Invariants and Monovariants

AoMP Team

March 2022

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1 Definition

A property that does not change under A particular transformation is called an **Invariant**. And a quantity that changes monotonically (either non-increasing or non-decreasing) or in a regular manner is a **Monovariant**.
Invariants and Monovariants can be extremely diverse, and if you pick the right property, it can explain a great deal about what the process is doing, and very difficult problems can become almost trivial! And there are numerous problems of international mathematical olympiads that can be solved by finding some very special Invariant or Monovariant.
When one is trying to solve olympiad problems (or any problem in mathematics), looking for theses properties is a fundamental strategy.

2 Important ideas

Usually an invariant problem is pretty easy once you find the right invariant (or monovariant), but finding it can be pretty tough! Actually, finding the right invariant is an art, and that is what makes many problems hard. Nonetheless, here are a few things that you should always be thinking about.

• **Colorings and Numberings:** Color all the squares in a grid with two or more colors. Or number each square with a certain value (usually the sum of their values is an invariant). Consider squares of each color separately.

• **Algebraic expressions:** Given a set of values, look at their differences, their sum, the sum of their squares, or occasionally their product. If you are working with integers, try looking at these values modulo n.

• **Corners and edges:** For grid-based problems, consider any shapes formed. How many boundary edges do they have? How many corners?

• **disorderings:** If you are permuting a sequence of numbers consider the number of disorderings, that is: pairs (i, j) such that i and j are listed in the wrong order. Both the number of disorderings and its parity are useful.

• **Integers and rationals:** Can you find a positive integer that keeps decreasing? Or does the denominator of a rational number keep decreasing?

• **Symmetries:** Can you ensure that after each step, a figure is symmetrical in some way? Perhaps two paired objects are always in the same state? Perhaps the problem can be divided into two essentially identical subproblems? This is especially useful for game-theory type problems.

3 Useful methods

Even though a large number of problems in combinatorics have a quick and/or easy solution, that does not mean the problem one has to solve is not hard. Many times the difficulty of a problem in combinatorics lies in the fact that the idea that works is very well "hidden", yet here are a few methods that may help you find these "hidden" solutions.

• **To prove a process cannot reach a specific configuration:** Find a quantity that changes predictably at each step. Calculate the quantity at the start and end configurations and look for a contradiction.

• **To prove a process terminates:** Find a quantity that goes down at each step. If the quantity has to be a positive integer then you are done, otherwise it might keep decreasing but never reach zero!

• **To prove a configuration is not possible:** Find one or more quantities that are easy to understand locally, and prove they cannot take on the value needed for the bad configuration.

• **To prove all configurations have some property:** Prove one configuration X has the property, and that you can get from any configuration to X via simple steps that don't change whether the property holds.

• **To prove one player has a winning strategy in a game:** Find an invariant that they can maintain at the end of each of their moves, and prove they cannot actually lose as long as this invariant holds.

4 Introductory Problems

When finding a problem, try to solve it by yourself before reading the solution. This is the only way to see the complications that particular problem brings. By doing this you will have an easier time understanding why the ideas in the solution work and why they should be natural.

Exercise 1: very easy Suppose the positive integer n is odd. First Omar writes the numbers $1, 2, \ldots, 2n$ on the blackboard. Then he picks any two numbers a, b and erases them, and writes, instead, $|a-b|$. Determine whether the final number on the board will be odd or even, and show that it does not depend on the manner in which the numbers were chosen at each step.

Solution: Suppose S is the sum of all the numbers still on the blackboard. Initially this sum $S = 1 + 2 + \cdots + 2n = n(2n+1)$, is an odd number. Each step reduces S by $2\,min(a, b)$, which is an even number. So the parity of S is an invariant. Initially the parity is odd. So it will also end odd. ■

Exercise 2: easy On every square of a 1997×1997 board is written either $+1$ or -1. For every row, we compute the product R_i of all numbers written in that row, and for every column, we compute the product C_i of all written in that column. Is it possible to arrange the numbers in such a way that

$$\sum_{i=1}^{1997} (R_i + C_i) = 0\ ?$$

Solution: let $\sum_{i=1}^{1997}(R_i+C_i) = S$ Now suppose by way of contradiction that we have found some arrangement for which $S = 0$. Let us replace each -1 with $+1$, one at a time. Each time we do this, we change some R_i and some C_j, and so S changes by $\pm 2 \pm 2 \equiv 0 \pmod 4$. Since $S \equiv 0 \pmod 4$ originally, it must still be 0 (mod 4) when done. However, at that point, we know each R_i and C_j equals 1, so $S = 2 \times 1997 \equiv 2 \pmod 4$, contradiction. ■

Colorado Math Olympiad 1997

Exercise 3: medium The numbers $1, 2, 3, \ldots, n$ are written in a row. It is permitted to swap any two numbers. If 2011 such operations are performed, is it possible that the final arrangement of numbers coincides with the original?

Solution: ***Main idea: Count all pairs (a, b) that aren't in order***.

$$\overbrace{b_1\, b_2\, b_3 \ldots b_{x-2}\, b_{x-1}\, b_x}^{x}\, m\, \overbrace{r_1\, r_2\, r_3 \ldots r_{y-2}\, r_{y-1}\, r_y}^{y}\, n\, \overbrace{g_1\, g_2\, g_3 \ldots g_{z-2}\, g_{z-1}\, g_z}^{z}$$

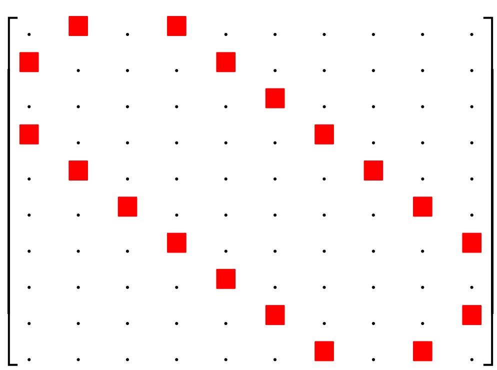
When swapping two numbers m and n notice that the pairs containing a "blue" and/or a "green" element, and the pairs where both of their elements are "red" don't switch order, while all the other $2y+1$ pairs switch order. Initially all pairs are in order, but after each step an odd number of them switch order, and we will an odd number of moves, so the number of pairs that aren't in their correct order in the final position must be odd. $\Rightarrow$ It is impossible to reach the original configuration after 2011 steps. ■

Exercise 4: hard There are some counters in some cells of a 100×100 board. Call a cell "nice" if there are an even number of counters in its adjacent cells (adjacent by side). Can exactly one cell be "nice"?

Solution: *A **good set** is a set S of cells such that any cell is adjacent to an even number of cells in S.*

Claim: For any cell C, there exists a **good set** S containing C with an even number of elements.

Proof: All sets of this shape are good (red squares mark the points of S):



Descriptively, the cells look like a "tilted rectangle". By varying the dimensions and orientation of the rectangle, we can cover any cell. (There is an edge case including just the major diagonal.) If there is exactly one nice cell C, then consider the set S and consider

$$\sum_{\text{counter } x} \#\,(\text{cells in } S \text{ adjacent to } x)\,.$$

On the one hand, it is even by the definition of a good set (because each counter doesn't change its parity, **this is the invariant**). On the other hand, counting by the cells, the total should be odd, contradiction. ■

All Rusian Olympiad 2011

Exercise 5: very hard To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution: Let the numbers in the pentagon be a, b, c, d, e, When one does the transformation, the sum of all numbers remains the same. Consider

$$S(a, b, c, d, e) = (a - c)^2 + (c - e)^2 + (e - b)^2 + (b - d)^2 + (d - a)^2$$

It is clear that $S(a, b, c, d, e) \geq 0$. Suppose that c is negative and we do the change with c as the middle number. We have that

$$\begin{aligned} S(a, b + c, -c, d + c, e) &= (a + c)^2 + (-c - e)^2 + (e - b - c)^2 + (b - d)^2 + (d + c - a)^2 \\ &= S(a, b, c, d, e) + 2c(a + b + c + d + e) \end{aligned}$$

Since $c < 0$ and that $a + b + c + d + e > 0$, we have that $2c(a + b + c + d + e) < 0$. Therefore, $S(a, b, c, d, e)$ is reduced after every transformation.
By infinite descent, there must be a moment when we can no longer do the transformation, which is what we wanted to prove. ■

Note: *It should be noted that this problem was solved completely by a very small number of students (as it should be expected for a Problem 3 at the IMO). The invariant S here might seem at first quite unnatural, but that is not the case. The first thing a student has to note is that the sum of all the numbers is constant and positive, and thus so is their average. This means that if every number was as close as possible to the average, then none would be negative. This makes it natural to search for functions that increase as the numbers are different from their average, or different from each other. Given two numbers x and y, $(x - y)^2$ is a good way to see how different they are. If we start with $(x - y)^2$ and try to apply it to 5 numbers, then the function S (or one of the same kind) arises naturally.*

IMO P3 1986 and AUO 1992

5 Training Problems

Every year there is at least one combinatorics problem in each major mathematical olympiad of international level. These problems need a very high level of creativity to find a solution. Even in the most recent competitions there are difficult problems that can be solved just by a single brilliant idea. However, to be able to attack these problems with comfort it is necessary to have faced problems of similar difficulty previously and to have a good knowledge of the techniques that are commonly used to solve them.
The following problems are arranged by our opinion on their difficulty ranging from very easy and trivial problems to problems that are of an international competition level, so you should not become discouraged if you find a particularly difficult one. And remember to chase your own ideas and to not restrict yourself to the tools and methods presented before. take it easy. Remember to that solving problems is a discipline that is learned with constant practice and Training.

Very Easy

problem 1) There are several signs $+$ and $-$ on a blackboard. You may erase two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. Prove that the last sign on the board does not depend on the order of steps.

problem 2) Waterloo Math Circles A hockey player has 3 pucks labeled A, B, C in an arena. He picks a puck at random, and fires it through the other 2. He keeps doing this. Can the pucks return to their original spots after 1987 hits?

problem 3) Anas has a 7×7 board, such that 5 random cells are colored black and the rest of the squares (44) are colored white, in each move Anas chooses a 2×2 square inside the board and switches each of its 4 cells (from black to white or from white to black), can the board be completely colored white after a finite number of steps?

problem 4) There are three piles with n tokens each. In every step we are allowed to choose two piles, take one token from each of those two piles and add a token to the third pile. Using these moves, is it possible to end up having only one token?

problem 5) A circle is divided into six sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written into the sectors (counterclockwise) You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?

problem 6) There is one stone at each vertex of a square. We are allowed to change the number of stones according to the following rule: We may take away any number of stones from one vertex and add twice as many stones to the pile at one of the adjacent vertices. Is it possible to get 2022, 2021, 2023, and 2022 stones at consecutive vertices after a finite number of moves?

problem 7) St. Petersburg City Olympiad 1996 Several positive integers are written on a blackboard. Azzam can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

problem 8) You have a chocolate bar consisting of squares arranged in an $n \times m$

rectangular pattern. Your task is to split the bar into squares (always breaking one piece at a time along a line between the squares) with a minimum number of breaks. How many breaks do you need?

problem 9) Let $d(n)$ be the digital sum of $n \in N$. Solve for n:
$n + d(n) + d(d(n)) = 2011$.

problem 10) Soviet Union 1961 Tricky Consider a rectangular array with m rows and n columns whose entries are real numbers. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations we can make the sum of numbers along each line (row or column) non-negative.

Easy

problem 11) Seven vertices of a cube are marked by 0 and one by 1. You may repeatedly select an edge and increase by 1 the numbers at the ends of that edge. Can you reach:(a) 8 equal numbers, (b) 8 numbers divisible by 3

problem 12) There exist 3 ants in a two dimensional plane. Each minute one of the ants moves parallel to the line connecting the other two ants. If they start at $(0, 0)(1, 0)(0, 1)$ can they eventually end at $(1, 1)(2, 2)(2, 0)$?

problem 13) There is a checker at point $(1, 1)$ of the lattice (x, y) with x, y positive integers. It moves as follows. At any move it may double one coordinate, or it may subtract the smaller coordinate from the larger. Which points of the lattice can the checker reach?

problem 14) We may extend a set S of space points by reflecting any point X of S at point A, $A \neq X$. Initially, S consists of the 7 vertices of a cube. Can you ever get the eight vertex of the cube into S?

problem 15) Alexa chooses a positive integer n and writes in a board $2n + 1$ numbers $n, n/2, \ldots, n/(2n + 1)$ Ali erases two numbers in the board a, b and writes, instead, $2ab - a - b + 1$. Ali reapeats this operation until one number is left on the board, find all possibilities for this number.

problem 16) On a single pile there are 1001 stones. In a single move one is allowed to remove one stone from a pile and split the remaining stones on the pile into two non-zero piles of arbitrary amount. Is it possible to end up only with piles containing three stones?

problem 17) In a 4×4 table suppose the bottom left corner's coordinate is $(1, 1)$. the cell $(1, 2)$ is colored black while every other cell in the table is colored white. in each step you may switch the color of all cells of a row, column, or a parallel to one of the diagonals. In particular, you may switch the color of each corner cell. Prove that at least one white cell will remain in the table.

problem 18) Each of the numbers $a_1, \ldots, a_n$ is 1 or -1, and we have
$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0$ Prove that $4|n$.

problem 19) The numbers $1, 2, \ldots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2 + b^2 + c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

problem 20) USAMO 1997 To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $\frac{1}{3}$, for all $n \geq 6$.

problem 21) All Russian Olympiad 2012 101 wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion. The rest do not change. Prove that at one point they will all stop changing opinions.

problem 22) There is an integer in each square of an 8×8 chessboard. In one move, you may choose any 4×4 or 3×3 square and add 1 to each integer of the chosen square. Can you always get a table with each entry divisible by (a) 2, (b) 3?

problem 23) Slovenia National MO 1999 The numbers $1, 2, \ldots, n$ are written, in each step Adeeb removes two numbers a, b and replaces them with $ab + a + b$. Will the last number remaining always be the same?

problem 24) Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.
(a) Does the game necessarily end?
(b) Does there exist a winning strategy for the starting player?

problem 25) IMO 2012 C1 Several positive integers are written in a row. Iteratively, Abdulrahman chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that Abdulrahman can perform only finitely many such iterations.

problem 26) It is given 5 numbers 1, 3, 5, 7, 9. Each step we arbitrary take 4 numbers (out of current 5 numbers) a, b, c, d and replace them with $\dfrac{a+b+c-d}{2}, \dfrac{a+b-c+d}{2}, \dfrac{a-b+c+d}{2}, \dfrac{-a+b+c+d}{2}$ Can we, with repeated iterations, get numbers: 3, 4, 5, 6 and 7

problem 27) n numbers are written on a blackboard. In one step you may erase any two of the numbers, say a, b, and write, $(a+b)/4$. Repeating this step $n-1$ times, there will be one number left. Prove that, initially, if there were n ones on the board, at the end, a number, which is not less than $1/n$ will remain.

problem 28) Putnam 1979 Tricky Given n red points and n blue points in the plane, no three collinear, prove that we can draw n segments, each joining a red point to a blue point, such that no segments intersect.

problem 29) All Rusian Olympiad 1998 Two distinct positive integers a, b are written on the board. The smaller of them is erased and instead of it the number $\frac{ab}{|a-b|}$ is written. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

Medium

problem 30) IMO 2014 C2 We have 2^m sheets of paper, with the number 1 written on each of them. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

problem 31) Consider a $n \times n$ board. There is one token in each 1×1 cell. An operation is as follows - if we have two 1×1 cells of any rectangle with tokens; one on the upper left corner and the other one on the lower right corner, then these are changed to the lower left corner and to the upper right corner. Prove that eventually, after a finite number of operations all the tokens will be in the diagonal of the $n \times n$ board.

problem 32) IOI 2002, USAMO 1998 A $(2n+1) \times (2n+1)$ board is colored in the chessboard fashion. One is in one move allowed to select a rectangle on the board and reverse all colors. Find the minimum number of moves needed to make the entire board be of the same color.

problem 33) IMO 1994 C5 1994 boys are seated at a round table. Initially one boy holds n tokens. Each turn a boy who is holding more than one token passes one token to each of his neighbors. Show that if $n = 1994$ it cannot terminate.

problem 34) The integers $1, \ldots, n$ are arranged in order. In one step you may take any four integers and interchange the first with the fourth and the second with the third. Prove that, if $n = 0, 1$ modulo 4, then by means of such steps you may reach the arrangement $n, n-1, \ldots, 1$. But if $n = 2, 3$ modulo 4, you cannot reach this arrangement.

problem 35) Tricky Hadi infects $n-1$ cells of a $n \times n$ board. And each minute, Hadi infects the cells with at least two infected neighbors (having a common side). Can Hadi eventually infect the whole board?

problem 36) Given a permutation $(a_1, a_2, ..., a_n)$ of the numbers $1, 2, ..., n$ one may interchange any two consecutive "blocks" i.e.

$$(a_1, a_2, ..., a_i, \overbrace{a_{i+1}, ...a_{i+p}}^{A}, \overbrace{a_{i+p+1}, ..., a_{i+q}}^{B}, ..., a_n)$$

into

$$(a_1, a_2, ..., a_i, \overbrace{a_{i+p+1}, ..., a_{i+q}}^{B}, \overbrace{a_{i+1}, ...a_{i+p}}^{A}, ..., a_n)$$

by interchanging the "blocks" A and B. Find the least number of such changes which are needed to transform $(n, n-1, ..., 1)$ into $(1, 2, ..., n)$

problem 37) MOP 1998 A circle has been cut into 2000 sectors. There are 2001 frogs inside these sectors. There will always be some two frogs in the same sector; two such frogs jump to the two sectors adjacent to their original sector (in opposite directions). Prove that, at some point, at least 1001 sectors will be inhabited.

problem 38) NICE MO 2021 Let n be a positive integer. Azzam and Adeeb play a game with $2n$ lamps numbered 1 to $2n$ from left to right. Initially, all lamps numbered 1 through n are on, and all lamps numbered $n+1$ through $2n$ are off. They play with the following rules, where they alternate turns with Azzam going first:
On Azzam's turn, he can choose two adjacent lamps i and $i+1$, where lamp i is on and lamp $i+1$ is off, and toggle both.
On Adeeb's turn, he can choose two adjacent lamps which are either both on or both off, and toggle both.
Players must move on their turn if they are able to, and if at any point a player is not able to move on his turn, then the game ends. Determine all n for which Adeeb can turn off all the lamps before the game ends, regardless of the moves that Azzam makes.

problem 39) Kosovo IMO TST Each term in the sequence $1, 0, 1, 0, 1, 0, 3\dots$ is the sum of the last 6 terms modulo 10. Prove that the sequence $\dots, 0, 1, 0, 1, 0, 1, \dots$ never occurs.

problem 40) In a 4×4 board the numbers from 1 to 15 are arranged in the following way:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

In a move we can move some number that is in a square sharing a side with the empty square to that square. Is it possible to reach the following position using these moves?

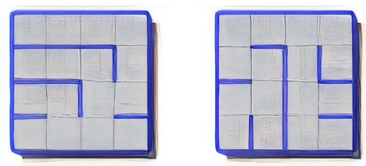
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

problem 41) Tricky n squares in an infinite grid are colored black; the rest are colored white. When a square is the opposite color from 2 or more of its 4 neighbors, its color may be switched. Eventually, we get to having 2022 black squares, no two of which border along an edge, and all other squares white. Prove that $n \geq 2022$.

problem 42) IMO 2013 C3 Omar discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. Omar has found a way to perform the following two kinds of operations with these particles, one operation at a time.
(i) If some imon is entangled with an odd number of other imons in the lab, then omar can destroy it.
(ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I'

becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.
Prove that Omar may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

problem 43) MOP, Canadian Math Olympiad 2012 A maze consists of a finite grid of squares where the boundary and some internal edges are "walls" that cannot be crossed. For example:



Two mazes are given, each with a robot in the top-left square. You may give a list of directions (up, down, left, or right) to the robots. Both robots will independently follow the same list of directions. For each direction, the robot will move one square in that direction if it can, or do nothing if there is a wall in the way. It will then proceed to the next direction, and repeat until it has gone through the whole list. Suppose that there is a list of directions that will get each robot individually from the top-left corner to the bottom-right corner of its maze. Prove there is also a list of directions that will get both robots to the bottom-right corner at the same time. In the example above, you could give the directions 'Right', 'Right', 'Down', 'Down', 'Down', 'Right', 'Down', 'Down', 'Left', 'Down', 'Right'.

problem 44) IMO 2018 C3 Let n be a given positive integer. Alzahrani performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Alzahrani chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Alzahrani's aim is to move all n stones to square n. Prove that Alzahrani cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

problem 45) On a blackboard the numbers $2, 3, 4$ are written, in each step Abdulrahman removes a and replaces it with $1/a(b+c)$. can he reach $1, 2, 3$ after a finite number of steps?

problem 46) USAMO 1994 The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, , . . . , , red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, , . . . , , red, yellow, blue?

problem 47) AMSP CP The numbers from 1 through 2008 are written on a blackboard. Every second, Hamza erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

problem 48) All Russian Olympiad 1997 On an infinite (in both directions) strip of squares, indexed by the integers, are placed several stones (more than one may be placed on a single square). We perform a sequence of moves of one of the following types:
(a) Remove one stone from each of the squares $n-1$ and n and place one stone on square $n+1$.
(b) Remove two stones from square n and place one stone on each of the squares $n+1$, $n-2$.
Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves

problem 49) Mexico National Olympiad 2020 Let $n \geq 3$ be an integer. Two players, Anas and Abdulrahman, play the following game. Anas tags the vertices of a regular n-gon with the numbers from 1 to n, in any order he wants. Every vertex must be tagged with a different number. Then, we place a turkey in each of the n vertices. These turkeys are trained for the following. If Abdulrahman whistles, each turkey moves to the adjacent vertex with greater tag. If Abdulrahman claps, each turkey moves to the adjacent vertex with lower tag. Abdulrahman wins if, after some number of whistles and claps, he gets to move all the turkeys to the same vertex. Anas wins if he can tag the vertices so that Abdulrahman can't do this. For each $n \geq 3$, determine which player has a winning strategy.

problem 50) Argentina 2009 Consider an $a \times b$ board, with a and b integers greater than or equal to two. Initially all the squares are painted white and black as a chessboard. The allowed operation is to choose two unit squares that share one side and recolor them in the following way: Any white square is painted black, any black square is painted green and any green square is painted white. Determine for which values of a and b it is possible, using this operation several times, to get all the original black squares to be painted white and all the original white squares to be painted black.

problem 51) IMO 1994 C3 Aldubaisi has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Can Aldubaisi always transfer all his money into two accounts?

Hard

problem 52) We have n coins placed on one square of an infinite one-dimensional grid. In each move we select two consecutive squares such that one on the left has at least 2 coins more and move a coin from the left square to the right square. The process continues until no more moves can be made. Show that the final configuration of coins is independent of the sequence of moves made.

problem 53) On a blackboard n non-negative integer numbers are written. One is allowed to select two numbers from the board x and y such that $x \geq y$ and replace them with $x-y$ and $2y$. Determine for which n-tuples of initial numbers is it possible to increase the number of zeros on the blackboard to $n-1$.

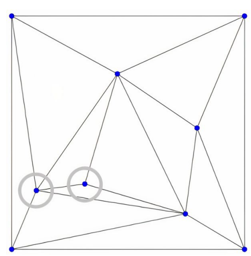
problem 54) IMO 1993 P3 The following game is played on an infinite chessboard. Initially, each cell of an $n \times n$ square is occupied by a chip. A move consists in a jump of a chip over a chip in a horizontal or vertical direction onto a free cell directly behind it. The chip jumped over is removed. Find all values of n, for which the game ends with one chip left over.

problem 55) Kyiv City MO 2022 2022 points are arranged in a circle, one of which is colored in black, and others in white. In one operation, Hamza can do one of the following actions:
1) Choose two adjacent points of the same color and flip the color of both of them (white becomes black, black becomes white)
2) Choose two points of opposite colors with exactly one point in between them, and flip the color of both of them
Is it possible for Hamza to achieve a configuration where one point is white and all other points are black?

problem 56) John Horton Conway In any way you please, fill up the lattice points below or on the x-axis by chips. By solitaire jumps can you get one chip to $(0, 5)$? A solitaire jump is a horizontal or vertical jump of any chip over its neighbor to a free point with the chip jumped over removed. For instance, with (x, y) and $(x, y + 1)$ occupied and $(x, y + 2)$ free, a jump consists in removing the two chips on (x, y) and $(x, y + 1)$ and placing a chip onto $(x, y + 2)$.

problem 57) USAMO 2008 Three non-negative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

problem 58) TOT spring 2003 A square is triangulated in a way similar to what is shown bellow. If a triangle touches another triangle or the square, they must either share exactly one vertex or they must share a whole edge and both surrounding vertices. Prove at least one vertex has an odd number of edges coming out of it.



problem 59) USAMO 2003 At the vertices of a regular hexagon are written six non-negative integers whose sum is 2003^{2003}. Ali-R is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Ali-R can make a sequence of moves, after which the number 0 appears at all six vertices.

problem 60) South Korea TST 2009 2008 white stones and 1 black stone are in a row. An 'action' means the following: select one black stone and change the color of neighboring stone(s). Find all possible initial position of the black stone, to make all stones black by finite actions.

problem 61) IMO 2005 C5 There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n-1$ is not divisible by 3.

Very Hard

problem 62) APMO 1997 P5 n people are seated in a circle. A total of nk coins have been distributed among them, but not necessarily equally. A move is the transfer of a single coin between two adjacent people. Find an algorithm for making the minimum possible number of moves resulting in everyone ending up with the same number of coins.

problem 63) IMO 2011 P2 Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

problem 64) IMO 1998 C7 A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

problem 65) IMO 2009 C5 Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Anas and Azzam go through a sequence of rounds: At the beginning of every round, Azzam takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Anas chooses a pair of neighboring buckets, empties them to the river and puts them back. Then the next round begins. Azzam's goal is to make one of these buckets overflow. Anas's goal is to prevent this. Can Azzam enforce a bucket overflow?

problem 66) IMO 2014 C7 Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

problem 67) IMO 2000 P3 Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$. Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

problem 68) IMO 2014 C9 There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Adeeb the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Adeeb always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*. Suppose that Adeeb's path entirely covers all circles. Prove that n must be odd.

problem 69) IMO 2017 P3 A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, $B0$ are the same. After $n-1$ rounds of the game, the rabbit is at point $An-1$ and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: i. he rabbit moves invisibly to a point An such that the distance between $An-1$ and A_n is exactly 1. ii. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. iii. The hunter moves visibly to a point Bn such that the distance between $Bn-1$ and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

problem 70) (Japan 2012, John Conway's "Angel problem") Tricky Omar is playing a game with Hamza. Omar begins by standing at the origin of the coordinate plane. Omar and Hamza take turns, starting with Hamza. On his turn, Hamza places lava on a lattice point of his choosing, preventing Omar from going there. Then on Omar's turn, Omar can move m times, each time going from the point (x, y) to the point $(x+1, y)$ or $(x, y+1)$ (but he can never go to a lattice point with lava). Hamza's goal is to make it so that Omar can't move. For which positive integers m can Hamza guarantee victory?

problem 71) Unsolved (Solve the problem for each positive integer n separately) $2n$ cells were marked on an infinite triangular grid, is it always possible to find a triangle (made by the grid lines) that contains exactly n marked cells?

6 Solutions

Some problems have more than one solution. This does not mean that these solutions are the only ones. Chase your own ideas and
you will probably find solutions that differ from the ones presented here and perhaps are even better.
There'll be some errors in the proofs, for which we take full responsibility.
Don't be surprised if you find a problem that seems too difficult compared to the other problems in its section; it is very difficult to rate a problem's difficulty. Even the IMO jury, now consisting of more than 75 highly skilled problem solvers, commits grave errors in rating the difficulty of the problems it selects. The over 500 IMO contestants are also an unreliable guide. Too much depends on the previous training by an ever-changing set of hundreds of trainers.
A problem changes from impossible to trivial if a related problem was solved in training.

There are several signs $+$ and $-$ on a blackboard. You may erase two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. Prove that the last sign on the board does not depend on the order of steps.

Solution 1

First solution: Replace each $+$ by $+1$ and each $-$ by -1, and form the product P of all the numbers. Obviously, P is an invariant and does not depend on the order of steps. Then the last sign on the board also does not depend on the order of steps. ■
Second solution: Note that after each step the parity of the number of $-$ signs is an invariant. If the number of $-$ signs are odd at the start then the last sign would be $-$, and If the number of $-$ signs are even at the start then the last sign would be $+$. ■

A hockey player has 3 pucks labeled A, B, C in an arena. He picks a puck at random, and fires it through the other 2. He keeps doing this. Can the pucks return to their original spots after 1987 hits?

Solution 2

No. If you read the pucks in clockwise order starting with A, you will get either ABC or ACB. And after each move it switches, so after an odd number of moves its position will be different from its original position. ■

Anas has a 7×7 board, such that 5 random cells are colored black and the rest of the squares (44) are colored white, in each move Anas chooses a 2×2 square inside the board and switches each of its 4 cells (from black to white or from white to black), can the board be completely colored white after a finite number of steps?

Solution 3

No.
Claim: the number of black squares will stay odd after every step.

Proof: the claim is obviously true in the beginning because we have 5 squares. If Anas chooses a 2×2 square with x black squares and $4 - x$ white squares, then the number of black square will decrease by $x - (4 - x) = 2x - 4 \equiv 0$ modulo 2, that means that the parity of the number of black squares remains the same after each step. That means that our claim is true! So there will be at least one black square. ■

There are three piles with n tokens each. In every step we are allowed to choose two piles, take one token from each of those two piles and add a token to the third pile. Using these moves, is it possible to end up having only one token?

Solution 4

No. At the start all piles have the same parity (modulo 2) and in each step we subtract 1 from two piles and add 1 to the other pile so the piles will still have the same parity (this is the invariant) but if we end up with 1 token then two piles will have an even parity and one pile will have an odd parity. Contradiction! ■

A circle is divided into six sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written into the sectors (counterclockwise) You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?

Solution 5

No. Suppose $s_1, s_2, s_3, s_4, s_5, s_6$ are the numbers currently on the sectors. Then $K = s_1 - s_2 + s_3 - s_4 + s_5 - s_6$ is an invariant. Initially $K = 2$ that means that the goal $K = 0$ cannot be reached. ■

There is one stone at each vertex of a square. We are allowed to change the number of stones according to the following rule: We may take away any number of stones from one vertex and add twice as many stones to the pile at one of the adjacent vertices. Is it possible to get 2022, 2021, 2023, and 2022 stones at consecutive vertices after a finite number of moves?

Solution 6

No. Suppose p_1, p_2, p_3, p_4 are the number of stones in each pile. Then $K = p_1 - p_2 + p_3 - p_4$ (modulo 3) is an invariant. Since the configuration $(1, 1, 1, 1)$ has the invariant equal to 0 and the configuration $(2022, 2021, 2023, 2022)$ has the invariant equal to 2, we can't reach our wanted configuration. ■

Several positive integers are written on a blackboard. Azzam can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

Solution 7

Step 1: The product of all numbers written on the blackboard is an invariant $= P$.
Step 2: Take S as the sum of all numbers written on the blackboard. Obviously, after each step S increases by at least 1. (unless Azzam erased two numbers $\{x, y\}$ such that x is a multiple of y or vise versa, *but in that case the numbers will not change*)

Step 3: If the numbers never change then S will never stop increasing, but from "step 1" we can claim that there should be a limit for the sum of numbers S. Contradiction. ■

You have a chocolate bar consisting of squares arranged in an $n \times m$ rectangular pattern. Your task is to split the bar into squares (always breaking one piece at a time along a line between the squares) with a minimum number of breaks. How many breaks do you need?

Solution 8

$mn - 1$, because Each break increases the number of pieces by 1, and you need to go from 1 piece to mn pieces. ■

Let $d(n)$ be the digital sum of $n \in N$. Solve: $n + d(n) + d(d(n)) = 2011$.

Solution 9

The transformation d leaves the remainder on division by 3 invariant. Hence, modulo 3 the equation has the form $0 \equiv 2$. There is no solution. ■

Consider a rectangular array with m rows and n columns whose entries are real numbers. It is permissible to reverse the signs of all the numbers in any row or column. Prove that after a number of these operations we can make the sum of numbers along each line (row or column) non-negative.

Solution 10

First solution: First, observe that there are a finite number of arrangements of numbers since there are only 2 possible numbers for each cell. Let S be the sum of all the numbers in the table. For each move, assume that there is one column or row that is negative (if not then we are done), and switch the signs of all the numbers in that row/column. Since the sum of the numbers in that row/column were originally negative, the sum now becomes positive, and S increases. Hence, for each move, S is strictly increasing. Furthermore, since there are a finite number of arrangements, after a finite number of moves, we will either reach a position where the sums of all the rows/columns are non-negative. ■

Second solution: Let S be the sum of all numbers in the cells of the table. From all the possible configurations (a configuration is the table after some steps) we will choose the configuration for which S is maximal (this is possible because there are a finite number of arrangements since there are only 2 possible numbers for each cell). Let's suppose that there is a column of that configuration with the propriety that the sum of the numbers inscribed in the column's cells is negative. Let $a_1, a_2, ..., a_n$ be the numbers from that column. Then $a_1 + a_2 + \cdots + a_n < 0$. In one step we change the signs of the numbers of that column.Then the new sum S' of all the numbers of that configuration is equal to $S - 2(a_1 + a_2 + \cdots + a_n) > S$, contradicting the maximality of S! Thus there is no column such that the sum of the number inscribed in the cells of that column is negative. In the same manner we can prove that there will be no row such that the sum of the numbers inscribed in the cells of that row is negative. ■

Seven vertices of a cube are marked by 0 and one by 1. You may repeatedly select an edge and increase by 1 the numbers at the ends of that edge. Can you reach:(a) 8 equal numbers, (b) 8 numbers divisible by 3

Solution 11

Select four vertices such that no two are joined by an edge. Let X be the sum of the numbers at these vertices, and let Y be the sum of the numbers at the remaining four vertices. Initially, $S = x - y = \pm 1$. A step does not change S. So neither (a) nor (b) can be attained. ■

Another solution for (a): Notice that the sum of all numbers is always odd, that means that we cannot reach 8 equal numbers. ■

There exist 3 ants in a two dimensional plane. Each minute one of the ants moves parallel to the line connecting the other two ants. If they start at $(0,0)(1,0)(0,1)$ can they eventually end at $(1,1)(2,2)(2,0)$?

Solution 12

No. Consider the area of the triangle made by the 3 ants after each move. Notice that it is an invariant. At the start the area is $1/2$ but it should end at 1. Contradiction. ■

There is a checker at point $(1,1)$ of the lattice (x,y) with x,y positive integers. It moves as follows. At any move it may double one coordinate, or it may subtract the smaller coordinate from the larger. Which points of the lattice can the checker reach?

Solution 13

Claim 1: We can't reach The point (x,y) if $gcd(x,y) \neq 2^k$.
Proof: The permitted moves either leave $gcd(x,y)$ invariant or double it.

Claim 2: We can reach any $(2^n a, 2^m b)$, $gcd(a,b) = 1$.
Proof:
step 1: $(1,1) \to (a,b)$.
lets go backwards, start with (a,b) and try to reach $(1,1)$ by adding the smaller coordinate to the bigger one and by dividing by 2.
(only even numbers. Otherwise, we would get fractions and this is not possible because a,b always $\in N$).
Method:
(1) Add the smaller one to the bigger one.
(2) Repeatedly divide the bigger number by 2 until it is odd.
(3) Repeat.
this method will always reach $(1,1)$ because if $a \neq b$ then $max(a,b)$ will always decrease (this is the monovariant) until $a = b = 1$ [Notice that $a \neq b$ because $gcd(a,b) = 1$].
step 2: $(a,b) \to (2^n a, 2^m b)$.
Method: Obviously, double a (n times) and double b (m times).

$\Rightarrow$ From "claim 1" and "claim 2":
The point (x,y) can be reached from $(1,1)$ iff $gcd(x,y) = 2^n$, $n \in N$.

We may extend a set S of space points by reflecting any point X of S at point A, $A \neq X$. Initially, S consists of the 7 vertices of a cube. Can you ever get the eight vertex of the cube into S

Solution 14

Place a coordinate system so that the seven given points have coordinates $(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,1,0),(1,0,1),(0,1,1)$. We observe that a point preserves the parity of its coordinates on reflection. Thus, we never get points with all three coordinates odd. Hence the point $(1,1,1)$ can never be reached. ■
This follows from the mapping formula $X \to 2A - X$, or in coordinates $(x,y,z) \to (2a-x, 2b-y, 2c-z)$, where $A=(a,b,c)$ and $X=(x,y,z)$. The invariant is the parity pattern of the coordinates of the points in S.

Alexa chooses a positive integer n and writes in a board $2n+1$ numbers $n, n/2, \ldots, n/2n+1$ Ali erases two numbers in the board a, b and writes, instead, $2ab-a-b+1$. Ali reapeats this operation until one number is left on the board, find all possibilities for this number.

Solution 15

The last number will always be $\frac{1}{2}$
First solution:
Claim: The number $\frac{1}{2}$ will always stay on the board.
Proof: Notice that the number $\frac{n}{2n} = \frac{1}{2}$ is initially written on the board. If Ali picks two numbers $(\frac{1}{2}, b)$ then after he erases them he will write, instead, $2\frac{1}{2}b - \frac{1}{2} - b + 1 = \frac{1}{2}$ so the number $\frac{1}{2}$ will remain! Thus, our claim has been proven. ■
Second solution: notice that $f(x) = (2x_1-1)(2x_2-1)\ldots(2x_{2n+1}-1)$ is invariant, such that the numbers x_i, $1 \leq i \leq 2n+1$ are the numbers written on the board. Initially $f(x) = (2n-1)(2\frac{n}{2}-1)\ldots(2\frac{n}{2n}-1)(2\frac{n}{2n+1}-1) \Rightarrow f(x) = (2k-1) = 0 \Rightarrow k = \frac{1}{2}$, such that k is the last number remaining on the board. ■
This solution may not seem natural, but it actually shows a deep understanding of the problem!

On a single pile there are 1001 stones. In a single move one is allowed to remove one stone from a pile and split the remaining stones on the pile into two non-zero piles of arbitrary amount. Is it possible to end up only with piles containing three stones?

Solution 16

No. Take S as the number of stones and P as the number of piles at a specific moment. $S+P$ is an invariant. At the start $S+P=1002$, in the end it should be on the form $3k+k=4k$ such that k is the number of piles. But $4k$ can't be equal to 1002. ■

In a 4×4 table suppose the bottom left corner's coordinate is $(1,1)$. the cell $(1,2)$ is colored black while every other cell in the table is colored white. in each step you may switch the color of all cells of a row, column, or a parallel to one of the diagonals. In particular, you may switch the color of each corner cell. Prove that at least one white cell will remain in the table.

Solution 17

the number of white squares in $(1,2)$, $(1,3)$, $(2,1)$, $(2,4)$, $(3,1)$, $(3,4)$, $(4,2)$, $(4,3)$ is always odd, this is the invariant. ■

$$\begin{bmatrix} \bullet & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \bullet & \bullet & \blacksquare \\ \blacksquare & \bullet & \bullet & \blacksquare \\ \bullet & \blacksquare & \blacksquare & \bullet \end{bmatrix}$$

Each of the numbers $a_1, \ldots, a_n$ is 1 or -1, and we have
$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0$ Prove that $4|n$.

Solution 18

First solution: If we replace any a_i by $-a_i$, then S does not change mod 4 since four cyclically adjacent terms change their sign. Indeed, if two of these terms are positive and two negative, nothing changes. If one or three have the same sign, S changes by ± 4. Finally, if all four are of the same sign, then S changes by ± 8. Initially, we have $S = 0$ which implies $S \equiv 0 \bmod 4$. Now, step-by-step, we change each negative sign into a positive sign. This does not change S mod 4. At the end (when all of them are positive), we still have $S \equiv 0 \bmod 4$, but also $S = n$, $\Rightarrow 4|n$. ■

Second solution: Take $b_i = a_ia_{i+1}a_{i+2}a_{i+3}$ such that $a_{n+j} = a_j$.

Claim 1: The number of $b_i = 1$ is equal to the number of $b_i = -1$.

proof: From that statement of the problem we get that $\sum_{i=1}^{n} b_i = 0$ this obviously proves our claim.

Claim 2: The number of $b_i = -1$ is even.

proof: $\prod_{i=1}^{n} b_i = (a_1a_2 \ldots a_n)^4 = 1$ and this obviously proves our claim.

From "claim 1" and "claim 2" we get that the number of b_i's is divisible by 4. $\Rightarrow 4|n$. ■

The numbers $1, 2, \ldots, 10$ are written on a board. Every minute, one can select three numbers a, b, c on the board, erase them, and write $\sqrt{a^2 + b^2 + c^2}$ in their place. This process continues until no more numbers can be erased. What is the largest possible number that can remain on the board at this point?

Solution 19

Claim 1: The sum of the squares of the last two numbers must be 385.

Proof: Let us take the numbers $a_1, a_2, \ldots, a_k$ and replace a_1, a_2, a_3. Then, clearly, our new sequence becomes:

$$\sqrt{a_1^2 + a_2^2 + a_3^2}, a_4, a_5, ..., a_k$$

in which the sum of the squares of the numbers is indeed the same. Therefore, at the end of the game, the sum of the squares of the remaining two numbers must be $1^2 + 2^2 + 3^2 + \ldots + 10^2 = 385$.

Claim 2: A bound on the largest number is $\sqrt{384} = 8\sqrt{6}$

Proof: Clearly, the smallest possible value for the square of a number on the blackboard is 1. Therefore, the a bound on the largest possible number is $\sqrt{385-1} = 8\sqrt{6}$.
Claim 3: It is possible to achieve $8\sqrt{6}$
Proof: Take the following sequence of steps:

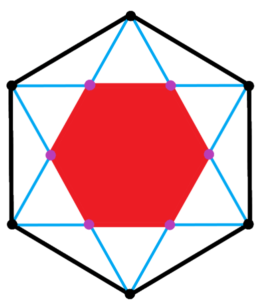
$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$$
$$\left(1, 2, 3, 4, 5, 6, 7, \sqrt{245}\right) \to \left(1, 2, 3, 4, 5, \sqrt{330}\right) \to \left(1, 2, 3, \sqrt{371}\right) \to \left(1, \sqrt{384}\right)$$

Therefore, our answer is $\boxed{8\sqrt{6}}$ ∎

To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments $AM, MN,$ and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $\frac{1}{3}$, for all $n \geq 6$.

Solution 20

We must always retain one point on each side of the original hexagon, so the result contains a hexagon "inscribed" in P_6 with area equal to $\frac{1}{3}$. ∎



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101 *wise men stand in a circle. Each of them either thinks that the Earth orbits Jupiter or that Jupiter orbits the Earth. Once a minute, all the wise men express their opinion at the same time. Right after that, every wise man who stands between two people with a different opinion from him changes his opinion. The rest do not change. Prove that at one point they will all stop changing opinions.*

Solution 21

For simplicity, let 0 and 1 be the people who think that the Earth orbits Jupiter and the people who think that Jupiter orbits the Earth respectively.
Now from the condition, if a number is different from both of its neighbors, then it changes in the next minute. We define a block of 0 to be at least 2 consecutive 0s in the circle. Define a block of 1 analogously. The key observation is
Claim 1: The numbers in a block of 0 will never change, and The numbers in a block of 1 will never change.

Proof: This is very simple. Indeed, a number changes if and only if it is different from both of its neighbors. However, each of the 0s in the middle of the block has two 0s as its neighbors. Therefore it will never change. On the other hand, each of the 0s at the side of the block has one 0 as its neighbors, so it never changes as well. Similarly, the numbers in a block of 1 will never change.
Now since there are 101 numbers which is an odd number, there must be a block initially.
Claim 2: After finitely many operations, the numbers can be decomposed into blocks.
Proof: It's suffices to show that there are no elements which do not belong to a block. We perform the following algorithm: We start with a block, WLOG assume that it is a block of 0. Consider the element next to it, then by definition it must be 1, for example:

$$100\ldots0$$

consider the element next to 1, if it is a 1, then there is a block of 1. If it is a 0 then in the next minute the 1 will become 0, that is

$$0100...0 \to 0000...0$$

Notice that this algorithm must terminate, and that its final state must be the state in which the numbers can be decomposed into blocks. ■

There is an integer in each square of an 8×8 chessboard. In one move, you may choose any 4×4 or 3×3 square and add 1 to each integer of the chosen square. Can you always get a table with each entry divisible by:
(a) 2
(b) 3

Solution 22

Solution for (a): No. Let S be the sum of all numbers except the third and sixth row. S modulo 2 is invariant. If $S \equiv 1 \pmod 2$ initially, then odd numbers will remain on the board. ■

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\blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare \\
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\blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare
\end{bmatrix}$$

Solution for (b): No. Let S be the sum of all numbers except the forth and eight row. S modulo 3 is invariant. If $S \equiv 1 \pmod 3$ initially, then there will always be numbers on the board which are not divisible by 3. ■

$$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \\ \blacksquare & \blacksquare & \blacksquare & \bullet & \blacksquare & \blacksquare & \blacksquare & \bullet \end{bmatrix}$$

The numbers $1, 2, \ldots, n$ are written, in each step Adeeb removes two numbers a, b and replaces them with $ab + a + b$. Will the last number remaining always be the same?

Solution 23

Yes. Take

$$S(x_1, x_2, x_3 \ldots x_n) = (x_1 + 1)(x_2 + 1)(x_3 + 1)\ldots(x_n + 1)$$

This value (S) is a constant, because WLOG if Adeeb changes a, b to $ab + a + b$ then S doesn't change, because $(a + 1)(b + 1) = (ab + a + b + 1)$. Initially, ($S$) is equal to $2^n \Rightarrow k + 1 = S = 2^n$, such that k is the last number remaining. Adeeb will always end with $2^n - 1$. ■

This solution is similar to the "second solution" in problem 15, and some of the following problems.

Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.
(a) Does the game necessarily end?
(b) Does there exist a winning strategy for the starting player?

Solution 24

Solution for (a): Yes. Label the gold cards with 1 and the black cards with 0. Then consider the binary number formed with 0's and 1's. Each move decreases this number, so the game has to end. ■

Solution for (b): No. Consider f as the number of gold cards in the cards numbered $10, 60, 110, \ldots, 1960$ (Now the cards are numbered serially from left to right). Initially $f = 40$. It is easy to note that after every move f either increases or decreases by 1. So after odd number of moves f is always odd and consequently ≥ 1. The second player always takes a turn after an odd number of moves, so he always finds a gold card among the set, and consequently, always has a move. Since the game has to end necessarily, the second player has a winning strategy. ■

Several positive integers are written in a row. Iteratively, Abdulrahman chooses two *adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that Abdulrahman can perform only finitely many such iterations*

Solution 25

First solution: Let the numbers on the line be $a_1, a_2, \ldots, a_n$. Consider the sum $S = a_1 + 2a_2 + \ldots + na_n$.

Claim: each operation increases S.

proof: Suppose we replace (x, y) with $(y + 1, x)$. As

$$nx + (n + 1)y < n(y + 1) + (n + 1)x \iff y < n + x$$

which is true as $x > y$, it follows that replacing (x, y) with $(y + 1, x)$ decreases S. Similarly, suppose we replace (x, y) with $(x - 1, x)$. As

$$nx + (n + 1)y < n(x - 1) + (n + 1)x \iff ny + y < n(x - 1) + x$$

which is true as $x - 1 \geq y$ and $x > y$. Hence, both operations strictly increase S. Now, consider the maximal element m in the initial sequence written on the board. It is easy to see that neither operation changes the maximal number on the board. Hence, we have $S \leq m + 2m + \ldots + nm$, which is finite. As S takes only integer values, it cannot grow without bound. Thus, the process terminates, as desired. ■

Second solution: By induction.

Basis: If we have 1 number no moves can be made.

Inductive steps: Suppose that any n numbers must terminate after a finite number of moves. Consider any set of $n + 1$ numbers and let M denote the maximum of all of the numbers. Notice that M will never be effected by an operation.

Claim: M can be operated on finitely many times.

proof: Every time we operate on M, it moves one to the right. But, it can only move right finitely many times. Thus, we can operate on M finitely many times. Proving the claim.

Final step: Suppose we stop operating on M. We are left with n numbers which terminates after a finite number of steps by our hypothesis. Completing the induction. ■

It is given 5 numbers 1, 3, 5, 7, 9. Each step we arbitrary take 4 numbers (out of current 5 numbers) a, b, c, d and replace them with
$\frac{a+b+c-d}{2}, \frac{a+b-c+d}{2}, \frac{a-b+c+d}{2}, \frac{-a+b+c+d}{2}$ *Can we, with repeated iterations, get numbers: 3, 4, 5, 6 and 7*

Solution 26

No. Consider the sum of the squares of the five numbers. Observe that $\left(\frac{a+b+c-d}{2}\right)^2 + \left(\frac{a+b-c+d}{2}\right)^2 + \left(\frac{a-b+c+d}{2}\right)^2 + \left(\frac{-a+b+c+d}{2}\right)^2 = a^2 + b^2 + c^2 + d^2$ this means that the sum of squares is invariant. Initially the sum of squares is 165, and it should reach 135. Contradiction. ■

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n numbers are written on a blackboard. In one step you may erase any two of the numbers, say a, b, and write, $(a + b)/4$. Repeating this step $n - 1$ times, there will be one

number left. Prove that, initially, if there were n ones on the board, at the end, a number, which is not less than $1/n$ will remain.

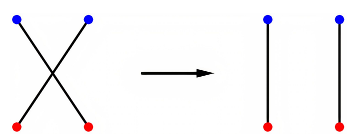
Solution 27

From the inequality $1/a + 1/b \geq 4/(a+b)$ which is equivalent to $(a+b)/2 \geq 2ab/(a+b)$, we conclude that the sum S of the inverses (switching the numerator and the denominator) of the numbers does not increase. Initially, we have $S = n$. Hence, at the end, we have $S \leq n$. For the last number $1/S$, we have $1/S \geq 1/n$. ■

Given n red points and n blue points in the plane, no three collinear, prove that we can draw n segments, each joining a red point to a blue point, such that no segments intersect.

Solution 28

First solution: Consider the greedy algorithm in which, given any two segments that intersect, we "un-cross" them. For example, if segments R_1B_1 and R_2B_2 intersect, we can uncross them into segments R_1B_2 and R_2B_1. Note that by the Triangle Inequality, the total length of all the segments decreases, so this algorithm must eventually terminate (because there are a finite number of pairings), which means that there are no more crossed segments.



Second solution: Define S as the sum of lengths of segments of a specific pairing. Because there are a finite number of pairings, we can take the pairing with the least S. Call this pairing X. If there are no two segments that intersect in X then we are done. Otherwise, take any two segments that intersect and "un-cross" them (same as in the first solution), creating pairing Y. Note that by the Triangle Inequality, the total length of all the segments decreases, so $S(X) > S(Y)$ but this contradicts out definition of X. $\Rightarrow$ there are no two segments that intersect in X. ■

Two distinct positive integers a, b are written on the board. The smaller of them is erased and instead of it the number $\frac{ab}{|a-b|}$ is written. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

Solution 29

Let us write numbers on a second board. Every time numbers x, y appear on the first board, we will write down numbers $\frac{ab}{x}$, $\frac{ab}{y}$ on the second board. Then when numbers x, y are on the first board, $x < y$, and the operation is performed, then on the other board the pair $(\frac{ab}{x}, \frac{ab}{y})$ will be replaced by $(\frac{ab}{y}, \frac{ab}{x} - \frac{ab}{y})$ This is just Euclid's algorithm, so eventually both numbers on the second board will be equal to $gcd(a, b)$. At that point both numbers on the first board will be equal. ■

We have 2^m sheets of paper, with the number 1 written on each of them. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we *erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.*

Solution 30

Consider the product of the 2^m numbers. Each step multiplies it by at least 4 since $(x+y)^2 \geq 4xy$ for all real x, y, so the product at the end of all the steps is at least

$$4^{m2^{m-1}} = 2^{m2^m}$$

By AM-GM, the sum is then at least

$$2^m \sqrt[2^m]{2^{m2^m}} = 2^m \cdot 2^m = 4^m \quad \blacksquare$$

Consider a $n \times n$ board. There is one token in each 1×1 cell. An operation is as follows - if we have two 1×1 cells of any rectangle with tokens; one on the upper left corner and the other one on the lower right corner, then these are changed to the lower left corner and to the upper right corner. Prove that eventually, after a finite number of operations all the tokens will be in the diagonal of the $n \times n$ board.

Solution 31

Numbering: Start with the bottom-left corner of the n by n square as $(1,1)$ and label up and right. Now label each square (ℓ, m) with $(\ell - m)^2$.

Claim 1: The sum of the labels of the squares the tokens are on is always decreasing after every move.

Proof: When we do the move with two tokens, we move them from, say $(x+m, y)$ and $(x, y+k)$ to $(x+m, y+k)$ and (x, y). So the sum of the labels of the tokens change from

$$(x+m-y)^2 + (x-k-y)^2 = 2(x-y)^2 + 2(x-y)m - 2(x-y)k + m^2 + k^2$$

to

$$(x-y)^2 + (x-y+m-k)^2 = 2(x-y)^2 + 2(x-y)(m-k) + (m-k)^2$$

so it always decreases by exactly $2mk$.

Claim 2: There's always a move available when the tokens aren't on the diagonal

Proof: Note that there's always n tokens in every row and in every column. If there's a token off the diagonal, say row a_1, column a_2, then there is at most $n-1$ tokens on spot (a_2, a_2), so that there's at least one token in row a_2 not on the diagonal, say row a_2, column a_3, and so on. In this way we form a sequence $a_1, a_2, a_3, \ldots a_k$ with no two consecutive a_i equal, and $(ai, ai+1)$ has a token (where indices are taken $\mod k$). Look at (one copy of) the maximum number among these ai: it must be greater than its two neighbors $ai-1$ and $ai+1$. So that means that the token on $(ai-1, a_i)$ is above and to the left of the token on $(ai, ai+1)$. Thus, our claim is proven.

$\Rightarrow$ There's always a move available when the tokens aren't on the diagonal, thus, this monovariant will always decrease, forcing the position to get closer to the diagonal in this sense, until eventually it must be on the diagonal, as we require. $\blacksquare$

A $(2n+1) \times (2n+1)$ board is colored in the chessboard fashion. One is in one move allowed to select a rectangle on the board and reverse all colors. Find the minimum number of moves needed to make the entire board be of the same color.

Solution 32

WLOG assume that the corners of the board are colored white. Call a vertex "good" if it is adjacent to an even number of black squares and "bad" otherwise. Initially we have $8n$ "bad" vertices (the vertices on the sides but not on the corners), and in each move 4 vertices are switched from "bad" to "good" or vise versa (the vertices on the corners of the rectangle). The "bad" vertices on the sides but not on the corners when the entire board is entirely the same color is 0. So we need at least $8n/4 = 2n$ moves. Example: switch the even rows and columns. ■

1994 boys are seated at a round table. Initially one boy holds n tokens. Each turn a boy who is holding more than one token passes one token to each of his neighbors. Show that if $n = 1994$ it cannot terminate.

Solution 33

Let the 1994 boys have labels $B_1, B_2, \ldots B_{1994}$. Assign a weight of i to each coin B_i. Let B_1 start with 1994 tokens originally. Notice that in each turn, the sum of the weights is invariant mod 1994. However, in order for the game to terminate, each boy must have exactly 1 token. The sum of the weights in this case is $1 + 2 + \cdots + 1994$, which is <u>not</u> congruent to 0 mod 1994. Hence, the game can never terminate. ■

The integers $1, \ldots, n$ are arranged in order. In one step you may take any four integers and interchange the first with the fourth and the second with the third. Prove that, if $n = 0, 1$ modulo 4, then by means of such steps you may reach the arrangement $n, n-1, \ldots, 1$. But if $n = 2, 3$ modulo 4, you cannot reach this arrangement.

Solution 34

Count the pairs (a, b) that are not in order with respect to the one we want to reach. Initially all pairs are in their correct order, but after each step an even number of pairs are switched. If $n = 2, 3[mod4]$ then the total number of pairs $(n)(n-1)/2$ is also odd, but in each step we change an even number of pairs so we get a contradiction. if $n = 0, 1[mod4]$ then we can reach $n, n-1, \ldots, 1$ by the following steps: split the n integers into pairs of neighbors (leaving the middle integer unmatched for odd n). Then form quadruplets from the first, last, second, second from behind, etc. ■

Hadi infects $n-1$ cells of a $n \times n$ board. And each minute, Hadi infects the cells with at least two infected neighbors (having a common side). Can Hadi eventually infect the whole board?

Solution 35

No. Denote S as the number of edges between an infected and uninfected square (consider all cells outside of the board uninfected). Initially $S \leq 4n - 4$. Notice that when a new cell gets infected, the number of edges between an infected and uninfected square dose not increase (this is the monovariant) $\Rightarrow S$ will always be at most $4n-4$, but for the whole board to get infected S should reach $4n$ (the perimeter of the board). Contradiction. ■

Given a permutation $(a_1, a_2, ..., a_n)$ of the numbers $1, 2, ..., n$ one may interchange any two consecutive "blocks" i.e.

$$(a_1, a_2, ..., a_i, \overbrace{a_{i+1}, ...a_{i+p}}^{A}, \overbrace{a_{i+p+1}, ..., a_{i+q}}^{B}, ..., a_n)$$

into

$$(a_1, a_2, ..., a_i, \overbrace{a_{i+p+1}, ..., a_{i+q}}^{B}, \overbrace{a_{i+1}, ...a_{i+p}}^{A}, ..., a_n)$$

by interchanging the "blocks" A and B. Find the least number of such changes which are needed to transform $(n, n-1, ..., 1)$ into $(1, 2, ..., n)$

Solution 36

Claim: The desired number of interchanges is $\lceil \frac{n+1}{2} \rceil$.
proof: Consider the number of pairs (a_i, a_{i+1}) such that $a_i < a_{i+1}$. Call such pairs good. In the initial sequence the number of good pairs is 0. In the final sequence the number of good pairs is $n-1$.
lemma 1: In the first interchange we can only increase the number of good pairs by 1.
Proof: Suppose that in the sequence $\ldots, k, x, \ldots, y, a, \ldots, b, m, \ldots$ we want to interchange blocks $x, \ldots, y$ and $a, \ldots, b$. Because we haven't made any interchanges yet, it holds $k > x > y > a > b > m$. After interchanging the two blocks, our sequence becomes $\ldots, k, a, \ldots, b, x, \ldots, y, m, \ldots$. The only good pair of numbers created is (b, x). Thus, our lemma is proven and so our claim has also been proven! ■

A circle has been cut into 2000 sectors. There are 2001 frogs inside these sectors. There will always be some two frogs in the same sector; two such frogs jump to the two sectors adjacent to their original sector (in opposite directions). Prove that, at some point, at least 1001 sectors will be inhabited.

Solution 37

Numbering: Number the sectors $1, 2, \ldots, 2000$.
Claim 1: Every sector will eventually be visited by a frog.
Proof: WLOG suppose a frog never goes in sector 2000. Take S as the sum of squares of the frog positions. Notice that after each step S increases [unless a jump occurred at secter 2000]. Thus, S will forever increase. Contradiction.
Claim 2: If a frog ever gets to sector n, then sectors n and $n+1$ can never again both be empty.
Proof: Obviously, when a frog goes to sector 2000 then it will stay there unless a jumped occurred in that sector, in that case a frog will jump to sector 2001, and that frog will stay there unless a jumped occurred in that sector, but then a frog will jump to sector 2000. And repeat!
Now, after each sector has been visited, we have two cases!

Case 1: There are two adjacent sectors which are both inhabited.
Solution: WLOG assume these are sectors $1, 2$. Now pair up sectors $i, i+1$ [for $i = 3, 5, 7, \ldots, 1999$]. Notice that from "claim 2" there will be at least one inhabited sector for each pair. That means that there are at least $2 + 999 = 1001$ sectors that are inhabited. ■
Case 2: There are no two adjacent sectors which are both inhabited.
Solution: Obviously, there are at most 1000 inhabited sectors. By The Pigeonhole Principle there will be at least one sector with at least 3 frogs. After two of the frogs from this sector jump, we will have three consecutive sectors that are inhabited. Notice that this is "Case 1". ■

Let n be a positive integer. Azzam and Adeeb play a game with $2n$ lamps numbered 1 to $2n$ from left to right. Initially, all lamps numbered 1 through n are on, and all lamps numbered $n+1$ through $2n$ are off. They play with the following rules, where they alternate turns with Azzam going first:
On Azzam's turn, he can choose two adjacent lamps i and $i+1$, where lamp i is on and lamp $i+1$ is off, and toggle both.
On Adeeb's turn, he can choose two adjacent lamps which are either both on or both off, and toggle both.
Players must move on their turn if they are able to, and if at any point a player is not able to move on his turn, then the game ends. Determine all n for which Adeeb can turn off all the lamps before the game ends, regardless of the moves that Azzam makes.

Solution 38

Claim 1: n is even.
Proof: Since, in every move the parity of the number of on/off lamps never changes, and in the end, there must be an even number of 'on' lamps (zero), which means there were an even number of 'on' lamps to begin with.
Claim 2: n is divisible by 4.
Proof: Consider the sum of the numbers on all the lamps that are on. Initially, it is $\frac{n(n+1)}{2}$. Suppose $4 \nmid n$. Then $\frac{n(n+1)}{2}$ is odd. In every move, this sum either increases by one (in Azzam's turn) or increases/decreases by $2k+1$ (in Adeeb's turn). Hence the sum always changes by an odd quantity. Initially it is odd, and in the end it must be even (zero), so an odd number of moves must be made before there are no 'on' lamps remaining, which means the last move made must be Azzam's, since Azzam makes moves on every odd turn. But this is a contradiction since Azzam cannot reduce the number of 'on' lamps. Hence $4|n$.
Strategy: On Adeeb's turn, Adeeb toggles the leftmost pair of adjacent on lamps unless the rightmost two lamps are both on, in which case he toggles those. If there exists no such pair, he toggles the rightmost pair of adjacent off lamps that is to the left of the first on lamp.
Proof that the game eventually ends: Consider an 'on' lamp as 1 and an 'off' lamp as 0. Notice that the binary number created always decreases.
Thus, Adeeb can turn off all the lamps for $4|n$. ■

Each term in the sequence $1, 0, 1, 0, 1, 0, 3\ldots$ is the sum of the last 6 terms modulo 10. Prove that the sequence $\ldots, 0, 1, 0, 1, 0, 1, \ldots$ never occurs.

Solution 39

$$V(x_1, x_2, ..., x_6) = 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 12x_6 \; mod \, 10$$

is the invariant. Starting with $V(1, 0, 1, 0, 1, 0) = 8$, the goal $V(0, 1, 0, 1, 0, 1) = 4$ cannot be reached. ■

Motivation: Guess $x_1a + x_2b + x_3c + x_4d + x_5e + x_6f \; (mod \, 10)$ as an invariant where a, b, c, d, e, f are consecutive six terms, $S = a + b + c + d + e + f$ and use the problem condition :

$$x_1a + x_2b + x_3c + x_4d + x_5e + x_6f \equiv x_1b + x_2c + x_3d + x_4e + x_5f + x_6S \; (mod \, 10)$$

So it's enough to make the coefficients congruent with $mod \, 10$. In other words, $x_1 \equiv x_6, x_1 + x_6 \equiv x_2$, and so on. Now just find any solution to this congruence equation.

In a 4×4 board the numbers from 1 to 15 are arranged in the following way:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

In a move we can move some number that is in a square sharing a side with the empty square to that square. Is it possible to reach the following position using these moves?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Solution 40

No. Count the pairs (a, b) of numbers $a < b$ that are not in order with respect to the one we want to reach. A horizontal movement of the empty space does not change the number of pairs. A vertical movement changes an odd number of pairs. However, since we want the empty space to end in its original position, the number of times it goes up must be equal to the number of times it goes down. Thus the number of vertical movements must be even. This means that the parity of the number of pairs must be preserved. Since it was 1 in the beginning, it cannot be 0 in the end. ■

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n squares in an infinite grid are colored black; the rest are colored white. When a square is the opposite color from 2 or more of its 4 neighbors, its color may be switched. Eventually, we get to having 2022 black squares, no two of which border along an edge, and all other squares white. Prove that $n \geq 2022$.

Solution 41

We will say an edge is "special" if it is between a white square and a black square. Every time a square changes color, then all four of its edges swap whether they are special or not. We are only allowed to change the color of a square if at least two of the neighbors are opposite color, which means we must have had at least 2 special edges around the square originally and at most 2 when done. Therefore, the number of special edges can never increase (this is the monovariant). In the final configuration, the number of special edges is exactly 4×2022. In the starting configuration, the number of boundary edges is at most $4n$, and so we must have $n \geq 2022$. ■

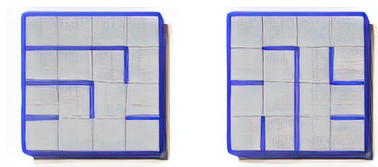
Omar discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. Omar has found a way to perform the following two kinds of operations with these particles, one operation at a time.
(i) If some imon is entangled with an odd number of other imons in the lab, then omar can destroy it.
(ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.
Prove that Omar may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

Solution 42

We'll go by **strong induction** on the number of vertices n.
Base case: $n = 1$ is obvious.
Inductive steps: Suppose we have an n-vertex graph. If any vertices have odd degree, remove it and use the inductive hypothesis. If all vertices have degree 0, we're already done.
Otherwise, all vertices have even degree, and we have to double the graph. Let the copy of a vertex v in the original be v'. All vertices of our doubled graph have odd degree. Start by arbitrarily deleting vertices from the original graph until we can't delete anymore. Let S be the set of the remaining vertices of the original graph.
Case 1: S is empty (we got rid of the entire original). Undo the last delete of a vertex, v. Delete v's copy v' instead. v is now an isolated vertex, so we can effectively ignore it and apply the inductive hypothesis to the $n - 1$ vertices that remain.
Case 2: S is nonempty. Each $v \in S$ currently has even degree, while the paired vertex v' still has odd degree. We'll say $v \in S$ is strange if its degree plus the degree of its copy is odd. Initially, S has all strange vertices. Now perform the following algorithm.
Algorithm:
1. Choose a strange vertex $v \in S$ not yet deleted.
2a. v has odd degree, then v' has even degree. Remove v, then remove v'.

2b. v has even degree, then v' has odd degree. Remove v', then remove v.
3. Notice that the degrees of all remaining vertices in S have changed by the same amount as their copies, so any strange vertices are still strange.
4. Go back to step 1.
It's easy to see that all vertices of S will be strange each time we get to step 1, so we'll in fact be able to delete all of them and their copies by this procedure. The end result is that we've removed $n + |S|$ vertices. Since S is nonempty, the inductive hypothesis finishes the problem. ■
Notice that the **strong induction** can be considered as a **monovariant** that always decreases. But it's usually easier writing the proof when we use strong induction.

A maze consists of a finite grid of squares where the boundary and some internal edges are "walls" that cannot be crossed. For example:



Two mazes are given, each with a robot in the top-left square. You may give a list of directions (up, down, left, or right) to the robots. Both robots will independently follow the same list of directions. For each direction, the robot will move one square in that direction if it can, or do nothing if there is a wall in the way. It will then proceed to the next direction, and repeat until it has gone through the whole list. Suppose that there is a list of directions that will get each robot individually from the top-left corner to the bottom-right corner of its maze. Prove there is also a list of directions that will get both robots to the bottom-right corner at the same time. In the example above, you could give the directions 'Right', 'Right', 'Down', 'Down', 'Down', 'Right', 'Down', 'Down', 'Left', 'Down', 'Right'.

Solution 43

Algorithm:
1. Give a minimal list of directions that will get "Robot 1" to the bottom right corner from its current position.
2. If "Robot 2" is not also in the bottom-right corner, give a minimal list of directions that will get it to the bottom right corner from its current position.
3. If "Robot 1" is no longer in the bottom-right corner, repeat from Step 1.
After each phase of the algorithm, we will have one robot R at the bottom-right corner while the other robot R' is x steps away for some x. During the next phase of the algorithm, we will take exactly x steps and move R' to the bottom-right corner. Note that R will end up below or right of where it started.
Let's consider what happened to R during this time. It started at the bottom-right corner and attempted x steps. If it never hit a wall, then it would have moved the same amount right and down as R' did and so in particular, it would also need to end up below or right of where it started. However, that is impossible since it started in the

bottom-right corner. Thus, R must have successfully taken at most $x-1$ steps during this phase, and so must now be at most $x-1$ steps from the bottom-right corner. Thus, x goes down during each phase (this is the monovariant). Since it is a non-negative integer, it must eventually reach 0 and we are done. ■

Let n be a given positive integer. Alzahrani performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Alzahrani chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Alzahrani's aim is to move all n stones to square n. Prove that Alzahrani cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

Solution 44

Numbering: Label the stones as $1, 2, \ldots, n$.
Each time Alzahrani moves, WLOG let him take the stone with the highest number (the stones are indistinguishable). Then stone i can only move by at most i each time (this is the monovariant); this means that we need at least $\lceil \frac{n}{i} \rceil$ moves. ■

On a blackboard the numbers $2, 3, 4$ are written, in each step Abdulrahman removes a and replaces it with $1/a(b+c)$. can he reach $1, 2, 3$ after a finite number of steps?

Solution 45

No. Take

$$S = ab + bc + ca + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

as an invariant. At the start $S = \frac{77}{6}$ but it should end at $S = \frac{121}{12}$ and this is a contradiction. ■

The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, $, \ldots, ,$ red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, $, \ldots, ,$ red, yellow, blue?

Solution 46

Label red, blue, yellow as $1, 0, -1$ respectively. So, from $1 \to 99$, we have a chain of

$$\underbrace{1, 0, 1, 0, \ldots, 1, 0}_{\text{98 vertices}}, -1$$

say in the clockwise direction. We compute the difference of each side in the clockwise direction; that is, we have differences of

$$\underbrace{1, -1, 1, -1, \ldots, 1, -1}_{\text{96 sides}}, 1, 1, -2$$

which have signs of

$$\underbrace{+,-,+,-,\ldots,+,-}_{\text{96 signs}},+,+,-.$$

Upon a move, the number of each sign does not change; no moves are ever made on a vertex who is surrounded by two different colors because then there would be two adjacent sides of the same color violating the problem's constraint.

$$1,0,1 \to 1,-1,1 \iff +,- \to +,-$$

$$0,1,0 \to 0,-1,0 \iff -,+ \to +,-$$

$$-1,1,-1 \to -1,0,-1 \iff -,+ \to -,+$$

and vice versa. Our original color configuration, when number-coded, has 50 sides whose clockwise difference is positive. The final, expected color configuration, when number coded, is

$$\underbrace{1,0,1,0,\ldots,1,0}_{\text{96 vertices}},1,-1,0$$

and has clockwise side differences of

$$\underbrace{1,-1,1,-1,\ldots,1,-1}_{\text{96 sides}},2,-1,-1$$

of which there are 49 positive numbers. $50 \neq 49$ so we can't reach our goal. ■

The numbers from 1 through 2008 are written on a blackboard. Every second, Hamza erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

Solution 47

Notice that the sum and the sum of squares is an invariant

$$a+b+c+(a+b+c) = (a+b)+(b+c)+(c+a)$$

$$a^2+b^2+c^2+(a+b+c)^2 = (a+b)^2+(b+c)^2+(c+a)^2$$

We see that after k seconds, by Cauchy-Schwarz,
$2017-k \geq \frac{(\sum a_i)^2}{\sum a_i^2} = \frac{4141847733409}{2737280785} \approx 1513.12 \Rightarrow k < 504$ ■

On an infinite (in both directions) strip of squares, indexed by the integers, are placed several stones (more than one may be placed on a single square). We perform a sequence of moves of one of the following types:
(a) Remove one stone from each of the squares $n-1$ and n and place one stone on square $n+1$.
(b) Remove two stones from square n and place one stone on each of the squares $n+1$, $n-2$.

Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves

Solution 48

Notice that this is the Fibonacci sequence.
Solution for (a): Give a stone in square k weight φ^k with $\varphi = \frac{1+\sqrt{5}}{2}$, so that the total weight of all stones remains invariant due to the relation $\varphi^k = \varphi^{k-1} + \varphi^{k-2}$. Suppose the process never terminates, from which there exists at least one square that is operated on infinitely many times. Consider the square of least value that is operated on finitely many times, and note that the square to the left of it is operated on infinitely, so arbitrarily many stones build up on the former square, so it must be operated on infinitely, contradiction. Thus all squares are operated on infinitely, so the index of the rightmost square that contains a stone grows arbitrarily large. At some point, the weight of the stone in that square exceeds the initial total weight, contradicting the invariant. ■
Solution for (b): Note that the initial total weight W is invariant throughout the process. Then when the algorithm terminates, each stone is on a square such that no two consecutive squares have a stone and at most one stone is on each square. So it suffices to show for any good A, B satisfying

$$\sum_{i \in A} \alpha^i = \sum_{i \in B} \alpha^i$$

we must have $A = B$. Assume contrary. Choose the largest k which lies in exactly one of A, B, say A. Then we would obtain

$$\sum_{k > i \in B} \alpha^i < \alpha^{k-1} + \alpha^{k-3} + \alpha^{k-5} + \cdots = \alpha^k$$

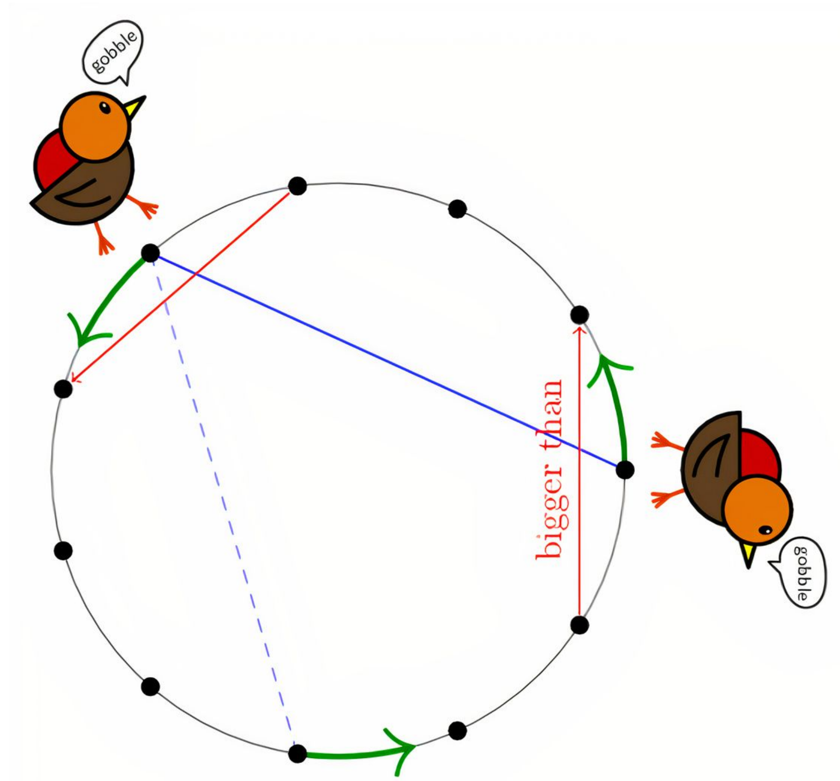
So we obtain our desired contradiction. ■

Let $n \geq 3$ be an integer. Two players, Anas and Abdulrahman, play the following game. Anas tags the vertices of a regular n-gon with the numbers from 1 to n, in any order he wants. Every vertex must be tagged with a different number. Then, we place a turkey in each of the n vertices. These turkeys are trained for the following. If Abdulrahman whistles, each turkey moves to the adjacent vertex with greater tag. If Abdulrahman claps, each turkey moves to the adjacent vertex with lower tag. Abdulrahman wins if, after some number of whistles and claps, he gets to move all the turkeys to the same vertex. Anas wins if he can tag the vertices so that Abdulrahman can't do this. For each $n \geq 3$, determine which player has a winning strategy.

Solution 49

The answer is that Abdulrahman wins if n is an odd prime, and otherwise Anas wins.
Case 1: If n is even, Anas wins no matter what the tagging is, because when coloring the vertices of the n-gon in alternating colors, we find that the number of turkeys on one color always equals the number of turkeys of the other color.
Case 2: If n is odd.
Fact: For Abdulrahman to win, it is necessary and sufficient for Abdulrahman to be able to merge any two given turkeys.

Notations: Given two turkeys with odd n, there is a notion of even distance d between them (by drawing the arc of even length). Hence one of left or right turkey. Note that:
• There is always a choice of move that moves the right turkey counterclockwise; hence it cannot decrease the even distance. We call this a safe move.
• Abdulrahman could try to decrease the even distance by repeatedly applying safe moves. If at any point one of the safe moves causes the left turkey to move clockwise (rather than counterclockwise), the even distance will decrease. If he is always able to do this, for any $d > 0$, he can eventually merge the turkeys.
• On the other hand, suppose there is an even distance d for which safe moves do not ever decrease the distance. Then both turkeys will traverse the entire circle.
With that notation, let's work through both situations. In what follows, the vertices are labeled A_0, A_1, ..., A_{n-1} in counterclockwise order, indices modulo n.
Proof that Abdulrahman wins whenever n is prime: Suppose for contradiction that there is an even distance d such that Abdulrahman cannot get two turkeys of even distance d closer.
Consider any two vertices, say A_0 and A_d which are d apart. WLOG, suppose that clapping is a safe move (in other words, $A_{n-1} > A_1$). Then the clap makes the other turkey move away, meaning that $A_{d-1} > A_{d+1}$. This situation is illustrated for $d = 4$ below, with green arrows marking the clap direction.



Now imagine the turkeys are situated at A_d and A_{2d}. We have seen that clapping is the safe move for A_d; hence clapping is the safe move for A_{2d} as well. Repeating this argument, and noting that $d < n$ means $\gcd(d, n) = 1$, we find that clapping is the safe move at every vertex. And that means we have the inequality chain

$$A_{n-1} > A_1 > A_3 > A_5 > \dots$$

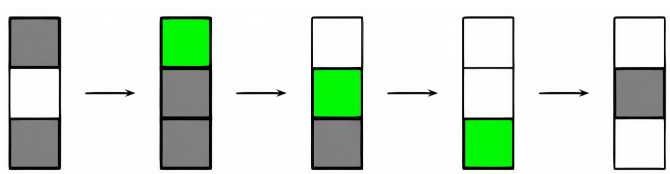
but this chain is cyclic and we get a contradiction. ∎

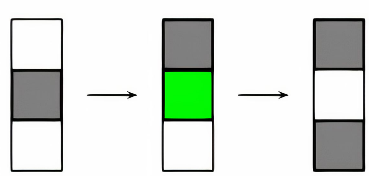
Proof that Abdulrahman loses whenever n is composite: On the other hand if n is an odd composite number, let p be the smallest prime divisor of n, and focus on $d = 2 \cdot \frac{n}{p}$. Then label $A_1, A_3, \dots, A_{n-2}, A_n, A_2, \dots, A_{2n-2}$ as the following numbers, in order: $1, p+1, 2p+1, \dots, n-p+1$ $2, p+2, 2p+2, \dots, n-p+2$ $l \dots$ $p, 2p, 3p, \dots, n$. Then, every $\frac{n}{p}$'th vertex will have safe move 'whistle', and the rest will have safe move 'clap'. This completes the solution. ∎

Consider an $a \times b$ board, with a and b integers greater than or equal to two. Initially all the squares are painted white and black as a chessboard. The allowed operation is to choose two unit squares that share one side and recolor them in the following way: Any white square is painted black, any black square is painted green and any green square is painted white. Determine for which values of a and b it is possible, using this operation several times, to get all the original black squares to be painted white and all the original white squares to be painted black.

Solution 50

Consider the set X of squares which were originally white and the set Y of squares that were originally black. Assign a 0 to each white square, a 1 to each black square and a 2 to each green square. Note that each time we are applying the operation we are using one square of X and one of Y . Thus the sum modulo 3 of the numbers assigned to the squares in X grows by 1 after each operation, and so does the sum of the squares in Y . We also want each square in X to be 1 at the end, so if x is the number of squares of X, the number of operations we need is congruent to x modulo 3. Since we want each square of Y to end up with 0, the corresponding numbers must go down by 1 modulo 3 each. If there were y squares in Y originally, the number of moves has to be congruent to $-y$ modulo 3. That is, $x \equiv -y \pmod 3$, which is the same as $x + y \equiv 0 \pmod 3$. But $x + y$ is the total number of squares, so we need one of the numbers a, b to be a multiple of 3. If any of a, b is a multiple of 3 we can divide the board into 3×1 strips. In each of these we can swap the colors in the following way:





We conclude that the change can be done iff one of a and b is a multiple of 3. ■

Aldubaisi has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Can Aldubaisi always transfer all his money into two accounts?

Solution 51

WLOG let the three accounts in the bank belong to Aldubaisi's sons: "$Omar$", "$Anas$" and "$Adeeb$" WLOG suppose $Adeeb \geq Anas \geq Omar$ we claim that we can always operate such that min $(Omar, Anas, Adeeb)$ Decreases (monovariant). Write $Anas$ in terms of $Omar$. $Anas = Omar(X_1 + 2X_2 + \ldots + 2^k X_k) + r$ such that $Omar > r$ and for all X_k, $X_k = 0$ or 1. Note that Anas can give Omar a power of two at every turn (working up from $1, 2, 4, \ldots, 2^k$). Say Anas has $X_k = 0$, then Adeeb can give $Omar(2^k X_k)$ to Omar. This will ensure that Anas will become reduced to r. Note now, min $(Omar, Anas, Adeeb) = r < Omar$ Now repeat this process until it terminates: When the min is 0. ■

We have n coins placed on one square of an infinite one-dimensional grid. In each move we select two consecutive squares such that one on the left has at least 2 coins more and move a coin from the left square to the right square. The process continues until no more moves can be made. Show that the final configuration of coins is independent of the sequence of moves made.

Solution 52

Notice that the number of stones in each square is non-increasing from left to the right.
Claim 1: There cannot be three equal squares at any moment.
Proof: What move results in this configuration? Obviously none, because when we have five squares a, b, c, d, e and b, c, d are equal then if the move leading to this configuration was between:

- $(a, b) \rightarrow b$ should have been smaller than c. Contradiction.
- $(b, c) \rightarrow c$ should have been smaller than d. Contradiction.
- $(c, d) \rightarrow b$ should have been smaller than c. Contradiction.
- $(d, e) \rightarrow c$ should have been smaller than d. Contradiction.

Claim 2: We cannot have two consecutive squares with difference larger than 1 in the final configuration.
Proof: We can obviously make a move between these two squares (thus, it could not have been the final configuration).
Claim 3: We cannot have two pairs of equal squares in the final configuration.
Proof: To reach:

$$\ldots, \mathbf{x,x}, \underbrace{x-1, \ldots, y+1}_{x-y-1}, \mathbf{y,y}, \ldots$$

the previous configuration should have been either:

$$\ldots, x+1, \mathbf{x\text{-}1,x\text{-}1}, \underbrace{x-2, \ldots, y+1}_{x-y-2}, \mathbf{y,y}, \ldots$$

or

$$\ldots, \mathbf{x,x}, \underbrace{x-1, \ldots, y+2}_{x-y-2}, \mathbf{y+1,y+1}, y-1\ldots$$

Notice the the 'gap' always decreases (towards the past) until we reach three equal squares (which we cannot reach because of 'claim 1').
From 'claim 1,2,3' the final configuration has differences of $1, 0$ and at most one difference if 0. Thus, the final configuration can only be:

$$x, x-1, \ldots, y+1, y, y, y-1, \ldots, 2, 1$$

if $n = \frac{(x)(x+1)}{2} + y$ such that $y \leq x$. ■

On a blackboard n non-negative integer numbers are written. One is allowed to select two numbers from the board x and y such that $x \geq y$ and replace them with $x-y$ and $2y$. Determine for which n-tuples of initial numbers is it possible to increase the number of zeros on the blackboard to $n-1$.

Solution 53

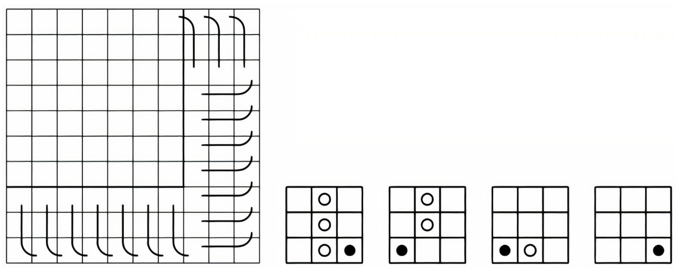
Call $(x_1, \ldots, x_n)$ 'good' if it can be reduced to $(x_1 + x_2 + \cdots + x_n, 0, \ldots, 0)$.
$(x_1, \ldots, x_n)$ is good iff $(Cx_1, Cx_2, \ldots, Cx_n)$ is good for $C \in \mathbb{N}$.
Thus, we can assume $\text{GCD}(x_1, \ldots, x_n) = 1$.
If odd prime $p | x-y, 2y \Rightarrow p | x, y$
$\Rightarrow p$ dividing all the numbers on the board is invariant!
$\Rightarrow$ If $p | x_1 + x_2 + \cdots + x_n \Rightarrow p | x_1, \ldots, x_n \Rightarrow p | 1$ since $\text{GCD}(x_1, \ldots, x_n) = 1$.
If $(x_1, \ldots, x_n)$ is good then $x_1 + x_2 + \cdots + x_n = 2^k$
Claim: If $x_1 + x_2 + \cdots + x_n = 2^k$ then $(x_1, \ldots, x_n)$ is good
Proof: By induction.
Base case: $k = 1$ is trivial $x_1 = 1$, $x_2, \ldots, x_n = 0$.
Inductive steps: Assume the claim is true for $k-1$.
$x_1 + x_2 + \cdots + x_n = 2^k$ for $(x_1, \ldots, x_n) \rightarrow$ there is an even number of x_i.
Pair up the odd numbers: x, y odd $\rightarrow x-y, 2y$ even.
From $(x_1, \ldots, x_n)$ we can obtain 'good' $(2y_1, 2y_2, \ldots, 2y_n)$ iff $(y_1, y_2, \ldots, y_n)$ is good, but $y_1 + y_2 + \cdots + y_n = 2^{k-1}$ so $(y_1, y_2, \ldots, y_n)$ is good by the inductive hypothesis.
$\Rightarrow (x_1, \ldots, x_n)$ is good iff $x_1 + x_2 + \cdots + x_n = 2^k \text{GCD}(x_1, \ldots, x_n) = 1$ for some $k \in \mathbb{N}_0$. ■

The following game is played on an infinite chessboard. Initially, each cell of an $n \times n$ square is occupied by a chip. A move consists in a jump of a chip over a chip in a horizontal or vertical direction onto a free cell directly behind it. The chip jumped over is removed. Find all values of n, for which the game ends with one chip left over.

Solution 54

The following picture shows how to reduce an L-tetromino occupied by chips to one square by using one free cell which is the reflection of the black square at the center of the first horizontal square. Applying this operation repeatedly we can reduce any $n \times n$ square to a 1×1, 2×2, or 3×3 square. A 1×1 square is already done. It is trivial to see how we can reduce a 2×2 square to one occupied square.

The reduction of a 3×3 square to one occupied square does not succeed. We are left with at least two chips on the board. But maybe another reduction not necessarily using L-tetrominoes will succeed. To see that this is not so, we start with any n divisible by 3, and we color the $n \times n$ board diagonally with three colors A, B, C. Denote the number of occupied cells of colors A, B, C by a, b, c, respectively. Initially, $a = b = c$, i.e., $a \equiv b \equiv c$ mod 2. That is, all three numbers have the same parity. If we make a jump, two of these numbers are decreased by 1, and one is increased by 1. After the jump, all three numbers change parity, i.e., they still have the same parity. Thus, we have found the invariant $a \equiv b \equiv c \bmod 2$. This relation is violated if only one chip remains on the board. ■



2022 points are arranged in a circle, one of which is colored in black, and others in white. In one operation, Hamza can do one of the following actions:
1) Choose two adjacent points of the same color and flip the color of both of them (white becomes black, black becomes white)
2) Choose two points of opposite colors with exactly one point in between them, and flip the color of both of them
Is it possible for Hamza to achieve a configuration where one point is white and all other points are black?

Solution 55

No. Label the positions of the points consecutively $1, -1, 1, -1, \dots$ and let $\ell_i, i = 1, \dots, 2022$ be the values of these labels. Set $c_i = 1$ if the color of the i-th point is white, otherwise $c_i = -1$. Consider $S = \sum_{i=1}^{2022} \ell_i c_i$. This value is invariant under the allowed recolorings. But S is different for the initial and desired configurations. ■

In any way you please, fill up the lattice points below or on the x-axis by chips. By solitaire jumps can you get one chip to $(0, 5)$? A solitaire jump is a horizontal or vertical jump of any chip over its neighbor to a free point with the chip jumped over removed. For instance, with (x, y) and $(x, y + 1)$ occupied and $(x, y + 2)$ free, a jump consists in removing the two chips on (x, y) and $(x, y + 1)$ and placing a chip onto $(x, y + 2)$.

Solution 56

No. We can get a chip to $(0, 4)$, but not to $(0, 5)$.
We introduce the 'norm' of a point (x, y) as follows: $n(x, y) = |x| + |y - 5|$.
We define the 'weight' of that point by α^n, where α is the positive root of

$\alpha^2 + \alpha - 1 = 0$. The weight of a set S of chips will be defined by

$$W(S) = \sum_{p \in S} \alpha^n$$

Cover all the lattice points for $y \leq 0$ by chips. The weight of the chips with $y = 0$ is $\alpha^5 + 2\alpha^4$. By covering the half plane with $y \leq 0$, we have the total weight

$$(\alpha^5 + 2\alpha^4)(1 + \alpha + \alpha^2 + \dots) = \frac{\alpha^5 + 2\alpha^4}{1 - \alpha} = \alpha^3 + 2\alpha^2 = 1$$

We make the following observations: A horizontal solitaire jump toward the y-axis leaves total weight unchanged. A vertical jump up leaves total weight unchanged. Any other jump decreases total weight. The weight of $(0, 5)$ is 1 And the Initial weight is also 1. thus, for us to reach $(0, 5)$ we would have to use all of our chips. Hence, the goal cannot be reached by finitely many chips. ■

Three non-negative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

Solution 57

Case 1: Two of the a_i vanish.
WLoG, say a_2 and a_3, then r_1 must be zero and we are done. ■
Case 2: Any one a_i vanishes.
WLoG, say a_3, then $\frac{r_2}{r_1} = -\frac{a_1}{a_2}$ is a non-negative rational number. Write this number in lowest terms as $\frac{p}{q}$, and put $r = \frac{r_2}{p} = \frac{r_1}{q}$. We can then write $r_1 = qr$ and $r_2 = pr$. Performing the Euclidean algorithm on r_1 and r_2 will ultimately leave r and 0 on the blackboard. Thus we are done again. ■
Case 3: None of the a_i vanishes.
In this case we will show that we can perform an operation to obtain r'_1, r'_2, r'_3 for which either one of r'_1, r'_2, r'_3 vanishes, or there exist integers a'_1, a'_2, a'_3, not all zero, with $a'_1r'_1 + a'_2r'_2 + a'_3r'_3 = 0$ and

$$|a'_1| + |a'_2| + |a'_3| < |a_1| + |a_2| + |a_3|.$$

After finitely many steps we must arrive at a case where one of the a_i vanishes, in which case we finish as above.
Case 3a: Two of the r_i are equal.
Then we are immediately done by choosing them as x and y. ■
Case 3b: $0 < r_1 < r_2 < r_3$.
Since we are free to negate all the a_i, we may assume $a_3 > 0$. Then either $a_1 < -\frac{1}{2}a_3$ or $a_2 < -\frac{1}{2}a_3$ (otherwise $a'_1r'_1 + a'_2r'_2 + a'_3r'_3 > (a_1 + \frac{1}{2}a_3)r_1 + (a_2 + \frac{1}{2}a_3)r_2 > 0$). WLoG, we may assume $a_1 < -\frac{1}{2}a_3$. Then choosing $x = r_1$ and $y = r_3$ gives the triple

$(r_1', r_2', r_3') = (r_1, r_2, r_3 - r_1)$ and $(a_1', a_2', a_3') = (a_1 + a_3, a_2, a_3)$. Since $a_1 < a_1 + a_3 < \frac{1}{2}a_3, -a_1$, we have $|a_1'| = |a_1| + |a_3| < |a_1|$ and hence this operation has the desired effect. ■

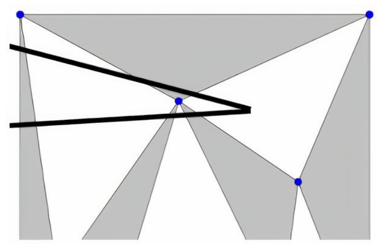
A square is triangulated in a way similar to what is shown bellow. If a triangle touches another triangle or the square, they must either share exactly one vertex or they must share a whole edge and both surrounding vertices. Prove at least one vertex has an odd number of edges coming out of it.

Solution 58

We give a proof by contradiction. Suppose that every vertex has an even number of edges coming out of it.

Claim: The triangles (and the outside region) can be colored white and black so that each triangle is adjacent only to triangles of the opposite color.

Proof: To see this, let's consider some point P inside the square. Draw a line segment L from P to the outside of the square and not going through any vertices. Define $f(L)$ to be the number of edges that L passes through. Now consider what happens if you rotate L slightly about P, as shown in the picture.



If you do not rotate L across a vertex, then L continues to intersect the same edges and so $f(L)$ stays constant. If you do cross a vertex, then L will start by touching some of the edges adjacent to the vertex, and end by touching the other edges adjacent to the vertex. This may cause $f(L)$ to change, but since all vertices have an even number of edges coming out, the parity of $f(L)$ will stay the same. So for every line L going from P to the outside, either $f(L)$ is even or $f(L)$ is odd. We will color P white in the first case and black in the second case. A very similar argument shows that if we move P around inside a single triangle, the color of P doesn't change. Therefore, we really have given a consistent coloring to each triangle. Finally, if two triangles share an edge, choose points X and Y on opposite sides of that edge such that no vertices lie on line XY. Then, if Z is a point outside the square on line XY, we have $f(XZ) = f(YZ) \pm 1$. This means X and Y have different colors, and so the coloring does indeed have the desired property! And now, we are almost done. Let us suppose there are B black triangles and W white triangles. Every edge must be adjacent to one black triangle and either one white triangle or the outside region. Therefore, the total number of edges must equal both $3B$ and $3W + 4$. Thus, $3B = 3W + 4$, which is impossible! Contradiction. ■

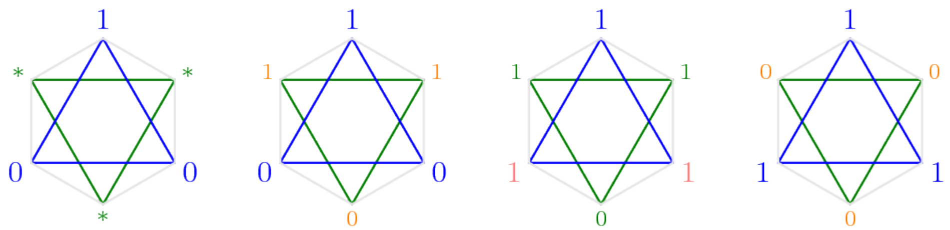
At the vertices of a regular hexagon are written six non-negative integers whose sum is 2003^{2003}*. Ali-R is allowed to make moves of the following form: he may pick a vertex* *and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Ali-R can make a sequence of moves, after which the number 0 appears at all six vertices.*

Solution 59

If $a \le b \le c$ are odd integers, the configuration which has $(a, b-a, b, c-b, c, c-a)$ around the hexagon in some order (up to cyclic permutation and reflection) is said to be great (picture below).

Claim 1: We can reach a great configuration from any configuration with odd sum.

Proof: We should be able to find an equilateral triangle whose vertices have odd sum. If all three vertices are odd, then we are already done. Otherwise, operate as in the following picture (modulo 2).

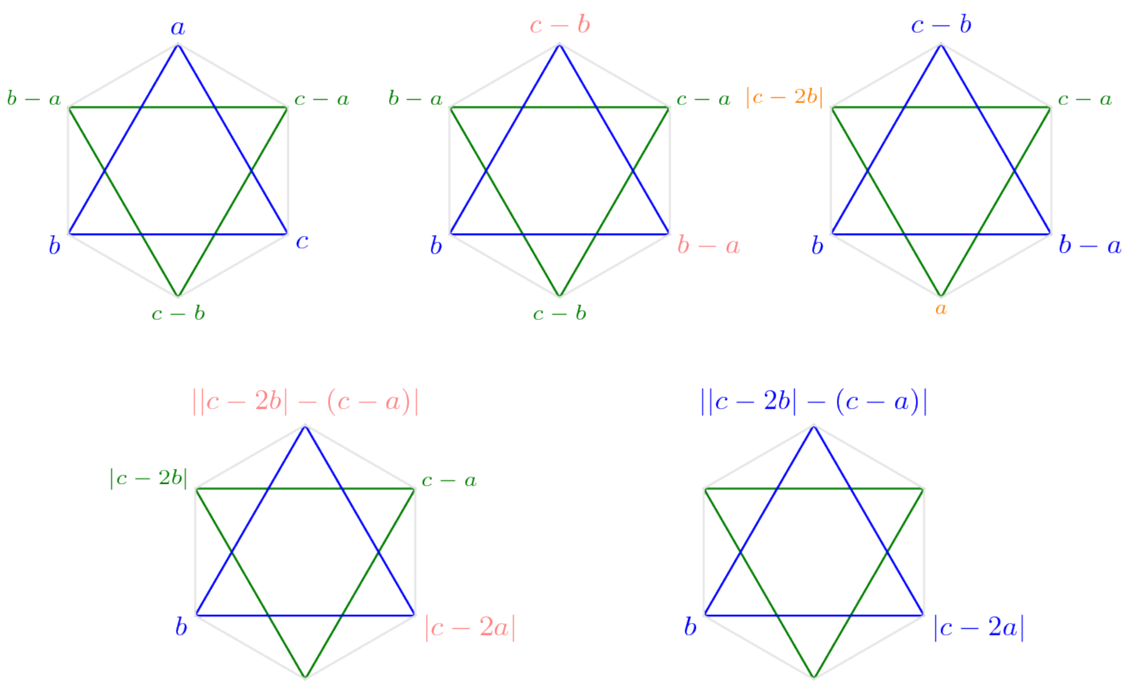


Thus we arrived at a great configuration.

Claim 2: Ali-R's goal is possible for all great configurations.

Proof: If $a = b = c$ then we have $(t, 0, t, 0, t, 0)$ which is obviously winnable.

Otherwise, perform six moves as shown in the diagram to reach a new great configuration whose odd entries are b, $|c-2a|$, $||c-2b|-(c-a)|$ (and perform three more moves to get the even numbers). The idea is to show the largest odd entry has decreased.



This is annoying, but straightforward. Our standing assumption is $a \neq c$ (but possibly $b = c$). It's already obvious that $|c - 2a| < c$, so focus on the last term. If $c > 2b$, then $|(c - 2b) - (c - a)| = |2b - a| < c$ as well for $a \neq c$. When $c \leq 2b$ we instead have $|(2b - c) - (c - a)| \leq \max(2b - c, c - a)$ with equality if and only if $c - a = 0$; but $2b - c \leq c$ as needed. Thus, in all situations we have

$$c \neq a \implies \max(||c - 2b| - (c - a)|, |c - 2a|) < c.$$

Now denote the new odd entries by $a' \leq b' \leq c'$ (in some order). If $b < c$ then $c' < c$, while if $b = c$ then $c' = b$ but $b' < c = b$. Thus (c', b', a') precedes (c, b, a) lexicographically, and we can induct down. ■

2008 white stones and 1 black stone are in a row. An 'action' means the following: select one black stone and change the color of neighboring stone(s). Find all possible initial position of the black stone, to make all stones black by finite actions.

Solution 60

Let the stones be $a_1, a2.....a2009$ let $f(i)$ be the number of black stones to the right of the i'th stone, define g as follows: $g(a_i) = 0$ iff a_i is black $g(a_i) = (-1)^{f(i)}$ iff a_i is white note that $S = \sum a_i$ is invariant so the only possible initial position is

$$\overbrace{www\ldots www}^{1004}b\overbrace{www\ldots www}^{1004} \quad ■$$

There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that, by a finite sequence of such steps, one can achieve a state with only two markers remaining if and only if $n - 1$ is not divisible by 3.

Solution 61

Inductive Construction: Let k be a positive integer. We provide an inductive construction for $n = 3k + 2$ and $n = 3k$.
Note that the base cases $n = 2$ and $n = 3$ are trivial. Denote a white marker by W, and a black marker by B. The original state is

$$WWWWW\ldots$$

We will prove that we can reduce the number of white markers by 3 and still keep all the markers white. This is achievable as follows:

$$WWWWW\cdots \to BBWW\cdots \to BWB\cdots \to WW\ldots$$

completing the induction.

Proof for $3|n-1$: We set up an invariant. For each marker located x places from the left, consider the function

$$f(x) = \begin{cases} 1 & \text{if marker } x \text{ is white and the number of black markers to the left of } x \text{ is even} \\ 0 & \text{if marker } x \text{ is black} \\ -1 & \text{if marker } x \text{ is white and the number of black markers to the left of } x \text{ is odd} \end{cases}$$

Let S be the sum of $f(i)$ for all $1 \leq i \leq n$. Note that $S \equiv 1 \pmod 3$. The key observation is that after each step, S is invariant $\pmod 3$.
Suppose that marker i was removed. Note that there are only three values of $f(i)$ that are changed: $f(i-1), f(i)$, and $f(i+1)$. Moreover, note that since $f(i-1), f(i)$ and $f(i), f(i+1)$ have opposite parities, the sum S is decreased by $3f(i)$, and so stays the same $\pmod 3$.
Now suppose for the sake of contradiction that we were able to reach a state with 2 markers. Note that the total number of black markers is always even, so in the end the two markers must either be both black or both white. Now $S \equiv 0 \pmod 3$ or $S \equiv 2 \pmod 3$, and we have our desired contradiction. ■
Notice that this solution is equivalent to the solution of the previous problem.

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n people are seated in a circle. A total of nk coins have been distributed among them, but not necessarily equally. A move is the transfer of a single coin between two adjacent people. Find an algorithm for making the minimum possible number of moves which result in everyone ending up with the same number of coins.

Solution 62

Note that allowing negative numbers of objects does not matter, since we can rearrange the moves to avoid the negative numbers.
We want each person to end up with k coins. Let the people be labeled from $1, 2, \ldots, n$ in order (note that n is next to 1 since they are sitting in a circle). Suppose person i has c_i coins. We introduce the variable $d_i = c_i - k$, since this indicates how close a person is to having the desired number of coins. Consider the quantity

$$X = |d_1| + |d_1 + d_2| + |d_1 + d_2 + d_3| + \ldots + |d_1 + d_2 + \ldots + d_{n-1}|$$

Clearly, $X = 0$ if and only if everyone has k coins, so our goal is to make $X = 0$. The reason for this choice of X is that moving a coin between person j and person $j+1$ for $1 \leq j \leq n-1$ changes X by exactly 1 as only the term $|d_1 + d_2 + \ldots + d_j|$ will be affected. Hence X is a monovariant and is fairly easy to control (except when moving a coin from 1 to n or vice versa). Let $s_j = d_1 + d_2 + \cdots + d_j$.
Claim: As long as $X > 0$ it is always possible to reduce X by 1 by a move between j and $j+1$ for some $1 \leq j \leq n-1$.
Proof:
Algorithm: Assume WLoG $d_1 \geq 1$. Take the first j such that $d_{j+1} < 0$. If $s_j > 0$, then simply make a transfer from j to $j+1$. This reduces X by one since it reduces the term $|s_j|$ by one. The other possibility is $s_j = 0$, which means $d1 = d2 = \ldots = dj = 0$ (recall that d_{j+1} is the first negative term). In this case, take the first $m > i+1$ such that

$d_m \geq 0$. Then $d_{m-1} < 0$ by the assumption on m, so we move a coin from m to $m-1$. Note that all terms before d_m were either 0 or less than 0 and $d_{m-1} < 0$, so s_{m-1} was less than 0. Our move has increased s_{m-1} by one, and has hence decreased $|s_{m-1}|$ by one, so we have decreased X by one.
Thus at any stage we can always decrease X by at least one by moving between j and $j+1$ for some $1 \leq j \leq n-1$.
We have not yet considered the effect of a move between 1 and n.
Final Algorithm: At any point of time, if we can decrease X by moving a coin from 1 to n or n to 1, do this. Otherwise, decrease X by 1 by the algorithm described above paragraph. ■
Sometimes while creating algorithms that monotonically decrease (or increase) a quantity, we run into trouble in particular cases and our algorithm doesn't work. We can often get around these difficulties as follows. Suppose we want to monotonically decrease a particular quantity. Call a position good if we can decrease the monovariant with our algorithm. Otherwise, call the position bad. Now create an algorithm that converts bad positions into good positions, without increasing our monovariant. We use the first algorithm when possible, and then if we are stuck in a bad position, use the second algorithm to get back to a good position. Then we can again use the first algorithm.

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Solution 63

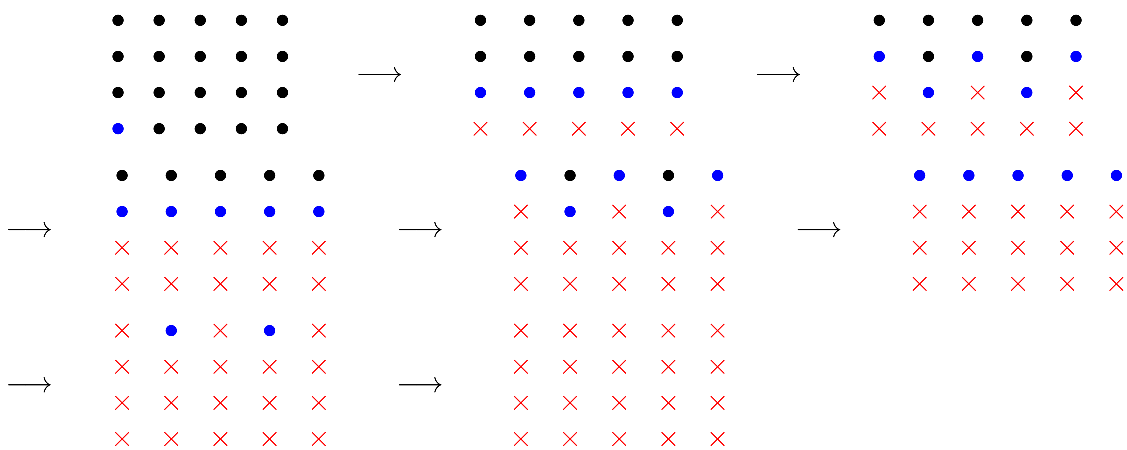
Consider the set of all lines that go through two points of S. Choose a direction I that is not parallel to any of those lines. Then, find a line J parallel to I that goes through a point $P \in S$ and such that the absolute difference of the number of points on the two sides of J is at most 1. That is, the closest we can get to dividing the points of S in half. Consider one of the half-planes determined by a line as the red half-plane and the other as the blue half-plane.
Start the windmill with J until it makes half of a complete turn. At this point consider its position T. T is parallel to J, and the positions of the red and blue half-planes have changed. Note that the absolute difference between the number of points on the two sides of T is also at most 1 because it remains invariant when rotated. Thus there are no points of S between J and T. This means that, other than the pivots of J and T, every point that was on the red half-plane is now on the blue half-plane and vice versa. Thus, the windmill used each point of S at least once. Since it will continue to use each point at least once per half turn, we are done. ■
Grant Sanderson has a beautiful explanation of the problem and solution on his YouTube channel 3Blue1Brown.

A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

Solution 64

All markers can be removed if and only if at least one of (m, n) is odd.

Example: Below we show the procedure for case $(m, n) = (5, 4)$, which easily generalizes ($\bullet$ means a marker is there but cannot be removed, $\bullet$ means a marker is there and can also be removed, $\times$ means no marker is there):



Proof for when m, n are even: Consider the case where m and n are both even. We construct a graph with mn vertices representing the markers and connect two vertices by an edge if and only if the markers they represent occupy adjacent squares (by side). Assign -1 to each edge and to each vertex representing a marker with its white side up, and 1 to each vertex representing a marker with its black side up. Let P be the product of all these numbers, on the edges as well as on the vertices.

Claim: P remains unchanged as the game progresses.

Proof: We can remove a marker only if its black side is up. If the vertex representing it is isolated, P is unchanged. Suppose this vertex is adjacent to another. Deleting the edge Changes the sign of P. However, since the marker represented by the other vertex is flipped over, the change in sign is negated. Thus the claim is correct.

Note that as markers are removed, we delete the vertices representing them, along with the edges incident to these vertices.

Initially, the number of -1's on the vertices is $mn - 1$, and the number of -1's on edges is $m(n-1) + n(m-1)$. Since the total $3mn - m - n - 1$ is odd, $P = -1$. If we have succeeded in removing all markers, the last move must involve the removal of the last marker, which must have its black side up. At that point, we have an isolated vertex with a 1, so $P = 1$. This is a contradiction. ■

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Anas and Azzam go through a sequence of rounds: At the beginning of every round, Azzam takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Anas chooses a pair of neighboring buckets, empties them to the river and puts them back. Then the next round begins. Azzam's goal is to make one of these buckets overflow. Anas's goal is to prevent this. Can Azzam enforce a bucket overflow?

Solution 65

First solution: No, Azzam cannot enforce a bucket overflow and Anas can keep playing forever. Throughout we denote the five buckets by B_0, B_1, B_2, B_3, and B_4, where B_k is adjacent to bucket B_{k-1} and B_{k+1} ($k = 0, 1, 2, 3, 4$) and all indices are taken modulo 5. Anas enforces that the following three conditions are satisfied at the beginning of every round:
(1) Two adjacent buckets (say B_1 and B_2) are empty.
(2) The two buckets standing next to these adjacent buckets (here B_0 and B_3) have total contents at most 1.
(3) The remaining bucket (here B_4) has contents at most 1. These conditions clearly hold at the beginning of the first round, when all buckets are empty.
Assume that Anas manages to maintain them until the beginning of the r-th round ($r \geq 1$). Denote by x_k ($k = 0, 1, 2, 3, 4$) the contents of bucket B_k at the beginning of this round and by y_k the corresponding contents after Azzam has distributed his liter of water in this round.
By the conditions, we can assume $x_1 = x_2 = 0, x_0 + x_3 \leq 1$ and $x_4 \leq 1$. Then, since Azzam adds one liter, we conclude $y_0 + y_1 + y_2 + y_3 \leq 2$. This inequality implies $y_0 + y_2 \leq 1$ or $y_1 + y_3 \leq 1$. For reasons of symmetry, we only consider the second case. Then Anas empties buckets B_0 and B_4. At the beginning of the next round B_0 and B_4 are empty (condition (1) is fulfilled), due to $y_1 + y_3 \leq 1$ condition (2) is fulfilled and finally since $x_2 = 0$ we also must have $y_2 \leq 1$ (condition (3) is fulfilled).
Therefore, Anas can indeed manage to maintain the three conditions (1)-(3) also at the beginning of the $(r+1)$-th round. By induction, he thus manages to maintain them at the beginning of every round. In particular he manages to keep the contents of every single bucket at most 1 liter. Therefore, the buckets of 2-liter capacity will never overflow. ■

Second solution: We prove that Anas can maintain the following two conditions and hence he can prevent the buckets from overflow:
(1) Every two non-adjacent buckets contain a total of at most 1.
(2) The total contents of all five buckets is at most $\frac{3}{2}$.
We use the same notations as in the first solution. The two conditions again clearly hold at the beginning. Assume that Anas maintained these two conditions until the beginning of the r-th round. A pair of non-neighboring buckets (B_i, B_{i+2}), $i = 0, 1, 2, 3, 4$ is called critical if $y_i + y_{i+2} > 1$. By condition (2), after Azzam has distributed his water we have $y_0 + y_1 + y_2 + y_3 + y_4 \leq \frac{5}{2}$. Therefore,

$$(y_0 + y_2) + (y_1 + y_3) + (y_2 + y_4) + (y_3 + y_0) + (y_4 + y_1) = 2(y_0 + y_1 + y_2 + y_3 + y_4) \leq 5,$$

and hence there is a pair of non-neighboring buckets which is not critical, say (B_0, B_2). Now, if both of the pairs (B_3, B_0) and (B_2, B_4) are critical, we must have $y - 1 < \frac{1}{2}$ and Anas can empty the buckets B_3 and B_4. This clearly leaves no critical pair of buckets and the total contents of all the buckets is then $y_1 + y_0 + y_2 \leq \frac{3}{2}$ Therefore, conditions (1) and (2) are fulfilled.
Now suppose that without loss of generality the pair (B_3, B_0) is not critical.
case 1: $y_0 \leq \frac{1}{2}$
Then one of the inequalities $y_0 + y_1 + y_2 \leq \frac{3}{2}$ or $y_0 + y_3 + y_4 \leq \frac{3}{2}$ must hold. But then Anas can empty B_3 and B_4 or B_1 and B_2, respectively and clearly fulfill the conditions.
case 2: $y_0 > \frac{1}{2}$
By $y_0 + y_1 + y_2 + y_3 + y_4 \leq \frac{5}{2}$, at least one of the pairs (B_1, B_3) and (B_2, B_4) is not critical. Without loss of generality let this be the pair (B_1, B_3). Since the pair (B_3, B_0) is not critical and $y_0 > \frac{1}{2}$, we must have $y_3 \leq \frac{1}{2}$. But then, as before, Anas can maintain the two conditions at the beginning of the next round by either emptying B_1 and B_2 or B_4 and B_0. ■

A comments on GREEDY approaches. A natural approach for Anas would be a GREEDY strategy as for example: *Always remove as much water as possible from the system.* It is straightforward to prove that GREEDY can avoid buckets of capacity $\frac{5}{2}$ from overflowing: If before Azzam move one has $x_0 + x_1 + x_2 + x_3 + x_4 \leq \frac{3}{2}$ then after his move the inequality $Y = y_0 + y_1 + y_2 + y_3 + y_4 \leq \frac{5}{2}$ holds. If now Anas removes the two adjacent buckets with maximum total contents he removes at least $\frac{2Y}{5}$ and thus the remaining buckets contain at most $\frac{3}{5}Y \leq \frac{3}{2}$.
But GREEDY is in general not strong enough to settle this problem as can be seen in the following example:

- In an initial phase, Azzam brings all the buckets (after his move) to contents of at least $\frac{1}{2} - 2\epsilon$, where ϵ is an arbitrary small positive number. This can be done by always splitting the 1 liter he has to distribute so that all buckets have the same contents. After his r-th move the total contents of each of the buckets is then c_r with $c_1 = 1$ and $c_{r+1} = 1 + \frac{3}{5}c_r$ and hence $c_r = \frac{5}{2} - 32(\frac{3}{5})^{r-1}$. So the contents of each single bucket indeed approaches $\frac{1}{2}$ (from below). In particular, any two adjacent buckets have total contents strictly less than 1 which enables Azzam to always refill the buckets that Anas just emptied and then distribute the remaining water evenly over all buckets.
- After that phase GREEDY faces a situation like this $(\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon)$ and leaves a situation of the form $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, 0, 0)$.
- Then Azzam can add the amount $(0, \frac{1}{4} + \epsilon, \epsilon, \frac{3}{4} - 2\epsilon, 0)$ to achieve a situation like this: $(y_0, y_1, y_2, y_3, y_4) = (\frac{1}{2} - 2\epsilon, \frac{3}{4} - \epsilon, \frac{1}{2} - \epsilon, \frac{3}{4} - 2\epsilon, 0)$.
- Now B_1 and B_2 are the adjacent buckets with the maximum total contents and thus GREEDY empties them to yield $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, 0, 0, \frac{3}{4} - 2\epsilon, 0)$.
- Then Azzam adds $(\frac{5}{8}, 0, 0, \frac{3}{8}, 0)$, which yields $(\frac{9}{8} - 2\epsilon, 0, 0, \frac{9}{8} - 2\epsilon, 0)$.
- Now GREEDY can only empty one of the two nonempty buckets and in the next step Azzam adds his liter to the other bucket and brings it to $\frac{17}{8} - 2\epsilon$, i.e an overflow. ■

A harder variant of the problem: *Five identical empty buckets of capacity b stand at the vertices of a regular pentagon. Anas and Azzam go through a sequence of rounds:*

At the beginning of every round, Azzam takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Anas chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. Azzam's goal is to make one of these buckets overflow. Anas's goal is to prevent this. Determine all bucket capacities b for which Azzam can enforce a bucket to overflow.

Solution: The answer is $b < 2$.

The previous proof shows that for all $b \geq 2$ Azzam cannot enforce overflowing. Now if $b < 2$, let R be a positive integer such that $b < 2 - 2^{1-R}$. In the first R rounds Azzam now ensures that at least one of the (nonadjacent) buckets B_1 and B_3 have contents of at least $1 - 2^{1-r}$ at the beginning of round r ($r = 1, 2, \ldots, R$). This is trivial for $r = 1$ and if it holds at the beginning of round r, he can fill the bucket which contains at least $1 - 2^{1-r}$ liters with another 2^{-r} liters and put the rest of his water $-1 - 2^{-r}$ liters $-$ in the other bucket. As Anas now can remove the water of at most one of the two buckets, the other bucket carries its contents into the next round.

At the beginning of the R-th round there are $1 - 2^{1-R}$ liters in B_1 or B_3. Azzam puts the entire liter into that bucket and produces an overflow since $b < 2 - 2^{1-R}$. ■

Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

Solution 66

A line is said to be *red* if it contains two points of M. As no three points of M are collinear, each red line determines a unique pair of points of M. Moreover, there are precisely $\binom{n}{2} < \frac{n^2}{2}$ red lines. By *the value of a segment* we mean the number of red lines intersecting it in its interior, and *the value of a set of segments* is defined to be the sum of the values of its elements. We will prove that

(i) the value of the initial set of segments is smaller than $\frac{n^3}{2}$.

(ii) Each step decreases the value of the set of segments present by at least 2.

Since such a value can never be negative, these two assertions imply the statement of the problem.

To show **(i)** we just need to observe that each segment has a value that is smaller than $\frac{n^2}{2}$. Thus the combined value of the n initial segments is indeed below $n(\frac{n^2}{2}) = \frac{n^3}{2}$.

It remains to establish **(ii)**. Suppose that at some moment we have two segments AB and CD sharing an interior point S, and that at the next moment we have the two segments AC and BD instead. Let X_{AB} denote the set of red lines intersecting the segment AB in its interior and let the sets X_{AC}, X_{BD}, and X_{CD} be defined similarly. We are to prove that $|X_{AC}| + |X_{BD}| + 2 \leq |X_{AB}| + |X_{CD}|$.

As a first step in this direction, we claim that

$$|X_{AC} \cup X_{BD}| + 2 \leq |X_{AB} \cup X_{CD}|. \quad (1)$$

Indeed, if g is a red line intersecting, e.g. the segment AC in its interior, then it has to intersect the triangle ACS once again, either in the interior of its side AS, or in the interior of its side CS, or at S, meaning that it belongs to X_{AB} or to X_{CD} (see Figure 1). Moreover, the red lines AB and CD contribute to $X_{AB} \cup X_{CD}$ but not to $X_{AC} \cup X_{BD}$. Thereby **(1)** is proved.

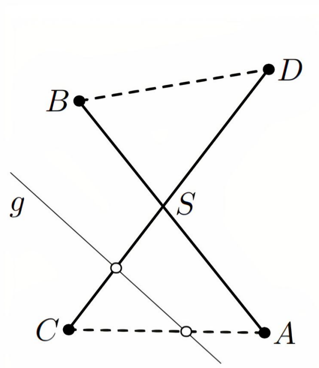


Figure 1

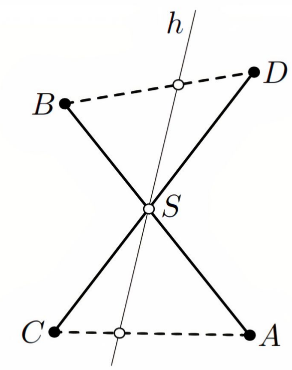


Figure 2

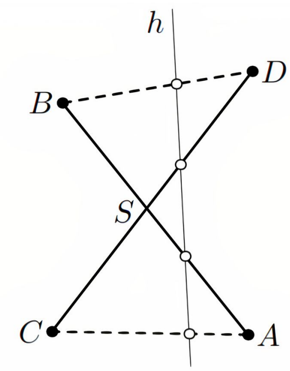


Figure 3

Similarly but more easily one obtains

$$|X_{AC} \cap X_{BD}| + 2 \leq |X_{AB} \cap X_{CD}|. \quad \textbf{(2)}$$

Indeed, a red line h appearing in $X_{AC} \cap X_{BD}$ belongs, for similar reasons as above, also to $X_{AB} \cap X_{CD}$. To make the argument precise, one may just distinguish the cases $S \in h$ (see Figure 2) and $S \notin h$ (see Figure 3). Thereby **(2)** is proved.
Adding (1) and (2) we obtain the desired conclusion, thus completing the solution of this problem. ■

Comment 1. There are some other essentially equivalent ways of presenting the same solution. E.g., put $M = \{A_1, A_2, \ldots, A_n\}$, denote the set of segments present at any moment by $\{e_1, e_2, \ldots, e_n\}$, and call a triple (i, j, k) of indices with $i \neq j$ intersecting, if the line A_iA_j intersects the segment e_k. It may then be shown that the number S of intersecting triples satisfies $0 \leq S < n^3$ at the beginning and decreases by at least 4 in each step.

Comment 2. There is a problem belonging to the folklore (problem 28 here), in the solution of which one may use the same kind of operation:
Given n red points and n blue points in the plane, no three collinear, prove that we can draw n segments, each joining a red point to a blue point, such that no segments intersect.
A standard approach to this problem consists in taking n arbitrary segments connecting the red points with the blue points, and to perform the same operation as in the above proposal whenever an intersection occurs. Now each time one performs such a step, the total length of the segments that are present decreases due to the triangle inequality. So, as there are only finitely many possibilities for the set of segments present, the process must end at some stage.
however, in the above proposal, considering the sum of the Euclidean lengths of the segment that are present does not seem to help much, even though it shows that the process must necessarily terminate after some finite number of steps, it does not seem to easily yield any upper bound on the number of these steps that grows polynomially.

One may regard the concept of the value of a segment in the first solution as an appropriately discretised version of Euclidean length suitable for obtaining such a bound. The IMO Problem Selection Committee still believes the problem to be sufficiently original for the competition.

Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move as choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C so that $\frac{BC}{AB} = \lambda$. Determine all values of λ such that, for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

Solution 67

The answer is $\lambda \geq \frac{1}{n-1}$.

We change the problem be replacing the fleas by bowling balls B_1, B_2, $\ldots$, B_n in that order. Bowling balls aren't exactly great at jumping, so the operation now works as follows.

• Choose any indices $i < j$.

• Then ball B_i moves to B_{i+1}'s location, B_{i+1} moves to B_{i+2}'s location, and so on; until B_{j-1} moves to B_j's location,

• Finally, B_j moves some distance at most $\lambda \cdot |B_jB_i|$ forward, but it may not pass B_{j+1}.

Claim 1: If $\lambda < \frac{1}{n-1}$ the bowling balls have bounded movement.

Proof: Let $a_i \geq 0$ denote the initial distance between B_i and B_{i+1}, and let Δ_i denote the distance traveled by ball i. Of course we have $\Delta_1 \leq a_1 + \Delta_2$, $\Delta_2 \leq a_2 + \Delta_3$, $\ldots$, $\Delta_{n-1} \leq a_{n-1} + \Delta_n$ by the relative ordering of the bowling balls. Finally, distance covered by B_n is always λ times distance travelled by other bowling balls, so

$$\begin{aligned}\Delta_n &\leq \lambda \sum_{i=1}^{n-1} \Delta_i \leq \lambda \sum_{i=1}^{n-1} \left((a_i + a_{i+1} + \cdots + a_{n-1}) + \Delta_n\right) \\ &= (n-1)\lambda \cdot \Delta_n + \sum_{i=1}^{n-1} i a_i\end{aligned}$$

and since $(n-1)\lambda > 1$, this gives an upper bound.

Remark. Equivalently, you can phrase the proof without bowling balls as follows: if $a1 < \ldots < an$ are the positions of the fleas, the quantity

$$L = a_n - \lambda(a_1 + \ldots + a_{n-1})$$

is a **monovariant** which never increases; i.e. L is bounded above. Since $L > (1 - (n-1)\lambda)a_n$, it follows $\lambda < \frac{1}{n-1}$ is enough to stop the fleas.

Claim 2: When $\lambda \geq \frac{1}{n-1}$, it suffices to always jump the leftmost flea over the rightmost flea.

Proof: If we let x_i denote the distance traveled by B_1 in the ith step, then $x_i = a_i$ for $1 \leq i \leq n-1$ and $x_i = \lambda(x_{i-1} + x_{i-2} + \cdots + x_{i-(n-1)})$.

In particular, if $\lambda \geq \frac{1}{n-1}$ then each x_i is at least the average of the previous $n-1$ terms. So if the a_i are not all zero, then $\{x_n, \ldots, x_{2n-2}\}$ are all positive and thereafter $x_i \geq \min\{x_n, \ldots, x_{2n-2}\} > 0$ for every $i \geq 2n-1$. So the partial sums of x_i are unbounded, as desired. ■

Remark. Other inductive constructions are possible. Here is the idea of one of them, although the details are more complicated.

Claim: Given $n-1$ fleas at 0 and one flea at 1, we can get all the fleas arbitrarily close to $\frac{1}{1-(n-1)\lambda}$ (or as far as we want if $\lambda > \frac{1}{n-1}$.).

Proof: By induction for $n \geq 2$.

Base case: For $n = 2$ we get a geometric series.

Inductive steps: For $n \geq 3$, we leave one flea at zero and move the remainder close to $\frac{1}{1-(n-2)\lambda}$, then jump the last flea to $\frac{1+\lambda}{1-(n-2)\lambda}$.

Now we're in the same situation, except we shifted $\frac{1}{1-(n-2)\lambda}$ right and have then scaled everything by $r = \frac{\lambda}{1-(n-2)\lambda}$. If we repeat this process again and check the geometric series, we see the fleas converge to

$$\frac{1}{1-(n-2)\lambda}(1+r+r^2+r^3+\ldots) = \frac{1}{1-(n-2)\lambda}\left(\frac{1}{1-r}\right) = \frac{1}{1-(n-1)\lambda}.$$

There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Adeeb the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Adeeb always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or vice versa. Suppose that Adeeb's path entirely covers all circles. Prove that n must be odd.

Solution 68

First solution: Replace every cross (i.e. intersection of two circles) by two small circle arcs that indicate the direction in which the snail should leave the cross (see Figure 1.1). Notice that the placement of the small arcs does not depend on the direction of moving on the curves; no matter which direction the snail is moving on the circle arcs, he will follow the same curves (see Figure 1.2). In this way we have a set of curves, that are the possible paths of the snail. Call these curves *snail orbits* or just *orbits*. Every snail orbit is a simple closed curve that has no intersection with any other orbit.

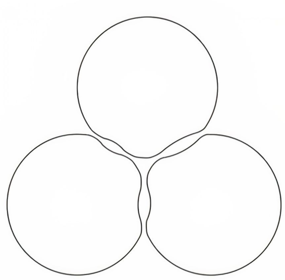


Figure 1.1

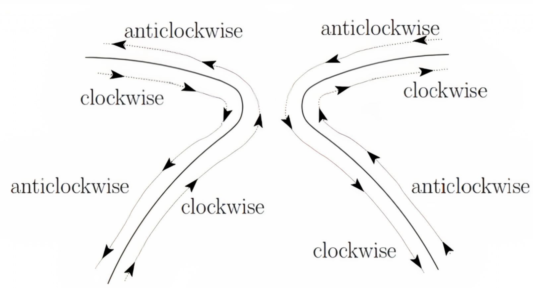


Figure 1.2

We prove the following, more general statement.
(*) In any configuration of n circles such that no two of them are tangent, the number of snail orbits has the same parity as the number n. (Note that it is not assumed that all circle pairs intersect.)
This immediately solves the problem.
Let us introduce the following operation that will be called *flipping a cross.* At a cross, remove the two small arcs of the orbits, and replace them by the other two arcs. Hence, when the snail arrives at a flipped cross, he will continue on the other circle as before, but he will preserve the orientation in which he goes along the circle arcs (see Figure 2).

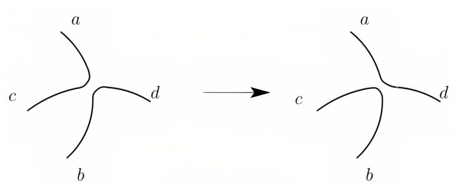


Figure 2

Consider what happens to the number of orbits when a cross is flipped. Denote by a, b, c, and d the four arcs that meet at the cross such that a and b belong to the same circle. Before the flipping a and b were connected to c and d, respectively, and after the flipping a and b are connected to d and c, respectively.
The orbits passing through the cross are closed curves, so each of the arcs a, b, c, and d is connected to another one by orbits outside the cross. We distinguish three cases.
Case 1: a is connected to b and c is connected to d by the orbits outside the cross (see Figure 3.1).
We show that this case is impossible. Remove the two small arcs at the cross, connect a to b, and connect c to d at the cross. Let γ be the new closed curve containing a and b, and let δ be the new curve that connects c and d. These two curves intersect at the cross. So one of c and d is inside γ and the other one is outside γ. Then the two closed curves have to meet at least one more time, but this is a contradiction, since no orbit can intersect itself.

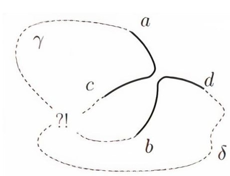


Figure 3.1

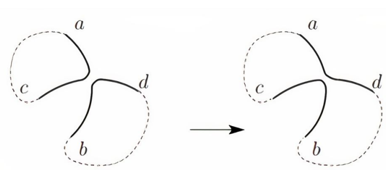


Figure 3.2

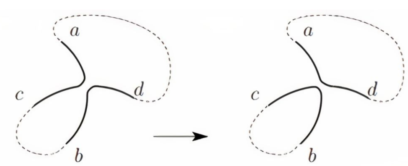


Figure 3.3

Case 2: a is connected to c and b is connected to d (see Figure 3.2).
Before the flipping a and c belong to one orbit and b and d belong to another orbit. Flipping the cross merges the two orbits into a single orbit. Hence, the number of orbits decreases by 1.
Case 3: a is connected to d and b is connected to c (see Figure 3.3).
Before the flipping the arcs a, b, c, and d belong to a single orbit. Flipping the cross splits that orbit in two. The number of orbits increases by 1.

As can be seen, every flipping decreases or increases the number of orbits by one, thus changes its parity.
Now flip every cross, one by one. Since every pair of circles has 0 or 2 intersections, the number of crosses is even. Therefore, when all crosses have been flipped, the original parity of the number of orbits is restored. So it is sufficient to prove *(*)* for the new configuration, where all crosses are flipped. Of course also in this new configuration the (modified) orbits are simple closed curves not intersecting each other.
Orient the orbits in such a way that the snail always moves anticlockwise along the circle arcs. Figure 4 shows the same circles as in Figure 1 after flipping all crosses and adding orientation. (Note that this orientation may be different from the orientation of the orbit as a planar curve; the orientation of every orbit may be negative as well as positive, like the middle orbit in Figure 4.) If the snail moves around an orbit, the total angle change in his moving direction, the *total curvature*, is either $+2\pi$ or -2π, depending on the orientation of the orbit. Let P and N be the number of orbits with positive and negative orientation, respectively. Then the total curvature of all orbits is $(P - N)2\pi$.

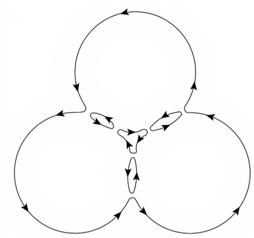


Figure 4

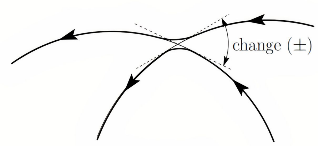


Figure 5

Double-count the total curvature of all orbits. Along every circle the total curvature is 2π. At every cross, the two turnings make two changes with some angles having the same absolute value but opposite signs, as depicted in Figure 5. So the changes in the direction at the crosses cancel out. Hence, the total curvature is $n(2\pi)$.
Now we have $(P - N)2\pi = n(2\pi)$, so $P - N = n$. The number of (modified) orbits is $P + N$, that has a same parity as $P - N = n$.
Second solution: We present a different proof of *(*)*.
We perform a sequence of small modification steps on the configuration of the circles in such a way that at the end they have no intersection at all (see Figure 6.1). We use two kinds of local changes to the structure of the orbits (see Figure 6.2):
- *Type-1 step:* An arc of a circle is moved over an arc of another circle; such a step creates or removes two intersections.
- *Type-2 step:* An arc of a circle is moved through the intersection of two other circles.

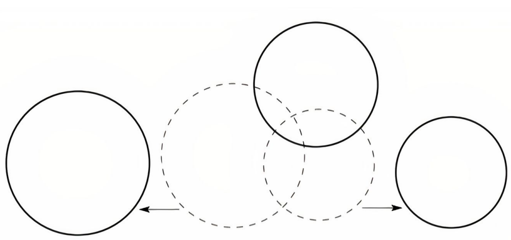


Figure 6.1

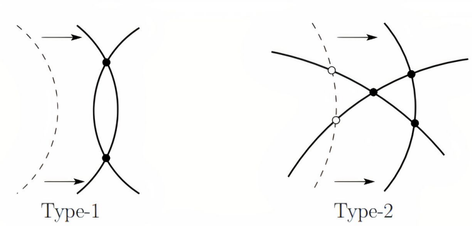


Figure 6.2

We assume that in every step only one circle is moved, and that this circle is moved over at most one arc or intersection point of other circles.
We will show that the parity of the number of orbits does not change in any step. As every circle becomes a separate orbit at the end of the procedure, this fact proves (*).
Consider what happens to the number of orbits when a Type-1 step is performed. The two intersection points are created or removed in a small neighborhood. Denote some points of the two circles where they enter or leave this neighborhood by a, b, c, and d in this order around the neighborhood; let a and b belong to one circle and let c and d belong to the other circle. The two circle arcs may have the same or opposite orientations. Moreover, the four end-points of the two arcs are connected by the other parts of the orbits. This can happen in two ways without intersection: either a is connected to d and b is connected to c, or a is connected to b and c is connected to d. Altogether we have four cases, as shown in Figure 7.

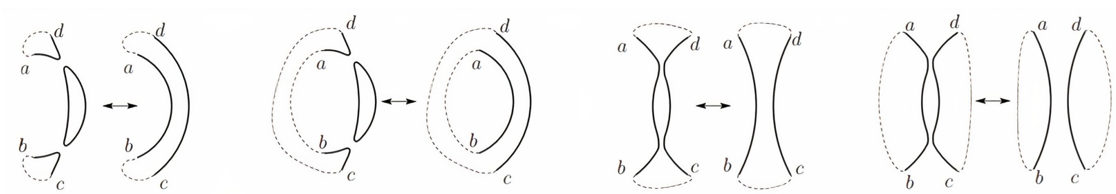


Figure 7

We can see that the number of orbits is changed by -2 or $+2$ in the leftmost case when the arcs have the same orientation, a is connected to d, and b is connected to c. In the other three cases the number of orbits is not changed. Hence, Type-1 steps do not change the parity of the number of orbits.
Now consider a Type-2 step. The three circles enclose a small, triangular region; by the step, this triangle is replaced by another triangle. Again, the modification of the orbits is done in some small neighborhood; the structure does not change outside. Each side of the triangle shaped region can be convex or concave; the number of concave sides can be $0, 1, 2$ or3, so there are 4 possible arrangements of the orbits inside the neighborhood, as shown in Figure 8.

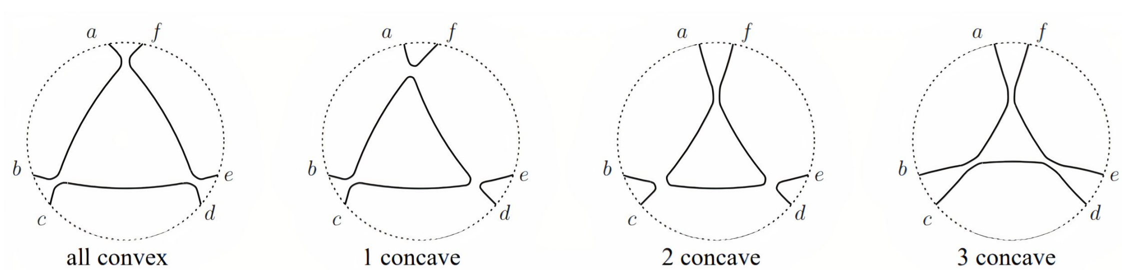


Figure 8

Denote the points where the three circles enter or leave the neighborhood by a, b, c, d, e, and f in this order around the neighborhood. As can be seen in Figure 8, there are only two essentially different cases; either a, c, e are connected to b, d, f, respectively, or a, c, e are connected to f, b, d, respectively. The step either preserves the set of connections or

switches to the other arrangement. Obviously, in the earlier case the number of orbits is not changed; therefore we have to consider only the latter case.
The points a, b, c, d, e, and f are connected by the orbits outside, without intersection. If a was connected to c, say, then this orbit would isolate b, so this is impossible. Hence, each of a, b, c, d, e and f must be connected either to one of its neighbors or to the opposite point. If say a is connected to d, then this orbit separates b and c from e and f, therefore b must be connected to c and e must be connected to f. Altogether there are only two cases and their reverses: either each point is connected to one of its neighbors or two opposite points are connected and the the remaining neigh boring pairs are connected to each other. See Figure 9.

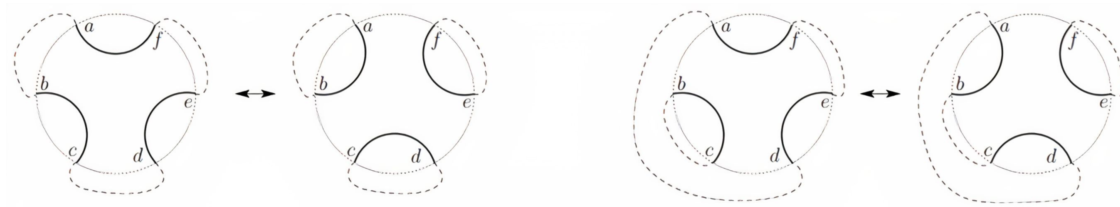


Figure 9

We can see that if only neighboring points are connected, then the number of orbits is changed by $+2$ or -2. If two opposite points are connected (a and d in the figure), then the orbits are re-arranged, but their number is unchanged. Hence, Type-2 steps also preserve the parity. This completes the proof of *(*)*.

Third solution: Like in the previous solutions, we do not need all circle pairs to intersect but we assume that the circles form a connected set. Denote by C and P the sets of circles and their intersection points, respectively.
The circles divide the plane into several simply connected, bounded regions and one unbounded region. Denote the set of these regions by R. We say that an intersection point or a region is *odd* or *even* if it is contained inside an odd or even number of circles, respectively. Let P_{odd} and R_{odd} be the sets of odd intersection points and odd regions, respectively.
Claim:

$$|R_{odd}| - |P_{odd}| \equiv n \pmod 2. \quad \textbf{(1)}$$

For each circle c, apply *Euler*'s polyhedron theorem to the (simply connected) regions in c. There are $2X_c$ intersection points on c; they divide the circle into 2Xc arcs. The polyhedron theorem yields

$$(R_c + 1) + (P_c + 2X_c) = (A_c + 2X_c) + 2$$

, considering the exterior of$c$ as a single region. Therefore,

$$R_c + P_c = A_c + 1. \quad \textbf{(2)}$$

Moreover, we have four arcs starting from every interior points inside c and a single arc starting into the interior from each intersection point on the circle. By double-counting the end-points of the interior arcs we get $2A_c = 4P_c + 2X_c$, so

$$A_c = 2P_c + X_c. \quad \textbf{(3)}$$

The relations **(2)** and **(3)** together yield

$$R_c - P_c = X_c + 1. \quad \textbf{(4)}$$

By summing up **(4)** for all circles we obtain

$$\sum_{c \in C} R_c - \sum_{c \in C} P_c = \sum_{c \in C} X_c + |C|,$$

which yields

$$|R_{odd}| - |P_{odd}| \equiv \sum_{c \in C} X_c + n \pmod 2. \quad \textbf{(5)}$$

Notice that in $\sum_{c \in C} X_c$ each intersecting circle pair is counted twice, i.e., for both circles in the pair, so

$$\sum_{c \in C} X_c \equiv 0 \pmod 2,$$

which finishes the proof of the Claim.

Now insert the same small arcs at the intersections as in the first solution, and suppose that there is a single snail orbit b.

First we show that the odd regions are inside the curve b, while the even regions are outside. Take a region $r \in R$ and a point x in its interior, and draw a ray y, starting from x, that does not pass through any intersection point of the circles and is neither tangent to any of the circles. As is well-known, x is inside the curve b if and only if y intersects b an odd number of times (see Figure 10). Notice that if an arbitrary circle c contains x in its interior, then c intersects y at a single point; otherwise, if x is outside c, then c has 2 or 0 intersections with y. Therefore, y intersects b an odd number of times if and only if x is contained in an odd number of circles, so if and only if r is odd.

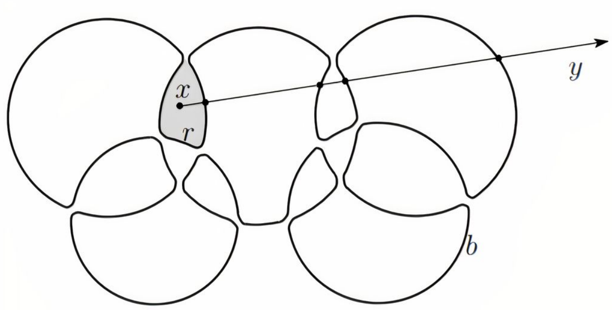


Figure 10

Now consider an intersection point p of two circles c_1 and c_2 and a small neighborhood around p. Suppose that p is contained inside k circles.

We have four regions that meet at p. Let r_1 be the region that lies outside both c_1 and c_2, let r_2 be the region that lies inside both c_1 and c_2, and let r_3 and r_4 be the two remaining regions, each lying inside exactly one of c_1 and c_2. The region r_1 is contained inside the same k circles as p; the region r_2 is contained also by c_1 and c_2, so by $k + 2$

circles in total; each of the regions r_3 and r_4 is contained inside $k+1$ circles. After the small arcs have been inserted at p, the regions r_1 and r_2 get connected, and the regions r_3 and r_4 remain separated at p (see Figure 11). If p is an odd point, then r_1 and r_2 are odd, so two odd regions are connected at p. Otherwise, if p is even, then we have two even regions connected at p.

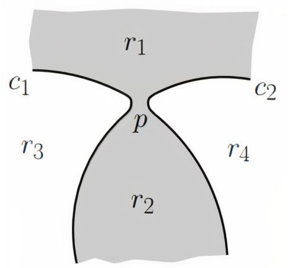


Figure 11

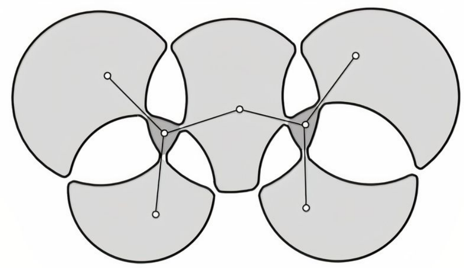


Figure 12

Consider the system of odd regions and their connections at the odd points as a graph. In this graph the odd regions are the vertices, and each odd point establishes an edge that connects two vertices (see Figure 12). As b is a single closed curve, this graph is connected and contains no cycle, so the graph is a tree. Then the number of vertices must be by one greater than the number of edges, so

$$|R_{odd}| - |P_{odd}| = 1. \quad \textbf{(6)}$$

The relations **(1)** and **(6)** together prove that n must be odd.

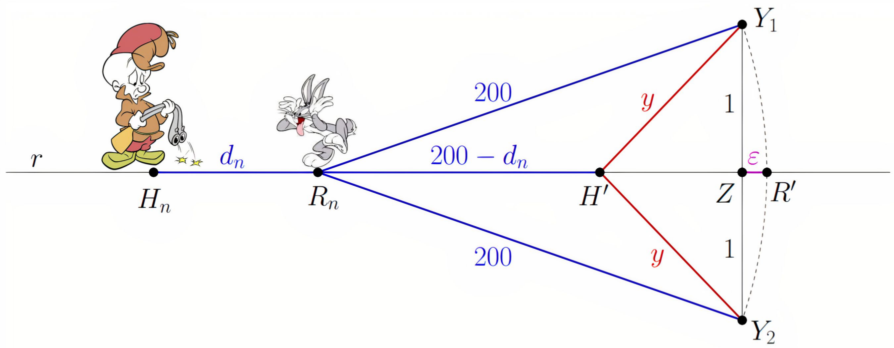
Comment. For every odd n there exists at least one configuration of n circles with a single snail orbit. In general, if a circle is rotated by $k\frac{360}{n}$. ($k = 1, 2, \ldots, n-1$) around an interior point other than the center, the circle and its rotated copies together provide a single snail orbit.

A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B0 are the same. After $n-1$ rounds of the game, the rabbit is at point $An-1$ and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: i. he rabbit moves invisibly to a point An such that the distance between $An-1$ and A_n is exactly 1. ii. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. iii. The hunter moves visibly to a point Bn such that the distance between $Bn-1$ and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

Solution 69

There is no such strategy for the hunter. The rabbit "wins".
If the answer were "yes", the hunter would have a strategy that would "work", no matter how the rabbit moved or where the radar pings R'_n appeared. We will show the

opposite; with bad luck from the radar pings there is no strategy for the hunter that guarantees that the distance stays between 100 in 10^9 rounds.
So, let d_n be the distance between the hunter and the rabbit after n rounds. Of course, if $d_n \geq 100$ for any $n < 10^9$, the rabbit has won - it just needs to moves straight away from the hunter, and the distance will be kept at or above 100 thereon.
We will now show that, while $d_n < 100$, whatever given strategy the hunter follows, the rabbit has a way of increasing d_n^2 by at least $\frac{1}{2}$ every 200 round (as long as the radar pings are lucky enough for the rabbit). This way d_n^2 will reach 10^4 in less than $2 \times 10^4 \times 200 = 4 \times 10^6 < 10^9$ rounds, and he rabbit wins.
Suppose the hunter is at H_n and the rabbit is at R_n. Suppose even that the rabbit *reveals* its position at this moment to the hunter (this allows us to ignore all information from previous radar pings). Let r be the line H_nR_n, and Y_1 and Y_2 be points which are 1 unit away from r and 200 units away from R_n , as in the figure below.



The rabbit's plan is simply to choose one of the points Y_1 or Y_2 and hop 200 rounds straight towards it. Since all hops stay within 1 distance unit from r, it is possible that all radar pings stay on r. In particular, in this case, the hunter has no way of knowing whether the rabbit chose Y_1 or Y_2.
Looking at such pings, what is the hunter going to do? If the hunter's strategy tells him to go 200 rounds straight to the right, he ends up at point H' in the figure. Note that the hunter does not have a better alternative! Indeed, after these 200 rounds he will always end up at a point to the left of H'. If his strategy took him to a point above r, he would end up even further from Y_2; and if his strategy took him below r, he would end up even further from Y_1. In other words, no matter what strategy the hunter follows, he can never be sure his distance to the rabbit will be less than $y = H'Y_1 = H'Y_2$ after these 200 rounds.
To estimate y^2, we take Z as the midpoint of segment Y_1Y_2, we take R' as a point 200 units to the right of R_n and we define $\epsilon = ZR'$ (note that $H'R' = d_n$). Then

$$y^2 = 1 + (H'Z)^2 = 1 + (d_n - \epsilon)^2$$

where

$$\epsilon = 200 - R_nZ = 200 - \sqrt{200^2 - 1} = \frac{1}{200 + \sqrt{200^2 - 1}} > \frac{1}{400}.$$

In particular, $\epsilon^2 + 1 = 400\epsilon$, so

$$y^2 = d_n^2 - 2\epsilon d_n + \epsilon^2 + 1 = d_n^2 + \epsilon(400 - 2d_n).$$

Since $\epsilon > \frac{1}{400}$ and we assumed $d_n < 100$, this shows that $y^2 > d_n^2 + \frac{1}{2}$. So, as we claimed, with this list of radar pings, no matter what the hunter does, the rabbit might achieve $d_{n+200}^2 > d_n^2 + \frac{1}{2}$. The rabbit wins.

Omar is playing a game with Hamza. Omar begins by standing at the origin of the coordinate plane. Omar and Hamza take turns, starting with Hamza. On his turn, Hamza places lava on a lattice point of his choosing, preventing Omar from going there. Then on Omar's turn, Omar can move m times, each time going from the point (x, y) to the point $(x + 1, y)$ or $(x, y + 1)$ (but he can never go to a lattice point with lava). Hamza's goal is to make it so that Omar can't move. For which positive integers m can Hamza guarantee victory?

Solution 70

This is Conway's famous **Angel problem.**
Hamza is the devil and Omar is the angel (a k-power angel) and (x, y) represents the unit square with (x, y) as it's lower left corner.
From the problem statement there are three differences between this game and the Angel's game:

• In the Angel game, the angel can move in any manner that a king in chess could (up, down, left, right); here the angel can only move up and right.
• In the Angel game, the angel can "move over" marked squares, as long as the final destination is at most k moves away from the starting position; here the angel cannot cross a marked square.
• In the Angel game, the angel must move k times between two consecutive turns of the devil; here the angel can move any number of moves between 1 and k.

It turns out these three differences are negligible in the long run - indeed the first two differences make it easier.
In this paper (Games of No Chance MSRI Publications Volume 29, 1996) (Conway assumed $k = 1000$, but this is of no importance), Conway defines a *Plain Fool* to be an angel that only increases it's y-coordinate at the end of the k moves. He later defines a *Lax Fool* to be angel that never decreases it's y-coordinate at the end of the k moves.
Proof that a devil has a winning strategy against the k-power Plain Fool, found by Conway and Blass (Theorem 3.1): If the Fool is ever at some point P, he will be at all subsequent times in the "upward cone" from P, whose boundary is defined by the two upward rays of slope $\pm\frac{1}{k}$ through P (a diagram is given in the paper). Then we counsel the Devil to act as follows (Figure 2): he should truncate this cone by a horizontal line AB at a very large height H above the Fool's starting position, and use his first few moves to eat one out of every M squares along AB, where M is chosen so that this task will be comfortably finished when the Angel reaches a point Q on the halfway line that's distant $\frac{1}{2}H$ below AB (we'll arrange H to be exactly divisible by a large power of two).
At subsequent times, the Devil knows that the Fool will be safely ensconced in the smaller cone QCD, where CD is a subinterval of AB of exactly half its length, and for the next few of his moves, he should eat the second one of every M squares along the segment CD. He will have finished this by the time the Fool reaches a point R on the

horizontal line $\frac{1}{4}H$ below AB. At later times, the Fool will be trapped inside a still smaller cone REF, with $EF = \frac{1}{4}CD = \frac{1}{4}AB$, and the Devil should proceed to eat the third one of every M squares along the segment EF of AB.

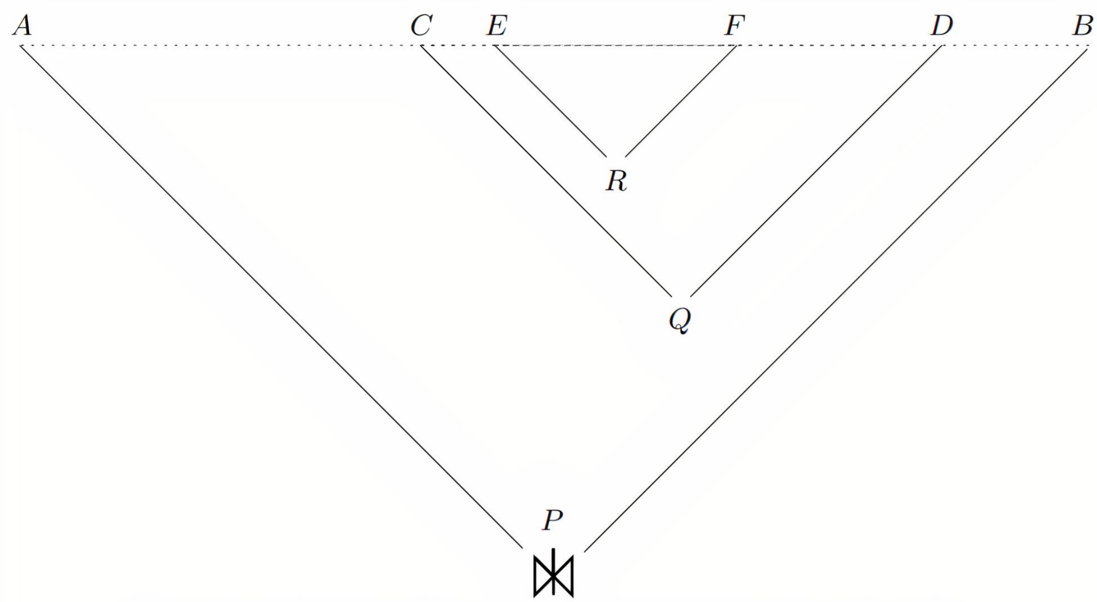


Figure 2. A Fool travelling north with the Devil eating along AB.

If he proceeds in this way, then by the time the Fool reaches the horizontal line at distance $H' = 2^{-M}H$ below AB, the Devil will have eaten every square of the subsegment of AB that might still be reached by the Fool (note: we can stop here for the game in the problem). The Devil should then continue, by eating the first out of every M squares on the segment AB just below this one, a task which will be finished before the Fool reaches the horizontal line distant $\frac{1}{2}H'$ below AB , when he should start eating the second of every M squares on the portion CD of AB that is still accessible, and so on. We see that if we take H of the form $k \times 2N$, where $N > kM$, then before the Fool crosses the horizontal line that is k units below AB, the Devil will have eaten all squares between this line and AB that the Fool might reach, and so the Fool will be unable to move.

Proof that a Lax fool can be treated as a Plain fool, found by Conway and Blass (Theorem 4.1): The Devil uses his odd-numbered moves to convert the Lax Fool into a Plain Fool (of a much higher power). He chooses two squares L and R a suitable distance D to the left and right of the Fool's starting position, and eats inward alternately from L and R, so long as the Fool stays on the starting line (Figure 3).

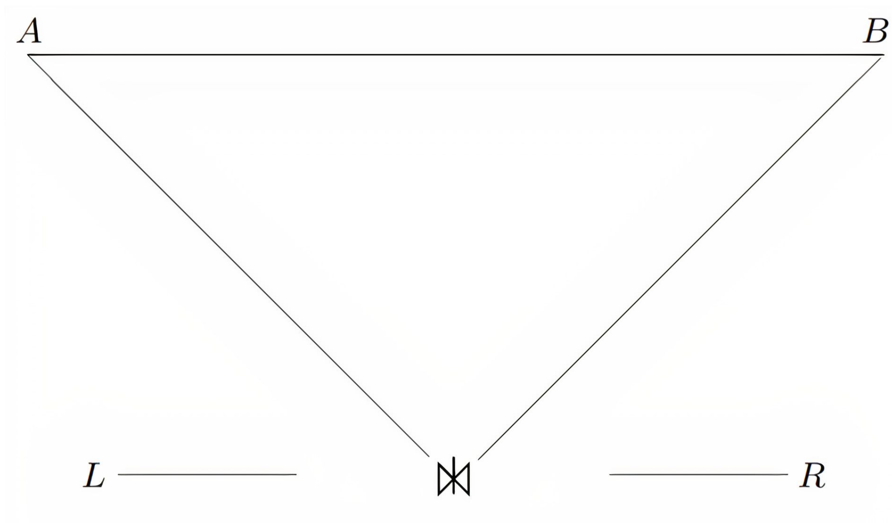


Figure 3. A Lax Fool travelling north.

Whenever the Fool makes an upwards move, the Devil takes the horizontal line through his new position as a new starting line.
Suppose that the Fool stays on the starting line for quite some time. Then he can use the four moves he has between two consecutive bites of the Devil at the left end of this line to move at most $4k$ places. So, if we take $D = 4k^2$, the Devil will have eaten k consecutive squares at the left end of this line before the Fool can reach them. We now see that the Fool can stay on the starting line for at most $8k^2$ moves, since in $8k^2 + 1$ moves the Devil can eat every square between L and R.
If the Devil adopts this strategy, then the Fool must strictly increase his y coordinate at least once in every $8k^2$ moves. If we look at him only at these instants, we see him behaving like a Plain Fool of power $8k^3$, since each of his moves can have carried him at most k places. So if the Devil uses the strategy that catches an $8k^3$-power Plain Fool on his even-numbered moves, he'll safely trap the k-power Lax Fool.
It follows, therefore, that the Devil has a winning strategy against any Lax Fool.
Remark. Now, the angel described in the game here acts as a Plain Fool, but a more restricted one in the sense that it cannot move left at all. This condition only disadvantages the angel and any strategy has against the Plain Fool will also work against this "One-Sided" Plain Fool. Moreover, the angel's ability to move any number of moves between 1 and k makes no impact - it will optimally choose k moves each time. The last difference is the fact that this angel cannot "move over" any of the marked squares, but this again only impedes the angel in this game. It follows that for any positive integer k, Hamza can win regardless of how Omar moves.
What about the general case? When the angel can move in any direction?
It turns out the devil has a winning strategy against a 1-power angel. However, Kloster and Máthé proved, more or less simultaneously in 2007, that a 2-power angel wins (and so any angel of a higher power also wins). Bowditch also had a proof published in the same year for a power of at least 4.

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