Mock Olympiad #2

1. Let p be the product of two consecutive integers greater than 2. Show that there are no integers $x_1, x_2, \ldots, x_p$ satisfying the equation

$$\sum_{i=1}^{p} x_i^2 - \frac{4}{4 \cdot p + 1} \left(\sum_{i=1}^{p} x_i \right)^2 = 1.$$

2. Let S $= \{x_1, x_2, \ldots, x_{k+l}\}$ be a $(k+l)$-element set of real numbers contained in the interval $[0, 1]$; k and l are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{l} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k+l}{2kl}$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+l} \dbinom{k+l}{k}$.

3. In $\triangle ABC$, let AA_0, BB_0, CC_0 be altitudes. Let A_1 be a point inside $\triangle ABC$ such that

$$\angle A_1BC = \angle A_1AB, \ \angle A_1CB = \angle A_1AC.$$

Let A_2, B_2, C_2 be midpoints of AA_1, BB_1, CC_1 respectively. Prove that A_2A_0, B_2B_0, C_2C_0 are concurrent.