Transformations: Rotations and Inversions

1. (IMO 2014, #4) Points P and Q lie on side BC of acute-angled triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the lines BM and CN intersect on the circumcircle of triangle ABC.
2. (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle CBA = \angle DCA = \angle EDA$. Diagonals BD and CE meet at P. Prove that line AP bisects side CD.
3. (China 1992) Convex quadrilateral ABCD is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90°$.
4. Let ABCD be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M, N and O be the midpoints of AC, BD and O_1O_2. Show that O is the circumcenter of MPN.
5. (USAMO 1993/2). Let ABCD be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.
6. Points A,B,C are given on a line in this order. Semicircles ω,ω1,ω2 are drawn on AC,AB, BC respectively as diameters on the same side of the line. A sequence of circles k_n is constructed as follows: k_0 is the circle determined by ω2 and k_n is tangent to $\omega, \omega_1, k_{n-1}$ for $n \geq 1$. Prove that the distance from the center of k_n to AB is 2n times the radius of k_n.
7. (USAMO 1995,2) Given a non-isosceles, non-right triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that OAA_1 is similar to OA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent.
8. (IMO 2015/3). Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.