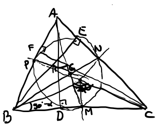
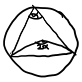
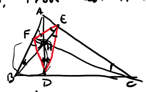


H = orthocenter G = centroid
O = circumcenter
α, β, γ = the angles
$\sphericalangle BOC = 2\alpha$
$\sphericalangle BHC = 180^\circ - \alpha$
$\sphericalangle OBC = \sphericalangle OCB = 90^\circ - \alpha$
$\sphericalangle HBA = \sphericalangle HCA = 90^\circ - \alpha$



O, G, H lie on the Euler line.

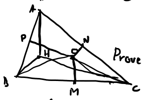
1. Prove that H is the incenter of $\triangle DEF$



$\sphericalangle FDH = \sphericalangle FBH = \sphericalangle ABE = 90^\circ - \alpha = \sphericalangle ECH = \sphericalangle EDH$
$\Rightarrow DH$ bisects $\sphericalangle FDE$.

Where are the 3 excenters of $\triangle DEF$?
A, B, C

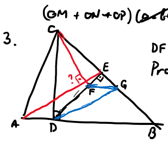
2.



R = the circumradius
r = the inradius
Prove: $AH + BH + CH = 2(R+r)$.
$\underline{AH = 2OM}$, $BH = 2ON$, $CH = 2OP$

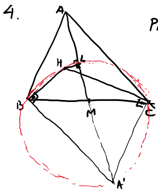
$OM + ON + OP = R + r$?
$ANOP$ is cyclic: Ptolemy's theorem $\Rightarrow AP \cdot ON + AN \cdot OP = AO \cdot NP$
$\frac{1}{2} c \cdot ON + \frac{1}{2} b \cdot OP = \frac{1}{2} a \cdot R$
$c \cdot ON + b \cdot OP = a \cdot R$
$a \cdot OP + c \cdot OM = b \cdot R$ Apply the same on $BMOP$, $CMON$
$b \cdot OM + a \cdot ON = c \cdot R$
$OM(b+c) + ON(c+a) + OP(a+b) = (a+b+c)R$
$OM \cdot a + ON \cdot b + OP \cdot c = 2[ABC] = (a+b+c) \cdot r$
$(OM + ON + OP)\cancel{(a+b+c)} = \cancel{(a+b+c)}(R+r)$

3.



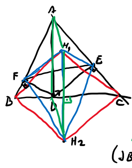
$DF : FE = BD : DA$
Prove that $CF \perp AE$
$BG : GE = BD : DA = DF : FE$
$DG \parallel AE$, $GF \parallel AB$
F is the orthocenter of CDG $\Rightarrow CF \perp DG \parallel AE$

4.



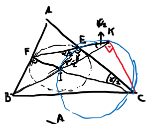
Prove that B, C, H and L lie on a circle.
B, C, L lie on the circle with diameter $A'L$.

5.



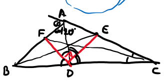
H_1, H_2 = orthocenters of AEF, DEF.
$[ABC] = S$
$[BH_1CH_2] = ?$ S
DEH_1F, AEH_2F
$AD = H_1H_2$
(JBMO 2018)

6.

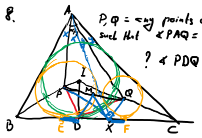


Prove: $\sphericalangle BKC = 90^\circ$.

7.



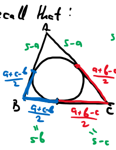
8.



P, Q = any points on BI and CI such that $\sphericalangle PAQ = \frac{\alpha}{2}$
? $\sphericalangle PDQ = 90^\circ$.

Take $X \in BC$ so that AP bisects $\sphericalangle BAX$
P = incenter of ABX, Q = incenter of ACX
$\Rightarrow \sphericalangle PXQ = 90^\circ$
We want to show that $PQXD$ is cyclic.
The center of this circle should be M = the midpoint of PQ
We want to show that $MX = MD$, i.e. that $ED = XF$.

Recall that:



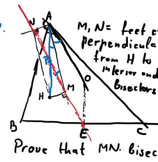
$s = \frac{a+b+c}{2}$
$s-a$, $s-a$
$s-b = \frac{a+c-b}{2}$
$\frac{a+b-c}{2} = s-c$
$\frac{a+c-b}{2} = s-b$
$\frac{a+b-c}{2} = s-c$

$XF = \frac{AX + XC - AC}{2}$
$ED = BD - BE = \frac{\cancel{AB} + BC - AC}{2} - \frac{\cancel{AB} + BX - AX}{2}$
$= \frac{CX + AX - AC}{2} = XF$

9.

$\frac{180^\circ - b}{2}$
I = incenter
F = midpoint of AB
Suppose: $\sphericalangle BIF = 90^\circ$
Prove that $b + c = 3a$.
E = midpoint of AC
$\Rightarrow \sphericalangle BFI = \sphericalangle IFE = \frac{180^\circ - b}{2}$
FI bisects $\sphericalangle BFE$
$\Rightarrow I$ is the incenter of $BCEF$
$\frac{3a}{2} = BC + EF = BF + CE = \frac{c}{2} + \frac{b}{2}$

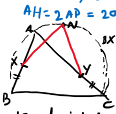
10.



M, N = feet of perpendiculars from H to the interior and exterior bisectors of $\sphericalangle BAC$.

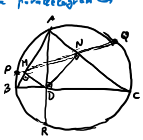
Prove that MN bisects BC.
$AMHN$ is a rectangle $\Rightarrow$ MN passes through the midpoint P of AH
$MN \parallel AO$, so it remains to show that $EP \parallel AO$
$AH = 2AP = 2OE \Rightarrow APEO$ is a parallelogram

11.

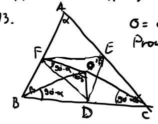


$BX = CY$
N = midpoint of arc BAC
Prove that $NX = NY$

12.

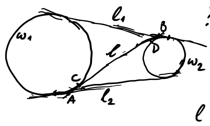


13.



O = circumcenter of ABC, $\triangle DEF \sim \triangle ABC$
Prove that O is the orthocenter of DEF.
Define O' as the orthocenter of DEF.
Then $BDO'F$ is cyclic
$\sphericalangle O'BC = \sphericalangle O'CB = 90^\circ - \alpha = \sphericalangle OBC = \sphericalangle OCB$
$\Rightarrow O' = O$.

14.



? $AC = BD$.
ω_1, ω_2 = circles
ℓ = interior common tangent
ℓ_1, ℓ_2 = exterior common tangents
ℓ meets ℓ_1 at B, ℓ_2 at A
meets ω_1 at C, ω_2 at D