The Online Math Open Spring Contest
March 18 - 29, 2016

Online Math Open

Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.
Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.
Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2016 Spring Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams cannot communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.
After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com`. (Include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Let A_n denote the answer to the nth problem on this contest ($n = 1, \ldots, 30$); in particular, the answer to this problem is A_1. Compute $2A_1(A_1 + A_2 + \cdots + A_{30})$.

2. Let x, y, and z be real numbers such that $x + y + z = 20$ and $x + 2y + 3z = 16$. What is the value of $x + 3y + 5z$?

3. A store offers packages of 12 pens for \$10 and packages of 20 pens for \$15. Using only these two types of packages of pens, find the greatest number of pens \$173 can buy at this store.

4. Given that x is a real number, find the minimum value of $f(x) = |x+1|+3|x+3|+6|x+6|+10|x+10|$.

5. Let ℓ be a line with negative slope passing through the point $(20, 16)$. What is the minimum possible area of a triangle that is bounded by the x-axis, y-axis, and ℓ?

6. In a round-robin basketball tournament, each basketball team plays every other basketball team exactly once. If there are 20 basketball teams, what is the greatest number of basketball teams that could have at least 16 wins after the tournament is completed?

7. Compute the number of ordered quadruples of positive integers (a, b, c, d) such that
$$a! \cdot b! \cdot c! \cdot d! = 24!.$$

8. Let $ABCDEF$ be a regular hexagon of side length 3. Let X, Y, and Z be points on segments AB, CD, and EF such that $AX = CY = EZ = 1$. The area of triangle XYZ can be expressed in the form $\dfrac{a\sqrt{b}}{c}$ where a, b, c are positive integers such that b is not divisible by the square of any prime and $\gcd(a, c) = 1$. Find $100a + 10b + c$.

9. Let $f(n) = 1 \times 3 \times 5 \times \cdots \times (2n-1)$. Compute the remainder when $f(1) + f(2) + f(3) + \cdots + f(2016)$ is divided by 100.

10. Lazy Linus wants to minimize his amount of laundry over the course of a week (seven days), so he decides to wear only three different T-shirts and three different pairs of pants for the week. However, he doesn't want to look dirty or boring, so he decides to wear each piece of clothing for either two or three (possibly nonconsecutive) days total, and he cannot wear the same outfit (which consists of one T-shirt and one pair of pants) on two different (not necessarily consecutive) days. How many ways can he choose the outfits for these seven days?

11. For how many positive integers x less than 4032 is $x^2 - 20$ divisible by 16 and $x^2 - 16$ divisible by 20?

12. A 9-*cube* is a nine-dimensional hypercube (and hence has 2^9 vertices, for example). How many five-dimensional faces does it have?

    (An n dimensional hypercube is defined to have vertices at each of the points $(a_1, a_2, \cdots, a_n)$ with $a_i \in \{0, 1\}$ for $1 \le i \le n$.)

13. For a positive integer n, let $f(n)$ be the integer formed by reversing the digits of n (and removing any leading zeroes). For example $f(14172) = 27141$. Define a sequence of numbers $\{a_n\}_{n \ge 0}$ by $a_0 = 1$ and for all $i \ge 0$, $a_{i+1} = 11a_i$ or $a_{i+1} = f(a_i)$. How many possible values are there for a_8?

14. Let ABC be a triangle with $BC = 20$ and $CA = 16$, and let I be its incenter. If the altitude from A to BC, the perpendicular bisector of AC, and the line through I perpendicular to AB intersect at a common point, then the length AB can be written as $m + \sqrt{n}$ for positive integers m and n. What is $100m + n$?

15. Let a, b, c, d be four real numbers such that $a + b + c + d = 20$ and $ab + bc + cd + da = 16$. Find the maximum possible value of $abc + bcd + cda + dab$.

16. Jay is given a permutation $\{p_1, p_2, \ldots, p_8\}$ of $\{1, 2, \ldots, 8\}$. He may take two dividers and split the permutation into three non-empty sets, and he concatenates each set into a single integer. In other words, if Jay chooses a, b with $1 \le a < b < 8$, he will get the three integers $\overline{p_1 p_2 \ldots p_a}$, $\overline{p_{a+1} p_{a+2} \ldots p_b}$, and $\overline{p_{b+1} p_{b+2} \ldots p_8}$. Jay then sums the three integers into a sum $N = \overline{p_1 p_2 \ldots p_a} + \overline{p_{a+1} p_{a+2} \ldots p_b} + \overline{p_{b+1} p_{b+2} \ldots p_8}$. Find the smallest positive integer M such that no matter what permutation Jay is given, he may choose two dividers such that $N \le M$.

17. A set $S \subseteq \mathbb{N}$ satisfies the following conditions:

    (a) If $x, y \in S$ (not necessarily distinct), then $x + y \in S$.

    (b) If x is an integer and $2x \in S$, then $x \in S$.

    Find the number of pairs of integers (a, b) with $1 \le a, b \le 50$ such that if $a, b \in S$ then $S = \mathbb{N}$.

18. Kevin is in kindergarten, so his teacher puts a 100×200 addition table on the board during class. The teacher first randomly generates distinct positive integers $a_1, a_2, \ldots, a_{100}$ in the range $[1, 2016]$ corresponding to the rows, and then she randomly generates distinct positive integers $b_1, b_2, \ldots, b_{200}$ in the range $[1, 2016]$ corresponding to the columns. She then fills in the addition table by writing the number $a_i + b_j$ in the square (i, j) for each $1 \le i \le 100$, $1 \le j \le 200$.

    During recess, Kevin takes the addition table and draws it on the playground using chalk. Now he can play hopscotch on it! He wants to hop from $(1, 1)$ to $(100, 200)$. At each step, he can jump in one of 8 directions to a new square bordering the square he stands on a side or at a corner. Let M be the minimum possible sum of the numbers on the squares he jumps on during his path to $(100, 200)$ (including both the starting and ending squares). The expected value of M can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

19. Let $\mathbb{Z}_{\ge 0}$ denote the set of nonnegative integers.

    Define a function $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ with $f(0) = 1$ and

    $$f(n) = 512^{\lfloor n/10 \rfloor} f(\lfloor n/10 \rfloor)$$

    for all $n \ge 1$. Determine the number of nonnegative integers n such that the hexadecimal (base 16) representation of $f(n)$ contains no more than 2500 digits.

    Then the condition about $50^2 = 2500$ digits means that $\lfloor \log_{16} f(n) \rfloor + 1 \le 2500$. This is equivalent to $\left\lfloor \frac{n - s(n)}{4} \right\rfloor \le 2499$, which in turn is equivalent to $\frac{n-s(n)}{9} < 1111 + \frac{1}{9}$. But note that $n - s(n)$ is always divisible by 9, so the LHS must be an integer, implying that $\frac{n-s(n)}{9} \le 1111$. Now, if we let $n = \sum_{i=0}^{\infty} a_i 10^i$ for integers $a_i \in [0, 9]$, then $\frac{n-s(n)}{9} = a_1 + 11a_2 + 111a_3 + \ldots$. This shows that $n - s(n)$ is always nonnegative. We can then easily determine that for every integer $k \in [0, 1111]$ except for 1110 and those in $[0, 1109]$ that are $10, 21, 32, 43, 54, 65, 76, 87, 98, 109, 110 \pmod{111}$, there exist exactly 10 solutions to $\frac{n-s(n)}{9} = k$. This is a total of 1001 numbers, so $\boxed{10010}$ solutions.

20. Define $A(n)$ as the average of all positive divisors of the positive integer n. Find the sum of all solutions to $A(n) = 42$.

21. Say a real number r is *repetitive* if there exist two distinct complex numbers z_1, z_2 with $|z_1| = |z_2| = 1$ and $\{z_1, z_2\} \ne \{-i, i\}$ such that

    $$z_1(z_1^3 + z_1^2 + rz_1 + 1) = z_2(z_2^3 + z_2^2 + rz_2 + 1).$$

    There exist real numbers a, b such that a real number r is *repetitive* if and only if $a < r \le b$. If the value of $|a| + |b|$ can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p and q, find $100p + q$.

22. Let ABC be a triangle with $AB = 5$, $BC = 7$, $CA = 8$, and circumcircle ω. Let P be a point inside ABC such that $PA : PB : PC = 2 : 3 : 6$. Let rays $\overrightarrow{AP}$, $\overrightarrow{BP}$, and $\overrightarrow{CP}$ intersect ω again at X, Y, and Z, respectively. The area of XYZ can be expressed in the form $\dfrac{p\sqrt{q}}{r}$ where p and r are relatively prime positive integers and q is a positive integer not divisible by the square of any prime. What is $p + q + r$?

23. Let S be the set of all 2017^2 lattice points (x, y) with $x, y \in \{0\} \cup \{2^0, 2^1, \cdots, 2^{2015}\}$. A subset $X \subseteq S$ is called BQ if it has the following properties:

    (a) X contains at least three points, no three of which are collinear.
    (b) One of the points in X is $(0, 0)$.
    (c) For any three distinct points $A, B, C \in X$, the orthocenter of $\triangle ABC$ is in X.
    (d) The convex hull of X contains at least one horizontal line segment.

    Determine the number of BQ subsets of S.

24. Bessie and her 2015 bovine buddies work at the Organic Milk Organization, for a total of 2016 workers. They have a hierarchy of bosses, where obviously no cow is its own boss. In other words, for some pairs of employees (A, B), B is the boss of A. This relationship satisfies an obvious condition: if B is the boss of A and C is the boss of B, then C is also a boss of A. Business has been slow, so Bessie hires an outside organizational company to partition the company into some number of groups. To promote growth, every group is one of two forms. Either no one in the group is the boss of another in the group, or for every pair of cows in the group, one is the boss of the other. Let G be the minimum number of groups needed in such a partition. Find the maximum value of G over all possible company structures.

25. Given a prime p and positive integer k, an integer n with $0 \le n < p$ is called a (p, k)-Hofstadterian residue if there exists an infinite sequence of integers $n_0, n_1, n_2, \ldots$ such that $n_0 \equiv n$ and $n_{i+1}^k \equiv n_i \pmod{p}$ for all integers $i \ge 0$. If $f(p, k)$ is the number of (p, k)-Hofstadterian residues, then compute $\displaystyle\sum_{k=1}^{2016} f(2017, k)$.

26. Let S be the set of all pairs (a, b) of integers satisfying $0 \le a, b \le 2014$. For any pairs $s_1 = (a_1, b_1), s_2 = (a_2, b_2) \in S$, define

    $$s_1 + s_2 = ((a_1 + a_2)_{2015}, (b_1 + b_2)_{2015}) \text{ and } s_1 \times s_2 = ((a_1a_2 + 2b_1b_2)_{2015}, (a_1b_2 + a_2b_1)_{2015}),$$

    where n_{2015} denotes the remainder when an integer n is divided by 2015.

    Compute the number of functions $f : S \to S$ satisfying

    $$f(s_1 + s_2) = f(s_1) + f(s_2) \text{ and } f(s_1 \times s_2) = f(s_1) \times f(s_2)$$

    for all $s_1, s_2 \in S$.

27. Let ABC be a triangle with circumradius 2 and $\angle B - \angle C = 15^\circ$. Denote its circumcenter as O, orthocenter as H, and centroid as G. Let the reflection of H over O be L, and let lines AG and AL intersect the circumcircle again at X and Y, respectively. Define B_1 and C_1 as the points on the circumcircle of ABC such that $BB_1 \parallel AC$ and $CC_1 \parallel AB$, and let lines XY and B_1C_1 intersect at Z. Given that $OZ = 2\sqrt{5}$, then AZ^2 can be expressed in the form $m - \sqrt{n}$ for positive integers m and n. Find $100m + n$.

28. Let N be the number of polynomials $P(x_1, x_2, \ldots, x_{2016})$ of degree at most 2015 with coefficients in the set $\{0, 1, 2\}$ such that $P(a_1, a_2, \cdots, a_{2016}) \equiv 1 \pmod{3}$ for all $(a_1, a_2, \cdots, a_{2016}) \in \{0, 1\}^{2016}$.

    Compute the remainder when $v_3(N)$ is divided by 2011, where $v_3(N)$ denotes the largest integer k such that $3^k | N$.

29. Yang the Spinning Square Sheep is a square in the plane such that his four legs are his four vertices. Yang can do two different types of *tricks*:

    (a) Yang can choose one of his sides, then reflect himself over the side.

    (b) Yang can choose one of his legs, then rotate $90°$ counterclockwise around the leg.

    Yang notices that after 2016 tricks, each leg ends up in exactly the same place the leg started out in! Let there be N ways for Yang to perform his 2016 tricks. What is the remainder when N is divided by 100000?

30. In triangle ABC, $AB = 3\sqrt{30} - \sqrt{10}$, $BC = 12$, and $CA = 3\sqrt{30} + \sqrt{10}$. Let M be the midpoint of AB and N be the midpoint of AC. Denote l as the line passing through the circumcenter O and orthocenter H of ABC, and let E and F be the feet of the perpendiculars from B and C to l, respectively. Let l' be the reflection of l in BC such that l' intersects lines AE and AF at P and Q, respectively. Let lines BP and CQ intersect at K. X, Y, and Z are the reflections of K over the perpendicular bisectors of sides BC, CA, and AB, respectively, and R and S are the midpoints of XY and XZ, respectively. If lines MR and NS intersect at T, then the length of OT can be expressed in the form $\frac{p}{q}$ for relatively prime positive integers p and q. Find $100p + q$.