47. Solve the system of equations

$$\begin{cases} \sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6 \\ 9x^2+5y^2=45. \end{cases}$$

$$\sqrt{(x+2)^2+y^2} = 6-\sqrt{(x-2)^2+y^2}$$

$$\Rightarrow (x+2)^2+\cancel{y^2} = 36 + (x-2)^2+\cancel{y^2} - 12\sqrt{(x-2)^2+y^2}$$

$$\Rightarrow 8x - 36 = -12\sqrt{(x-2)^2+y^2}$$

$$\Rightarrow \frac{2}{3}x - 3 = -\sqrt{(x-2)^2+y^2}$$

$$\Rightarrow \frac{4}{9}x^2+9-4x = (x-2)^2+y^2 \Rightarrow \frac{4}{9}x^2+5 = x^2+y^2$$

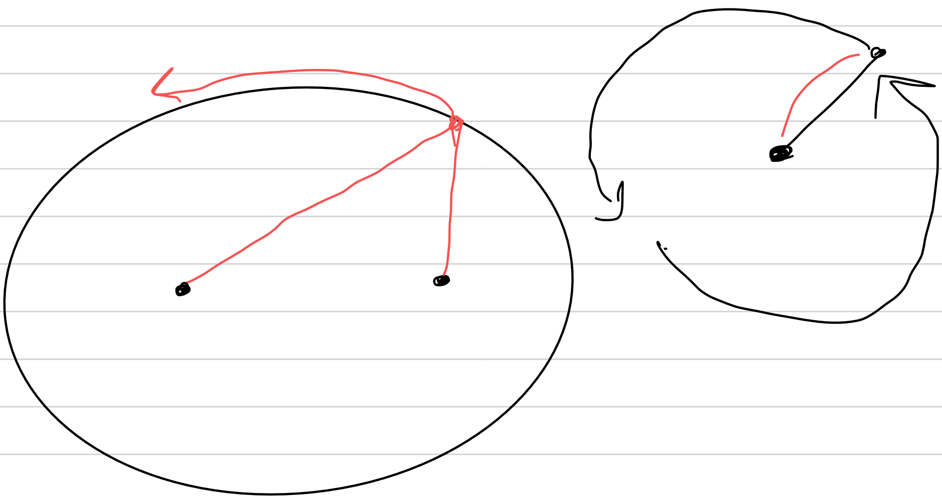
$$\Rightarrow \frac{5x^2}{9}+y^2=5 \Rightarrow \frac{x^2}{9}+\frac{y^2}{5}=1 \text{ or } 5x^2+9y^2=45 \ (2)$$

from (1) and (2)

$$\Rightarrow x^2=y^2 \Rightarrow 14x^2=45 \Rightarrow x=\pm\sqrt{\frac{45}{14}}$$

$$y=\pm\sqrt{\frac{45}{14}}$$

$$\boxed{\frac{x^2}{9}+\frac{y^2}{5}=1}$$



47. Solve the system of equations

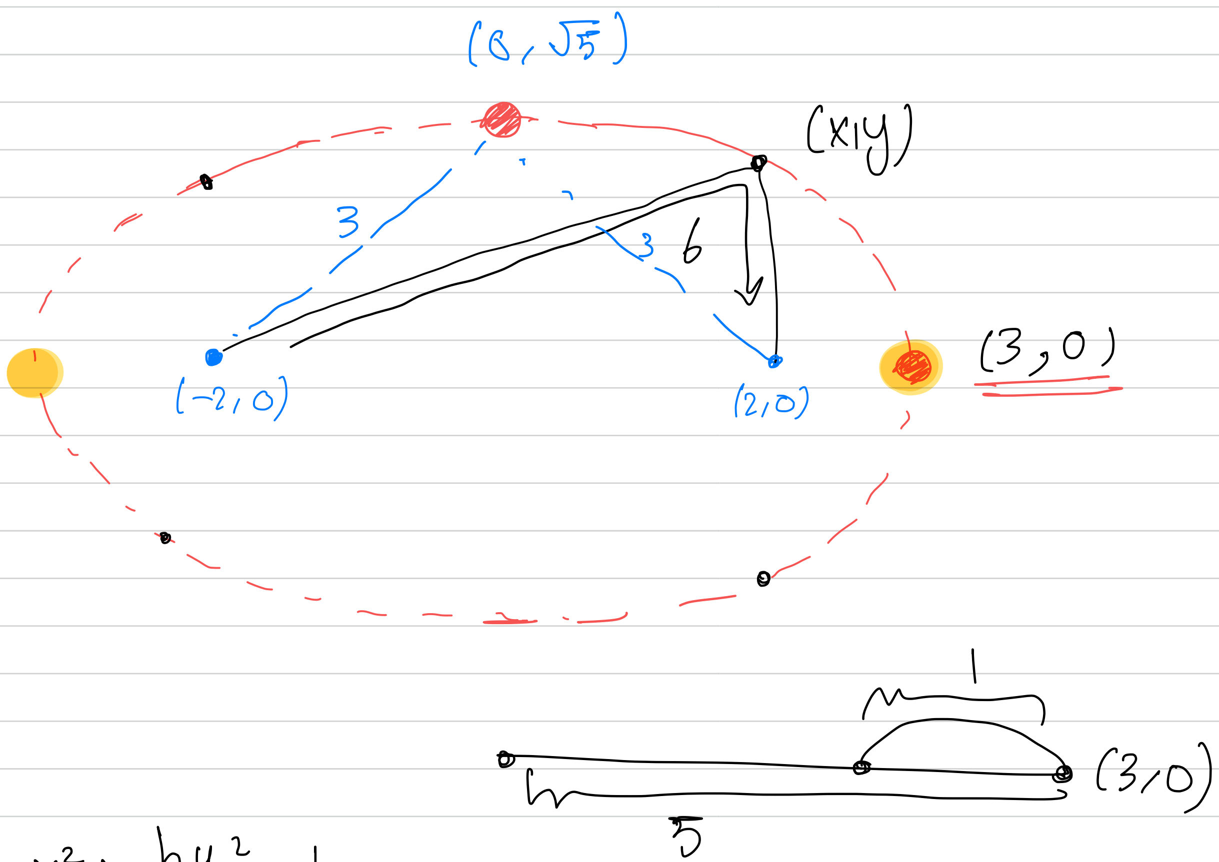
$$\begin{cases} \sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6 \\ 9x^2+5y^2=45. \end{cases} \longrightarrow \frac{x^2}{9}+\frac{y^2}{5}=1$$

المعادلة الأولى:

$\sqrt{(x-2)^2+y^2}$ = المسافة بين (x,y) و $(2,0)$

$\sqrt{(x+2)^2+y^2}$ = المسافة بين (x,y) و $(-2,0)$

مجموع المسافة بين (x,y) و $(-2,0)$ و $(2,0)$ ثابت وقيمته 6



$$a x^2+by^2=1$$

$y=0,\ x=3$

$$a(3)^2=1 \Rightarrow a=\frac{1}{9}$$

$y=\sqrt{5},\ x=0$

$$b\,5=1 \Rightarrow b=\frac{1}{5}$$

$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

48. Solve the equation

$$\sqrt[n]{1+x}+2\sqrt[n]{1-x}=3\sqrt[2n]{1-x^2}.$$

Clearly $x \neq \pm 1$

$$\sqrt[n]{1+x}+2\sqrt[n]{1-x}=3\sqrt[2n]{(1-x)(1+x)}$$

$$\frac{\sqrt[n]{1+x}}{\sqrt[2n]{(1+x)(1-x)}}+\frac{2\sqrt[n]{1-x}}{\sqrt[2n]{(1+x)(1-x)}}=3$$

$$\frac{\sqrt[2n]{1+x}}{\sqrt[2n]{1-x}}+2\frac{\sqrt[2n]{1-x}}{\sqrt[2n]{1+x}}=3$$

Let $y=\dfrac{\sqrt[2n]{1+x}}{\sqrt[2n]{1-x}}$ $\Rightarrow$ $y+\dfrac{2}{y}=3$

$\Rightarrow y^2-3y+2=0$

$\Rightarrow (y-1)(y-2)=0$

$\Rightarrow y=1$ or 2

- if $y=1$ then $1+x=1-x \Rightarrow x=0$
  which's a solution
- if $y=2$ then

$$\sqrt[2n]{\frac{1+x}{1-x}}=2 \Rightarrow \frac{1+x}{1-x}=2^{2n}=4^n$$

$$4^n+1=\frac{1+x}{1-x}+1=\frac{2}{1-x}$$

$$\Rightarrow 1-x=\frac{2}{4^n+1} \Rightarrow x=1-\frac{2}{4^n+1}=\frac{4^n-1}{4^n+1}$$

Another solution :

$$\sqrt[n]{1+x} + 2\sqrt[n]{1-x} = 3\sqrt[2n]{(1-x)(1+x)}$$

let $a = \sqrt[n]{1+x}$
$b = \sqrt[n]{1-x}$ $\Rightarrow$ $a + 2b = 3\sqrt{ab}$

quadratic equation in $\sqrt{a}, \sqrt{b}$
homogeneous

by dividing by b:

$$\frac{a}{b} + 2 = 3\frac{\sqrt{ab}}{b} = 3\sqrt{\frac{a}{b}}$$

let $s = \frac{a}{b}$ $\Rightarrow$ $s + 2 = 3\sqrt{s}$ $\Rightarrow$ $\sqrt{s} = 1$ or 2

$\Rightarrow$ — — the same as the first solution

3^{rd} solution

$$a + 2b = 3\sqrt{ab}$$

Note that $1 + 2 = 3$

$$a - \sqrt{ab} = 2\sqrt{ab} - 2b$$

$$\Rightarrow \sqrt{a}(\sqrt{a} - \sqrt{b}) = 2\sqrt{b}(\sqrt{a} - \sqrt{b})$$

$$\Rightarrow \sqrt{a} = \sqrt{b} \quad \text{or} \quad \sqrt{a} = 2\sqrt{b}$$

$\Rightarrow$ the rest is the same as solution 1

51. Solve in real numbers the system of equations

$$\begin{cases} x^2 - 2y^2 = \sqrt{y(x^3 - 4y^3)} \\ x^2 + 2y^2 = 2y\sqrt{y(5x - y)}. \end{cases}$$

$$\begin{cases} (x^2 - 2y^2)^2 = y(x^3 - 4y^3) \\ (x^2 + 2y^2)^2 = 4y^2 (y(5x - y)) \end{cases}$$

$$\Rightarrow \begin{cases} x^4 + 4y^4 - 4x^2y^2 = x^3y - 4y^4 \\ x^2 + 4y^2 + 4x^2y^2 = 20xy^3 - 4y^4 \end{cases}$$

$$8x^2y^2 = 20xy^3 - x^3y$$

- if $xy \neq 0$:

$$8xy = 20y^2 - x^2$$

$$x^2 + 8xy + 16y^2 = 36y^2$$

$$(x + 4y)^2 = 36y^2 \Rightarrow x + 4y = \pm 6y$$

$$\Rightarrow x = 2y \text{ or } \boxed{x = -10y}$$

- if $x = 0 \Rightarrow 2y^2 = y\sqrt{-y^2} \Rightarrow y = 0$ because $-y^2 \geq 0$
- if $y = 0 \Rightarrow x = 0$
- if $x = -10y \Rightarrow 2y\sqrt{y(5x - y)} \geq 0$

$$\Rightarrow 2y\sqrt{y(-51y)} \geq 0$$

$$\Rightarrow y = 0 \Rightarrow \Leftarrow$$

- if $x = 2y \Rightarrow$ solution

$$(x, y) = (2t, t) \quad \forall t \in \mathbb{R}$$

Homework:

49. Let a and b be nonnegative real numbers such that

$$2a^2 + 3ab + 2b^2 \le 7.$$

Prove that $\max(2a+b, 2b+a) \le 4$.

Hint for 4:

Assume that $a \geqslant b$, $2a+b = \max(2a+b, 2b+a)$

$$\frac{7}{16}(2a+b)^2 \leqslant 2a^2+3ab+2b^2 \leqslant 7$$

50. Let a, b, c be positive real numbers. Prove that

$$\frac{1+a(b+c)}{(1+b+c)^2} + \frac{1+b(c+a)}{(1+c+a)^2} + \frac{1+c(a+b)}{(1+a+b)^2} \ge 1.$$

53. Let a, b, c, d be real numbers greater than 0 satisfying $abcd = 1$. Prove that

$$\frac{1}{a+b+2} + \frac{1}{b+c+2} + \frac{1}{c+d+2} + \frac{1}{d+a+2} \le 1.$$

Example 9.9. Find the minimum possible value of

$$\max\{a+b+c, b+c+d, c+d+e, d+e+f, e+f+g\}.$$

over all nonnegative real numbers a, b, c, d, e, f, g such that

$$a+b+c+d+e+f+g = 1.$$