Invariants and Colouring

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1 Introduction

Invariants, monovariants, and colourings are examples of a common and important technique in mathematics:

> **Main idea:**
>
> If you are studying something complex, *don't try to understand everything at once*. Is there one specific piece that you can focus on that is easier to understand? If so, focus on that one thing and see what you learn!

These notes revolve around this principle and some interesting applications of it. Let's start with an example from a recent IMO shortlist:

Example 1. 2017 *cards, each having one red side and one white side, are arranged in a row. Initially all cards show their red sides. Two players play a game with alternating moves. Each move consists of choosing a block of* 50 *consecutive cards, the leftmost of which is showing red, and turning them all over, so those which showed red now show white and vice-versa. The last player who can make a legal move wins.*

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?[1]

Solution. (a) Given a game position, let's define its "size" to be the 2017-digit number obtained by reading cards from left-to-right and writing down a 1 for each red card and a 0 for each white card. A move will cause a 1 in the size to switch to a 0, and the next 49 digits will swap between 0 and 1. Since the leftmost digit goes down, the overall size is guaranteed to decrease every turn. The size is always a positive integer that is less than 10^{2017}, so there will be at most 10^{2017} turns and hence the game *will* end.

(b) Let's define a card to be "cool" if it is exactly $50n$ spots from the right-most end for some n. Given a game position, let's define its "cool count" to be the number of cool cards that are red. Each sequence of 50 consecutive cards contains one and only one cool card. Therefore, a position's cool count either decreases by one or increases by one each turn. In particular, its parity changes. The cool count starts at 40 - an even number - and so it will always be odd on the second player's turn. This means that there will be at least one red cool card, and so the second player always has

[1] IMO Shortlist 2009 C1

at least one legal move. Since this is always true, it's *impossible* for the second player to lose, and so, no the first player does not have a winning strategy. □

This is a quintessential monovariant / invariant problem. The total number of configurations is enormous and you can't possibly do case analysis or brute force to understand them all. However, the size and cool count are easy to understand and tell you all you need to know. The size is a number that always go down, which is called a *monovariant*, and the cool count parity is something that always stays the same, which is called an *invariant*. These are just fancy terms though – the important thing is taking a complicated process and focusing on one very simple part of it at a time. We used two main ideas here:

To prove a process terminates:

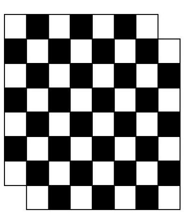
Find a quantity that goes down at each step. If the quantity has to be a positive integer, you are done!

To prove one player has a winning strategy in a game:

Find an invariant that they can maintain at the end of each of their moves, and prove they cannot actually lose as long as this invariant holds.

The idea of decreasing size comes up all the time, and taking a "lexicographical ordering" (interpreting a series of digits as one big number) is very common. The cool count is a little trickier though. Why would somebody try that? Let's compare with one of the most famous colouring problems:

Example 2. *A standard* 8×8 *chessboard has two diagonally opposite corners removed, leaving 62 squares. Is it possible to place 31 dominoes of size* 2×1 *so as to cover all of these squares?*[2]



Solution. No. Consider the normal black-and-white colouring of a chessboard as shown. A domino will cover exactly one white square and exactly one black squares, so 31 dominoes will cover exactly 31 white squares and exactly 31 black squares. But there are only 30 black squares, so it is impossible to place 31 dominoes. □

Do you see how this argument relates to Example 1? Don't think about it just as a colouring that magically solves everything. Think about it instead as an interesting quantity (the number of black squares covered by dominoes) that changes in a simple way as you place each domino. The cool count in the IMO shortlist problem is very similar – it's just that we are placing blocks of size 50 there so we should look at every 50th square instead of every 2nd square.

[2] This is called the mutilated chessboard problem, and dates back all the way to 1946!

2 More examples

Invariants and monovariants can be used in different ways. I will give a few examples here that illustrate some of these ways.

The single most common application is the following:

To prove a process cannot reach a specific configuration:

Find a quantity that changes predictably at each step. Calculate the quantity at the start and end configurations and look for a contradiction.

Example 3. *n squares in an infinite grid are coloured black; the rest are coloured white. When a square is the opposite colour from 2 or more of its 4 neighbours, its colour may be switched. Eventually, we get to having 2017 black squares, no two of which border along an edge, and all other squares white. Prove that $n \geq 2017$.*[3]

Solution. We will say an edge is a "boundary" if it is between a white square and a black square. Every time a square changes colour, then all four of its edges swap whether they are boundaries. We are only allowed to change the colour of a square if at least two of the neighbours are opposite colour, which means we must have had at least 2 boundary edges around the square originally and at most 2 when done. Therefore, the number of boundary edges can never increase.

In the final configuration, the number of boundary edges is exactly $4 \cdot 2017$. In the starting configuration, the number of boundary edges is at most $4n$, and so we must have $n \geq 2017$. □

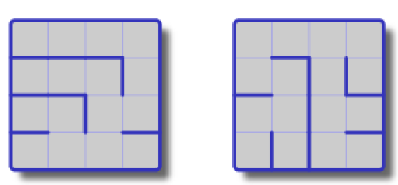
Other than periodic colourings, boundaries and corners are the best things to look at on grids. Parity can also be especially good to focus on.

The next application is very similar to the Extremal Principle:

To prove a configuration exists:

Find an "error" quantity that measures how far you are from that configuration. Come up with a method to decrease the error at each step. This works best if the error is a positive integer; otherwise it might keep decreasing but never reach 0!

Example 4. *A maze consists of a finite grid of squares where the boundary and some internal edges are "walls" that cannot be crossed. For example:*



Two mazes are given, each with a robot in the top-left square. You may give a list of directions (up, down, left, or right) to the robots. Both robots will independently follow the same list of

[3]Berkeley Math Circle 2000

directions. For each direction, the robot will move one square in that direction if it can, or do nothing if there is a wall in the way. It will then proceed to the next direction, and repeat until it has gone through the whole list. Suppose that there is a list of directions that will get each robot individually from the top-left corner to the bottom-right corner of its maze. Prove there is also a list of directions that will get both robots to the bottom-right corner at the same time.

In the example mazes above, you could give the directions 'Right', 'Right', 'Down', 'Down', 'Down', 'Right', 'Down', 'Down', 'Left', 'Down', 'Right'.[4]

Solution. The following algorithm works:

1. Give a minimal list of directions that will get robot 1 to the bottom right corner from its current position.

2. If Robot 2 is not also in the bottom-right corner, give a minimal list of directions that will get it to the bottom right corner from its current position.

3. If Robot 1 is no longer in the bottom-right corner, repeat from Step 1.

After each phase of the algorithm, we will have one robot R at the bottom-right corner while the other robot R' is x steps away for some x. During the next phase of the algorithm, we will take exactly x steps and move R' to the bottom-right corner. Note that R' will end up below or right of where it started.

Let's consider what happened to R during this time. It started at the bottom-right corner and attempted x steps. If it never hit a wall, then it would have moved the same amount right and down as R' did and so in particular, it would also need to end up below or right of where it started. However, that is impossible since it started in the bottom-right corner. Thus, R must have successfully taken at most $x - 1$ steps during this phase, and so must now be at most $x - 1$ steps from the bottom-right corner.

Thus, x goes down during each phase. Since it is a non-negative integer, it must eventually reach 0 and we are done. □

The next application is a little rarer, but can be powerful. It is similar to induction, and may be familiar to you if you got C4 on the camp warmup problems.

To prove all configurations have some property:

Prove one configuration X has the property, and that you can get from any configuration to X via simple steps that don't change whether the property holds.

Example 5. *On every square of a 2017×2017 board, either a +1 or -1 is written. For every row, we compute the product R_i of all numbers written in that row, and for every column, we compute the product C_i of all numbers written in that column. Is it possible to arrange the numbers in such a way that*

$$\sum_{i=1}^{2017} (R_i + C_i) = 0?^5$$

[4] MOP, except the question there had N mazes. That version is much harder and requires a different approach! The version here is very similar to Canadian Math Olympiad 2012 #4.

[5] Colorado Math Olympiad 1997

Solution. Let $S = \sum_{i=1}^{2017}(R_i + C_i)$. Suppose by way of contradiction that we have found some arrangement for which $S = 0$. Let us replace each -1 with +1, one at a time. Each time we do this, we negate some R_i and some C_j, and so S changes by $\pm 2 \pm 2 \equiv 0 \pmod 4$. Since we assumed $S \equiv 0 \pmod 4$ originally, it must still be $0 \pmod 4$ when done. However, at that point, we know each R_i and C_j equals 1, so $S = 2 \cdot 2017 \equiv 2 \pmod 4$, contradiction. □

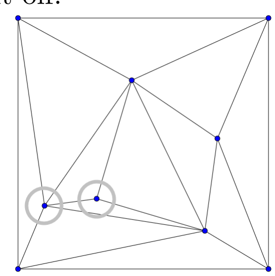
Finally, the application used by most colouring problems:

To prove a configuration is not possible:

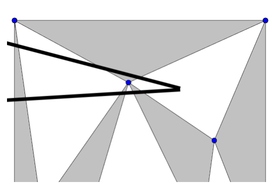
Find one or more quantities that are easy to understand locally, and prove they cannot take on the value needed for the bad configuration.

I'll give a harder example this time, one where some additional invariant ideas are needed to show the colouring even exists, and where double-counting is needed to finish it off.

Example 6. *A square is partitioned into triangles in a way similar to what is shown to the right. If a triangle touches another triangle or the square, they must either share exactly one vertex or they must share a whole edge and both surrounding vertices. Prove at least one vertex has an odd number of edges coming out of it.*



Solution. We give a proof by contradiction. Suppose that every vertex has an even number of edges coming out of it. We claim that the triangles (and the outside region) can be coloured white and black so that each triangle is adjacent only to triangles of the opposite colour.



To see this, let's consider some point P inside the square. Draw a line segment ℓ from P to the outside of the square and not going through any vertices. Define $f(\ell)$ to be the number of edges that ℓ passes through. Now consider what happens if you rotate ℓ slightly about P, as shown in the picture to the left. If you do not rotate ℓ across a vertex, then ℓ continues to intersect the same edges and so $f(\ell)$ stays constant. If you do cross a vertex, then ℓ will start by touching some of the edges adjacent to the vertex, and end by touching the other edges adjacent to the vertex. This may cause $f(\ell)$ to change, but since all vertices have an even number of edges coming out, the *parity* of $f(\ell)$ will stay the same. So for every line ℓ going from P to the outside, either $f(\ell)$ is even or $f(\ell)$ is odd. We will colour P white in the first case and black in the second case.

A very similar argument shows that if we move P around inside a single triangle, the colour of P doesn't change. Therefore, we really have given a consistent colouring to each triangle. Finally, if two triangles share an edge, choose points X and Y on opposite sides of that edge such that no vertices lie on line XY. Then, if Z is a point outside the square on line XY, we have $f(XZ) = f(YZ) \pm 1$. This means X and Y have different colours, and so the colouring does indeed have the desired property!

And now, we are almost done. Let us suppose there are B black triangles and W white triangles. Every edge must be adjacent to one black triangle and either one white triangle or the outside region. Therefore, the total number of edges must equal both $3B$ and $3W+4$. Thus, $3B = 3W+4$, which is impossible! Contradiction. □

3 Beyond Olympiads!

Believe it or not, there is actually math beyond Olympiads, and invariants are useful there too! I will give one really cool example here. Note: this section is just for fun, so it's okay if you find it confusing. But it *is* cool...

We will define a "loop" to be a path on the plane that returns to where it started and does not pass through the origin. Here are a few examples:

Now imagine that loops X and X' represent two positions of a rubber band around a tube. We will say X can be "continuously deformed" into X' if it is possible to move the rubber band from position X to position X' without taking it off the tube. If you try playing with it a little, you should see that the third loop in the example above can be continuously deformed into the second loop but not into the first or fourth.

This relates to a general invariant. For a loop X, define its "winding number" $W(X)$ to be the number of times it passes over the negative x-axis going down minus the number of times it passes over the negative x-axis going up. The loops in the example above have winding numbers -2, 0, 0, and 3 respectively.

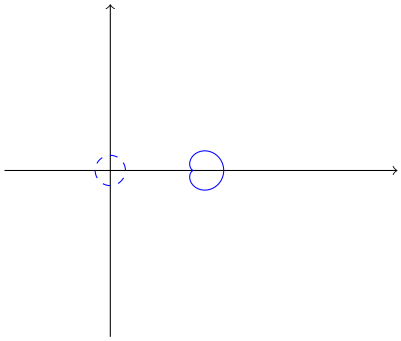
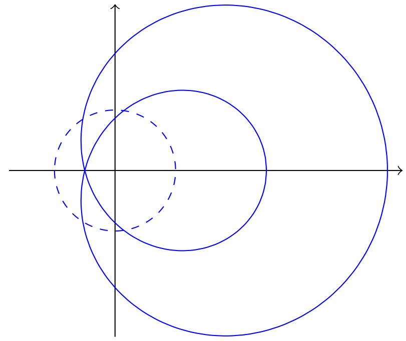
Fact: *Winding number is invariant under continuous deformation. In other words, if two loops X and X' have different winding numbers, it is not possible to continuously deform X into X'.*

I'm going to skip the rigorous proof here but it should hopefully be intuitively clear. Just try it with a real rubber band and you'll see!

Example 7. *The Fundamental Theorem of Algebra: Every non-constant polynomial with complex coefficients has at least one complex root.*

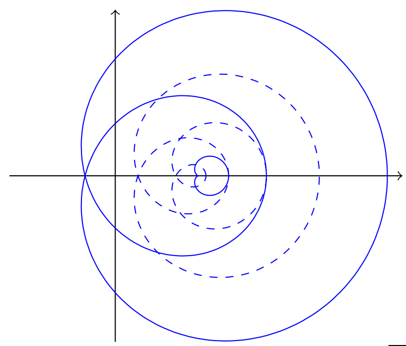
Solution. Suppose by way of contradiction that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is a non-constant polynomial with no complex roots. Note that $a_0 \neq 0$ since otherwise $x = 0$ is a root. Since the roots do not change when P is multiplied by a constant, we can assume without loss of generality that a_0 is a positive real number.

Now let's apply P to every point on a circle on the complex plane centered at the origin with some radius r. This will give us a loop X_r. Note that the loop does not pass through the origin since P is assumed to have no roots. This is shown below for the polynomial $P(x) = x^2 + x + 3$ with $r = 0.5$ and $r = 2$:

Now let ϵ be a very small positive number. If $|x| = \epsilon$, then $P(x) \approx a_0$, a positive real number. In particular, if ϵ is small enough, $P(x)$ cannot get all the way to the negative real-axis, and so the winding number of X_ϵ must be zero. Conversely, let R be a very large positive number. If $|x| = R$, then $P(x) \approx a_n x^n$. But if you graph $a_n x^n$, you will see it has winding number n, and so the winding number of X_R must also be n. Indeed, in the example above you can see that $X_{0.5}$ has winding number 0 and X_2 has winding number 2.

However, it turns out we have an easy way to continuously deform X_ϵ to X_R. Just gradually increase r from ϵ to R, as shown to the right. Since P is assumed to not have roots, each X_r avoids the origin and so is a valid loop. So this gives a way of continuously going from X_ϵ to X_R. However, this is a contradiction since X_ϵ and X_R have different winding number. We conclude that our initial assumption must have been incorrect, and P does indeed have a complex root.



□

4 Problems

1. You have a chocolate bar consisting of squares arranged in an $n \times m$ rectangular pattern. Your task is to split the bar into squares (always breaking one piece at a time along a line between squares) with a minimum number of breaks. How many breaks do you need?

2. Write the numbers $1, 2, \ldots, 2017$ on the board. Choose any 2 numbers a and b, erase them, and write $|a - b|$. Determine whether the final number on the board will be odd or even, and show that it does not depend on the manner in which the numbers were chosen at each step.

3. Consider a rectangular array with m rows and n columns whose entries are real numbers. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations we can make the sum of numbers along each line (row or column) non-negative.

4. A hockey player has 3 pucks labeled A, B, C in an arena (as in all math, an infinite area!). He picks a puck at random, and fires it through the other 2. He keeps doing this. Can the pucks return to their original spots after 2017 hits?

5. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

6. Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0 \\ \left\{ \frac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0 \end{cases}$$

where $\{x\}$ denotes the fractional part of x. (The fractional part of x is given by $x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

7. A natural number is written on the blackboard. Whenever number x is written, one can write any of the numbers $2x + 1$ and $\frac{x}{x+2}$. At some moment the number 2008 appears on the blackboard. Show that it was there from the very beginning.

8. Is it possible for a chess knight to pass through all the squares of a $4 \times N$ board having visited each square exactly once, and return to the initial square?

9. Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a, f_2 = b$, and letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

10. To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon P_6 of area 1 is clipped to obtain a heptagon P_7. Then P_7 is clipped (in one of the seven possible ways) to obtain an octagon P_8, and so on. Prove that no matter how the clippings are done, the area of P_n is greater than $\frac{1}{3}$, for all $n \geq 6$.

11. (Matthew's Worst Nightmare) We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

12. The numbers from 1 through 2017 are written on a blackboard. Every second, Dr. Math erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

13. Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?

14. A circle has been cut into 2000 sectors. There are 2001 frogs inside these sectors. There will always be some two frogs in the same sector; two such frogs jump to the two sectors

adjacent to their original sector (in opposite directions). Prove that, at some point, at least 1001 sectors will be inhabited.

15. Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

16. A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

    - If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.
    - At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

    Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

17. You may put a checker on any lattice point (point with integer coordinates) in the plane with y-coordinate less than or equal to 0 (i.e. lattice points on and below the x-axis). Note that you may place infinitely many checkers to start. The only legal moves are horizontal or vertical jumping – a checker can leap over a neighbor, ending up 2 units up, down, right, or left of its original position, provided that the destination point is unoccupied. After the jump is complete, the checker that was jumped over is removed. Prove that no matter how many jumps you do, it is impossible to get a checker to $y = 5$.

18. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in S$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line ℓ going through P such that the resulting windmill uses each point of S infinitely many times.

19. We assign an integer to each vertex of a regular pentagon, so that the sum of all is positive. If three consecutive vertices have assigned numbers x, y, z, respectively, and $y < 0$, we are allowed to change the numbers (x, y, z) to $(x + y, y, z + y)$. This transformation is made as long as one of the numbers is negative. Decide if this process always comes to an end.

20. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

21. The liar's guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

    At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

    After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

    (a) If $n \geq 2^k$, then B can guarantee a win.

    (b) For all sufficiently large k, there exists an integer $n \geq (1.99)^k$ such that B cannot guarantee a win.

22. James is playing a game with Lavaman. James begins by standing at the origin of the coordinate plane. James and Lavaman take turns, starting with Lavaman. On his turn, Lavaman places lava on a lattice point of his choosing, preventing James from going there. Then on James's turn, James can move m times, each time going from the point $(x,\ y)$ to the point $(x+1,\ y)$ or $(x,\ y+1)$ (but he can never go to a lattice point with lava). Lavaman's goal is to make it so that James can't move. For which positive integers m can Lavaman guarantee victory?

23. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*. Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

5 Hints

1. (Traditional) Each break increases the number of pieces by 1, and you need to go from 1 piece to nm pieces.

2. (Traditional) Look at the sum of numbers mod 2.

3. (Soviet Union 1961) The sum goes up each time, but that's not enough. Since we have real numbers here, not integers, you need to show the sum can take on only finitely many values. For this, prove at most 2^{m+n} configurations are attainable.

4. (Waterloo Math Circles) If you read the pucks in clockwise order starting with A, you will get either ABC or ACB. What happens after each hit?

5. (St. Petersburg 1996) Arrange the numbers lowest to highest, and show the list must get "lexicographically smaller" each time. (In other words, if the numbers are interpreted as digits of a base-B number for some big B, that number must go down.)

6. (USAMO 1997 #1) If x is rational, prove its denominator decreases at each step. If x is irrational, prove it never becomes rational.

7. (Russa 2008 Grade 9 #7) If you start with $\frac{a}{b}$, you get $\frac{2a+b}{b}$ or $\frac{a}{a+2b}$. You want to show that the numerator plus denominator (after reducing the fraction to lowest terms) stays the same or doubles at each step. Watch out though: $\gcd(2a+b, b)$ might not be 1 even if $\gcd(a, b) = 1$.

8. (Russian Math Circles) It is not. Prove that the knight must be in row 2 or 3 on every other turn. Now look at a chessboard colouring.

9. (USAMO 1993 #4) To prove f_n becomes constant, show that $\max(f_{n-1}, f_n)$ never increases and regularly decreases. To find the constant value, note that $\gcd(f_{n-1}, f_n)$ is invariant.

10. (USAMO 1997 #4) We must always retain one point on each side of the original hexagon, so the result contains a hexagon "inscribed" in P_6.

11. (IMO Shortlist 2014 C2) What happens to the product of the numbers at each step? The main challenge here is thinking to try a product in the first place. The intuition is we want *all* moves to significantly improve our monovariant, whether it be $(1, 1) \to (2, 2)$ or $(1000, 1000) \to (2000, 2000)$. The way to do this is to note each move makes a pair 4 times bigger, and to then ask how we can aggregate that in a useful way.

12. (Original) The sum and sum of squares are both invariant. What formulas do you know that relate a sum and sum of squares?

13. (IMO Shortlist 1994 C3) Try to decrease the minimum amount of money in a bank account. It might help to start with the case where one account has \$1.

14. (MOP 1998 but see also China 2005 #3) First suppose a frog never goes in sector 2000. Then you can show the sum of squares of frog positions is increasing and get a contradiction. Now note that if a frog ever gets to sector n, then sectors n and $n + 1$ can never again both be empty.

15. (IMO Shortlist 2009 C5) Cinderella wins by ensuring the water in adjacent buckets a, b, c, d, e satisfies $d = e = 0, b \leq 1$ and $a+c \leq 1$ at the end of her turn. You can come up with this idea by working backwards. What position would let the stepmother win, what position would let her get there, and so on?

16. (IMO Shortlist 2013 C3) The problem is describing a graph. The chromatic number of a graph is defined to be the minimum number of colours needed to colour every vertex such that adjacent vertices have different colour. Show how to decrease the chromatic number.

17. (John Conway) Try assigning a checker at (x, y) a weight of $\phi^{|x|+y}$ where ϕ satisfies $\phi^2 = \phi+1$. You should be able to show the sum of the weights is non-increasing and hence that no checker can ever reach $(0, 5)$.

18. (IMO 2011 #2) Show that the number of points to the right of the line never changes. Choose ℓ and P such that half the points are to the right.

19. (IMO 1986 #3) For each sequence of consecutive vertices, calculate the absolute value of the sum of the numbers on those vertices. Add these up.

20. (IMO Shortlist 2014 C7) Let L be the set of $n \cdot (n-1)$ lines that contains two lines just on either side of the line joining every pair of points in M. Show that the number of intersections between L and the n segments goes down by 4 each time.

21. (IMO 2012 #3) Part (b) is the invariant problem. Let's say B wants to eliminate one of $\{1, 2, \ldots, n+1\}$. Let a_i denote the number of consecutive answers that A has given that are inconsistent with the secret number being i. Forget about actually picking the secret number in advance – this is a game where B wants to get some a_i up to $n+1$ and A wants to stop him. Consider $\sum_i \alpha^{a_i}$ for some $\alpha \approx 2$.

22. (Japan 2012 #5 but see also John Conway's "Angel problem") Lavaman has a winning strategy for any m, playing only on the diagonal $x + y = m \cdot N$ for some big N. In $\frac{N}{2}$ moves, Lavaman can cover $\frac{1}{2m}$ fraction of all squares James can reach. How many moves does it take Lavaman to cover the next $\frac{1}{2m}$ fraction?

23. (IMO Shortlist 2014 C9) Define an "orbit" to be the number of different cycles that the snail could follow. Prove that the number of cycles has the same parity as n. To do this, gradually move the circles apart until they are all disjoint.