The Online Math Open Fall Contest
Official Solutions
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Acknowledgements

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1. Jingoistic James wants to teach his kindergarten class how to add in Chimese. The Chimese also use base-10 number system, but have replaced the digits 0-9 with ten of its own characters. For example, the two-digit number 十六 represents 16. What is the sum of the two-digit numbers 十六 and 六十?

   Proposed by James Lin.

   Answer. $\boxed{77}$.

   Solution. The digit 十 represents 1 and the digit 六 represents 6, so it follows that 十六+六十= $16 + 61 = 77$. $\square$

2. The numbers a, b, c, d are $1, 2, 2, 3$ in some order. What is the greatest possible value of $a^{b^{c^d}}$?

   Proposed by Yannick Yao and James Lin.

   Answer. $\boxed{512}$.

   Solution. Since a 1 in any other position other than d will render the numbers on top of it useless, to maximize the value we need $d = 1$. Then it is easy to verify that $2^{3^2} = 512 > 2^{2^3} = 256 > 3^{2^2} = 81$, so the maximum value is 512, achieved when $a = 2, b = 3, c = 2, d = 1$. $\square$

3. The USAMO is a 6 question test. For each question, you submit a positive integer number p of pages on which your solution is written. On the ith page of this question, you write the fraction i/p to denote that this is the ith page out of p for this question. When you turned in your submissions for the 2017 USAMO, the bored proctor computed the sum of the fractions for all of the pages which you turned in. Surprisingly, this number turned out to be 2017. How many pages did you turn in?

   Proposed by Tristan Shin.

   Answer. $\boxed{4028}$.

   Solution. Let a_i denote the number of pages used in problem i. Then your fractions for problem i were to be $\frac{j}{a_i}$, $j = 1, 2, \ldots, a_i$. The sum of these is $\frac{a_i+1}{2}$. The sum of this over all i is $\frac{a_1+a_2+\ldots+a_6+6}{2} = 2017$. Then $a_1 + a_2 + \ldots + a_6 = 4028$. But the left hand side is the total number of pages, so you turned in 4028 pages.

   In general, if you turned in P pages and there were Q questions, then the sum of all of the fractions would be $\frac{P+Q}{2}$. $\square$

4. Steven draws a line segment between every two of the points

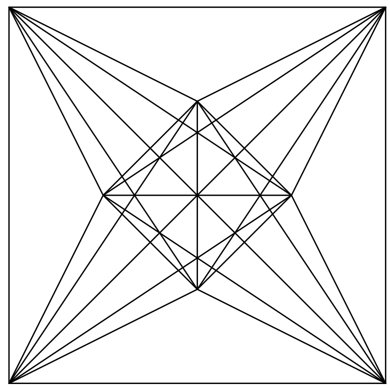
   $$A(2, 2), B(-2, 2), C(-2, -2), D(2, -2), E(1, 0), F(0, 1), G(-1, 0), H(0, -1).$$

   How many regions does he divide the square $ABCD$ into?

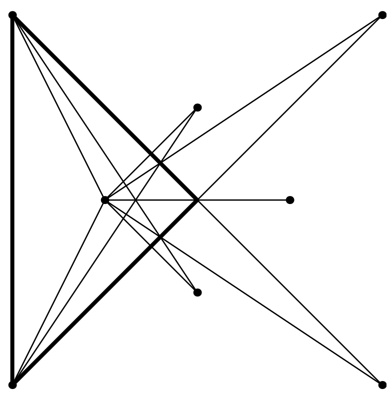
   Proposed by Michael Ren.

   Answer. $\boxed{60}$.

   Solution. Draw the diagram (note concurrences)



One actually only needs to draw any segments passing through the left triangle (between B, C, and the origin) and multiply the number of regions by 4:



There are 15 regions here, so there are a total of 60 regions in square $ABCD$. $\square$

5. Henry starts with a list of the first 1000 positive integers, and performs a series of steps on the list. At each step, he erases any nonpositive integers or any integers that have a repeated digit, and then decreases everything in the list by 1. How many steps does it take for Henry's list to be empty?

   Proposed by Michael Ren.

   Answer. $\boxed{11}$.

   Solution. First of all, 10 will be decremented 10 times before reaching 0 and being erased at the 11th step, so Henry needs at least 11 steps. To show that 11 steps is sufficient, we note that among the numbers $00, 01, 02, \dots, 98, 99$, there does not exist 11 consecutive numbers that have no repeated digits, as $00, 11, 22, \dots, 99$ separate the list into 10-number blocks. Therefore the answer is 11. $\square$

6. A convex equilateral pentagon with side length 2 has two right angles. The greatest possible area of the pentagon is $m + \sqrt{n}$, where m and n are positive integers. Find $100m + n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{407}$.

   Solution. There are two configurations. If the right angles next to each other, then the pentagon is the combination of a square and an equilateral triangle, both of side length 2, so the area is $4 + \sqrt{3}$. If the right angles not next to each other, then the pentagon can be separated into two right isosceles triangle with leg lengths 2 and an isosceles triangle with side lengths $2\sqrt{2}, 2\sqrt{2}$, and 2. The height of the third triangle is $\sqrt{7}$, so the area of the pentagon is $4 + \sqrt{7}$. The second configuration is clearly larger so the answer is 407. $\square$

7. Let S be a set of 13 distinct, pairwise relatively prime, positive integers. What is the smallest possible value of $\max_{s\in S} - \min_{s\in S}$?

   Proposed by James Lin.

   Answer. $\boxed{32}$.

   Solution. Let $T \subset S$ be the elements of S relatively prime to 30, and let T have maximum element M. At most one element of S can be divisible by each of $2, 3, 5$, so at least $|T| \geq 10$. Since there are 8 residue values of 30 relatively prime to 30, there are two elements of $T\setminus\{M\}$ that are congruent modulo 30. Since M is at least two greater than any element in $T\setminus\{M\}$, it follows that $\max_{s\in S} - \min_{s\in S} \geq 32$, and equality is achieved for $11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 16, 27, 25$. $\square$

8. A permutation of $\{1, 2, 3, \ldots, 16\}$ is called *blocksum-simple* if there exists an integer n such that the sum of any 4 consecutive numbers in the permutation is either n or $n + 1$. How many blocksum-simple permutations are there?

   Proposed by Yannick Yao.

   Answer. $\boxed{48}$.

   Solution. Call the numbers in the permutation $a_1, a_2, \ldots, a_{16}$. Since $a_1 + a_2 + a_3 + a_4$ and $a_2 + a_3 + a_4 + a_5$ differ by at most one, it follows that a_1 and a_5 are consecutive integers. Similarly, for any $k = 1, 2, \ldots, 12$, we have $|a_k - a_{k+4}| = 1$. Consequently, $A_1 = (a_1, a_5, a_9, a_{13})$ is an arithmetic sequence of consecutive integers (either ascending or descending), and similar for the other three quadruples $A_2 = (a_2, a_6, a_{10}, a_{14}), A_3 = (a_3, a_7, a_{11}, a_{15}), A_4 = (a_4, a_8, a_{12}, a_{16})$, which means that A_1, A_2, A_3, A_4 partition the set into $\{1, 2, 3, 4\}, \ldots, \{13, 14, 15, 16\}$. Moreover, since $a_1+a_2+a_3+a_4$ and $a_3+a_4+a_5+a_6$ differ by at most one as well, this implies that $a_1 + a_2 = a_5 + a_6$ since $|a_1 - a_5| = |a_2 - a_6| = 1$. As a result, if A_1 is increasing, then A_2 is decreasing and vice versa, and similar for A_3 and A_4. There are 2 ways to choose the ascending/descending arrangements and 4! ways to choose which numbers each of A_1, A_2, A_3, A_4 have, so in total there are $2 \cdot 4! = 48$ possible permutations. $\square$

9. Let a and b be positive integers such that $(2a + b)(2b + a) = 4752$. Find the value of ab.

   Proposed by James Lin.

   Answer. $\boxed{520}$.

   Solution. WLOG let $a \geq b$ and define $c = 2a + b$, $d = 2a + b$ such that $cd = 4752 = 2^4 \cdot 3^3 \cdot 11$ and $c \geq d$.

   Observe that 3 divides $c + d = 3\,(a + b)$, so since 3 divides cd, 3 divides both c and d.

   Next, note that $a = \frac{2c-d}{3}$ and $b = \frac{2d-c}{3}$, so $2d \geq c$. Thus,

   $$d^2 \leq cd = 4752 \leq 2d^2,$$

   so d is between 49 and 68, inclusive. Furthermore, d is a divisor of 4752 and a multiple of 3. The only d in this range that satisfy these conditions are 54 and 66. If $d = 54$, then $c = 88$, not a multiple of 3. If $d = 66$, then $c = 72$, so $a = 26$ and $b = 20$ for $ab = 520$. $\square$

10. Determine the value of $-1 + 2 + 3 + 4 - 5 - 6 - 7 - 8 - 9 + ... + 10000$, where the signs change after each perfect square.

    Proposed by Michael Ren.

    Answer. $\boxed{1000000}$.

    Solution. Note that if $f(n) = n^2 + 1 + n^2 + 2 + ... + n^2 + 2n + 1 = (2n + 1)(n^2 + n + 1)$ then $f(2k + 1) - f(2k) = 24k^2 + 24k + 8$. The sum we want is then $\sum_{k=0}^{49}(24k^2 + 24k + 8) = 1000000$. $\square$

11. Let $\{a, b, c, d, e, f, g, h, i\}$ be a permutation of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\gcd(c, d) = \gcd(f, g) = 1$ and

$$(10a + b)^{c/d} = e^{f/g}.$$

Given that $h > i$, evaluate $10h + i$.

Proposed by James Lin.

Answer. $\boxed{65}$.

Solution. We can rewrite the equation as

$$A^{c \cdot g} = B^{d \cdot f},$$

where $A = \overline{ab}$ and $B = e$. Therefore there must be a number p such that $A = p^m$ and $B = p^n$ for some positive integer m and n. Since $B < 10$, we have $p < 10$ and we can assume that p is not a power itself. Using the fact that A and B do not share digits, we can narrow the possibilities down to $p = 2, 3, 7$.

When $p = 7$, we must have $A = 49$ and $B = 7$, and $c, g, d, f \in \{1, 2, 3, 5, 6, 8\}$. To get $df = 2cg$, the only possibility is $\{d, f\} = \{3, 8\}$ and $\{c, g\} = \{2, 6\}$. But since 8 is not relatively prime to either 2 or 6 this is impossible.

When $p = 3$, we have $A = 27$ or 81 and $B = 3$ or 9.

- When $(A, B) = (27, 3)$ we have $c, g, d, f \in \{1, 4, 5, 6, 8, 9\}$ and $df = 3cg$, which happens only when $\{d, f\} = \{8, 9\}$ and $\{c, g\} = \{4, 6\}$ or $\{d, f\} = \{4, 6\}$ and $\{c, g\} = \{1, 8\}$. In both cases, because 8 is not relatively prime to 4 and 6, there are no valid solutions.
- When $(A, B) = (27, 9)$ we have $c, g, d, f \in \{1, 3, 4, 5, 6, 8\}$ and $2df = 3cg$, which happens only when $\{d, f\} = \{3, 4\}$ and $\{c, g\} = \{1, 8\}$. For the relatively prime condition to hold, we have $(c, d, f, g) = (1, 4, 3, 8)$ or $(3, 8, 1, 4)$. In either case we have $(h, i) = (6, 5)$.
- When $(A, B) = (81, 3)$ we have $c, g, d, f \in \{2, 4, 5, 6, 7, 9\}$ and $df = 4cg$, but since $5, 6, 7, 9$ cannot be used there are no valid solutions.
- When $(A, B) = (81, 9)$ we have $c, g, d, f \in \{2, 3, 4, 5, 6, 7\}$ and $df = 2cg$, but since 5 and 7 cannot be used and $2 \cdot 3 \cdot 4 \cdot 6$ is a perfect square (and not twice of a perfect square) there are no valid solutions.

When $p = 2$, we have $A = 16, 32$, or 64 and $B = 2, 4$, or 8.

- When $(A, B) = (16, 2)$ we have $c, g, d, f \in \{3, 4, 5, 7, 8, 9\}$ and $df = 4cg$, but since $3, 5, 7, 9$ cannot be used there are no valid solutions.
- When $(A, B) = (16, 4)$ we have $c, g, d, f \in \{2, 3, 5, 7, 8, 9\}$ and $df = 2cg$, but since $3, 5, 7, 9$ cannot be used there are no valid solutions.
- When $(A, B) = (16, 8)$ we have $c, g, d, f \in \{2, 3, 4, 5, 7, 9\}$ and $3df = 4cg$, but since 5 and 7 cannot be used, and $2 \cdot 3 \cdot 4 \cdot 9$ is not 12 times a perfect square there are no valid solutions.
- When $(A, B) = (32, 4)$ we have $c, g, d, f \in \{1, 5, 6, 7, 8, 9\}$ and $2df = 5cg$, but since $6, 7, 9$ cannot be used there are no valid solutions.
- When $(A, B) = (32, 8)$ we have $c, g, d, f \in \{1, 4, 5, 6, 7, 9\}$ and $3df = 5cg$, but since $4, 6, 7$ cannot be used there are no valid solutions.
- When $(A, B) = (64, 2)$ we have $c, g, d, f \in \{1, 3, 5, 7, 8, 9\}$ and $df = 6cg$, but since $5, 7, 8$ cannot be used there are no valid solutions.
- When $(A, B) = (64, 8)$ we have $c, g, d, f \in \{1, 2, 3, 5, 7, 9\}$ and $df = 2cg$, but since $3, 5, 7, 9$ cannot be used there are no valid solutions.

Therefore, the only possible value of $10h + i$ is 65. $\square$

12. Bill draws two circles which intersect at X, Y. Let P be the intersection of the common tangents to the two circles and let Q be a point on the line segment connecting the centers of the two circles such that lines PX and QX are perpendicular. Given that the radii of the two circles are $3, 4$ and the distance between the centers of these two circles is 5, then the largest distance from Q to any point on either of the circles can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

   Proposed by Tristan Shin.

   Answer. $\boxed{4807}$.

   Solution. Let A, B be the centers of the circles (circle centered at A is smaller), and let their radii be r_A, r_B, respectively. Observe that $\angle PXA = \angle QXB$. Let their common angle be θ. Then we see that

$$\frac{QA}{QB} = \frac{XA}{XB} \cdot \frac{\sin(90^\circ - \theta)}{\sin\theta} = \frac{XA}{XB} \cdot \cot\theta$$

   and

$$\frac{PA}{PB} = \frac{XA}{XB} \cdot \frac{\sin\theta}{\sin(90^\circ + \theta)} = \frac{XA}{XB} \cdot \tan\theta$$

   so

$$\frac{PA}{PB} \cdot \frac{QA}{QB} = \left(\frac{XA}{XB}\right)^2.$$

   But $\frac{PA}{PB} = \frac{r_A}{r_B} = \frac{XA}{XB}$, so $\frac{QA}{QB} = \frac{XA}{XB}$ also, so XQ is the angle bisector of $\angle AXB$, meaning that $AQ = \frac{15}{7}$ and $BQ = \frac{20}{7}$. Then the largest distance from Q to any point on either circle is $\frac{20}{7} + 4 = \frac{48}{7}$, so $100m + n = 4807$.

   Alternate solution: Use the same notation as the previous solution. Observe that $PXQY$ cyclic. But P, X, Y all are on the Appolonian circle of AB with ratio $\frac{r_A}{r_B}$, so Q also has this ratio. But then Q is the insimilicenter of the two circles, so the largest distance from Q' to any point on either of the circles is $\frac{4}{7}(3 + 5 + 4) = \frac{48}{7}$, so the answer is 4807. □

13. We define the sets of lattice points $S_0, S_1, \ldots$ as $S_0 = \{(0,0)\}$ and S_k consisting of all lattice points that are exactly one unit away from exactly one point in S_{k-1}. Determine the number of points in S_{2017}.

   Proposed by Michael Ren.

   Answer. $\boxed{16384}$.

   Solution. Show by induction that $S_{2^k} = \{(2^k, 0), (0, 2^k), (-2^k, 0), (0, -2^k)\}$, and S_{2^k+a} $(0 \le a < 2^k)$ is 4 copies of S_a centered on the four points in S_{2^k}. Hence, $|S_{2017}| = 4|S_{993}| = 4^2|S_{481}| = 4^3|S_{225}| = 4^4|S_{97}| = 4^5|S_{33}| = 4^6|S_1| = 4^7 = 16384$. In general, there are $4^{s_2(n)}$ points in S_n, where $s_2(n)$ is the sum of the digits of n in binary. □

14. Let S be the set of all points $(x_1, x_2, x_3, \ldots, x_{2017})$ in $\mathbb{R}^{2017}$ satisfying $|x_i| + |x_j| \le 1$ for any $1 \le i < j \le 2017$. The volume of S can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $100m + n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{201}$.

   Solution. We consider two cases. Case 1: the absolute values of all coordinates do not exceed $\frac{1}{2}$, then all these points are in S, and the volume is that of a unit 2017-cube, which is 1.

   Case 2: The absolute value of one coordinate (WLOG let it be x_1) is greater than $\frac{1}{2}$, then all other coordinates must have an absolute value of less than or equal to $1 - |x_1|$. One can see that if $x_1 > \frac{1}{2}$, then the volume of all the points satisfying the requirement is a 2017-dimensional pyramid with the base being a unit 2016-cube and height being $1 - \frac{1}{2} = \frac{1}{2}$, the volume of this pyramid is therefore

$\frac{1\cdot\frac{1}{2}}{2017} = \frac{1}{4034}$. Since there are 2017 ways to determine this coordinate and 2 ways to determine whether it is positive or negative, there are $2 \cdot 2017 = 4034$ such pyramids, and the sum of their volumes is 1.

Summing up the two cases, the volume of S is $1 + 1 = 2 = \frac{2}{1}$. And our answer is $2 \cdot 100 + 1 = 201$. $\square$

15. Find the number of integers $1 \le k \le 1336$ such that $\binom{1337}{k}$ divides $\binom{1337}{k-1}\binom{1337}{k+1}$.

Proposed by Tristan Shin.

Answer. $\boxed{1222}$.

Solution. Observe that

$$\frac{\binom{n}{k-1}\binom{n}{k+1}}{\binom{n}{k}} + \frac{1}{n+2}\binom{n+2}{k+1} = \binom{n}{k},$$

so $\binom{n}{k}$ divides $\binom{n}{k-1}\binom{n}{k+1}$ if and only if $n+2$ divides $\binom{n+2}{k+1}$. Thus, it suffices to count the number of binomial coefficients $\binom{1339}{m}$, $m = 2, 3, \ldots, 1337$, that are divisible by $1339 = 13 \cdot 103$. By Lucas' Theorem, since $1339_{10} = 7C0_{13}$, $\binom{1339}{m}$ is divisible by 13 if and only if m is not. Similarly, since $1339_{10} = D0_{103}$, $\binom{1339}{m}$ is divisible by 103 if and only if m is not. Thus, we need m to be relatively prime to 1339. Since 1 and 1338 are relatively prime to m, we get that the count should be $\varphi(1339) - 2 = 1222$. $\square$

16. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two parabolas with distinct directrices ℓ_1 and ℓ_2 and distinct foci F_1 and F_2 respectively. It is known that $F_1F_2||\ell_1||\ell_2$, F_1 lies on $\mathcal{P}_2$, and F_2 lies on $\mathcal{P}_1$. The two parabolas intersect at distinct points A and B. Given that $F_1F_2 = 1$, the value of AB^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $100m + n$.

Proposed by Yannick Yao.

Answer. $\boxed{1504}$.

Solution. By definition, a point on parabola is equidistant to the focus and the diretrix.

Since $F_1F_2 = 1$, and F_2 lies on $\mathcal{P}_1$, the distance from F_2 to ℓ_1 is 1, and similarly the distance from F_1 to ℓ_2 is 1. Since F_1F_2 is parallel to both directrices (that are distinct), both points must lie halfway between these two lines. Without loss of generality, assume that $\ell_1 : y = 1, \ell_2 : y = -1, F_1 = (\frac{1}{2}, 0), F_2 = (-\frac{1}{2}, 0)$. A point of intersection $P = (x, y)$ must satisfy both $(x - \frac{1}{2})^2 + y^2 = (y-1)^2$ and $(x + \frac{1}{2})^2 + y^2 = (y+1)^2$. Solving this system of equations give $(x, y) = \pm(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4})$, so $AB^2 = (\sqrt{3})^2 + (\frac{\sqrt{3}}{2})^2 = \frac{15}{4}$, and our answer is 1504. $\square$

17. For a positive integer n, define $f(n) = \sum_{i=0}^{\infty} \frac{\gcd(i,n)}{2^i}$ and let $g : \mathbb{N} \to \mathbb{Q}$ be a function such that $\sum_{d|n} g(d) = f(n)$ for all positive integers n. Given that $g(12321) = \frac{p}{q}$ for relatively prime integers p and q, find $v_2(p)$.

Proposed by Michael Ren.

Answer. $\boxed{12324}$.

Solution. I claim that

$$f(n) = \sum_{d|n} \sum_{i=0}^{\infty} \frac{\varphi(d)}{2^{id}}.$$

Consider all of the possible ways to write a nonnegative integer k in the form id with d a divisor of n. Then $\frac{1}{2^k}$ appears for each d that is a divisor of k and a divisor of n, so it appears for every divisor of $\gcd(k, n)$. Then the sum of the coefficients of $\frac{1}{2^k}$ is

$$\sum_{d|\gcd(k,n)} \varphi(d) = \gcd(k, n)$$

by well-known sums. Thus, this representation of f works.

Then

$$f(n) = \sum_{d|n} \sum_{i=0}^{\infty} \frac{\varphi(d)}{2^{id}} = \sum_{d|n} \frac{2^d \varphi(d)}{2^d - 1}$$

so the answer is $12321 + v_2(\varphi(12321)) = 12324$. Observe that g is unique by MAbius inversion. $\square$

18. Let a, b, c be real nonzero numbers such that $a + b + c = 12$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = 1.$$

Compute the largest possible value of $abc - (a + 2b - 3c)$.

Proposed by Tristan Shin.

Answer. $\boxed{56}$.

Solution. Note that the second condition is equivalent to

$$bc + ca + ab + 1 = abc.$$

Then

$$\begin{aligned} abc - (a + 2b - 3c) &= bc + ca + ab - a - 2b + 3c + 1 \\ &= \frac{1}{2}(a + b + c)^2 - \frac{1}{2}a^2 - \frac{1}{2}b^2 - \frac{1}{2}c^2 - a - 2b + 3c + 1 \\ &= \frac{1}{2}(a + b + c)^2 - \frac{1}{2}a^2 + 3a - \frac{1}{2}b^2 + 2b - \frac{1}{2}c^2 + 7c - 4(a + b + c) + 1 \\ &= 56 - \frac{1}{2}(a - 3)^2 - \frac{1}{2}(b - 2)^2 - \frac{1}{2}(c - 7)^2 \\ &\leq 56. \end{aligned}$$

Equality holds when $(a, b, c) = (3, 2, 7)$. $\square$

19. Tessa the hyper-ant is at the origin of the four-dimensional Euclidean space $\mathbb{R}^4$. For each step she moves to another lattice point that is 2 units away from the point she is currently on. How many ways can she return to the origin for the first time after exactly 6 steps?

Proposed by Yannick Yao.

Answer. $\boxed{725568}$.

Solution. Imagine another ant named Asset at the origin. If both Tessa and Asset moves 3 steps and end up at the same point (without returning to the origin), then Tessa can return to the origin in 6 steps by retracing the steps of Asset. It then suffices to consider the points that Tessa can reach in 3 steps and the number of ways to reach each of them.

The number of points to consider is enormous, but using symmetry can significantly simplify the calculation. In fact, there are only eight classes of lattice points that need to be considered listed below in order of increasing distance from the origin. (Note: in each state, when a point (a, b, c, d) is mentioned, all of its coordinate permutations will also be included.)

(1) The points $(\pm1, \pm1, \pm1, \pm1)$ and $(\pm2, 0, 0, 0)$. (24 vertices of a 24-cell)

(2) The points $(\pm2, \pm2, 0, 0)$. (24 vertices of a larger 24-cell)

(3) The points $(\pm3, \pm1, \pm1, \pm1)$ and $(\pm2, \pm2, \pm2, 0)$. (96 vertices of a rectified 24-cell)

(4) The points $(\pm4, 0, 0, 0)$ and $(\pm2, \pm2, \pm2, \pm2)$. (24 vertices of an even larger 24-cell)

(5) The points $(\pm 3, \pm 3, \pm 1, \pm 1)$ and $(\pm 4, \pm 2, 0, 0)$. (144 vertices of a runcinated 24-cell)

(6) The points $(\pm 4, \pm 2, \pm 2, 0)$. (96 vertices of a larger rectified 24-cell)

(7) The points $(\pm 5, \pm 1, \pm 1, \pm 1)$, $(\pm 3, \pm 3, \pm 3, \pm 1)$, and $(\pm 4, \pm 2, \pm 2, \pm 2)$. (192 vertices of a truncated 24-cell)

(8) The points $(\pm 6, 0, 0, 0)$ and $(\pm 3, \pm 3, \pm 3, \pm 3)$. (24 vertices of a yet even larger 24-cell)

(Some of the polytopes mentioned here are not actually uniform, but are instead the generalized (isogonal) form that preserves vertex transitivity. Since all of them share the symmetry of a 24-cell, all the points in one group, or the movements between a point in a group to a "nearby" point in another group (or the same group) can be considered equivalent. The equivalence can be verified by working through the number of ways to get from a point from class i to class j, see below.)

Then, from we compute the number of ways for one to move from a *given* point in one class to *any* point in another class. The results are tabulated in the following table. We skip the calculation here as all of them can be obtained by simple computations. (Rows are the originating class and columns are the destination class. Note that only the first four classes are relevant as movement origins as they are the only ones that can be reached in 2 steps from the origin.)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1)	8	6	8	1	-	-	-	-
(2)	6	-	12	-	6	-	-	-
(3)	2	3	6	2	6	3	2	-
(4)	1	-	8	-	6	-	8	1

From the table, we can compute the number of ways to reach a class *from a given point in class (1)* in 2 steps:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
117	72	192	24	90	24	24	1

There are two things to note: first, since class (1) contains 24 points and after 1 step Tessa and Asset can each end up on any one of them, we need to multiply 24 to the number of ways to each ant; second, if both ants end up in the same class, in only $\frac{1}{N}$ of the cases do they actually end up on the same point, where N is the number of points in the class, so we need to divide this number for each class.

As a result, we compute the answer as $24^2 \cdot (\frac{117^2}{24} + \frac{72^2}{24} + \frac{192^2}{96} + \frac{24^2}{24} + \frac{90^2}{144} + \frac{24^2}{96} + \frac{24^2}{192} + \frac{1^2}{24}) = 725568$. $\square$

20. Let $p = 2017$ be a prime. Suppose that the number of ways to place p indistinguishable red marbles, p indistinguishable green marbles, and p indistinguishable blue marbles around a circle such that no red marble is next to a green marble and no blue marble is next to a blue marble is N. (Rotations and reflections of the same configuration are considered distinct.) Given that $N = p^m \cdot n$, where m is a nonnegative integer and n is not divisible by p, and r is the remainder of n when divided by p, compute $pm + r$.

Proposed by Yannick Yao.

Answer. $\boxed{3913}$.

Solution. First we show that no valid configuration can be rotated to obtain itself. In fact, the only way to place 2017 blue marbles to have rotational symmetry they must be vertices of a regular 2017-gon, and the same goes for red and green. However, this requires all red marbles to be next to a green marble, so it cannot be valid. Therefore, we can assume without loss of generality that the "first" position is a blue marble, which comprises of $\frac{p}{3p} = \frac{1}{3}$ of all the ways.

The 2017 blue marbles will leave out 2017 gaps in between them. By caseworking on how many gaps are filled with red marbles and how many are filled with green marbles, we get that

$$N = 3 \cdot \sum_{i=1}^{2016} \binom{2017}{i}\binom{2016}{i-1}\binom{2016}{2016-i}.$$

Since $\binom{2017}{i}$ is always divisible by 2017, N is divisible by 2017. Moreover, since $\binom{2016}{i} \equiv (-1)^i \pmod{2017}$, we have

$$\frac{N}{2017} = 3\cdot\sum_{i=1}^{2016} \frac{1}{2017}\binom{2017}{i}\binom{2016}{i-1}\binom{2016}{2016-i} \equiv -\frac{3}{2017}\cdot\sum_{i=1}^{2016}\binom{2017}{i}\cdot(-1)^{2017} = -\frac{3}{2017}(2^{2017}-2) \pmod{2017}.$$

It suffices to compute 2^{2017} modulo 2017^2, which comes out to be 2793547 (one can compute this directly by computing $2^1, 2^2, 2^4, 2^8, \dots, 2^{1024}$ modulo 2017^2). Therefore, $\frac{N}{2017} \equiv -3 \cdot \frac{2793547-2}{2017} = -3 \cdot 1385 \equiv 3 \cdot 632 = 1896 \pmod{2017}$. Thus, $m = 1$ and $r = 1896$, and $pm + r = 2017 + 1896 = 3913$. ☐

21. Iris has an infinite chessboard, in which an 8×8 subboard is marked as Sacred. In order to preserve the Sanctity of this chessboard, her friend Rosabel wishes to place some indistinguishable Holy Knights on the chessboard (not necessarily within the Sacred subboard) such that:

- No two Holy Knights occupy the same square;
- Each Holy Knight attacks at least one Sacred square;
- Each Sacred square is attacked by exactly one Holy Knight.

In how many ways can Rosabel protect the Sanctity of Iris' chessboard? (A Holy Knight works in the same way as a knight piece in chess, that is, it attacks any square that is two squares away in one direction and one square away in a perpendicular direction. Note that a Holy Knight does *not* attack the square it is on.)

Proposed by Yannick Yao.

Answer. $\boxed{7056}$.

Solution. For convenience, we denote each square by its coordinates, with $(0, 0)$ being the lower left corner of the Sacred subboard, and $(7, 7)$ being the upper right corner. Moreover, we color a square black if its sum of coordinates is even and white otherwise (like a normal chessboard). Since a Holy Knight (called HK from now on) on a black square attack only white squares and vice versa, it suffices to consider the number of ways for all the white squares to be attacked by HKs (on black squares) and square this number to get the answer.

We use the following two observations (proof omitted as they are almost entirely casework):

Observation 1: If a HK is placed on a black square (x, y) where $1 \le x, y \le 6$, then for its neighboring Sacred white squares to be attacked, then either there are two knights placed at $(x + 2, y + 2)$ and $(x - 2, y - 2)$ or $(x + 2, y - 2)$ and $(x - 2, y + 2)$, *with one exception*: if the HK is placed on $(1, 1)$ (or $(6, 6)$), then the four neighbors can be covered by HKs on $(3, 3)$, $(-2, 0)$, and $(0, -2)$ (or $(4, 4)$, $(9, 7)$, and $(7, 9)$).

Observation 2: Suppose that (x, y) is not yet attacked but all of $(x-1, y+3), (x, y+2), (x+1, y+1), (x+2, y), (x+3, y-1)$ are, then the square can only be attacked by an HK on $(x - 2, y - 1)$ or $(x - 1, y - 2)$. (The same applies for the rotated/reflected arrangements.)

With these two observations we can consider various possibilities for arrangements. In particular, consider the HK that attacks $(3, 4)$ (WLOG assume that the HK is below the main diagonal.)

Case 1: The HK is on $(2, 2)$, then by Observation 1, there are two HKs on $(0, 4)$ and $(4, 0)$ or $(0, 0)$ and $(4, 4)$.

Case 1.1: In the first possibility, for $(0,5)$ to be attacked without attacking other Sacred squares that are already attacked, an HK must be placed on $(-2,6)$, and similar for $(5,0)$. This will cover all black squares on or below the line $x+y=7$. By Observation 2, for $(4,5)$ to be attacked an HK must be placed on $(6,6)$ or $(5,7)$, and for $(5,4)$ to be attacked an HK must be placed on $(6,6)$ or $(7,5)$.

Case 1.1.1: An HK is placed on $(6,6)$. By Observation 1, since HK canat be placed on $(4,4)$ anymore, two HKs must be placed on $(4,8)$ and $(8,4)$. This takes care of all the Sacred white squares.

Case 1.1.2: Two HKs are placed on $(5,7)$ and $(7,5)$. This leaves $(2,7),(4,7),(7,2),(7,4)$ unattacked. The only valid way is to place two HKs at $(3,9)$ and $(9,3)$ respectively.

Case 1.2: In the second possibility, applying Observation 1 to $(4,4)$ show that $(6,6)$ must contain an HK. As mentioned in the exception to Observation 1, the two squares $(6,7),(7,6)$ can be attacked by the same HK at $(8,8)$ or two different HKs at $(7,9)$ and $(9,7)$, and either way does not affect the other white Sacred squares. We need now to consider the unattacked squares above $y-x=4$ and below $x-y=4$. Applying Observation 2 to $(1,6)$ gives that one HK must be placed at $(-1,7)$ or $(0,8)$, and it is not difficult to see that another HK must be placed at $(1,9)$ or $(-2,6)$ respectively to cover the remaining squares on the upper-left corner. The same applies for the squares on the lower-right corner.

We have 2 ways from Case 1.1 and $2\cdot 2\cdot 2=8$ ways from Case 1.2, for 10 ways in this case.

Case 2: The HK is on $(1,3)$, then by Observation 1, there are two HKs on $(-1,5)$ and $(3,1)$ or $(-1,1)$ and $(3,5)$.

Case 2.1: In the first possibility, by Observation 1 again on $(3,1)$ an HK must be placed on $(5,-1)$, and these HKs combined attack all the squares on or below the main diagonal. The rest is identical to Case 1.1.

Case 2.2: In the second possibility, applying Observation 1 on $(3,5)$ gives that an HK must be placed on $(5,7)$. For $(6,7)$ to be attacked, an HK must be placed on $(7,9)$. This leaves a single square $(0,7)$ and all the squares below $x-y=2$ unattacked. There are two ways to attack $(0,7)$ without attacking other Sacred squares: an HK at $(-2,8)$ or $(-1,9)$. For the lower-right corner, applying Observation 2 on $(4,1),(5,2),(6,3)$ gives that two HKs must be placed on $(5,-1)$ and $(7,1)$ or $(6,0)$ and $(8,2)$. In the former case we still need to attack $(7,2)$ and $(7,4)$, which can only be done by an HK at $(9,3)$, and the latter case is symmetric to this case.

We have 2 ways from Case 2.1 and $2\cdot 2=4$ ways from Case 2.2, for 6 ways in this case.

Case 3: The HK is on $(4,2)$, then by Observation 1, there are two HKs on $(2,4)$ and $(6,0)$ or $(2,0)$ and $(6,4)$.

Case 3.1: In the first possibility, by Observation 1 on $(2,4)$ forces placement of another HK at $(0,6)$. There are two ways to attack the squares $(0,1)$ and $(1,0)$ (same as $(6,7)$ and $(7,6)$ in Case 1.2). There are two ways to attack each of the unattacked squares $(0,7)$ and $(7,0)$ (see Case 2.2). It now remains to consider the squares above $x+y=10$. Applying Observation 2 on $(5,6)$ and $(6,5)$ gives two cases.

Case 3.1.1: They are attacked by the same HK on $(7,7)$. Then to cover $(4,7),(6,7),(7,4),(7,6)$ we need to place two more HKs on $(5,9)$ and $(9,5)$.

Case 3.1.2: They are attacked by two different HKs on $(6,8)$ and $(8,6)$. Then all the white Sacred squares are taken care of.

Case 3.2: In the second possibility, Observation 1 on $(6,4)$ forces placement of another HK on $(8,6)$. For $(1,0)$ to be attacked, an HK must be placed on $(0,-2)$. The arrangement of remaining squares ($(7,0)$ and all squares above $y-x=2$) is symmetric to Case 2.2.

We have $2\cdot 2^2\cdot 2=16$ ways from Case 3.1 and 4 ways from Case 3.2, for 20 ways in this case.

Case 4: The HK is on $(1,5)$, then by Observation 1, there are two HKs on $(-1,7)$ and $(3,3)$ or $(-1,3)$ and $(3,7)$.

Case 4.1: In the first possibility, applying Observation 1 on $(3,3)$ forces an HK on $(5,1)$ and applying it again on $(5,1)$ forces an HK on $(7,-1)$. The rest is identical to Case 3.1 (after $(7,0)$ and $(0,7)$ are covered).

Case 4.2: In the second possibility, for $(4,7)$ to be attacked an HK must be placed on $(5,9)$, which also attacks $(6,7)$. Therefore, it remains to consider the squares below the line $y=x$. Applying Observation 2 to $(4,3)$ gives that an HK is on either $(5,1)$ or $(6,2)$. (Note that the two cases are symmetric.) If an HK is on $(5,1)$, then by Observation 1 there are two HKs on $(3,-1)$ and $(7,5)$. This leaves $(7,4)$ and $(7,6)$, which must be attacked by the same HK on $(9,5)$.

This gives $2\cdot 2=4$ ways from Case 4.1 and 2 ways from Case 4.2, for 6 ways in this case.

Summing these cases up gives $10+6+20+6=42$ ways for the HK to be below the diagonal, so there are $42\cdot 2=84$ ways to cover all Sacred white squares. Squaring this number, we get $84^2=7056$ ways for Rosabel to protect the Sanctity of Irisa chessboard. $\square$

22. Given a sequence of positive integers $a_1, a_2, a_3, \ldots, a_n$, define the *power tower function*

$$f(a_1, a_2, a_3, \ldots, a_n) = a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_n}}}}}}.$$

Let $b_1, b_2, b_3, \ldots, b_{2017}$ be positive integers such that for any i between 1 and 2017 inclusive,

$$f(a_1, a_2, a_3, \ldots, a_i, \ldots, a_{2017}) \equiv f(a_1, a_2, a_3, \ldots, a_i + b_i, \ldots, a_{2017}) \pmod{2017}$$

for all sequences $a_1, a_2, a_3, \ldots, a_{2017}$ of positive integers greater than 2017. Find the smallest possible value of $b_1 + b_2 + b_3 + \cdots + b_{2017}$.

Proposed by Yannick Yao.

Answer. $\boxed{6072}$.

Solution. We first show the following: for any positive integers $n > 1$ and N, the smallest positive integer b for which $a^c \equiv (a+b)^c \pmod{n}$ for any positive integer a and $c > N$ is n.

Proof: let c be one more than a multiple of $n!$ that is larger than N, then by Euler's Theorem, for any a relatively prime to n, $a^c \equiv a \pmod{n}$ since $c \equiv 1 \pmod{\phi(n)}$. If $a+b$ is not relatively prime to n, then $(a+b)^c$ is not relatively prime to n, so it cannot be congruent to a modulo n. Therefore, $a+b$ is also relatively prime to n, which means $(a+b)^c \equiv a+b \pmod{n}$. This means that b must be a multiple of n, and $b=n$ clearly works, so it's the smallest one.

Therefore, we can begin with $b_1 = 2017$. Since there exists a primitive root g modulo 2017, if we let $a_1 = g$, we see that $f(a_2, a_3, \ldots, a_{2017}) \equiv f(a_2 + b_2, a_3, \ldots, a_{2017}) \pmod{2016}$, so b_2 is at least 2016. Since for all a relatively prime to 2017, $\mathrm{ord}_{2017}(a)$ is a divisor of 2016, and for all a that are divisible by 2017, a^c is always a multiple of 2017, we see that $b_2 = 2016$ is sufficient.

Now we break $2016 = 2^5 \cdot 3^2 \cdot 7$ into prime powers and look at them separately. For $2^5 = 32$, it does not have a primitive root (that has order 16), but we do have $\mathrm{ord}_{32}(3) = 8$, and all other orders must be a factor of 8. If the number is an even number, then since all of $a_3, a_4, \ldots$ are greater than 2017, $f(a_2, a_3, \ldots, a_{2017})$ will be a multiple of 32. (Similarly, in our later discussions, we can disregard all numbers that are not relatively prime to the prime power we are looking at.) So for modulo 32, we need $f(a_3, a_4, \ldots, a_{2017}) \equiv f(a_3 + b_3, a_4, \ldots, a_{2017}) \pmod{8}$.

Since there is a primitive root modulo 9 and 7, we also need $f(a_3, a_4, \ldots, a_{2017}) \equiv f(a_3+b_3, a_4, \ldots, a_{2017}) \pmod{6}$. (Note that $\phi(9) = \phi(7) = 6$.) Therefore, we need $f(a_3, a_4, \ldots, a_{2017}) \equiv f(a_3+b_3, a_4, \ldots, a_{2017}) \pmod{24 (= \mathrm{lcm}(8,6))}$, and $b_3 = 24$ is the smallest possible value.

Now we break $24 = 2^3 \cdot 3$ into prime powers. For $2^3 = 8$, there is no primitive root modulo 8 (with order 4), but $\mathrm{ord}_8(3) = 2$. For modulo 3, there is a primitive root with order 2 as well. So we need $f(a_4, a_5, \ldots, a_{2017}) \equiv f(a_4 + b_4, a_5, \ldots, a_{2017}) \pmod{2}$, and thus $b_4 = 2$ is the smallest possible value.

The parity of a power tower only depends on the parity of its base, so we can conclude that $b_5 = b_6 = \cdots = b_{2017} = 1$ works. Therefore the smallest possible sum is $2017+2016+24+2+2013\cdot 1 = 6072$. $\square$

23. Call a nonempty set V of nonzero integers *victorious* if there exists a polynomial $P(x)$ with integer coefficients such that $P(0) = 330$ and that $P(v) = 2|v|$ holds for all elements $v \in V$. Find the number of victorious sets.

    Proposed by Yannick Yao.

    Answer. $\boxed{217}$.

    Solution. Obviously $0 \notin V$. We consider two cases regarding the victorious set V:

    Case 1: V contains two elements of different signs. Suppose that these two elements are a and $-b$ for some positive integers a and b. WLOG assume that $a \geq b$. Suppose that $L(x)$ is a linear polynomial that passes through $(a, 2a)$ and $(-b, 2b)$, then we can write the any polynomial $P(x)$ as $(x-a)(x+b)R(x) + L(x)$ for some polynomial $R(x)$. Since P has integer coefficients, so does R and therefore L. In particular, this means that the slope of L, which is $\frac{2a-2b}{a+b}$, is an integer. By our assumption this slope is nonnegative and less than 2, so it must be 0 or 1. The former case implies that $a = b$, which gives $L(x) = 2b$, and the latter case implies that $a = 3b$, which gives $L(x) = x + 3b$.

    In the first case, We have $P(0) = -b^2R(0)+2b = 330$, and since $R(0)$ is an integer, we have $b^2 \mid 330-2b$. It is not difficult to check that the only solutions are $b = 1, 3, 165$. (Notice that b has to be a divisor of 330.) This means that we have the sets $\{1, -1\}, \{3, -3\}, \{165, -165\}$. These sets are victorious because we can simply set $R(x)$ to be a constant polynomial.

    In the second case, we have $P(0) = -3b^2 + 3b = 330$, and similarly we have $b^2 \mid 110 - b$. It is not difficult to check that the only solutions are $b = 1, 2, 10, 110$. This means that we have the sets $\{3, -1\}, \{6, -2\}, \{30, -10\}, \{330, -110\}$, as well as their opposites $\{1, -3\}, \{2, -6\}, \{10, -30\}, \{110, -330\}$.

    It remains to check whether a subset of $\{3, 1, -1, -3\}$ with two elements of opposite signs other than the ones mentioned before is victorious. The quadratic that passes through $(-1, 2), (1, 2)$, and $(3, 6)$ is $\frac{1}{2}x^2 + \frac{3}{2}$, which means that no polynomials with integer coefficients passes through these three points. Similarly, the quadratic that passes through $(-3, 6), (1, 2)$, and $(3, 6)$ is also $\frac{1}{2}x^2 + \frac{3}{2}$, which means that no polynomials with integer coefficients passes through these three points either. Therefore, we only have $3 + 8 = 11$ victorious sets in this case.

    Case 2: All elements of V are the same sign. WLOG assume that they are all positive, and that $V = \{v_1, v_2, \ldots, v_n\}$. This means that $(x - v_1)(x - v_2)\ldots(x - v_n) \mid P(x) - 2x$. And thus by plugging in $x = 0$, we see that we need $v_1v_2\ldots v_n \mid 330$. Since whether 1 is in the set or not is inconsequential (unless the set is $\{1\}$), we only consider nontrivial factors of $330 = 2 \cdot 3 \cdot 5 \cdot 11$.

    If there is only 1 factor used, then there are $2^4 - 1 = 15$ such sets since there are 15 nontrivial factors. If there are 2 factors used, then there are $\binom{4}{2} = 6$ sets such that each uses only one prime factor, $4 \cdot \binom{3}{2} = 12$ sets such that one uses one prime and the other uses two, 4 sets such that one uses one prime and the other uses three, and $\binom{4}{2}/2 = 3$ sets such that each uses two primes. So there are $6 + 12 + 4 + 3 = 25$ sets in this case. If there are 3 factors used, then there are $\binom{4}{3} = 4$ sets such that each uses only one prime factor and $\binom{4}{2} = 6$ sets such that one of them uses two. So there are $4 + 6 = 10$ sets in this case. If there are 4 factors used, then there is only 1 such set since it must be $\{2, 3, 5, 11\}$.

    Taking into consideration of the usage of 1 and the set $\{1\}$ itself, there are $2\cdot(15+25+10+1)+1 = 103$ sets with only positive elements, and similarly there are 103 sets with only negative elements, so in total there are $103 \cdot 2 = 206$ sets in this case.

    Summing up the two cases, we see that there are $11 + 206 = 217$ victorious sets in total. □

24. Senators Sernie Banders and Cedric "Ced" Truz of OMOrica are running for the office of Price Dent. The election works as follows: There are 66 states, each composed of many adults and 2017 children, with only the latter eligible to vote. On election day, the children each cast their vote with equal probability to Banders or Truz. A majority of votes in the state towards a candidate means they "win" the state, and the candidate with the majority of won states becomes the new Price Dent.

Should both candidates win an equal number of states, then whoever had the most votes cast for him wins.

Let the probability that Banders and Truz have an unresolvable election, i.e., that they tie on both the state count and the popular vote, be $\frac{p}{q}$ in lowest terms, and let m, n be the remainders when p, q, respectively, are divided by 1009. Find $m + n$.

Proposed by Ashwin Sah.

Answer. $\boxed{96}$.

Solution. Clearly the total number of election outcomes is $2^{2017\cdot 66}$. We'll count the number of outcomes which result in an unresolvable election. First of all we choose the states that vote for Banders; there are $\binom{66}{33}$ ways to do this. Next we choose the amount of votes Banders gets. Suppose he gets $x_1, x_2, ...x_{33}$ votes in the states he loses in and $x_{34} + 1009, x_{35} + 1009, ...x_{66} + 1009$ votes in the states he wins in, where $0 \le x_i \le 1008$. Then clearly $\sum x_i = 2017 \cdot 33 - 1009 \cdot 33 = 1008 \cdot 33$. For $1 \le i \le 33$ there are $\binom{2017}{x_i}$ ways for the ith state's voters to choose to vote in that particular way; similarly there are $\binom{2017}{1009+x_i}$ ways for the ith state's voters for $34 \le i \le 66$. So the total number of ways for this to happen is $\binom{66}{33}$ times the coefficient of $x^{1008\cdot 33}$ in the polynomial $\left(\binom{2017}{0} + \binom{2017}{1}x + ... + \binom{2017}{1008}x^{1008}\right)^{33}\left(\binom{2017}{1009} + \binom{2017}{1010}x + ... + \binom{2017}{2017}x^{1009}\right)^{33}$. Defining $P(x) = \binom{2017}{0} + \binom{2017}{1}x + ... + \binom{2017}{1008}x^{1008}$, it's not hard to see it's equivalent to find the constant term of $P(x)^{33}P(\frac{1}{x})^{33}$.

First I claim that $1009|p$. Indeed, first note that 1009 doesn't divide the denominator, which is a power of two, hence we only need to show the constant term of the above expression is divisible by 1009. To do this we note by Lucas's theorem that for $i \le 1008$, $\binom{2017}{i} \equiv \binom{1}{0}\binom{1008}{i} \equiv (-1)^i$ modulo 1009, so modulo 1009 it's enough to get the $x^{1008\cdot 33}$ term of $\left(1 - x + x^2 - .... + x^{1008}\right)^{66} = \dfrac{(1 + x^{1009})^{66}}{(1 + x)^{66}} = \sum_{0\le i\le 66} x^{1009i}\binom{66}{i}\sum_{i\ge 0}\binom{66+i}{66}x^i$. Since $1008 \cdot 33$ is between $1009 \cdot 32, 1009 \cdot 33$ we just need $\displaystyle\sum_{0\le i\le 32}\binom{66}{i}\binom{1008 \cdot 33 - 1009i + 66}{66}$. But the latter coefficient equals

$$\frac{(1008 \cdot 33 - 1009i + 66)(1008 \cdot 33 - 1009i + 65)...(1008 \cdot 33 - 1009i + 1)}{66!}$$

. The denominator is clearly not divisible by 1009, while the numerator has a term of $1008 \cdot 33 - 1009i + 33$, which is clearly divisible by 1009, hence the entire coefficient is divisible by 1009, so $1009|p$.

Then to compute q modulo 1009, we only need to find the largest factor of 2 which divides $\binom{66}{33} \cdot [x^0]P(x)^{33}P(\frac{1}{x})^{33}$, because knowing this will allow us to determine q.

Now we claim that the largest factor of 2 dividing $\binom{66}{33}[x^0]P(x)^{33}P(\frac{1}{x})^{33}$ is 16. By Legendre's Formula $v_2(\binom{66}{33}) = 2$, hence it's enough to show $[x^0]P(x)^{33}P(\frac{1}{x})^{33} \equiv 4 \bmod 8$.

Next we'll write $P(x)P(\frac{1}{x}) = f(x)+2g(x)$, where $f(x)$ is some power series in x, all of whose coefficients are in $\{0, 1\}$, and $g(x)$ is the "quotient" when the left side is divided by 2. Then we'd like to evaluate $(f(x) + 2g(x))^{33}$ modulo 8, but by the Binomial Theorem this is $f(x)^{33} + f(x)^{32} \cdot 2g(x) \cdot 33 + f(x)^{31} \cdot 4g(x) \cdot 33 * 16 + ...$ It's clear that every term other than the first two vanishes modulo 8, so we're left with $f(x)^{33} + 2f(x)^{32}g(x)$.

First we claim $f(x)^{33}[x^0] = 0$. Indeed the coefficient of x^i in f is just $\displaystyle\sum_{a-b=i}\binom{2017}{a}\binom{2017}{b}$ mod 2. But $2017 = \overline{11111100001}_2$, so by Lucas's theorem $\binom{2017}{j}$ is nonzero mod 2 iff $j \equiv 0, 1$ mod 32. Now suppose i is even. Then we claim $\displaystyle\sum_{a-b=i}\binom{2017}{a}\binom{2017}{b}$ is even. This is because the only nonzero terms are ones where both $a, b \equiv \{0, 1\}$ modulo 32, and since $a - b = i$ is even, we require $a \equiv b$, hence there is a bijection between valid pairs $(a, b) = (32k, 32l)$ and $(32k + 1, 32l + 1)$. This implies there are an even number of nonzero terms in the coefficient of x^i, so $[x^i]f(x) = 0$, so $[x^0]f(x)^{33} = 0$, since an odd power series raised to an odd power has a zero constant term.

Now we'd like to show $[x^0]2f^{32}(x)g(x) \equiv 4 \bmod 8$, or alternatively that $[x^0]f^{32}(x)g(x) \equiv 2 \bmod 4$.

First we deal with $f^{32}(x)$. Write $f(x) = \sum x^{a_i}$ for some distinct integers a_i. The coefficients in the expansion of $f^{32}(x)$ will then be things of the form $\binom{32}{x_1, x_2, \ldots x_n}$ for some nonzero x_i summing to 32. But by Legendre's formula $v_2(\binom{32}{x_1, x_2, \ldots x_n}) = \sum s_2(x_i) - s_2(32) \geq n - 1 \geq 2$ whenever $n \geq 3$, hence all such terms disappear. Meanwhile when $n = 2$, we're left with $s_2(x_1) + s_2(x_2) - s_2(32) = s_2(x_1) + s_2(x_2) - 1$. This is ≤ 1 only when $s_2(x_1) = s_2(x_2) = 1 \implies x_1 = x_2 = 16$, in which case the coefficient is 2 modulo 4. Clearly when $n = 1$ the coefficient is always 1 mod 4.

So $f^{32}(x)$ expands as $\sum x^{32a_i} + \sum_{i \neq j} x^{16a_i + 16a_j}$.

Now we need to figure out what the a_i terms are, i.e. when f has odd coefficients. Recall $[x^{a_i}]f(x) \equiv \sum_{a-b=a_i} \binom{2017}{a}\binom{2017}{b}$. Recall again that $a, b \in \{0, 1\}$ mod 32 and that $2|a_i$ yields a coefficient of zero, hence $a - b = a_i$ must be ± 1 mod 32.

Let's assume $a_i = 32k + 1$ for $k \geq 0$. Then $a \equiv 1$ mod 32 and $b \equiv 0$. Since a, b are bounded from above by 1008, we have $(a, b) = (32k + 1 + 32l, 32l)$ for $0 \leq l \leq 31 - k$, yielding $32 - k$ valid solutions, hence the coefficient is odd iff k is odd, or $a_i = 64k + 33$ for $k \geq 0$. By similar methods, we find for $a_i = 32k + 31$ that $(a, b) = (32k + 32l + 32, 32l + 1)$, which has $31 - k$ solutions, hence the coefficient is odd iff k is even, or $a_i = 64k + 31$. Similar arguments work for $k < 0$. So f only has coefficients of 1 when $a_i \equiv 32 \pm 1$ mod 64.

Now recall since $p(x)p(-x)$ has terms in the range $[x^{-1008}, x^{1008}]$ that all of g's terms are also in this range. So to get the constant term of $f(x)^{32}g(x)$, we only care about terms of $f(x)^{32}$ which are in this range.

OK let's figure out which terms are in this range. We'll deal with x^{32a_i} terms first. These are easy-only $x^{-31 \cdot 32}, x^{31 \cdot 32}$ fall in the right range, so we have $x^{-992} + x^{992}$ from those terms.

Now we do $x^{16a_i + 16a_j}$ terms. Note from $a_i, a_j \in \{31, 33\}$ mod 64 that $a_i + a_j \in \{-2, 0, 2\}$ mod 64. So the only terms in this range are x^{16k} where $k = 0, \pm 2, \pm 62$. OK we'll count $k = 62$ first; $k = -62$ will follow by symmetry. Then $a_i + a_j = 62$. It's not hard to see $(a_i, a_j) = (64u + 31, -64u + 31)$, where u ranges from -15 to 15 and doesn't include 0, hence we have 30 possible combinations, for a total term of $30(x^{992} + x^{-992})$. Next we do $k = 2$; $k = -2$ will follow by symmetry. If $a_i + a_j = 2$, it's not hard to see that $(a_i, a_j) = (64u + 33, -64u - 31)$, so $-16 \leq u \leq 15$, for a total of 32 solutions. Then modulo 4 these terms all vanish. Similarly if $k = 0$, we have $a_i + a_j = 0$, and trivially the coefficient of x^0 is 64, which vanishes modulo 4. So in the end we're left with $(x^{992} + x^{-992}) + 30(x^{992} + x^{-992}) = 31(x^{992} + x^{-992})$. This is the only thing left of $f^{32}(x)$ when taken modulo 4.

Now suppose $c = [x^{992}]g(x) = [x^{-992}]g(x)$. If we show c is odd, then $[x^0]f^{32}(x)g(x)$ will be $62c \equiv 2$ mod 4 and we will be done. Since $P(x)P(\frac{1}{x}) = f + 2g$ and $[x^{992}]f(x) = 0$, it's enough to show $[x^{992}]P(x)P(\frac{1}{x})$ is 2 mod 4. Recall this is just $\displaystyle\sum_{a-b=992} \binom{2017}{a}\binom{2017}{b}$. Clearly $a \equiv b$ mod 32. Now if $a \notin \{0, 1\}$, by Lucas we know $2|\binom{2017}{a}, \binom{2017}{b}$, so the product vanishes modulo 4. Then clearly we only need to consider the case $a \in \{0, 1\}$ mod 32, or alternatively $(a, b) = (993, 1), (992, 0)$, which yields $\binom{2017}{993} \cdot 2017 + \binom{2017}{992} \equiv \binom{2017}{993} + \binom{2017}{992} = \binom{2018}{993}$. By Legendre's Formula this is 2 mod 4, so we have successfully shown c is odd.

As a result the largest power of two dividing $[x^0](P(x)P(\frac{1}{x})^{33}$ is 2, so $\binom{66}{33}$ times this coefficient is divisible by 16 and not 32. But from earlier $m = 0$ while $q = 2^{2017 \cdot 66 - 4} \equiv 2^{66-4} = 2^{62} \equiv 96$, so $n = 96 \implies m + n = 96$ as desired. $\square$

25. For an integer k let T_k denote the number of k-tuples of integers $(x_1, x_2, \ldots x_k)$ with $0 \leq x_i < 73$ for each i, such that $73|x_1^2 + x_2^2 + \ldots + x_k^2 - 1$. Compute the remainder when $T_1 + T_2 + \ldots + T_{2017}$ is divided by 2017.

Proposed by Vincent Huang.

Answer. $\boxed{2}$.

Solution. We'll work in $\mathbb{Z}/73\mathbb{Z}$ for convenience.

For each k also let S_k be the number of k-tuples of integers $(x_1, x_2, ... x_k)$, $x_j \in Z/73Z$ such that $p | x_1^2 + x_2^2 + ... + x_k^2$. We will first determine a useful recursion for S_k.

Let i be a positive integer less than 73 with $73 | i^2 + 1$. Suppose that $x_k = 0$. Then the number of valid tuples $(x_1, x_2, .. x_k)$ is just S_{k-1}. Now if $x_k \neq 0$ then we can multiply each x_j by ix_k^{-1} to generate a solution to $73 | x_1^2 + x_2^2 + ... + x_{k-1}^2 - 1$, and this process is clearly reversible, so we have T_{k-1} solutions for each $x_k \neq 0$. Putting this together yields $S_k = S_{k-1} + 72T_{k-1}$.

It follows that $S_k = S_{k-1} + 72T_{k-1} = S_{k-2} + 72T_{k-1} + 72T_{k-2} = ... = S_1 + 72(T_{k-1} + T_{k-2} + ... + T_1)$. Then our desired answer is just $(S_{2018} - S_1)72^{-1}$ mod 2017.

We will first compute S_{2017} mod 2017. Note that for any $(x_1, x_2, ... x_{2017})$ with not all x_j equal, we can perform cyclic shifts to pair each solution with 2017 other solutions, while if all the x_j are equal to some a then $2017a^2 \equiv 0 \bmod 73$, implying $a = 0$, hence there is 1 solution in this case, yielding $S_{2017} \equiv 1 \bmod 2017$. A similar process for T_{2017} yields that we have $T_{2017} \equiv 1 + \left(\frac{2017}{73}\right) \equiv 2$ solutions, hence $S_{2018} = S_{2017} + 72T_{2017} \equiv 145 \bmod 2017$.

Then since $S_1 = 1$, the expression we want is just $144 \cdot 72^{-1} \equiv 2 \bmod 2017$, as desired.

Alternative solution:

We proceed with roots of unity. Define $P_k(x) = (x^{0^2} + x^{1^2} + ... + x^{72^2})^k$. It's not hard to see that T_k equals the sum of coefficients of all terms in $P_k(x)$ with degrees that are one modulo 73. Then we can apply root of unity filter to the polynomial $x^{72}P_k(x)$ to evaluate T_k. By root of unity filter, $T_k = \frac{1}{73}\sum_{j=0}^{72} \omega^{72j} P_k(\omega^j)$, where $\omega = e^{\frac{2i\pi}{73}}$.

Then our desired answer is $\sum_{k=1}^{2017} \frac{1}{73} \sum_{j=0}^{72} \omega^{72j} P_k(\omega^j) = \frac{1}{73} \sum_{k=1}^{2017} \left(73^k + \sum_{\text{j is QR}} \omega^{72j} P_k(\omega^j) + \sum_{\text{j is NQR}} \omega^{72j} P_k(\omega^j) \right)$, where QR denotes a nonzero quadratic residue modulo 73 while NQR denotes a nonzero non-quadratic residue modulo 73.

Note that the Gauss sum $\sum_{k=0}^{72} e^{\frac{2ik^2\pi}{73}}$ is equal to $\sqrt{73}$ since 73 is a prime which is 1 mod 4. It follows that $P_k(\omega^j) = \sqrt{73}^k$ if j is a QR, and $P_k(\omega^j) = (-\sqrt{73})^k$ if j is a NQR. So we're left with $\frac{1}{73}\sum_{k=1}^{2017}\left(73^k + \sum_{\text{j is QR}} \omega^{-j}\sqrt{73}^k + \sum_{\text{j is NQR}} \omega^{-j}(-\sqrt{73})^k\right)$.

But note that $\sqrt{73} = \sum_{k=0}^{72} e^{\frac{2ik\pi}{73}} = 1 + 2\sum_{\text{j is QR}} \omega^j = 1 + 2\sum_{\text{j is QR}} \omega^{-j}$ where the last equality holds since 73 is a prime which is 1 mod 4, so $\sum_{\text{j is QR}} \omega^{-j} = 0.5(\sqrt{73} - 1)$ and similarly $\sum_{\text{j is NQR}} \omega^{-j} = 0.5(-\sqrt{73} - 1)$. So we're left with $\frac{1}{73}\sum_{k=1}^{2017}\left(73^k + \sqrt{73}^k \cdot 0.5(\sqrt{73} - 1) + (-\sqrt{73})^k \cdot 0.5(-\sqrt{73} - 1)\right)$.

This last sum telescopes to $\frac{1}{73}\left(\sum_{k=1}^{2017} 73^k + 0.5\left(\sqrt{73}^{2018} - \sqrt{73}\right) + 0.5\left((-\sqrt{73})^{2018} - (-\sqrt{73})\right)\right) = \frac{1}{73}\left(\sum_{=1}^{2017} 73^k\right) + \frac{1}{73}73^{1009}$. By the geometric series formula along with Fermat's Little Theorem the first sum is just $73^{2016} \equiv 1$, while since 73 is a quadratic residue modulo 2017, the second term is $73^{1008} \equiv 1$ as well, hence our final answer is 2. $\square$

26. Define a sequence of polynomials $P_0, P_1, \ldots$ by the recurrence $P_0(x) = 1, P_1(x) = x, P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$. Let $S = \left|P'_{2017}\left(\frac{i}{2}\right)\right|$ and $T = \left|P'_{17}\left(\frac{i}{2}\right)\right|$, where i is the imaginary unit. Then $\frac{S}{T}$ is a rational number with fractional part $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

Proposed by Tristan Shin.

Answer. $\boxed{4142}$.

Solution. Solution by Tristan Shin

Let L_n and F_n denote the nth Lucas and Fibonacci numbers ($L_0 = 2, L_1 = 1, F_0 = 0, F_1 = 1$).

Compute $P_0 = 1, P_1 = x, P_2 = 2x^2 - 1, P_3 = 4x^3 - 3x$. Define $a_i = P_i\left(\frac{i}{2}\right)$ and $b_i = P'_i\left(\frac{i}{2}\right)$. Then $a_0 = 1, a_1 = \frac{i}{2}, a_2 = -\frac{3}{2}, a_3 = -2i$. Note that $a_{n+1} = ia_n - a_{n-1}$, so

$$\frac{2a_{n+1}}{i^{n+1}} = \frac{2a_n}{i^n} + \frac{2a_{n-1}}{i^{n-1}}.$$

Since $\frac{2a_0}{i^0} = 2 = L_0, \frac{2a_1}{i^1} = 1 = L_1$, and $L_{n+1} = L_n + L_{n-1}$, we deduce that $\frac{2a_n}{i^n} = L_n$ for all integers n, so $a_n = \frac{L_n i^n}{2}$.

Next, differentiate the recursion for P using the chain rule to deduce that

$$P'_{n+1}(x) = 2P_n(x) + 2xP'_n(x) - P'_{n-1}(x),$$

so $b_{n+1} = ib_n - b_{n-1} + 2a_n$. Compute $b_0 = 0, b_1 = 1, b_2 = 2i, b_3 = -6$. I claim that $b_n = nF_n i^{n-1}$. This is true for $n = 0, 1$. Assume that it is true for $n = 0, 1, \ldots, k$ with k a positive integer. We prove it for $n = k + 1$.

First, we show that $L_k = F_k + 2F_{k-1}$. Observe that the sequences for the left and right hand side satisfy the same second-degree linear recursion and the initial conditions match up as $L_1 = 1 = 1+0 = F_1 + 2F_0$ and $L_2 = 3 = 1 + 2 = F_2 + 2F_1$. Thus, the identity is true.

Observe that

$$\begin{aligned} b_{k+1} &= ib_k - b_{k-1} + L_k i^k \\ &= ikF_k i^{k-1} - (k-1)F_{k-1} i^{k-2} + L_k i^k \\ &= i^k (kF_k + (k-1)F_{k-1} + L_k) \\ &= i^k ((k+1)F_{k+1} - F_k - 2F_{k-1} + L_k) \\ &= (k+1)F_{k+1} i^k, \end{aligned}$$

as desired.

Thus, we deduce that $b_n = nF_n i^{n-1}$.

Because of this, $\frac{S}{T} = \frac{2017F_{2017}}{17F_{17}}$.

First, we compute $F_{2017} \pmod{17}$. If ϕ, φ are the roots of $x^2 - x - 1$, then we know that $F_n = \frac{\phi^n - \varphi^n}{\phi - \varphi}$. Since 5 is not a quadratic residue modulo 17, ϕ, φ are not in $\mathbb{F}_{17}$ but are in $\mathbb{F}_{17^2}$. Work in $\mathbb{F}_{17^2}$. By the Frobenius Endomorphism, $\phi^{289} = \phi$ and $\varphi^{289} = \varphi$. Then

$$F_{2017} = \frac{\phi^{2017} - \varphi^{2017}}{\phi - \varphi} = \frac{\phi^{289 \cdot 7 - 6} - \varphi^{289 \cdot 7 - 6}}{\phi - \varphi} = \frac{\phi - \varphi}{\phi - \varphi} = 1.$$

In particular, F_{2017} is not divisible by 17.

Since $F_{17} = 1597$ is relatively prime to 2017, note that

$$\gcd(2017F_{2017}, 17F_{17}) = \gcd(F_{2017}, 17F_{17}) = \gcd(F_{2017}, F_{17}) = F_{\gcd(2017,17)} = F_1 = 1,$$

so m is just the remainder when $2017F_{2017}$ is divided by $17F_{17}$. Observe that F_{17} is also relatively prime to 17, so we can CRT the remainders for 17 and F_{17}.

First, we compute the remainder modulo 17: we have $2017 \equiv 11$ and $F_{2017} \equiv 1$, so $2017F_{2017} \equiv 11 \pmod{17}$.

Now, we compute the remainder modulo F_{17}. First note that $2017 \equiv 420 \pmod{F_{17}}$. It suffices to compute $F_{2017} \pmod{F_{17}}$.

Consider the matrix
$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$
Also define
$$J_n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$
Observe that $M = J_1$ and $J_{n+1} = MJ_n$ for all integers n, so $M^n = J_n$. Thus, we wish to compute M^{2017} modulo F_{17}. Note that
$$M^{17} = J_{17} \equiv \begin{bmatrix} F_{18} & 0 \\ 0 & F_{16} \end{bmatrix} \equiv \begin{bmatrix} F_{16} & 0 \\ 0 & F_{16} \end{bmatrix} \pmod{F_{17}}.$$
Then
$$M^{2006} \equiv F_{16}^{118} I \pmod{F_{17}},$$
where I is the identity matrix. But note that $F_{16}^2 = F_{15}F_{17} - 1 \equiv -1 \pmod{F_{17}}$, so $F_{16}^{118} \equiv -1 \pmod{F_{17}}$. Thus, $M^{2006} \equiv -I \pmod{F_{17}}$. Thus, $J_{2017} \equiv M^{2017} \equiv -M^{11} \equiv -J_{11} \pmod{F_{17}}$, so $F_{2017} \equiv -F_{11} \equiv -89 \pmod{F_{17}}$.

Thus, $2017F_{2017} \equiv 948 \pmod{F_{17}}$.

Now, to CRT everything together, observe that $1597 \equiv -1 \pmod{17}$ and $948 \equiv 13 \pmod{17}$, so $1597 \cdot 2 + 948 \equiv 11 \pmod{17}$, so m is 4142.

Alternate solution (Vincent Huang): Here's an alternative method for showing that $P_n'(0.5i) = i^{n-1}nF_n$ using some more calculus.

Note that $P_n(x)$ is the Chebyshev polynomial of the first kind, hence $P_n(\cos\theta) = \cos n\theta$. Perform the substitution $\cos\theta = 0.5(z + \frac{1}{z}) = u$ so that $P_n = \cos n\theta = 0.5(z^n + \frac{1}{z^n})$. Then by the Chain Rule, $\frac{dP_n}{du}\frac{du}{dz} = \frac{dP_n}{dz} = 0.5(nz^{n-1} - \frac{n}{z^{n+1}})$. But it's easy to see $\frac{du}{dz} = 0.5(1 - \frac{1}{z^2})$. So it follows that $\frac{dP_n}{du}$ reduces to $n\frac{z^n - \frac{1}{z^n}}{z - \frac{1}{z}}$.

Now we want $P_n'(0.5i)$. Solving $u = 0.5i$ in terms of z yields the solutions $z = \pm i\phi$ where ϕ is the golden ratio. By Binet's formula, however, $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right)$. Plugging $z = i\phi$ into the expression for P_n' yields $n\frac{i^n\phi^n - \frac{1}{i^n\phi^n}}{i\phi - \frac{1}{i\phi}} = ni^{n-1}\frac{\phi^n - \left(-\frac{1}{\phi}\right)^n}{\phi - \left(-\frac{1}{\phi}\right)} = \frac{ni^{n-1}F_n}{F_1} = ni^{n-1}F_n$ as desired. From here we proceed as in the first solution. □

27. For a graph G on n vertices, let $P_G(x)$ be the unique polynomial of degree at most n such that for each $i = 0, 1, 2, \ldots, n$, $P_G(i)$ equals the number of ways to color the vertices of the graph G with i distinct colors such that no two vertices connected by an edge have the same color. For each integer $3 \le k \le 2017$, define a *k-tasty* graph to be a connected graph on 2017 vertices with 2017 edges and a cycle of length k. Let the *tastiness* of a k-tasty graph G be the number of coefficients in $P_G(x)$ that are odd integers, and let t be the minimal tastiness over all k-tasty graphs with $3 \le k \le 2017$. Determine the sum of all integers b between 3 and 2017 inclusive for which there exists a b-tasty graph with tastiness t.

Proposed by Vincent Huang.

Answer. $\boxed{2022}$.

Solution. Write $P_G(x)$ as $P(G,x)$ for convenience. For the remainder of the solution we work in $\mathbb{F}_2[x]$.

Let T_n be an arbitrary tree on n vertices.

Lemma 1: $P(T_n,x) = x(x-1)^{n-1}$ for all positive integers x (including $x > n$).

Proof: We'll evaluate $P(T_n,k)$ for each k. Pick an arbitrary vertex of the tree as the root. It has k possible ways to be colored. Now at each step we pick an uncolored vertex whose parent is colored. The only restriction on the color of the child is that it cannot equal the parent's color, hence each child has $k-1$ color choices, yielding a final result of $P(T_n,k) = k(k-1)^{n-1}$. Since $k(k-1)^{n-1}$ is a degree n polynomial which equals $P(T_n,k)$ for $0 \le k \le n$, we see $P(T_n,x) = x(x-1)^{n-1}$.

Because of this lemma, for the remainder of the solution I will use T_n to refer to any member of the class of graphs consisting of a tree on n vertices, rather than an individual tree.

Now, we'll define a function Q such that $Q(G,k)$ gives the number of valid colorings of G with at most k colors for all k (inclunding $k > |G|$).

Lemma 2: If $\overline{uv}$ is an edge of G, then $Q(G,k) = Q(G-\overline{uv},k) - Q(G_{u=v},k)$, where $G_{u=v}$ is the graph formed by merging vertices u and v, while $G-\overline{uv}$ is the graph formed by removing edge $\overline{uv}$.

Proof: Clearly any coloring of $G-\overline{uv}$ is either a valid coloring of G, or else vertices u and v have the same color. If the second case is true then by merging vertices u and v, we get a valid coloring of $G_{u=v}$. So it's clear the identity holds.

Now, the key observation is that by inducting on $|E|+|V|$, we can recursively reduce $Q(G,k)$ until it is defined only in terms of values of the form $Q(T_n,k)$ via the equation in lemma 2. This reduction does not depend on k, so since $Q(T_n,k) = k(k-1)^{n-1}$ for all k, it follows that $Q(G,k)$ is actually a fixed polynomial in k. Furthermore, since $Q(T_n,x)$ is of degree exactly n, by running the recursion backwards it's easy to see $Q(G,k)$ will always have degree exactly $|G|$. Then it's not difficult to see we must have $P(G,k) = Q(G,k)$. In particular, we see $P(G,k)$ satisfies the recursion of lemma 2. (Note that this was NOT possible to conclude previously, because $|G_{u=v}| = |G|-1$, hence it was possible to verify $P(G,x) = P(G-\overline{uv},x) - P(G_{u=v},x)$ for $x = 0,1,2,...|G|-1$, but not $x = |G|$).

Now, we'll write a connected graph on n vertices with n edges and a cycle of length k as $T_{n,k}$. Once again, I'll use $T_{n,k}$ to refer to any element of the class of all such graphs rather than an individual graph. Note that if $\overline{uv}$ is an edge in the cycle of $G = T_{2017,k}$, then $G-\overline{uv} = T_{2017}$ while $G_{u=v} = T_{2016,k-1}$. By lemma 2, we have $P(T_{2017,k},x) = P(T_{2017},x) - P(T_{2016,k-1},x) \equiv P(T_{2017},x) + P(T_{2016,k-1},x)$ since all manipulations are in $\mathbb{F}_2[x]$.

Similarly, $P(T_{2016,k-1},x) = P(T_{2016},x) + P(T_{2015,k-2},x)$, $P(T_{2015,k-2},x) = P(T_{2015},x) + P(T_{2014,k-3},x), ...,$ until we reduce $P(T_{2020-k,3},x) = P(T_{2020-k},x) + P(T_{2019-k,2},x) = P(T_{2020-k},x) + P(T_{2019-k},x)$.

Summing all such equations and cancelling yields $P(T_{2017,k},x) = P(T_{2017},x) + P(T_{2016},x) + ... + P(T_{2019-k},x)$, showing $P(T_{2017,k},x)$ is the same regardless of which graph $T_{2017,k}$ is actually chosen.

By our first formula, the RHS equals $x(x-1)^{2016} + ... + x(x-1)^{2018-k} = x(x-1)^{2018-k}[(x-1)^{k-2} + (x-1)^{k-3} + ... + 1] = x(x-1)^{2018-k}\dfrac{(x-1)^{k-1}-1}{(x-1)-1}$, which via $\mathbb{F}_2$ reduction equals $(x-1)^{2018-k}[(x-1)^{k-1}-1] = (x-1)^{2017} - (x-1)^{2018-k}$.

Now we compute the answer. Note that the coefficient of x^i in $P(T_{2017,k},x)$ equals $\binom{2017}{i} - \binom{2018-k}{i}$. Recall by Lucas's Theorem that $\binom{\overline{a_n a_{n-1} \dots a_0}_2}{\overline{b_n b_{n-1} \dots b_0}_2} = \prod \begin{pmatrix} a_j \\ b_j \end{pmatrix}$, where $a_i, b_i \in \{0,1\}$.

Clearly $3 \le k \le 2017$, so $2018-k$ ranges from 1 to 2015. We'll let $2017 = \overline{a_{10}a_9a_8...a_0}_2, 2018-k = \overline{b_{10}b_9...b_0}_2$, possibly with leading zeroes. Let W be the number of tuples (a_i,b_i) equal to $(1,0)$. Similarly define x,y,z for the tuples $(1,1),(0,1),(0,0)$. Finally, write $i = \overline{c_{10}c_9...c_0}_2$.

If the coefficient of x^i is odd, then either its coefficient in $(x-1)^{2017}$ is even while its coefficient in $(x-1)^{2018-k}$ is odd or vice versa. In the first case, the fact that the first coefficient is even implies there was a $\binom{0}{1}$ term somewhere in the multiplicative expression of Lucas's theorem. Meanwhile, since the coefficient of $(x-1)^{2018-k}$ was odd, we must have $\binom{b_i}{c_i} = 1$ for each i, hence $b_i \geq c_i$ for each i. Putting all this together, there must be at least one (a_i, b_i, c_i) tuple of the form $(0, 1, 1)$, and the only other restriction is that whenever $b_i = 0$, c_i is also 0. There are no restrictions when $(a_i, b_i) = (1, 1)$. Then clearly there are $(2^y - 1) \cdot 2^x$ possible values for the sequence $(c_0, c_1, ...c_{10})$, resulting in that many nonzero coefficients. All such values i will be valid (i.e. ≤ 2017), since if $i > 2017$ both binomial coefficients are zero. Similarly, in the other case there are $(2^w - 1) \cdot 2^x$ total values for i.

So we deduce that the number of odd coefficients equals $2^x(2^w + 2^y - 2)$. Since $2017 = 11111100001_2$, we're subject to the constraints $w + x = 7, y + z = 4$. In addition, $(w, x, y, z) = (0, 7, 0, 4)$ is not possible because it yields $2018 - k = 2017$.

Suppose $(w, x) \neq (0, 7)$. Then to minimize the number of odd coefficients, which equals $2^x(2^w + 2^y - 2)$, obviously we wish to minimize 2^y by setting $(y, z) = (0, 4)$. Then the expression reduces to $2^x(2^w - 1) = 2^{w+x} - 2^x = 128 - 2^x$,whose minimal value is 64, occurring at $(w, x, y, z) = (1, 6, 0, 4)$. Meanwhile, if $(w, x) = (0, 7)$, we end up with $128(2^y - 1) \geq 128$. So it's clear the minimum number of odd coefficients is 64. This value of a is attainable only at $(w, x, y, z) = (1, 6, 0, 4)$, which is equivalent to creating $2018 - k$ by replacing one of the ones in $2017 = \overline{11111100001}_2$ with a zero. However, since $1 \leq k \leq 2015$, replacing the last one with a zero is not a valid choice, meaning we can only remove one of the first six digits.

Then $2018 - k \in \{2017 - 1024, 2017 - 512, 2017 - 256, 2017 - 128, 2017 - 64, 2017 - 32\} \implies k \in \{1025, 513, 257, 129, 65, 33\}$, hence $\sum b_i$ is 2022. $\square$

28. Let ABC be a triangle with $AB = 7, AC = 9, BC = 10$, circumcenter O, circumradius R, and circumcircle ω. Let the tangents to ω at B, C meet at X. A variable line ℓ passes through O. Let A_1 be the projection of X onto ℓ and A_2 be the reflection of A_1 over O. Suppose that there exist two points Y, Z on ℓ such that $\angle YAB + \angle YBC + \angle YCA = \angle ZAB + \angle ZBC + \angle ZCA = 90^\circ$, where all angles are directed, and furthermore that O lies inside segment YZ with $OY * OZ = R^2$. Then there are several possible values for the sine of the angle at which the angle bisector of $\angle AA_2O$ meets BC. If the product of these values can be expressed in the form $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with b squarefree and a, c coprime, determine $a + b + c$.

Proposed by Vincent Huang.

Answer. $\boxed{567}$.

Solution. From $\angle XA_1O = 90^\circ$ we know $A_1 \in (BOC)$. Then if A_0 is where ℓ meets BC, it follows that A_0, A_1 are inverses in ω, hence A_0, A_2 are inverses in γ, the imaginary circle centered at O with radius iR.

Let AA_2 meet ω at P. By Desargues' Involution Theorem on quadrilateral $ABCP$ and conic ω, we conclude that if ℓ meets any conic at two points, these points must be inverses in γ. Now, note that due to the given angle condition, Y and Z lie on the McCay cubic of ABC. Then if Y', Z' are the isogonal conjugates of Y, Z, by properties of the McCay cubic we know $Y' \in OY, Z' \in OZ$. Since this cubic can only meet ℓ at three points, one of which is O, we know $Y' = Z, Z' = Y$. Furthermore, by the converse of Desargues' Involution Theorem, since Y, Z are inverses in γ, we know $ABCPYZ$ lie on a conic. Then the isogonal conjugate of this conic is obviously the line YZ. Denote P' as the isogonal conjugate of P. Then clearly $P' \in YZ$, implying that the line through A parallel to ℓ and AP are isogonal, hence the angle bisectors of $\angle AA_2O$ are always parallel to one of the A-angle bisectors of $\triangle ABC$.

But now with standard techniques we can compute the sines of the angles at which the A-internal and external angle bisectors meet BC, and their product is $\dfrac{16\sqrt{26}}{525}$, and the answer is 567. $\square$

29. Let $p = 2017$. If A is an $n \times n$ matrix composed of residues $\pmod p$ such that $\det A \not\equiv 0 \pmod p$ then let $\text{ord}(A)$ be the minimum integer $d > 0$ such that $A^d \equiv I \pmod p$, where I is the $n \times n$ identity matrix. Let the maximum such order be a_n for every positive integer n. Compute the sum of the digits when $\sum_{k=1}^{p+1} a_k$ is expressed in base p.

Proposed by Ashwin Sah.

Answer. $\boxed{6048}$.

Solution. Solution by Ashwin Sah

All equalities are over $\overline{\mathbb{F}_p}$, the algebraic closure of $\mathbb{F}_p$. For all $x \in \overline{\mathbb{F}_p}$ we let $c(x)$ be the degree of the minimal polynomial of x.

Take n by n matrix A over $\mathbb{F}_p$ and nonzero determinant. Consider its eigenvalues over $\overline{\mathbb{F}_p}$. Then compute the characteristic polynomial of A and its eigenvalues, and diagonalize into the Jordan Normal Form.

We find

$$A = P^{-1}DP,$$

where D is a diagonal matrix or has diagonal blocks of the form $\lambda 10\ 0\lambda 1\ 00\lambda$, for example. Notice that

$$A^d = P^{-1}D^dP,$$

and this equals I if and only if $D^d = I$. Now since diagonal matrices have powers derived by simply taking the powers of the diagonal elements and since the nth power of the above block gives $\lambda^n, n\lambda^{n-1}, n(n-1)\lambda^{n-2}\ 0, \lambda^n, n\lambda^{n-1}\ 0, 0, \lambda^n$ for rows; we can thus find the order simply. If there is at least one block, we need $p|d$. But beyond that, we just need d to be divisible by the orders of all the eigenvalues, none of which involve factors of p. Thus d will definitely be at most

$$p \prod_{i=c(\lambda_j)} (p^i - 1),$$

with the factor of p only occurring if there is a multiple root which has differing geometric and algebraic multiplicities. Now notice that the values of i above sum to at most n since the characteristic polynomial of A has degree n, and they sum to at most $n-1$ when there is a multiple root. Thus multiple roots can be easily seen to give a maximum order of

$$p(p^{n-1} - 1) < p^n - 1$$

while distinct roots give a maximum order of

$$p^n - 1.$$

Equality occurs if and only if matrix A has a characteristic polynomial with roots primitive roots of F_{p^n}. This can be achieved using, for example, a companion matrix to a minimal polynomial of such a primitive root.

Thus we want the sum of the digits of $\sum_{i=1}^{p+1}(p^i - 1) = (p-1)p^0 + (p-1)p^1 + \sum_{i=3}^{p+1} 1p^i$ in base p, which is $3p - 3 = 6048$. $\square$

30. We define the bulldozer of triangle ABC as the segment between points P and Q, distinct points in the plane of ABC such that $PA \cdot BC = PB \cdot CA = PC \cdot AB$ and $QA \cdot BC = QB \cdot CA = QC \cdot AB$. Let XY be a segment of unit length in a plane $\mathcal{P}$, and let $\mathcal{S}$ be the region of $\mathcal{P}$ that the bulldozer of XYZ sweeps through as Z varies across the points in $\mathcal{P}$ satisfying $XZ = 2YZ$. Find the greatest integer that is less than 100 times the area of $\mathcal{S}$.

Proposed by Michael Ren.

Answer. $\boxed{129}$.

Solution. Let Ω be the locus of Z, which is an Appolonius circle with respect to X and Y of radius $\frac{2}{3}$ and center O. Let $f : \mathcal{P} \to \mathcal{P}$ be the projective transformation fixing Ω and XY and taking O to O' with $OO' = \frac{8}{15}$ and O' on segment YZ.

The key observation is that any bulldozer of XYZ has the property that $f^{-1}(\mathcal{T})$ is a $120°$ chord of Ω. The desired region is thus $f(\mathcal{S}')$ where S' is the area between Ω and ω, the image of Ω under a homothety centered at O with ratio $\frac{1}{2}$. This is because over all $120°$ chords on a circle, the set of points on their perimeters is the area between the circle and the smaller circle that all of them are tangent to.

Note first that ZPQ is clearly inscribed in Ω as it is on the Z Appolonius circle of XYZ and P and Q are the isodynamic points of XYZ. Let $x = \angle XZY$. From well-known facts about isodynamic points, $\angle XPY, XQY = 60° + x, -60° + x$, where the angles are directed. Thus, embedding this in the complex plane, we have that the cross ratio $(X, Y; P, Q) = \omega$, a third root of unity. Taking a projective transformation fixing Ω and taking X to O takes Y to the point at infinity along XY, so $(X, Y; P, Q)$ becomes the ratio $\frac{p}{q}$, where p, q are the complex numbers that represent P, Q. As projective transformations preserve cross ratios, this takes bulldozers to $120°$ chords. It is not too hard to compute that f^{-1} is this projective transformation.

To compute the desired area, we use homogenous coordinates, first scaling Ω up to a unit circle, setting Ω at $x^2 + y^2 - z^2 = 0$ and thus ω at $4x^2 + 4y^2 - z^2 = 0$. If we find the area of the image of ω under f, we can subtract it from Ω to find the area. As f takes $(0 : 0 : 1)$ to $(4 : 0 : 5)$, $(1 : 0 : 1)$ and $(-1 : 0 : 1)$ to themselves, and Ω to itself, we obtain its transformation matrix as $\begin{pmatrix} 5 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{pmatrix}$ by setting up some equations. This takes $4x^2 + 4y^2 - z^2 = 0$ to $4(5x + 4z)^2 + 4(3y)^2 - (4x + 5z)^2 = 0$, or $84x^2 + 36y^2 + 39z^2 + 120xz = 0$ which simplifies to $28x^2 + 12y^2 + 13 + 40x = 0$. We can easily compute now that this has area $\frac{3\sqrt{21}}{196}\pi$ through ellipse formulas, so $\mathcal{S}$ has area $\frac{4}{9}\left(1 - \frac{3\sqrt{21}}{196}\right)\pi = \frac{196 - 3\sqrt{21}}{441}\pi$. This gives our answer of 129. $\square$