Preparation for Saudi Arabia Team 2021
June Session: Junior Balkan Mathematics Olympiad

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Lesson 5
Elementary geometry

Problems:

1. Let the incircle of $\triangle ABC$ touch BC, CA and AB respectively at D, E and F. Let K be a point on the same side of FE with respect to A such that $\measuredangle KFE = \measuredangle ACB$ and $\measuredangle KEF = \measuredangle ABC$. Show that $KD \perp BC$.

2. In an acute triangle ABC, let O be the circumcenter and P a point on the opposite side of BO with respect to point A such that $AO = AP$ and $AP \perp BO$. Let Q be the midpoint of OP and let R be the point on the opposite side of AB with respect to C such that $AR = RB$ and $\measuredangle ARB = 90^o$. Prove that $PR = \sqrt{2} \cdot BQ$.

3. Given a triangle ABC let the angular bisector of $\measuredangle BAC$ intersect BC at D. Let M be the midpoint of BD. Let a circle through A and tangent to line BC at D intersect the second time AM at P and AC at Q $(P, Q \neq A)$. Prove that B, P and Q are collinear.

4. Let $ABCD$ be a convex quadrilateral such that $\measuredangle BCD = \measuredangle CDA = 120^o$. We construct equilateral triangles ACE and BDF such that E and D are on the same side of AC and F and C are on the same side of BD. Let AE and DF intersect in X and let CE and BF intersect in Y. Prove that AC, BD and XY are concurrent.

5. Let $ABCD$ be a cyclic quadrilateral (inscribed in a circle) such that neither $\triangle ABC$ not $\triangle ADC$ are equilateral. Let E be the midpoint of the diagonal AC. Let the line through D perpendicular to the Euler line of $\triangle ABC$ and the line through B perpendicular to the Euler line of $\triangle ADC$ intersect in F. Let G be the point on the segment EF such that $GF = 2GE$. Prove that $GB = GD$.

6. Let k be the circumcircle of a triangle ABC. A circle l with center O passes through B and C and meets the segments AC and AB again at D and E respectively. Let $P \neq A$ be the point at which the circumcircle of $\triangle ADE$ meets k. Prove that $AP \perp PO$.

7. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ touch each other externally at D, and touch a circle ω internally at B and C, respectively. Let A be an intersection point of ω and the common tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ at D. Lines AB and AC meet $\mathcal{C}_1$ and $\mathcal{C}_2$ again at K and L, respectively, and the line BC meets $\mathcal{C}_1$ again at M and $\mathcal{C}_2$ again at N. Prove that the lines AD, KM, LN are concurrent.

8. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.