Saudi Arabia – Online Math Camp
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Bijections

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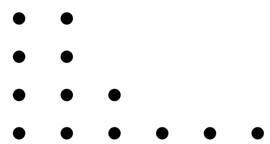
Problems – June 10

- A *partition* of a positive integer n is a representation of n as a sum of positive integer summands (or “parts”), where the order does not matter.

  A partition can be represented graphically as an array of dots, by the so-called Ferrer’s diagram. For example, the partition $13 = 4 + 4 + 2 + 1 + 1 + 1$ looks like this:

  Reflect this diagram in the line $x = y$. You will obtain a Ferrer’s diagram of another partition of number n, which we call the *conjugate* partition. In the above case, the conjugate partition is $13 = 6+3+2+2$ (read the dots horizontally instead of vertically).

  Conjugation is obviously a reversible operation, as each partition is conjugate to its conjugate.

1. For any $n, k \in \mathbb{N}$, prove that the number of partitions of n into k parts equals the number of partitions of n whose largest part is equal to k.

2. The *Catalan number* C_n can be defined as the number of lattice paths from point $(0, 0)$ to point (n, n) using only unit steps up or right that entirely lie in the region $x \geqslant y$. Prove that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

3. Prove that the number of sequences of digits $0, 1, 2, 3$ of length n with an even total number of 0’s and 1’s is equal to the number of binary sequences (i.e. sequences of digits $0, 1$) of length $2n$ with an even number of 1’s.

4. We are given one weight of each of the masses $1, 3, 5, 7, 9, \dots$ For a positive integer n, define A_n to be the number of ways to choose a set of weights with the total mass n. Prove that $A_n \leqslant A_{n+1}$ for $n > 1$.

5. Let n be a positive integer. Each point (x, y) in the coordinate plane, where x and y are non-negative integers with $x+y < n$, is colored red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leqslant x$ and $y' \leqslant y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

6. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfies the following condition: for each positive integer $k \geqslant 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

7. There are 100 towns on Mars, and there are 1000 highways, each connecting two towns, projected in such a way that from any town one can reach any other town. The government intends to close some highways (maybe all) in such a way that at each town there are an even number of highways (0 is even). In how many ways can this be done?

8. Prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

9. Fix $2n$ points on a circle. Find the number of ways to connect these points with n arrows (i.e. oriented line segments) having no common points in such a way that there are no two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in which the points A, B, C, D appear in the clockwise order around the circle (not necessarily consecutively).

10. Let n and $k \geqslant n$ be positive integers of the same parity. We are given $2n$ lamps labelled $1, 2, \ldots, 2n$, which can be either on or off. Initially all the lamps are off. We consider sequences of k switches (each applied on one lamp):

    (1) Denote by N the number of such sequences resulting in the state where lamps $1, \ldots, n$ are all on and lamps $n+1, \ldots, 2n$ are all off.
    (2) Denote by M the number of such sequences resulting in the state where lamps $1, \ldots, n$ are all on and lamps $n+1, \ldots, 2n$ are all off, but where none of the lamps $n+1, \ldots, 2n$ is ever switched on.

    Determine N/M.

11. Let P be a regular 1000-gon. A diagonal is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called good. Suppose P has been dissected into triangles by 997 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration. *[IMO 2006]*

12. We call a partition of a positive integer n *self-conjugate* if it is conjugate to itself. Prove that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

13. An n-*diamond* is a symmetric figure consisting of n rows with $1, 3, \ldots, 2n-3, 2n-1, 2n-3, \ldots, 3, 1$ unit squares. Let n and k be positive integers with $2 \leqslant k \leqslant (n-1)^2$. Prove that the number of ways to place k non-overlapping 1×2 dominoes (which can be rotated) on an n-diamond is greater than the number of ways to place k non-overlapping 2×2 squares on a $(2n-1) \times (2n-1)$ square board.

14. For a positive integer n, denote by A_n the number of partitions of the set $\{1, 2, \ldots, n\}$ into subsets in which every two adjacent numbers in ascending order have different parity, and by B_n the number of partitions of $\{1, 2, \ldots, n\}$ into subsets with all elements of the same parity. Prove that $A_n = B_{n+1}$.

[E.g. for $n = 7$, $\{1, 4\} \cup \{2, 5, 6, 7\} \cup \{3\}$ is a partition of the first type, and $\{1, 3, 7\} \cup \{2, 6\} \cup \{4\} \cup \{5\}$ is of the second type.]