The Online Math Open Spring Contest
Official Solutions
March 22 – April 2, 2019

Acknowledgments

Tournament Director

- Vincent Huang

Problem Authors

- Ankan Bhattacharya
- James Lin
- Michael Ren
- Luke Robitaille
- Tristan Shin
- Edward Wan
- Brandon Wang
- Yannick Yao

Website Manager

- Evan Chen
- Douglas Chen

LaTeX/Python Geek

- Evan Chen

OMO Spring 2019
Official Solutions

1. Daniel chooses some distinct subsets of $\{1, \ldots, 2019\}$ such that any two distinct subsets chosen are disjoint. Compute the maximum possible number of subsets he can choose.

   Proposed by Ankan Bhattacharya.

   Answer. $\boxed{2020}$.

   Solution. The union of all chosen sets must have at most 2019 elements, and everything other than the empty set contributes an element, so there are at most 2020 sets. An example is $\emptyset, \{1\}, \ldots, \{2019\}$. $\square$

2. Let $A = (0,0), B = (1,0), C = (-1,0)$, and $D = (-1,1)$. Let $\mathcal{C}$ be the closed curve given by the segment AB, the minor arc of the circle $x^2 + (y-1)^2 = 2$ connecting B to C, the segment CD, and the minor arc of the circle $x^2 + (y-1)^2 = 1$ connecting D to A. Let $\mathcal{D}$ be a piece of paper whose boundary is $\mathcal{C}$. Compute the sum of all integers $2 \le n \le 2019$ such that it is possible to cut $\mathcal{D}$ into n congruent pieces of paper.

   Proposed by Vincent Huang.

   Answer. $\boxed{2039189}$.

   Solution. All n work. Let $P_1, P_2, \ldots, P_{n-1}$ be equally spaced points on arc AD and $Q_1, Q_2, \ldots, Q_{n-1}$ be equally spaced points on arc CB. Then cutting $\mathcal{D}$ along each line segment P_iQ_i produces n congruent pieces. $\square$

3. Compute the smallest positive integer that can be expressed as the product of four distinct integers.

   Proposed by Yannick Yao.

   Answer. $\boxed{4}$.

   Solution. The answer is $4 = 1(-1)(2)(-2)$. These are the four nonzero integers with lowest absolute value, so any number n that is the product of 4 distinct integers must satisfy $|n| \ge 4$. $\square$

4. Compute $\left\lceil \displaystyle\sum_{k=2018}^{\infty} \frac{2019! - 2018!}{k!} \right\rceil$. (The notation $\lceil x \rceil$ denotes the least integer n such that $n \ge x$.)

   Proposed by Tristan Shin.

   Answer. $\boxed{2019}$.

   Solution. For $k \ge 2020$, write

$$\frac{1}{k!} = \frac{1}{2018!} \prod_{i=2019}^{k} \frac{1}{i} < \frac{1}{2018!} \prod_{i=2019}^{k} \frac{1}{2019} = \frac{1}{2018! \cdot 2019^{k-2018}}$$

   so we can write the sum as

$$\begin{aligned}
\sum_{k=2018}^{\infty} \frac{2018 \cdot 2018!}{k!} &= 2018 + \frac{2018}{2019} + \sum_{k=2020}^{\infty} \frac{2018 \cdot 2018!}{k!} \\
&< 2018 + \frac{2018}{2019} + \sum_{k=2020}^{\infty} \frac{2018}{2019^{k-2018}} \\
&= \frac{2018}{1 - \frac{1}{2019}} \\
&= 2019
\end{aligned}$$

   and clearly the sum is at least 2018 so the ceiling is 2019 as desired. $\square$

5. Consider the set S of lattice points (x, y) with $0 \le x, y \le 8$. Call a function $f : S \to \{1, 2, \ldots, 9\}$ a *Sudoku function* if:

   - $\{f(x, 0), f(x, 1), \ldots, f(x, 8)\} = \{1, 2, \ldots, 9\}$ for each $0 \le x \le 8$ and $\{f(0, y), f(1, y), \ldots, f(8, y)\} = \{1, 2, \ldots, 9\}$ for each $0 \le y \le 8$.
   - For any integers $0 \le m, n \le 2$ and any $0 \le i_1, j_1, i_2, j_2 \le 2$, $f(3m+i_1, 3n+j_1) \ne f(3m+i_2, 3n+j_2)$ unless $i_1 = i_2$ and $j_1 = j_2$.

   Over all Sudoku functions f, compute the maximum possible value of $\sum_{0\le i\le 8} f(i, i) + \sum_{0\le i\le 7} f(i, i+1)$.

   Proposed by Brandon Wang.

   Answer. $\boxed{123}$.

   Solution. The definition of a Sudoku function is simply that the 9×9 grid whose rows and columns are zero-indexed with $(0, 0)$ in the upper left and whose entry in the ith row and jth column is equal to $f(i, j)$ is a valid solution to the puzzle *Sudoku*. The sum, then, is the sum of the entries along the main diagonal and the 8 squares directly above it. If we identify the grid as

$$\begin{matrix} ABC \\ DEF \\ GHI, \end{matrix}$$

   where each variable represents a 3×3 grid, then we see that the set of 17 entries has 5 entries in each of A, E, I, and the lower left corners of each of B, F.

   The two entries in B, F are at most 9 each. The 5 entries in A have sum at most $5+6+7+8+9 = 35$ since they're distinct; the same thing holds for the sum of the entries in E, I, so the total sum is at most $9 + 9 + 3(35) = 123$. To show that this can actually be attained, set

$$A = E = I = \begin{array}{|c|c|c|} \hline 7 & 5 & 1 \\ \hline 2 & 9 & 6 \\ \hline 3 & 4 & 8 \\ \hline \end{array},$$

$$B = F = G = \begin{array}{|c|c|c|} \hline 4 & 8 & 3 \\ \hline 5 & 1 & 7 \\ \hline 9 & 6 & 2 \\ \hline \end{array},$$

   and

$$C = D = H = \begin{array}{|c|c|c|} \hline 6 & 2 & 9 \\ \hline 8 & 3 & 4 \\ \hline 1 & 7 & 5 \\ \hline \end{array},$$

   which can be verified to work. □

6. Let $A, B, C, ..., Z$ be 26 nonzero real numbers. Suppose that $T = TNYWR$. Compute the smallest possible value of

$$\left\lceil A^2 + B^2 + \cdots + Z^2 \right\rceil.$$

   (The notation $\lceil x \rceil$ denotes the least integer n such that $n \ge x$.)

   Proposed by Luke Robitaille.

   Answer. $\boxed{5}$.

   Solution. We have $NYWR = 1$, so $N^2Y^2W^2R^2 = 1$ and $N^2 + Y^2 + W^2 + R^2 \ge 4\sqrt[4]{N^2Y^2W^2R^2} = 4$. Since A^2, B^2, etc. are all positive, we must have that the sum of squares is greater than four, so its ceiling is at least 5. A construction is $N = Y = W = R = 1$ and all other variables equal $\frac{1}{100}$. □

7. Let $ABCD$ be a square with side length 4. Consider points P and Q on segments AB and BC, respectively, with $BP = 3$ and $BQ = 1$. Let R be the intersection of AQ and DP. If BR^2 can be expressed in the form $\frac{m}{n}$ for coprime positive integers m, n, compute $m + n$.

   Proposed by Brandon Wang.

   Answer. $\boxed{177}$.

   Solution. Note that AQ, DP are $90°$ rotations of each other so $\angle PRQ = 90°$. Now we can note by the Extended Law of Sines that $BR = PQ \sin \angle BQR = \sqrt{10} \cdot \dfrac{4}{\sqrt{17}}$, which gives the answer. Alternatively, we can compute $PR = \dfrac{1}{\sqrt{17}}, RQ = \dfrac{13}{\sqrt{17}}$ and use Ptolemy's Theorem on cyclic quadrilateral $PRQB$ to finish. $\square$

8. In triangle ABC, side AB has length 10, and the A- and B-medians have length 9 and 12, respectively. Compute the area of the triangle.

   Proposed by Yannick Yao.

   Answer. $\boxed{72}$.

   Solution. Let G be the centroid, so $AG = \frac{2}{3} \cdot 12 = 8$ and $BG = \frac{2}{3} \cdot 9 = 6$. Thus the area of $\triangle BGC$ equals $\frac{1}{2} \cdot 6 \cdot 8 = 24$, and the area of $\triangle ABC$ equals thrice this, for an answer of 72. $\square$

9. Susan is presented with six boxes $B_1, \ldots, B_6$, each of which is initially empty, and two identical coins of denomination 2^k for each $k = 0, \ldots, 5$. Compute the number of ways for Susan to place the coins in the boxes such that each box B_k contains coins of total value 2^k.

   Proposed by Ankan Bhattacharya.

   Answer. $\boxed{32}$.

   Solution. For n boxes $B_1, \ldots, B_n$, we claim Susan has exactly 2^{n-1} ways to place the coins. We proceed by induction on n, with $n = 1$ clear.

   Suppose the statement is proven for n, and consider $n + 1$. Susan has two coins of denomination 2^n, which may only be placed in boxes B_n and B_{n+1}. Thus, Susan may place one in both B_n and B_{n+1}, or both in B_{n+1}. In both situations, Susan has boxes with values $2^1, \ldots, 2^n$ and coins of denominations $2^0, 2^0, 2^1, 2^1, \ldots, 2^{n-1}, 2^{n-1}$ remaining, so the total number of ways for Susan to accomplish the task is $2 \cdot 2^{n-1} = 2^n$, as expected.

   When $n = 6$, the answer is $2^5 = 32$. $\square$

10. When two distinct digits are randomly chosen in $N = 123456789$ and their places are swapped, one gets a new number N' (for example, if 2 and 4 are swapped, then $N' = 143256789$). The expected value of N' is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the remainder when $m + n$ is divided by 10^6.

    Proposed by Yannick Yao.

    Answer. $\boxed{962963}$.

    Solution. Each digit has $\frac{28}{36}$ chance of not being moved and $\frac{1}{36}$ chance of being swapped to a given position. This means that $E(N') = \frac{1}{36}(111111111)(1+2+\cdots+9)+(\frac{28-1}{36})(123456789) = \frac{5}{4}(111111111)+\frac{3}{4}(123456789) = \frac{925925922}{4} = \frac{462962961}{2}$. Therefore $m+n = 462962963$ and the remainder is 962963. $\square$

11. Jay is given 99 stacks of blocks, such that the ith stack has i^2 blocks. Jay must choose a positive integer N such that from each stack, he may take either 0 blocks or exactly N blocks. Compute the value Jay should choose for N in order to maximize the number of blocks he may take from the 99 stacks.

    Proposed by James Lin.

    Answer. $\boxed{4489}$.

    Solution. It's clear that Jay must choose $N = i^2$, and for each stack with at least N blocks Jay will choose exactly N blocks. Hence, we need to maximize $f(i) = i^2(100 - i) = \frac{1}{2}i^2(200 - 2i)$. There are several ways to do this. One is to note that by AM-GM, $f(i)$ is maximal at $i = \frac{200}{3}$, and then we finish by comparing $f(66), f(67)$. Another is to note that $f(i) \leq f(i + 1)$ reduces to $3i^2 \leq 197i + 99$, which is true for $i = 0, 1, \ldots, 66$, hence we choose $i = 67$ and $N = 67^2$. □

12. A set D of positive integers is called *indifferent* if there are at least two integers in the set, and for any two distinct elements $x, y \in D$, their positive difference $|x - y|$ is also in D. Let $M(x)$ be the smallest size of an indifferent set whose largest element is x. Compute the sum $M(2) + M(3) + \cdots + M(100)$.

    Proposed by Yannick Yao.

    Answer. $\boxed{1257}$.

    Solution. In fact, for any two elements $x, y \in D$, by Euclidean Algorithm we can see that $\gcd(x, y) \in D$. As a result, all elements in D are a multiple of the smallest element. Therefore, for all $x > 1$, $M(x)$ is equal to the smallest (prime) divisor of x that is greater than 1.

    When $M(x)$ is 2, x is an even number, so there are 50 possibilities. When $M(x)$ is 3, x is odd and a multiple of 3, so there are $\frac{33-1}{2} + 1 = 17$ possibilities. When $M(x)$ is 5, x is relatively prime to $2 \cdot 3$ and divisible by 5, so the possibilities are $5 \cdot 1, 5 \cdot 5, 5 \cdot 7, 5 \cdot 11, 5 \cdot 13, 5 \cdot 17, 5 \cdot 19$, for 7 possibities. When $M(x)$ is 7, x is relatively prime to $2 \cdot 3 \cdot 5$ and divisible by 7, so the possibilities are $7 \cdot 1, 7 \cdot 7, 7 \cdot 11, 7 \cdot 13$, for 4 possibilities. When $M(x)$ is a prime number greater than 10, the only possibility is x itself.

    Therefore, the sum is $2 \cdot 50 + 3 \cdot 17 + 5 \cdot 7 + 7 \cdot 4 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 + 47 + 53 + 59 + 61 + 67 + 71 + 73 + 79 + 83 + 89 + 97 = 1257$. □

13. Let $S = \{10^n + 1000 : n = 0, 1, \ldots\}$. Compute the largest positive integer not expressible as the sum of (not necessarily distinct) elements of S.

    Proposed by Ankan Bhattacharya.

    Answer. $\boxed{34999}$.

    Solution. Note that S contains the numbers $1001, 1010, 1100, 2000, 11000$. The last two numbers allow us to make any multiple of 1000 which is at least 10000. Meanwhile, the smallest number attainable which ends in $\overline{xyz}$ is obviously $x \cdot 1100 + y \cdot 1010 + z \cdot 1001 = 1000(x + y + z) + (100x + 10y + z)$. This implies that all numbers which are of the form $1000n + 100x + 10y + z$ with $0 \leq x, y, z \leq 9$ and $n \geq x + y + z + 10 \geq 37$ is attainable, so all numbers which are at least 37000 obviously work.

    However, 36999 is not the answer, because we have $36999 = 19 \cdot 1001 + 8 \cdot 1010 + 9 \cdot 1100$. Also, if $\overline{xyz} \neq 999$, then our previous bound becomes $n \geq x + y + z + 10 \geq 36$, so these observations show that all numbers at least 36000 are attainable.

    Similar constructions show that $35999, 35998, 35989, 35899$ (eg. $35999 = 9 \cdot 1001 + 9 \cdot 1010 + 9 \cdot 1100 + 4 \cdot 2000$ and $35998 = 18 \cdot 1001 + 8 \cdot 1010 + 9 \cdot 1100$) are all attainable, and if $\overline{xyz} \notin \{999, 998, 989, 899\}$, our bound becomes $n \geq x + y + z + 10 \geq 35$, so these observations now show that all numbers at least 35000 are attainable.

    Finally, it remains to show that 34999 fails. It's clear that if $a \cdot 1001 + b \cdot 1010 + c \cdot 1100$ ends in $\overline{999}$, then $a + b + c \geq 27$ and the second-smallest value of $a + b + c$ is 36, so we require $a = b = c = 9$, giving a sum of 27999, and adding 2000s to 27999 does not produce 34999, so we're done. □

14. The sum
$$\sum_{i=0}^{1000} \frac{\binom{1000}{i}}{\binom{2019}{i}}$$
can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.

Proposed by James Lin.

Answer. $\boxed{152}$.

Solution. Consider shuffling a deck of 2019 cards, 1019 of which are golden. We will count the expected location of the topmost golden card, where we count down from the top. Let $E[X]$ be this expected location, and let $P[X > i]$ be the probability that the topmost golden card has at least i cards above it, so it is beyond the ith position. Then,

$$\begin{aligned} E[X] &= \sum_{i=0}^{1000} P[X > i] \\ &= \frac{(1000)(999)\cdots(1000-i)}{(2019)(2018)\cdots(2019-i)} \\ &= \frac{\binom{1000}{i}}{\binom{2019}{i}}. \end{aligned}$$

Now, consider 1020 blocks of non-golden cards, 1018 of which are between consecutive golden cards and the other 2 which are at the ends of the stack. Each non-golden card is equally likely to be in any of the 1020 regions, so there are expected to be $\frac{1000}{1020}$ cards before the first golden card. Hence, $E[X] = \frac{1000}{1020} + 1 = \frac{2020}{1020} = \frac{101}{51}$, so the answer is $101 + 51 = 152$.

Alternate solutions include evaluating the sum with algebraic manipulations and the Hockey Stick Identity. Also, on the Art of Problem Solving forums, user hukilau17 posted a solution using integration, which can be found at this link: artofproblemsolving.com/community/c487h1814861p12103390 $\square$

15. Evan has 66000 omons, particles that can cluster into groups of a perfect square number of omons. An omon in a cluster of n^2 omons has a potential energy of $\frac{1}{n}$. Evan accurately computes the sum of the potential energies of all the omons. Compute the smallest possible value of his result.

Proposed by Michael Ren and Luke Robitaille.

Answer. $\boxed{284}$.

Solution. This wants us to minimize $a_1 + a_2 + \cdots + a_k$ given $a_1^2 + a_2^2 + \cdots + a_k^2 = 66000$. Clearly $a_1 = 256, a_2 = 20, a_3 = 8$ gives a result of 284. I claim this is optimal.

Since $f(x) = \sqrt{x}$ is concave, by Karamata we see $\sum f(a_i^2) \le \sum f(b_i^2)$ if $\{a_i^2\}$ majorizes $\{b_i^2\}$. Therefore, for some sequence $\{b_i^2\}$ to attain a smaller result than $\{256^2, 20^2, 8^2\}$, it must not be majorized by $\{256^2, 20^2, 8^2\}$.

It follows that $b_1 \ge 100$. (If all $b_i \le 100$ obviously the b_i will be majorized). Now note that $b_2 + \cdots + b_k \ge \sqrt{b_2^2 + \cdots + b_k^2} = \sqrt{66000 - b_1^2}$ so $\sum b_i \ge \sqrt{66000 - b_1^2} + b_1$. Since $b_1 \ge 100$, once again by Karamata we see that the expression is minimized when b_1 approaches $\sqrt{66000}$. If $b_1 \le 255$ then $\sqrt{66000 - b_1^2} + b_1 > 284$, so we can assume $b_1 = 256$.

Now $b_2^2 + \cdots + b_k^2 = 464$. By a similar argument we see that $b_2 \in \{20, 21\}$. If $b_2 = 20$ then the optimal sequence is the one we have already considered; if $b_2 = 21$ then $b_3^2 + b_4^2 + \cdots = 23$ so the optimal sequence of b_i is $4, 2, 1, 1, 1$ or $3, 3, 2, 1$. Each of these produces a final result of 286 which is more than 284, so 284 is the answer. $\square$

16. In triangle ABC, $BC = 3, CA = 4$, and $AB = 5$. For any point P in the same plane as ABC, define $f(P)$ as the sum of the distances from P to lines AB, BC, and CA. The area of the locus of P where $f(P) \le 12$ is $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

Proposed by Yannick Yao.

Answer. $\boxed{92007}$.

Solution. We first observe the following: If X and Y are on the same side of lines AB, BC, CA, and we move a point P from X to Y along the segment, then $f(P)$ is a linear function of the distance it moved along the segment, since the distance of P from each of the three lines varies linearly. In particular, if $f(X) = f(Y) = 12$, then $f(P) = 12$ for all P along the segment. This means that we only need to consider the points that are on one of the three lines that achieves $f(P) = 12$, since connecting these points will form a polygon that is the locus. It's clear that $f(P) < 12$ for all P inside (or on the boundary of) the triangle. Therefore, the critical points must lie on the six rays. We consider each of them separately:

- Ray CA: If $AP = d$, then $f(P) = (d + 4) + 0 + \frac{3}{5}d = \frac{8}{5}d + 4$, so $d = 5$.
- Ray CB: If $BP = d$, then $f(P) = 0 + (d + 3) + \frac{4}{5}d = \frac{9}{5}d + 3$, so $d = 5$.
- Ray AC: If $CP = d$, then $f(P) = d + 0 + \frac{3}{5}(d + 4) = \frac{8}{5}d + \frac{12}{5}$, so $d = 6$.
- Ray AB: If $BP = d$, then $f(P) = \frac{4}{5}d + \frac{3}{5}(d + 5) + 0 = \frac{7}{5}d + 3$, so $d = \frac{45}{7}$.
- Ray BC: If $CP = d$, then $f(P) = 0 + d + \frac{4}{5}(d + 3) = \frac{9}{5}d + \frac{12}{5}$, so $d = \frac{16}{3}$.
- Ray BA: If $AP = d$, then $f(P) = \frac{4}{5}(d + 5) + \frac{3}{5}d + 0 = \frac{7}{5}d + 4$, so $d = \frac{40}{7}$.

Therefore, if we add up the seven components of the locus, we get the area of

$$\frac{3\cdot 4}{2} + \frac{(3+5)\cdot(4+5) - 3\cdot 4}{2} + \frac{5\cdot\frac{45}{7}}{2}\cdot\frac{4}{5} + \frac{(\frac{45}{7}+5)\cdot(6+4) - 5\cdot 4}{2}\cdot\frac{3}{5}$$
$$+\frac{6\cdot\frac{16}{3}}{2} + \frac{(\frac{16}{3}+3)\cdot(\frac{40}{7}+5) - 3\cdot 5}{2}\cdot\frac{4}{5} + \frac{\frac{40}{7}\cdot 5}{2}\cdot\frac{3}{5} = \frac{920}{7}.$$

So the answer is 92007. $\square$

17. Let $ABCD$ be an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$. The incircle of $\triangle ABC$ has center I and is tangent to $\overline{BC}$ at P. The incircle of $\triangle ABD$ has center J and is tangent to $\overline{AD}$ at Q. If $PI = 8$, $IJ = 25$, and $JQ = 15$, compute the greatest integer less than or equal to the area of $ABCD$.

Proposed by Ankan Bhattacharya.

Answer. $\boxed{1728}$.

Solution. The main claim is that P, I, J, Q are collinear. This can be easily proven by noting that $BP - PC = AQ - QD$ by incircle lengths, whence $\overline{PQ}$ is perpendicular to the bases of $ABCD$, but a more "conceptual" solution is to invoke the Japanese theorem for $ABCD$, which gives that $\overline{IJ}$ is perpendicular to the bases (which is also sufficient).

With this, the problem is easily solved. Let the angle bisectors AJ and BI meet at K. By angle chasing $\angle AKB = 90^\circ$, whence we obtain similar triangles

$$\triangle AJQ \sim \triangle IJK \sim \triangle IBP \sim \triangle BKA$$

(the last one from equal angles formed by the bisectors).

Now we are in business. Write $IK = x$ and $JK = y$ (thus $x^2 + y^2 = 25^2 = 625$). Then $BI = \frac{200}{x}$ and $AJ = \frac{375}{y}$, so

$$\frac{x}{y} = \frac{IK}{JK} = \frac{AK}{BK} = \frac{y + \frac{375}{y}}{x + \frac{200}{x}}$$

and thus $x^2 + 200 = y^2 + 375$, which forces $x^2 = 400$ and $y^2 = 225$.

It follows that $BK = 30$ and $AK = 40$, so $AB = 50$.

Here is a nice way to complete the solution, due to Vincent Huang. Let M be the midpoint of minor arc AB on $(ABCD)$ so it's well-known that M is the circumcenter of $AJIB$. By Ptolemy's Formulas on cyclic quadrilateral $AJIB$ we find $AI \cdot BJ = (25 \cdot 10 + 25 \cdot 50)$ and $\dfrac{AI}{BJ} = \dfrac{25 \cdot 10 + 25 \cdot 50}{25 \cdot 25 + 10 \cdot 50} = \dfrac{4}{3}$, hence $AI = 20\sqrt{5}, BJ = 15\sqrt{5}$. Now by computing the area of $[AIJ]$ in two ways, once as $\frac{1}{2} \cdot 25 \cdot 20$ and once as $\frac{AI \cdot AJ \cdot IJ}{4R}$, we find the radius of circle $(AJIB)$ is $\frac{25\sqrt{5}}{2} = AM = BM$.

Now we can easily find that $[AMB] = \dfrac{625}{2} = \dfrac{AM \cdot MB \cdot AB}{4R'}$, where R' is the radius of $(ABCD)$. This in turn yields $R' = \dfrac{125}{4}$. Now clearly $\sin \frac{\angle A}{2} = \frac{3}{5} \implies \sin \angle A = \frac{24}{25}$ so $BD = 2R' \sin A = 60$, as is AC. Now we know already that $AD - BC = 28$, and by Ptolemy we see $AD \cdot BC + AB \cdot CD = AC \cdot BD \implies AD \cdot BC = 1100$, so $AD = 50$ and $BC = 22$. It easily follows that $[ABCD] = 1728$ and we're done. $\square$

18. Define a function f as follows. For any positive integer i, let $f(i)$ be the smallest positive integer j such that there exist pairwise distinct positive integers $a, b, c,$ and d such that $\gcd(a, b)$, $\gcd(a, c)$, $\gcd(a, d)$, $\gcd(b, c)$, $\gcd(b, d)$, and $\gcd(c, d)$ are pairwise distinct and equal to $i, i+1, i+2, i+3, i+4,$ and j in some order, if any such j exists; let $f(i) = 0$ if no such j exists. Compute $f(1) + f(2) + \cdots + f(2019)$.

    Proposed by Edward Wan.

    Answer. $\boxed{1871}$.

    Solution. For a sextuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ of distinct positive integers, we will call it **vengeful** if there exists a quadruple (a, b, c, d) of integers for which $\gcd(a, b), \gcd(a, c), \gcd(a, d), \gcd(b, c), \gcd(b, d)$, and $\gcd(c, d)$ are equal to a_1, a_2, a_3, a_4, a_5 and a_6 in some order. In that case, we will say that (a, b, c, d) **avenges** $(a_1, a_2, a_3, a_4, a_5, a_6)$. We will now show a lemma which will later prove useful:

    Lemma 1. A sextuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ of distinct positive integers is vengeful only if there exists no positive integer n for which there are exactly two multiples of n among the a_i's.

    Proof. Assume, for contradiction, that there exists a vengeful sextuple $\Gamma = (a_1, a_2, a_3, a_4, a_5, a_6)$ and a positive integer n dividing exactly two of the a_i's. Then, if we consider a quadruple (a, b, c, d) avenging Γ, and let k be the number of multiples of n among a, b, c, d, then we know that $\binom{k}{2} = 2$. However, this is absurd, contradiction. $\blacksquare$

    With Lemma 1, let's return to the problem. When $i = 1$, Lemma 1 implies that if $f(1) > 0$, then $2|f(1), 3 \nmid f(1), 4 \nmid f(1)$, and $5 \nmid f(1)$. These conditions, together with $f(1) \notin \{1, 2, 3, 4, 5\}$, imply that $f(1) = 0$ or $f(1) \geq 14$. Since $(4, 15, 42, 140)$ is a quadruple which avenges $(1, 2, 3, 4, 5, 14)$, we have that $f(1) = 14$. When $i = 2$, Lemma 1 implies that $3|f(2)$, which together with $f(2) \notin \{2, 3, 4, 5, 6\}$ gives that $f(2) = 0$ or $f(2) \geq 9$. Therefore, since $(12, 18, 20, 45)$ is a quadruple which avenges $(2, 3, 4, 5, 6, 9)$, we have that $f(2) = 9$. The following result characterizes the value of $f(n)$ for all $n \geq 3$:

    Lemma 2. Set $x_0 = 0, x_1 = 2, x_2 = 3, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 3, x_7 = 2, x_8 = 0, x_9 = 0, x_{10} = 1$, and $x_{11} = 0$. Then, if we let $\upsilon(n)$ denote the remainder of n upon division by 12, then we have that $f(n) = x_{\upsilon(n)}$ for $n \geq 3$.

    Proof. We will conduct casework on $\upsilon(n)$.

- $v(n) = 0$: Let $n = 12k$ for $k \geq 1$. Then by Lemma 1, we know that if $f(n) > 0$ then $2 \nmid f(n)$, but $4|f(n)$, clearly absurd. Hence $f(n) = 0$ in this case.
- $v(n) = 1$: Let $n = 12k+1$ for $k \geq 1$. We know by Lemma 1 that $2|f(n)$, and since $((12k+1)(12k+3)(12k+5), (12k+1)(12k+2), (12k+4)(6k+1)(12k+5), (12k+4)(12k+3))$ is a quadruple that avenges $(12k+1, 12k+2, 12k+3, 12k+4, 12k+5, 2)$, we have that $f(n) = 2$ in this case.
- $v(n) = 2$: Let $n = 12k+2$ for $k \geq 1$. We know by Lemma 1 that $3|f(n)$, and since $((12k+4)(6k+1)(12k+5), (12k+6)(6k+2), (12k+6)(4k+1)(6k+1), (12k+3)(12k+5))$ is a quadruple that avenges $(12k+2, 12k+3, 12k+4, 12k+5, 12k+6, 3)$, we have $f(n) = 3$ in this case.
- $v(n) = 3$: Let $n = 12k+3$ for $k \geq 0$. We know by Lemma 1 that if $f(n) > 0$, then $2|f(n), 3|f(n)$, and $6 \nmid f(n)$, which is absurd. Hence, $f(n) = 0$ in this case.
- $v(n) = 4$: Let $n = 12k+4$ for $k \geq 0$. We know by Lemma 1 that if $f(n) > 0$ then $2 \nmid f(n), 4|f(n)$, clearly absurd. Hence, $f(n) = 0$ in this case.
- $v(n) = 5$: Let $n = 12k+5$ for $k \geq 0$. We know by Lemma 1 that if $f(n) > 0$ then $2|f(n), 3|f(n)$, and $6 \nmid f(n)$, clearly absurd. Hence, $f(n) = 0$ in this case.
- $v(n) = 6$: Let $n = 12k+6$ for $k \geq 0$. By Lemma 1 we know that $3|f(n)$, so since $((12k+7)(12k+8)(6k+5), (12k+8)(12k+9), (12k+10)(6k+3), (12k+7)(12k+9))$ avenges $(12k+6, 12k+7, 12k+8, 12k+9, 12k+10, 3)$, we know that $f(n) = 3$ in this case.
- $v(n) = 7$: Let $n = 12k+7$ for $k \geq 0$. By Lemma 1 we know that $2|f(n)$, so since $((12k+7)(12k+9)(12k+11), (12k+7)(12k+10), (12k+8)(12k+9), (12k+8)(6k+5)(12k+11))$ avenges $(12k+7, 12k+8, 12k+9, 12k+10, 12k+11, 2)$, we have that $f(n) = 2$ in this case.
- $v(n) = 8$: Let $n = 12k+8$ for $k \geq 0$. By Lemma 1 we know that if $f(n) > 0$ then $2 \nmid f(n)$ and $4|f(n)$, clearly absurd. Hence, $f(n) = 0$ in this case.
- $v(n) = 9$: Let $n = 12k+9$ for $k \geq 0$. By Lemma 1 we know that if $f(n) > 0$ then $2|f(n), 3|f(n)$, and $6 \nmid f(n)$, clearly absurd. Hence $f(n) = 0$ in this case.
- $v(n) = 10$: Let $n = 12k+10$ for $k \geq 0$. Since $((12k+11)(12k+13), (12k+10)(12k+11)(6k+7), (12k+12)(12k+13)(6k+5), (12k+12)(6k+7))$ is a quadruple that avenges $(12k+10, 12k+11, 12k+12, 12k+13, 12k+14, 1)$, we have that $f(n) = 1$ in this case.
- $v(n) = 11$: Let $n = 12k+11$ for $k \geq 0$. By Lemma 1 we know that if $f(n) > 0$ then $2|f(n), 3|f(n)$, and $6 \nmid f(n)$, clearly absurd. Hence $f(n) = 0$ in this case.

In conclusion, as we've exhausted all 12 cases for $v(n)$, we know that $f(n) = x_{v(n)}$ for all $n \geq 3$, as desired. ■

Due to Lemma 2, we can easily compute the sum in the question as $14 + 9 + 0 + 168 * 11 = 1871$. □

19. Arianna and Brianna play a game in which they alternate turns writing numbers on a paper. Before the game begins, a referee randomly selects an integer N with $1 \leq N \leq 2019$, such that i has probability $\frac{i}{1+2+\cdots+2019}$ of being chosen. First, Arianna writes 1 on the paper. On any move thereafter, the player whose turn it is writes $a+1$ or $2a$, where a is any number on the paper, under the conditions that no number is ever written twice and any number written does not exceed N. No number is ever erased. The winner is the person who first writes the number N. Assuming both Arianna and Brianna play optimally, the probability that Brianna wins can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m+n$.

Proposed by Edward Wan.

Answer. $\boxed{1274114}$.

Solution. We claim that Arianna wins if and only if N is odd, expressible as $4k+2$ for $k \geq 1$, or 4. Otherwise, Brianna wins.

First of all, notice that Arianna always wins for odd numbers, since if Brianna wrote x on her previous move, she can just write $x+1$. The reason this strategy wins is because Brianna always writes only even numbers and Arianna only odds. Hence, Arianna wins because Brianna cannot possibly ever write the odd number N, and Arianna never writes anything larger than N.

Now, we will deal with the cases where N is even. When $N = 2$, it's clear that Brianna wins by simply writing 2 on her first move. When $N = 4$, Arianna wins by simply writing 4 on her second move (as Brianna must write 2 on her first move). Otherwise, we will suppose that $N = 2k$, where $k > 2$. Firstly, we can observe that the first player to write k or $2k - 1$ loses, as the next player simply writes $2k$ on her next move. Hence, one player winning is equivalent to "forcing" their opponent to write one of the "traps" $k, 2k - 1$. Hence, the first player to write a number which is not a member of the set $\{1, 2, \cdots, k-1, k+1, \cdots, 2k-2\}$ loses. Therefore, since both players play optimally, we may assume that neither player moves outside of this set unless they are forced to.

Let's first consider the case where k is odd, i.e., $4|N - 2$. Observe then that it is always possible to play in $\{1, 2, \cdots, k-1, k+1, \cdots, 2k-2\}$ unless all of these numbers have already been played. To see this, simply note that we can always just play the smallest unused element e of this set, since $e - 1$ has been played, except for when $e = k + 1$, in which case $\frac{k+1}{2}$ has been played. Therefore, since this set has cardinality $2k - 3$, an odd number, Arianna is the one who plays the last unused number in this set, and so hence Brianna is the first one who is forced to play in $\{k, 2k - 1\}$, and hence Arianna wins.

Now, let's consider the case when k is even, i.e. $4|N$. With a similar argument as before, we know that it is always possible to play in $\{1, 2, \cdots, k-1, k+1, \cdots, 2k-2\}/\{k + 1\}$ whenever there is an unused element in this set. However, the difference when k is even is that under optimal strategy, neither Arianna nor Brianna should ever play the number $k + 1$. To see this, note that since $k + 1$ is odd, if it's possible to write $k + 1$, then k must have been written already, and hence it's possible to immediately win by writing $N = 2k$. Therefore, since the set $\{1, 2, \cdots, k-1, k+1, \cdots, 2k-2\}/\{k+1\}$ has cardinality $2k - 4$, an even number, we know that Brianna is the last person to play in this set. Therefore, Arianna is eventually forced to play either k or $2k - 1$, and so hence Brianna wins in this case.

All that remains now is to compute $\frac{2+8+12+16+\cdots+2016}{1+2+\cdots+2019} = \frac{509038}{2039190} = \frac{254519}{1019595}$, so that $m + n = 1274114$. $\square$

20. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. Suppose X and Y are points such that

- BC and XY are parallel
- BX and CY intersect at a point P on the circumcircle of $\triangle ABC$
- the circumcircles of $\triangle BCX$ and $\triangle BCY$ are tangent to AB and AC, respectively.

Then AP^2 can be written in the form $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $100p+q$.

Proposed by Tristan Shin.

Answer. $\boxed{230479}$.

Solution. First note that $\angle CBY = 180^\circ - \angle ACY = 180^\circ - \angle ACP$. Similarly we have $\angle BCX = 180^\circ - \angle ABP$. It follows that $\angle CBY + \angle BCX = 180^\circ$, so $BY||CX$ and $BCXY$ is a parallelogram.

Now, by the Extended Law of Sines applied to $(ABC), (BCY)$ we have that $\dfrac{R_{ABC}}{R_{BCY}} = \dfrac{AB}{BC} = \dfrac{BP}{BY}$. Similarly we have that $\dfrac{AC}{BC} = \dfrac{CP}{CX}$, hence combining these equations and using $BY = CX$ yields that $\dfrac{BP}{CP} = \dfrac{AB}{AC}$, so AP is the A-symmedian of $\triangle ABC$.

Now let M be the midpoint of BC. Then $AM = \dfrac{\sqrt{2 \cdot 4^2 + 2 \cdot 6^2 - 5^2}}{2} = \dfrac{\sqrt{79}}{2}$. Since P, M are inverses in $\sqrt{bc}$−inversion, we see $AP = \dfrac{24}{AM} = \dfrac{48}{\sqrt{79}}$, so $AP^2 = \dfrac{2304}{79}$, yielding an answer of 230479. $\square$

21. Define a sequence by $a_0 = 2019$ and $a_n = a_{n-1}^{2019}$ for all positive integers n. Compute the remainder when

$$a_0 + a_1 + a_2 + \cdots + a_{51}$$

is divided by 856.

Proposed by Tristan Shin.

Answer. $\boxed{108}$.

Solution. Let $p = 107$, $q = 53$ so that $p - 1 = 2q$. Also set $n = 2019$. Observe that n is a primitive root mod both p and q since n^2 and n^{53} are not 1 (mod 107) and n^4 and n^{26} are not 1 (mod 53). Then n is also a primitive root mod $2q$ since powers of n range all residues mod q and are all 1 (mod 2). Observe that $a_i = n^{(n^i)}$, so we can reduce $n^i \pmod{2q}$ and the sum remains the same mod p. But the n^i range all odd integers from 1 to $2q - 1 = p - 2$ inclusive except for q because n is a primitive root mod $2q$, so the sum is

$$n^1 + n^3 + n^5 + \ldots + n^{p-2} - n^{\frac{p-1}{2}} \pmod{p}.$$

But observe that

$$n^1 + n^3 + n^5 + \ldots + n^{p-2} = \frac{n^p - n}{n^2 - 1} \equiv 0 \pmod{p},$$

so we just need to compute $-n^{\frac{p-1}{2}} \pmod{p}$. But since n is a primitive root mod p, this is just 1 (mod p). Then clearly the sum is $52 \cdot 3 \equiv 4 \pmod{8}$, so we deduce that it is $108 \pmod{856}$. □

22. For any set S of integers, let $f(S)$ denote the number of integers k with $0 \le k < 2019$ such that there exist $s_1, s_2 \in S$ satisfying $s_1 - s_2 = k$. For any positive integer m, let x_m be the minimum possible value of $f(S_1) + \cdots + f(S_m)$ where $S_1, \ldots, S_m$ are nonempty sets partitioning the positive integers. Let M be the minimum of $x_1, x_2, \ldots$, and let N be the number of positive integers m such that $x_m = M$. Compute $100M + N$.

Proposed by Ankan Bhattacharya.

Answer. $\boxed{202576}$.

Solution. The main claims are that $M = 2019$ and $N = 676$.

First we prove $M \ge 2019$. Let a be any positive integer, and consider the segment $[a, a + 2018]$ of 2019 integers. Consider any subset $S \subseteq [a, a + 2018]$; it is clear that $f(S) \ge |S|$ with equality iff S is an arithmetic progression. Summing this inequality over all $S_1, \ldots, S_m$, it follows that

$$f(S_1) + \cdots + f(S_m) \ge f(S_1 \cap [a, a+2018]) + \cdots + f(S_m \cap [a, a+2018]) \ge \sum_{k=1}^{m} |S_k \cap [a, a+2018]| = 2019.$$

This is clearly possible (e.g. when $m = 1$).

Now we analyze the equality cases; suppose that $S_1 \sqcup \cdots \sqcup S_m = \mathbb{N}$ gives equality. The first claim is that if $a \in S_i$ then $a + 2019 \in S_i$. Indeed, as above

$$\begin{aligned} f(S_1) + \cdots + f(S_m) &= \sum_{k \ne i} f(S_k \cap [a+1, a+2019]) + f(S_m \cap [a+1, a+2019]) \\ &= \sum_{k \ne i} f(S_k \cap [a, a+2018]) + f(S_m \cap [a+1, a+2019]) = 2019 \\ &= \sum_{k \ne i} f(S_k \cap [a, a+2018]) + f(S_m \cap [a, a+2018]) \end{aligned}$$

and the result follows. Now we may interpret each S_k as an arithmetic progression in $\mathbb{Z}/2019\mathbb{Z}$; conversely any such arithmetic partition works.

Thus, we only need to enumerate the number of possible m for which $\mathbb{Z}/2019\mathbb{Z}$ can be split into m cyclic arithmetic progressions. Since $2019 = 3 \cdot 673$, each set must have size 1, 3, 673, or 2019, and both 3 and 673 cannot occur (else they will share an element).

Now it is easy to see that the possible values of m are $1, 3, 673, 675, \ldots, 2019$, for a total of 676 possible values. Thus $N = 676$ and $100M + N = 202576$. $\square$

23. Let a_1, a_2, a_3, a_4, and a_5 be real numbers satisfying

$$\begin{aligned} a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_1 &= 20, \\ a_1a_3 + a_2a_4 + a_3a_5 + a_4a_1 + a_5a_2 &= 22. \end{aligned}$$

Then the smallest possible value of $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$ can be expressed as $m + \sqrt{n}$, where m and n are positive integers. Compute $100m + n$.

Proposed by Ankan Bhattacharya.

Answer. $\boxed{2105}$.

Solution. Here is a solution by Vincent Huang. The minimum possible value is $21+\sqrt{5}$, for an answer of 2105. To prove that this is a lower bound, set $\omega = e^{2\pi i/5}$, and note that

$$\begin{aligned} 0 &\le |a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4|^2 \\ &= (a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4)(a_0 + a_1\omega^{-1} + a_2\omega^{-2} + a_3\omega^{-3} + a_4\omega^{-4}) \\ &= \sum_{k=0}^{4} a_k^2 + 2a_ka_{k+1}\operatorname{Re}(\omega) + 2a_ka_{k+2}\operatorname{Re}(\omega^2) \\ &= \left(a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2\right) + 2 \cdot 20 \cdot \tfrac{-1+\sqrt{5}}{4} + 2 \cdot 22 \cdot \tfrac{-1-\sqrt{5}}{4} \end{aligned}$$

which gives $a_0^2 + \cdots + a_4^2 \ge 21 + \sqrt{5}$, as desired. (Here indices are taken modulo 5.)

Equality is possible if and only if $a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 = 0$. Note that $\omega^2 + \frac{1-\sqrt{5}}{2}\omega + 1 = 0$, so $(a_0, a_1, a_2, a_3, a_4) = (s + t, s \cdot \frac{1-\sqrt{5}}{2} + t, s + t, t, t)$ gives equality for any real s and t.

We claim that we may choose s and t in such a way so that both conditions are satisfied. Indeed, by computing, note that

$$\begin{aligned} a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_1 &= (1 - \sqrt{5})s^2 + (5 - \sqrt{5})st + 5t^2, \\ a_1a_3 + a_2a_4 + a_3a_5 + a_4a_1 + a_5a_2 &= s^2 + (5 - \sqrt{5})st + 5t^2. \end{aligned}$$

Thus, we need $\sqrt{5} \cdot s^2 = 2$. The other necessary condition is $5t^2 + (5 - \sqrt{5})st + (s^2 - 22) = 0$, so as a quadratic in t, its discriminant is

$$(5 - \sqrt{5})^2 s^2 - 20(s^2 - 22) = 440 - 10\sqrt{5} \cdot s^2 = 420 > 0,$$

and thus there exists a real solution for t, as desired. $\square$

24. We define the binary operation $\times$ on elements of $\mathbb{Z}^2$ as

$$(a, b) \times (c, d) = (ac + bd, ad + bc)$$

for all integers a, b, c, and d. Compute the number of ordered six-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of integers such that

$$[[[[(1, a_1) \times (2, a_2)] \times (3, a_3)] \times (4, a_4)] \times (5, a_5)] \times (6, a_6) = (350, 280).$$

Proposed by Michael Ren and James Lin.

Answer. $\boxed{8}$.

Solution. (Solution by Vincent Huang.) The below solution is casework-intensive, but it's really not that bad with the right organization and optimizations and definitely doable in 90 minutes.

Notice that $(a+bt)\times(c+dt)\equiv(ac+bd)+(ad+bc)t \pmod{t^2-1}$. This motivates a natural algebraic interpretation of the conditions:

Let $P(t)=(1+a_1t)(2+a_2t)(3+a_3t)(4+a_4t)(5+a_5t)(6+a_6t)\equiv 350+280t \pmod{t^2-1}$, so we wish to find the number of solutions to $P(1)=(1+a_1)(2+a_2)(3+a_3)(4+a_4)(5+a_5)(6+a_6)=630$ and $P(-1)=(1-a_1)(2-a_2)(3-a_3)(4-a_4)(5-a_5)(6-a_6)=70$.

Note that if $3\mid 3+a_3$ then $3\mid 3-a_3\mid 70$, contradiction, so $3+a_3, 3-a_3\mid 70$. Similarly we have $6+a_6, 6-a_6\mid 70$. By examining all the divisors of 70, it follows that $(3+a_3, 3-a_3)=(7,-1),(5,1),(1,5),(-1,7)$. Call these options A, B, C, D. Similarly we find that $(6+a_6, 6-a_6)=(14,-2),(10,2),(7,5),(5,7),(2,10),(-2,14)$. Call these options A,B,C,D,E,F. For the rest of this solution, we'll use eg. case AE to refer to the case where $(3+a_3,3-a_3)=(7,-1)$ and $(6+a_6,6-a_6)=(2,10)$. In addition, we'll use the notation $(1245)_+$ to refer to the product $(1+a_1)(2+a_2)(4+a_4)(5+a_5)$ and define $(1245)_-$ similarly; other notation like $(124)_+, (12)_+$ will mean the same thing.

Cases AA, AC, BB, BD, CC, CE, DD, DF: In each of these cases we either get that $(3+a_3)(6+a_6)\nmid 70$ or that $(3-a_3)(6-a_6)\nmid 70$, contradiction.

Case AB: We have $(1245)_+=9, (1245)_-=-35$. If $5\mid 5-a_5$ then $5\mid 5+a_5\mid 9$, contradiction, hence $5-a_5=\pm1,\pm7$. Then for $5+a_5\mid 9$, we require $(5+a_5,5-a_5)=(9,1),(3,7)$. In the first case we get $(124)_+=1,(124)_-=-35$, which has no solutions. In the second case we get $(124)_+=3,(124)_-=-5$. Since one of $4+a_4, 4-a_4$ is ≥ 4, we see $4-a_4=5$, so $(12)_+=1,(12)_-=-1$, which has no solutions.

Case AD: We have $(1245)_+=18,(1245)_-=-10$. By reasoning similar to case AB, we see $(5+a_5,5-a_5)=(9,1)$. Thus $(124)_+=2,(124)_-=-10$ and we see $(4+a_4,4-a_4)=(-2,10)$, so $(12)_+=-1,(12)_-=-1$, contradiction.

Case AE: We have $(1245)_+=45,(1245)_-=-7$. Then by similar reasoning, we see $(5+a_5,5-a_5)=(9,1),(3,7)$. The first case yields $(124)_+=5,(124)_-=-7$, from which we obtain the solution $(0,3,4,-3,4,-4)$. In the second case, $(124)_+=15,(124)_=-1$, and there are no possible values for a_4, contradiction.

Case AF: We have $(1245)_+=-45,(1245)_-=-5$, from which we deduce $(5+a_5,5-a_4)=(5,5),(9,1),(15,-5)$. Thus $((124)_+,(124)_-)=(-9,-1),(-5,-5),(-3,1)$, and in each case we find no solutions.

Case BA: We have $(1245)_+=9,(1245)_-=-35$. These equations are identical to case AB, so there are no solutions.

Case BC: We have $(1245)_+=18,(1245)_-=14$. Then casework yields that $(5+a_5,5-a_5)=(3,7),(9,1)$. In the first case we get $(124)_+=6,(124)_-=2$, from which there are no solutions. In the second case we get $(124)_+=2,(124)_-=14$, from which we obtain the solution $(0,0,2,-3,4,1)$.

Case BE: We have $(1245)_+=63,(1245)_-=7$. Then casework shows $(5+a_5,5-a_5)=(3,7),(9,1)$. In the first case we get $(124)_+=21,(124)_-=1$, from which we obtain the solution $(0,1,2,3-,2-4)$. In the second case we get $(124)_+=(124)_-=7$, and there are no solutions.

Case BF: We have $(1245)_+=-63,(1245)_-=5$. Casework shows $(5+a_5,5-a_5)=(9,1)$, from which we get $(124)_+=-7,(124)_-=5$, and we get the solution $(0,-3,2,3,4,-8)$.

Case CA: We have $(1245)_+=45,(1245)_-=-7$. These equations are identical to case AE, so we get the solution $(0,3,-2,-3,4,8)$.

Case CB: We have $(1245)_+=63,(1245)_-=7$. These equations are identical to case BE, from which we get the solution $(0,1,-2,3,-2,4)$.

Case CD: We have $(1245)_+=126,(1245)_-=2$. Casework show $(5+a_5,5-a_5)=(9,1)$, from which we get $(124)_+=14,(124)_-=2$, and we get the solution $(0,0,-2,3,4,-1)$.

Case CF: We have $(1245)_+=-315,(1245)_-=1$. Casework shows $(5+a_5,5-a_5)=(9,1)$, from which we get $(124)_+=-35,(124)_-=1$, and we find there are no solutions.

Case DA: We have $(1245)_+=-45,(1245)_-=-5$. These equations are identical to case AF, so there are no solutions.

Case DB: We have $(1245)_+ = -63, (1245)_- = 5$. These equations are identical to case BF, so we get the solution $(0, -3, -4, 3, 4, 4)$.

Case DC: We have $(1245)_+ = -90, (1245)_- = 2$. Casework shows that $(5 + a_5, 5 - a_5) = (9, 1)$, so $(124)_+ = -10, (124)_- = 2$, and we find there are no solutions.

Case DE: We have $(1245)_+ = -315, (1245)_- = 1$. Casework shows that $(5 + a_5, 5 - a_5) = (9, 1)$, from which we get $(124)_+ = -35, (124)_- = 1$, and there are no solutions.

Thus we have exhausted all cases, and there are 8 total solutions. □

25. Let S be the set of positive integers not divisible by p^4 for all primes p. Anastasia and Bananastasia play a game.

At the beginning, Anastasia writes down the positive integer N on the board. Then the players take moves in turn; Bananastasia moves first. On any move of his, Bananastasia replaces the number n on the blackboard with a number of the form $n - a$, where $a \in S$ is a positive integer. On any move of hers, Anastasia replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bananastasia wins if the number on the board becomes zero.

Compute the second-smallest possible value of N for which Anastasia can prevent Bananastasia from winning.

Proposed by Brandon Wang and Vincent Huang.

Answer. $\boxed{625}$.

Solution. The main claim is that if, at the end of Bananastasia's turn, the number on the board isn't 0 or 1, then Anastasia can always prevent Banastasia from writing a 0 or 1 on the following turn. By continuing this process indefinitely, Anastasia never loses.

Indeed, let $n > 1$ be the number on the board. Then we may as well assume $n > 9$, because if $n \le 9$ Anastasia can just pretend n is n^4 and make the same moves. Now, if $n - 1$ is a power of two then $16 \mid n^4 - 1$, so clearly $n^4 - 1, n^4$ are not in S, hence if Anastasia chooses $k = 4$ and replaces n with n^4, Banastasia cannot make a 0 or 1 on his next turn. If $n - 1$ isn't a power of two then we can find an odd prime $p \mid n - 1$ so that $v_p(n^{p^3} - 1) = v_p(n - 1) + v_p(p^3) \ge 4$, hence once again $n^{p^3} - 1, n^{p^3}$ are not in S, so Anastasia can choose $k = p^3$ this time and achieve the same result.

Therefore, if Anastasia writes N such that $N, N - 1$ are both divisible by fourth powers of primes, she cannot lose. Thus the second-smallest value of N is 625. □

26. There exists a unique prime $p > 5$ for which the decimal expansion of $\frac{1}{p}$ repeats with a period of exactly 294. Given that $p > 10^{50}$, compute the remainder when p is divided by 10^9.

Proposed by Ankan Bhattacharya.

Answer. $\boxed{572857143}$.

Solution. The conditions on p are equivalent to $p \mid 10^{294} - 1$ but $p \nmid 10^k - 1$ for $k < 294$. It follows that $p \mid \Phi_{294}(10)$.

We claim that the only prime dividing both $\Phi_{294}(10)$ and one of $10^1 - 1, \dots, 10^{293} - 1$ is 7. Let q be such a prime, and let d be the order of 10 modulo q, so $d \mid 294$. Then we require $\nu_q(10^{294} - 1) > \nu_q(10^d - 1)$. However by exponent lifting this forces $q \mid \frac{294}{d}$, so q equals 2, 3, or 7. Clearly $q \ne 2$ since $\Phi_{294}(10)$ is odd. To see $q \ne 3$, note that $\Phi_{294}(10) \equiv \Phi_{294}(1) = 1 \pmod 3$, as 294 is not a prime power.

Thus it follows that $\Phi_{294}(10)$ is a power of 7 times a power of p. We claim that $\nu_7\big(\Phi_{294}(10)\big) = 1$. Indeed, note that by exponent lifting,

$$\sum_{d \mid n} \nu_7\big(\Phi_d(10)\big) = \nu_7(10^n - 1) = \begin{cases} 1 + \nu_7(n) & 6 \mid n \\ 0 & \text{otherwise,} \end{cases}$$

from which $\nu_7\big(\Phi_{294}(10)\big) = 1$ quickly follows.

Thus $\frac{1}{7}\Phi_{294}(10)$ is a power of p. However, it is easy to see that $\Phi_{294}(10) < 2\cdot 10^{\varphi(294)} = 2\cdot 10^{84} < \left(10^{50}\right)^2$, so in fact $p = \frac{1}{7}\Phi_{294}(10)$.

Now to compute p modulo 10^9, note that $\Phi_{294}(10) = \Phi_{42}(10^7)$ by well-known cyclotomic polynomial properties. Thus it follows that

$$7p \equiv a_1 10^7 + a_0 \pmod{10^9},$$

where we write $\Phi_{42}(x) = a_{12}x^{12} + \cdots + a_0$. Recall that the coefficients of cyclotomic polynomials are symmetric, so $a_0 = a_{12} = 1$ and $a_1 = a_{11}$ is the negative of the sum of the roots of $\Phi_{42}(x)$. This is known to equal $\mu(42) = -1$, so $a_1 = 1$.

Finally, $p \equiv \frac{1}{7} \cdot \left(10^7 + 1\right) \equiv 572857143$ modulo 10^9. $\square$

27. Let G be a graph on n vertices $V_1, V_2, \ldots, V_n$ and let $P_1, P_2, \ldots, P_n$ be points in the plane. Suppose that, whenever V_i and V_j are connected by an edge, P_iP_j has length 1; in this situation, we say that the P_i form an *embedding* of G in the plane. Consider a set $S \subseteq \{1, 2, \ldots, n\}$ and a configuration of points Q_i for each $i \in S$. If the number of embeddings of G such that $P_i = Q_i$ for each $i \in S$ is finite and nonzero, we say that S is a *tasty* set. Out of all tasty sets S, we define a function $f(G)$ to be the smallest size of a tasty set. Let T be the set of all connected graphs on n vertices with $n-1$ edges. Choosing G uniformly and at random from T, let a_n be the expected value of $\frac{f(G)^2}{n^2}$. Compute $\left\lfloor 2019 \lim_{n\to\infty} a_n \right\rfloor$.

Proposed by Vincent Huang.

Answer. $\boxed{273}$.

Solution. For a fixed G and tasty set S, clearly S must contain all the leaves of G, as otherwise the points corresponding to the leaves can vary freely along some unit circle centered at a neighbor. Now consider two leaves v_i, v_j such that the path from $v_i \to v_j$ contains k edges. If we set $Q_iQ_j = k$ in our embedding, every intermediate vertex in the path has a fixed position, so all their positions are forced. In this manner we can choose the positions of the leaves such that the position of every other vertex in the embedding is forced, so $f(G)$ is just the total number of leaves.

Now let $x_1, x_2, \ldots, x_n$ be the indicator variables where x_i is 1 iff V_i is a leaf. Then $f(G)^2 = \sum x_i^2 + \sum x_ix_j$. Note that V_i is a leaf if removing it yields a tree, and there are $(n-1)^{n-3}$ possible trees on the other $n-1$ vertices as well as $n-1$ options for the parent of V_i, so the expected value of $x_i^2 = x_i$ is $\dfrac{(n-1)^{n-2}}{n^{n-2}}$ by Cayley's Formula. This tends to $\dfrac{1}{e}$ as $n \to \infty$. Similarly x_i, x_j are both leaves if removing them yields one of $(n-2)^{n-4}$ possible trees, and we can attach them in $(n-2)^2$ ways, so the expected value of x_ix_j is $\dfrac{(n-2)^{n-2}}{n^{n-2}}$, which tends to $\dfrac{1}{e^2}$. So the expected value of $\frac{f(G)^2}{n^2}$ tends to $\dfrac{1}{e^2}$, and the answer is 273. $\square$

28. Let ABC be a triangle. There exists a positive real number x such that $AB = 6x^2 + 1$ and $AC = 2x^2 + 2x$, and there exist points W and X on segment AB along with points Y and Z on segment AC such that $AW = x$, $WX = x + 4$, $AY = x + 1$, and $YZ = x$. For any line ℓ not intersecting segment BC, let $f(\ell)$ be the unique point P on line ℓ and on the same side of BC as A such that ℓ is tangent to the circumcircle of triangle PBC. Suppose lines $f(WY)f(XY)$ and $f(WZ)f(XZ)$ meet at B, and that lines $f(WZ)f(WY)$ and $f(XY)f(XZ)$ meet at C. Then the product of all possible values for the length of BC can be expressed in the form $a + \dfrac{b\sqrt{c}}{d}$ for positive integers a, b, c, d with c squarefree and $\gcd(b, d) = 1$. Compute $100a + b + c + d$.

Proposed by Vincent Huang.

Answer. $\boxed{413}$.

Solution. Let $E = f(WY), F = f(XY), G = f(XZ), H = f(WZ)$. Let T be the Miquel point of quadrilateral $EHGF$. Note that $TBCF$ is cyclic, but due to tangency we have $\angle XFB = \angle FCB = \angle GCB = \angle XGB$, hence $X \in (TBCF)$. Similarly we get that $(TGHZC), (TBWEH), (TCYEF)$ are cyclic.

Next note $\angle WXY = \angle BTF = \angle BGF = \angle CGH = \angle WZY$, so $WXYZ$ is cyclic. Now note $\angle XWZ = 180^\circ - \angle BTH = 180^\circ - \angle XTH - \angle XGB = 180^\circ - \angle XTH - \angle HTZ = 180^\circ - \angle XTZ$, so T also lies on this circle. By Miquel's Theorem applied to triangle AXZ with points B, G, C, we get $T \in (ABC)$.

Finally, let BG, CG meet (ABC) at B_1, C_1. By Reim's Theorem we have $B_1C_1 || XZ$. However, $\angle B_1BC = \angle GBC = \angle ZGC = \angle ZTC$, so T, Z, B_1 are collinear, and similarly T, X, C_1 are as well. Therefore a homothety centered at T sends (TXZ) to (TB_1C_1), so the circles are tangent.

Now from $AW \cdot AX = AY \cdot AZ$ we get $x(2x+4) = (x+1)(2x+1) \implies x = 1$. By Casey's Theorem applied to (ABC) and four circles $A, B, C, (WXYZT)$, we see that $a\sqrt{6} = b\sqrt{(c-6)(c-1)} + c\sqrt{(b-3)(b-2)}$. Plugging in $b = 4, c = 7$ gives $a = 4 + \frac{7\sqrt{3}}{3}$, so the answer is 413 and we're done. $\square$

29. Let n be a positive integer and let $P(x)$ be a monic polynomial of degree n with real coefficients. Also let $Q(x) = (x+1)^2(x+2)^2 \dots (x+n+1)^2$. Consider the minimum possible value m_n of $\sum_{i=1}^{n+1} \frac{i^2 P(i^2)^2}{Q(i)}$. Then there exist positive constants a, b, c such that, as n approaches infinity, the ratio between m_n and $a^{2n} n^{2n+b} c$ approaches 1. Compute $\lfloor 2019abc^2 \rfloor$.

Proposed by Vincent Huang.

Answer. $\boxed{4318}$.

Solution. Apply Lagrange Interpolation to P at the points $1^2, 2^2, \dots, (n+1)^2$ to deduce that

$$\sum_{i=1}^{n+1} \frac{P(i^2) \prod_{j \neq i}(x - j^2)}{\prod_{j \neq i}(i^2 - j^2)} = P(x).$$

Comparing x^n coefficients on both sides therefore yields

$$1 = \sum_{i=1}^{n+1} \frac{P(i^2)}{\prod_{j \neq i}(i+j) \prod_{j \neq i}(i-j)} = \frac{1}{n!} \sum_{i=1}^{n+1} \frac{P(i^2)}{\prod_{j \neq i}(i+j)} \binom{n}{i-1} (-1)^{n+1-i}.$$

Therefore, by Cauchy-Schwarz, we have

$$n!^2 = \left(\sum_{i=1}^{n+1} \frac{P(i^2)}{\prod_{j \neq i}(i+j)} \binom{n}{i-1} (-1)^{n+1-i} \right)^2 \leq \left(\sum_{i=1}^{n+1} \binom{n}{i-1}^2 \right) \left(\sum_{i=1}^{n+1} \frac{P(i^2)^2}{\prod_{j \neq i}(i+j)^2} \right).$$

This last expression simplifies as $\binom{2n}{n} \sum_{i=1}^{n+1} \frac{4i^2 P(i^2)^2}{Q(i)}$, hence we have that the desired expression is at least $\frac{n!^4}{4(2n)!}$. Equality can easily be achieved by examining the Cauchy-Schwarz equality case and selecting an appropriate P such that the $P(i^2)$ terms are in the proper ratio. By Stirling's Approximation we find the expression tends towards $n^{2n+3/2} \frac{\pi^{3/2}}{2} \frac{1}{(2e)^{2n}}$, so $a = \frac{1}{2e}, b = \frac{3}{2}, c = \frac{\pi\sqrt{\pi}}{2}$, and we can compute the answer is $\lfloor 2019 \cdot \frac{3}{16e} \pi^3 \rfloor = 4318$. $\square$

30. Let ABC be a triangle with symmedian point K, and let $\theta = \angle AKB - 90^\circ$. Suppose that θ is both positive and less than $\angle C$. Consider a point K' inside $\triangle ABC$ such that A, K', K, and B are concyclic and $\angle K'CB = \theta$. Consider another point P inside $\triangle ABC$ such that $K'P \perp BC$ and $\angle PCA = \theta$.

If $\sin \angle APB = \sin^2(C-\theta)$ and the product of the lengths of the A- and B-medians of $\triangle ABC$ is $\sqrt{\sqrt{5}+1}$, then the maximum possible value of $5AB^2 - CA^2 - CB^2$ can be expressed in the form $m\sqrt{n}$ for positive integers m, n with n squarefree. Compute $100m+n$.

Proposed by Vincent Huang.

Answer. $\boxed{802}$.

Solution. Let the projections of K' onto BC, CA, AB be P_A, P_B, P_C. Then the angle condition rewrites as $\angle K'AB + \angle K'BA + \angle K'CB = 90^\circ$, or equivalently that $\angle P_C P_A B = \angle P_C P_B P_A$, so $(P_A P_B P_C)$ is tangent to BC. It follows that if P' is the isogonal conjugate of K', then the six-point circle of K', P' is tangent to BC, so $K'P' \perp BC$ and CK', CP' are isogonal, hence $P = P'$.

Now $\angle APB = 180^\circ + \angle C - \angle AK'B = 90^\circ + \angle C - \theta$. Then letting $\phi = \angle C - \theta$, it follows from the condition that $\cos\phi = \sin^2\phi$, from which we easily get $\cos\phi = \dfrac{-1+\sqrt{5}}{2}$.

Finally, since P, Q are isogonal conjugates and $P \in (AKB)$, we have $Q \in (AGB)$ where G is the centroid, so applying Law of Cosines yields $AB^2 = AG^2 + BG^2 - 2AG\cdot BG\cos\angle AGB = AG^2 + BG^2 + 2AG\cdot BG \sin\phi$. Letting m_a, m_b be the lengths of medians from A and B and using the median formula yields $5c^2 - a^2 - b^2 = 8m_a m_b \sin\phi = 8\sqrt{\sqrt{5}+1}\sqrt{\frac{\sqrt{5}-1}{2}} = 8\sqrt{2}$, so the answer is 802. $\square$