Probabilistic Method

1. Show that one can colour any complete graph with 256 vertices in red and blue so that there is no monochromatic subgraph with 16 verticies and 120 edges.

2. At the IMO, there are n people, some of whom are students and some of whom are guides. Each person brought $k > log_2 n$ different colored shirts. To avoid confusion, the IMO wants to ensure that no guide is wearing the same colored shirt as a student. Prove that there is a choice of shirts which ensures this.

3. In the Duma, there are 1600 delegates who have formed 16000 committees of 80 persons each. Prove that one can find two committees having at least four common members.

4. Let p and q be nonnegative numbers that add up to 1. Let m and n be nonnegative integers. Prove that $(1 - p^m)^n + (1 - q^n)^m \geq 1$.

5. In a (round-robin) tournament, every player plays one game with every other player. A Hamiltonian path of the tournament is an ordering of the players from left to right so that every player (except the last) beat the player immediately to its right. Let n be a positive integer. Show that there is a tournament with n players that has at least $n!/2^{n-1}$ Hamiltonian paths.

6. In a public examination there are n subjects, each offered in Chinese and English. Candidates may sit for as many (or as few) subjects as they like, but each candidate may only choose one language version for each subject. For any two different subjects, there exists a candidate sitting for different language versions of the two subjects. If there are at most 10 candidates sitting for each subject, determine the maximum possible value of n.

7. Show that we can color the elements of the set $\{1, 2, \ldots, 1987\}$ with 4 colors so that any arithmetic progression of ten terms, each in the set, is not monochromatic.

8. Let T be a family of subsets of $\{1, \ldots, n\}$. If no set in T is a subset of another set in T, show that if $\sum_{S \in T} \binom{n}{|S|} \leq 1$.

9. A bishop and a knight are placed far apart on a chessboard. A meteor lands in each square on the chessboard independently and randomly with probability p. Fortunately, neither the bishop nor the knight are hit, but their movement may be obstructed by the meteors. For what value of p is the expected number of valid squares that the bishop can move to (in one move) equal to the expected number of squares that the knight can move to (in one move)?

10. Let A be a set of n residues mod n^2. Show that there is a set B of n residues mod n^2 such that at least half of the residues mod n^2 can be written as $a + b$ with $a \in A$ and $b \in B$.

11. Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

12. Suppose one is given n real numbers, not all zero, such that their sum is zero. Prove that one can label these numbers $a_1, a_2, \ldots, a_n$ in such a manner that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 < 0$.

13. A tramp has a coat of area 1 with 5 patches. Each patch has area at least $\frac{1}{2}$. Prove that there exist two patches with common area at least $\frac{1}{5}$.

14. A set of numbers is sum-free if the sum of every two numbers (possibly equal) in the set is not in the set. Let A be a set of $n \geq 1$ nonzero integers. Show that there is a sum-free subset of A with more than $\frac{n}{3}$ elements.

15. Show that there exists a partition of the set of positive integers into two classes such that neither class contains an infinite arithmetic progression and neither class contains 3 consecutive integers.

16. In an $n \times n$ array, each of the numbers $1, 2, \ldots, n$ appears exactly n times. Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct numbers.

17. Let F be the set of all n-tuples $(A_1, A_2, \ldots, A_n)$ where each $A_i, i = 1, 2, \ldots, n$, is a subset of $\{1, 2, \ldots, 1998\}$. Find $\sum_{(A_1,\ldots,A_n)\in F} |A_1 \cup \cdots \cup A_n|$.

18. Let n be a positive integer, $n \geq 6$. Show that given n points in the plane, no three of them colinear, at most two thirds of the triangles that can be formed with these points as vertices are acute.

19. Show that one can colour any complete graph with 600 vertices in red, blue and green so that the sum of the number of monochromatic complete 3-vertex red graphs, 4-vertex blue graphs and 5-vertex green graphs does not exceed 7000000.

20. If S is a set of nonnegative integers, and n is a nonnegative integer, define $R_S(n)$ to be the number of ways that n can be represented as the sum of two distinct integers in S. Show that there is a set S such that for all $n > 1$, we have $1 \leq R_S(n) \leq 1000 ln n$.