Combinatorics

C1 The pirate has five bags of coins with 30 coins each. He knows that one contains gold coins, the other contains silver coins, the third contains bronze coins, and each of the remaining two contains equal parts gold, silver and bronze. You can simultaneously take out any number of coins from any bags and see them (you can take out coins from multiple bags). What is the smallest number of coins you need to take to know for sure the contents of at least one bag?

C2 A field has a shape of a 41x41 square. A tank is concealed in one of the cells of the field. In one shot, a fighter airplane hits one of the cells. If a shot hits the tank, then the tank moves to a neighboring cell of the field, otherwise it stays in its cell (the cells are neighbours if they share a side). A pilot has no information about the tank and needs to hit it twice. Find the least number of shots sufficient to destroy the tank for sure.

C3 Consider a n-sided regular polygon, $n \geq 4$, and let V be a subset of r vertices of the polygon. Show that if $r(r-3) \geq n$, then there exist at least two congruent triangles whose vertices belong to V.

C4 There's a convex $3n$-polygon on the plane with a robot on each of it's vertices. Each robot fires a laser beam toward another robot. On each of your move, you can select a robot to rotate counter clockwise until it's laser points towards a new robot. Three robots A, B and C form a triangle if A's laser points at B, B's laser points at C, and C's laser points at A. Find the minimum number of moves that can guarantee n triangles on the plane.

Number Theory

N1 Let $X = \overline{A373512530649258635292477760B}$ be a 28-digit number, where A and B are non-zero digits, such that X^2 is a palindrome with 55 digits. Find (A, B).

N2 Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that $a_{100}^3 = a_1^3 + a_2^3 + \cdots + a_{99}^3$?

N3 Assume that the polynomial $(x+1)^n - 1$ is divisible by some polynomial $P(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \ldots + c_1 x + c_0$, whose degree k is even and whose coefficients c_{k-1}, c_{k-2}, ..., c_1, c_0 all are odd integers. Show that $k + 1 \mid n$.

N4 Prove that for any natural numbers a, b there exist infinitely many prime numbers p so that $Ord_p(a) = Ord_p(b)$.

Algebra

A1 Express $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$ as a rational number.

A2 Prove that there does not exist an infinite sequence $x_1, x_2, \ldots$ of positive real numbers satisfying the equations $x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n}$ for each natural number n.

A3 Let k be a positive integer, let $m = 2^k+1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that $(P(r))^2 + (Q(r))^2 = -1$.

A4 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers and let $b_1, b_2, b_3, \ldots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n}, \text{ for } n \geq 1$$

Show that, if there exists at least one term among every million consecutive terms of the sequence $b_1, b_2, b_3, \ldots$ that is an integer, then there exists some k such that $b_k > 2022^{2022}$.

Geometry

G1 Let $n \geq 4$ be a natural number. A convex n-gon $A_1A_2 \ldots A_n$ has the following property: $A_{i-1}A_iA_{i+1}$ is isosceles for all i (taking indices mod n, so that $A_{n-1}A_nA_1$ and $A_nA_1A_2$ are also isosceles). Prove that among any four sides of this n-gon, at least two are equal.

G2 Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

G3 A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

G4 A triangular pyramid $ABCD$ is given. A sphere ω_A is tangent to the face BCD and to the planes of other faces in points don't lying on faces. Similarly, sphere ω_B is tangent to the face ACD and to the planes of other faces in points don't lying on faces. Let K be the point where ω_A is tangent to ACD, and let L be the point where ω_B is tangent to BCD. The points X and Y are chosen on the prolongations of AK and BL over K and L such that $\angle CKD = \angle CXD + \angle CBD$ and $\angle CLD = \angle CYD + \angle CAD$. Prove that the distances from the points X, Y to the midpoint of CD are the same.