USAMO 1999 Solution Notes

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This is an compilation of solutions for the 1999 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

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§0 Problems

1. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

   (a) every square that does not contain a checker shares a side with one that does;

   (b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

   Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

2. Let $ABCD$ be a convex cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

3. Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\} + \left\{\frac{rb}{p}\right\} + \left\{\frac{rc}{p}\right\} + \left\{\frac{rd}{p}\right\} = 2$$

   for any integer r not divisible by p. (Here, $\{t\} = t - \lfloor t \rfloor$ is the fractional part.) Prove that at least two of the numbers $a + b$, $a + c$, $a + d$, $b + c$, $b + d$, $c + d$ are divisible by p.

4. Let $a_1, a_2, \ldots, a_n$ be a sequence of $n > 3$ real numbers such that

$$a_1 + \cdots + a_n \geq n \quad \text{and} \quad a_1^2 + \cdots + a_n^2 \geq n^2.$$

   Prove that $\max(a_1, \ldots, a_n) \geq 2$.

5. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

6. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

§1 USAMO 1999/1

Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

(a) every square that does not contain a checker shares a side with one that does;

(b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

Take a spanning tree on the set V of checkers where the $|V| - 1$ edges of the tree are given by orthogonal adjacency. By condition (a) we have

$$\sum_{v \in V} (4 - \deg v) \geq n^2 - |V|$$

and since $\sum_{v \in V} \deg v = 2(|V| - 1)$ we get

$$4|V| - (2|V| - 2) \geq n^2 - |V|$$

which implies $|V| \geq \frac{n^2 - 2}{3}$.

§2 USAMO 1999/2

Let $ABCD$ be a convex cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

Let the diagonals meet at P, and let $AP = pq$, $DP = pr$, $BP = qs$, $CP = rs$. Then set $AB = qx$, $CD = rx$, $AD = py$, $BC = sy$.

In this way we compute

$$|AC - BD| = |(p - s)(q - r)|$$

and

$$|AB - CD| = |q - r|\, x.$$

By triangle inequality on $\triangle AXB$, we have $x \geq |p - s|$. So $|AB - CD| \geq |AC - BD|$.

§2 USAMO 1999/2

§3 USAMO 1999/3

Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\left\{\frac{ra}{p}\right\} + \left\{\frac{rb}{p}\right\} + \left\{\frac{rc}{p}\right\} + \left\{\frac{rd}{p}\right\} = 2$$

for any integer r not divisible by p. (Here, $\{t\} = t - \lfloor t \rfloor$ is the fractional part.) Prove that at least two of the numbers $a + b$, $a + c$, $a + d$, $b + c$, $b + d$, $c + d$ are divisible by p.

First of all, we apparently have $r(a + b + c + d) \equiv 0 \pmod{p}$ for every prime p, so it automatically follows that $a + b + c + d \equiv 0 \pmod{p}$. By scaling appropriately, and also replacing each number with its remainder modulo p, we are going to assume that

$$1 = a \le b \le c \le d < p.$$

We are going to prove that $d = p - 1$, which will solve the problem.

> **Claim —** For each integer $r = 1, 2, \ldots, p - 1$ we have
> $$2(r - 1) = \left\lfloor \frac{rb}{p} \right\rfloor + \left\lfloor \frac{rc}{p} \right\rfloor + \left\lfloor \frac{rd}{p} \right\rfloor.$$

Proof. By plugging in $r = 1$ to the given we have $a + b + c + d = 2p$. Now, we have

$$2 = \sum_{\text{cyc}} \left(\frac{ra}{p} - \left\lfloor \frac{ra}{p} \right\rfloor \right)$$

and since $a + b + c + d = 2p$ the conclusion follows. $\square$

We vaguely outline the approach now, before giving a formalization. Imagine the interval $[0, 1]$. One by one, for each $r = 1, 2, 3, \ldots, p - 1$, we mark the fractions with denominator r on this number line; the resulting pictures may be better known as *Farey fractions*. At each step, we can place the three numbers b/p, c/p, d/p into one of the resulting sub-intervals. Our goal is to show that d/p is always in the rightmost interval, while b/p and c/p are always to the right of symmetrically mirrored points. An example of a possible diagram is shown below (not to scale).
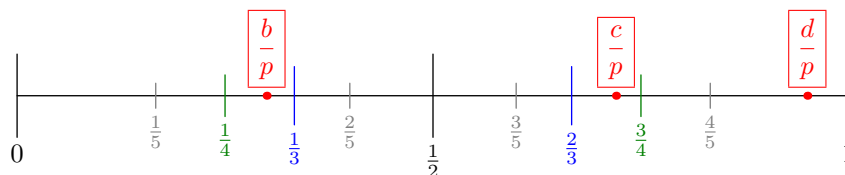
In symbols, it will be enough to prove the following.

> **Claim —** For each $r = 1, 2, \ldots, p - 2$ we have $\frac{r-1}{r} < \frac{d}{p} < 1$.
>
> Equivalently, for each $r = 1, 2, \ldots, p - 2$ we have $\left\lfloor \frac{rb}{p} \right\rfloor + \left\lfloor \frac{rc}{p} \right\rfloor = r - 1$.
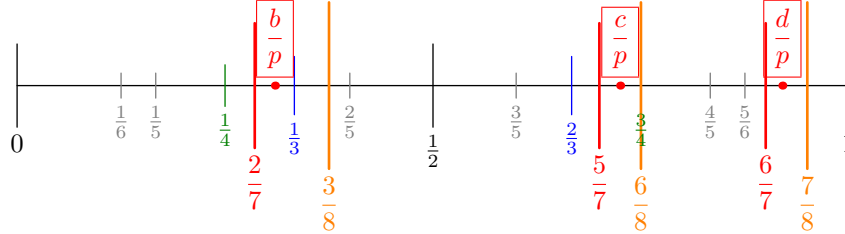
Proof. Assume this is not true and take the minimal counterexample $r > 1$. Then evidently

$$r - 1 > \left\lfloor \frac{rd}{p} \right\rfloor \ge \left\lfloor \frac{(r-1)d}{p} \right\rfloor = r - 2.$$

Now, we have that

$$2(r-1) = \left\lfloor \frac{rb}{p} \right\rfloor + \left\lfloor \frac{rc}{p} \right\rfloor + \underbrace{\left\lfloor \frac{rd}{p} \right\rfloor}_{=r-2}.$$

Thus $\left\lfloor \frac{rb}{p} \right\rfloor > \left\lfloor \frac{(r-1)b}{p} \right\rfloor$, and $\left\lfloor \frac{rc}{p} \right\rfloor > \left\lfloor \frac{(r-1)b}{p} \right\rfloor$. An example of this situation is illustrated below with $r = 7$ (not to scale).

Right now, $\frac{b}{p}$ and $\frac{c}{p}$ are just to the right of $\frac{u}{r}$ and $\frac{v}{r}$ for some u and v with $u + v = r$. The issue is that the there is some fraction just to the right of $\frac{b}{p}$ and $\frac{c}{p}$ from an earlier value of r, and by hypothesis its denominator is going to be strictly greater than 1.

It is at this point we are going to use the properties of Farey sequences. When we consider the fractions with denominator $r + 1$, they are going to lie outside of the interval they we have constrained $\frac{b}{p}$ and $\frac{c}{p}$ to lie in.

Indeed, our minimality assumption on r guarantees that there is no fraction with denominator less than r between $\frac{u}{r}$ and $\frac{b}{p}$. So if $\frac{u}{r} < \frac{b}{p} < \frac{s}{t}$ (where $\frac{u}{r}$ and $\frac{s}{t}$ are the closest fractions with denominator at most r to $\frac{b}{p}$) then Farey theory says the next fraction inside the interval $[\frac{u}{r}, \frac{s}{t}]$ is $\frac{u+s}{r+t}$, and since $t > 1$, we have $r + t > r + 1$. In other words, we get an inequality of the form

$$\frac{u}{r} < \frac{b}{p} < \underbrace{\text{something}}_{=s/t} \leq \frac{u+1}{r+1}.$$

The same holds for $\frac{c}{p}$ as

$$\frac{v}{r} < \frac{c}{p} < \text{something} \leq \frac{v+1}{r+1}.$$

Finally,

$$\frac{d}{p} < \frac{r-1}{r} < \frac{r}{r+1}.$$

So now we have that

$$\left\lfloor \frac{(r+1)b}{p} \right\rfloor + \left\lfloor \frac{(r+1)c}{p} \right\rfloor + \left\lfloor \frac{(r+1)d}{p} \right\rfloor \leq u + v + (r-1) = 2r - 1$$

which is a contradiction.   □

Now, since

$$\frac{p-3}{p-2} < \frac{d}{p} \implies d > \frac{p(p-3)}{p-2} = p - 1 - \frac{2}{p-2}$$

which for $p > 2$ gives $d = p - 1$.

§4 USAMO 1999/4

Let a_1, a_2, ..., a_n be a sequence of $n > 3$ real numbers such that

$$a_1 + \cdots + a_n \geq n \quad \text{and} \quad a_1^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, \ldots, a_n) \geq 2$.

Proceed by contradiction, assuming $a_i < 2$ for all i.

If all $a_i \geq 0$, then $n^2 \leq \sum_i a_i^2 < n \cdot 2^2$, contradiction.

Otherwise, assume at least one a_i is negative. Note that if $-x$ and $-y$ are both present in the sequence $(x, y > 0)$, then we can replace them with $-(x + y)$ and 0. So we may assume that there is exactly one negative term, say $a_n = -M$.

Now, smooth all the nonnegative a_i to be 2, making all inequalities strict. Now, we have that

$$2(n - 1) - M > n$$
$$4(n - 1) + M^2 > n^2.$$

This gives $n - 2 < M < n - 2$, contradiction.

Equality in the original occurs when $n - 1$ of the a_i are equal to 2 and the last one is equal to $-(n - 2)$.

§5 USAMO 1999/5

The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

The main insight is that a construct of the form

$$S \,\square\,\square\, S$$

(here the $\square$ is blank) will kill any player which plays inside it. We call this a *trap* accordingly.

> **Claim —** The second player can force a trap to exist; in this case the game will never end in a draw.

Proof. Actually the second player can construct a trap on her second turn by playing an S far enough away from the edges of the board and the first player's initial move. $\square$

> **Claim —** The second player always has a move which prevents her from losing.

Proof. Since there are an odd number of empty squares at the start of the second player's turn, there must be a square which is bordered by either two filled or two empty squares. The second player can then play O in this square, which is always safe. $\square$

Together these two claims finish the problem.

> **Remark.** Actually, one can show that the "only" way to lose is to be forced to play inside a trap. Indeed, suppose playing in a certain cell c loses. If we wrote O, that means c is bordered by exactly one S, with a blank cell on the neighbor. But we could also write S; checking cases we find c is part of a trap.
>
> Thus a player can lose only if all blank cells are in traps; ergo, the number of blank cells is even. This never happens for the second player. Thus this gives an alternative solution, and moreover a reason to believe that all correct solutions must involve traps.
>
> A similar proof shows that for large n, with a $1 \times n$ board, the first player has a winning strategy for odd n, and the second player has a winning strategy for even n.

§6 USAMO 1999/6

Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

Note E is contact point of A-excircle of $\triangle ACD$, so F is A-excenter. Hence CF is external angle bisector of $\angle ACG$ which implies $FA = FG$ (since F is the arc midpoint on the circumcircle of AFG).