark. The same technique works if n is chosen as the free variable except the algebra will be slightly more messy.

7 Generating Functions in Olympiad Questions

[China 1996] Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.
Solution. Let a_n be the answer to the problem. Define

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

to be the generating function for a_n. Let $P(x) = \sum_{i=0}^{k} c_i x^i$ where $c_i \in \{0, 1, 2, 3\}$. Observe that $P(2) = n$ if and only if $\sum_{i=0}^{k} c_i 2^i = n$. Since $t^n = \prod_{i=0}^{k} t^{2^i c_i}$ and $2^i c_i \in \{0, 2^i, 2 \cdot 2^i, 3 \cdot 2^i\}$ then $f(t)$ can be factored as

$$\begin{aligned} f(t) &= \prod_{i=0}^{\infty} \left(1 + t^{2^i} + t^{2 \cdot 2^i} + t^{3 \cdot 2^i}\right) \\ &= \prod_{i=0}^{\infty} \left(\frac{1 - t^{4 \cdot 2^i}}{1 - t^{2^i}}\right) \\ &= \frac{1}{1-t} \cdot \frac{1}{1-t^2} \\ &= \frac{1}{2}\left(\frac{1}{(1-t)^2} + \frac{1}{1-t^2}\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} k x^{k-1} + \sum_{k=0}^{\infty} x^{2k}\right) \\ &= \sum_{n=0}^{\infty} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) x^n \end{aligned}$$

Therefore, the number of such polynomials is $\left\lfloor \frac{n}{2} \right\rfloor + 1$.
[IMO 1995 P6 by Nikolay Nikolov] Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are there, the sum of whose elements is divisible by p?
Solution. Let ω be a primitive p-th root of unit then

$$\prod_{k=1}^{2p} \left(x - \omega^k\right) = (x^p - 1)^2 = x^{2p} - 2x^p + 1$$

Comparing the coefficient of x^p,

$$2 = \sum \omega^{k_1 + k_2 + \cdots + k_p} = \sum_{j=0}^{p-1} n_j \omega^j$$

where the first summation is over all subsets $\{k_1, k_2, \ldots, k_p\}$ of $\{1, 2, \ldots, 2p\}$ and n_j in the second summation is the number of such subsets such that

$$k_1 + k_2 + \cdots + k_p \equiv j \pmod{p}$$

Thus, ω is a root of

$$G(x) = (n_0 - 2) + \sum_{j=1}^{p-1} n_j x^j$$

which is a polynomial of degree $p - 1$. Recall that the minimal polynomial of ω in $\mathbb{Q}[x]$ is

$$F(x) = \sum_{j=0}^{p-1} x^j$$

which is also of degree $p - 1$. Since minimal polynomials are unique up to a scalar multiple then $G(x)$ is a scalar multiple of $F(x)$. This implies that

$$n_0 - 2 = n_1 = n_2 = \cdots = n_{p-1}$$

Since $\sum_{j=0}^{p} n_j = \binom{2p}{p}$ then

$$n_0 = \frac{1}{p}\left(\binom{2p}{p} - 2\right) + 2$$

8 String Generating Functions

A binary string of length n is a $n-$tuple $(a_1, \ldots, a_n)$ where $a_i \in \{0,1\}$. An empty string is denoted with ϵ. Let A and B be two sets of strings then

$$AB = \{ab : a \in A, b \in B\}$$

The set of binary strings of length 7 can be written as $\{0,1\}^7$. The set of all binary strings can be written as

$$\{0,1\}^* = \{\epsilon\} \cup \bigcup_{k=1}^{\infty} \{0,1\}^k = \bigcup_{k=0}^{\infty} \{0,1\}^k$$

An expression is **unambiguous** if there exists a unique way of writing every string according to the expression. For example: $\{0,00\}\{\epsilon,0\}$ is ambiguous.
Remark. $\{1\}^*\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*\{0\}^*$ is unambiguous and is equivalent to $\{0,1\}^*$. The longer version is actually much more useful.

Theorem. If $S = A \cup B$ unambiguously then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$.
If $S = AB$ unambiguously then $\Phi_S(x) = \Phi_A(x)\Phi_B(x)$.
If $S = A^*$ unambiguously then $\Phi_S(x) = \frac{1}{1-\Phi_A(x)}$.

Example. Prove that $\{1\}^*\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*\{0\}^* = \{0,1\}^*$.
Solution. The generating function for $\{0,1\}^*$ is

$$\Phi_{\{0,1\}^*}(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

Therefore, there are 2^n number of binary strings of length n.
The generating function for $A = \{1\}^*\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*\{0\}^*$ is

$$\Phi_A(x) = \frac{1}{1-x}\frac{1}{1-\frac{x}{1-x}\frac{x}{1-x}}\frac{1}{1-x} = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$

Therefore, there are 2^n number of such binary strings of the form A. Since this is the maximum number of such binary strings, $A = \{0,1\}^*$.

Example. Determine the number of binary strings such that all blocks has length at least 2.
Solution. The unambiguous expression is

$$S = (\epsilon \cup \{00\}0^*)(\{11\}1^*\{00\}0^*)^*(\epsilon \cup \{11\}1^*)$$

The generating function is

$$\Phi_S(x) = \left(1 + \frac{x^2}{1-x}\right)\left(\frac{1}{1-\frac{x^2}{1-x}\frac{x^2}{1-x}}\right)\left(1 + \frac{x^2}{1-x}\right) = \frac{1-x+x^2}{1-x-x^2}$$

The number of such binary strings is given by the coefficient of x^n, 1 when $n = 0$ and when $n \geq 1$,

$$\begin{cases} 1 & \text{if } n = 0 \\ \left(1 - \frac{1}{5}\sqrt{5}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(1 + \frac{1}{5}\sqrt{5}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n & \text{if } n \geq 1 \end{cases}$$

Remark. The set of binary strings can also be defined recursively as $\{0,1\}^* = \{\epsilon\} \cup \{0,1\}^*\{0,1\}$.
Example. Determine the generating function that models the number of of strings without 111.
Solution. Let S be the set of such strings. The unambiguous expression is $S = \{\epsilon, 1, 11\} \cup S\{0, 01, 011\}$. The corresponding generating function is

$$\Phi_S(x) = (1 + x + x^2) + \Phi_S(x)(x + x^2 + x^3) = \frac{1+x+x^2}{1-x-x^2-x^3}$$

9 Problem Set

1. Let $a_0 = 1, a_1 = 1,$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 3$. Use generating functions to find the closed form formula for a_n.
2. Evaluate $\sum_{k=0}^{\infty} \binom{k}{n-k}$.
3. Let $a_1 = 1, a_2 = 4,$ and $a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 3$. Use generating functions to find the closed form formula for a_n.
4. A binary string α is called self-avoiding provided that whenever $\alpha\beta = \gamma\alpha$ then $|\alpha| \leq |\beta|$. (The absolute values represent the length). For example, 01001 is not self avoiding because $(01001)(001) = (010)(01001)$. Determine the generating functio that models the number of self-avoiding strings of length n.
5. Evaluate $\sum_{k=0}^{\infty} \binom{n}{\lfloor \frac{k}{2} \rfloor} x^k$.
6. Find the number of ways n dollars can be changed into 1 or 2 dollar coins (regardless of order).
7. Let F_n be the n^{th} term in the Fibonacci sequence. Determine the value of $\sum_{n=0}^{\infty} \frac{F_n}{4^n}$.
8. Determine the generating function that models the number of of strings without 11101.
9. Evaluate $\sum_{k=0}^{\infty} \binom{n}{k} \binom{n-k}{\lfloor \frac{m-k}{2} \rfloor} (-2)^k$.
10. Let $x_0 = 1$ and $x_1 = 3$ and for $n \geq 2$, let $x_n = 6x_{n-1} - 10x_{n-2}$. Find the closed formula, which only involves real numbers.
11. Find the number of binary strings of length n in which each block of 0s has odd length and each block of 1s has length one or two.
12. Prove that $\sum_{k=0}^{n} \binom{2n}{2k} \binom{2k}{k} 2^{2n-2k} = \binom{4n}{2n}$.
13. For given n and p evaluate $\sum_{k=0}^{\infty} \binom{2n+1}{2p+2k+1} \binom{p+k}{k}$.
14. Find the number of binary strings of length n with exactly m blocks of 01.
15. Prove that the sequence defined by $a_1 = 1$, $a_{2n+2} = a_n + a_{n+1}$, and $a_{2n+1} = a_n$ counts the number of representations of n as a sum of powers of two using each power at most twice.
16. [Putnam 1992] Show that the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$ is $\sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}$.
17. [HMMT 2007] Let S be the set of triplets (i, j, k) of positive integers which satisfy $i + j + k = 17$. Compute
$$\sum_{(i,j,k) \in S} ijk$$
18. [AIME 2001] A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?
19. [Putnam 1991] Let p be an odd prime . Prove that
$$\sum_{n=0}^{p} \binom{p}{n} \binom{p+n}{n} \equiv 2^p + 1 \left(\bmod p^2\right)$$

20. Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be sets such that $A_1 = \emptyset$ and $B_1 = \{0\}$, and

$$A_{n+1} = \{x + 1 : x \in B_n\}$$
$$B_{n+1} = (A_n \cup B_n) \setminus (A_n \cap B_n)$$

for all positive integers n. Determine all positive integers n such that $B_n = \{0\}$.

21. Determine if the set of positive integers can be partitioned into more than one but finite number of arithmetic progressions with no two having the same common difference?

22. [Leningrad Mathematical Olympiad 1991] A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form $a_k + a_{k+p} + \cdots$ is the same for any $k = 1, 2, \ldots, p$. For instance the sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 3$, $a_6 = 2$ is 3-balanced. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$ then all its members are equal to zero.

23. [China 1999] For a set A, let $s(A)$ denote the sum of the elements of A. If $A = \emptyset$ then $s(A) = 0$. Let $S = \{1, 2, \ldots, 1999\}$. For $r = 0, 1, 2, \ldots, 6$, define

$$T_r = \{T | T \subseteq S, s(T) \equiv r \pmod 7\}$$

For each r, find the number of elements in T_r.

24. [Vietnam TST 1994] Evaluate

$$\sum \frac{1}{n_1! n_2! \cdots n_{1994}! \, (n_2 + 2n_3 + 3n_4 + \cdots + 1993 n_{1994})!}$$

where the sum is taken over all 1994$-$tuples of nonnegative integers $(n_1, n_2, \ldots, n_{1994})$ such that $\sum_{k=1}^{1994} k a_k = 1994$.

25. [IMO 1998 SL] Let $a_0, a_1, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$where i, j, k are not necessarily distinct. Determine a_{1998}.

26. [IMO 2008] Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labeled $1, 2, \ldots, 2n$ be given, each of which can be either on or off. Initially all lamps are off. We consider a sequence of steps: at each step one of the lamps is switched (from on to off or from off to on).
Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through $n$are all on, and lamps $n + 1$ through $2n$ are all off.
Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.
Determine the ratio $\frac{N}{M}$.

27. An $a \times b$ rectangle can be tiled by a number $p \times 1$ and $1 \times q$ tiles of rectangles, where a, b, p, q are fixed positive integers. Prove that a is divisible by p or b is divisible by q. Note that $k \times 1$ and $1 \times k$ are considered to be different tiles.

10 Reference/Further Reading

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