Inequalities

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Harmonic, Geometric and Arithmetic Mean Inequality:

Let $x_1,\ x_2,\ \dots,\ x_n$ be positive real numbers. Then

$$\frac{x_1 + x_2 + \dots + x_n}{n} \ \geq \ \sqrt[n]{x_1 x_2 \dots x_n} \ \geq \ \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}}.$$

The equalities hold if and only if $x_1 = x_2 = \dots = x_n$.

Useful Consequence:

1. Let $x_1,\ x_2,\ \dots,\ x_n$ be positive real numbers. Then

$$(x_1 + x_2 + \ \dots \ + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ \dots \ + \frac{1}{x_n}\right) \ \geq \ n^2.$$

Rearrangement Inequality:

Consider two collections of real numbers in increasing order,

$$a_1 \leq a_2 \leq \dots \leq a_n \qquad b_1 \leq b_2 \leq \dots \leq b_n.$$

For any permutation $(a'_1,\ a'_2,\ \dots,\ a'_n)$ of $(a_1,\ a_2,\ \dots,\ a_n)$, we have

$$a_1b_1 + a_2b_2 + \dots + a_nb_n \ \geq \ a'_1b_1 + a'_2b_2 + \dots + a'_nb_n \ \geq \ a_nb_1 + a_{n-1}b_2 + \dots + a_1b_n.$$

Moreover, the first equality holds if and only if $(a'_1,\ a'_2,\ \dots,\ a'_n) = (a_1,\ a_2,\ \dots,\ a_n)$, and the second equality holds if and only if $(a'_1,\ a'_2,\ \dots,\ a'_n) = (a_n,\ a_{n-1},\ \dots,\ a_1)$.

Useful Consequence:

2. For any permutation $(a'_1,\ a'_2,\ \dots,\ a'_n)$ of $(a_1,\ a_2,\ \dots,\ a_n)$, it follows that

$$\frac{a'_1}{a_1} + \frac{a'_2}{a_2} + \dots + \frac{a'_n}{a_n} \ \geq \ n.$$

3. Tchebyshev's Inequality: Let $a_1 \leq a_2 \leq ... \leq a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$. Then

$$\frac{a_1b_1 + a_2b_2 + \ ... \ + a_nb_n}{n} \ \geq \ \frac{a_1 + a_2 + \ ... \ + a_n}{n} \ \cdot \ \frac{b_1 + b_2 + \ ... \ + b_n}{n}.$$

The equality holds when $a_1 = a_2 = \ ... \ = a_n$ or $b_1 = b_2 = \ ... \ = b_n$.

4. Nesbitt's Inequality: For positive real numbers a, b, and c, we have that

$$\frac{a}{b+c} \ + \ \frac{b}{c+a} \ + \ \frac{c}{a+b} \ \geq \ \frac{3}{2}.$$

5. Quadratic Mean-Arithmetic Mean Inequality: Let $x_1, x_2, \ ... \ , x_n$ be positive real numbers. Then

$$\sqrt{\frac{x_1^2 + x_2^2 + ... + x_n^2}{n}} \ \geq \ \frac{x_1 + x_2 + ... + x_n}{n}.$$

Cauchy-Schwartz Inequality:

For real numbers $x_1, x_2, \ ... \ , x_n, y_1, y_2, \ ... \ , y_n$, the following inequality holds:

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \ \leq \ \left(\sum_{i=1}^{n} x_i^2\right) \left(\sum_{i=1}^{n} y_i^2\right).$$

The equality holds if and only if there exists some $c \in \mathbb{R}$ with $x_i = cy_i$ for all $i = 1, 2, \ ... \ , n$.

Useful Consequence:

6. Cauchy-Schwartz Inequality in Engel form: For all real numbers $a_1, a_2, ..., a_n$ and positive real numbers $x_1, x_2, \ ... \ , x_n$,

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \ ... \ + \frac{a_n^2}{x_n} \ \geq \ \frac{(a_1 + a_2 + \ ... \ + a_n)^2}{x_1 + x_2 + ... + x_n},$$

with equality if and only if

$$\frac{a_1}{x_1} = \frac{a_2}{x_2} = \ ... \ = \frac{a_n}{x_n}.$$

Schur's Inequality:
If x, y, z are positive real numbers and n is a positive integer, then

$$x^n(x-y)(x-z) \ + \ y^n(y-z)(y-x) \ + \ z^n(z-x)(z-y) \ \geq \ 0.$$

For the case $n = 1$, the inequality can take one of the following forms:

(a) $x^3 \ + \ y^3 \ + \ z^3 \ + \ 3xyz \ \geq \ xy(x+y) \ + \ yz(y+z) \ + \ zx(z+x)$

(b) $xyz \ \geq \ (x+y-z)(y+z-x)(z+x-y)$

(c) If $x+y+z=1$, $9xyz+1 \ \geq \ 4(xy+yz+zx)$

Definition: A function $f : [a,b] \to \mathbb{R}$ is called **convex** in the interval $I = [a,b]$ if for any $t \in [0,1]$ and for all $a \leq x < y \leq b$, the following inequality holds:

$$f(ty+(1-t)x) \ \leq \ tf(y) \ + \ (1-t)f(x).$$

Note that if f is twice differentiable and $f''(x) \geq 0$, then the function is convex.

Jensen's Inequality:

If f is convex in $[a,b]$, then for any $t_1, t_2, \ \ldots \ , t_n \in [0,1]$, with $\sum_{i=1}^n t_i = 1$, and for $x_1, x_2, \ \ldots \ , x_n \in [a,b]$, we have that

$$f(t_1x_1 \ + \ t_2x_2 \ + \ \ldots \ + \ t_nx_n) \ \leq \ t_1f(x_1) \ + \ t_2f(x_2) \ + \ \ldots \ + \ t_nf(x_n).$$

Useful Consequence:

7. Weighted AM-GM Inequality: If $x_1, x_2, \ \ldots \ , x_n, t_1, t_2, \ \ldots \ , t_n$ are positive numbers and $\sum_{i=1}^n t_i = 1$, then

$$x_1^{t_1} \, x_2^{t_2} \ \ldots \ x_n^{t_n} \ \leq \ t_1x_1 \ + \ t_2x_2 \ + \ \ldots \ + \ t_nx_n.$$

8. Holder's Inequality: Let $x_1, x_2, \ \ldots \ , x_n, y_1, y_2, \ \ldots \ , y_n$ be positive numbers and $a, b > 0$ such that $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\sum_{i=1}^n x_iy_i \ \leq \ \Big(\sum_{i=1}^n \ x_i^a\Big)^{1/a} \Big(\sum_{i=1}^n \ y_i^b\Big)^{1/b}.$$

Problems:

1. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{1}{a_i b_i} \sum_{i=1}^{n} (a_i + b_i)^2 \geq 4n^2.$$

2. (Russia, 1991) For all non-negative real numbers x, y, z, prove that

$$\frac{(x+y+z)^2}{3} \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}.$$

3. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $s = a_1 + a_2 + \ldots + a_n$. Prove that

$$\frac{s}{s-a_1} + \frac{s}{s-a_2} + \ldots + \frac{s}{s-a_n} \geq \frac{n^2}{n-1}.$$

4. For positive real numbers a, b, c such that $a+b+c=1$, prove that

$$ab + bc + ca \leq \frac{1}{3}.$$

5. (Russia, 1992) For any real numbers $x, y > 1$, prove that

$$\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8.$$

6. Let x, y, z be positive real numbers. Prove that

$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \geq \frac{9}{x+y+z}.$$

7. Let x, y, z be positive real numbers. Prove that

$$\frac{x^3}{x^3+2y^3} + \frac{y^3}{y^3+2z^3} + \frac{z^3}{z^3+2x^3} \geq 1.$$

8. (Russia, 2004) If $n > 3$ and $x_1, x_2, \ldots, x_n$ are positive real numbers with $x_1 x_2 \ldots x_n = 1$, prove that

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \ldots + \frac{1}{1+x_n+x_nx_1} > 1.$$

9. (China, 1989) Let $a_1, a_2, \dots, a_n$ be positive real numbers such that $a_1 + a_2 + \dots + a_n = 1$. Prove that

$$\frac{a_1}{\sqrt{1-a_1}} + \dots + \frac{a_n}{\sqrt{1-a_n}} \geq \frac{1}{\sqrt{n-1}} (\sqrt{a_1} + \dots + \sqrt{a_n}).$$

10. (Romania, 2008) Let a, b, c be positive real numbers with $abc = 8$. Prove that

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leq 0.$$

11. (Poland, 2006) Let a, b, c be positive real numbers such that $ab + bc + ca = abc$. Prove that

$$\frac{a^4+b^4}{ab(a^3+b^3)} + \frac{b^4+c^4}{bc(b^3+c^3)} + \frac{c^4+a^4}{ca(c^3+a^3)} \geq 1.$$

12. (Russia, 2000) For real numbers x, y such that $0 \leq x, y \leq 1$, prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+xy}}$$

13. If a, b, c are positive real numbers, prove that

$$\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}.$$

14. Prove that if I is an interval and $f : I \to \mathbb{R}$ is a convex function, then for $a, b, c \in I$ the following inequality holds:

$$\frac{2}{3}\left[f\Big(\frac{a+b}{2}\Big) + f\Big(\frac{b+c}{2}\Big) + f\Big(\frac{c+a}{2}\Big) \right] \leq \frac{f(a)+f(b)+f(c)}{3} + f\Big(\frac{a+b+c}{3}\Big).$$

Hints:

Make sure you spend a good amount of time thinking about a problem before looking at its hint!

1. Use the fact that $(a+b)^2 \geq 4ab$ and consequence 1 to AM-GM inequality.

2. Apply AM-GM to $\{xy, yz, zx\}$ and to $\{x^2, x^2, y^2, z^2\}$.

3. Apply the rearrangement inequality $n-1$ times to $\{a_1, a_2, \dots, a_n\}$ and $\{\frac{1}{s-a_1}, \frac{1}{s-a_2}, \dots, \frac{1}{s-a_n}\}$.

4. Use the Quadratic Mean -Arithmetic Mean inequality and the expansion of $(a+b+c)^2$.

5. Use the AM-GM inequality.

6. Use Cauchy-Schwarz in Engel form.

7. Use the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$, and Cauchy-Schwarz in Engel form.

8. Use the substitution $x_1 = \frac{a_2}{a_1}$, $x_2 = \frac{a_3}{a_2}$, $\dots$, $x_n = \frac{a_1}{a_n}$.

9. Note that $\sum_{i=1}^{n} \frac{a_i}{\sqrt{1-a_i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{1-a_i}} - \sum_{i=1}^{n} \sqrt{1-a_i}$.

10. Use the substitution $a = \frac{2x}{y}$, $b = \frac{2y}{z}$, $c = \frac{2z}{x}$. You will also need Cauchy-Schwarz in Engel form.

11. Use the substitution $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$, and Tchebyshev's inequality.

12. Write $x = e^{-u}$ and $y = e^{-v}$ and use Jensen's Inequality.

13. Note that the inequality is homogeneous and try to use Jensen's Inequality.

14. Try to write $\frac{c+a}{2}$ and $\frac{b+c}{2}$ in the form $\lambda c + (1-\lambda)(\frac{a+b+c}{3})$, and use Jensen's Inequality.

Reference: This handout was made using the book *Inequalities* by Radmila Bulajich Manfrino, Jose Antonio Gomez Ortega, and Rogelio Valdez Delgado.