Divisibility and GCD

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1 Definitions

- The $gcd(a, b)$ is the greatest common divisor between a and b.

- The $lcm(a, b)$ is the least common multiple between a and b.

- We say that $a|b$, if and only if a divides b, or in other words b is divisible by a.

- The $v_p(n)$, is the greatest power of a prime p that divides n. So $v_p(n) = k$ if and only if, $p^k|n$, $p^{k+1} \not| n$. For example, $v_2(24) = 3$, $v_3(24) = 1$, $v_5(24) = 0$.

2 Theorems

- The euclidean algorithm states that: $gcd(a, b) = gcd(a, a - b)$.

- $v_p(gcd(a, b)) = min(v_p(a), v_p(b))$.

- $v_p(lcm[a, b]) = max(v_p(a), v_p(b))$.

- Bezout: If a, b are positive integers, there exists integers x, y such that $ax + by = gcd(a, b)$

3 Problems

Introductory Problems:

1. Let a, b be positive integers, prove that $gcd(a, b) \cdot lcm(a, b) = a \cdot b$.

2. If $gcd(a, b) = 1$, prove that $gcd(ab, a - b) = 1$.

3. Let x, a, b be positive integers, show that $gcd(x^a - 1, x^b - 1) = x^{gcd(a,b)} - 1$.

4. Show that the fraction

$$\frac{21n + 4}{14n + 3}$$

is irreducible for all positive integers n.

5. Let a, b, c be positive integers. Show that a divides bc if and only if $\frac{a}{gcd(a,b)}$ divides c.

6. Let m, n be positive integers, prove that if $mn|m^2 + n^2$ then $m = n$.

7. Prove that if $gcd(a, b) = 1$, and $x^2 = a \cdot b$ then a, b are both squares for $a, b, x \in N$.

Easy Problems:

8. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

9. Show that if a, b are relatively prime integers, then $gcd(a + b, a^2 - ab + b^2) = 1$ or 3.

10. Let n be a positive integer.(a) Find n consecutive composite numbers. (b) Find n consecutive positive integers, none of which is a power of a prime.

11. Show that if a and b are relatively prime integers, and p is an odd prime,
then $gcd(a + b, \frac{a^p+b^p}{a+b}) = 1$ or p.

12. Let $a, b, n > 1$ be positive integers. Show that $a^n + b^n$ is not divisible by $a^n - b^n$.

13. Let a, b, c be positive integers, such that $gcd(a, b, c) = 1$, and $\frac{ab}{a-b} = c$. Prove that $a - b$ is a whole square.

14. Let k be an odd positive integer, prove that $(1 + 2... + n)|(1^k + 2^k + ... + n^k)$.

15. Find all positive integers n such that $3^{n-1} + 5^{n-1}|3^n + 5^n$.

16. Prove that $8 \mid 3^m + 7^m - 2$ for every positive integer m.

17. Let a, b be positive integers such that

$$\frac{lcm(a, b)}{gcd(a, b)} = a - b.$$

Prove that $lcm(a, b) = gcd(a, b)^2$.

Medium Problems:

18. Suppose that each of the positive integers a, b, c, d is divisible by $ad - bc$. Find all possible values of$ad - bc$.

19. Let a and b be positive integers such that $a|b^2, b^2|a^3, a^3|b^4$.... prove that $a = b$.

20. Find all $n \in N$ for which the sum of digits of $n!$ is equal to 9.

21. Let a, b, c be positive integers, prove that:

$$\frac{gcd(a, b, c)^2}{gcd(a, b)gcd(b, c)gcd(c, a)} = \frac{lcm(a, b, c)^2}{lcm(a, b)lcm(b, c)lcm(c, a)}$$

22. Let a, b, c, d be positive integers such that $b < c$ and $a + b + c + d = ab - cd$. Prove that $a + c$

is a composite number.

23. Solve in integers the following equation: $x^2 + y^2 = 3z^2$.

24. Find all prime numbers a, b, c and positive integers k satisfying the equation

$$a^2 + b^2 + 16c^2 = 9k^2 + 1.$$

25. Find all primes p such that there exist positive integers x, y that satisfy

$$x(y^2 - p) + y(x^2 - p) = 5p.$$

26. The numbers in the sequence $101, 104, 109, 116, ...$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, ...$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

27. Prove there exist infinitely many positive integers divisible by 2021 and each of them containing the same number of digits $0, 1, ..., 9$.

28. Solve the following equation in positive integers: $x^2 - y^2 = 2xyz$.

29. Solve in integers the following equation: $x^2 + y^2 + z^2 = 2xyz$.

30. Let m, n be positive integers, and m is odd. that $gcd(2^m - 1, 2^n + 1) = 1$.

31. Let a, b, c, d be positive integers such that $ad = b^2 + bc + c^2$. Prove that $a^2 + b^2 + c^2 + d^2$ is composite.

32. Let x, y, z be positive integers such that gcd(x,y,z) = 1, and $x^2 + y^2 + z^2 = 2xy + 2yz + 2zx$. Prove that x, y, z are all squares.

33. Let $Fk = 2^{2k} + 1$, $k \geq 0$. Prove that for $m \neq n$, $gcd(F_m, F_n) = 1$.

Hard Problems:

34. Find all positive integers n and d with $d|n$ such that $n^2 + d^2$ is divisible by $nd + 1$.

35. Let $a, b, c > 1$ be distinct integers such that $gcd(a, b, c) = 1$. Find all possible values of

$$gcd(a^2b + b^2c + c^2a, ab^2 + bc^2 + ca^2, a + b + c).$$

36. If a, b, c are positive integers, prove that

$$gcd(a, b-1) \cdot gcd(b, c-1) \cdot gcd(c, a-1) \leq a(b-1) + b(c-1) + c(a-1) + 1.$$

Show that equality occurs for infinitely many triples (a, b, c).

37. If a, b are positive integers such that $lcm[a, b] + lcm[a+2, b+2] = 2 \cdot lcm[a+1, b+1]$, prove that $a|b$ or $b|a$.

38. Find all the pairs positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{lcm[x, y]} + \frac{1}{gcd(x, y)} = \frac{1}{2}.$$

39. Solve in non negative integers the equation $2^a \cdot 3^b + 9 = c^2$.

40. Prove that for any 2 positive integers m, n with $m > n$ holds the following inequality

$$lcm[m, n] + lcm[m+1, n+1] > \frac{2mn}{\sqrt{m-n}}.$$

Hint: Use AM-GM: $a + b \geq 2\sqrt{ab}$.

Challenging Problems:

41. Find all integers m and n such that $m^5 - n^5 = 16mn$.

42. Find all pairs of positive integers (x, y) such that $2^x + 3^y$ is a perfect square.

43. Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

44. Find all pairs of positive integers a and b such that $lcm(a+1, b+1) = a^2 - b^2$.

45. Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

46. Find all positive integers n that cannot be written as $n = [a, b] + [b, c] + [c, a]$ for some $a, b, c \in N$.