0.1 Inequalities from 2021 Olympiads

Problem 0.1.1. *If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are real numbers satisfying $a_1^2+\cdots+a_n^2 \leq 1$ and $b_1^2+\cdots+b_n^2 \leq 1$, show that:*

$$(1-(a_1^2+\cdots+a_n^2))(1-(b_1^2+\cdots+b_n^2)) \leq (1-(a_1b_1+\cdots+a_nb_n))^2$$

Problem 0.1.2. *Let $(a_n)_{n=1}^{+\infty}$ be a sequence defined recursively as follows: $a_1 = 1$ and*

$$a_{n+1} = 1 + \sum_{k=1}^{n} k a_k$$

For every $n > 1$, prove that $\sqrt[n]{a_n} < \frac{n+1}{2}$.

Problem 0.1.3. *Let a, b, c, d real numbers such that:*

$$a+b+c+d = 0 \text{ and } a^2+b^2+c^2+d^2 = 12$$

Find the minimum and maximum possible values for $abcd$, and determine for which values of a, b, c, d the minimum and maximum are attained.

Problem 0.1.4. *For arbitrary positive reals $a \geq b \geq c$ prove the inequality:*

$$\frac{a^2+b^2}{a+b} + \frac{a^2+c^2}{a+c} + \frac{c^2+b^2}{c+b} \geq (a+b+c) + \frac{(a-c)^2}{a+b+c}$$

Problem 0.1.5. *Let $a, b, c \geq 0$ and $a+b+c = 1$. Prove that*

$$\frac{a}{2a+1} + \frac{b}{3b+1} + \frac{c}{6c+1} \leq \frac{1}{2}.$$

Problem 0.1.6. *1400 real numbers are given. Prove that one can choose three of them like x, y, z such that :*

$$\left|\frac{(x-y)(y-z)(z-x)}{x^4+y^4+z^4+1}\right| < 0.009$$

Problem 0.1.7. *Let a, b, c are non-negative numbers such that*

$$2(a^2+b^2+c^2) + 3(ab+bc+ca) = 5(a+b+c)$$

then prove that $4(a^2+b^2+c^2) + 2(ab+bc+ca) + 7abc \leq 25$

Problem 0.1.8. *Let a, b and c be positive real numbers satisfying $abc = 1$. Prove that*

$$\frac{1}{a^3+2b^2+2b+4} + \frac{1}{b^3+2c^2+2c+4} + \frac{1}{c^3+2a^2+2a+4} \leq \frac{1}{3}.$$

Problem 0.1.9. *Suppose $x_1, x_2, ..., x_{60} \in [-1, 1]$. Find the maximum of*

$$\sum_{i=1}^{60} x_i^2(x_{i+1} - x_{i-1}),$$

where $x_{i+60} = x_i$.

Problem 0.1.10. *Let a, b and c be positive real numbers such that $a^5+b^5+c^5 = ab^2+bc^2+ca^2$. Prove the inequality:*

$$\frac{a^2+b^2}{b} + \frac{b^2+c^2}{c} + \frac{c^2+a^2}{a} \geq 2(ab+bc+ca).$$