Preparation for Saudi Arabia Team 2021

June Session: Junior Balkan Mathematics Olympiad

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Lesson 4

Problems with tiling

Problems:

1. A rectangle board $m \times n$, where m is an odd number, is tiled with 2×2 squares and L-tetraminoes. Find the minimum number of L-tetraminoes required.

2. A *configuration* of an tetramino is an arrangement of a tetramino so that the sides of the unit squares are aligned horizontally and vertically. Two configurations are only considered equal if and only if one can be brought into the other through translation (rotations and reflections are *not* allowed). For example, a 4×1 rectangle is not the same configuration as a 1×4 rectangle. Find the set of all possible configurations for the set of tetraminoes and prove that one cannot tile any rectangle with this set (wherein each configuration is only allowed to be moved by translation).

3. Is it possible to tile a 5×5 rectangle with 16 3×1 rectangles such that each field (unit square) is covered at most twice.

4. For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

5. A 6×6 board is covered in dominos. Show that there is a line that cuts the board into two without cutting any domino.

6. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

7. A rectangle $\mathcal{R}$ is tiled with rectangles such that each rectangle has an integer side. Prove that $\mathcal{R}$ also has an integer side.

8. Let n a positive integer. In a $2n \times 2n$ board, $1 \times n$ and $n \times 1$ pieces are arranged without overlap. Call an arrangement maximal if it is impossible to put a new piece in the board without overlapping the previous ones. Find the least k such that there is a maximal arrangement that uses k pieces.

9. Let $a, b > 1$ be odd positive integers. A board with a rows and b columns without fields $(2,1), (a-2,b)$ and (a,b) is tiled with 2×2 squares and 2×1 dominoes (that can be rotated). Prove that the number of dominoes is at least:

$$\frac{3}{2}(a+b) - 6.$$