BALKAN MATHEMATICAL OLYMPIAD

TEAM SELECTION TEST

Day 1 , October 14, 2020

Problem 1. For all non-negative real numbers x, y, z with $x \geq y$, prove the inequality

$$\frac{x^3 - y^3 + z^3 + 1}{6} \geq (x - y)\sqrt{xyz}.$$

Problem 2. Let ABC be a scalene and acute triangle, with circumcentre O. Let ω be the circle with centre A, tangent to BC at D. Suppose there are two points F and G on ω such that $FG \perp AO$, $\angle BFD = \angle DGC$ and the couples of points (B, F) and (C, G) are in different halfplanes with respect to the line AD. Show that the tangents to ω at F and G meet on the circumcircle of ABC.

Problem 3. Cells of 11×11 table are colored with n colors (each cell is colored with exactly one color). For each color, the total amount of the cells of this color is not less than 7 and not greater than 13. Prove that there exists at least one row or column which contains cells of at least four different colors.

Problem 4. Let $a_1 \in \mathbb{Z}$, $a_2 = a_1^2 - a_1 - 1$, $\ldots$, $a_{n+1} = a_n^2 - a_n - 1$. Prove that a_{n+1} and $2n + 1$ are coprime.