Incomplete Information

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1 Introduction

Microeconomics is all just game theory, according to my economics professor last semester. Game theory in economics, at least at the first-year level, deals a lot with complete information, but games are so much more interesting when we have to extrapolate from incomplete information.

In these games, you want to make sure that each move gives you as much information as possible, in order to guarantee that you land at the answer in the quickest manner. Also, unlike some other types of combo problems, it is extremely important to find some global (as opposed to local) structure/properties for these problems as you cannot solve them using induction or other local methods. Then, to solve the problem, it usually requires an exploitation of the global structure to get as much information from each query as possible for querying problems or be able to find a bijective mapping (or as close as possible to a bijection) from each set of outcomes to starting configurations.

2 Examples

2.1 First Example

We have an invisible space ship in n-dimensional space that starts at some rational point with some constant rational velocity (each component is rational). Prove that, if every second, we can make a guess as to a random point where the spaceship will be, we can guess in a way that guarantees we succeed in finite time.

Solution:
This problem is equivalent to saying that the number of distinct combinations of starting points and velocities is "countable" (can be injectively mapped to the set $\mathbb{N}$). This is because we can order the guesses as guess 0, guess 1, $\cdots$, where at each guess, we guess the point the spaceship would be given this starting point and velocity. If the actual starting point and velocity maps to some finite number k, then the k-th guess will be correct, so if we can find such a mapping, we'd solve the problem.

Now, the key lemma that proves the problem is this: The number of pairs (a, b), where $a \in A, b \in B$ and A, B are both countable sets, is countable.
Proof: Since A, B are countable, we can refer to each element of these sets as the i-th element. Thus, every pair (a, b) can be mapped to some pair (x, y) where $x, y \in \mathbb{N}$. Here is a nice way of counting all pairs of natural numbers: First count the number of pairs which sum to 0 starting from $(0, 0)$ and going to $(0, 0)$, then the number of pairs which sum to 1 starting from $(0, 1)$ and going to $(1, 0)$, and so on. This covers all pairs of natural numbers exactly once and in finite time, so it proves the lemma.

Now, we can repeatedly apply this lemma to solve the problem. The rationals are just pairs of natural numbers, the numerator and the denominator, so it's countable. Similarly, in n dimensional space, repeating this lemma proves that all n-tuples of rationals is countable, so the number of starting points and velocities is countable. Finally, this means the number of pairs of starting points and velocities is countable, so we're done.

Remark: This problem shows how to solve incomplete information problems by finding a global mapping (in this case to $\mathbb{N}$).

2.2 Second Example

A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \ldots, 13$. The audience chooses some five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges the other four cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, he has to determine the face down card (both suite and number). Explain how the trick can be completed.

Solution:
By pigeonhole, there must be two cards of the same suit. Pick two cards from this suit. Since the rank goes from 1 to 13, the two ranks r_1, r_2 has either $r_1 + d \equiv r_2 \pmod{13}$ or $r_2 + d \equiv r_1 \pmod{13}$ for some $1 \leq d \leq 6$. WLOG $r_1 + d \equiv r_2 \pmod{13}$. Now, the first part of the strategy is to always put the card with rank r_2 face down. Now, he rearranges the other 4 cards such that the left-most card with rank r_1 (and the same suite as the face-down card). Now, it suffices to use the ordering of the other 3 cards to specify the value of d. This can be done by just considering the "size" of the last 3 cards by assigning each card in the deck a number from 1 to 52 beforehand. Now, there are 6 different ways to order these 3 cards which can correspond to an integer from 1 to 6. For example, ordering the last three cards from smallest to largest gives $d = 1$ while ordering them from largest to smallest gives $d = 6$, and other orderings correspond to integers d from 2 to 5.

Remark: This question is another one but now based on finite mapping. The key part is noticing that we have only 4 suites but 5 cards, so we can use the first face-up card to tell the magician the suite of the face-down card. Similarly, the ordering of the last three face-up cards can correspond to the rank of the face-down card, but there are 12 other possible ranks and only 6 unique orderings. This motivates the relative rank system.

2.3 Third Example

Let's say you are trying to get from intersection X to intersection Y in a grid street layout with north-south streets being significantly wider than east-west streets. There are traffic lights at every intersection. What is the best way of going from X to Y?

Solution: First, this problem is not from an actual contest, but from one of Po-Shen Loh's handouts. Essentially, we model this as the traffic light being randomly determined when we arrive at an intersection, but it stays green longer for the wider street. The strategy is to always start walking towards Y by first walking east-west. This means every time we get to an intersection, we either cross if possible, or move 1 block north-south (if that gets us closer to the destination) and then attempt to cross there. This is the optimal method because it gives us as many attempts to cross the wider streets before we actually have to wait for the light to change, and crossing the wider streets is harder than crossing the smaller streets.

Remark: This problem is obviously not a contest problem, but it is a good example of how we actually encounter these incomplete information problems in daily life. It shows how we can apply mathematical theory to get to our destination faster (at least decrease the expected time it takes to arrive).

2.4 Fourth Example

We have 13 coins such that 12 have equal weight and the last one has a different weight. We have a balance that can be used 3 times. Devise a strategy to find the coin with different weight.

Solution: First, we eliminate moves that don't give any information. Let L,R,C represent when the balance tilts to the left, right, or is balanced. This includes LLL or RRR as they will never tell us which one it's in (how do we know if it's the coin that's always on the left side, or the coin that's always on the right side). Also, we notice that the complement of a set of results gives you the complement of the original coins, so if one is lighter, it should be heavier. This means that we must have the following mapping.

So now, we can consider the following table, where L, C, R represents the heavier side. Also, I numbered the coins from 1 to 13

RLL 1 light	LRR 1 heavy
LRL 2 light	RLR 2 heavy
LLR 3 light	RRL 3 heavy
RLC 4 light	LRC 4 heavy
CRL 5 light	CLR 5 heavy
LCR 6 light	RCL 6 heavy
RRC 7 light	LLC 7 heavy
CRR 8 light	CLL 8 heavy
RCR 9 light	LCL 9 heavy
LCC 10 light	RCC 10 heavy
CLC 11 light	CRC 11 heavy
CCL 12 light	CCR 12 heavy
CCC 13 either lighter or heavier	

Now, we just have to map these to possible outcomes which we can do as follows:
First: 1,4,7,9 - 2,3,6,10
Second: 2,5,7,8 - 1,3,4,11
Third: 3,6,8,9 - 1,2,5,12
It's clear that these balances ensure that each outcome follows the mapping specific in the table above.

Remark: Again, this problem first requires you to think about some mapping argument of each outcome to a specific starting configuration. Yet, this one also needs some insights into what information is conveyed from each outcome and the structure of the problem. A lot of the harder problems will require something similar to this, where you first have to get some insight into the structure of the problem and then devise a mapping/strategy to solve it.

3 Problems

Recommended problems are:

- 1,2,4,5,6,8,9,12,13,14

1. Prior to the game John selects an integer greater than 100. Then Mary calls out an integer $d > 1$. If John's integer is divisible by d, then Mary wins. Otherwise, John subtracts d from his number and the game continues (with the new number). Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Does Mary have a winning strategy?

2. Adithya and Bill are playing a game on a connected graph with $n > 2$ vertices, two of which are labeled A and B, so that A and B are distinct and non-adjacent and known to both players. Adithya starts on vertex A and Bill starts on B. Each turn, both players move simultaneously: Bill moves to an adjacent vertex, while Adithya may either move to an adjacent vertex or stay at his current vertex. Adithya loses if he is on the same vertex as Bill, and wins if he reaches B alone. Adithya cannot see where Bill is, but Bill can see where Adithya is. Given that Adithya has a winning strategy, what is the maximum possible number of edges the graph may have? (Your answer may be in terms of n.)

3. Given 365 cards, in which distinct numbers are written. We may ask for any three cards, the order of numbers written in them. Is it always possible to find out the order of all 365 cards by 2000 such questions?

4. The King called two wizards. He ordered First Wizard to write down 100 positive real numbers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these numbers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct numbers, each of which is either one of the numbers on the cards or a sum of some of these numbers. He is not allowed to tell which numbers are on the cards and which numbers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of numbers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

5. The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least 22 cards?

6. Now, consider the previous question, but after each guess, the magician can see the suit of the card. Prove that he can now guess at least 24 correct.

7. You are tasked with surveying a town of N people. Your goal is to identify all of the town's celebrities. A celebrity is a person that is known by everyone but knows no one. You are only allowed to ask questions of the form "Does A know B?" where A and B are distinct people (bear in mind that knowing is directional, so A knowing B doesn't imply that B knows A). Find with proof the minimum number of questions that you must ask to guarantee that you can identify all of the town's celebrities?

8. A magician and his assistant present the following trick. Thirteen empty closed boxes are placed in a row. Then, the magician leaves the stage, and a random person from the audience is selected to put two coins into two boxes of their choice, one coin in each box, in front of the magician's assistant, i.e. the assistant knows which boxes contain coins. Then, the magician returns and his assistant is allowed to open one box that does not contain a coin. After that the magician must choose four boxes to be opened simultaneously. The goal of the magician is to open both boxes with coins. Construct a scheme by which the magician and his assistant can perform the trick successfully every time.

9. a) There are $2n+1$ $(n>2)$ batteries. We don't know which batteries are good and which are bad but we know that the number of good batteries is greater by 1 than the number of bad batteries. A lamp uses two batteries, and it works only if both of them are good. What is the least number of attempts sufficient to make the lamp work?

   b) The same problem but the total number of batteries is $2n$ $(n>2)$ and the numbers of good and bad batteries are equal.

10. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn.

    How many ways are there to put the cards in the three boxes so that the trick works?

11. Let T be the set of ordered triples (x,y,z), where x,y,z are integers with $0 \leq x,y,z \leq 9$. Players A and B play the following guessing game. Player A chooses a triple (x,y,z) in T, and Player B has to discover A's triple in as few moves as possible. A move consists of the following: B gives A a triple (a,b,c) in T, and A replies by giving B the number $|x+y-a-b|+|y+z-b-c|+|z+x-c-a|$. Find the minimum number of moves that B needs to be sure of determining A's triple.

12. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

13. Alice has a map of Wonderland, a country consisting of $n \geq 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

    Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

14. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

1. If $n \geq 2^k$, then B can guarantee a win.
2. For all sufficiently large k, there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.

4 Hints

1. Consider the possible remainders modulo 12
2. The only information that Adithya knows is Bill always moves to a new vertex.
3. Divide and Conquer
4. Powers of 2
5. Can the assistant help "optimize" the guessing of the magician for cards that he must guess?
6. For this, we can group cards into groups of 3.
7. Consider how the sets of people who can still be a celebrity and those who cannot changes over time.
8. Think about the second example involves differences between ranks. Can we use differences here to create some bijective map?
9. Structure the problem as a graph theory problem!
10. Induction!
11. Can you prove that it's impossible to do in 2 moves?
12. How can the rabbit give the impression that it is moving along a line when it actually isn't?
13. What is the worst final position do we have to brute force at? How do we avoid this?
14. We need to find some way of structuring this problem so that A always knows when to lie or not.