JMO 2017 Solution Notes

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This is an compilation of solutions for the 2017 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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has infinitely many solutions in positive integers, and characterize all the solutions.

3. Let ABC be an equilateral triangle and P a point on its circumcircle. Set $D = \overline{PA} \cap \overline{BC}$, $E = \overline{PB} \cap \overline{CA}$, $F = \overline{PC} \cap \overline{AB}$. Prove that the area of triangle DEF is twice the area of triangle ABC.

4. Are there any triples (a, b, c) of positive integers such that $(a-2)(b-2)(c-2)+12$ is a prime number that properly divides the positive number $a^2+b^2+c^2+abc-2017$?

5. Let O and H be the circumcenter and the orthocenter of an acute triangle ABC. Points M and D lie on side BC such that $BM = CM$ and $\angle BAD = \angle CAD$. Ray MO intersects the circumcircle of triangle BHC in point N. Prove that $\angle ADO = \angle HAN$.

6. Let P_1, P_2, ..., P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, ..., R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, ..., B_n. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering R_1, ..., R_n of the red points.

§1 JMO 2017/1, proposed by Gregory Galperin

Prove that there exist infinitely many pairs of relatively prime positive integers $a, b > 1$ for which $a + b$ divides $a^b + b^a$.

One construction: let $d \equiv 1 \pmod 4$, $d > 1$. Let $x = \frac{d^d+2^d}{d+2}$. Then set

$$a = \frac{x+d}{2}, \qquad b = \frac{x-d}{2}.$$

To see this works, first check that b is odd and a is even. Let $d = a - b$ be odd. Then:

$$\begin{aligned} a+b \mid a^b + b^a &\iff (-b)^b + b^a \equiv 0 \pmod{a+b} \\ &\iff b^{a-b} \equiv 1 \pmod{a+b} \\ &\iff b^d \equiv 1 \pmod{d+2b} \\ &\iff (-2)^d \equiv d^d \pmod{d+2b} \\ &\iff d+2b \mid d^d + 2^d. \end{aligned}$$

So it would be enough that

$$d + 2b = \frac{d^d+2^d}{d+2} \implies b = \frac{1}{2}\left(\frac{d^d+2^d}{d+2} - d\right)$$

which is what we constructed. Also, since $\gcd(x, d) = 1$ it follows $\gcd(a, b) = \gcd(d, b) = 1$.

> **Remark.** Ryan Kim points out that in fact, $(a, b) = (2n - 1, 2n + 1)$ is always a solution.

§2 JMO 2017/2, proposed by Titu Andreescu

Show that the Diophantine equation

$$\left(3x^3+xy^2\right)\left(x^2y+3y^3\right)=(x-y)^7$$

has infinitely many solutions in positive integers, and characterize all the solutions.

Let $x=da$, $y=db$, where $\gcd(a,b)=1$ and $a>b$. The equation is equivalent to

$$(a-b)^7 \mid ab\left(a^2+3b^2\right)\left(3a^2+b^2\right) \qquad (\star)$$

with the ratio of the two becoming d. Note that

- If a and b are both odd, then $a^2+3b^2 \equiv 4 \pmod 8$. Similarly $3a^2+b^2 \equiv 4 \pmod 8$. Hence 2^4 exactly divides right-hand side, contradiction.
- Now suppose $a-b$ is odd. We have $\gcd(a-b,a)=\gcd(a-b,b)=1$ by Euclid, but also
$$\gcd(a-b,a^2+3b^2)=\gcd(a-b,4b^2)=1$$
and similarly $\gcd(a-b,3a^2+b^2)=1$. Thus $a-b$ is coprime to each of a, b, a^2+3b^2, $3a^2+b^2$ and this forces $a-b=1$.

Of course $(\star)$ holds whenever $a-b=1$ as well, and thus $(\star) \iff a-b=1$. This describes all solutions.

Remark. For cosmetic reasons, one can reconstruct the curve explicitly by selecting $b=\frac{1}{2}(n-1)$, $a=\frac{1}{2}(n+1)$ with $n>1$ an odd integer. Then $d=ab(a^2+3b^2)(3a^2+b^2)=\frac{(n-1)(n+1)(n^2+n+1)(n^2-n+1)}{4}=\frac{n^6-1}{4}$, and hence the solution is

$$(x,y)=(da,db)=\left(\frac{(n+1)(n^6-1)}{8},\frac{(n-1)(n^6-1)}{8}\right).$$

The smallest solutions are $(364,182)$, $(11718,7812)$, ...

§3 JMO 2017/3, proposed by Titu Andreescu, Luis Gonzalez, Cosmin Pohoata

Let ABC be an equilateral triangle and P a point on its circumcircle. Set $D = \overline{PA} \cap \overline{BC}$, $E = \overline{PB} \cap \overline{CA}$, $F = \overline{PC} \cap \overline{AB}$. Prove that the area of triangle DEF is twice the area of triangle ABC.

First solution (barycentric) We invoke barycentric coordinates on ABC. Let $P = (u : v : w)$, with $uv + vw + wu = 0$ (circumcircle equation with $a = b = c$). Then $D = (0 : v : w)$, $E = (u : 0 : w)$, $F = (u : v : 0)$. Hence

$$\begin{aligned} \frac{[DEF]}{[ABC]} &= \frac{1}{(u+v)(v+w)(w+u)} \det \begin{bmatrix} 0 & v & w \\ u & 0 & w \\ u & v & 0 \end{bmatrix} \\ &= \frac{2uvw}{(u+v)(v+w)(w+u)} \\ &= \frac{2uvw}{(u+v+w)(uv+vw+wu) - uvw} \\ &= \frac{2uvw}{-uvw} = -2 \end{aligned}$$

as desired (areas signed).

Second solution ("nice" lengths) WLOG $ABPC$ is convex. Let $x = AB = BC = CA$. By Ptolemy's theorem and strong Ptolemy,

$$\begin{aligned} PA &= PB + PC \\ PA^2 &= PB \cdot PC + AB \cdot AC = PB \cdot PC + x^2 \\ \implies x^2 &+ PB^2 + PB \cdot PC + PC^2. \end{aligned}$$

Also, $PD \cdot PA = PB \cdot PC$ and similarly since $\overline{PA}$ bisects $\angle BPC$ (causing $\triangle BPD \sim \triangle APC$).

Now P is the Fermat point of $\triangle DEF$, since $\angle DPF = \angle FPE = \angle EPD = 120^\circ$. Thus

$$\begin{aligned} [DEF] &= \frac{\sqrt{3}}{4} \sum_{\text{cyc}} PE \cdot PF \\ &= \frac{\sqrt{3}}{4} \sum_{\text{cyc}} \left(\frac{PA \cdot PC}{PB} \right) \left(\frac{PA \cdot PB}{PC} \right) \\ &= \frac{\sqrt{3}}{4} \sum_{\text{cyc}} PA^2 \\ &= \frac{\sqrt{3}}{4} \left((PB+PC)^2 + PB^2 + PC^2 \right) \\ &= \frac{\sqrt{3}}{4} \cdot 2x^2 = 2[ABC]. \end{aligned}$$

§4 JMO 2017/4, proposed by Titu Andreescu

Are there any triples (a, b, c) of positive integers such that $(a-2)(b-2)(c-2)+12$ is a prime number that properly divides the positive number $a^2+b^2+c^2+abc-2017$?

No such (a, b, c).

Assume not. Let $x = a-2$, $y = b-2$, $z = c-2$, hence $x, y, z \ge -1$.

$$\begin{aligned} a^2+b^2+c^2+abc-2017 &= (x+2)^2+(y+2)^2+(z+2)^2 \\ &\quad + (x+2)(y+2)(z+2) - 2017 \\ &= (x+y+z+4)^2 + (xyz+12) - 45^2. \end{aligned}$$

Thus the divisibility relation becomes

$$p = xyz+12 \mid (x+y+z+4)^2 - 45^2 > 0$$

so either

$$\begin{aligned} p = xyz+12 &\mid x+y+z-41 \\ p = xyz+12 &\mid x+y+z+49 \end{aligned}$$

Assume $x \ge y \ge z$, hence $x \ge 14$ (since $x+y+z \ge 41$). We now eliminate several edge cases to get $x, y, z \ne -1$ and a little more:

Claim — We have $x \ge 17$, $y \ge 5$, $z \ge 1$, and $\gcd(xyz, 6) = 1$.

Proof. First, we check that neither y nor z is negative.

- If $x > 0$ and $y = z = -1$, then we want $p = x+12$ to divide either $x-43$ or $x+47$. We would have $0 \equiv x-43 \equiv -55 \pmod p$ or $0 \equiv x+47 \equiv 35 \pmod p$, but $p > 11$ contradiction.
- If $x, y > 0$, and $z = -1$, then $p = 12 - xy > 0$. However, this is clearly incompatible with $x \ge 14$.

Finally, obviously $xyz \ne 0$ (else $p = 12$). So $p = xyz + 12 \ge 14 \cdot 1^2 + 12 = 26$ or $p \ge 29$. Thus $\gcd(6, p) = 1$ hence $\gcd(6, xyz) = 1$.

We finally check that $y = 1$ is impossible, which forces $y \ge 5$. If $y = 1$ and hence $z = 1$ then $p = x+12$ should divide either $x+51$ or $x-39$. These give $39 \equiv 0 \pmod p$ or $25 \equiv 0 \pmod p$, but we are supposed to have $p \ge 29$. □

In that situation $x+y+z-41$ and $x+y+z+49$ are both even, so whichever one is divisible by p is actually divisible by $2p$. Now we deduce that:

$$x+y+z+49 \ge 2p = 2xyz+24 \implies 25 \ge 2xyz - x - y - z.$$

But $x \ge 17$ and $y \ge 5$ thus

$$\begin{aligned} 2xyz - x - y - z &= z(2xy-1) - x - y \\ &\ge 2xy - 1 - x - y \\ &> (x-1)(y-1) > 60 \end{aligned}$$

which is a contradiction. Having exhausted all the cases we conclude no solutions exist.

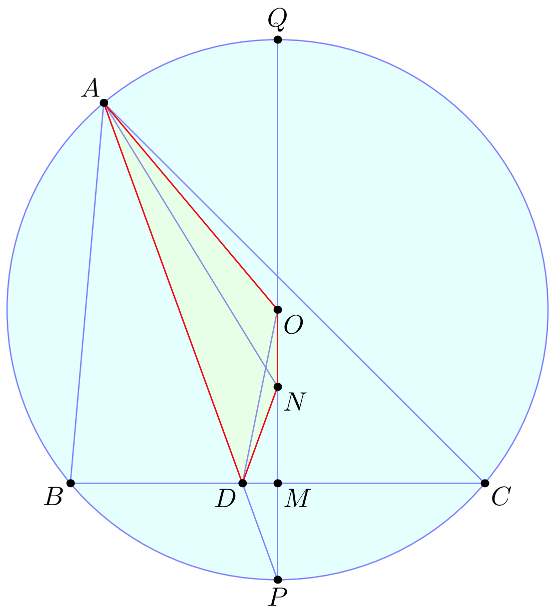
The condition that $x+y+z-41 > 0$ (which comes from "properly divides") cannot be dropped. Examples of solutions in which $x+y+z-41 = 0$ include $(x, y, z) = (5, 5, 31)$ and $(x, y, z) = (1, 11, 29)$.

§5 JMO 2017/5, proposed by Ivan Borsenco

Let O and H be the circumcenter and the orthocenter of an acute triangle ABC. Points M and D lie on side BC such that $BM = CM$ and $\angle BAD = \angle CAD$. Ray MO intersects the circumcircle of triangle BHC in point N. Prove that $\angle ADO = \angle HAN$.

It's known that N is the reflection of the arc midpoint P across M.

The main claim is that $ADNO$ is cyclic. To see this let P and Q be the arc midpoints of $\widehat{BC}$, so that $ADMQ$ is cyclic (as $\measuredangle QAD = \measuredangle QMD = 90°$) . Then $PN \cdot PO = PM \cdot PQ = PD \cdot PA$ as advertised.



To finish, note that $\measuredangle HAN = \measuredangle ONA = \measuredangle ODA$.

> **Remark.** The orthocenter H is superficial and can be deleted basically immediately. One can reverse-engineer the fact that $ADNO$ is cyclic from the truth of the problem statement.

> **Remark.** One can also show $ADNO$ concyclic by just computing $\measuredangle DAO = \measuredangle PAO$ and $\measuredangle DNO = \measuredangle DPN = \measuredangle APQ$ in terms of the angles of the triangle, or even more directly just because
>
> $$\measuredangle DNO = \measuredangle DNP = \measuredangle NPD = \measuredangle OPD = \measuredangle ONA = \measuredangle HAN.$$

§6 JMO 2017/6, proposed by Maria Monks

Let P_1, P_2, $\dots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\dots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points B_1, $\dots$, B_n. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering R_1, $\dots$, R_n of the red points.

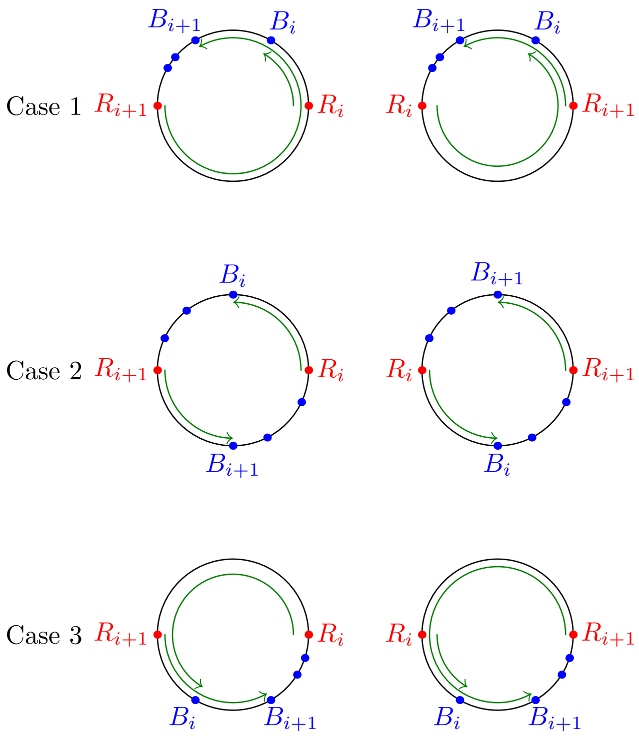
We present two solutions, one based on swapping and one based on an invariant.

First "local" solution by swapping two points Let $1 \le i < n$ be any index and consider the two red points R_i and R_{i+1}. There are two blue points B_i and B_{i+1} associated with them.

> **Claim** — If we swap the locations of points R_i and R_{i+1} then the new arcs $R_i \to B_i$ and $R_{i+1} \to B_{i+1}$ will cover the same points.

Proof. Delete all the points R_1, $\dots$, R_{i-1} and B_1, $\dots$, B_{i-1}; instead focus on the positions of R_i and R_{i+1}.

The two blue points can then be located in three possible ways: either 0, 1, or 2 of them lie on the arc $R_i \to R_{i+1}$. For each of the cases below, we illustrate on the left the locations of B_i and B_{i+1} and the corresponding arcs in green; then on the right we show the modified picture where R_i and R_{i+1} have swapped. (Note that by hypothesis there are no other blue points in the green arcs).

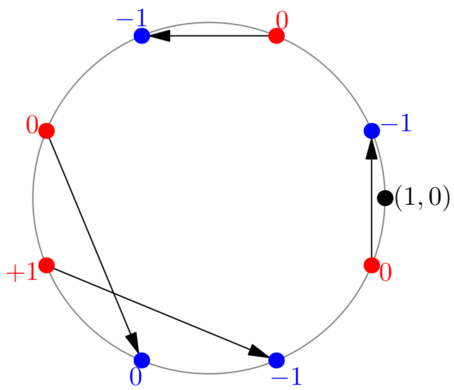


Observe that in all cases, the number of arcs covering any given point on the circumference is not changed. Consequently, this proves the claim. □

Finally, it is enough to recall that any permutation of the red points can be achieved by swapping consecutive points (put another way: $(i\ i+1)$ generates the permutation group S_n). This solves the problem.

> **Remark.** This proof does *not* work if one tries to swap R_i and R_j if $|i-j| \neq 1$. For example if we swapped R_i and R_{i+2} then there are some issues caused by the possible presence of the blue point B_{i+1} in the green arc $R_{i+2} \to B_{i+2}$.

Second longer solution using an invariant Visually, if we draw all the segments $R_i \to B_i$ then we obtain a set of n chords. Say a chord is *inverted* if satisfies the problem condition, and *stable* otherwise. The problem contends that the number of stable/inverted chords depends only on the layout of the points and not on the choice of chords.



In fact we'll describe the number of inverted chords explicitly. Starting from $(1,0)$ we keep a running tally of $R-B$; in other words we start the counter at 0 and decrement by 1 at each blue point and increment by 1 at each red point. Let $x \le 0$ be the lowest number ever recorded. Then:

> **Claim** — The number of inverted chords is $-x$ (and hence independent of the choice of chords).

This is by induction on n. I think the easiest thing is to delete chord R_1B_1; note that the arc cut out by this chord contains no blue points. So if the chord was stable certainly no change to x. On the other hand, if the chord is inverted, then in particular the last point before $(1,0)$ was red, and so $x<0$. In this situation one sees that deleting the chord changes x to $x+1$, as desired.