0.1 Diophantine Equations from 2021 Olympiads

Problem 0.1.1. *Call a positive integer n "Fantastic" if none of its digits are zero and it is possible to remove one of its digits and reach to an integer which is a divisor of n . (for example , 25 is fantastic , as if we remove digit 2 , resulting number would be 5 which is divisor of 25) Prove that the number of Fantastic numbers is finite.*

Problem 0.1.2. *Find all triples (a, b, c) of natural numbers a, b and c, for which $a^{b+20}(c-1) = c^{b+21} - 1$ is satisfied.*

Problem 0.1.3. *Find all pairs of integers (a, b) so that each of the two cubic polynomials*

$$x^3 + ax + b \text{ and } x^3 + bx + a$$

has all the roots to be integers.

Problem 0.1.4. *Given are positive integers $n > 20$ and $k > 1$, such that k^2 divides n. Prove that there exist positive integers a, b, c, such that $n = ab + bc + ca$.*

Problem 0.1.5. *Find all triples (x, y, z) of positive integers such that*

$$x^2 + 4^y = 5^z.$$

Problem 0.1.6. *Given positive integers a, b, c which are pairwise coprime. Let $f(n)$ denotes the number of the non-negative integer solution (x, y, z) to the equation*

$$ax + by + cz = n.$$

Prove that there exists constants $\alpha, \beta, \gamma \in \mathbb{R}$ such that for any non-negative integer n,

$$|f(x) - \left(\alpha n^2 + \beta n + \gamma\right)| < \frac{1}{12}(a + b + c).$$

Problem 0.1.7. *Let n be an integer greater than 1 such that n could be represented as a sum of the cubes of two rational numbers, prove that n is also the sum of the cubes of two non-negative rational numbers.*

Problem 0.1.8. *Determine whether there are infinitely many triples (u, v, w) of positive integers such that u, v, w form an arithmetic progression and the numbers $uv + 1, vw + 1$ and $wu + 1$ are all perfect squares.*