Warmup Problems

Algebra

1. Do there exist three different quadratic trinomials $f(x), g(x), h(x)$ such that the roots of the equation $f(x) = g(x)$ are 1 and 4, the roots of the equation $g(x) = h(x)$ are 2 and 5, and the roots of the equation $h(x) = f(x)$ are 3 and 6?

2. Let a, b and c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{a+b+c}{3} \geq \frac{a}{a^2b+2} + \frac{b}{b^2c+2} + \frac{c}{c^2a+2}.$$

3. For any positive integer n, define

$$c_n = \min_{z_1,z_2,\ldots,z_n \in \{-1,1\}} \left| z_1 \cdot 1^{2019} + z_2 \cdot 2^{2019} + \cdots + z_n \cdot n^{2019} \right|.$$

   Is the sequence (c_n) bounded?

4. Find all non-decreasing functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$f\left(\frac{x+f(x)}{2} + y\right) = 2x - f(x) + f(f(y))$$

   for each $x, y \in \mathbb{R}_{\geq 0}$.

Combinatorics

1. Students on red, blue and green teams are arranged in a circle. When a teacher asked students on the red team with at least one green team neighbor to raise their hands, 20 children raised their hands. When she asked students on the blue team with at least one green team neighbor to raise their hands, 25 children raised their hands. Prove that at least one student who raised his hand had two neighbors on the green team.

2. There are 100 distinct real numbers given. Prove that it is possible to arrange these numbers in a 10×10 table so that the difference between any two numbers in cells with common side is not equal to 1.

3. David and Sarah play a game on a 100×100 board. First, David writes an integer from 1 to 10000 in each square of the board so that each number is used exactly once. Then Sarah chooses a square in the leftmost column and places a token on this square. She makes a number of moves in order to reach the rightmost column. In each move, the token is moved to a square adjacent by side or by vertex. For each visited square (including the starting one) Sarah pays David the number of coins equal to the number written in that square. Sarah wants to pay as little as possible, whereas David wants to write the numbers in such a way to maximize the amount he will receive. How much money will Sarah pay David if both players follow their best strategies?

4. A snake of length k is an animal which occupies an ordered k-tuple $(s_1, \ldots, s_k)$ of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. If the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead. The snake has turned around if it occupied $(s_1, s_2, \ldots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \ldots, s_1)$ instead. Determine whether there exists an integer $n > 1$ such that: one can place some snake of length $0.9n^2$ in an $n \times n$ grid which can turn around.

Geometry

1. Let $ABCD$ be a rectangle and K be a point on the ray DC with $DK = BD$. If M is the midpoint of BK, prove that AM is the angle bisector of $\angle BAC$.

2. A circle is tangent to side AB of a triangle ABC at A, is tangent to BC at P and intersects AC at Q. The reflection of the line PQ with respect to AC meets AP at X. Prove that $PC = CX$.

3. A point P on the side AB of a triangle ABC and points S and T on the sides AC and BC are such that $AP = AS$ and $BP = BT$. The circumcircle of PST meets the sides AB and BC again at Q and R, respectively. The lines PS and QR meet at L. Prove that the line CL bisects the segment PQ.

4. Let ABC be a triangle and D be the foot of the altitude from A. Let ℓ be the line that passes through the midpoints of BC and AC, and let E be the reflection of D over ℓ. Prove that the circumcentre of $\triangle ABC$ lies on the line AE.

Number Theory

1. The positive integers $1, 2, \ldots, 121$ are arranged in the squares of a 11×11 table. Dima wrote down the product of numbers in each row and Sasha wrote down the product of the numbers in each column. Could they have written down the same set of 11 numbers?

2. A prime p and a positive integer n are given. The product

$$(1^3 + 1)(2^3 + 1) \cdots ((n-1)^3 + 1)(n^3 + 1)$$

is divisible by p^3. Prove that $p \leq n + 1$.

3. Prove that for every positive integer $d > 1$ and positive integer m the sequence $a_n = 2^{2^n} + d$ contains two terms a_k and a_l with $k \neq l$ such that $\gcd(a_k, a_l) \geq m$.

4. Let n be a positive integer and d be a divisor of n. Consider the set S of n-tuples $(x_1, x_2, \ldots, x_n)$ of nonnegative integers satisfying that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n$ and d divides $x_1 + x_2 + \cdots + x_n$. Prove that exactly half of the n-tuples in S satisfy $x_n = n$.