Winter Camp January 2021
Functional Equations

1. Solve $f(\lfloor x \rfloor y) = f(x)\lfloor f(y) \rfloor$ over $\mathbb{R}$.

2. Solve over $\mathbb{R}$:
$$f(x^2 - y^2) = xf(x) - yf(y).$$

3. Solve $f(t^2 + u) = tf(t) + f(u)$ over $\mathbb{R}$.

4. Solve the functional equation
$$f(xf(x) + f(y)) = y + f(x)^2$$
for all $x, y \in \mathbb{R}$.

5. Determine all continuous functions from $\mathbb{R} \to \mathbb{R}$ which satisfy
$$f(x+y) + f(x-y) = 2(f(x) + f(y))$$
for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$(n-1)^2 < f(n)f(f(n)) < n^2 + n$$
for every positive integer n.

7. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying both conditions
    - f takes all real values
    - $f(f(x)) = (x-1)f(x) + 2$ for all $x \in \mathbb{R}$?

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(xy - 1) + f(x)f(y) = 2xy - 1$$
holds for all $x, y \in \mathbb{R}$.

9. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the equation
$$f(x + f(xy)) = xf(1 + f(y)).$$

10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
    - $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and
    - $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$ for all $x \neq 0$.