# Sequences – Bounding and Combinatorics

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Generally, a sequence problem starts with a function $f : \mathbb{N} \to \mathbb{R}$ with some properties, typically divisibility relations, inequalities, and sometimes (conditional) equalities. In many problems, the entire sequence can be defined uniquely from the initial terms – in others, the conditions imposed on the sequence leave plenty of possibilities.

Alex Remorov's article (found at http://www.mit.edu/∼alexrem/Sequences.pdf) covers many examples of sequences defined recursively by their initial values. Usually, for easier problems, a nice closed form for the entire sequence can be determined. On the other hand, it would be important to find the key properties hidden within the definition.

This article will focus on questions where a sequence with some properties that are not equalities are given. A common goal of these problems (and precisely what many *functional equation* problems ask) is a closed form for the sequence. On the other hand, for other questions the closed form is not so useful because there are too many possible sequences – usually those questions ask to prove another property of the sequence given some other properties.

One common technique is to consider something *extermal*. Things to consider include (but are not limited to):

- When trying to prove something, assume contradiction and give a minimal counterexample.
- Consider a minimal term that satisfies some nice property (especially if the sequence properties include conditional equalities).
- Consider a minimum value (or maximum, if bounded) of the range of the sequence.
- Consider the first (or last, if definitely finitely many) time a critical equality holds.

Additionally, in number theoretic-flavoured problems (involving divisiblity and number-theoretic functions), it could be useful to consider primes and numbers with few divisors to give stronger bounds.

# 1 Example Problems and Solutions

**Problem:** (IMO Shortlist 2015) Determine all positive integers $M$ such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for}\, k = 0, 1, 2, \cdots$$

contains at least one integer term.

**Solution:** Assume $M - 1 = 2^N \cdot P$, where $N$ is a nonnegative integer and $P$ is odd. Then, $a_1 = M(M + \frac{1}{2}) = \frac{1}{2} + 2^{2N} \cdot P^2 + 5 \cdot 2^{N-1} \cdot P + 1 = \frac{1}{2} + 1 + (2^{N-1})(2^{N+1}P^2 + P)$. If $N = 0$, we are done as this is an integer. Otherwise, we reduced it to the case where $N$ is one less.

**Problem:** (IMO Shortlist 2004) Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$\left(f(m)^2 + f(n)\right) \mid \left(m^2 + n\right)^2$$

for any two positive integers $m$ and $n$.

**Solution:** Start with plugging in $m, n = 1$, which gives $f(1) = 1$ immediately. Now, for every prime $p$, we can plug in $m = 1, n = p - 1$, which gives $f(p - 1) = p - 1$ or $f(p - 1) = p^2 - 1$. In the latter case, plugging in $m = p^2 - 1, n = 1$ gives a contradiction.

To finish the problem, let $m = p - 1$, and then we get $(p - 1)^2 + f(n)|((p - 1)^2 + n)^2$, which implies that $(p - 1)^2 + f(n)|(f(n) - n)^2$. Now we get that $f(n) = n$ by taking $p$ sufficiently large.

**Problem:** (IMO Shortlist 2001) Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}(1 + n^{-1})$ holds for infinitely many positive integers $n$.

**Solution:** Proceed by contradiction. Assume that for all $n > N$, we have $1 + a_n < a_{n-1}(1 + n^{-1})$. Now $a_{N+1} < a_N \left(\frac{N+2}{N+1}\right) - 1$, $a_{N+2} < a_N \left(\frac{N+3}{N+1}\right) - \frac{N+3}{N+2} - 1$, and more generally

$$a_{N+M} < a_N \left(\frac{N+M+1}{N+1}\right) - \left(\sum_{i=1}^{M} \frac{N+M+1}{N+i+1}\right)$$

Now we finish by showing that this becomes negative eventually.

## 2 Inequalities

1. (Matt Brennan) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions $f$, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer $n$.

2. (IMO 2014) Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

3. (Canada 2000) Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$\begin{aligned} 0 \leq a_{100} \leq a_{99} \leq \cdots \leq a_2 &\leq a_1, \\ a_1 + a_2 &\leq 100 \\ a_3 + a_4 + \cdots + a_{100} &\leq 100. \end{aligned}$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

4. (IMO Shortlist 2008) Let $a_1$, $a_2$, …, $a_n$ be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices $i$ and $j$ such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, …, $3a_n$.

5. (USA 2007) Let $n$ be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting $a_k$ be the unique integer in the range $0 \leq a_k \leq k-1$ for which $a_1 + a_2 + ... + a_k$ is divisible by $k$. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, ....$ Prove that for any $n$ the sequence $a_1, a_2, ...$ eventually becomes constant.

6. (IMO Shortlist 2015) Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence $a_n$ is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

7. (USA 1997) Suppose the sequence of nonnegative integers $a_1, a_2, \ldots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number $x$ such that $a_n = \lfloor nx \rfloor$ (the greatest integer $\leq nx$) for all $1 \leq n \leq 1997$.

8. (IMO Shortlist 2002) Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number $c$ with $0 \leq a_i \leq c$ for all $i$, such that

$$|a_i - a_j| \geq \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \neq j.$$

Prove that $c \geq 1$.

9. (IMO Shortlist 2008) Let $a_0,\ a_1,\ a_2,\ \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

10. (Iran 2009) Let $a_1 < a_2 < \cdots < a_n$ be positive integers such that for every distinct $1 \leq i, j \leq n$ we have $a_j - a_i$ divides $a_i$. Prove that

$$ia_j \leq ja_i \qquad \text{for } 1 \leq i < j \leq n$$

11. (IMO Shortlist 2007) Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

12. (IMO Shortlist 2013) Let $n$ be a positive integer, and consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \cdots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

13. (IMO Shortlist 2007) Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \geq 1,$$

and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

14. (IMO 2015) The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

(i) $1 \leq a_j \leq 2015$ for all $j \geq 1$, (ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

Prove that there exist two positive integers $b$ and $N$ for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

for all integers $m$ and $n$ such that $n > m \geq N$.

## 3 Not Inequalities

1. (IMO Shortlist 2005) Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer $n$ the numbers $a_1, a_2, \ldots, a_n$ leave $n$ different remainders upon division by $n$. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

2. (Canada 2008) Determine all functions $f$ defined on the natural numbers that take values among the natural numbers for which
$$(f(n))^p \equiv n \mod f(p)$$
for all $n \in \mathbf{N}$ and all prime numbers $p$.

3. (USA 2012) Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of nonzero integers such that the equality
$$a_k + 2a_{2k} + \ldots + na_{nk} = 0$$
holds for every positive integer $k$.

4. (IMO Shortlist 2004) Let $a_0,\ a_1,\ a_2,\ \ldots$ be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where $a_0$ and $a_1$ are two different positive reals. Prove that the sequence is unbounded.

5. (USA 1995) Suppose $q_0,\ q_1,\ q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

   (i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$, (ii) there is a polynomial $P$ such that $|q_n| < P(n)$ for all $n$

   Prove that there is a polynomial $Q$ such that $q_n = Q(n)$ for all $n$.

6. (IMO Shortlist 2012) Let $r$ be a positive integer, and let $a_0, a_1, \cdots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers $m$ and $s$ there exists a positive integer $n \in [m+1, m+r]$ such that
$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}$$
Prove that the sequence is periodic, i.e. there exists some $p \geq 1$ such that $a_{n+p} = a_n$ for all $n \geq 0$.

7. (Iran TST 2008) $k$ is a given natural number. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ the following holds:
$$f(m) + f(n) | (m+n)^k$$

8. (Iran 2011) Find all increasing sequences $a_1, a_2, a_3, \ldots$ of natural numbers such that for each $i, j \in \mathbb{N}$, number of the divisors of $i + j$ and $a_i + a_j$ is equal. (an increasing sequence is a sequence that if $i \leq j$, then $a_i \leq a_j$.)

9. (IMO Shortlist 2007) Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime $p$, the number $f(m+n)$ is divisible by $p$ if and only if $f(m) + f(n)$ is divisible by $p$.