Number theory 2

September 1, 2020

Problem 1. *It is known that k consecutive numbers have a prime factor, which does not exceed 13. Find the maximum possible value of k.*

Problem 2. *Prove that if p is a prime number, then it has at most one representation of form $a^2 + b^2$, where $0 < a < b$ are integers.*

Problem 3. *Prove that for each prime p, there exist integers x, y with $0 \leq x \leq \frac{p-1}{2}$, $0 \leq y \leq \frac{p}{2}$ so that $p|x^2 + y^2 + 1$.*

Problem 4. *Find all the triples of positive integers (x, y, p) where p is prime and*

$$p + 1 = 2x^2, p^2 + 1 = 2y^2.$$

Problem 5. *A class has 20 students, and each student has equal number of friends in the class. The students wrote a test in mathematics and received different marks. A student is smart if his mark was higher, then the mark of majority of his friends. Find the maximal number of smart students.*

Problem 6. *Let n points be given in the interior of the rectangle R, such that no two points lie on a line parallel to the sides of R. The rectangle is partitioned into several rectangles, so that none of n points is in the interior of a small rectangle. Prove that R is partitioned into at least $n + 1$ parts.*