GENERATING FUNCTIONS

Initial conditions:

(1) Find a (non-recursive) formmula for the Fibonacci sequence.
(2) Determine $\sum_{k=0}^{\lceil \frac{n}{3} \rceil} \binom{n}{3k}$.
(3) Show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.
(4) (CGMO 2004) A deck has 32 cards. 30 of them are in red, yellow and blue, with 10 cards in each colour, given the numbers $1, 2, \ldots, 10$ respectively. 2 of them are different jokers, both with the number 0. A card with the number k counts for 2^k points. Call a group of cards a 'good group' if the sum of their points is 2004. Calculate the number of 'good groups'.

Degenerate problems:

(1) Find the coefficient of x^{2021} in the power series of each of the following:
  (a) $(1-3x)^{5000}$
  (b) $(2-x)^{-10}$
  (c) $\frac{1}{x^3-3x+2}$
(2) For each sequence below, find a general expression for the terms of the sequence.
  (a) $a_0 = 1, a_1 = 2, a_n = 5a_{n-1} - 6a_{n-2}$.
  (b) $a_0 = 1,\ a_n = 2a_{n-1} + n$.
  (c) $a_0 = 1, a_1 = 4, a_n = 4a_{n-1} - 4a_{n-2}$.

Functional problems:

(1) Given a pair of dice, find a way to relabel their faces with positive integers (not both with 1 through 6) so that the probability of rolling each of $2, 3, \ldots, 12$ as a sum is unchanged.
(2) A two-dimensional checkerboard has a checker placed at every square below the x-axis. A checker may jump over another checker, removing the one jumped over in the process. What is the maximal y-coordinate that a checker can attain?
(3) There are 10000 identical red balls, 10000 identical yellow balls and 10000 identical green balls. In how many different ways can we select 2021 balls so that the number of red balls is even or the number of yellow balls is odd?
(4) Find a general expression for the terms of the following sequence: $a_0 = 1, a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i}$.

Problems generated by olympiads:

(1) (Romania 2003) How many n-digit numbers, whose digits are in the set $\{2, 3, 7, 9\}$, are divisible by 3?
(2) (Putnam 2003) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S, s_2 \in S, s_1 \neq s2 and s1 + s2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?
(3) (2017 B3) Suppose that $f(x) = \sum_{i=1}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(\frac{2}{3}) = \frac{3}{2}$, then $f(\frac{1}{2})$ must be irrational.
(4) (ISL 1998) Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine $a_1 998$.
(5) (IMO 1995) Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are there such that the sums of its elements are divisible by p?