PROOF WRITING with Sarah

Prove Me Wrong (Proof by Contradiction!)

January 6, 2017

1. Let a, b, c be integers such that $a^6 + 2b^6 = 4c^6$. Show that $a = b = c = 0$.

2. (USAJMO 2011, Problem 1) Find, with proof, all positive integers n for which $2^n + 12^n + 2011^n$ is a perfect square.

3. (USAJMO 2013, Problem 1) Are there integers a and b such that $a^5b + 3$ and $ab^5 + 3$ are both perfect cubes of integers?

4. Let $a_1, a_2, \ldots, a_{2000}$ be natural numbers such that
$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{2000}} = 1$$
Prove that at least one of the a_k's is even.

5. Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ be a polynomial with integral coefficients. Suppose that there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

6. Let $p(x)$ be a polynomial with integer coefficients. If $p(0) = p(1) = 2017$, show that p has no integer zeroes.

7. (Hungary Mathematical Olympiad 1999, Problem 10) Let $n > 1$ be an arbitrary positive integer, and let k be the number of positive prime numbers less than or equal to n. Select $k + 1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k + 1$ that is bigger than n.

8. (German Mathematical Olympiad 1985, Problem 4) Every point in $\mathbb{R}^3$ is colored either red, green, or blue. Prove that one of the colors attains all distances, i.e., every positive real number represents the distance between two points of this color.

9. (USAMO 1973, Problem 5) Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arthimetic progression.

10. (USAMO 1991, Problem 3) Show that, for any fixed integer $n \geq 1$, the sequence
$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots (\mod n)$$
is eventually constant.

11. (USA 1999, Problem 4) Let $a_1, a_2, \ldots, a_n (n > 3)$ be real numbers such that $a_1 + a_2 + \ldots + a_n \geq n$ and $a_1^2 + a_2^2 + \ldots + a_n^2 \geq n^2$. Prove that $\max a_1, a_2, \ldots, a_n \geq 2$

12. (USAMO 2000, Problem 1) Call a real-valued function *very convex* if:

$$\frac{f(x)+f(y)}{2} \geq f(\frac{x+y}{2}) + |x-y|$$

holds for all real numbers x and y. Prove that no *very convex* function exists.

13. (USAMO 2003, Problem 1) Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

14. (British Math Olympiad, Problem 1) Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$, and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^o$.

15. (IMO 1959 - 1966 Longlist) Given $n > 3$ points in the plane such that no three of the points are collinear, does there exist a circle passing through (at least) 3 of the given points and not containing any other of the n points in its interior?

16. (IMO 1959, Problem 1) Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

17. (IMO 1987, Problem 4) Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

18. (IMO 1988, Problem 6) Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Prove that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

19. (IMO 2001, Shortlist) Let $a_0, a_1, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers n.

20. (IMO 2001, Problem 1) Consider an acute triangle $\triangle ABC$. Let P be the foot of the altitude of triangle $\triangle ABC$ issuing from the vertex A, and let O be the circumcenter of triangle $\triangle ABC$. Assume that $\angle C \geq \angle B + 30^o$. Prove that $\angle A + \angle COP < 90^o$