Graph Theory

Howard Halim

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Note: In case you're not familiar with some of the terminology, you can ask one of the trainers or search up the definition online.

Introduction

Graph theory questions show up all the time in math olympiads, but they aren't usually phrased in terms of vertices and edges. The first step is being able to recognize a graph theory problem when you see one.

Once you convert the problem into graph theoretic terms, the next step is solving it! There are 3 methods that work especially well for graph problems:

- Induction (induct on the number of vertices, number of edges, or any other reasonable metric for the size of a graph)
- Double Counting (compute the sum of some quantity over all vertices in the graph by rewriting the sum in another way)
- Extremal Principle (focus on the vertex with the maximum or minimum degree)

Of course, these aren't the only possible ways to solve a graph theory question, but they are some of the approaches that you should try first. You'll also come across solutions that combine several of the above ideas.

Induction

When a problem asks you to prove that all graphs have a certain property, that's the perfect time to use induction! This is because removing a vertex or edge from a graph will still give you a graph (just smaller).

Example 1 (CMO 2010 P4)

Each vertex of a finite graph can be coloured either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the colour of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a sequence of operations of this type?

Solution: [I'll add this later]

Induction works well because the statement is true for all graphs. But you can still use it for graphs satisfying a certain condition, as long as you reduce it to a smaller graph that also satisfies the condition.

Example 2 (BAMO 2005 P4)

Let G be a connected graph on n vertices, where n is even. Prove that there is a subset S of the edges such that every vertex is an endpoint of an odd number of edges in S.

Solution: [I'll add this later]

A couple things to keep in mind when using induction for graphs:

- Base case depends on how you induct
  - Inducting on vertices: base case is a graph with 1 vertex
  - Inducting on edges: base case is a graph with n vertices, but no edges
- There are many other ways to get smaller graphs (induced subgraph of any subset of vertices, edge contractions, etc.)

Double Counting

Graphs usually have some symmetry that you can make use of. Usually this involves counting something in two different ways, like the solution below:

Example 3 (Handshake Lemma)

In any graph with E edges, the sum of degrees is

$$\sum_v \deg(v) = 2E$$

Solution:

We'll say that a vertex v and an edge e form a *valid* pair (v, e) if e is one of the edges adjacent to v. Let P be the number of valid pairs in the graph. We can compute P in two ways:

1. By focusing on one vertex v at a time. There are $\deg(v)$ valid pairs that involve our chosen vertex v (one for each of v's edges), so the total number of valid pairs is
$$\sum_v \deg(v)$$
2. By focusing on one edge e at a time. There are 2 valid pairs that involve our chosen edge e (one for each endpoint of e), so there are $2E$ valid pairs.

Therefore,

$$\sum_v \deg(v) = P = 2E$$

Example 4

Suppose a graph has E edges and T triangles. Prove that

$$9T^2 \leq 2m^3$$

Solution:

We prove that

$$3T \leq \sum_v \binom{\deg(v)}{2}$$

by double counting the number of pairs (v, t), where v is a vertex and t is a triangle containing v.

It is equal to $3T$, because for each triangle t there are 3 ways to choose v.

But if each vertex v is part of N_v triangles, then we can also write it as $\sum_v N_v$.

There are $\binom{\deg(v)}{2}$ triangles that contain v, since we just need to pick two vertices a and b out of the $\deg(v)$ vertices adjacent to v. Therefore

$$N_v \leq \binom{\deg(v)}{2}$$

giving us the inequality above (it's an inequality because a and b might not be adjacent).

But there are certain scenarios where $\binom{\deg(v)}{2}$ is a very weak bound, like a star graph with v as the center vertex. We also know that $N_v \leq E$, since there are at most E possible choices for the edge (a, b). Therefore,

$$N_v \leq \min\left(\binom{\deg(v)}{2}, E\right)$$

From here, we get that

$$3T = \sum_v N_v \leq \sum_v \min\left(\binom{\deg(v)}{2}, E\right) \leq \sqrt{\binom{\deg(v)}{2} \cdot m} \leq \deg(v)\sqrt{m/2} = m\sqrt{2m}$$

and squaring both sides finishes the problem.

Extremal Principle

Sometimes it helps to consider extremal objects, such as:

- The vertex with maximal/minimal degree
- The largest clique/independent set
- The longest/shortest cycle in the graph

You can use them in a proof by contradiction (by showing that the object wasn't actually maximal/minimal), or as part of a construction, since it's usually easier to prove things about these extremal objects.

Example 5 (Russia 2011)

Suppose a graph G has no K_4 (complete subgraph on 4 vertices) and is not 3-colorable. Prove that the vertices of G can be partitioned into two sets A and B such that the induced subgraph of A is not 2-colorable and the induced subgraph of B is not 1-colorable.

Solution:

Let A be the smallest odd cycle in G, and B be the remaining vertices. Assume for the sake of contradiction that B is an independent set. Then each vertex of B is adjacent to at most 2 vertices in A, otherwise it contradicts the K_4 condition (if A is a triangle) or contradicts minimality of A, by forming a smaller odd cycle. But then we can 3-color G by coloring the cycle A, then assigning colors to each vertex of B (there will always be a valid color since their degrees are less than 3).

Practice Problems

1. (Graph Fundamentals) Prove these facts about graphs. They are all useful lemmas to know!

   (a) Let G be a graph with n vertices. Then any 2 of these properties imply the third:

   - G is connected
   - G has no cycles
   - G has exactly $n-1$ edges.

   (graphs with these 3 properties are called trees)

   (b) Every tree contains a vertex of degree exactly 1, which is called a *leaf*.

   (c) A graph is bipartite if and only if it has no odd cycles.

   (d) If a graph G has maximum degree d, then it can be colored with $d+1$ colors.

   (e) If an n-vertex graph has at least n edges, then it has a cycle.

   (f) A connected graph (with at least 2 vertices) has an Euler circuit if and only if all degrees are even.

   (g) Every connected graph G has a spanning tree.

   (h) Every tournament (complete directed graph) has a Hamiltonian path.

2. (PUMaC 2019) A finite graph G is drawn on a blackboard. The following operation is permitted: pick any cycle of C of G, draw a new vertex v, connect it to all vertices of C, and finally erase all the edges of C. Prove that this operation can only be done a finite number of times.

3. (EGMO 2017 P4) Let $n \geq 1$ be an integer and let $t_1 < t_2 < \cdots < t_n$ be positive integers. In a group of t_n+1 people, some games of chess are played. Two people can play each other at most once. Prove that it is possible for the following two conditions to hold at the same time:

   - The number of games played by each person is one of $t_1, t_2, \ldots, t_n$.
   - For every i with $1 \leq i \leq n$, there is someone who has played exactly t_i games of chess.

4. Let T be a tree with n edges. Prove that any graph G with minimum degree at least n has a subgraph isomorphic to T.

5. Prove that a tournament has a (directed) Hamiltonian cycle if and only if it is strongly connected.

6. (Zarankewicz) Let G be a graph with no $(k+1)$-clique. Prove that some vertex has degree at most $\frac{k-1}{k}n$.

7. (Russia 1999) In a country, there are N airlines that offer two-way flights between pairs of cities. Each airline offers exactly one flight from each city in such a way that it is possible to travel between any two cities in the country through a sequence of flights, possibly from more than one airline. If $N-1$ flights are cancelled, all from different airlines, show that it is still possible to travel between any two cities.

8. Let G be a finite simple graph with $m > 0$ edges and $n > 1$ vertices. Show that one can delete some number of vertices of G to obtain a graph with at least one vertex whose minimum degree is at least m/n.

9. (USA TSTST 2011) In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

10. (Ivan Borsenco, Mathematical Reflections) Prove that a $3n$-vertex graph with no K_4 has at most n^3 triangles.

11. (USA TSTST 2021 P5) Let T be a tree on n vertices with exactly k leaves. Suppose that there exists a subset of at least $\frac{n+k-1}{2}$ vertices of T, no two of which are adjacent. Show that the longest path in T contains an even number of edges.

12. (CMO 2019 P5) A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.

13. (Korea 2015 P6) There are 2015 distinct circles of radius 1 in the plane. Prove that one may choose 27 of them, such that either any two intersect, or any two don't intersect.

14. (Russia 2015) Suppose 110 teams participated in a volleyball tournament; every two teams played against each other exactly one game (in volleyball, there are no draws). It happened that in every group of 55 teams there exists a team which has lost to at most four out of the other 54 teams in this group. Prove that in the whole tournament there exists a team which has lost to at most four out of the other 109 teams.

15. A graph has n vertices and m edges. If the edges are assigned the labels $1, 2, \ldots, m$ prove that there exists a path consisting of at least $\frac{2m}{n}$ vertices such that the labels of the edges along the path are in increasing order.

16. (CMO 2020 P5) Let G be a graph with 19998 vertices such that any 9999-vertex subgraph of G has at least 9999 edges. Determine the minimum possible number of edges in G

17. (ISL 2013 C3) A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

    (i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

    (ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

    Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.