Day 1

Problem 1. *Let (a_n) be the integer sequence which is defined by $a_1 = 1$ and*

$$a_{n+1} = a_n^2 + na_n, \ \forall n \geq 1.$$

Denote $\mathcal{S}$ is the set of the prime p, which is the divisor of some term of the sequence (a_n). Prove that $\mathcal{S}$ is an infinite set but not the set of all primes.

Solution. First, we shall show that $3 \notin \mathcal{S}$ by proving that

$$a_{3k-2} \equiv 2 \pmod 3, \ a_{3k-1} \equiv 2 \pmod 3 \text{ and } a_{3k} \equiv 2 \pmod 3.$$

Since $a_1 = 1, a_2 = 2, a_3 = 2^2 + 2 \cdot 2 = 8 \equiv 2 \pmod 3$ so the claim is true for $k = 1$. Suppose that the claim holds for $k = n$. We have

$$\begin{aligned} a_{3n+1} &= a_{3n}(a_{3n} + 3n) \equiv 2 \cdot 2 \equiv 1 \pmod 3, \\ a_{3n+2} &= a_{3n+1}(a_{3n+1} + 3n + 1) \equiv 1(1+1) \equiv 2 \pmod 3, \\ a_{3n+3} &= a_{3n+2}(a_{3n+2} + 3n + 2) \equiv 2(2+2) \equiv 2 \pmod 3. \end{aligned}$$

Thus, the claim is also true for $k = n+1$. So by induction, the claim is proved.

Now suppose on the contrary that $\mathcal{S}$ is finite, denote $\mathcal{S} = \{p_1, p_2, ..., p_k\}$. Note that $a_n | a_{n+1}$ so $a_n | a_m$ for all $m \geq n$. By the definition of $\mathcal{S}$, there are some index t such that $p_1 | a_t$, thus $p_1 | a_{t'}$ for all $t' \geq t$. Similarly for $p_2, p_3, ..., p_k$ so there exist N big enough such that $p_1 p_2 ... p_k | a_n$ for all $n \geq N$. Taking the integer $\ell > N + 1$ such that $\ell \equiv 2 \pmod{p_1 p_2 ... p_k}$. Since $a_\ell = a_{\ell-1}(a_{\ell-1} + \ell - 1)$, we get

$$a_{\ell-1} + \ell - 1 \equiv 1 \pmod{p_1 p_2 ... p_k}$$

so $a_{\ell-1} + \ell - 1$ is coprime to all primes in $\mathcal{S}$, which implies that is has some prime divisor that not belong to $\mathcal{S}$, a contradiction. Hence, $\mathcal{S}$ is an infinite set. $\square$

P2 Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Solution 1. Let the segments TE and TF cross AC at P and Q, respectively. Since $PE \parallel CD$ and ED is tangent to the circumcircle of $ABCD$, we have

$$\angle EPA = \angle DCA = \angle EDA,$$

and so the points A, P, D, and E lie on some circle α. Similarly, the points C, Q, D, and F lie on some circle γ.

We now want to prove that the line DT is tangent to both α and γ at D. Indeed, since $\angle FCD + \angle EAD = 180°$, the circles α and γ are tangent to each other at D. To prove that T lies on their common tangent line at D (i.e., on their radical axis), it suffices to check that $TP \cdot TE = TQ \cdot TF$, or that the quadrilateral $PEFQ$ is cyclic. This fact follows from

$$\angle QFE = \angle ADE = \angle APE.$$

Since $TD = TK$, we have $\angle TKD = \angle TDK$. Next, as TD and DE are tangent to α and Ω, respectively, we obtain

$$\angle TKD = \angle TDK = \angle EAD = \angle BDE,$$

which implies $TK \parallel BD$.

Next we prove that the five points T, P, Q, D, and K lie on some circle τ. Indeed, since TD is tangent to the circle α we have

$$\angle EPD = \angle TDF = \angle TKD,$$

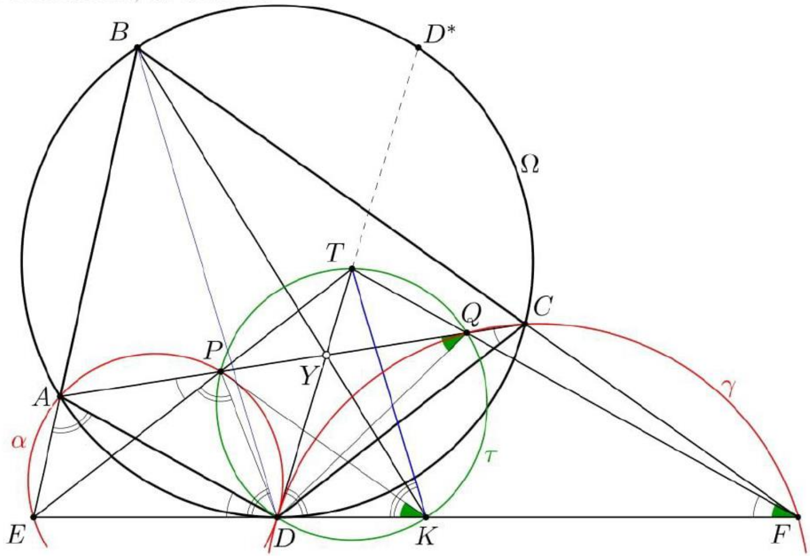
which means that the point P lies on the circle (TDK). Similarly, we have $Q \in (TDK)$.

Finally, we prove that $PK \parallel BC$. Indeed, using the circles τ and γ we conclude that

$$\angle PKD = \angle PQD = \angle DFC,$$

which means that $PK \parallel BC$.

Triangles TPK and DCB have pairwise parallel sides, which implies the fact that TD, PC and KB are concurrent, as desired.



Comment 1. There are several variations of the above solution.

E.g., after finding circles α and γ, one can notice that there exists a homothety h mapping the triangle TPQ to the triangle DCA; the centre of that homothety is $Y = AC \cap TD$. Since

$$\angle DPE = \angle DAE = \angle DCB = \angle DQT,$$

the quadrilateral $TPDQ$ is inscribed in some circle τ. We have $h(\tau) = \Omega$, so the point $D^* = h(D)$ lies on Ω.

Finally, by

$$\angle DCD^* = \angle TPD = \angle BAD,$$

the points B and D^* are symmetric with respect to the diameter of Ω passing through D. This yields $DB = DD^*$ and $BD^* \parallel EF$, so $h(K) = B$, and BK passes through Y.

Solution 2. Consider the spiral similarity ϕ centred at D which maps B to F. Recall that for any two points X and Y, the triangles $DX\phi(X)$ and $DY\phi(Y)$ are similar.

Define $T' = \phi(E)$. Then

$$\angle CDF = \angle FBD = \angle \phi(B)BD = \angle \phi(E)ED = \angle T'ED,$$

so $CD \parallel T'E$. Using the fact that DE is tangent to (ABD) and then applying ϕ we infer

$$\angle ADE = \angle ABD = \angle T'FD,$$

so $AD \parallel T'F$; hence T' coincides with T. Therefore,

$$\angle BDE = \angle FDT = \angle DKT,$$

whence $TK \parallel BD$.

Let $BK \cap TD = X$, $AC \cap TD = Y$, and $AC \cap TF = Q$. Notice that $TK \parallel BD$ implies

$$\frac{TX}{XD} = \frac{TK}{BD} = \frac{TD}{BD}.$$

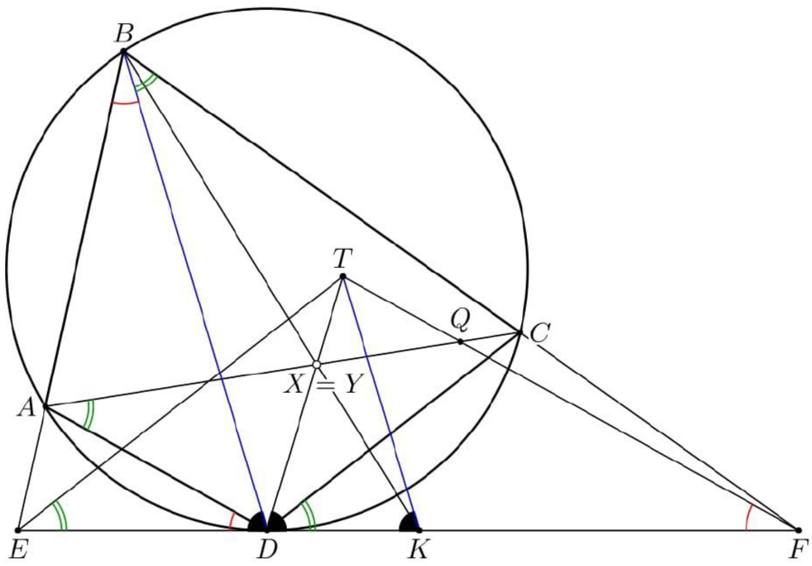
So we wish to prove that $\frac{TY}{YD}$ is equal to the same ratio.

We first show that $\phi(A) = Q$. Indeed,

$$\angle DA\phi(A) = \angle DBF = \angle DAC,$$

and so $\phi(A) \in AC$. Together with $\phi(A) \in \phi(EB) = TF$ this yields $\phi(A) = Q$. It follows that

$$\frac{TQ}{AE} = \frac{TD}{ED}.$$



Now, since $TF \parallel AD$ and $\triangle EAD \sim \triangle EDB$, we have

$$\frac{TY}{YD} = \frac{TQ}{AD} = \frac{TQ}{AE} \cdot \frac{AE}{AD} = \frac{TD}{ED} \cdot \frac{ED}{BD} = \frac{TD}{BD},$$

which completes the proof.

Comment 2. The point D is the Miquel point for any 4 of the 5 lines BA, BC, TE, TF and AC. Essentially, this is proved in both solutions by different methods.

Problem 3 . *Find all non-constant functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfy*

$$f(ab + bc + ca) = f(a)f(b) + f(b)f(c) + f(c)f(a), \quad \forall a, b, c \in \mathbb{Q}^+.$$

Solution. Put $c = 1$ in the given condition, we have

$$f(ab + a + b) = f(a)f(b) + f(a)f(1) + f(b)f(1); \quad \forall a, b \in \mathbb{R}^+, \quad (1)$$

Put $b = 3$ into (1), we have

$$f(4a + 3) = f(a)f(3) + f(a)f(1) + f(3)f(1); \quad \forall a \in \mathbb{R}^+.$$

Put $b = 1$ into (1), we have

$$f(2a + 1) = 2f(a)f(1) + f(1)^2; \quad \forall a \in \mathbb{R}^+.$$

Thus

$$f(4a+3) = 2f(2a+1)f(1) + f(1)^2 = 4f(1)^2 f(a) + 2f(1)^3 + f(1)^2; \quad \forall a \in \mathbb{R}^+.$$

From these, we can conclude that

$$[f(3) + f(1)]\, f(a) + f(3)f(1) = 4f(1)^2 f(a) + 2f(1)^3 + f(1)^2; \ \forall a \in \mathbb{R}^+.$$

If $f(3) + f(1) \neq 4f(1)^2$ then f is constant. Thus $f(3) + f(1) = 4f(1)^2$, otherwise f will be constant. So we must have

$$f(3) + f(1) = 4f(1)^2 \text{ and } f(3)f(1) = 2f(1)^3 + f(1)^2.$$

Thus $f(3), f(1)$ are solutions of the quadratic equation $t^2 - 2f(1)t + 2f(1)^3 + f(1)^2 = 0$, thus

$$f(1)^2 - 4f(1)^2 + 2f(1)^3 + f(1)^2 = 0 \Leftrightarrow f(1)^2(f(1) - 1) = 0.$$

So we must have $f(1) = 1$ and then $f(3) = 3$. Put $c = 1$ into (1), we have

$$f(ab + a + b) = f(a)f(b) + f(a) + f(b); \quad \forall a, b \in \mathbb{R}^+ \quad (2)$$

Continue to put $b = 1$ and $b = 3$, we get

$$f(4a+3) = 4f(a) + 3 \text{ and } f(2a+1) = 2f(a) + 1.$$

Put $a = b = c = \frac{1}{3}$ into the given condition, $f\left(\frac{1}{3}\right) = 3f\left(\frac{1}{3}\right)^2$ so $f\left(\frac{1}{3}\right) = \frac{1}{3}$.
Put $a = 2$ and $b = \frac{1}{3}$ into (2), we have $f(3) = f(2)f\left(\frac{1}{3}\right) + f\left(\frac{1}{3}\right) + f(2)$, thus$f(2) = 2$.

Put $b = c = 2$ into the given condition, $f(4a+4) = 4f(a) + 4; \quad \forall a \in \mathbb{R}^+$ thus

$$4f(a) + 4 = f(4a+4) = f\left(4\left(a + \frac{1}{4}\right) + 3\right) = 4f\left(a + \frac{1}{4}\right) + 3.$$

From these, we can conclude that $f\left(a + \frac{1}{4}\right) = f(a) + \frac{1}{4}$, thus

$$f(4a+4) = 4f(a) + 4; \quad \forall a \in \mathbb{R}^+.$$

Hence, by induction, one can show that $f(x+n) = f(x) + n$ for all positive integer n and positive real number x; thus $f(n) = n, \quad \forall n \in \mathbb{Z}^+$.

Finally, put $b \to n$ and $a \to \frac{m}{n+1}$ for some $m, n \in \mathbb{Z}^+$ into (2), we get

$$f(m+n) = f(n)f\left(\frac{m}{n+1}\right) + f\left(\frac{m}{n+1}\right) + f(n) \to f\left(\frac{m}{n+1}\right) = \frac{m}{n+1}.$$

Thus $f(x) = x$ for all $x \in \mathbb{Q}^+$. It is easy to check this function satisfies the condition. $\square$