LINEAR ALGEBRA IN OLYMPIADS

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1. Introduction

Linear algebra is the study of finite abelian groups. Linear algebra is also the study of linear maps.

Linear algebra comes in handy in olympiad problems in many scenarios. The short list (in order of popularity) goes something like:

(1) Incidences of sets via linear algebra over $\mathbb{F}_2$.

(2) Problems where larger configurations are "built up" by "linear combinations" of smaller units.

(3) Actual linear algebra (kernels, dimension arguments, etc.) This sort of argument works when you have a situation where you have a lot of linear constraints, and a very small number of unknowns.

2. Linear Algebra

Here's a brief theoretical minimum. Things mostly behave how you expect them to behave. For more, check out Evan Chen's *Napkin* or, for a more exhaustive treatment, Sheldon Axler's *Linear Algebra Done Right*.

A **vector space** over a field K is a set V equipped with a binary operation $+$ such that:

(a) There is an additive identity $0 \in V$ such that for any $v \in V$ we have $0+v = v+0 = v$.
(b) The operation $+$ is commutative.
(c) The operation is associative $(a+b)+c = a+(b+c)$ for any $a, b, c \in V$.
(d) Additive inverses exist: for each $v \in V$, there is an element we denote as $-v$ such that $v + (-v) = 0$.

We can also multiply vectors in V by element of K to obtain a new vector in V: this is called multiplication by a scalar. Scalar multiplication satisfies the following properties:

(a) Scalar multiplication is compatible with field multiplicaction: $a(bv) = (ab)v$.
(b) The identity is preserved: $1v = v$.
(c) It distributes over vector addition: $a(u+v) = au + av$.
(d) It distributes over scalar addition: $(a+b)v = av + bv$.

The really important idea is the notion of **dimension** of a vector space. Roughly, it is the "size" of your vector space; you can also think about it as the number of degrees of freedom you have.

Formally, we need the following definitions.

A set of vectors $\{v_1, \ldots, v_k\}$ is said to be **linearly independent** if $\sum_i a_i v_i = 0$ implies that all a_i are equal to 0.

A set of vectors S are said to **span** the vector space V if any vector in V can be written as a linear combination of vectors in S.

It's an exercise to show that if m vectors span V, and there are n linearly independent vectors, then $m \geq n$.

It follows that every set that is both linearly independent and spans V has the same cardinality, which we call the dimension of V. We call such a set a **basis**.

These facts combined often provide useful inequalities.

The last important idea is that of linear maps and dimensions. If we have a vector space V of dimension n, and a system of n equations in n unknowns, each of which can be thought of as the coordinate of a basis vector, then we can treat the resulting matrix as a linear map $A : V \to V$. The set of vectors v such that $Av = 0$ is called the **kernel**, and its dimension is the **nullity**. Likewise, the dimension of the image of the map A is called the **rank**. We have the famous *rank-nullity theorem*:

$$\text{rank} + \text{nullity} = n.$$

3. Examples

What does any of this have to do with solving combinatorics problems? Hopefully, the examples make it all very clear.

Lecture Problem 3.1 (Oddtown)

In a certain town with n citizens, a number of clubs are set up. No two clubs have exactly the same set of members. Determine the maximum number of clubs that can be formed if every club has an odd number of members, and every pair of clubs share an even number of members.

Lecture Problem 3.2

Let $a_1, a_2, \ldots, a_{2n+1}$ be real numbers, such that for any $1 \leq i \leq 2n+1$, we can remove a_i and separate the remaining $2n$ numbers into two groups of n numbers with equal sums. Show that $a_1 = a_2 = \ldots = a_{2n+1}$.

Lecture Problem 3.3

Let $S_1, \dots, S_n$ be subsets of $\{1, 2, \dots, n\}$ each containing an even number of elements. Prove that there exists i, j such that $|S_i \cap S_j|$ is even.

Lecture Problem 3.4

At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

4. Problems

Problem 4.1 (Putnam 2003)**.** Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Problem 4.2 (TSTST 2014)**.** Let $P(x)$ and $Q(x)$ be arbitrary polynomials with real coefficients, and let d be the degree of $P(x)$. Assume that $P(x)$ is not the zero polynomial. Prove that there exist polynomials $A(x)$ and $B(x)$ such that:

(1) both A and B have degree at most $d/2$

(2) at most one of A and B is the zero polynomial.

(3) $\frac{A(x)+Q(x)B(x)}{P(x)}$ is a polynomial with real coefficients. That is, there is some polynomial $C(x)$ with real coefficients such that $A(x) + Q(x)B(x) = P(x)C(x)$.

Problem 4.3. A *derangement* is a permutation with no fixed points. For which integers $n \geq 2$ is the number of derangements of $\{1, 2, \dots, n\}$ even?

Problem 4.4 (Non-uniform Fisher inequality)**.** Let $A_1, \dots, A_m$ be distinct subsets of $\{1, 2, \dots, n\}$. Suppose that there is an integer $1 \leq \lambda < n$ such that $|A_i \cap A_j| = \lambda$ for all $i \neq j$. Prove that $m \leq n$.

Problem 4.5. Let G be a graph with n elements. For every pair of vertices in G, there are an even number of vertices sharing edges with both of them. Prove that n is odd.

Problem 4.6. There are $2n$ people at a party. Each person has an even number of friends at the party. (Here, friendship is a mutual relationship.)

Prove that there are two people who have an even number of common friends at the party.

Problem 4.7 (CMO 2010)**.** In a finite simple graph G, all vertices can be black or red. In a move, you can pick a vertex v and toggle the colors of v and its neighbors. Initially, all vertices are black; prove that it's possible to make all vertices red.

Problem 4.8 (APMO 2017 adapted)**.** We call a 5-tuple of *real numbers* arrangeable if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, ..., n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

Problem 4.9. Let $n \geq 2$ be an even integer. There are n boys and n girls at a dance party. Each girl dances with an even number of boys. Prove that there exist two girls for which an even number of boys danced with both.

Problem 4.10 (Russia 1998)**.** Each square of a $(2^n - 1) \times (2^n - 1)$ board contains either $+1$ or -1. Such an arrangement is called successful if each number is the product of its neighbors (squares sharing a common side with the given square). Find the number of successful arrangements.

Problem 4.11 (TSTST 2018)**.** In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

Problem 4.12 (Russia 2001)**.** A contest with n question was taken by m contestants. Each question was worth a certain (positive) number of points, and no partial credits were given. After all the papers have been graded, it was noticed that by reassigning the scores of the questions, any desired ranking of the contestants could be achieved. What is the largest possible value of m?

Problem 4.13 (Iran 2006)**.** Let B be a subset of $\mathbb{Z}_3^n$ with the property that for every two distinct members $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ of B there exist $1 \leq i \leq n$ such that $a_i \equiv b_i + 1 \pmod 3$. Prove that $|B| \leq 2^n$.

Problem 4.14 (Frankl-Wilson)**.** Let $A_1, \dots, A_m$ be distinct subsets of $\{1, 2, \dots, n\}$. Let L be the set of numbers that occur as $|A_i \cap A_j|$ for some $i \neq j$ and suppose that $|L| = s$. Show that

$$m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

Problem 4.15 (Ray-Chaudhuri-Wilson)**.** Let $0 \leq k \leq n$ be a positive integer, and let $A_1, \ldots, A_m$ be distinct $k-$element subsets of $\{1, 2, \ldots, n\}$. Let L be the set of numbers that occcur as $|A_i \cap A_j|$ for some $i \neq j$. Prove that $m \leq \binom{n}{s}$.