The Online Math Open Fall Contest
October 25 – November 5, 2019

Online Math Open
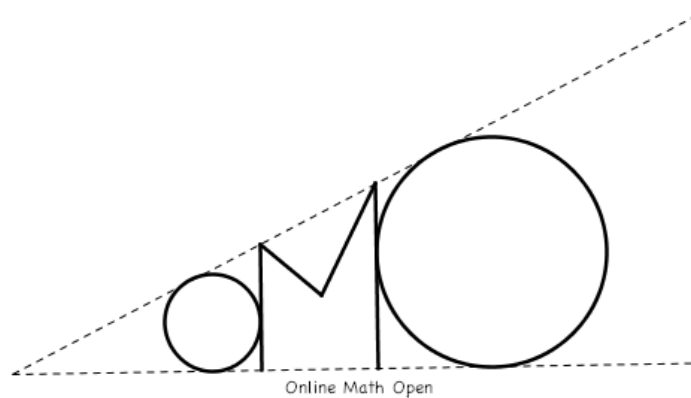
Acknowledgments

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student may belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.

Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2019 Fall Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs, are prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams may not communicate with each other about the contest while the contest is running.

4. Anything that may be considered offensive or inappropriate is not allowed as a team name.

5. Team member names must be the actual team members' names. In particular, names of other people (including celebrities) or phrases that do not represent actual people are not allowed. This includes names of mathematical objects and multiple names concatenated into one name entry.

6. At our discretion, we may rename teams or team members that do not satisfy these rules.

7. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

8. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

Contest Information

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com` (include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

1. Compute the sum of all positive integers n such that the median of the n smallest prime numbers is n.

2. Let A, B, C, and P be points in the plane such that no three of them are collinear. Suppose that the areas of triangles BPC, CPA, and APB are 13, 14, and 15, respectively. Compute the sum of all possible values for the area of triangle ABC.

3. Let k be a positive real number. Suppose that the set of real numbers x such that $x^2 + k|x| \le 2019$ is an interval of length 6. Compute k.

4. Maryssa, Stephen, and Cynthia played a game. Each of them independently privately chose one of Rock, Paper, and Scissors at random, with all three choices being equally likely. Given that at least one of them chose Rock and at most one of them chose Paper, the probability that exactly one of them chose Scissors can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

5. Compute the number of ordered pairs (m, n) of positive integers that satisfy the equation $\operatorname{lcm}(m, n) + \gcd(m, n) = m + n + 30$.

6. An ant starts at the origin of the Cartesian coordinate plane. Each minute it moves randomly one unit in one of the directions up, down, left, or right, with all four directions being equally likely; its direction each minute is independent of its direction in any previous minutes. It stops when it reaches a point (x, y) such that $|x| + |y| = 3$. The expected number of moves it makes before stopping can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

7. At a concert 10 singers will perform. For each singer x, either there is a singer y such that x wishes to perform right after y, or x has no preferences at all. Suppose that there are n ways to order the singers such that no singer has an unsatisfied preference, and let p be the product of all possible nonzero values of n. Compute the largest nonnegative integer k such that 2^k divides p.

8. There are three eight-digit positive integers which are equal to the sum of the eighth powers of their digits. Given that two of the numbers are 24678051 and 88593477, compute the third number.

9. Convex equiangular hexagon $ABCDEF$ has $AB = CD = EF = 1$ and $BC = DE = FA = 4$. Congruent and pairwise externally tangent circles γ_1, γ_2, and γ_3 are drawn such that γ_1 is tangent to side $\overline{AB}$ and side $\overline{BC}$, γ_2 is tangent to side $\overline{CD}$ and side $\overline{DE}$, and γ_3 is tangent to side $\overline{EF}$ and side $\overline{FA}$. Then the area of γ_1 can be expressed as $\frac{m\pi}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

10. Let k be a positive integer. Marco and Vera play a game on an infinite grid of square cells. At the beginning, only one cell is black and the rest are white.

   A turn in this game consists of the following. Marco moves first, and for every move he must choose a cell which is black and which has more than two white neighbors. (Two cells are neighbors if they share an edge, so every cell has exactly four neighbors.) His move consists of making the chosen black cell white and turning all of its neighbors black if they are not already. Vera then performs the following action exactly k times: she chooses two cells that are neighbors to each other and swaps their colors (she is allowed to swap the colors of two white or of two black cells, though doing so has no effect). This, in totality, is a single turn. If Vera leaves the board so that Marco cannot choose a cell that is black and has more than two white neighbors, then Vera wins; otherwise, another turn occurs.

   Let m be the minimal k value such that Vera can guarantee that she wins no matter what Marco does. For $k = m$, let t be the smallest positive integer such that Vera can guarantee, no matter what Marco does, that she wins after at most t turns. Compute $100m + t$.

11. Let ABC be a triangle with incenter I such that $AB = 20$ and $AC = 19$. Point $P \ne A$ lies on line AB and point $Q \ne A$ lies on line AC. Suppose that $IA = IP = IQ$ and that line PQ passes through the midpoint of side BC. Suppose that $BC = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

12. Let $F(n)$ denote the smallest positive integer greater than n whose sum of digits is equal to the sum of the digits of n. For example, $F(2019) = 2028$. Compute $F(1) + F(2) + \cdots + F(1000)$.

13. Compute the number of subsets S with at least two elements of $\{2^2, 3^3, \ldots, 216^{216}\}$ such that the product of the elements of S has exactly 216 positive divisors.

14. The sequence of nonnegative integers $F_0, F_1, F_2, \ldots$ is defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 0$. Let d be the largest positive integer such that, for all integers $n \geq 0$, d divides $F_{n+2020} - F_n$. Compute the remainder when d is divided by 1001.

15. Let A,B,C, and D be points in the plane with $AB = AC = BC = BD = CD = 36$ and such that $A \neq D$. Point K lies on segment AC such that $AK = 2KC$. Point M lies on segment AB, and point N lies on line AC, such that D, M, and N are collinear. Let lines CM and BN intersect at P. Then the maximum possible length of segment KP can be expressed in the form $m + \sqrt{n}$ for positive integers m and n. Compute $100m + n$.

16. Let ABC be a scalene triangle with inradius 1 and exradii r_A, r_B, and r_C such that

$$20 \left(r_B^2 r_C^2 + r_C^2 r_A^2 + r_A^2 r_B^2\right) = 19 \left(r_A r_B r_C\right)^2 .$$

If

$$\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = 2.019,$$

then the area of $\triangle ABC$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

17. For an ordered pair (m, n) of distinct positive integers, suppose, for some nonempty subset S of $\mathbb{R}$, that a function $f : S \to S$ satisfies the property that $f^m(x) + f^n(y) = x + y$ for all $x, y \in S$. (Here $f^k(z)$ means the result when f is applied k times to z; for example, $f^1(z) = f(z)$ and $f^3(z) = f(f(f(z)))$.) Then f is called (m, n)-*splendid*. Furthermore, f is called (m, n)-*primitive* if f is (m, n)-splendid and there do not exist positive integers $a \leq m$ and $b \leq n$ with $(a, b) \neq (m, n)$ and $a \neq b$ such that f is also (a, b)-splendid. Compute the number of ordered pairs (m, n) of distinct positive integers less than 10000 such that there exists a nonempty subset S of $\mathbb{R}$ such that there exists an (m, n)-primitive function $f : S \to S$.

18. Define a *modern artwork* to be a nonempty finite set of rectangles in the Cartesian coordinate plane with positive areas, pairwise disjoint interiors, and sides parallel to the coordinate axes. For a modern artwork S, define its *price* to be the minimum number of colors with which Sean could paint the interiors of rectangles in S such that every rectangle's interior is painted in exactly one color and every two distinct touching rectangles have distinct colors, where two rectangles are *touching* if they share infinitely many points. For a positive integer n, let $g(n)$ denote the maximum price of any modern artwork with exactly n rectangles. Compute $g(1) + g(2) + \cdots + g(2019)$.

19. Let ABC be an acute triangle with circumcenter O and orthocenter H. Let E be the intersection of BH and AC and let M and N be the midpoints of HB and HO, respectively. Let I be the incenter of AEM and J be the intersection of ME and AI. If $AO = 20$, $AN = 17$, and $\angle ANM = 90°$, then $\frac{AI}{AJ} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

20. Define a *crossword puzzle* to be a 15×15 grid of squares, each of which is either black or white. In a crossword puzzle, define a *word* to be a sequence of one or more consecutive white squares in a row or column such that the squares immediately before and after the sequence both are either black or nonexistent. (The latter case would occur if an end of a word coincides with an end of a row or column of the grid.) A crossword puzzle is *tasty* if every word consists of an even number of white squares. Compute the sum of all nonnegative integers n such that there exists a tasty crossword puzzle with exactly n white squares.

21. Let p and q be prime numbers such that $(p - 1)^{q-1} - 1$ is a positive integer that divides $(2q)^{2p} - 1$. Compute the sum of all possible values of pq.

22. For finite sets A and B, call a function $f : A \to B$ an *antibijection* if there does not exist a set $S \subseteq A \cap B$ such that S has at least two elements and, for all $s \in S$, there exists exactly one element s' of S such that $f(s') = s$. Let N be the number of antibijections from $\{1, 2, 3, \ldots 2018\}$ to $\{1, 2, 3, \ldots 2019\}$. Suppose N is written as the product of a collection of (not necessarily distinct) prime numbers. Compute the sum of the members of this collection. (For example, if it were true that $N = 12 = 2 \times 2 \times 3$, then the answer would be $2 + 2 + 3 = 7$.)

23. Let v and w be real numbers such that, for all real numbers a and b, the inequality
$$(2^{a+b} + 8)(3^a + 3^b) \le v(12^{a-1} + 12^{b-1} - 2^{a+b-1}) + w$$
holds. Compute the smallest possible value of $128v^2 + w^2$.

24. Let ABC be an acute scalene triangle with orthocenter H and circumcenter O. Let the line through A tangent to the circumcircle of triangle AHO intersect the circumcircle of triangle ABC at A and $P \ne A$. Let the circumcircles of triangles AOP and BHP intersect at P and $Q \ne P$. Let line PQ intersect segment BO at X. Suppose that $BX = 2$, $OX = 1$, and $BC = 5$. Then $AB \cdot AC = \sqrt{k} + m\sqrt{n}$ for positive integers k, m, and n, where neither k nor n is divisible by the square of any integer greater than 1. Compute $100k + 10m + n$.

25. The sequence $f_0, f_1, \ldots$ of polynomials in $\mathbb{F}_{11}[x]$ is defined by $f_0(x) = x$ and $f_{n+1}(x) = f_n(x)^{11} - f_n(x)$ for all $n \ge 0$. Compute the remainder when the number of nonconstant monic irreducible divisors of $f_{1000}(x)$ is divided by 1000.

26. Let $p = 491$ be prime. Let S be the set of ordered k-tuples of nonnegative integers that are less than p. We say that a function $f \colon S \to S$ is k-*murine* if, for all $u, v \in S$, $\langle f(u), f(v) \rangle \equiv \langle u, v \rangle \pmod{p}$, where $\langle (a_1, \ldots, a_k), (b_1, \ldots, b_k) \rangle = a_1 b_1 + \cdots + a_k b_k$ for any $(a_1, \ldots a_k), (b_1, \ldots b_k) \in S$.

Let $m(k)$ be the number of k-murine functions. Compute the remainder when $m(1) + m(2) + m(3) + \cdots + m(p)$ is divided by 488.

27. A *complex set*, along with its *complexity*, is defined recursively as the following:

- The set $\mathbb{C}$ of complex numbers is a complex set with complexity 1.
- Given two complex sets C_1, C_2 with complexity c_1, c_2 respectively, the set of all functions $f : C_1 \to C_2$ is a complex set denoted $[C_1, C_2]$ with complexity $c_1 + c_2$.

A *complex expression*, along with its *evaluation* and its *complexity*, is defined recursively as the following:

- A single complex set C with complexity c is a complex expression with complexity c that evaluates to itself.
- Given two complex expressions E_1, E_2 with complexity e_1, e_2 that evaluate to C_1 and C_2 respectively, if $C_1 = [C_2, C]$ for some complex set C, then (E_1, E_2) is a complex expression with complexity $e_1 + e_2$ that evaluates to C.

For a positive integer n, let a_n be the number of complex expressions with complexity n that evaluate to $\mathbb{C}$. Let x be a positive real number. Suppose that
$$a_1 + a_2 x + a_3 x^2 + \cdots = \frac{7}{4}.$$
Then $x = \frac{k\sqrt{m}}{n}$, where k,m, and n are positive integers such that m is not divisible by the square of any integer greater than 1, and k and n are relatively prime. Compute $100k + 10m + n$.

28. Let S be the set of integers modulo 2020. Suppose that $a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c$ are arbitrary elements of S. For any $x_1, x_2, ..., x_{2020} \in S$, define $f(x_1, x_2, ..., x_{2020})$ to be the 2020-tuple whose ith coordinate is $x_{i-2} + a_i x_{2019} + b_i x_{2020} + c x_i$, where we set $x_{-1} = x_0 = 0$. Let m be the smallest positive

integer such that, for some values of $a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c$, we have, for all $x_1, x_2, ..., x_{2020} \in S$, that $f^m(x_1, x_2, ..., x_{2020}) = (0, 0, ..., 0)$. For this value of m, there are exactly n choices of the tuple $(a_1, a_2, ..., a_{2020}, b_1, b_2, ..., b_{2020}, c)$ such that, for all $x_1, x_2, ..., x_{2020} \in S$, $f^m(x_1, x_2, ..., x_{2020}) = (0, 0, ..., 0)$. Compute $100m + n$.

29. Let ABC be a triangle. The line through A tangent to the circumcircle of ABC intersects line BC at point W. Points $X, Y \neq A$ lie on lines AC and AB, respectively, such that $WA = WX = WY$. Point X_1 lies on line AB such that $\angle AXX_1 = 90°$, and point X_2 lies on line AC such that $\angle AX_1X_2 = 90°$. Point Y_1 lies on line AC such that $\angle AYY_1 = 90°$, and point Y_2 lies on line AB such that $\angle AY_1Y_2 = 90°$. Let lines AW and XY intersect at point Z, and let point P be the foot of the perpendicular from A to line X_2Y_2. Let line ZP intersect line BC at U and the perpendicular bisector of segment BC at V. Suppose that C lies between B and U. Let x be a positive real number. Suppose that $AB = x + 1$, $AC = 3$, $AV = x$, and $\frac{BC}{CU} = x$. Then $x = \frac{\sqrt{k}-m}{n}$ for positive integers $k, m,$ and n such that k is not divisible by the square of any integer greater than 1. Compute $100k + 10m + n$.

30. For a positive integer n, we say an *n-transposition* is a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that there exist exactly two elements i of $\{1, 2, \ldots, n\}$ such that $\sigma(i) \neq i$.

Fix some four pairwise distinct n-transpositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Let q be any prime, and let $\mathbb{F}_q$ be the integers modulo q. Consider all functions $f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q$ that satisfy, for all integers i with $1 \leq i \leq n$ and all $x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n, y, z \in \mathbb{F}_q^n$,

$$f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, y + z, x_{i+1}, \ldots, x_n),$$

and that satisfy, for all $x_1, \ldots, x_n \in \mathbb{F}_q^n$ and all $\sigma \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$,

$$f(x_1, \ldots, x_n) = -f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

(Note that the equalities in the previous sentence are in $\mathbb{F}_q$. Note that, for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}_q$, we have $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$, where $a_1 + b_1, \ldots, a_n + b_n \in \mathbb{F}_q$.)

For a given tuple $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n$, let $g(x_1, \ldots, x_n)$ be the number of different values of $f(x_1, \ldots, x_n)$ over all possible functions f satisfying the above conditions.

Pick $(x_1, \ldots, x_n) \in (\mathbb{F}_q^n)^n$ uniformly at random, and let $\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be the expected value of $g(x_1, \ldots, x_n)$. Finally, let

$$\kappa(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = -\lim_{q \to \infty} \log_q \left(-\ln \left(\frac{\varepsilon(q, \sigma_1, \sigma_2, \sigma_3, \sigma_4) - 1}{q - 1} \right) \right).$$

Pick four pairwise distinct n-transpositions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ uniformly at random from the set of all n-transpositions. Let $\pi(n)$ denote the expected value of $\kappa(\sigma_1, \ldots, \sigma_4)$. Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients such that $q(-3) \neq 0$ and such that $\pi(n) = \frac{p(n)}{q(n)}$ for infinitely many positive integers n. Compute $\frac{p(-3)}{q(-3)}$.