Set 1

JBMO Level

July 2022

1 Theorems

* Try to prove these first! *

1. We denote $v_p(n)$ as the greatest power of p that divides n.

2. *(Legendre)* $v_p(n!) = \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor$

3. *(LTE)* If we have integers a, b such that $p|a-b$ then $v_p(a^n - b^n) = v_p(a-b) + v_p(n)$.(Is this true for $p = 2$, is their an extra condition?)

4. We say that the order of an element a is $d \pmod{n} \iff$ d is the smallest number such that $a^d \equiv 1 \pmod{n}$. We also denote d by $\text{ord}_n(a)$.

5. Let $d = \text{ord}_n(a)$ then $a^k \equiv 1 \pmod{n} \iff d|k$. Therefore $d|\phi(n)$

2 Problems

1.*(Lemma)* Let n be a natural number. Prove that $\tau(n) < 2\sqrt{n}$. Where $\tau(n)$ is the number of divisors of n.

2. Let a, b, c be positive reals such that $3 \leq a + b + c \leq 6$. Prove $\frac{a}{2+bc} + \frac{b}{2+ca} + \frac{c}{2+ab} \geq 1$

3. Let a, b, c be positive integers with $gcd(a, b, c) = 1$ and $a^2 + b^2 + c^2 = 2(ab + bc + ca)$. Prove that a, b, c are perfect squares.

4. Let $0 < x < 1$. The sequence $x_0, x_1, x_2...$ is given by $x_0 = 1$ and $x_{n+1} = x^{x_n}$ for every $n \geq 0$. Now fix an $n > 1$, find the number of indices $k < n$ satisfying $x_k < x_n$.

5. Find all pairs (n, k) of non-negative integers satisfying:
$$n^k + 1 = (n-2)!$$

6. Find all positive integers n such that $n = 5 * \tau(n)$. Where $\tau(n)$ is the number of divisors of n.

7. Each point on the plane has been colored one of 2022 colors, prove that there is a rectangle with 4 points all of the same color.

8*. *(Lemma)* Suppose that $a > b \geq 3$ are integers. Prove that $b^a > a^b$.

9. Do there exist four different natural number such that $ad = bc$ and
$$n^2 \leq a, b, c, d < (n+1)^2?$$

10. Find all prime number p and q such that
$$1 + \frac{p^q - q^p}{p+q}$$
is a prime number.

11. Let $a, b, c > 0$. Prove:
$$\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

12*. Find all prime numbers p such that $p^2 - p + 1$ is a perfect cube.

13. It is given that $a + b + c \leq 4 \leq ab + bc + ca$. Prove that at least two of the following quantities are not more than 2:
$$|a - b|, |b - c|, |c - a|.$$

14. Let p be a prime number, $a \geq 2, m \geq 1, a^m \equiv 1 \pmod{p}$, $a^p \equiv 1 \pmod{p^2}$. Prove that $a^m \equiv 1 \pmod{p^2}$.

15. Let p be a prime number, a a fixed number not divisible by p. Prove that the sequence $(a^n - n)$, $n \geq 1$ has infinitely many terms divisible by p.

16 Let x, y, z be positive integers. Find all solutions (x, y, z) satisfying $x^2 + y^2 = 3z^2$.

17**. *(Pell's equation)* Prove that there are infinitely many solutions to the equation $a^2 - 2b^2 = 1$.

18**. Five positive reals a, b, c, d, e have a product of 1. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{d^2} + \frac{d^2}{e^2} + \frac{e^2}{a^2} \geq a + b + c + d + e$$