IMO Winter Camp 2010 Buffet Contest

Algebra

A1 From an infinite arithmetic sequence $a_1, a_2, a_3 \cdots$ of positive real numbers, some (possibly infinitely many) terms are deleted, obtaining an infinite geometric sequence $1, r, r^2, r^3, \cdots$ for some real number $r > 0$. Prove that r is an integer.

A2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

A3 Find all finite sets A of distinct non-negative real numbers for which:

(a) the set A contains at least four numbers.

(b) for any 4 distinct numbers $a, b, c, d \in A$, the number $ab + cd \in A$.

Number Theory

N1 Let a, b be rational numbers such that $a + b$ and $a^2 + b^2$ are integers. Prove that a, b are both integers.

N2 Find all pairs of positive integers (a, b) such that the sequence of positive integers $a_1, a_2, a_3, \cdots$, formed by $a_1 = a, a_2 = b$ and

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}, a_{n-2})},$$

for $n \geq 3$, is bounded. (A bounded sequence is a sequence for which there exists a positive real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$.)

N3 For a positive integer n, let $f(n)$ be the largest prime divisor of n. Prove that there are infinitely many positive integers n such that

$$f(n) < f(n + 1) < f(n + 2).$$

Combinatorics

C1 Let $n \geq 2$ be a positive integer. An $n \times n$ grid is filled with the integers $1, 2, \cdots, n^2$. Let t be the maximum of the (positive) difference of the entries of two neighbouring squares, where two squares are said to be neighbours if they share at least one vertex. Determine the minimum possible value of t in terms of n.

C2 A chessboard is tiled with 32 dominoes. Each domino covers two adjacent squares, a white and a black square. Show that the number of horizontal dominoes with the white square on the left of the black square equals the number of horizontal dominoes with the white square on the right of the black square.

C3 Let $n > 1$ be a positive integer. On each of $2n$ points around a circle we place a disk with one white side and one black side. We may perform the following move: select a black disk, and reverse its two neighbors. Find all initial configurations from which some sequence of such moves leads to a position where all disks but one are white.

Geometry

G1 The altitude from A of triangle ABC intersects the side BC at D. A circle is tangent to BC at D, intersects AB at M and N, and intersects AC at P and Q. Prove that

$$\frac{AM + AN}{AC} = \frac{AP + AQ}{AB}.$$

G2 Let $\mathcal{P}$ be a convex 2010-gon. The 1005 diagonals connecting opposite vertices and the 1005 lines connecting the midpoints of opposite sides are concurrent. (i.e. all 2010 lines are concurrent.) Prove that the opposite sides of $\mathcal{P}$ are parallel and have the same length.

G3 Two circles meet at A and B. Line ℓ passes through A and meets the circles again at C and D respectively. Let M and N be the midpoints of arcs BC and BD which do not contain A, and let K be the midpoint of CD. Prove that $\angle MKN = 90°$.