28. Find all triples (x, y, z) of real numbers such that

$$x^2 + y^2 + z^2 + 1 = xy + yz + zx + |x - 2y + z|.$$

$$|x-2y+z| \leqslant |x-y| + |z-y|$$

$$x^2+y^2+z^2+1 \leqslant xy+yz+zx+|x-y|+|z-y|$$

$$2x^2+2y^2+2z^2+2-2xy-2yz-2zx-2|x-y|-2|z-y| \leqslant 0$$

$$(x^2+y^2-2xy-2|x-y|+1)+(y^2+z^2-2yz-2|z-y|+1)$$

$$+(z^2+x^2-2xz) \leqslant 0$$

$$(|x-y|^2-2|x-y|+1)+(|y-z|^2-2|y-z|+1)+(x-z)^2 \leqslant 0$$

$$(|x-y|-1)^2+(|y-z|-1)^2+(x-z)^2 \leqslant 0$$

$$\Longrightarrow \begin{cases} |x-y|=1 \\ |y-z|=1 \\ x=z \end{cases} \Rightarrow (x,y,z)=$$

$$(k, k+1, k), (k, k-1, k)$$

37. Let p be an odd prime and let

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \ldots + \frac{1}{q(q+1)(q+2)},$$

where $q = \frac{3p-5}{2}$. Assume that $\frac{1}{p} - 2S_q = \frac{m}{n}$, for some integers m and n. Prove that $m \equiv n \pmod{p}$.

الفكرة: إعادة صياغة المجموع

$$\frac{1}{(k-1)\,k\,(k+1)} = \frac{1}{k+1}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{(k+1)(k-1)} - \frac{1}{k(k+1)}$$

$$= \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) - \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \frac{1}{2}\,\frac{1}{k-1} + \frac{1}{2}\,\frac{1}{k+1} - \frac{1}{k}$$

$$= \frac{1}{2}\left(\frac{1}{k-1} + \frac{1}{k} + \frac{1}{k+1}\right) - \frac{3}{2}\,\frac{1}{k}$$

$$\frac{2}{(k-1)\,k\,(k+1)} = \left(\frac{1}{k-1} + \frac{1}{k} + \frac{1}{k+1}\right) - \frac{3}{k} \qquad (*)$$

Using $(*)$: $$2S_q = \sum_{\substack{k=3i\\ i=1}}^{\frac{q+1}{3}} \frac{2}{(k-1)\,k\,(k+1)}$$

$$2S_q = \sum_{\substack{i=1\\ k=3i}}^{\frac{q+1}{3}} \frac{1}{k-1} + \frac{1}{k} + \frac{1}{k+1} - \frac{3}{k}$$

$$2S_q = \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{q+2}\right) - 3\left(\frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{q+1}\right)$$

$$2S_q = \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{q+2}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{\frac{q+1}{3}}\right)$$

$$\Rightarrow \quad 1 + 2S_q = \frac{1}{\frac{q+1}{3} + 1} + \cdots + \frac{1}{q+2}$$

37. Let p be an odd prime and let

$$S_q = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7} + \ldots + \frac{1}{q(q+1)(q+2)},$$

where $q = \frac{3p-5}{2}$. Assume that $\frac{1}{p} - 2S_q = \frac{m}{n}$, for some integers m and n. Prove that $m \equiv n \pmod p$.

$$2S_q = \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{q+2}\right) - \left(1+\frac{1}{2}+\cdots+\frac{1}{\frac{q+1}{3}}\right)$$

$$\Rightarrow \quad 1+2S_q = \frac{1}{\frac{q+1}{3}+1} + \cdots + \frac{1}{q+2}$$

but $\frac{q+1}{3} = \frac{3p-3}{3\cdot 2} = \frac{p-1}{2}$, $q+2 = \frac{3p-1}{2}$

$$1+2S_q = \frac{1}{\frac{p+1}{2}} + \cdots + \frac{1}{\frac{3p-1}{2}}$$

$$1-\frac{m}{n}+\frac{1}{p} = \frac{1}{\frac{p+1}{2}} + \cdots + \frac{1}{\frac{3p-1}{2}}$$

$$\Rightarrow \quad \frac{n-m}{n} = \frac{1}{\frac{p+1}{2}} + \cdots + \frac{1}{p-1} + \frac{1}{p+1} + \cdots + \frac{1}{\frac{3p-1}{2}}$$

$$= \left(\frac{1}{\frac{p+1}{2}} + \frac{1}{\frac{3p-1}{2}}\right) + \cdots + \left(\frac{1}{p-1} + \frac{1}{p+1}\right)$$

$$\frac{n-m}{n} = \frac{2p}{\frac{p+1}{2}\left(\frac{3p-1}{2}\right)} + \cdots + \frac{2p}{(p-1)(p+1)}$$

Note that $p \nmid \frac{p+1}{2}, \frac{3p-1}{2}, \ldots, p-1, p+1$

$$\Rightarrow \quad p \mid n-m \quad \Rightarrow \quad n \equiv m \pmod p$$

40. Find all triples (x, y, z) of positive real numbers for which there is a positive real number t such that the following inequalities hold simultaneously:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+t \leq 4, \quad x^2+y^2+z^2+\frac{2}{t} \leq 5.$$

الفكرة: تخمين الحل الصحيح ثم استعمال متباينات

لإيجاده ق الحل الصحيح $x=y=z=t=1$

By using AM_GM:

$$4\sqrt[4]{\frac{t}{xyz}} \leqslant \frac{1}{x}+\frac{1}{y}+\frac{1}{z}+t$$

$$\Rightarrow 4\sqrt[4]{\frac{t}{xyz}} \leqslant 4 \Rightarrow \frac{t}{xyz} \leqslant 1$$

$$\Rightarrow t \leqslant xyz \quad (1)$$

$$5\sqrt[5]{\frac{x^2y^2z^2}{t^2}} \leqslant x^2+y^2+z^2+\frac{1}{t}+\frac{1}{t}$$

$$\Rightarrow 5\sqrt[5]{\frac{x^2y^2z^2}{t^2}} \leqslant 5 \Rightarrow \frac{xyz}{t} \leqslant 1$$

$$\Rightarrow t \geqslant xyz \quad (2)$$

From (1), (2) $t = xyz$

$\Rightarrow$ Equality holds in (1) $\Rightarrow \frac{1}{x}=\frac{1}{y}=\frac{1}{z}=t$

$$\Rightarrow x=y=z=t=1$$

The only solution is $(x,y,z)=(1,1,1)$ with $t=1$