The Online Math Open Spring Contest
Official Solutions
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1. Let ℓ be a line and let points A, B, C lie on ℓ so that $AB = 7$ and $BC = 5$. Let m be the line through A perpendicular to ℓ. Let P lie on m. Compute the smallest possible value of $PB + PC$.

   Proposed by Ankan Bhattacharya and Brandon Wang.

   Answer. $\boxed{9}$

   Solution. We clearly need $P = M$, so we want to minimize $PB + PC$, which happens when C is between A, B and $PB + PC = 7 + 2 = 9$. $\square$

2. Po writes down five consecutive integers and then erases one of them. The four remaining integers sum to 153. Compute the integer that Po erased.

   Proposed by Ankan Bhattacharya.

   Answer. $\boxed{37}$

   Solution. The integers written are 40, 39, 38, and 36. $\square$

3. Given that the answer to this problem can be expressed as $a \cdot b \cdot c$, where a, b, and c are pairwise relatively prime positive integers with $b = 10$, compute $1000a + 100b + 10c$.

   Proposed by Ankit Bisain.

   Answer. $\boxed{203010}$

   Solution. We obtain the equation

$$10ac = 1000a + 1000 + 10c$$

$$ac - 100a - c = 100$$

$$(a-1)(c-100) = 200.$$

   $a - 1$ is nonnegative, so $c - 100$ must also be nonnegative. The pairwise relatively prime condition forces $1 = \gcd(c, b) = \gcd(c, 10) = \gcd(c - 100, 10)$, but since $c - 100$ is a nonnegative divisor of 200, it must be 1. Thus, $c = 101$ and $a - 1 = 200$, so $a = 201$. Thus, the answer is $201 \cdot 10 \cdot 101 = 201 \cdot 1000 + 10 \cdot 100 + 101 \cdot 10 = 203010$. $\square$

4. Let $ABCD$ be a square with side length 16 and center O. Let $\mathcal{S}$ be the semicircle with diameter AB that lies outside of $ABCD$, and let P be a point on $\mathcal{S}$ so that $OP = 12$. Compute the area of triangle CDP.

   Proposed by Brandon Wang.

   Answer. $\boxed{136}$

   Solution. Construct O' so that $AOBO'$ is a square. Then, P lies on the circle with diameter OO', so $OP = 12, OO' = 16, PO' = 4\sqrt{7}$. If X is the foot of the perpendicular from P to OO', and Y the perpendicular from P to AB, then $16PX = 12 \cdot 4\sqrt{7}$, so $PX = 3\sqrt{7}$, and so $PY = \sqrt{64 - (3\sqrt{7})^2} = 1$. Thus, the distance from P to CD is 17 and CDP has area $\frac{1}{2} \cdot 16 \cdot 17 = 136$. $\square$

5. Compute the smallest positive integer n such that there do not exist integers x and y satisfying $n = x^3 + 3y^3$.

   Proposed by Luke Robitaille.

   Answer. $\boxed{6}$

Solution. We first compute

$$\begin{aligned}1 &= 1^3 + 3 \cdot 0^3\\ 2 &= (-1)^3 + 3 \cdot 1^3\\ 3 &= 0^3 + 3 \cdot 1^3\\ 4 &= 1^3 + 3 \cdot 1^3\\ 5 &= 2^3 + 3 \cdot (-1)^3.\end{aligned}$$

Now, note that if $6 = x^3 + 3y^3$, then $x = 3z$ for some integer z, so $2 = 9z^3 + y^3$, or $y^3 \equiv 2 \pmod 9$, which is impossible. Thus $n = 6$. □

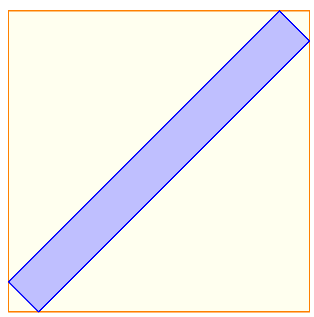
6. Alexis has 2020 paintings, the ith one of which is a $1 \times i$ rectangle for $i = 1, 2, \dots, 2020$. Compute the smallest integer n for which they can place all of the paintings onto an $n \times n$ mahogany table without overlapping or hanging off the table.

Proposed by Brandon Wang.

Answer. $\boxed{1430}$

Solution. The answer is 1430. The lower bound is since the smallest square in which a 1×2020 rectangle can be inscribed is one with sides of length $\frac{2021}{\sqrt{2}} > 1429$.

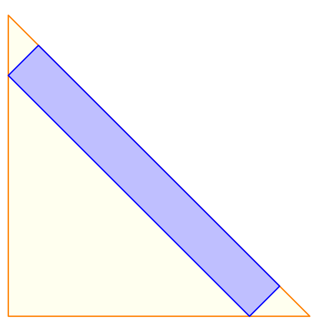
Now I claim all these rectangles can be inscribed in a square with sides of length $1011\sqrt{2} < 1430$, which will complete the proof. Place the 1×2020 rectangle in the square as shown.



Let S_k denote the set of positive integers not exceeding k and with the same parity as k. The main claim:

Claim. $1 \times i$ rectangles, for $i \in S_k$, fit in an isosceles right triangle with legs of length $\frac{k+2}{\sqrt{2}}$

Proof. We induct with step size 2. The base cases $k = 1, 2$ may be promptly verified. Now assume the hypothesis holds for $k - 2$; we will prove it holds for k.



Inscribe a $1 \times k$ rectangle in a isosceles right triangle with legs $\frac{k+2}{\sqrt{2}}$ as shown above. Then the bottom-left isosceles right triangle has sides of length $\frac{k}{\sqrt{2}}$, so by the inductive hypothesis, the $1 \times i$ rectangle for all the other i in S_k fit in the bottom-left triangle, as claimed. $\square$

In the first diagram, the upper-left and bottom-right isosceles right triangles have leg length $\frac{2021}{\sqrt{2}}$, so by the claim, we can fit $1 \times i$ rectangles for $i = 1, 3, \ldots, 2019$ in one of them and $1 \times i$ rectangles for $i = 2, 4, \ldots, 2018$ in the other. $\square$

7. On a 5×5 grid we randomly place two *cars*, which each occupy a single cell and randomly face in one of the four cardinal directions. It is given that the two cars do not start in the same cell. In a *move*, one chooses a car and shifts it one cell forward. The probability that there exists a sequence of moves such that, afterward, both cars occupy the same cell is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $100m + n$.

   Proposed by Sean Li.

   Answer. $\boxed{1148}$

   Solution. two cases: - two cars are in the same row/column (25*8). then at least one car must face toward other car. - two cars do not share a row or column (25*16). then there are two locations that the cars can meet. $\square$

8. Let $a > b$ be positive integers. Compute the smallest possible integer value of $\frac{a!+1}{b!+1}$.

   Proposed by Sean Li.

   Answer. $\boxed{103}$

   Solution. The answer is 103, achieved when $(a, b) = (6, 3)$; it remains to show its minimality.

   First, note that
   $$b! + 1 \mid a! + 1 \implies b! + 1 \mid a! - b! \implies b! + 1 \mid \frac{a!}{b!} - 1,$$
   the second implication following from $\gcd(b!, b!+1) = 1$. Because $\frac{a!}{b!} - 1 > 0$, we must have $b!+1 \leq \frac{a!}{b!} - 1$, or $b! \leq \frac{a!}{b!+2} < \frac{a!+1}{b!+1}$. Moreover, we have $b \neq 4$, as $4! + 1 = 25$ and $5 \nmid a! + 1$ for $a \geq 5$. Thus, $\frac{a!+1}{b!+1} > 120$ if $b > 3$.

   It is then a finite case check to show $(a, b) = (6, 3)$ is the only solution for $b \leq 3$. $\square$

9. A magician has a hat that contains a white rabbits and b black rabbits. The magician repeatedly draws pairs of rabbits chosen at random from the hat, without replacement. Call a pair of rabbits *checkered* if it consists of one white rabbit and one black rabbit. Given that the magician eventually draws out all the rabbits without ever drawing out an unpaired rabbit and that the expected value of the number of checkered pairs that the magician draws is 2020, compute the number of possible pairs (a, b).

   Proposed by Ankit Bisain.

   Answer. $\boxed{16}$

   Solution. $2020 = EV = \frac{a+b}{2} \cdot \frac{2ab}{(a+b)(a+b-1)} = \frac{ab}{a+b-1}$ so $(a - 2020)(b - 2020) = 2020 \cdot 2019 = 4 * 5 * 101 * 3 * 673$, and since $a + b$ must be even, the answer turns out to 16. $\square$

10. Compute the number of functions $f : \{1, \ldots, 15\} \to \{1, \ldots, 15\}$ such that, for all $x \in \{1, \ldots, 15\}$,
    $$\frac{f(f(x)) - 2f(x) + x}{15}$$
    is an integer.

    Proposed by Ankan Bhattacharya.

Answer. $\boxed{375}$

Solution. The condition means each orbit of f is an arithmetic progression modulo 15. It is easy to see that each orbit of f has size 1, 3, 5, or 15. If f has an orbit of size 15, then f must be a shift $x \to x + a$, and there are $\varphi(15) = 8$ choices for a.

Otherwise, each orbit of f has size 1, 3, or 5. It is easy to see that f cannot have both an orbit of size 3 and one of size 5 (if so, they will share exactly one element by CRT). Thus we split into two cases.

Case 1: f has no orbit of size 3. Then there are 5 ways to determine f on each of the sets $\{0, 3, 6, 9, 12\}$, $\{1, 4, 7, 10, 13\}$, $\{2, 5, 8, 11, 14\}$, so there are $5^3 = 125$ possible functions f in this case. Case 2: f has no orbit of size 5. Then there are 3 ways to determine f on each of the sets $\{0, 5, 10\}$, $\{1, 6, 11\}$, $\{2, 7, 12\}$, $\{3, 8, 13\}$, $\{4, 9, 14\}$, so there are $3^5 = 243$ possible functions f in this case.

We have counted the identity function twice, so there are a total of $8 + 125 + 243 - 1 = 375$ functions satisfying the requested property. □

11. A mahogany bookshelf has four identical-looking books which are 200, 400, 600, and 800 pages long. Velma picks a random book off the shelf, flips to a random page to read, and puts the book back on the shelf. Later, Daphne also picks a random book off the shelf and flips to a random page to read. Given that Velma read page 122 of her book and Daphne read page 304 of her book, the probability that they chose the same book is $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m+n$.

    Proposed by Sean Li.

    Answer. $\boxed{6425}$

    Solution. $\frac{61}{325}$, basic conditional probability □

12. Convex pentagon $ABCDE$ is inscribed in circle γ. Suppose that $AB = 14$, $BE = 10$, $BC = CD = DE$, and $[ABCDE] = 3[ACD]$. Then there are two possible values for the radius of γ. The sum of these two values is $\sqrt{n}$ for some positive integer n. Compute n.

    Proposed by Luke Robitaille.

    Answer. $\boxed{417}$

13. For nonnegative integers p, q, r, let

$$f(p, q, r) = (p!)^p (q!)^q (r!)^r.$$

    Compute the smallest positive integer n such that for any triples (a, b, c) and (x, y, z) of nonnegative integers satisfying $a + b + c = 2020$ and $x + y + z = n$, $f(x, y, z)$ is divisible by $f(a, b, c)$.

    Proposed by Brandon Wang.

    Answer. $\boxed{6052}$

14. Let S and T be non-empty, finite sets of positive integers. We say that $a \in \mathbb{N}$ is *good* for $b \in \mathbb{N}$ if $a \geq \frac{b}{2} + 7$. We say that an ordered pair $(a, b) \in S \times T$ is *satisfiable* if a and b are good for each other.

    A subset R of S is said to be *unacceptable* if there are less than $|R|$ elements b of T with the property that there exists $a \in R$ such that (a, b) is satisfiable. If there are no unacceptable subsets of S, and S contains the elements 14, 20, 16, 32, 23, and 31, compute the smallest possible sum of elements of T given that $|T| \geq 20$.

    Proposed by Tristan Shin.

    Answer. $\boxed{219}$

Solution. By Hall's Marriage Theorem, there exists an injection $f : S \to T$ with $(a, f(a))$ satisfiable for all $a \in S$. Thus, we need six distinct elements of T that correspond to the 6 provided elements of S. The smallest that we can get is $14, 17, 15, 23, 19, 24$. We need 14 more elements, so we choose $1 \to 13$ and 16. The sum of these numbers is 219. $\square$

15. Let ABC be a triangle with $AB = 20$ and $AC = 22$. Suppose its incircle touches $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively, and P is the foot of the perpendicular from D to $\overline{EF}$. If $\angle BPC = 90^\circ$, then compute BC^2.

Proposed by Ankan Bhattacharya.

Answer. $\boxed{84}$

Solution. The condition is equivalent to $\angle B = 90^\circ$ or $\angle C = 90^\circ$, so $BC = 2\sqrt{21}$. $\square$

16. Compute the number of ordered pairs (m, n) of positive integers such that $(2^m - 1)(2^n - 1) \mid 2^{10!} - 1$.

Proposed by Luke Robitaille.

Answer. $\boxed{5509}$

Solution. Sketch: The key lemma is that for all m, n, the minimum x with $(2^m - 1)(2^n - 1) \mid 2^x - 1$ is $x = \operatorname{lcm}(m, n)(2^{\gcd(m,n)} - 1)$. Thus we want $\operatorname{lcm}(m, n)(2^{\gcd(m,n)} - 1) \mid 10!$. Doing casework on $\gcd(m, n)$ (the possible values are 1,2,3,4 and 6) we get that the answer is $2295 + 1575 + 595 + 819 + 225 = 5509$. $\square$

17. Compute the number of integers $1 \le n \le 1024$ such that the sequence $\lceil n \rceil, \lceil n/2 \rceil, \lceil n/4 \rceil, \lceil n/8 \rceil, \ldots$ does not contain any multiple of 5.

Proposed by Sean Li.

Answer. $\boxed{351}$

Solution. consider in reverse $(n \to 2n - 1, 2n)$ and use recursion. oeis A005251. $\square$

18. Vincent has a fair die with sides labeled 1 to 6. He first rolls the die and records it on a piece of paper. Then, every second thereafter, he re-rolls the die. If Vincent rolls a different value than his previous roll, he records the value and continues rolling. If Vincent rolls the same value, he stops, does *not* record his final roll, and computes the average of his previously recorded rolls. Given that Vincent first rolled a 1, let E be the expected value of his result. There exist rational numbers $r, s, t > 0$ such that $E = r - s \ln t$ and t is not a perfect power. If $r + s + t = \frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by Sean Li.

Answer. $\boxed{13112}$

Solution. $\sum_{n=0}^{\infty} \frac{1}{6} \cdot \left(\frac{5}{6}\right)^n \cdot \frac{1}{n+1} \left((4n+1) - 3\left(\frac{n}{6} - \frac{5}{36}\left(1 - \left(-\frac{1}{5}\right)^n\right)\right)\right) = \frac{42 - 5\ln 7}{12}$ $\square$

19. Let ABC be a scalene triangle. The incircle is tangent to lines BC, AC, and AB at points D, E, and F, respectively, and the A-excircle is tangent to lines BC, AC, and AB at points D_1, E_1, and F_1, respectively. Suppose that lines AD, BE, and CF are concurrent at point G, and suppose that lines AD_1, BE_1, and CF_1 are concurrent at point G_1. Let line GG_1 intersect the internal bisector of angle BAC at point X. Suppose that $AX = 1$, $\cos\angle BAC = \sqrt{3} - 1$, and $BC = 8\sqrt[4]{3}$. Then $AB \cdot AC = \frac{j + k\sqrt{m}}{n}$ for positive integers j, k, m, and n such that $\gcd(j, k, n) = 1$ and m is not divisible by the square of any integer greater than 1. Compute $1000j + 100k + 10m + n$.

Proposed by Luke Robitaille and Brandon Wang.

Answer. $\boxed{3173}$

20. Reimu invented a new number base system that uses exactly five digits. The number 0 in the decimal system is represented as 00000, and whenever a number is incremented, Reimu finds the leftmost digit (of the five digits) that is equal to the "units" (rightmost) digit, increments this digit, and sets all the digits to its right to 0. (For example, an analogous system that uses three digits would begin with 000, 100, 110, 111, 200, 210, 211, 220, 221, 222, 300, ...) Compute the decimal representation of the number that Reimu would write as 98765.

    Proposed by Yannick Yao.

    Answer. $\boxed{1727}$

    Solution. This base system is essentially putting all the unordered quintuples of nonnegative integers in lexicographical order, so the number $\overline{abcde}$ represents $\binom{a+4}{5} + \binom{b+3}{4} + \binom{c+2}{3} + \binom{d+1}{2} + \binom{e}{1}$. □

21. For positive integers $i = 2, 3, \ldots, 2020$, let

$$a_i = \frac{\sqrt{3i^2 + 2i - 1}}{i^3 - i}.$$

Let $x_2, \ldots, x_{2020}$ be positive reals such that $x_2^4 + x_3^4 + \cdots + x_{2020}^4 = 1 - \frac{1}{1010 \cdot 2020 \cdot 2021}$. Let S be the maximum possible value of

$$\sum_{i=2}^{2020} a_i x_i (\sqrt{a_i} - 2^{-2.25} x_i)$$

and let m be the smallest positive integer such that S^m is rational. When S^m is written as a fraction in lowest terms, let its denominator be $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for prime numbers $p_1 < \cdots < p_k$ and positive integers α_i. Compute $p_1\alpha_1 + p_2\alpha_2 + \cdots + p_k\alpha_k$.

Proposed by Edward Wan, Brandon Wang.

Answer. $\boxed{1254}$

Solution. We have

$$S = \frac{3}{2^{1.75}} \left(\frac{1}{2} - \frac{1}{2020^2 \cdot 2021} \right),$$

so S^4 has denominator $2^7 \cdot 2020^8 \cdot 2021^4 = 2^{23} 5^8 101^8 43^4 47^4$. □

22. Let ABC be a scalene triangle with incenter I and symmedian point K. Furthermore, suppose that $BC = 1099$. Let P be a point in the plane of triangle ABC, and let D, E, F be the feet of the perpendiculars from P to lines BC, CA, AB, respectively. Let M and N be the midpoints of segments EF and BC, respectively. Suppose that the triples (M, A, N) and (K, I, D) are collinear, respectively, and that the area of triangle DEF is 2020 times the area of triangle ABC. Compute the largest possible value of $\lceil AB + AC \rceil$.

Proposed by Brandon Wang.

Answer. $\boxed{2205}$

Solution. We use barycentric coordinates. Note that the MAN collinear if and only if P lies on AO, and so the area condition simply collapses into $\overrightarrow{OP} = \pm t\overrightarrow{OA}$, where $t = \sqrt{8081}$. Let O lie at the origin. Since K is $[a^2 : b^2 : c^2]$ and I is $[a : b : c]$, so $D = BC \cap KI$ must be $[0 : b(a-b) : c(a-c)]$. In particular, this means that

$$\pm t(b(a-b) + c(a-c))\overrightarrow{OA} - (b(a-b))\overrightarrow{OB} - (c(a-c))\overrightarrow{OC} \perp \overrightarrow{BC}.$$

By Strong EFFT, this is equivalent to

$$0 = a^2(c(a-c) - b(a-b)) \pm t(b(a-b) + c(a-c))(c^2 - b^2),$$

or

$$\pm t(a(b+c)-b^2-c^2)(b+c) = a^2(b+c-a).$$

Scale down so that $a = 1$. If $b + c = 2 + \varepsilon$,

$$\pm t(2+\varepsilon-b^2-c^2) = \frac{1+\varepsilon}{2+\varepsilon},$$

so

$$b^2 + c^2 = 2 + \varepsilon \mp \frac{(1+\varepsilon)}{(2+\varepsilon)\sqrt{8081}}.$$

On the other hand, we require $b^2 + c^2 \geq 2(1 + \varepsilon/2)^2 = 2 + 2\varepsilon + \varepsilon^2/2$, so

$$\varepsilon(\varepsilon+2)^2 \cdot \sqrt{8081} \leq 2 + 2\varepsilon,$$

or

$$2\varepsilon + \frac{\varepsilon^3}{2+2\varepsilon} \leq \frac{1}{\sqrt{8081}}.$$

So, $BC = 2198 + 1099\varepsilon \leq 2198 + \frac{1099}{2\sqrt{8081}} \leq 2205$. (It is easy to check that $6/1099$ satisfies the above inequality. $\square$

23. In the Bank of Shower, a bored customer lays n coins in a row. Then, each second, the customer performs "The Process." In The Process, all coins with exactly one neighboring coin heads-up before The Process are placed heads-up (in its initial location), and all other coins are placed tails-up. The customer stops once all coins are tails-up.

    Define the function f as follows: If there exists some initial arrangement of the coins so that the customer never stops, then $f(n) = 0$. Otherwise, $f(n)$ is the average number of seconds until the customer stops over all initial configurations. It is given that whenever $n = 2^k - 1$ for some positive integer k, $f(n) > 0$.

    Let N be the smallest positive integer so that

    $$M = 2^N \cdot \left(f(2^2 - 1) + f(2^3 - 1) + f(2^4 - 1) + \cdots + f(2^{10} - 1)\right)$$

    is a positive integer. If $M = \overline{b_k b_{k-1} \cdots b_0}$ in base two, compute $N + b_0 + b_1 + \cdots + b_k$.

    Proposed by Israel 2015/6, Edward Wan, Brandon Wang.

    Answer. $\boxed{1040}$

    Solution. $f(n) = (n-1) + \frac{1}{2^n}$ if $n = 2^k - 1$, and 0 otherwise. We get $N = 1023, \sum b_i = 17$.

    Question: For which n does the process necessarily terminate?

    The answer is all integers n of the form $2^k - 1$, where k is any positive integer. Let 0 mean "off" and 1 mean "on." As usual, we will work in $\mathbb{F}_2^n$. For a vector $v \in \mathbb{F}_2^n$, we will let $f(v)$ be the vector which corresponds to the result when the operation described in the problem. For instance $f(011010010) = 111001101$ and $f(0100010110101) = 1010100110000$. It's easily checked that f is additive. The problem asks for all values of n for which there exists a sufficiently large positive integer x, so that for every vector $v \in \{0,1\}^n$, we have that $f^{(x)}(v) = 000\cdots 0$. Call such n [b]tangy[/b]. Call a vector $v \in \mathbb{F}_2^n$ [b] symmetric [/b] if it is preserved under the reversing of its digits.

    Lemma 1. No even numbers are tangy, and $2k+1$ is tangy if and only if k is.

    Proof. To see that no even numbers are tangy, simply notice that for every symmetric vector v, $f(v)$ is not $000\cdots 0$. This implies that if we start by turning on any symmetric subset of the lamps (first and last, for example), there will never be a time when all the lamps are off.

    Now, let's show the second part of the lemma. Let's first show that if $2k+1$ is tangy, then so is k. Indeed, this is easily seen since for every vector $v \in \mathbb{F}_2^k$, we can consider the vector $w \in \mathbb{F}_2^{2k+1}$ which is v, followed by 0, followed by the reverse of v. Then, when we apply operation once, w will change into the vector which is $f(v)$, then 0, then the reverse of $f(v)$. For example, we have that when $v = 110$, $f(v) = 111$ and $f(1100011) = 1110111$.

    Now, let's show that if k is tangy, then so is $2k+1$. Consider any vector $w \in \mathbb{F}_2^{2k+1}$. If we let w_r be the reverse of w, then observe that $w + w_r$ is symmetric and has 0 as its middle digit. Therefore, there exists a vector v such that $w + w_r$ is composed of v, then 0, then the reverse of v. From the previous observation and the fact that k is tangy, it's clear that there is some x so that $f^{(x)}(w + w_r) = 000\cdots 0$, and so therefore by the additivity of f we've that $f^{(x)}(w) = f^{(x)}(w_r)$. By symmetry, it's clear that $f^{(t)}(w), f^{(t)}(w_r)$ are reverses of each other for all $t \in \mathbb{N}$. Therefore, we've that $f^{(x)}(w)$ is the reverse of itself, and hence is symmetric. Finally, it's clear that $f^{(x+1)}(w)$ is symmetric and has middle digit 0. Hence, there is a vector $u \in \mathbb{F}_2^k$ so that $f^{(x+1)}(w)$ is composed of u, then 0, and then the reverse of u. Then, selecting large enough $h \in \mathbb{N}$ so that $f^{(h)}(u) = 00\cdots 0$, it becomes apparent that $f^{(x+1+h)}(w) = 0$. Since w was arbitrary, the lemma is proven. □

    Now say n is tangy. We'll compute $f(n)$. So, if for $v \in \mathbb{F}_2^n$ we let the order of v be the minimal k so that $T^k v = 0$, then we want the average of the orders of v. Note that $w = 1010\cdots 01$ is the only element with order 1. In general, note that $T^{-1}v$ has at most two elements. So, this means that for $1 \le m \le n$ there are at most 2^{m-1} elements with order m, since this is just $(T^{-1})^{m-1}w$. If there are a_i elements with order a_i, we get $a_0 = 1$, $a_1 \le 1, a_2 \le 2, \ldots, a_n \le 2^{n-1}$.

    But $a_0 + \cdots + a_n = 2^n$, so equality must hold everywhere so $a_m = 2^{m-1}$. This gives the desired value of $f(n)$.

Remark. It is possible to answer the original question using the linear algebra approach. Note that the answer extraction did not require $n = 2^k - 1$, only that $T^n = 0$. So, suppose $T^n = 0$ first. We note that $T^m \neq 0$ for $m < n$, so if $P(z)$ is the minimal polynomial in $\mathbb{F}_2[z]$ of T, then $P \mid z^n \implies P(z) = z^k$, and so $k = n$ is forced.

Thus, in particular T has characteristic polynomial z^n. This is equivalent to

$$Q_n(z) \stackrel{\text{def}}{=} \det \begin{bmatrix} z & 1 & & & & & \\ 1 & z & 1 & & & & \\ & 1 & z & & & & \\ & & & \ddots & & & \\ & & & & z & 1 & \\ & & & & 1 & z & 1 \\ & & & & & 1 & z \end{bmatrix} = z^n$$

Note, however, that $Q_n(z) = zQ_{n-1}(z) + Q_{n-2}(z)$. So, $Q_0(z) = 1$, $Q_1(z) = z$, $Q_2(z) = z^2 + 1$, $Q_3(z) = z^3$, etc. Let $R_n(z) \in \mathbb{Z}[z]$ be defined similarly, with $R_0(z) = 1R_1(z) = z$, $R_n(z) = zR_{n-1} + R_{n-2}(Z)$. Then, a simple induction shows that

$$R_n(z) = \sum_{0 \le 2k \le n} \binom{n-k}{n-2k} z^{n-k},$$

so since $Q_n(z)$ is the reduction of $R_n(z)$ mod 2, we see that n is tangy if and only if $\binom{n-k}{n-2k} = \binom{n-k}{k}$ is even for all $2 \le 2k \le n$. Say $n = \overline{b_\ell \cdots b_0}$ in base two with $b_\ell = 1$, and say $k = c_t \cdots c_0$ with $t < \ell$.

If n is one less than a power of two, then $n-k$'s digit in the 2^ts place is 0 while k's is 1, so by Lucas, $\binom{n-k}{k} = 0 \pmod 2$. Thus, $Q_n(z) = z^n$ indeed, so n is tangy. Otherwise, say $b_s = 0$, $b_{s+1} = 1$, then let $k = 2^s$, and in k the only digit that is 1 is the 2^s's placed one, which is also 1 in $n-k$, so $\binom{n-2^s}{2^s}$ is odd, which means $Q_n(z) \neq z^n$.

Thus the only tangy integers are the ones which are one less than a power of two indeed. $\square$

24. Let A, B be opposite vertices of a unit square with circumcircle Γ. Let C be a variable point on Γ. If $C \notin \{A, B\}$, then let ω be the incircle of triangle ABC, and let I be the center of ω. Let C_1 be the point at which ω meets $\overline{AB}$, and let D be the reflection of C_1 over line CI. If $C \in \{A, B\}$, let $D = C$. As C varies on Γ, D traces out a curve $\mathfrak{C}$ enclosing a region of area $\mathcal{A}$. Compute $\lfloor 10^4 \mathcal{A} \rfloor$.

    Proposed by Brandon Wang.

    Answer. $\boxed{1415}$

    Solution. $\pi - 3$ $\square$

25. Let $\mathcal{S}$ denote the set of positive integer sequences (with at least two terms) whose terms sum to 2019. For a sequence of positive integers $a_1, a_2, \ldots, a_k$, its *value* is defined to be

$$V(a_1, a_2, \ldots, a_k) = \frac{a_1^{a_2} a_2^{a_3} \cdots a_{k-1}^{a_k}}{a_1! a_2! \cdots a_k!}.$$

    Then the sum of the values over all sequences in $\mathcal{S}$ is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute the remainder when $m + n$ is divided by 1000.

    Proposed by Sean Li.

    Answer. $\boxed{401}$

    Solution. let $N = 2019$. then the answer is $\frac{(N+1)^{N-1}-1}{N!}$: $2019! \cdot V(a_1, \ldots, a_k)$ counts the number of labeled trees with a_i vertices with depth $i+1$, then use cayley's.

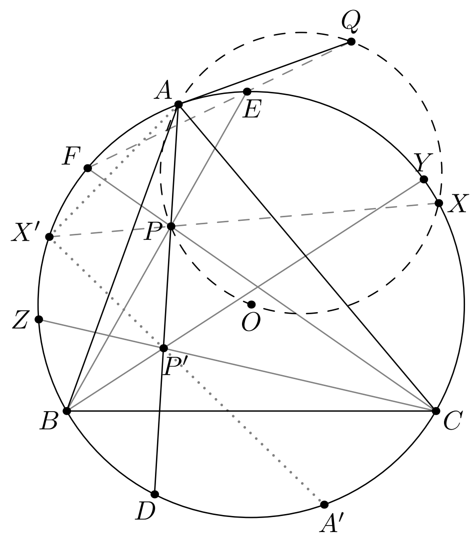
    moreover, if $p | 2020^{2018} - 1$ and $p \le 2019$, then $\text{ord}_p(2020) \in \{1, 2, 1009, 2018\}$, so either $p | 2020^2 - 1$ or $1009 | p - 1$. so $m = \frac{2020^{2018}-1}{2019 \cdot 2021}$ and $n = \frac{2019!}{2019 \cdot 2021}$, then use mods $\square$

26. Let ABC be a triangle with circumcircle ω and circumcenter O. Suppose that $AB = 15$, $AC = 14$, and P is a point in the interior of $\triangle ABC$ such that $AP = \frac{13}{2}$, $BP^2 = \frac{409}{4}$, and P is closer to $\overline{AC}$ than to $\overline{AB}$. Let E, F be the points where $\overline{BP}$, $\overline{CP}$ intersect ω again, and let Q be the intersection of $\overline{EF}$ with the tangent to ω at A. Given that $AQOP$ is cyclic and that CP^2 is expressible in the form $\frac{a}{b} - c\sqrt{d}$ for positive integers a, b, c, d such that $\gcd(a, b) = 1$ and d is not divisible by the square of any prime, compute $1000a + 100b + 10c + d$.

Proposed by Edward Wan.

Answer. $\boxed{365492}$

Solution. Let P' be the reflection of A over P, and let $(APOQ)$ intersect ω again at X. Also let $\overline{AP}$, $\overline{BP'}$, $\overline{CP'}$, $\overline{XP}$ intersect ω again at D, Y, Z, X'. Finally let A' be the antipode of A on ω.



The key claim is this:

Claim. $\angle ABP' = \angle ACP'$.

Proof. It will suffice to prove that $AYA'Z$ is a kite, i.e. $-1 = (AA'; YZ)$.

Several observations to note:

- $\angle OXQ = \angle OAQ = 90^\circ$, so $\overline{QX}$ is also tangent to (ABC).
- Since $OA = OX$, we have $\measuredangle OPD = \measuredangle OPA = \measuredangle XPO$, thus $PD = PX$.
- Since $PD = PX$, we have $PX' = PA = PP'$, so $\angle AXP' = 90^\circ$ and X', P', A' collinear.

From this, we have

$$-1 = (AX; EF) \stackrel{P}{=} (DX'; BC) \stackrel{P'}{=} (AA'; YZ),$$

as claimed. $\square$

The rest is routine computation: in fact $\angle ACP' = \angle ABP' = 60^\circ$ by median formula and Law of Cosines, so $CP^2 = \frac{365}{4} - 7\sqrt{22}$, and the requested sum is 365492. $\square$

27. The *equatorial algebra* is defined as the real numbers equipped with the three binary operations $\natural$, $\sharp$, $\flat$ such that for all $x, y \in \mathbb{R}$, we have

$$x \natural y = x + y, \quad x \sharp y = \max\{x, y\}, \quad x \flat y = \min\{x, y\}.$$

An *equatorial expression* over three real variables x, y, z, along with the *complexity* of such expression, is defined recursively by the following:

- x, y, and z are equatorial expressions of complexity 0;
- when P and Q are equatorial expressions with complexity p and q respectively, all of $P \natural Q$, $P \sharp Q$, $P \flat Q$ are equatorial expressions with complexity $1 + p + q$.

Compute the number of distinct functions $f : \mathbb{R}^3 \to \mathbb{R}$ that can be expressed as equatorial expressions of complexity at most 3.

Proposed by Yannick Yao.

Answer. $\boxed{588}$

Solution. For ease of notation, let us define some more notations representing certain equatorial expressions:

$$a = \min(y, z), b = \min(z, x), c = \min(x, y), A = \max(y, z), B = \max(z, x), C = \max(x, y),$$

$$m = \min(x, y, z), M = \max(x, y, z), k^+ = \max(k, 0), k^- = \min(k, 0).$$

The main difficulty in this problem comes from the many identities these opertions satisfy. We begin our enumeration with the easiest cases, making sure that in later cases we discard expressions that reduce to an earlier or the same case.

Case 1. *Only addition is used.* When the complexity is n, there are $\binom{n+2}{3}$ different possible functions (by Stars and Bars), so we have $3 + 6 + 10 + 15 = \mathbf{34}$ functions in this case.

Case 2. *Only max and min are used.* When the complexity is 1, the possibilities are a, b, c, A, B, C.

When the complexity is 2, we note that $\max(x, b) = x$ by absorption law, and also $\max(x, A) = M, \max(x, B) = B$, so the possibilities are

$$m, M, \max(x, a), \max(y, b), \max(z, c), \min(x, A), \min(y, B), \min(z, C).$$

When the complexity is 3, we consider combining either a complexity-0 expression and a complexity-2 expression or two complexity-1 expressions.

- For a complexity-0 expression and a complexity-2 expression, note that if the latter is m or M then nothing new will be created, so let us suppose that it is $\max(x, a)$. Max-ing or min-ing it with x does not create anything new, so we consider $\max(y, \max(x, a))$ and $\min(y, \max(x, a))$. The former is simply $\max(x, y) = c$, and the latter is equal to $\min(y, \max(x, z))$ because $\min(y, \max(x, a)) = \max(\min(y, x), \min(y, a)) = \max(\min(y, x), \min(y, z)) = \min(y, \max(x, z))$ by distributive laws. Therefore, nothing new can be created in this case.
- For two complexity-1 expressions, we note that min-ing two min expressions gives m, and max-ing two min expressions like $\max(a, b)$ gives $\min(z, C)$ from distributive law, so we must combine a min-expression and a max-expression, say a and B. However, since $a \le z \le B$ always holds, nothing new will be created either.

Therefore, there are $6 + 8 = \mathbf{14}$ functions in this case.

Case 3. *One addition and one max/min.* We WLOG consider expression that has one addition and one max.

- If the max contains the addition, then there are four possible forms: $\max(x, 2x) = x + x^+$, $\max(x, 2y)$, $\max(x, y + z)$ or $\max(x, x + y) = x + y^+$. These possibilities give $3, 6, 3, 6$ functions respectively.
- If the addition contains the max, then there are again two possible forms $x + A$ or $x + B$. The former gives 3 functions and the latter gives 6 functions.

It is not difficult to verify that these functions (as well as the version with addition and min) are all distinct and new. Therefore we have $2 \cdot (3 + 6 + 3 + 6 + 3 + 6) = \mathbf{54}$ functions in this case.

Case 4. *Two additions and one max/min.* Again we WLOG suppose that the expression has one max.

- If the max contains both additions, then we consider the two subcases:
  - The two expressions in the max each has one addition. The only forms we need to avoid is when both expressions share a common variable like $\max(x + y, x + z) = x + \max(y, z)$. Therefore, there are two possible forms: $\max(2x, y + z)$ and $\max(2x, 2y)$. There are 3 and 3 functions for each form.
  - One of the two expressions in the max have two additions. There are 10 ways to choose this complexity-2 sum and 3 ways to choose the complexity-0 expression, so 30 functions in this case.
- If the max contains one addition, then from previous case it has the form of $k_0 + \max(x, y + z)$, $k_0 + \max(x, 2y)$, or $k_0 + k_1 + k_2^+$ where $k_0, k_1, k_2 \in \{x, y, z\}$. There are $9, 18, 18$ functions for each form.
- If the max contains no additions, then it has the form $k_0 + k_1 + A$. There are $6 \cdot 3 = 18$ functions in this form.

Once again, it is not difficult to verify that these function and their min-analog are all distinct and new, so we have $2 \cdot (3 + 3 + 30 + 9 + 18 + 18 + 18) = \mathbf{198}$ functions in this case.

Case 5. *One addition and two maxes (or two mins).* There are the following possible forms:

- The addition is outside both maxes, like $A + B$ or $x + M$. Observe that $2A = \max(2y, 2z)$ which is already covered before, so there are 3 and 3 functions in each form.
- The addition is inside both maxes, like $\max(x, \max(y, k_0 + k_1)) = \max(x, y, k_0 + k_1)$. There are $3 \cdot 6 = 18$ functions in this form.
- The addition is in one of the maxes and outside the other, like $\max(k_0 + A, k_1)$ (note that the other possibility $\max(k_0 + k_1, A)$ is covered in the previous form). There are $3 \cdot 3 \cdot 3 = 27$ functions in this form.

Therefore, we have $2 \cdot (3 + 3 + 18 + 27) = \mathbf{102}$ functions in this case.

Case 6. *One addition, one max, and one min.* Consider the following subcases:

- The outermost operation is addition.
  - The two inner expressions are both complexity-1. The only case we need to avoid is $A + a = y + z$, so there are 6 functions.
  - One of the two expressions is complexity-2. There are $3 \cdot 6 = 18$ functions in this case.
- The outermost opertaion is max (or min).
  - The two inner expressions are each of complexity 1, then the expression looks like $\max(k_0 + k_1, a)$. Hence there are $6 \cdot 3 = 18$ functions in this case.
  - One of the two expressions has complexity 2 where the addition is outside the min (or max), then the expression looks like $\max(k_0, k_1 + a)$. There are $3 \cdot 3 \cdot 3 = 27$ functions in this case.
  - One of the two expressions has complexity 2 where the addition is inside the min (or max), then the expression looks like $\max(x, \min(y, k_1 + k_2))$. (Note that $\max(x, \min(x, k_1 + k_2)) = x$.) There are $3 \cdot 2 \cdot 6 = 36$ functions in this case.

  Therefore, we have $6 + 18 + 2 \cdot (18 + 27 + 36) = \mathbf{186}$ functions in this case.

In conclusion, we get $34 + 14 + 54 + 198 + 102 + 186 = \boxed{588}$ functions in total.

I'm sorry. $\square$

28. Let A_0BC_0D be a convex quadrilateral inscribed in a circle ω. For all integers $i \geq 0$, let P_i be the intersection of lines A_iB and C_iD, let Q_i be the intersection of lines A_iD and BC_i, let M_i be the midpoint of segment P_iQ_i, and let lines M_iA_i and M_iC_i intersect ω again at A_{i+1} and C_{i+1}, respectively. The circumcircles of $\triangle A_3M_3C_3$ and $\triangle A_4M_4C_4$ intersect at two points U and V.

If $A_0B = 3$, $BC_0 = 4$, $C_0D = 6$, $DA_0 = 7$, then UV can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c such that $\gcd(a, c) = 1$ and b is squarefree. Compute $100a + 10b + c$.

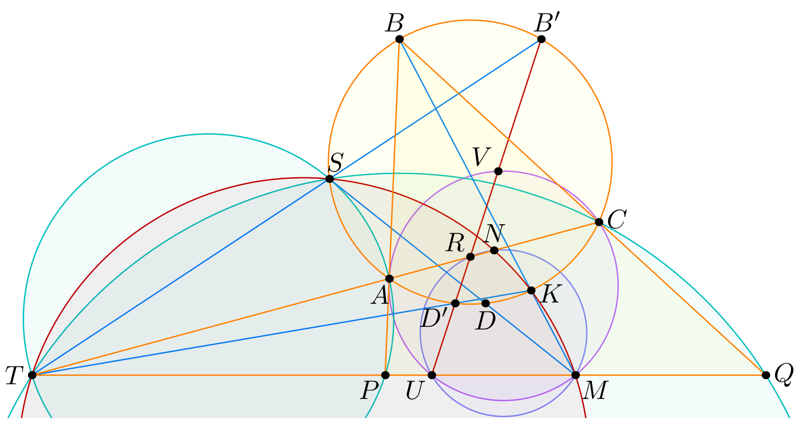
Proposed by Eric Shen.

Answer. $\boxed{4375}$

Solution. The circles $(A_iM_iC_i)$ are coaxial for all nonnegative integers i. Let $R = \overline{A_0C_0} \cap \overline{BD}$. Since $\overline{A_iA_{i+1}} \cap \overline{C_iC_{i+1}} = M_i$ for all i, by Brokard's theorem $\overline{A_iC_i} \cap \overline{A_{i+1}C_{i+1}}$ lies on the polar of M_i. Hence we have by induction that all $\overline{A_iC_i}$ concur on at R. Thus $\overline{P_iQ_i}$ is always the polar of R, so the points P_i and Q_i are all collinear on a line ℓ.

Claim. Let $ABCD$ be a quadrilateral with circumcircle ω and circumcenter O, and let $P = \overline{AB} \cap \overline{CD}$, $Q = \overline{AD} \cap \overline{BC}$, $R = \overline{AC} \cap \overline{BD}$. The reflection of $\overline{BD}$ in $\overline{OR}$ intersects $\overline{PQ}$ at U, and M is the midpoint of $\overline{PQ}$. Let V be the foot from O to $\overline{RU}$. Then $CMUAV$ is cyclic.

Proof. Let $T = \overline{AC} \cap \overline{PQ}$. Let S be the Miquel point of $APQC$ and K the Miquel point of $AQPC$. Then the spiral similarity at S sending $\overline{AC}$ to $\overline{PQ}$ sends N to M, and the spiral similarity at K sending $\overline{AC}$ to $\overline{QP}$ sends N to M. Hence S, K, T, M, N are concyclic.



Let B' and D' be the reflections of B and D aross $\overline{OR}$. Note that $\measuredangle STP = \measuredangle SAP = \measuredangle SAB = \measuredangle SB'B$, so T, S, B' collinear. Similarly T, K, D' collinear. By Reim's theorem on ω, (MNT), we have B, K, M collinear and D, S, M collinear.

Hence $\measuredangle RNM = \measuredangle TNM = \measuredangle TSM = \measuredangle B'SD = \measuredangle D'B'B = \measuredangle RUM$, thus $RNMU$ is cyclic. Since $-1 = (AC; RT)$, we have $TA \cdot TC = TR \cdot TN = TU \cdot TM$, so $CMUA$ is cyclic.

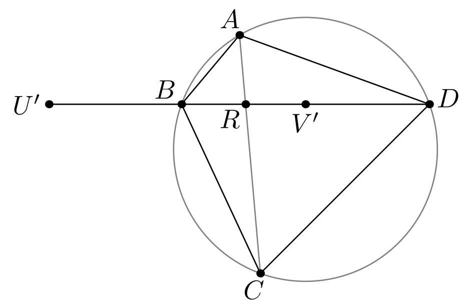
Finally V is the midpoint of $\overline{B'D'}$. Since $-1 = (B'D'; RU)$, we have $RU \cdot RV = RB' \cdot RD' = RA \cdot RC$, as desired. $\square$

Let $A = A_0$ and $C = C_0$. In summary, to construct U and V, we let O be the circumcenter, $R = \overline{AC} \cap \overline{BD}$, and let B_0 and D_0 be the reflections of B and D across $\overline{OR}$. Then U is the intersection of $\overline{B_0D_0}$ and the polar of R, and V is the midpoint of $\overline{B_0D_0}$.

Reflect U and V across $\overline{OR}$, so that U' is the intersection of $\overline{BD}$ and the polar of R, and V' is the midpoint of $\overline{BD}$. The task is to compute $U'V'$.

However $-1 = (BD; RU')$, so $RB \cdot RD = RU' \cdot RV'$. Thus

$$U'V' = RV' + \frac{RB \cdot RD}{RU'} = RV' + \frac{(BD/2)^2 - RV'^2}{RV'} = \frac{BD^2}{4RV'}.$$



The rest is just a routine computation. Let $x = \cos\angle BAD$. By Law of Cosines on $\triangle ABD$ and $\triangle BCD$, we have BD^2 is equal to the two quantities

$$3^2 + 7^2 - 2\cdot 3\cdot 7x = 4^2 + 6^2 + 2\cdot 4\cdot 6x$$
$$\implies 58 - 42x = 52 + 48x \implies x = 1/15.$$

Hence $BD^2 = 276/5$. Now

$$\frac{BR}{RD} = \frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} = \frac{AB\cdot BC}{AD\cdot DC} = \frac{2}{7},$$

so $BR/BD = 2/9$ and $RV'/BD = 5/18$. Thus

$$U'V' = \frac{BD}{4\cdot RV'/BD} = \frac{9}{10}BD = \frac{9\sqrt{345}}{25},$$

and the requested sum is $900 + 3450 + 25 = 4375$. □

29. Let $x_0, x_1, \ldots, x_{1368}$ be complex numbers. For an integer m, let $d(m)$, $r(m)$ be the unique integers satisfying $0 \le r(m) < 37$ and $m = 37d(m) + r(m)$. Define the 1369×1369 matrix $A = \{a_{i,j}\}_{0\le i,j\le 1368}$ as follows:

$$a_{i,j} = \begin{cases} x_{37d(j)+d(i)} & r(i) = r(j),\ i \ne j \\ -x_{37r(i)+r(j)} & d(i) = d(j),\ i \ne j \\ x_{38d(i)} - x_{38r(i)} & i = j \\ 0 & \text{otherwise} \end{cases}.$$

We say A is r-*murine* if there exists a 1369×1369 matrix M such that r columns of $MA - I_{1369}$ are filled with zeroes, where I_{1369} is the identity 1369×1369 matrix. Let $\text{rk}(A)$ be the maximum r such that A is r-murine. Let S be the set of possible values of $\text{rk}(A)$ as $\{x_i\}$ varies. Compute the sum of the 15 smallest elements of S.

Proposed by Brandon Wang.

Answer. $\boxed{3312}$

Solution. Let $n = 37$. It is n^2 minus the following numbers n^2, $(n-1)^2+1$, $(n-2)^2+2$, $(n-2)^2+4$, $(n-3)^2+3$, $(n-3)^2+5$, $(n-3)^2+9$, $(n-4)^2+4$, $(n-4)^2+6$, $(n-4)^2+8$, $(n-4)^2+10$, $(n-4)^2+16$, $(n-5)^2+13$, $(n-5)^2+17$, $(n-5)^2+25$. □

30. Let c be the smallest positive real number such that for all positive integers n and all positive real numbers $x_1, \ldots, x_n$, the inequality

$$\sum_{k=0}^{n} \frac{(n^3+k^3-k^2n)^{3/2}}{\sqrt{x_1^2+\cdots+x_k^2+x_{k+1}+\cdots+x_n}} \le \sqrt{3}\left(\sum_{i=1}^{n} \frac{i^3(4n-3i+100)}{x_i}\right) + cn^5 + 100n^4$$

holds. Compute $\lfloor 2020c \rfloor$.

Proposed by Luke Robitaille.

Answer. $\boxed{777}$

Solution. $c = \frac{2\sqrt{3}}{9}$ □