Winter Camp 2009
Pre-camp problem set

1. Prove that $n-1 < \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{3}{\sqrt{2}+\sqrt{5}} + \frac{5}{\sqrt{5}+\sqrt{10}} + \ldots + \frac{2n-1}{\sqrt{(n-1)^2+1}+\sqrt{n^2+1}} < n$ for every positive integer n.

2. Prove that $\frac{a+b}{c+d}$ is irreducible if $ad - bc = 1$.

3. The first five terms of a sequence are $1, 2, 3, 4, 5$. From the sixth term on, each term is 1 less than the product of all the preceding ones. Prove that the product of the first 70 terms is equal to the sum of their squares.

4. Prove that the equation $y^2 = x^5 - 4$ has no integer solutions.

5. Let ABC be an equilateral triangle of altitude 1. A circle, with radius 1, and center on the same side of AB as C, rolls along the segment AB. Prove that the length of the arc of the circle that is inside the triangle remains constant.

6. What real-valued functions f satisfy the inequalities

$$f(x) \leq x, \qquad f(x+y) \leq f(x) + f(y)$$

   for all real x, y?

7. Let S be a finite set of points in the plane, and let r be a positive real number. Suppose it is possible to place several circular discs of radius r on the plane such that: (a) every point in S is covered by exactly one disc, and (b) the center of every disc is a point in S. Prove that, no matter how this is done, the number of discs used will be constant.

8. Let a, b, c, d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$\frac{a^3}{b+c} + \frac{b^3}{c+d} + \frac{c^3}{d+a} + \frac{d^3}{a+b} \geq \frac{1}{8}.$$

9. Fix a prime $p > 2$. An integer x is called a quadratic residue mod p if there exists an integer y so that $y^2 - x$ is a multiple of p.

   (a) Determine the number of integers x in $\{0, 1, 2, \ldots, p-1\}$, for which both x and $x-1$ are quadratic residues mod p.

   (b) Prove that -1 is a quadratic residue mod p if and only if $\frac{p-1}{4}$ is an integer.

10. Prove that there exists a unique function f defined on the positive reals such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all positive x.

11. Let O be the circumcenter of an acute-angled triangle ABC and let A_1 be a point on the smaller arc BC of the circumcircle of $\triangle ABC$. Let A_2 and A_3 be points on the sides AB and AC respectively, such that $\angle BA_1A_2 = \angle OAC$ and $\angle CA_1A_3 = \angle OAB$. Show that the line A_2A_3 passes through the orthocenter of $\triangle ABC$.

12. The numbers $1, 2, \ldots, 2^{2008}$ are stored in 2009 memory locations of a computer. Two programmers take turns choosing five memory locations, and then subtracting 1 from each of these locations. If any location ever acquires a negative number, the computer breaks and the guilty programmer pays for the repairs. Which programmer can ensure himself financial security, and how?

13. (*Pascal's Theorem*) Let A, B, C, D, E, F be points on a circle, and let P, Q, R respectively be the intersection of AB and DE, BC and EF, CD and FA respectively.

    (a) Let ω be the circumcircle of $\triangle CFR$, and let G and H be the second intersections of lines BC and EF with ω. Prove that triangles RGH and PBE have parallel sides.

    (b) Prove that P, Q, and R lie on a line.

14. A positive integer is written in each square of an 8×8 chessboard. One is allowed to choose a 3×3 or a 4×4 square on the chessboard and increase all numbers in it by 1. Is it always possible, applying such operations several times, to arrive at a situation where all squares contain multiples of 10?

15. Let P and Q be points in the plane and let ω_1, ω_2, and ω_3 be circles passing through both. If A, B, C, D, E, and F are points on a line in that order so that A and D lie on ω_1, B and E lie on ω_2, and C and F lie on ω_3, prove that $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

16. Given $x, y, z \geq 0$ with $xy + yz + zx = 2$. Prove that

$$7(x+y+z)^3 - 9(x^3+y^3+z^3) \leq 108.$$

    Determine when equality holds.

17. Find all positive integers m, n such that $m|n^2+1$ and $n|m^2+1$.

18. For every finite set of positive integers A, define the collection A_2 to be

$$A_2 = \{a+b | a, b \in A, a \neq b\}.$$

    For example, if $A = \{1, 2, 3, 4\}$, then $A_2 = \{3, 4, 5, 5, 6, 7\}$. (Note that we are allowed to repeat elements in a collection.) Suppose there exist two finite sets A, B such that $A_2 = B_2$ but $A \neq B$. Prove that the number of elements in A and in B are the same and this number is a power of 2.

Solutions

1. Note that $\frac{2k-1}{\sqrt{(k-1)^2+1}+\sqrt{k^2+1}} = \sqrt{k^2+1} - \sqrt{(k-1)^2+1}$. Therefore,

$$\sum_{k=1}^{n} \frac{2k-1}{\sqrt{(k-1)^2+1}+\sqrt{k^2+1}} = \sum_{k=1}^{n} \sqrt{k^2+1} - \sqrt{(k-1)^2+1} = \sqrt{n^2+1} - 1,$$

and the result follows from the fact that $n < \sqrt{n^2+1} < n+1$.

2. $d(a+b) - b(c+d) = ad - bc = 1$, which implies $a+b$ and $c+d$ are relatively prime.

3. Let x_n denote the n^{th} term in the sequence, and let $s_n = \sum_{i=1}^{n} x_i^2$, $p_n = \prod_{i=1}^{n} x_i$. For $n \geq 6$,

$$\begin{aligned} s_n - p_n &= s_{n-1} + (p_{n-1}-1)^2 - p_{n-1}(p_{n-1}-1) \\ &= s_{n-1} - p_{n-1} + 1 \end{aligned}$$

Since $s_5 - p_5 = 55 - 120 = -65$, it follows that $s_{70} - p_{70} = 0$.

4. For any integers x, y, one can check $y^2 \in \{0, 1, 3, 4, 5, 9\}$ mod 11 and $x^5 \in \{-1, 0, 1\}$ mod 11. Therefore, the equation has no solutions mod 11, and hence no integer solutions.

5. Suppose the circle is centered at O. We assume without loss of generality that O is closer to B than A. Since O and C are both distance 1 from AB, we know CO is parallel to AB, and hence $\angle OCB = 60°$.

   Now let P and Q be where the circle hits CA and CB respectively. Also let Q' be the point on $\overrightarrow{CB}$ such that $\angle POQ' = 60°$. Note that $\angle POQ' = 60° = \angle PCQ'$, so $PCOQ'$ is cyclic. Therefore, $\angle OPQ' = \angle OCQ' = \angle OCB = 60°$, which implies that $\triangle OPQ'$ is equilateral and $OQ' = OP = 1$. However, we know Q is the unique point on $\overrightarrow{CB}$ such that $OQ = 1$, which means $Q' = Q$, and hence $\angle POQ = \angle POQ' = 60°$.

   Therefore, the arc contained within $\triangle ABC$ is always one-sixth of the circumference, regardless of where O is.

6. Setting $x = y = 0$, we have $f(0) \leq f(0) + f(0) \implies f(0) \geq 0$. We are given $f(0) \leq 0$, so it follows that $f(0) = 0$. Setting $y = -x$, we have $0 = f(0) \leq f(-x) + f(x)$. On the other hand, $f(-x) + f(x) \leq 0$ with equality only if $f(-x) = -x$ and $f(x) = x$. It follows that $f(x) = x$ for all x.

7. Let X and Y be two sets of discs, each fulfilling the given requirements. We define a function $f : X \to Y$ as follows. Fix a disc $C \in X$. Then, its center must be covered by some unique disc D in Y. Define $f(C) = D$. Suppose $f(C) = f(C') = D$ for some discs C, C'. Then, the center of D is within distance r of the centers of C and C', meaning C and C' both cover the center of D, which is impossible. Therefore, f is one-to-one and $|Y| \geq |X|$. Similarly, $|X| \geq |Y|$, and the result follows.

8. By the Cauchy-Schwarz inequality applied to the vectors $\left(\sqrt{\frac{a^3}{b+c}}, \sqrt{\frac{b^3}{c+d}}, \sqrt{\frac{c^3}{d+a}}, \sqrt{\frac{d^3}{a+b}}\right)$ and

$(\sqrt{b+c}, \sqrt{c+d}, \sqrt{d+a}, \sqrt{a+b})$, we have

$$\begin{aligned} \frac{a^3}{b+c} + \frac{b^3}{c+d} + \frac{c^3}{d+a} + \frac{d^3}{a+b} &\geq \frac{\left(a^{1.5}+b^{1.5}+c^{1.5}+d^{1.5}\right)^2}{(b+c)+(c+d)+(d+a)+(a+b)} \\ &= 8 \cdot \left(\frac{a^{1.5}+b^{1.5}+c^{1.5}+d^{1.5}}{4}\right)^2, \end{aligned}$$

which, by the power-mean inequality is at least $8 \cdot \left(\frac{a+b+c+d}{4}\right)^3 = \frac{1}{8}$.

9. Let n denote the number of consecutive quadratic residues mod p, and let m denote the number of solutions (x, y) to $x^2 - y^2 \equiv 1 (\text{mod } p)$. Since every quadratic residue except for 0 has exactly two square roots mod p, every pair of consecutive quadratic residues corresponds to 4 solutions to $x^2 - y^2 \equiv 1 (\text{mod } p)$, unless one of the residues is 0, in which case it corresponds to 2 solutions. Therefore, $m = 4n - 4$ if -1 is a quadratic residue and $m = 4n - 2$ otherwise.

   We now calculate m. Note that $x^2 - y^2 \equiv 1 (\text{mod } p)$ is equivalent to there existing $z \neq 0$ so that $x + y \equiv z (\text{mod } p)$ and $x - y \equiv z^{-1} (\text{mod } p)$, which in turn is equivalent to $x \equiv \frac{z+z^{-1}}{2} (\text{mod } p)$ and $y \equiv \frac{z-z^{-1}}{2} (\text{mod } p)$. Thus, there is one unique solution (x, y) for each $z \neq 0$, and $m = p-1$.

   Therefore, $n = \frac{p+3}{4}$ or $\frac{p+1}{4}$ depending on whether -1 is a quadratic residue. Since n is an integer, -1 is a quadratic residue if and only if $p \equiv 1 (\text{mod } 4)$, and either way, $n = \left\lceil \frac{p}{4} \right\rceil$.

10. For any $x > 0$, define the sequence $\{x_n\}$ by $x_0 = x$ and $x_n = f(x_{n-1})$ for $n > 0$. Then, $f(f(x_{n-2})) = 6x_{n-2} - f(x_{n-2}) \implies x_n = 6x_{n-2} - x_{n-1}$ for all n. It follows[1] that there exist constants A, B so that $x_n = A \cdot 2^n + B \cdot (-3)^n$. Since $x_n > 0$ for all n, we must have $B = 0$, and since $x_0 = x$, we then have $A = x$. Therefore, $f(x) = x_1 = 2x$. Conversely, it is easy to check $f(x) = 2x$ does satisfy the given condition.

11. Let B' and C' be where the altitudes from B and C hit the circumcircle of $\triangle ABC$. Then, $\angle BA_1A_2 = \angle OAC = 90° - \angle B = \angle BCC' = \angle BA_1C'$, so A_2 is the intersection of AB and A_1C'. Similarly, A_3 is the intersection of AC and A_1B'. Now, Pascal's theorem (see #13), applied to the "hexagon" $ABB'A_1C'C$, states that A_2, H, and A_3 are collinear.

    Remark: Instead of applying Pascal's theorem, one can show $HA_2 = C'A_2$ and $HA_3 = B'A_3$, and then angle-chase.

12. Let us call the programmers Alice and Biff. Also label the first 2005 memory locations as good, and the remaining ones as bad. Alice can guarantee a win as follows. On her first turn, she chooses the first good memory location and the four bad ones. From then on, she always copies Biff's last move.

    This ensures that at the end of every one of Alice's turns, all good memory locations have even parity. On the other hand, there are only 4 bad memory locations, so each turn must decrease a good memory location. Therefore, a good location will become negative in at most $1 + 1 + 2 + 4 + \ldots + 2^{2004} = 2^{2005}$ turns. Conversely, no bad location can become negative that quickly.

    Since all good memory locations are even at the end of every one of Alice's turns, it follows that Biff must be the first one to turn one of these location negative.

[1] We are using a general theorem here: let $c_1, c_2, \ldots, c_n$ be constants, and let $\{x_1, x_2, \ldots\}$ be a sequence satisfying $x_m = \sum_{i=1}^{n} c_i x_{m-i}$ for $m \geq n$. Suppose the polynomial $x^n = \sum_{i=1}^{n} c_i x^{n-i}$ has n distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then, there exist constants $C_1, C_2, \ldots, C_n$ such that $x_m = \sum_{i=1}^{n} C_i \cdot \alpha_i^m$ for all m.

13. It is important to correctly handle all possible configurations while doing part (a).

    (a) *Claim:* Let circles ω_1 and ω_2 intersect at X and Y. If AB is a chord on ω_1, and C and D are the second intersections of AX and BY with ω_2, then AB and CD are parallel.

        Proof: Using directed angles[2], we have $\angle XAB = \angle XYB = \angle XYD = \angle XCD$. Therefore, lines AB and CD make the same angle with line AXC, and thus are parallel. □

        The result follows from applying this claim to chords BA, BE, and DE.

    (b) Since triangles RGH and PBE have parallel sides, PR, BG, and EH all meet at a common point, namely Q, so Q lies on PR.

14. It is not always possible. We consider all positions modulo 10. In this setting, adding 1 to a square 10 times has no effect.

    If a position A can reach the zero position by adding 1 to various squares, the zero position can reach A by subtracting 1 from the same squares. Since there are $5^2 + 6^2 = 61$ squares to choose from, and each square can be subtracted from between 0 and 10 times, a total of $10^{61} < 10^{64}$ positions can be reached from the zero position in this way.

    However, there are 10^{64} positions altogether modulo 10, so not all of them can be reached.

15. Let $R = PQ \cap AF$, and let $a = AR, b = BR, c = CR, d = DR, e = ER, f = FR$ denote signed distances along the line $\overrightarrow{AB}$. Also let x denote the signed product $PR \cdot QR$. By power of a point on each circle, we have $ad = be = cf = x$. Now:

$$\begin{aligned} AB \cdot CD \cdot EF - BC \cdot DE \cdot FA &= (b-a)\cdot(d-c)\cdot(f-e) - (c-b)\cdot(e-d)\cdot(f-a) \\ &= bdf - bde - bcf + bce - adf + ade + acf - ace - \\ &\quad\; cef + ace + cdf - acd + bef - abe - bdf + abd \\ &= bdf - dx - bx + cx - fx + ex + ax - ace - \\ &\quad\; ex + ace + dx - cx + fx - ax - bdf + bx = 0. \end{aligned}$$

    Remark: A slicker solution is to perform an inversion around P, and then use angle-Ceva's theorem to exploit the fact that lines $A'D', B'E'$, and $C'F'$ concur at Q'.

16. Let $s = x + y + z$ and $p = xyz$. Then,

$$x^3 + y^3 + z^3 = (x+y+z)^3 - 3(x+y+z)(xy+yz+xz) + 3xyz = s^3 - 6s + 3p,$$

    which implies:

$$\begin{aligned} 108 - 7(x+y+z)^3 + 9(x^3+y^3+z^3) &= 108 + 2s^3 - 54s + 27p \\ &= 2(s-3)^2(s+6) + 27p \geq 0. \end{aligned}$$

    Equality holds iff $s = 3$ and $p = 0$, or equivalently, x, y, z are the roots of $w^3 - 3w^2 + 2w = 0$. This means equality holds iff $(x, y, z) = (0, 1, 2)$ or a permutation thereof.

[2] We define the directed angle ABC to be the counter-clockwise angle (mod $180°$) one has to rotate A around B by in order to make A lie on line BC. We use the following key properties: $\angle ABD = \angle ACD$ iff A, B, C, D are concyclic, and $\angle ABC = \angle ABD$ iff B, C, D are collinear. These properties hold regardless of configuration, which is a big improvement over traditional angle-chasing!

17. Let F_k denote the Fibonacci sequence. Then, the full set of solutions for (m, n) is:

$$T = \{(1,1), (F_{2k-1}, F_{2k+1}), (F_{2k+1}, F_{2k-1})\}.$$

That these solutions are all valid follows immediately from the fact that $F_{2i+1}^2 + 1 = F_{2i-1} \cdot F_{2i+3}$. (This identity can be proven directly by substituting $F_i = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$, on both sides and then expanding).

Now suppose there is a solution not in T. Take a minimal such solution (m, n), and assume without loss of generality that $m \leq n$. If $m = n$, then $\gcd(n, m^2+1) = 1$, implying $m = n = 1$, and giving a contradiction. Otherwise, let $x = \frac{m^2+1}{n}$, which we assumed to be an integer. Note $x \leq \frac{m^2+1}{m+1} \leq m < n$. On the other hand, working modulo m, we have $x^2 + 1 \equiv x^2 + (m^2+1)^2 \equiv x^2 + n^2x^2 \equiv x^2(n^2+1) \equiv 0$. Therefore, (x, m) is a smaller solution than (m, n).

Since $x \leq m$, and since (m, n) was assumed to be the smallest solution not in T, we have $\left(\frac{m^2+1}{n}, m\right) = \{1, 1\}$ or $\left(\frac{m^2+1}{n}, m\right) = \{F_{2k-1}, F_{2k+1}\}$. In the former case, $(m, n) = (1, 2)$. In the latter case, $(m, n) = (F_{2k+1}, F_{2k+3})$. Either way, we have contradicted the assumption that $(m, n) \notin T$, and the proof is complete.

18. Suppose $A_2 = B_2$. Then $|A_2| = \binom{|A|}{2}$ and $|B_2| = \binom{|B|}{2}$ so $|A| = |B|$. Let $n = |A| = |B|$.

Let $f(x)$ and $g(x)$ be the polynomials $\sum_{a \in A} x^a$ and $\sum_{b \in B} x^b$. Factor $f(x) - g(x)$ as $(x-1)^t \cdot h(x)$ for a non-negative integer t and a polynomial h satisfying $h(1) \neq 0$. Then,

$$\begin{aligned}
(x-1)^t \cdot h(x) \cdot (f(x) + g(x)) &= f^2(x) - g^2(x) \\
&= \sum_{a \in A} x^{2a} + 2\sum_{a \in A_2} x^a - \sum_{b \in B} x^{2b} - 2\sum_{b \in B_2} x^b \\
&= \sum_{a \in A} x^{2a} - \sum_{b \in B} x^{2b} \\
&= f(x^2) - g(x^2) \\
&= (x^2-1)^t \cdot h(x^2) \\
&= (x-1)^t \cdot h(x^2) \cdot (x+1)^t.
\end{aligned}$$

Dividing through by $(x-1)^t$, we have $h(x) \cdot (f(x) + g(x)) = h(x^2) \cdot (x+1)^t$. Now, letting $x = 1$ and dividing through by $h(1)$ (which we know is not 0), we have $2n = f(1) + g(1) = 2^t$. It follows that n is a power of 2.