2018 Winter Camp Mock Olympiad Solutions

1. Suppose that initially there is a path of friends between some two people A and B. We claim at any point, there is a a path of friends between A and B not using any people that have already hosted a party other than possibly A and B. Call this property P. Note that P is initially true since no one has hosted a party yet. Consider the first point at which P is not true. Since P was true before the most recent party, there must be a path from A to B consisting of exactly one person C who has hosted a party. This person C must have two friends D and E on this path. However, D and E must now be friends and thus we can remove C from the path, yielding one using no one who has hosted a party. Thus P is always true. After everyone has hosted a party, if A and B were initially connected by a path of friends then A and B must be friends since P is true. Thus if A and B are not friends, they could not have been connected by such a path to begin with. However, the process of hosting parties can never connect two people by a path who were not connected to begin with. Thus A and B will never be introduced at a party.

2. Let M be the midpoint of side AC. Since G divides AM in a ratio $2:1$, it follows that the distance of A to XY is twice the distance of M to XY. Thus $[XBY] = 2[XMY]$. Now note that since M is the midpoint of AC, the distance of M to XY is the average of the distances of A and C to XY. Thus $2[XMY] = [XAY] + [XCY]$. Now note that since $BXACY$ is cyclic, we have that $\angle XBY = 180° - \angle XAY = 180° - \angle XCY$. Thus $s = \sin \angle XBY = \sin \angle XAY = \sin \angle XCY$. It now follows that
$$\frac{s}{2} \cdot BX \cdot BY = [XBY] = [XAY] + [XCY] = \frac{s}{2} \cdot AX \cdot AY + \frac{s}{2} \cdot CX \cdot CY$$
which proves the result since $s \neq 0$.

3. Let P be the product of the a_i with $a_i \geq 2$ and S be their sum. Let k be the number of a_i with $a_i = 1$. If $k = n$, then the left side is n, which is not possible. Also note that since $a_1 + a_2 + \cdots + a_n \geq n$, we have that
$$P = a_1 a_2 \cdots a_n \leq 100$$
Furthermore, if $P = 100$ then it must follow that $a_1 + a_2 + \cdots + a_n = n$, which implies that $k = n$ and is impossible. Thus $P \leq 99$. Let $b_1 + 1, b_2 + 1, \dots, b_k + 1$ be the a_i with $a_i \geq 2$ where each $b_j \geq 1$. Expanding the product P and noting that there are 2^k terms, each of which is at least 1, yields that
$$P = (b_1+1)(b_2+1)\cdots(b_k+1) \geq b_1 + \cdots + b_k + 2^k - k \geq (b_1+1) + \cdots + (b_k+1) = S$$
since $2^k - k \geq k$ for all $k \geq 1$. Therefore $P \geq S$ and it follows that
$$99(99+k) \geq P(P+k) \geq P(S+k) = a_1 a_2 \cdots a_n (a_1 + a_2 + \cdots + a_n) = 100n \geq 100(k+1)$$
Rearranging gives that $k \leq 99^2 - 100$ and therefore
$$n \leq \frac{99(99+k)}{100} \leq 9702$$
This n can be achieved by taking $a_1 = 99$ and $a_2 = a_3 = \cdots = a_{9702} = 1$. Note that
$$a_1 a_2 \cdots a_n (a_1 + a_2 + \cdots + a_n) = 99 \cdot (9701 + 99) = 100 \cdot 9702 = 100n$$

4. The answer is $n(n-1)$, which is attained when $x_1 = 0$ and $x_2 = \cdots = x_n = n$. We now prove that this is optimal. Without loss of generality let $x_1 \le x_2 \le \cdots \le x_n$. If $x_2 \le n/2$ we are done, because then we have $x_1 + x_2 \le n$ and $x_3 + \cdots + x_n \le n(n-2)$. If $x_2 > n/2$, we have that $x_i > n - x_i$ for $i = 3, 4, \ldots, n$. This implies that $x_1 x_2 \le (n - x_1)(n - x_2)$, and thus $x_1 + x_2 \le n$. Again, since $x_3 + \cdots + x_n \le n(n-2)$, we have that the sum is at most $n(n-1)$.

5. We can assume $n \ge 2$. Let $G(x, y)$ denote the grid of squares (x', y') such that $x' \equiv x \pmod n$ and $y' \equiv y \pmod n$. We now prove the following key lemma.

   Lemma 1. *Any grid $G(x, y)$ either has monochromatic rows or monochromatic columns.*

   Proof. Suppose that (a, b) and $(a+n, b)$ have different colours. Let these colours be c_1 and c_2, respectively. Let S be the $n-1$ colours in the subrow $(a+1, b), \ldots (a+n-1, b)$ and T be the $n-1$ colours in $(a+1, b+n), \ldots (a+n-1, b+n)$. Let c_3 and c_4 be the colours in $(a, b+n)$ and $(a+n, b+n)$, respectively. Since the $n \times n$ subgrids with lower left corners (a, b) and $(a, b+1)$ each contain all n^2 colours, the set of colours among the subrow $(a, b), \ldots, (a+n-1, b)$ and among the subrow $(a, b+n), \ldots, (a+n-1, b+n)$ must be equal. Similarly, the set of colours among the subrow $(a+1, b), \ldots, (a+n, b)$ and the subrow $(a+1, b+n), \ldots, (a+n, b+n)$ must be equal. Therefore $S \cup \{c_1\} = T \cup \{c_3\}$ and $S \cup \{c_2\} = T \cup \{c_4\}$. If $c_1 \ne c_3$, then $c_1 \in T$. Since $c_1 \notin S$ and $c_2 \ne c_1$, it follows that $S \cup \{c_2\} = T \cup \{c_4\}$ does not contain c_1, which is a contradiction. Thus $c_1 = c_3$. Similarly, we have that $c_2 = c_4$. Repeating this argument up and down the columns of $G(x, y)$ with $x = a$ and $x = a + n$ yields that these two columns are each monochromatic and contain the colours $c_1 \ne c_2$, respectively.

   The argument above shows that if there is a row of $G(x, y)$ that is not monochromatic, it must change colours between two squares whose x coordinates differ by exactly n. Thus there must be two columns that are monochromatic and of different colours. Similarly, if there is a column of $G(x, y)$ that is not monochromatic, there must be two monochromatic rows of different colours. However, these cannot both happen, proving the lemma. □

   Note that there are exactly n^2 distinct grids $G(x, y)$. Call a grid $G(x, y)$ horizontal if it has all monochromatic rows and vertical if it has all monochromatic columns. By the lemma, every grid $G(x, y)$ is either horizontal or vertical. Consider a row $y = r$ containing at least $n^2 - n + 1$ colours. Let S_i be the set of colours in the subcolumn $(i, r), (i, r+1), \ldots, (i, r+n-1)$. By considering $n \times n$ subgrids with lower left corners on r, it follows that the disjoint union of the $S_i, S_{i+1}, \ldots, S_{i+n-1}$ is the set of all colours and consequently that $S_i = S_{i+n}$. Now fix some i and suppose that there is some y such that the grid $G(i, y)$ is horizontal. If $y \equiv r \pmod n$ then the intersection of the row r with S_i is only one colour and thus the column omits at least $n - 1 \ge 1$ colours of S_i. If $y \not\equiv r \pmod n$. Then consider the row of $G(i, y)$ between r and $r + n - 1$. This row contains a single colour of S_i that cannot be in the row r. In either case, if a column i contains a horizontal grid $G(x, y)$, then the row r omits a colour in S_i. If every column contains a horizontal grid $G(x, y)$, then r must omit at least n colours, one in each of $S_1, S_2, \ldots, S_{n-1}$, which is a contradiction. Thus there must be column consisting of entirely of vertical grids $G(x, y)$. This column must contain exactly n colours, as desired.