2017 IMO Winter Camp: Warm-up Problems

Algebra

1. The sequence a_n is defined by $a_1 = \frac{1}{2}$ and for all positive integers $m > 1$,
$$a_m = \frac{a_{m-1}}{2ma_{m-1} + 1}$$
Determine the value of $a_1 + a_2 + \cdots + a_k$ in terms of k.

2. Prove that the decimal representation of $\sqrt{2}$ contains at least one nonzero digit between its one millionth and three millionth digits after the decimal point.

3. The number $a_i + b_j$ is written in the (i, j)th square of a 100×100 grid, where $a_1, a_2, \ldots, a_{100}$ and $b_1, b_2, \ldots, b_{100}$ are given distinct real numbers. If the products of the numbers written in each row is 1, prove that the products of the numbers written in each column is -1.

4. Find all functions f from the interval $(1, \infty)$ to $(1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \le y \le x^3$, then $(f(x))^2 \le f(y) \le (f(x))^3$.

Combinatorics

1. There are 30 teams in NBA and every team plays 82 games in the year. Bosses of the NBA want to divide the teams between the Western and Eastern Conferences and devise a game schedule such that number of games between teams from different conferences is half the total number of games. Can they do it?

2. There are three colleges in a town. Each college has n students. Any student of any college knows $n + 1$ students of the other two colleges. Prove that it is possible to choose a student from each of the three colleges so that all three students would know each other.

3. Let k be a positive integer. Some of the $2k$-element subsets of a given set are marked. Suppose that for any subset of cardinality less than or equal to $(k+1)^2$ all the marked subsets contained in it (if any) have a common element. Show that all the marked subsets have a common element.

4. There are $2n + 1$ points in a plane such that no four are concyclic and no three are collinear. A circle is called halving if it passes through three of these points and contains exactly $n - 1$ points in its interior.

(a) Prove that there are at least $\frac{2n^2+n}{3}$ halving circles.

(b) Prove that the number of halving circles has the same parity as n.

(c) Prove that there are exactly n^2 halving circles.

Number Theory

1. Is there a natural number n such that the number with decimal representation $\overline{anb}$ is divisible by $\overline{ab}$ for all digits a and b?

2. Suppose that p is a prime number and there are positive integers u and v such that p^2 is the mean of u^2 and v^2. Prove that $2p - u - v$ is either a square or twice a square.

3. Find all pairs of prime numbers (p, q) such that $5^p + 5^q$ is divisible by pq.

4. Does there exist a positive integer n greater than 10^{1000} such that n is not divisible by 10 and it is possible to exchange two distinct digits of n without changing its set of prime divisors?

Geometry

1. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

2. Given a parallelogram $ABCD$ with $AB < BC$, prove that the circumcircles of the triangles APQ share a common point apart from A as the points P and Q move on the sides BC and CD respectively while satisfying $CP = CQ$.

3. Let the circle ω_1 be internally tangent to another circle ω_2 at N. Take a point K on ω_1 and draw a tangent AB to ω_1 at K which intersects ω_2 at A and B. Let M be the midpoint of the arc AB such that N and M are on opposite sides of AB. Prove that, the circumradius of the $\triangle KBM$ does not depend on the choice of K.

4. The quadrilateral $ABCD$ is inscribed in a circle and contains points P and Q satisfying

$$\angle PDC + \angle PCB = \angle PAB + \angle PBC = \angle QCD + \angle QDA = \angle QBA + \angle QAD = 90^\circ$$

Prove that the line PQ forms equal angles with lines AD and BC.