Problem Solving Session

1. Each grid point of a cartesian plane is coloured with one of three colours and each colour is used on at least one point. Show that there is a right triangle with all three colours at its vertices.

2. In a large room, there are 2019 telephones. Every two telephones are connected by a cable that is coloured in one of four colours. There is at least one cable of each colour. Can Dorette always find a set of telephones such that exactly three colours appear among the cables connecting them?

3. Two numbers are written on each vertex of a convex 100-gon. Prove that it is possible to remove a number from each vertex so that the remaining numbers on any two adjacent vertices are different.

4. There are 101 distinct integers chosen between 0 and 1000. Prove that among the absolute values of their pairwise differences, there are ten different numbers not exceeding 100.

5. David walks around on the squares of a 15×15 chessboard, visiting each square at most one time and at each step moving to a square sharing a side with his current square. If David's path is symmetric with respect to a main diagonal of the chessboard, prove that he has visited at most 200 squares.

6. At James Rickards high school, a secret code consists of words of length between 1 and 13 using only the letters J and R. If no two words in the code can be written back to back to form another word in the code, what is the largest possible number of words in the code?

7. Show that there exist four integers a, b, c, d whose absolute values are all greater than 1000000 and which satisfy
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{abcd}$$

8. There are $n > 1$ cities in a country, some pairs of which are joined by a two-way flight. For each pair of cities, there is exactly one path of flights between them. The mayor of each city C counts the number of ways to number the cities from 1 to n such that every path beginning with C has numbers appearing in ascending order. If all mayors but one report a number divisible by 2016, prove that last mayor also reported a number divisible by 2016.

9. Find all positive integers n for which there are positive integers a, b, c, d with
$$a+b+c+d=n\sqrt{abcd}$$

10. Two players alternate writing down positive integers between 1 and 1000 so that they never write down the divisor of a previously written number. Who has a winning strategy?

11. Initially, a positive integer N is written on the blackboard. Every second, Sarah computes the product P of the nonzero digits of N and replaces N with $N+P$. Prove that there is a positive integer that will be P infinitely many times.

12. If $x^n + x^{n-1} + \cdots + 1 = f(x)g(x)$ where f and g are polynomials with nonnegative real coefficients, prove that each of the coefficients of f and g is either 0 or 1.

13. There are 999 scientists. Every two scientists are both interested in exactly one topic and for each topic there are exactly three scientists that are interested in that topic. Prove that it is possible to choose 250 topics such that every scientist is interested in at most one of the chosen topics.

14. On the concealed side of 2013 cards, distinct numbers are written. In a single move, Dorette is allowed to point to ten cards and Sarah will tell her the set of ten numbers written on them. What is the largest number N such that Dorette can deduce the numbers on the backs of N cards?

15. If $x_1, x_2, \ldots, x_n \in [0,1]$, prove that there are $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{-1, 1\}$ such that

$$\left| \sum_{i=1}^{k} \epsilon_i x_i - \sum_{i=k+1}^{n} \epsilon_i x_i \right| \leq 2$$

for all $1 \leq k \leq n$.