1. Let $p(x) \in \mathbb{Z}[x]$ have positive degree. Prove, for some $c \in \mathbb{Z}$, that $p(c)$ is not prime.

2. Prove that $1 + x + x^2 + x^3 + x^4 \mid 1 + x^{11} + x^{22} + x^{33} + x^{44}$.

3. (USO 1976) Suppose we have $p(x), q(x), r(x), s(x)$ such that $p(x^5) + xq(x^5) + x^2r(x^5) = (1 + x + x^2 + x^3 + x^4)s(x)$. Prove that $x - 1 \mid p(x)$.

4. Let $f(x) = (2 + x^{1957} + x^{1958})^{1959} = \sum_{i=0}^{1958\times 1959} a_i x^i$. Evaluate the sum
$$a_0 - \frac{a_1}{2} - \frac{a_2}{2} + a_3 - \frac{a_4}{2} - \frac{a_5}{2} + a_6 - \frac{a_7}{2} - \frac{a_8}{2} + a_9 \pm \cdots$$

5. (IMO 1977 Longlist) Find all $n \geq 1$ such that there exists a polynomial $p(x) \in \mathbb{Z}[x]$ which satisfies all of the following:
   - $\deg(p) = n$
   - The equation $p(x) = n$ has exactly n (integer) solutions
   - $p(0) = 0$.

6. (IMO 1974) Let $p(x) \in \mathbb{Z}[x]$. Let $n(p)$ equal the number of distinct solutions to $p(x) = \pm 1$. Prove that $n(p) \leq 2 + \deg(p)$. (**Remark**: This is certainly false for $p(x) \in \mathbb{R}[x]$.)

7. Let $a_1, \cdots, a_{2021}$ be distinct integers. Let $h(x) = (x - a_1)^2 \cdots (x - a_{2021})^2 + 1$. Prove that $h(x)$ is irreducible over $\mathbb{Z}$.

8. (IMO 1981 Shortlist) Let $p(z), q(z) \in \mathbb{C}[z]$, each non-constant. Assume that $\{z \in \mathbb{C} : p(z) = 0\} = \{z \in \mathbb{C} : q(z) = 0\}$ and that $\{z \in \mathbb{C} : p(z) = 1\} = \{z \in \mathbb{C} : q(z) = 1\}$. Prove that $p(z) = q(z)$.

9. (IMO 1979 Shortlist) Find all $p(x) \in \mathbb{R}[x]$ which satisfy $p(x)p(2x^2) = p(2x^3 + x)$ for all $x \in \mathbb{R}$.

10. Let $\mathbb{N}$ denote the set of positive integers. Let $p(x) \in \mathbb{Z}[x]$ be such that $p(n) > n$ for all $n \in \mathbb{N}$. Define a sequence by $a_1 = 1$ and $a_{m+1} = p(a_m)$ whenever $m \geq 1$. Suppose that for each $D \in \mathbb{N}$ there exists some $k \in \mathbb{N}$ so that $D \mid a_k$. Prove that $p(x) = x + 1$.

11. (IMO 1976) Let $p_1(x) = x^2 - 2$. For $n \geq 2$, recursively define $p_n(x) = p_1(p_{n-1}(x))$. For each $n \geq 1$, prove that $p_n(x) - x$ only has real roots, and that they are all distinct.

12. (IMO 1985) Given $p(x) \in \mathbb{Z}[x]$, define $w(p)$ to be the number of odd coefficients. For each non-negative integer i, define $q_i(x) = (1 + x)^i$. Consider a sequence of integers $0 \leq b_1 < \cdots < b_n$. Prove that $w(q_{b_1} + \cdots + q_{b_n}) \geq w(q_{b_1})$.

13. (Silly) Prove or Disprove the following conjecture. Let $n \in \mathbb{Z}$ such that $1959 \leq n \leq 2020$. Suppose that n appears in the text of an IMO problem. Then the year of that IMO problem was year n.