Spiral Similarities and Ratios

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A lot of olympiad geo is taught like this: You're given a problem with a bunch of random components. You don't know how to deal with the components. Well, what do we do? It turns out that just through some chance, all these random components fit together with this ONE similarity and this ONE pair of ratios that just solve all the angles and make everything nice. And before you can get to ask how anyone would ever think of comparing those to triangles, you're asked to solve a completely different problem with a completely different diagram.

It's hopeless, right?

Today's lecture on spiral similiarities is going to teach you that it's **not**. In fact, these similar triangles, 99 percent of the time, come from - you guessed it - spiral similarities.

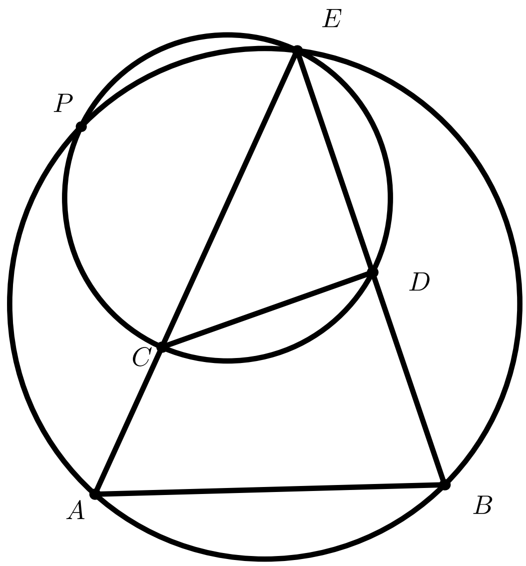
0.1 Notation

Term	Definition
$\cap$	Used to denote the **intersection** of two lines / circles, just like how it would usually denote the intersection of two sets

1 Main Lecture

1.1 Introducing Spiral Similarities

The basic idea between spiral similarities is this: between any two pairs of edges AB and CD, barring very extraneous cases (i.e. $A = B$) there exists a **unique** point P such that the triangle PAB is directly similar to PCD. The way we construct this is to consider $AC \cap BD = E$ - then this point P turns out to be $(EAB) \cap (ECD)$.



The proof that this point P indeed satisfies that $PAB \sim PCD$ is by angle chasing; a possible proof that it is unique is through complex numbers (roughly, don't worry about this; it works.)

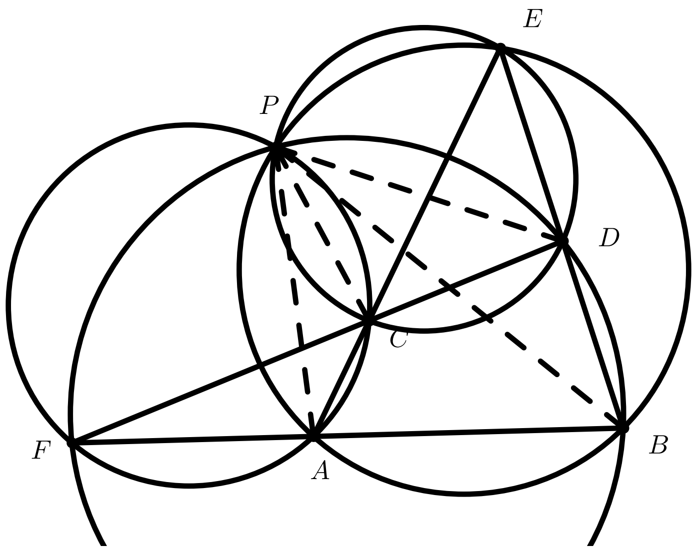
A nice property of this point P is that this spiral similarity is not isolated; specifically, we have that if P is the spiral center taking AB to CD, then it also takes AC to BD. This is because since

$$\begin{aligned} \angle APC &= \angle BPC - \angle APB \\ &= \angle BPC - \angle CPD \\ &= \angle BPD \end{aligned}$$

and

$$\frac{AP}{BP} = \frac{CP}{DP}$$
$$\frac{AP}{CP} = \frac{BP}{DP}$$

we have that $PAC \sim PBD$, as desired!



This is useful for two reasons. One, we just found another pair of similar triangles, that can help us tie together weird angles from easier ones! More however, we have that our "two circles" property still applies: if $AB \cap CD = F$, then $P = (ACF) \cap (BDF)$. So in fact this implies that there are four circles here - (ABE), (CDE), (ACF), and (BDF) - and they all share the common point P, which is also the spiral center! Now that's interesting.

An extra note: the spiral center P is also called the **Miquel point** of $ABDC$. In this case it's $ABDC$, as we're looking at the vertices that would make a convex quadrilateral ($ABCD$ would make a self-intersecting one!)

Example (USA TST 2007 4): Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.

Walkthrough:

1. Consider the quadrilateral $APDX$. What is Q called?
2. What circles do Q lie on as a result?
3. Using this, prove (by angle chasing or otherwise) that X, Q, and Z are collinear, and X, Q, and Y are collinear. This hence proves the problem.

1.1.1 Exercises

1. Let ABC be a triangle, and let $D \in BC$. Let O_1 and O_2 be the circumcenters of ABD and ABC, respectively. Then prove that $AO_1O_2 \sim ADC$.

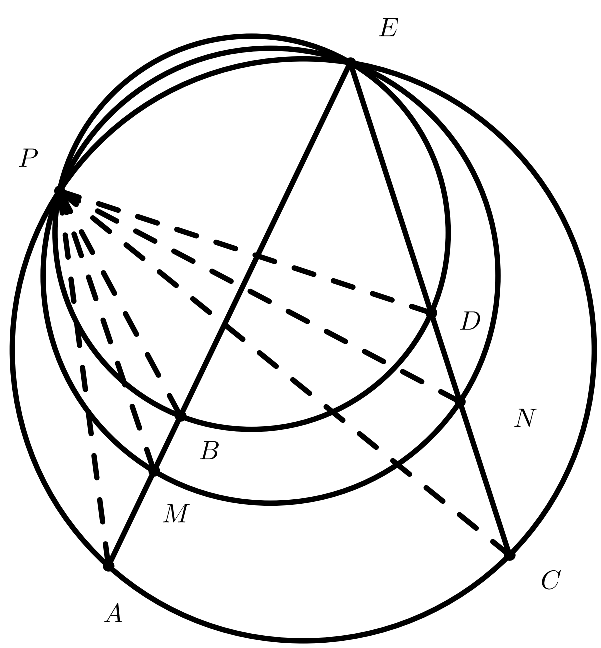
2. Consider an arbitrary quadrilateral $ABCD$ such that $AB \cap CD = P$, $BC \cap DA = Q$. Let the circumcenters of PBC, PDA, QAB, and QCD be O_1, O_2, O_3, and O_4, respectively. Prove that $O_1O_2O_3O_4$ is cylic, and the circumcircle of this cyclic quadrilateral passes through the miquel point of $ABCD$.

3. Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y . The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point Z distinct from point B. The lines ZD and ZY intersect ω again at V and W respectively. Prove that $AB = VW$.

4. Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

1.2 Ratios

One very useful property about spiral similarities is that if $PAB \sim PCD$, and M and N are points on AB and CD respectively such that $\frac{AM}{MB} = \frac{CN}{ND}$, then the spiral similarity centered at P taking AB to CD also takes M to N! In other words, we also have that $PAM \sim PCN$ and $PMB \sim PND$.



This property is especially useful because it really allows us to get a lot of difficult-to-reach angles from spiral similarities. Midpoints and the like are often very annoying to deal with - with this condition however, this allows us to work with their angles just like usual.

Here's a classic example:

Example (USAMO 2006 6): Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Walkthrough:

1. What is $(SAE) \cap (SBF)$ called?
2. What is $(TED) \cap (TFC)$ called?
3. Why are these the same point?

In particular, what this demonstrates is that spiral similarities work really well with **ratios**. In particular, since every pair of directly similar triangles gives you a new similarity, and these pairs can *themselves* be derived from previous ratios, this implies that often chasing ratios that may seem to be useful may work very well in tandem with spiral similarities.

Two key examples:

Example (Classical): Let ABC be a triangle and let D, E, F be the projections of the incenter of ABC onto BC, AC, AB respectively. Let the circumcircles of ABC and AEF intersect at K. Prove that KD bisects $\angle BKC$.

Walkthrough:

1. In terms of the angle bisector theorem, what would we need to prove in order to show that KD bisects $\angle BKC$?
2. K is a miquel point. Identify what quadrilateral it is the miquel point of, and give a few ratios that result from this spiral similarity.
3. Tie the ratios together to prove the claim.

Example (USA TSTST 2012 7): Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

Walkthrough:

1. Prove that the problem is equivalent to showing that $AL//MN$. This gets rid of H (the most mysterious point in the diagram anyway).
2. Let's now actually work with the problem. What does $(AMD) \cap (ABC)$ seem to be? Call this point S, and prove your proposition. You should have that $S \in ML$.
3. Consider the quadrilateral $BCPQ$. What is S in relation to this quadrilateral?
4. What triangles are SNM similar to?
5. Using this, prove that $\angle SMN = \angle SLA$, and use this to finish the problem.

Example (ISL 2005 G5): Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Walkthrough:

1. First, some setup. Let $Y = BH \cap AC$ and $Z = CH \cap AB$. Prove that if $K = (AYZ) \cap (ABC)$ and $K \neq A$, then K also lies on HM.
2. The main idea here is to prove that $AEKD$ is cyclic. Using spiral similarities, prove that this is equivalent to proving that
$$\frac{ZD}{DB} = \frac{YE}{EC}$$
3. Prove this ratio. It may help to recall that $ZC \cap YB = H$.

In a sense, what this implies is that a good way of thinking about these spiral similarities is as a sort of **algebraic manipulation**: by manipulating these ratios and spiral similarities, we can "reach" a lot of angles that would be previously unobtainable! In particular, some of the harder examples will demonstrate that this approach of algebraic manipulation to push for the exact similarities you want is exactly what is needed to solve many problems.

1.2.1 Exercises

1. In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ = BP/PC$.

2. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

   The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

3. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.

   Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

1.3 Similar Figures

Not only are the above results true for ratios, but they also work with similar figures! What I mean is this: if we have the spiral center P taking AB to CD, and E and F are points such that ABE and CDF themselves are directly similar, then this spiral similarity also takes E to F (as the two triangle similarities imply that $PABE \sim PCDF$.)

This allows us to tie together figures that are significantly more "arbitrary" into a similarity as well! A key example:

Example (2012 G3): In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Walkthrough:

1. Draw a good diagram :)

2. First off; there's no reason not to add the third incenter I_3 of CDE, so draw that in too. What circles do this incenter lie on (don't prove anything yet)?

3. Let's now prove that AI_1I_2B is cyclic. Prove that this is equivalent to proving that $I_1I_2F \sim ADF$.

4. Let's flip the spiral sim, and prove that $AI_1F \sim DI_2F$. Prove that $AEF \sim DBF$, and demonstrate that this implies $AI_1F \sim DI_2F$.

5. With the key claim proved, let's now move on to the problem. Prove that it suffices to show $CI \perp I_1I_2$, where I is the incenter of the triangle.

6. Prove this by angle chasing or otherwise.

7. Where did we use the similar figures argument?

1.4 Final Example

Here's a cool example that really demonstrates the power of combining spiral similarities with ratios:

Example (Original RMM 2019 4): Let there be an equilateral triangle ABC and a point P in its plane such that $AP < BP < CP$. Suppose that the lengths of segments AP, BP and CP uniquely determine the side length of ABC. Prove that P lies on the circumcircle of triangle ABC.

Walkthrough:

1. Let's prove the contrapositive; that if $P \notin (ABC)$, then if one such triangle exists, at least two exist. Consider an arrangement of P, A, B, and C, where $P \notin (ABC)$; we'll prove that there exists A', B', and C' such that $PA' = PA$, $PB = PB'$, $PC = PC'$, $A'B'C'$ is equilateral, and ABC and $A'B'C'$ have different side lengths. Here now comes your first task - prove that we may WLOG assume that $C = C'$, and $CA'B' \sim CAB$ (directly).

2. Let's now say A' and B' exist. Prove that if $K = AA' \cap BB'$, then $K \in (ABC)$.

3. Also prove that a working K would uniquely define an A' and B'. This demonstrates that we just need to determine K.

4. Let's now investigate this picture. The one thing that we haven't done yet is tie in P - we know that $PA = PA'$ and $PB = PB'$, but there doesn't seem to be a great way to use this condition directly. Let's now think - if we add the midpoints M_A and M_B of AA' and BB', then what can we say about C, M_A, M_B, and P?

5. Tie in K now (this is why the midpoints are useful), and prove that $\angle PCK = 90°$. This now gives us a guaranteed way of constructing K (and in fact proves that K, and hence the second triangle's side length, is unique)!

6. Prove that if K indeed satisfies that $\angle PCK = 90°$, then the A' and B' that are constructed satisfy that $CA'B'$ is an equilateral triangle with **different side length** than CAB. As a sanity check, also identify what goes *wrong* when $P \in (ABC)$.

1.5 Recap

1. The **miquel point** P of a quadrilateral $ABCD$ is the spiral center taking AB to DC and AD to BC. If $AB \cap CD = E$ and $AD \cap BC = F$, then P lies on (ADE), (BCE), (ABF), and (CDF).

2. If $M, N \in AB, CD$ respectively such that $\frac{AM}{MB} = \frac{DM}{MC}$, then we have that P also takes AMB to DMC. This property allows us to deal with the angles of points that aren't as tied-into the diagram as the other points.

   (a) The above also applies for points M, N satisfying that $AMB \sim DMC$.

3. To get the most out of spiral similarities, it can often help to often ratio-chase to really eek out the most information from the ratio property.

2 Problems

Here are some general problems regarding the materials of this handout that are less specific than the section exercises above. Note that the section exercises are intended to be easier than the problems below, so feel free to try them first if the following problems are a bit too challenging. Note that I've split the problems into four parts (as per Canada tradition-ish), with the As being IMO 1 level, the Bs being IMO 1-2 level, the Cs being IMO 2-3 level, and the Ds being very hard. I also have some stars below, to mark down which problems I especially like.

2.1 Part A

1. Let $\triangle ABC$ be an acute-angled triangle, and let D be the foot of the altitude from C. The angle bisector of $\angle ABC$ intersects CD at E and meets the circumcircle ω of triangle $\triangle ADE$ again at F. If $\angle ADF = 45°$, show that CF is tangent to ω.

2. ★ Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

3. We say that a triangle ABC is great if the following holds: for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB and AC, respectively, then the reflection of D in the line PQ lies on the circumcircle of the triangle ABC. Prove that triangle ABC is great if and only if $\angle A = 90°$ and $AB = AC$.

4. ★ Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.

5. Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

2.2 Part B

1. Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

2. Let ABC be a triangle such that $AB > BC$ and let D be a variable point on the line segment BC. Let E be the point on the circumcircle of triangle ABC, lying on the opposite side of BC from A such

that $\angle BAE = \angle DAC$. Let I be the incenter of triangle ABD and let J be the incenter of triangle ACE. Prove that the line IJ passes through a fixed point, that is independent of D.

3. ★ Let $ABCD$ be a parallelogram. Points P and Q lie inside $ABCD$ such that $\triangle ABP$ and $\triangle BCQ$ are equilateral. Prove that the intersection of the line through P perpendicular to PD and the line through Q perpendicular to DQ lies on the altitude from B in $\triangle ABC$.

4. ★ The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc $\overset{\frown}{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

   Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

5. Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

6. Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

7. Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

8. ★ Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

2.3 Part C

At this point the problems start requiring a lot more things than just spiral similarities.

1. ★ Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

2. Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through

M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

3. Let $ABCD$ be a quadrilateral with $AC = BD$. Diagonals AC and BD meet at P. Let ω_1 and O_1 denote the circumcircle and the circumcenter of triangle ABP. Let ω_2 and O_2 denote the circumcircle and circumcenter of triangle CDP. Segment BC meets ω_1 and ω_2 again at S and T (other than B and C), respectively. Let M and N be the midpoints of minor arcs $\widehat{SP}$ (not including B) and $\widehat{TP}$ (not including C). Prove that $MN \parallel O_1O_2$.

4. ★ Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

5. Oscar is drawing diagrams with trash can lids and sticks. He draws a triangle ABC and a point D such that DB and DC are tangent to the circumcircle of ABC. Let B' be the reflection of B over AC and C' be the reflection of C over AB. If O is the circumcenter of $DB'C'$, help Oscar prove that AO is perpendicular to BC.

6. Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

7. Let P be a point on the circumcircle of acute triangle ABC. Let D, E, F be the reflections of P in the A-midline, B-midline, and C-midline. Let ω be the circumcircle of the triangle formed by the perpendicular bisectors of AD, BE, CF.

   Show that the circumcircles of $\triangle ADP, \triangle BEP, \triangle CFP$, and ω share a common point.

8. ★ A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas. (solve this synthetically!)

9. Let ABC be a triangle with $AB < AC$, incenter I, and A excenter I_A. The incircle meets BC at D. Define $E = AD \cap BI_A$, $F = AD \cap CI_A$. Show that the circumcircle of $\triangle AID$ and $\triangle I_AEF$ are tangent to each other.

2.4 Part D

1. ★ Let ω be the incircle of the triangle ABC and with centre I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

2. Two circles ω_1 and ω_2 intersect at points A and B. Line l is tangent to ω_1 at P and to ω_2 at Q so that A is closer to l than B. Let X and Y be points on major arcs PA (on ω_1) and AQ (on ω_2), respectively, such that $AX/PX = AY/QY = c$. Extend segments PA and QA through A to R and S, respectively, such that $AR = AS = c \cdot PQ$. Given that the circumcenter of triangle ARS lies on line XY, prove that $\angle XPA = \angle AQY$.

3. ★ Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

4. ★ Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, let M be the midpoint of AC and let I_1, I_2 be the incentres of triangles ABM and CBM. Prove that points I_1, I_2, B, N lie on a circle.

5. ★ An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overleftrightarrow{MN}$ and $\overleftrightarrow{PQ}$. Denote by ℓ the angle bisector of $\angle MSQ$.

   Prove that $\overline{OI}$ is parallel to ℓ, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.

6. ★ In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$ $, CF = PB$. Let K be the intersection point of AP, OH. Prove that $\angle EKF = 90^\circ$

7. ★ Let $ABCD$ be a non-cyclical, convex quadrilateral, with no parallel sides. The lines AB and CD meet in E. Let $M \neq E$ be the intersection of circumcircles of ADE and BCE. The internal angle bisectors of $ABCD$ form an convex, cyclical quadrilateral with circumcenter I. The external angle bisectors of $ABCD$ form an convex, cyclical quadrilateral with circumcenter J.

   Show that I, J, M are collinear.

8. ★ Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of OH. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly. Prove that the common chords of ω_A,ω_B and ω_C are concurrent on line OH.

3 Sources

3.1 1.1 Exercises

1. Salmon's Theorem
2. Lemma 10.6 in Euclidean Geometry in Math Olympiads, by Evan Chen
3. 2015 APMO 1
4. 2002 ISL G4

3.2 1.2 Exercises

1. 2013 USAMO 1
2. 2006 ISL G3
3. 2005 IMO 5

3.3 Part A

1. 2015 EGMO 1
2. 2003 IMO 4
3. 2016 APMO 1
4. 2013 Canada 5
5. 2020 USOMO 1

3.4 Part B

1. 2017 APMO 2
2. Cyberspace Mathematical Competition 3
3. 2017 Canada 4
4. 2016 USAJMO 1
5. 2015 ISL G3
6. 2017 ISL G3
7. 2018 ISL G7
8. 2019 RMM 2

3.5 Part C

1. 2006 ISL G9
2. 2019 TST 1
3. 2012 TSTST 2
4. 2014 ARMO 11.4
5. 2016 ELMO 2
6. 2016 ISL G2
7. 2020 ISL G7
8. 1998 IMO 1
9. 2020 ISL G6

3.6 Part D

1. 2013 ARMO 11.8
2. 2011 TSTST 2
3. 2000 ISL G6
4. 2011 ARMO 11.8
5. 2016 USAMO 5
6. 2017 Iran TST 18
7. 2016 Brazil MO 6
8. 2020 USEMO 3

4 Solutions

USA TST 2007 4: Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.

Note that Q is the miquel point of $APDX$. This implies that $QDXC$ is a cyclic quadrilateral. Now the parallel condition gives us that $\angle DQX = \angle DCX = \angle DPZ = 180° - \angle DQZ$, so X, Q, and Z are collinear. Similarly, since $\angle YQC = \angle YPC = \angle XDC = \angle XQC$, we have that Y, X, and Q are collinear as well, thus solving the problem.

USAMO 2006 6: Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Consider the spiral center K taking AE to BF. Since $\frac{AE}{ED} = \frac{BF}{FC}$, we have that this spiral center also takes ED to FC. Hence K is the miquel point of both $ABFE$ and $EFCD$, and so $K \in (SAE), (SBF), (TED), and (TFC)$, thus solving the problem.

Classical: Let ABC be a triangle and let D, E, F be the projections of the incenter of ABC onto BC, AC, AB respectively. Let the circumcircles of ABC and AEF intersect at K. Prove that KD bisects $\angle BKC$.

Note that K is the spiral center taking FE to BC. This hence implies that:

$$\begin{aligned}\frac{KB}{KC} &= \frac{FB}{EC} \\ &= \frac{DB}{DC}\end{aligned}$$

with the last equality arising from the fact that $FB = DB$ and $EC = DC$ by incircle properties. This thus implies that DK bisects $\angle BKC$, as desired.

USA TSTST 2012 7: Triangle ABC is inscribed in circle Ω. The interior angle bisector of angle A intersects side BC and Ω at D and L (other than A), respectively. Let M be the midpoint of side BC. The circumcircle of triangle ADM intersects sides AB and AC again at Q and P (other than A), respectively. Let N be the midpoint of segment PQ, and let H be the foot of the perpendicular from L to line ND. Prove that line ML is tangent to the circumcircle of triangle HMN.

ISL 2005 G5: Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Let $Y = BH \cap AC$, $Z = CH \cap AB$, and A' be the antipode of A on (ABC). note that $BHCA'$ is a parallelogram (both BA' and CH are perpendicular to AB), so A' lies on HM. But now if $K = A'H \cap (ABC)$, then we know that $\angle AKH = \angle AKA' = 90^\circ$. Hence K lies on (AH), and so $K \in (AYZ)$. Thus, K is the spiral center taking ZY to BC.

Now, note that $\angle ADE = 90^\circ - \frac{1}{2}\angle A = \frac{1}{2}(90^\circ + 90^\circ - \angle A) = \frac{1}{2}(\angle AZC + \angle ABY)$. This thus implies that $\angle DHB = \frac{1}{2}\angle ZHB$, so thus DE bisects $\angle ZHB$. Hence, we have that by the Angle Bisector theorem,

$$\frac{ZD}{DB} = \frac{ZH}{HB}$$
$$\frac{YE}{EC} = \frac{YH}{HC}$$

But in fact $\angle HZB = \angle HYC = 90^\circ$, and $\angle HBZ = \angle HCY$ (as $BZYC$ is cyclic), so thus $HBZ \sim HCY$. This thus implies that:

$$\frac{ZH}{HB} = \frac{YH}{HC}$$

so thus:

$$\frac{ZD}{DB} = \frac{YE}{EC}$$

But now consider K; we have that since K takes BZ to CY through a spiral sim, it also takes BD to CE. But then K is the miquel point of $BDEC$, so since $BD \cap CE = A$ we have that $K \in (ADE)$. Since $K \in HM, (ABC)$ as well, this thus solves the problem!

2012 G3: In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

The main claim is that AI_1I_2B is cyclic. To prove this, note first that $AEF \sim DBF$ by angle chasing (in particular, this is a spiral similarity). Hence this implies that F takes the figure FAE to FDB, and in this spiral similarity also takes I_1 to I_2. So $FI_1I_2 \sim FAD$. Now this implies that:

$$\begin{aligned}
\angle I_1AB + \angle I_1I_2B &= \frac{1}{2}\angle A + \angle ADF + \angle FI_2B \\
&= \frac{1}{2}\angle A + \angle ACF + 90^\circ + \frac{1}{2}\angle FDB \\
&= \frac{1}{2}\angle A + 90^\circ - \angle A + 90^\circ + \frac{1}{2}\angle A \\
&= 180^\circ
\end{aligned}$$

which gives us that AI_1I_2B is cyclic, as desired. ■

Now, let I_3 be the incenter of CDE, and I the incenter of ABC. Note that AI_1, BI_2, and CI_3 all pass through CI. In addition, by an analogous proof to the one above, we have that BI_2I_3C and CI_3I_1A are both cyclic as well. this implies that $O_1O_2 \perp CI_3$. However, note that:

$$\begin{aligned}
\angle CIA &= 90^\circ + \frac{1}{2}\angle B \\
&= 90^\circ + \angle IBA \\
&= 90^\circ + \angle II_1I_2
\end{aligned}$$

This thus implies that $I_1I_2 \perp CI$ as well, and hence $I_1I_2 // O_1O_2$, as desired!

Original RMM 2019 4: Let there be an equilateral triangle ABC and a point P in its plane such that $AP < BP < CP$. Suppose that the lengths of segments AP, BP and CP uniquely determine the side length of ABC. Prove that P lies on the circumcircle of triangle ABC.

We prove that if P does not lie on the circumcircle of the triangle ABC in a valid A, B, C given by the lengths of AP, BP, and CP, then we may construct another equilateral triangle $A'B'C'$ with side length distinct from the side length of ABC such that $A'P = AP$, $B'P = BP$, and $C'P = CP$. Indeed, consider $K \in (ABC)$ such that $\angle KCP = 90^\circ$ (this is uniquely defined; if no such K exists, set $K = C$ - the details of this configuration are left to the reader) - note that this K is uniquely defined by P. Moreover, define A' as the intersection of KA with (P, PA'), and define B' similarly. We claim that $A'B'C$ now works as the desired construction for $A'B'C'$; indeed, it suffices to prove that $A'B'C$ is an equilateral triangle with side length distinct from the side length of ABC.

Indeed, consider the midpoints M_A and M_B of AA' and BB'; note that M_A and M_B are the feet from P to KA and KB. Now, note that M_A, M_B, and C all lie on (MK), so thus CM_AM_BK is cyclic. Since $CABK$ is cyclic as well, this implies that C is the spiral center taking AM_A to BM_B. Now, since $\frac{AA'}{AM_A} = \frac{BB'}{BM_B} = 2$, we have that C also takes AA' to BB'. Thus, this implies that $CA'B' \sim CAB$, and so thus $CA'B'$ is equilateral as well!

We are now almost done; it remains to prove that $CA \neq CA'$. Note that if $P \notin (ABC)$, then we have that both $\angle PAK$ and $\angle PBK$ are not right, as if either of them are, then since $\angle PCK = 90^\circ$ this would imply that PK is a diameter of (ABC), and hence $P \in (ABC)$. This hence implies that $A \neq A'$ and

$B \neq B'$. But then if $CA = CA'$, then $CB = CB'$ as well. Thus C is on both the perpendicular bisector of AA' and BB'. This either implies that $P = C$ (in which case it lies on (ABC); again a contradiction) or AA' and BB' share a perpendicular bisector. But this latter case is also impossible, as it would imply that $AA'//BB'$, when clearly AA' and BB' intersect at K (which is a point not at infinity). Thus all the cases have been exhausted, and hence we must have that $CA \neq CA'$ as desired. This thus demonstrates that $CA'B'$ is indeed a working construction, and hence we are done!