Summary of Some Concepts in Projective Geometry

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Intuitively, we understand how to project an image onto a surface. We take a light source, shine it through something translucent and a similar image appears on the surface. This assumes that the translucent film and the screen are parallel. For the purposes of this document, we will consider a projection to emanate **from a single point P** and that the **projection of point Q onto line ℓ is the intersection of ℓ with the line through P and Q**.

1.The Cross Ratio

Given four values on the real line, a, b, c, and d, we can define their cross ratio as:

$$\boxed{(a, b; c, d) = \frac{a-c}{b-c} \times \frac{b-d}{a-d}}$$

This can be generalized to any four collinear points A, B, C, and D by taking **directed lengths**:

$$(A, B; C, D) = \frac{|\overrightarrow{CA}|}{|\overrightarrow{CB}|} \div \frac{|\overrightarrow{DA}|}{|\overrightarrow{DB}|} = \frac{|\overrightarrow{CA}|}{|\overrightarrow{CB}|} \times \frac{|\overrightarrow{DB}|}{|\overrightarrow{DA}|}$$

2. Harmonic Conjugate Points

When $(A, B; C, D) = -1$, we say that C and D are **harmonic conjugates** with respect to A and B.

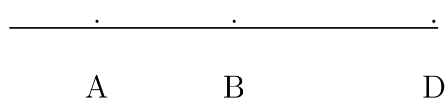
Note that if C and D are harmonic conjugates with respect to A and B, then

$$(C, D; A, B) = \frac{|\overrightarrow{AC}|}{|\overrightarrow{AD}|} \times \frac{|\overrightarrow{BD}|}{|\overrightarrow{BC}|} = \frac{|\overrightarrow{AC}|}{|\overrightarrow{BC}|} \times \frac{|\overrightarrow{AD}|}{|\overrightarrow{BD}|} = -1.$$

So A and B are harmonic conjugates with respect to C and D as well. For this reason, people sometimes refer to such a selection of four points as a {bfharmonic bundle.

Given collinear points A, B, and D, we can construct the harmonic conjugate to D with respect to A and B by performing the following steps:

Diagram:



<u>Step 1</u>: Label any point Q that is not collinear with A, B, and D.

<u>Step 2</u>: Draw any line through D that intersects QA at M and QB at N so that M and N are located between Q and A and between Q and B respectively.

<u>Step 3</u>: Let the intersection point of NA and MB be K. Then the line through Q and K intersects the original line at C.

Point C is the harmonic conjugate to D with respect to A and B.
Note: This is one configuration. Harmonic conjugates can also lie on a circle.

Question: What is the harmonic conjugate to the midpoint of A and B with respect to A and B?

3. Pencils

Let P be a point that is not collinear with A, B, C, and D. The the **pencil** $P(A, B, C, D)$ is the set of four lines: PA, PB, PC, and PD. When A, B, C, and D for a harmonic bundle, we say that $P(A, B, C, D)$ is harmonic.

Note: The term "pencil" often refers to an abstraction of the above definition and can also be generalized to more than two dimensions.

4. Fast Facts:

(1) The above diagram produces harmonic conjugate points C and D with respect to A and B.

(2) Given a triangle △QAB with points M, N, and C on sides QA, QB and AB respectively, and the intersection point between lines MN and AB called D, the points A, B, C, and D are harmonic if and only if QC, MB, and NA are concurrent.

(3) Let the lines in pencil $P(A, B, C, D) = \{PA, PB, PC, PD\}$ intersect line ℓ at A', B', C', and D' respectively. Then the cross ratios: $(A', B'; C', D') = (A, B; C, D)$.

(4) A, B, C, and D from (3), are harmonic if and only if A', B', C', and D' are harmonic.

(5) Let points A, B, C, and D be collinear and P a point not collinear with the others. Then the cross ratio:

$$(A, B; C, D) = \frac{\sin \angle APC}{\sin \angle APD} \cdot \frac{\sin \angle BPD}{\sin \angle BPC}$$

(6) If A, B, C, D form a harmonic bundle and M is the midpoint of AB, then $MC \cdot MD = MA^2$ and $DC \cdot DM = DB \cdot DA$.

(7) Let P, A, B, C, and D be points on a circle. Then PA, PB, PC, and PD form a harmonic pencil if and only if $\dfrac{AC}{BC} = \dfrac{AD}{BD}$ (i.e., if and only if $ACBD$ is a **harmonic quadrilateral** – a term that only applies to cyclic quadrilaterals). In this case, QA, QB, QC, and QD form a harmonic pencil for **any** point

Q on the circle.

(8) Points A, C, B, D lie on a line in this order. Let P be a point not on on this line. Then any two of the following conditions imply the third:
  (i) $(A, B; C, D)$ is harmonic.
  (ii) PB is the angle bisector of $\angle CPD$.
  (iii) $AP \perp PB$.

(9) A point P is outside or on a circle Γ. Let PC, PD be tangents to Γ, and ℓ be a line through P intersecting Γ at A, B (so that P, A, B are collinear in this order). Let AB intersect CD at Q. Then $ACBD$ is a harmonic quadrilateral and $(P, Q; A, B)$ is harmonic.

5. Poles and Polars

Given a circle Γ with centre O and a point P different from O, let P' is a point on the ray OP such that $OP \cdot OP' = r^2$. Such a point P' is the **inversion** of P through Γ where r is the radius of Γ. The line ℓ passing through P' that is perpendicular to OP is called the **polar** of point P with respect to Γ. Point P is called the **pole** of ℓ with respect to Γ.

(10) The locus of all conjugate points of D with respect to A and B is a line. In fact, it is the polar of D with respect to a circle through A and B. (Any such circle?)

(11) Two points M and N are conjugates iff one lies on the polar of the other

(12) If P is a point not on circle Γ, and if a variable line through P meets Γ at points A and B, then the variable harmonic conjugate of P with respect to A and B traces out a line. The point P is called the pole of that line of harmonic conjugates, and this line is called the polar line of P with respect to Γ.

(13) If three polars intersect in a point, then the three poles are collinear and vice versa.

(14) Consider a circle Γ and a point P outside of it. Let PC and PD be the tangent vectors from P to Γ. Then CD is the polar of P with respect to Γ.

(15) **La Hire's Theorem:**
Let Γ be a circle and P, Q points distinct from the centre of Γ. If the polar of P with respect to Γ passes through Q then the polar of Q with respect to Γ passes through P.

(16) Let Γ be a circle and D not on Γ and distinct from its centre. Let ℓ be the polar of D with respect to Γ. A line through D intersects ℓ at C, and Γ at A and B. Then $(A, B; C, D)$ is a harmonic bundle.

(17) Let Γ be a circle and P a point not lying on it and different from its centre. Let A, B be points on Γ such that the line AB is the polar of P . A line through P intersects the circle at points C and D. Then $ACBD$ is a harmonic cyclic quadrilateral.

(18) **Brockard's Theorem:**
The points A, B, C, D lie in this order on a circle Γ with centre O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of $\triangle PQR$. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to Γ.