

Mock Olympiad #4

July 9, 2009

1. Let a, b, c, d be positive real numbers such that

$$abcd = 1 \text{ and } a + b + c + d > \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Prove that

$$a + b + c + d < \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{a}{d}.$$

2. In an acute triangle ABC , segments BE and CF are altitudes. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q . Prove that the lines PE and QF intersect on the circumcircle of triangle AEF .
3. For every $n \in \mathbb{N}$, let $d(n)$ denote the number of (positive) divisors of n . Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ with the following properties:
- (a) $d(f(x)) = x$ for all $x \in \mathbb{N}$;
 - (b) $f(xy)$ divides $(x - 1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.