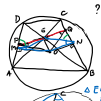
1.



? $EP = EQ$.

$\measuredangle OPE = \measuredangle OQE$

M, N = midpoints of AD, BC

$OEPM$, $OENQ$ - cyclic

$\measuredangle OPE = \measuredangle OME$, $\measuredangle OQE = \measuredangle ONE$.

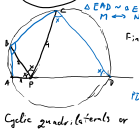
It remains to show that $\measuredangle OME = \measuredangle ONE$.

$\triangle EAD \sim \triangle EBC$

$M \leftrightarrow N \Rightarrow \measuredangle EMD = \measuredangle ENC$

Variant of the Butterfly Lemma

2.



Find AD.

$\triangle PAB \sim \triangle PBC$.

$\measuredangle ABC = \measuredangle ABP + \measuredangle PBC = \measuredangle ABP + \measuredangle PAB = 180 - r$

$\measuredangle APC = r$

$PD = PC = 4 \Rightarrow AD = 5$.

Cyclic quadrilaterals or geometric inequalities?

3.

$AD = BC$

$AM : MB = DN : NC$. Prove that $MN \le AD$.

$MP \parallel BC \Rightarrow AP : PC = AM : MB = DN : NC$

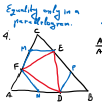
$\frac{AM}{AB} = \frac{DN}{DC} = k \Rightarrow MP \parallel AD$.

$MP = k \cdot BC$, $NP = (1-k) AD$

So $MN \le MP + NP = AD$.

Equality only in a parallelogram.

4.



$\frac{AD}{AB} = \frac{BE}{BC} = \frac{CF}{CA} = k > \frac{1}{2}$

Prove that $DE + EF + FD < k \cdot (AB + BC + CA)$.

$ME = (1-k) AB$, $MC = (1-k) AC = AF$

$MP = AC - MC - AF = (2k-1) AC$

$EF \le ME + MF = (1-k) AB + (2k-1) AC$

$FD \le NF + ND = (1-k) BC + (2k-1) AB$

$DE \le PD + PE = (1-k) AC + (2k-1) BC$

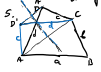
$DE + EF + FD \le ((1-k) + (2k-1))(AB + BC + CA) = k(AB + BC + CA)$

- Triangle ineq.: $AC \le AB + BC$

- $AC \ge BC \Leftrightarrow \measuredangle ABC > \measuredangle BAC$
- $P \le \frac{1}{2} AC \cdot BD$.

equality if $AC \perp BD$.

5.



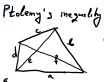
Prove that $P \le \frac{1}{2}(ac + bd)$.

$ACDD'$ is an isosceles trapezoid.

$P_{ABCD'} = P_{ABCD}$

$P_{DAD'} + P_{BCD'} \le \frac{1}{2} ac + \frac{1}{2} bd$.

Ptolemy's inequality



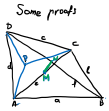
$ac + bd \ge cd$.

6. $a + c = 3$, $b + d = 4$. What is the maximum of $e + f$?

Of course, $e \le a + b$, $c + d$, $f \le b + c$, $d + a$

So $e + f \le a + b + c + d = 7$.

Some proofs



Ptolemy's inequality: $ac + bd \ge ef$

$\triangle DPA \sim \triangle DCB$

$\Rightarrow DP = \frac{d}{f} DC = \frac{cd}{f}$, $AP = \frac{bd}{f}$

$\measuredangle ADP = \measuredangle BDC \Rightarrow \measuredangle PDC = \measuredangle ADB$

$\triangle PDC \sim \triangle ADB \Leftarrow \frac{PD}{DC} = \frac{cd/f}{c} = \frac{d}{f} = \frac{AD}{DB}$

$PC = \frac{e}{f} \cdot AB = \frac{ce}{f} \Rightarrow AC \le AP + PC$

$c \le \frac{bd}{f} + \frac{ce}{f}$

Parallelogram inequality: $a^2 + b^2 + c^2 + d^2 \ge e^2 + f^2$

M, N = midpoints of AC, BD. Compute MN.

MN = median in MBD, so $MN^2 = \frac{2}{4} BM^2 + \frac{2}{4} DM^2 - \frac{1}{4} BD^2$

$BM^2 = \frac{1}{2} a^2 + \frac{1}{2} b^2 - \frac{1}{2} e^2$, $DM^2 = \frac{1}{2} c^2 + \frac{1}{2} d^2 - \frac{1}{2} e^2$

$\Rightarrow MN^2 = \frac{1}{2}\left(\frac{1}{2} a^2 + \frac{1}{2} b^2 - \frac{1}{2} e^2\right) + \frac{1}{2}\left(\frac{1}{2} c^2 + \frac{1}{2} d^2 - \frac{1}{2} e^2\right) - \frac{1}{4} f^2$

$= \frac{1}{4}\left(a^2 + b^2 + c^2 + d^2 - e^2 - f^2\right)$

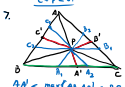
Let's complete problem 6: $(e + f)^2 = e^2 + f^2 + 2ef$

$e + f \le 7$

$\le a^2 + b^2 + c^2 + d^2 + 2ac + 2bd$

$= (a + c)^2 + (b + d)^2 = 9 + 16 = 25$

7.



side BC longest

P = any point inside $\triangle ABC$

Prove that $PA' + PB' + PC' < BC$.

$AA' + AC < BC$

$BB' < BC$

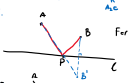
$CC' <$ max(AB, AC) $< BC$

$PA' < A_1A_2$

$PA_1 \parallel AB$, $PA_2 \parallel AC$

$PB' < PB_1 \le AC$, $PC' < PC_2 \le BA_1 \Rightarrow PA' + PB' + PC' < A_1A_2 + AC + BA_1 = BC$

8.



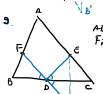
For which P is $AP + BP$ minimal?

B' = mirror of B in ℓ

$AP + BP = AP + B'P \ge AB'$

equality if P lies on AB'

9.

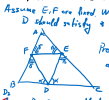


ABC acute

Find D, E, F for which $DE + EF + FD$ is minimal.

Assume E, F are fixed. Where should D be?

D should satisfy $\measuredangle FDB = \measuredangle EDC$.



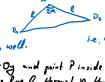
Probably, AD, BE, CF are the altitudes.

The optimal position must look like this:

D_1, D_2 = reflections of D in AB, AC

$DE + EF + FD = D_1E + EF + FD_2 \ge D_1D_2$

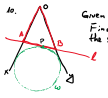
$AD = E = AD_1 = AD_2$



D_1D_2 is minimal when ℓ is minimal i.e. when AD is the altitude.

Equality is attained if BE, CF are altitudes as well.

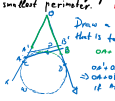
10.



Given $\angle xOy$ and point P inside it.

Find the line ℓ through P that cuts a triangle with the smallest perimeter.

min $\{OA + OB + AB\}$



Draw a circle ω through P that is tangent to x and y.

$OA + PB + AB = 2OC$

$OA' + OB' + A'B' = 2OC$

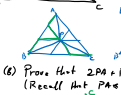
$\Rightarrow OA + OB + AB \ge 2OC$

if AB is not the tangent.

11.

$PA \le PB, PC$.

(5) Prove that $PA + PB + PC \le AB + BC + CA$.



We claim that $PB + PC < AB + AC$

$BP + PC < BP + PD + DC$

$= BD + DC < BA + AD + DC = BA + AC$

$PC + PA < BC + BA$

$PA + PB < CA + CB$

(6) Prove that $2PA + PB + PC \le AB + BC + CA$.

(Recall that $PA \le PB, PC$.)

Intermezzo:

An important lemma based on Δ-inequality

$AD_1D_2D_3 \ldots D_nB$ is convex, inside $\triangle ABC$

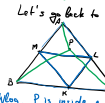
Prove that $AD_1 + D_1D_2 + \ldots + D_nD_n + D_nB < AC + CB$.

$AD_1 + D_1D_2 + D_2D_3 + D_3B < AD_1 + D_1E + EB < AC + CB$

Similarly, if P and Q are convex polygons and P is inside Q then the perimeter of P is less than the perimeter of Q.

Let's go back to problem 11(b).

K, L, M = midpoints of BC, CA, AB

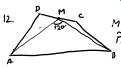


P lies inside four of the trapezoids $BCLM$, $CAMK$, $ABKL$.

Wlog. P is inside $AMKC$ and $ALKB$.

add these: $\begin{cases} AP + PB < AL + LK + KB = \frac{AB + BC + CA}{2} \\ AP + PC < AM + MK + KC \end{cases}$

12.



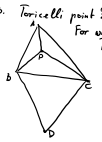
M = midpoint of CD, $\measuredangle AMB = 120°$

Prove that $AB \le AD + \frac{1}{2} DC + CB$.

13. Torricelli point? $\triangle ABC$ is acute

For which P is $AP + BP + CP$ minimal?

Take D so that BDC is equilateral.

Prove that $AP + BP + CP \ge AD$.

For which P does equality hold?