The Online Math Open Fall Contest
Official Solutions
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Online Math Open

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1. Leonhard has five cards. Each card has a nonnegative integer written on it, and any two cards show relatively prime numbers. Compute the smallest possible value of the sum of the numbers on Leonhard's cards.

   Note: Two integers are relatively prime if no positive integer other than 1 divides both numbers.

   Proposed by Michael Ren and Tristan Shin.

   Answer. $\boxed{4}$.

   Solution. Choosing $0, 1, 1, 1, 1$ gives a sum of four. It's not possible to have a sum less than four, as that would imply at least two of the numbers are 0, and 0 is not relatively prime to itself. Therefore 4 is the answer. $\square$

2. Let $(p_1, p_2, \dots) = (2, 3, \dots)$ be the list of all prime numbers, and $(c_1, c_2, \dots) = (4, 6, \dots)$ be the list of all composite numbers, both in increasing order. Compute the sum of all positive integers n such that $|p_n - c_n| < 3$.

   Proposed by Brandon Wang.

   Answer. $\boxed{16}$.

   Solution. Note that the sequence $\{(p_n, q_n)\}$ starts with

   $$(2,4),(3,6),(5,8),(7,9),(11,10),(13,12),(17,14),\dots,$$

   so $n = 1, 4, 5, 6$ work, while $p_7 = c_7+3$. Note that for $n \geq 7$, $p_{n+1} \geq p_n+2$ by parity, while $c_{n+1} \leq c_n+2$ (since one of c_n+1, c_n+2 is even and thus composite). Thus, $f(n) = p_n - c_n$ is nondecreasing so nothing else works. Hence the sum is $1 + 4 + 5 + 6 = 16$. $\square$

3. Katie has a list of real numbers such that the sum of the numbers on her list is equal to the sum of the squares of the numbers on her list. Compute the largest possible value of the arithmetic mean of her numbers.

   Proposed by Michael Ren.

   Answer. $\boxed{1}$.

   Solution. Let the numbers be $a_1, a_2, \dots, a_n$. By the Trivial Inequality,

   $$(a_1-1)^2 + (a_2-1)^2 + \dots + (a_n-1)^2 \geq 0,$$

   so

   $$\begin{aligned} 0 &\leq \left(a_1^2 + a_2^2 + \dots + a_n^2\right) - 2\left(a_1 + a_2 + \dots + a_n\right) + (1 + 1 + \dots + 1) \\ &= -\left(a_1 + a_2 + \dots + a_n\right) + n \end{aligned}$$

   and hence

   $$\frac{a_1 + a_2 + \dots + a_n}{n} \leq 1.$$

   Equality holds whenever $a_1 = a_2 = \dots = a_n = 1$. $\square$

4. Compute the largest integer that can be expressed in the form $3^{x(3-x)}$ for some real number x.

   Proposed by James Lin.

   Answer. $\boxed{11}$.

   Solution. Clearly the quadratic $x(3-x)$ takes on all values in the range $(-\infty, \frac{9}{4})$. This means $3^{x(3-x)}$ can take on all values in the range $(0, 3^{\frac{9}{4}})$. Since $11^4 < 3^9 < 12^4$, this makes the answer 11. $\square$

5. In triangle ABC, $AB = 8, AC = 9$, and $BC = 10$. Let M be the midpoint of BC. Circle ω_1 with area A_1 passes through A, B, and C. Circle ω_2 with area A_2 passes through A, B, and M. Then $\frac{A_1}{A_2} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

   Proposed by Luke Robitaille.

   Answer. $\boxed{16295}$.

   Solution. Let R_1 be the radius of ω_1, and let R_2 be the radius of ω_2. Then $\frac{A_1}{A_2} = \frac{\pi R_1^2}{\pi R_2^2} = (\frac{R_1}{R_2})^2$, and $\frac{R_1}{R_2} = \frac{2R_1 \sin\angle ABC}{2R_2 \sin\angle ABM} = \frac{AC}{AM}$, by the Extended Law of Sines. Thus $\frac{A_1}{A_2} = \frac{AC^2}{AM^2}$. Now $AM^2 = \frac{2AB^2+2AC^2-BC^2}{4} = \frac{95}{2}$, so $\frac{A_1}{A_2} = \frac{81}{(\frac{95}{2})} = \frac{162}{95}$, so the answer is $\boxed{16295}$. □

6. Patchouli is taking an exam with $k > 1$ parts, numbered Part $1, 2, \ldots, k$. It is known that for $i = 1, 2, \ldots, k$, Part i contains i multiple choice questions, each of which has $(i + 1)$ answer choices. It is known that if she guesses randomly on every single question, the probability that she gets exactly one question correct is equal to 2018 times the probability that she gets no questions correct. Compute the number of questions that are on the exam.

   Proposed by Yannick Yao.

   Answer. $\boxed{2037171}$.

   Solution. For each i, the probability that Patchouli gets exactly one question right in Part i is $i \cdot (\frac{1}{i+1}) \cdot (\frac{i}{i+1})^{i-1} = (\frac{i}{i+1})^i$, which is exactly the same as she getting all questions wrong in that part. Therefore, the probability that Patchouli gets exactly one question right overall is k times the probability that she gets no questions right. Hence $k = 2018$ and the number of questions is $k(k + 1)/2 = 2037171$. □

7. Compute the number of ways to erase 24 letters from the string "OMOMO$\cdots$OMO" (with length 27), such that the three remaining letters are O, M and O in that order. Note that the order in which they are erased does not matter.

   Proposed by Yannick Yao.

   Answer. $\boxed{455}$.

   Solution. By caseworking on the position M that remains, we see that the number of ways is equal to $1 \cdot 13 + 2 \cdot 12 + \cdots + 13 \cdot 1 = \binom{15}{3} = 455$. □

8. Let ABC be the triangle with vertices located at the center of masses of Vincent Huang's house, Tristan Shin's house, and Edward Wan's house; here, assume the three are not collinear. Let $N = 2017$, and define the A-*ntipodes* to be the points $A_1, \ldots, A_N$ to be the points on segment BC such that $BA_1 = A_1A_2 = \cdots = A_{N-1}A_N = A_NC$, and similarly define the B, C-ntipodes. A line ℓ_A through A is called a *qevian* if it passes through an A-ntipode, and similarly we define qevians through B and C. Compute the number of ordered triples (ℓ_A, ℓ_B, ℓ_C) of concurrent qevians through A, B, C, respectively.

   Proposed by Brandon Wang.

   Answer. $\boxed{6049}$.

   Solution. Let $M = N + 1 = 2p$, where p is prime. We note that by Ceva/Bary that it is equivalent to finding the number of solutions $(a, b, c) \in [N]^3$ to $abc = (M - a)(M - b)(M - c)$. In fact, we claim that for any such solution, we require $p \in \{a, b, c\}$. Indeed, since $abc \equiv -abc \pmod{p}$ (from $p \mid M$) we obtain $p \mid abc$, and so p divides one of a, b, c. Since $a, b, c \in [1, 2p - 1]$, one of a, b, c is p.

   If $a = p$, then we get that $b+c = M$, so there are N solutions for this case. Thus there are $3N-2 = 6049$ total solutions since (p, p, p) is counted thrice. □

9. Ann and Drew have purchased a mysterious slot machine; each time it is spun, it chooses a random positive integer such that k is chosen with probability 2^{-k} for every positive integer k, and then it outputs k tokens. Let N be a fixed integer. Ann and Drew alternate turns spinning the machine, with Ann going first. Ann wins if she receives at least N total tokens from the slot machine before Drew receives at least $M = 2^{2018}$ total tokens, and Drew wins if he receives M tokens before Ann receives N tokens. If each person has the same probability of winning, compute the remainder when N is divided by 2018.

   Proposed by Brandon Wang.

   Answer. $\boxed{5}$.

   Solution. We claim that in general the game is fair if $N = M + 1$. This implies that the answer is 5 by Fermat. Indeed, consider the following game: Ann and Drew flip a coin repeatedly, and Ann wants to get to N heads, while Drew wants to get to M tails. By replacing "runs" (i.e. maximal strings of only H or T) with a spin of a slot machine, we see that the two games are the same, so long as the first flip is heads, since the first run is heads. So, we need $N = M + 1$. □

10. Compute the largest prime factor of $357! + 358! + 359! + 360!$.

   Proposed by Luke Robitaille.

   Answer. $\boxed{379}$.

   Solution. Let $N = 357!+358!+359!+360!$. Then $N = 357!*(1+358+358*359+358*359*360) = N = 357!*359*(359+358*360)$. Now $359+358*360 = 359^2+359-1 = 360^2-360-1 = 360^2-19^2 = 341*379$. Thus $N = 357! * 359 * 341 * 379$, so since 379 is prime, the answer is $\boxed{379}$. □

11. Let an ordered pair of positive integers (m, n) be called *regimented* if for all nonnegative integers k, the numbers m^k and n^k have the same number of positive integer divisors. Let N be the smallest positive integer such that $\left(2016^{2016}, N\right)$ is regimented. Compute the largest positive integer v such that 2^v divides the difference $2016^{2016} - N$.

   Proposed by Ashwin Sah.

   Answer. $\boxed{10087}$.

   Solution. Consider any $m = \prod p_i^{e_i}$; then the number of divisors of m^k is just $\prod(ke_i + 1)$. Consider this expression as a polynomial in k, which we denote as $P_m(k)$- if the polynomials agree for some m, n and all nonnegative integers k, then clearly we must have $P_m = P_n$, so the multiset of exponents in the prime factorization of m matches the corresponding multiset in the prime factorization of n.

   Thus $2016^{2016} = 2^{10080} \cdot 3^{4032} \cdot 7^{2016}$ and $N = 2^{10080} \cdot 3^{4032} \cdot 5^{2016}$. Using LTE we find $v = 10080 + v_2\left(\frac{7^2-5^2}{2}\right) + v_2(2016) = 10080 + 2 + 5 = 10087$. □

12. Three non-collinear lattice points A, B, C lie on the plane $1 + 3x + 5y + 7z = 0$. The minimal possible area of triangle ABC can be expressed as $\frac{\sqrt{m}}{n}$ where m, n are positive integers such that there does not exists a prime p dividing n with p^2 dividing m. Compute $100m + n$.

   Proposed by Yannick Yao.

   Answer. $\boxed{8302}$.

   Solution. It's clear that the cross product of the two vectors AB and AC is a vector perpendicular to the plane, with its magnitude equal to twice the area of the triangle. This cross product is clearly a multiple of $[3, 5, 7]$, which means that it must have magnitude of at least $\sqrt{3^2 + 5^2 + 7^2} = \sqrt{83}$. Since we know that $[1, -2, 1] \times [2, 3, -3] = [3, 5, 7]$ the minimum actually possible (the two component vectors are clearly parallel to the plane). So the minimal area is $\sqrt{83}/2$ and the answer is 8302. □

13. Compute the largest possible number of distinct real solutions for x to the equation

$$x^6 + ax^5 + 60x^4 - 159x^3 + 240x^2 + bx + c = 0,$$

where a, b, and c are real numbers.

Proposed by Tristan Shin.

Answer. $\boxed{4}$.

Solution. Let P be the polynomial we are considering. Assume that P has only real roots. Then by Newton's Inequality,

$$\frac{159^2}{\binom{6}{3}^2} \geq \frac{60 \times 240}{\binom{6}{2}\binom{6}{4}},$$

which is not true, contradiction. Thus, P has at least one complex root. Because the coefficients of P are real, it has another complex root, so there are at most 4 real roots. To show that this is the maximum, we construct such as P. Consider

$$P(x) = x^6 - Ax^5 + 60x^4 - 159x^3 + 240x^2 + x$$

for A arbitrarily large. It has a real root of 0. Now, divide out x to get that this is

$$\frac{P(x)}{x} = Q(x) = x^5 - Ax^4 + 60x^3 - 159x^2 + 240x + 1.$$

We aim to show that this can have 3 real roots. Note that $Q(0) = 1$, so there exists a negative root of Q. Now, A can be constructed such that the inequality

$$x^5 + 60x^3 - 159x^2 + 240x + 1 < Ax^4$$

holds for some positive x, say $x = 1$. But then $Q(1) < 0$, so there exists a root of Q in $(0, 1)$ and some other root after 1, so there are at least 3 real roots of Q. Then there are at least 4 real roots of P, but there are at most 4 real roots of P, so there are exactly 4 real roots of P. Thus, the largest possible number of distinct real solutions to $P(x) = 0$ is 4. $\square$

14. In triangle ABC, $AB = 13, BC = 14, CA = 15$. Let Ω and ω be the circumcircle and incircle of ABC respectively. Among all circles that are tangent to both Ω and ω, call those that contain ω *inclusive* and those that do not contain ω *exclusive.* Let $\mathcal{I}$ and $\mathcal{E}$ denote the set of centers of inclusive circles and exclusive circles respectively, and let I and E be the area of the regions enclosed by $\mathcal{I}$ and $\mathcal{E}$ respectively. The ratio $\frac{I}{E}$ can be expressed as $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. Compute $100m + n$.

Proposed by Yannick Yao.

Answer. $\boxed{1558057}$.

Solution. It is well-known that the circumradius R and inradius r are $\frac{65}{8}$ and 4 respectively. By Euler's formula, we have that $OI' = d = \sqrt{R^2 - 2Rr} = \frac{\sqrt{65}}{8}$, wher I' denotes incenter. Notice that if an inclusive circle with center P has radius x, then $OP = R - x, I'P = x - r \Rightarrow OP + I'P = R - r$, and thus $\mathcal{I}$ is an ellipse with foci O and I' and major axis $R - r$. The minor axis of this ellipse is $\sqrt{(R-r)^2 - d^2} = \sqrt{r^2} = r$, which means that $I = \pi(R - r)r = \frac{33}{2}\pi$.

Similarly, if an exclusive circle has center P and radius x, then $OP = R - x, I'P = x + r \Rightarrow OP + IP = R + r$, and thus $\mathcal{E}$ is an ellipse with foci O and I' and major axis $R + r$. The minor axis of this ellipse is $\sqrt{(R+r)^2 - d^2} = \sqrt{r^2 + 4Rr}$, which means that $E = \pi(R + r)\sqrt{r^2 + 4Rr} = \frac{97\sqrt{146}}{8}\pi$.

Therefore, the ratio is equal to $\frac{132}{97\sqrt{146}}$, and its square is $\frac{8712}{686857}$, so our answer is 1558057. $\square$

15. Iris does not know what to do with her 1-kilogram pie, so she decides to share it with her friend Rosabel. Starting with Iris, they take turns to give exactly half of total amount of pie (by mass) they possess to the other person. Since both of them prefer to have as few number of pieces of pie as possible, they use the following strategy: During each person's turn, she orders the pieces of pie that she has in a line from left to right in increasing order by mass, and starts giving the pieces of pie to the other person beginning from the left. If she encounters a piece that exceeds the remaining mass to give, she cuts it up into two pieces with her sword and gives the appropriately sized piece to the other person.

    When the pie has been cut into a total of 2017 pieces, the largest piece that Iris has is $\frac{m}{n}$ kilograms, and the largest piece that Rosabel has is $\frac{p}{q}$ kilograms, where m, n, p, q are positive integers satisfying $\gcd(m, n) = \gcd(p, q) = 1$. Compute the remainder when $m + n + p + q$ is divided by 2017.

    Proposed by Yannick Yao.

    Answer. $\boxed{6}$.

    Solution. After the first two exchanges, Iris will have a $\frac{1}{2}$-kilogram piece and a $\frac{1}{4}$-kilogram piece, while Rosabel will have a $\frac{1}{4}$-kilogram piece. After that, Iris's largest piece will always exceed half of her total when it's her turn since her total amount of cake decreases each time, while Rosabel's largest piece will never exceed half or her total since her total amount of cake increases each time. So Iris will always cut the largest piece she has, while Rosabel will never need to do so. When there are 2017 pieces in total, there will have been 2016 exchanges so far, so Iris's largest piece will have a mass of $\frac{(2^{2015}+1)/3}{2^{2015}}$ kilograms (which is half of the amount of the cake she has after she gives cake to Iris for the 1008th time), while Rosabel's largest piece will have a mass of $\frac{1}{4}$ kilograms. Therefore we have $m = \frac{2^{2015}+1}{3}, n = 2^{2015}, p = 1, q = 4$, and thus $m + n + p + q = \frac{2^{2017}+16}{3} \equiv \frac{2+16}{3} = 6 \pmod{2017}$. □

16. Jay has a 24×24 grid of lights, all of which are initially off. Each of the 48 rows and columns has a switch that toggles all the lights in that row and column, respectively, i.e. it switches lights that are on to off and lights that are off to on. Jay toggles each of the 48 rows and columns exactly once, such that after each toggle he waits for one minute before the next toggle. Each light uses no energy while off and 1 kiloJoule of energy per minute while on. To express his creativity, Jay chooses to toggle the rows and columns in a random order. Compute the expected value of the total amount of energy in kiloJoules which has been expended by all the lights after all 48 toggles.

    Proposed by James Lin.

    Answer. $\boxed{9408}$.

    Solution. Linearity of expectation and grid symmetry means we can calculate $24^2 E[L]$, where L is amount of kilojoules consumed by the bottom left lightbulb. Notice that after all 48 toggles, each light will have been touched precisely twice, so will be off.

    Consider any specific lightbulb- it is toggled on at time t_1 and off at time t_2. Any other of the 46 toggles is equally likely to be before t_1, after t_2, or in between, so by linearity of expectation there are an expected $\frac{46}{3}$ toggles in between t_1, t_2; adding one gives an expected $\frac{49}{3}$ minutes between t_1 and t_2. Therefore the expected value is just $24^2 \cdot \frac{49}{3} = 9408$. □

17. A hyperbola in the coordinate plane passing through the points $(2, 5)$, $(7, 3)$, $(1, 1)$, and $(10, 10)$ has an asymptote of slope $\frac{20}{17}$. The slope of its other asymptote can be expressed in the form $-\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $100m + n$.

    Proposed by Michael Ren.

    Answer. $\boxed{14227}$.

    Solution. Assign $A(2, 5)$, $B(7, 3)$, $C(1, 1)$, and $D(10, 10)$. Apply Desargues' Involution Theorem on $ABCD$. The slopes of lines AB, BC, CD, DA, AC, BD are $-\frac{2}{5}, \frac{1}{3}, 1, \frac{5}{8}, 4, \frac{7}{3}$ respectively, so by Desargues we know $(-\frac{2}{5}, 1; \frac{1}{3}, \frac{5}{8}; 4, \frac{7}{3})$ are pairs of an involution f. Then this involution satisfies $(x+y)(x+f(y)) = c$

for some constants x, c and we simply solve for x, c. Since $(x+\frac{2}{5})(x-1) = (x-\frac{1}{3})(x-\frac{5}{8}) = c$, we get $x = \frac{73}{43}$ and $c = \frac{2706}{1849}$.

Hence, if the desired slope is $s = f\left(\frac{20}{17}\right)$, then $(\frac{73}{43} - \frac{20}{17})(\frac{73}{43} - s) = \frac{2706}{1849}$ so we conclude $s = -\frac{141}{127}$.

(Obviously it's also possible to just bash this problem, but this is the nice solution). $\square$

18. On Lineland there are 2018 bus stations numbered 1 through 2018 from left to right. A self-driving bus that can carry at most N passengers starts from station 1 and drives all the way to station 2018, while making a stop at each bus station. Each passenger that gets on the bus at station i will get off at station j for some $j > i$ (the value of j may vary over different passengers). Call any group of four distinct stations i_1, i_2, j_1, j_2 with $i_u < j_v$ for all $u, v \in \{1, 2\}$ a *good* group. Suppose that in any good group i_1, i_2, j_1, j_2, there is a passenger who boards at station i_1 and de-boards at station j_1, or there is a passenger who boards at station i_2 and de-boards at station j_2, or both scenarios occur. Compute the minimum possible value of N.

Proposed by Yannick Yao.

Answer. $\boxed{1017072}$.

Solution. Call a pair of stations (i, j) *bad* if there are no passengers going from station i to j.

First, we consider the least number of passengers that are on the bus between station 1009 and 1010. Suppose that there is $i \le 1009$ and $j \ge 1010$ such that (i, j) is bad, then by the problem statement all such bad (i', j') must have $i' = i$ or $j' = j$ or the good group i, i', j, j' violates the condition. Moreover, not both $i' = i$ and $j' = j$ can happen for the two different bad pairs because they will again form a good group that violates the condition. This means that all such bad pairs have the same i or all have the same j, meaning that there are at most 1009 bad pairs. This means that the number of passengers on the bus must be at least $1009^2 - 1009 = 1017072$.

We now show that $N = 1017072$ is possible. Suppose that for each pair (i, j) such that $1 \le i < j \le 2017$, there is a passenger from station i to j. Then on the trip from station s to $s+1$ $(s = 1, 2, \dots, 2017)$, there will be $s(2017 - s) \le 1008 \cdot 1009 = N$ passengers on the bus, so the bus can carry all of them. $\square$

19. Players $1, 2, \dots, 10$ are playing a game on Christmas. Santa visits each player's house according to a set of rules:

- Santa first visits player 1. After visiting player i, Santa visits player $i+1$, where player 11 is the same as player 1.
- Every time Santa visits someone, he gives them either a present or a piece of coal (but not both).
- The absolute difference between the number of presents and pieces of coal that Santa has given out is at most 3 at every point in time.
- If Santa has a choice between giving out a present and a piece of coal, he chooses with equal probability.

Let p be the probability that player 1 gets a present before player 2 does. If $p = \frac{m}{n}$ for relatively prime positive integers m and n, then compute $100m + n$.

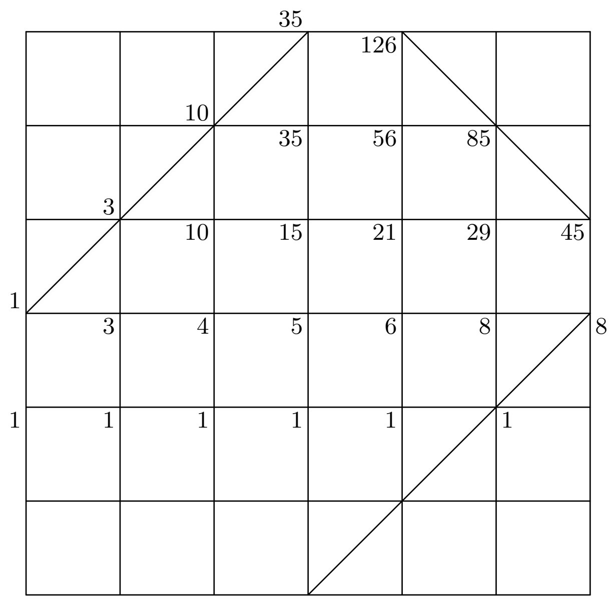
Proposed by Tristan Shin.

Answer. $\boxed{932774}$.

Solution. Suppose that it is currently player 1's turn and let $D = P - C$, where P is the number of presents and C is the number of pieces of coal. Let P_i $(i = 2, 0, -2)$ be the probability that player 1 wins when it is their turn and $D = i$.

Suppose that $D = -2$. If player 1 does not receive a present, then player 2 receives one immediately after, so $P_{-2} = \frac{1}{2}$.

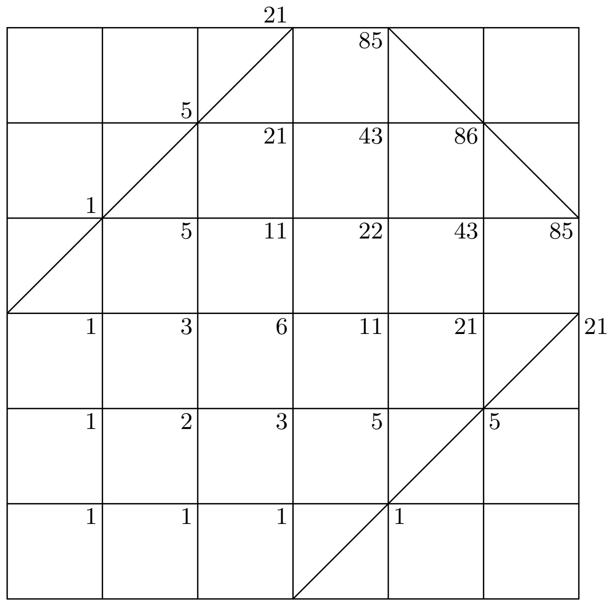
Suppose that $D = 0$. There is a $\frac{1}{2}$ chance that player 1 receives a present and wins. There is a $\frac{1}{4}$ chance that both player 1 and player 2 do not receive presents right now. Consider a Cartesian grid where Santa begins at $(0, 2)$ and moves one unit in the positive x direction if he gives a present and the positive y direction if he gives coal. Move Santa in this pattern until he goes back to player 1. Then Santa will stop somewhere on the line $x + y = 10$ such that $|x - y| \le 3$ for all positions that Santa is in. We can apply modified Pascal counting to count the number of ways that Santa can arrive at each of $(4, 6), (5, 5), (6, 4)$. We do this by starting with 1 at $(1, 1)$ and applying the usual Pascal counting except at a border point along $y = x \pm 3$, their value is doubled when added to get to a neighboring point. This works because the probability that Santa takes the path that is followed along this doubling is 1 rather than $\frac{1}{2}$ (which would be the probability for regular paths). Doing this gets us the following Pascal counting:



So the probability that Santa ends up at $(4, 6)$ is $\frac{126}{256}$, $(5, 5)$ is $\frac{85}{256}$, $(6, 4)$ is $\frac{45}{256}$. These states represent $D = -2, 0, 2$ respectively, so

$$P_0 = \frac{1}{2} + \frac{1}{4}\left(\frac{126}{256}P_{-2} + \frac{85}{256}P_0 + \frac{45}{256}P_2\right).$$

Suppose that $D = 2$. From the same logic as above, it suffices to do Pascal counting except starting at $(1, 1)$ this time. Doing this gets us the following Pascal counting:



So the probability that Santa ends up at $(4,6)$ is $\frac{85}{256}$, $(5,5)$ is $\frac{86}{256}$, $(6,4)$ is $\frac{86}{256}$. These states represent $D=-2,0,2$ respectively, so

$$P_2 = \frac{1}{2} + \frac{1}{4}\left(\frac{85}{256}P_{-2} + \frac{86}{256}P_0 + \frac{85}{256}P_2\right).$$

We can solve this system of equations to deduce that $P_0 = \frac{9185}{14274}$. Since the initial state is $D=0$, we have that $p = P_0$ and hence $100m+n = 932774$. $\square$

20. For positive integers k, n with $k \le n$, we say that a k-tuple $(a_1, a_2, \ldots, a_k)$ of positive integers is *tasty* if

- there exists a k-element subset S of $[n]$ and a bijection $f : [k] \to S$ with $a_x \le f(x)$ for each $x \in [k]$,
- $a_x = a_y$ for some distinct $x, y \in [k]$, and
- $a_i \le a_j$ for any $i < j$.

For some positive integer n, there are more than 2018 tasty tuples as k ranges through $2, 3, \ldots, n$. Compute the least possible number of tasty tuples there can be.

Note: For a positive integer m, $[m]$ is taken to denote the set $\{1, 2, \ldots, m\}$.

Proposed by Tristan Shin and Vincent Huang.

Answer. $\boxed{4606}$.

Solution. For $i = 1, \ldots, n$, let b_i be the number of a_j with $a_j = n+1-i$. Note that $b_1 + b_2 + \ldots + b_n = k \le n$. Note that this is also a bijection to the a_i because the k-tuples are ordered. We will say that a n-tuple $(b_1, b_2, \ldots, b_n)$ of nonnegative integers is *tasteful* if the corresponding k-tuple of a_i is tasty. It suffices to count the tasteful n-tuples.

I claim that there are $\frac{1}{n+2}\binom{2n+2}{n+1} - 2^n$ tasteful n-tuples. To prove this, we will establish that the setup is equivalent to the following:

i) $\displaystyle\sum_{i=1}^{m} b_i \le m$ for $m = 1, 2, \ldots, n$ and ii) $\displaystyle\max_{i=1,2,\ldots,n} b_i \ge 2$.

The first condition is equivalent to the first condition in the problem statement. We will prove this. Consider a sequence of a_i that satisfies the first condition in the problem statement. We wish to show that the number of a_j with $a_j \geq n+1-i$ is at most i. But we can see this as if there were more than i of them, then there are not enough terms of $\{n\}$ to compare with these a_j. Now, assume that we have a sequence of b_i that satisfies this first condition. Then the number of a_j with $a_j \geq n+1-i$ is at most i. Then it is obvious to see that we can compare these a_j with corresponding terms of $\{n\}$ with no problem and thus the first condition in the problem statement is satisfied.

The second condition is equivalent to the second condition in the problem statement. This is obvious upon inspection.

Putting this together, we get that our problem has been reformulated to finding the number of tasteful n-tuples.

Now, let $g(n)$ give the number of n-tuples satisfying the first condition. Then, the number of tasty n-tuples is $g(n)$ minus the number of n-tuples satisfying the first condition and having terms all 0 and 1. Notice that any n-tuple having terms all 0 and 1 must satisfy the first condition, so it suffices to count these. There are 2^n of them. Thus, we have that there are $g(n) - 2^n$ tasteful n-tuples.

Now to compute $g(n)$. I claim that this is $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$, the number of Dyck paths of length $2n+2$. On the ith move for $i = 1, 2, \ldots, n$, travel the vector $(1, b_i)$. On the $n+1$th move, travel the vector $(1, n+1-\sum_{i=1}^{n} b_i)$. Notice that this never goes above $y = x$ because $\sum_{i=1}^{k}(1, b_i) = \left(k, \sum_{i=1}^{k} b_i\right)$ and $\sum_{i=1}^{k} b_i \leq k$. Furthermore, every Dyck path can be decomposed into a corresponding sequence b_i that satisfies the inequality.

Thus, we have that there are $\frac{1}{n+2}\binom{2n+2}{n+1} - 2^n$ tasteful n-tuples and thus $\frac{1}{n+2}\binom{2n+2}{n+1} - 2^n$ tasty k-tuples. Now, it can be seen that this value is less than 2018 for $n = 1, 2, \ldots, 7$, while it is equal to 4606 for $n = 8$. $\square$

21. Suppose that a sequence $a_0, a_1, \ldots$ of real numbers is defined by $a_0 = 1$ and

$$a_n = \begin{cases} a_{n-1}a_0 + a_{n-3}a_2 + \cdots + a_0a_{n-1} & \text{if } n \text{ odd} \\ a_{n-1}a_1 + a_{n-3}a_3 + \cdots + a_1a_{n-1} & \text{if } n \text{ even} \end{cases}$$

for $n \geq 1$. There is a positive real number r such that

$$a_0 + a_1r + a_2r^2 + a_3r^3 + \cdots = \frac{5}{4}.$$

If r can be written in the form $\frac{a\sqrt{b}-c}{d}$ for positive integers a, b, c, d such that b is not divisible by the square of any prime and $\gcd(a, c, d) = 1$, then compute $a + b + c + d$.

Proposed by Tristan Shin.

Answer. $\boxed{1923}$.

Solution. Define

$$\begin{aligned} E(x) &= a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \ldots \\ O(x) &= a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \ldots \end{aligned}$$

so we want $E(r) + O(r) = \frac{5}{4}$. Observe that

$$\begin{aligned} E(x)^2 &= a_0a_0 + (a_2a_0 + a_0a_2)x^2 + (a_4a_0 + a_2a_2 + a_0a_4)x^4 \\ &= a_1 + a_3x^2 + a_5x^4 + \ldots \\ &= \frac{O(x)}{x} \end{aligned}$$

and

$$\begin{aligned} O(x)^2 &= a_1a_1x^2 + (a_3a_1 + a_1a_3)\,x^4 + (a_5a_1 + a_3a_3 + a_1a_5)\,x^6 + \dots \\ &= a_2x^2 + a_4x^4 + a_6x^6 + \dots \\ &= E(x) - a_0 \\ &= E(x) - 1 \end{aligned}$$

so

$$\left(\frac{5}{4} - E(r)\right)^2 = E(r) - 1$$

and thus

$$E(r)^2 - \frac{7}{2}E(r) + \frac{41}{16} = 0.$$

Then $E(r) = \frac{7\pm2\sqrt{2}}{4}$. If $E(r) = \frac{7+2\sqrt{2}}{4}$, then $O(r) = \frac{5}{4} - E(r) = -\frac{1+\sqrt{2}}{2}$, contradiction, so $E(r) = \frac{7-2\sqrt{2}}{4}$ and $O(r) = \frac{\sqrt{2}-1}{2}$. Then

$$r = \frac{O(r)}{E(r)^2} = \frac{232\sqrt{2}-8}{1681},$$

so $a+b+c+d = 1923$. $\square$

22. Let ABC be a triangle with $AB = 2$ and $AC = 3$. Let H be the orthocenter, and let M be the midpoint of BC. Let the line through H perpendicular to line AM intersect line AB at X and line AC at Y. Suppose that lines BY and CX are parallel. Then $[ABC]^2 = \frac{a+b\sqrt{c}}{d}$ for positive integers a, b, c and d, where $\gcd(a, b, d) = 1$ and c is not divisible by the square of any prime. Compute $1000a+100b+10c+d$.

Proposed by Luke Robitaille.

Answer. $\boxed{270382}$.

Solution. Let $D = AH \cap BC$, and let Z be such that $(H, D; A, Z) = -1$. Let $Z' = BY \cap AH; Z'' = CX \cap AH$. Claim: $Z = Z' = Z''$. Proof: Let $E = BH \cap AC; F = CH \cap AB; W = EF \cap BC$. Then it'swell-known that W, H, X, Y are collinear (for instance, by radical axis on $(BCEF)$, $(AEFH)$, (BHC) since the antipode to H on (BHC) is the reflection A' of A over M). Then $(H, D, A, Z') = (E, C; A, Y)$ by projecting through B onto line AC, and this in turn equals $(B, C; D, W) = -1$ by projecting through H onto BC. Then we must have $Z = Z'$ and similarly $Z = Z''$. $\blacksquare$ Then $Z = BY \cap CX$, so Z is a point at infinity, so A is the midpoint of HD. Then we find the answer using the well-known identity $AD*HD = BD*CD$. Specifically, set $x = AD^2$; then $2x = \sqrt{(4-x)(9-x)}$, so $3x^2+13x = 36$; then $x = \frac{\sqrt{601}-13}{6}$. Let $K = [ABC]$. Then $(2K)^2 = (\sqrt{x(4-x)} + \sqrt{x(9-x)})^2 = x(4-x) + x(9-x) + 2x\sqrt{(4-x)(9-x)}$, which equals $x(4-x)+x(9-x)+2x(2x) = 2x^2+13x = 36-x^2$, so $K^2 = 9 - \frac{x^2}{4} = \frac{263+13\sqrt{601}}{72}$, so the answer is $\boxed{270382}$. $\square$

23. Consider all ordered pairs (a, b) of positive integers such that $\frac{a^2+b^2+2}{ab}$ is an integer and $a \le b$. We label all such pairs in increasing order by their distance from the origin. (It is guaranteed that no ties exist.) Thus $P_1 = (1, 1), P_2 = (1, 3)$, and so on. If $P_{2020} = (x, y)$, then compute the remainder when $x + y$ is divided by 2017.

Proposed by Ashwin Sah.

Answer. $\boxed{52}$.

Solution. A standard Vieta jumping argument shows that $a^2+b^2+2 = 4ab$ and that $P_n = (a_{n-1}, a_n)$ if $a_0 = a_1 = 1, a_{n+2} = 4a_{n+1} - a_n$. We obtain

$$a_n = \frac{\left(3-\sqrt{3}\right)\left(2+\sqrt{3}\right)^n + \left(3+\sqrt{3}\right)\left(2-\sqrt{3}\right)^n}{6},$$

and thus since 3 is a quadratic residue $\pmod{2017}$ we conclude that $\left(2+\sqrt{3}\right)^{2016} = a + b\sqrt{3}$ where $b \equiv a - 1 \equiv 0 \pmod{2017}$. Then we conclude that $a_{2016} \equiv a_{2017} \equiv 1 \pmod{2017}$ and thus that $a_{2019} \equiv 11 \pmod{2017}$ and $a_{2020} \equiv 41 \pmod{2017}$. The answer is 52. □

24. Let $p = 101$ and let S be the set of p-tuples $(a_1, a_2, \ldots, a_p) \in \mathbb{Z}^p$ of integers. Let N denote the number of functions $f : S \to \{0, 1, \ldots, p-1\}$ such that

- $f(a+b) + f(a-b) \equiv 2\big(f(a) + f(b)\big) \pmod{p}$ for all $a, b \in S$, and
- $f(a) = f(b)$ whenever all components of $a - b$ are divisible by p.

Compute the number of positive integer divisors of N. (Here addition and subtraction in $\mathbb{Z}^p$ are done component-wise.)

Proposed by Ankan Bhattacharya.

Answer. $\boxed{5152}$.

Solution. In general there are $N = p^{p(p+1)/2}$ solutions, which are all of the quadratic forms:

$$f(x_1, \ldots, x_p) = \sum_{1 \le k \le \ell \le p} \lambda_{k\ell} x_k x_\ell$$

for constants $\lambda_{k\ell}$. These clearly work (enough to observe that $x_k x_\ell$ works for any k, ℓ).

Now let's show these solutions are all. Let $\mathbf{e}_1, \ldots, \mathbf{e}_p$ be the standard unit vectors, and let f be a function satisfying $f(a+b) + f(a-b) = 2\big(f(a) + f(b)\big)$. By subtracting a suitable quadratic form we may assume that $f(\mathbf{e}_k) = f(\mathbf{e}_k + \mathbf{e}_\ell) = 0$ for all $k \neq \ell$. This also implies every $f(2\mathbf{e}_k) = 0$.

The main lemma is that $f(a+b+c) + f(a) + f(b) + f(c) = f(a+b) + f(b+c) + f(c+a)$. To prove this, we first note that $a = b = 0$ yields $f(0) = 0$ and that $a = b$ yields $f(2a) = 4f(a)$. Now, note that $f(a+b) + f(a+c) = \frac{1}{2} f(2a+b+c) + \frac{1}{2} f(b-c)$. Therefore, $f(a+b+c) + f(a+b) + f(b+c) + f(c+a)$ equals $\frac{1}{2}[f(2a+b+c) + f(b+c)] + \frac{1}{2}[f(b+c) + f(b-c)] = \frac{1}{4}[f(2a) + f(2a+2b+2c)] + \frac{1}{4}[f(2b) + f(2c)]$, which equals $f(a+b+c) + f(a) + f(b) + f(c)$ as desired. Now this lemma implies that given $f(a)$, $f(b)$, $f(c)$, $f(a+b)$, $f(a+c)$, $f(b+c)$ we can determine $f(a+b+c)$, which allows us to determine every $f(\alpha_1 \mathbf{e}_1 + \cdots + \alpha_n \mathbf{e}_n)$ by induction on $\alpha_1 + \cdots + \alpha_n$. It follows that $f = 0$ and all the solutions are the ones claimed.

Now $N = p^{p(p+1)/2}$ has $\frac{p(p+1)}{2} + 1$ positive integer divisors. When $p = 101$, this quantity is equal to 5152. □

25. Given two positive integers x, y, we define $z = x \oplus y$ to be the bitwise XOR of x and y; that is, z has a 1 in its binary representation at exactly the place values where x, y have differing binary representations. It is known that $\oplus$ is both associative and commutative. For example, $20 \oplus 18 = 10100_2 \oplus 10010_2 = 110_2 = 6$. Given a set $S = \{a_1, a_2, \ldots, a_n\}$ of positive integers, we let $f(S) = a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n$. We also let $g(S)$ be the number of divisors of $f(S)$ which are at most 2018 but greater than or equal to the largest element in S (if S is empty then let $g(S) = 2018$). Compute the number of 1s in the binary representation of $\displaystyle\sum_{S \subseteq \{1,2,\ldots,2018\}} g(S)$.

Proposed by Brandon Wang and Vincent Huang.

Answer. $\boxed{10}$.

Solution. For each $i \le 2018$, let $h(i)$ be the number of subsets S of $\{1, 2, \ldots, i\}$ with $i \mid f(S)$. I claim that the desired sum is just $h(1) + h(2) + \cdots + h(2018)$. Indeed, for any $S \subseteq \{1, 2, \ldots, 2018\}$, let M be the largest element of S. Then for each divisor $M \le d \le 2018$ of $f(S)$, S will be counted once in $h(d)$, so the sums are equivalent and it's enough to compute $h(i)$.

For a fixed i, clearly there are 2^i possible corresponding subsets S. Let $k = \lfloor \log_2 i \rfloor + 1$ and consider any $j \le 2^k$; it follows that $f(S)$ can take on any value from 0 to $2^k - 1$. Consider any X, Y with

$f(X) = f(Y) = j$; then clearly if Z is the symmetric difference of X, Y, we have that $f(Z) = 0$; similarly, if we fix $f(X) = i$, then any Z with $f(Z) = 0$ yields a corresponding Y with $f(Y) = i$ simply by taking Y as the symmetric difference of X and Z. It follows that for each such j, the number of S with $f(S) = j$ is the same, hence this number is just $2^{i-\lfloor \log_2 i \rfloor - 1}$.

Now note that for $i \mid f(S)$ to occur, since $f(S)$ is the XOR of some elements between 1 and i, clearly $f(S) < 2i$, so $f(S) \in \{0, i\}$, meaning that $i \mid f(S)$ corresponds to two possible values of j, so $h(i) = 2 \cdot 2^{i-\lfloor \log_2 i \rfloor - 1} = 2^{i-\lfloor \log_2 i \rfloor}$.

As i goes from 1 to 2018, clearly $h(i+1) = 2h(i)$ except when $i + 1 = 2^k$, in which case we repeat a $2^{2^k - k}$ term. Therefore the sum of all $h(i)$ equals $\sum_{k=1}^{10} 2^{2^k - k} + 2^1 + 2^2 + \cdots + 2^{2008} = \sum_{k=2}^{10} 2^{2^k - k} + 2^{2009}$. Hence the number of 1s in the binary representation is 10 and we're done. □

26. Let $p = 2027$ be the smallest prime greater than 2018, and let $P(X) = X^{2031} + X^{2030} + X^{2029} - X^5 - 10X^4 - 10X^3 + 2018X^2$. Let $\mathrm{GF}(p)$ be the integers modulo p, and let $\mathrm{GF}(p)(X)$ be the set of rational functions with coefficients in $\mathrm{GF}(p)$ (so that all coefficients are taken modulo p). That is, $\mathrm{GF}(p)(X)$ is the set of fractions $\frac{P(X)}{Q(X)}$ of polynomials with coefficients in $\mathrm{GF}(p)$, where $Q(X)$ is not the zero polynomial. Let $D\colon \mathrm{GF}(p)(X) \to \mathrm{GF}(p)(X)$ be a function satisfying

$$D\left(\frac{f}{g}\right) = \frac{D(f) \cdot g - f \cdot D(g)}{g^2}$$

for any $f, g \in \mathrm{GF}(p)(X)$ with $g \neq 0$, and such that for any nonconstant polynomial f, $D(f)$ is a polynomial with degree less than that of f. If the number of possible values of $D(P(X))$ can be written as a^b, where a, b are positive integers with a minimized, compute ab.

Proposed by Brandon Wang.

Answer. $\boxed{4114810}$.

Solution. It's easy to show that this is equivalent to product rule- setting $f = 1$ yields $D\left(\frac{1}{g}\right) = \frac{-D(g)}{g^2}$, from which setting $h = \frac{1}{g}$ gives $D(fh) = hD(f) + fD(h)$. This means that any definition of D is defined by D on the irreducible terms, and that any definition of D on the irreducible terms satisfying the degree condition works. Now, we note that $P(X) = X^2(X^2 + X + 1)(X^{2027} - X + 2018)$. We claim the last term is irreducible; clearly it has no roots in $\mathrm{GF}(p)$. Suppose it has root α, then by Frobenius $\alpha + n$ is a root for any n.

Then, if it is divisible $Q(X) \in \mathrm{GF}(p)[X]$ with degree k, then Q factors in $\mathrm{GF}(p)(\alpha)$ (which exists since α is algebraic) as $(X - (\alpha + a_1))(X - (\alpha + a_2)) \cdots (X - (\alpha + a_k))$ in $\mathrm{GF}(p)(X)$, then considering the X^k coefficient we have that $k\alpha \in \mathrm{GF}(p)$, so $k = 0$ or $k = p$. So, we have p choices for $D(X)$, p^2 choices for $D(X^2 + X + 1)$, which is irreducible since $3 \nmid p - 1$, and p^p choices for $D(X^{2027} - X + 2018)$. Then, we have p^{p+3} solutions in total, so our answer is $p(p + 3)$. □

27. Let $p = 2^{16} + 1$ be a prime. Let N be the number of ordered tuples (A, B, C, D, E, F) of integers between 0 and $p - 1$, inclusive, such that there exist integers x, y, z not all divisible by p with p dividing all three of $Ax + Ez + Fy, By + Dz + Fx, Cz + Dy + Ex$. Compute the remainder when N is divided by 10^6.

Proposed by Vincent Huang.

Answer. $\boxed{282241}$.

Solution. We have $N = p^5 + p^3 - p^2$ for all primes $p > 2$. It is probably possible to count the solutions directly via elimination and/or substitution, but this is difficult and requires lots of casework.

First, we'll let A, B, C, D, E, F be elements of $\mathbb{F}_p$, the field of residues modulo p. This allows us to make the substitution $A, B, C \to 2A, 2B, 2C$. Now note that this is equivalent with $Mv = 0$, where $M = \begin{bmatrix} 2A & F & E \\ F & 2B & D \\ E & D & 2C \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq 0$.

From this point, one solution path is to compute the number of (A, B, C, D, E, F) for which the determinant of $\det M = 2(4ABC + DEF - AD^2 - BE^2 - CF^2)$, is $0 \pmod p$. This path is completely feasible with some careful casework, but we'll present a different solution.

Let $\mathcal{C}(x, y, z)$ be the conic given by $Ax^2 + By^2 + Cz^2 + Dyz + Eyz + Fzx = 0$. Then $\mathcal{C}$ can equivalently be expressed in the form $v^T Mv = 0$, hence the point $P = (x, y, z)$ lies on $\mathcal{C}$. Furthermore, the partial derivatives of $\mathcal{C}$ at P are given by the entries of Mv, so these derivatives vanish at P. If $\mathcal{C}$ is irreducible in $\overline{\mathbb{F}_p}[x, y, z]$ (where $\overline{\mathbb{F}_p}$ denotes the algebraic closure of $\mathbb{F}_p$), this is not possible, because $P \neq (0, 0, 0)$ and irreducible conics are "smooth" curves, so we must have that $\mathcal{C}$ actually splits into two lines, hence $\mathcal{C} = (u_1x + u_2y + u_3z)(v_1x + v_2y + v_3z)$ where the u_i, v_i lie in $\overline{\mathbb{F}_p}$.

Disregard the case $\mathcal{C} = 0$. Now $\mathcal{C}$ is the product of two lines q, r, and clearly either both are in $\mathbb{F}_p[x, y, z]$ or both are not. There are $p^2 + p + 1$ distinct nonzero lines in $\mathbb{F}_p$, so the number of ways to choose 2 such lines q, r, where order of selection doesn't matter, is $\binom{p^2+p+2}{2}$, though after selecting these 2 lines we must multiply by $p - 1$ to account for equivalent conics which produce different tuples (A, B, C, D, E, F). Hence we have $(p - 1)\binom{p^2+p+2}{2} + 1$ conics thus far.

In the remaining cases, we'll split by how many of A, B, C are zero.

If all three are zero, then $u_1v_1 = u_2v_2 = u_3v_3 = 0$. Both q, r must have some nonzero term, so WLOG let $u_1 \neq 0, v_2 \neq 0$. Then we can consider the equivalent conic $u_1v_2(u_1^{-1}q)(v_2^{-1}r)$. Clearly $u_1v_2 \in \mathbb{F}_p$, but since $u_1^{-1}q$ has leading coefficient 1, all its terms must be in $\mathbb{F}_p$, and similarly for $v_2^{-1}r$, so there are no solutions in this case.

If two of A, B, C are zero, WLOG $A \neq 0$ so $u_1, v_1 \neq 0$. This time we can write as $u_1v_1(u_1^{-1}q)(v_1^{-1}r)$, and the same reasoning as above shows all terms lie in $\mathbb{F}_p$.

If one of A, B, C is zero, WLOG $C = 0$ and we'll multiply by 3 later to account for the other cases. Once again we'll take out leading terms to end up with $u_1v_1(u_1^{-1}q)(v_1^{-1}r)$ and examine the two resulting linear factors, which are of the form $x + uy + u'z, x + vy + v'z$. WLOG let $v' = 0$; we're dealing with the case where both lines don't lie in $\mathbb{F}_p$, so it follows that $v \notin \mathbb{F}_p$. But $F = u + v, B = uv$ lie in $\mathbb{F}_p$, hence u, v are conjugates in $\mathbb{F}_{p^2}$, and $D = vu', E = u' \in \mathbb{F}_p$ implies $u' = 0$. It follows that the lines are $x + uy, x + vy$, where (u, v) is a pair of conjugates in $\mathbb{F}_{p^2}$ and not $\mathbb{F}_p$, of which $\frac{p^2-p}{2}$ such unordered pairs exist. Again scaling by $p - 1$, this gives a total of $\frac{3}{2}(p^2 - p)(p - 1)$ solutions.

In the final case, $ABC \neq 0$ so no u_i or v_i is zero. Again we consider $u_1v_1(u_1^{-1}q)(v_1^{-1}r)$. By a similar reasoning as earlier, the two lines in the factorization have representations $x + uy + vz, x + \overline{u}y + \overline{v}z$, where $\overline{w}$ represents the conjugate of w in $\mathbb{F}_{p^2}$. There are $p^2 - 1$ choices for each of u, v, giving $(p^2 - 1)^2$ total choices, but $(p - 1)^2$ of these choices have u, v both lying in $\mathbb{F}_p$, so we disregard them. Multiplying by $p - 1$ for scaling and dividing by 2 to account for when the two lines swap (i.e. $u \to \overline{u}, v \to \overline{v}$), we have $\frac{p-1}{2}[(p^2 - 1)^2 - (p - 1)^2]$ total solutions in this case.

Summing gives $p^5 + p^3 - p^2$ total solutions. Computing modulo 10^6 gives the answer of 282241 as desired. $\square$

28. Let ω be a circle centered at O with radius $R = 2018$. For any $0 < r < 1009$, let γ be a circle of radius r centered at a point I satisfying $OI = \sqrt{R(R - 2r)}$. Choose any $A, B, C \in \omega$ with AC, AB tangent to γ at E, F, respectively. Suppose a circle of radius r_A is tangent to AB, AC, and internally tangent to ω at a point D with $r_A = 5r$. Let line EF meet ω at P_1, Q_1. Suppose P_2, P_3, Q_2, Q_3 lie on ω such that $P_1P_2, P_1P_3, Q_1Q_2, Q_1Q_3$ are tangent to γ. Let P_2P_3, Q_2Q_3 meet at K, and suppose KI meets AD at a point X. Then as r varies from 0 to 1009, the maximum possible value of OX can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that b is not divisible by the square of any prime and $\gcd(a, c) = 1$. Compute $10a + b + c$.

Proposed by Vincent Huang.

Answer. $\boxed{10096}$.

Solution. By Poncelet's Porism and Euler's result on the length of OI, ABC has incircle γ.

The key claim is that KB, KC are both tangent to ω. To see this, let $K = BB \cap CC$, and we'll show $K \in P_2P_3, Q_2Q_3$.

Let P_1P_2, P_1P_3 meet γ at S_3, S_2. Since $S_2S_2 \cap S_3S_3 \in EF$, we know $(S_2, S_3; E, F)$ is harmonic. Consider the positive homothety $\mathcal{H}$ sending γ to ω; it's well-known that if $Y, Z \in \omega$ with YZ tangent to γ at point T, then $\mathcal{H}$ sends T to the midpoint of arc YZ on the opposite side of YZ as point I. It follows that $\mathcal{H}$ sends $(S_2, S_3; E, F)$ to $(M_1, M_2; M_3, M_4)$, where M_1, M_2, M_3, M_4 are the midpoints of arcs P_1P_3, P_1P_2, AC, AB on opposite sides of their respective chords as I. Projecting $(M_1, M_2; M_3, M_4)$ through I back onto γ sends this bundle to $(P_2, P_3; B, C)$, since by Poncelet's Porism $P_1P_2P_3$ and $Q_1Q_2Q_3$ both have incircle γ. This final bundle tells us $K = BB \cap CC$ lies on P_2P_3, and similarly K lies on Q_2Q_3, as desired.

Now, by Brokard we know that KI, which contains both $BB \cap CC$ and $BM_3 \cap CM_4$, is the polar of the point $BC \cap M_3M_4$; it follows that KI contains the point determined by $X' = M_3M_3 \cap M_4M_4$. Now $\mathcal{H}^{-1}$ sends M_3M_3 to AC and M_4M_4 to AB, hence it follows that X', A, X_{56} are collinear, where X_{56} is the center of $\mathcal{H}$. But by Monge on the A-mixtillinear incircle, ω, γ, we know that A, X_{56}, D are collinear, hence $X' = AD \cap KI = X$.

Now clearly X is the inverse of the midpoint M_5 of M_3M_4 in ω, hence $OX = \dfrac{R^2}{OM_5} = \dfrac{R^2}{R\cos\frac{B+C}{2}} = \dfrac{R}{\sin\frac{A}{2}}$. Meanwhile, it's well-known that $r_A = \dfrac{r}{\cos^2\frac{A}{2}}$, hence $\cos^2\frac{A}{2} = \dfrac{r}{r_A}$, from which we get that $OX = \dfrac{R}{\sqrt{1-\frac{r}{r_A}}}$. Plugging in the given information, we're left with $OX = \dfrac{1009\sqrt{5}}{1}$, so the answer is 10096 as desired. $\square$

29. For integers $0 \le m, n \le 2^{2017} - 1$, let $\alpha(m,n)$ be the number of nonnegative integers k for which $\lfloor m/2^k \rfloor$ and $\lfloor n/2^k \rfloor$ are both odd integers. Consider a $2^{2017} \times 2^{2017}$ matrix M whose (i,j)th entry (for $1 \le i, j \le 2^{2017}$) is
$$(-1)^{\alpha(i-1,j-1)}.$$
For $1 \le i, j \le 2^{2017}$, let $M_{i,j}$ be the matrix with the same entries as M except for the (i,j)th entry, denoted by $a_{i,j}$, and such that $\det M_{i,j} = 0$. Suppose that A is the $2^{2017} \times 2^{2017}$ matrix whose (i,j)th entry is $a_{i,j}$ for all $1 \le i, j \le 2^{2017}$. Compute the remainder when $\det A$ is divided by 2017.

Proposed by Michael Ren and Ashwin Sah.

Answer. $\boxed{1382}$.

Solution. Note that $\alpha(m,n)$ is just the number of 1s in the result of the binary operator m AND n. Replace 2017 with k and let M_k, A_k be the matrices corresponding to M, A for a general value of k. It's not hard to show that we have $M_{k+1} = \begin{bmatrix} M_k & M_k \\ M_k & -M_k \end{bmatrix}$.

Now by subtracting the top half of M_{k+1} from the bottom half, we get $\det M_{k+1} = \det \begin{bmatrix} M_k & M_k \\ 0 & -2M_k \end{bmatrix} = \det M_k \det(-2M_k) = (-2)^{2^k}(\det M_k)^2$. An easy recursion gives $\det M_k = 2^{k2^{k-1}}$.

Now I claim the entries of A_k are just $1 - 2^k$ times the entries of M_k. Indeed, consider any $M_{i,j}$ where the (i,j)th entry is just $(1-2^k)a_{i,j}$. Consider all occurences of -1 in the jth column; we'll negate every row containing such a -1. It's not hard to show that after performing this operation, each column has sum zero, meaning the sum of all rows is the zero vector, so the row vectors of $M_{i,j}$ are linearly dependent and $\det M_{i,j} = 0$ as desired.

Now clearly $\det A_k = (1-2^k)^{2^k} \det M_k = (2^k-1)^{2^k} \cdot 2^{k2^{k-1}}$. We'd like to compute this modulo 2017 for $k = 2017$. By FLT, the first term just equals one, so we just compute $2^{2017 \cdot 2^{2016}}$, which after some computations is just 1382. $\square$

30. Let ABC be an acute triangle with $\cos B = \frac{1}{3}, \cos C = \frac{1}{4}$, and circumradius 72. Let ABC have circumcenter O, symmedian point K, and nine-point center N. Consider all non-degenerate hyperbolas $\mathcal{H}$ with perpendicular asymptotes passing through A, B, C. Of these $\mathcal{H}$, exactly one has the property that there exists a point $P \in \mathcal{H}$ such that NP is tangent to $\mathcal{H}$ and $P \in OK$. Let N' be the reflection of N over BC. If AK meets PN' at Q, then the length of PQ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are positive integers such that c is not divisible by the square of any prime. Compute $100a + b + c$.

Proposed by Vincent Huang.

Answer. $\boxed{7132}$.

Solution. We'll begin with a proof of a little-known result known as Yiu's Theorem.

Let DEF be the orthic triangle of ABC so A, B, C are the excenters of triangle DEF. Let the feet of the F, E altitudes in $\triangle AEF$ be P_A, Q_A; define P_B, Q_B, P_C, Q_C similarly. Let $X = P_BQ_B \cap P_CQ_C$ and similarly define Y, Z. Clearly triangles BQ_CP_B, HEF are homothetic with center A, so $P_BQ_C \parallel EF$. Then Since EFP_AQ_A is cyclic with diameter EF, by Reim's Theorem $P_AQ_AP_BQ_C$ is cyclic. Similarly we deduce $P_BQ_BQ_AP_C, P_CQ_CP_AQ_B$ are cyclic. If these three circles are distinct, radical axis yields that AB, BC, CA concur, contradiction, hence all six points are concyclic on some circle Γ. Now radical axis on $\Gamma, (DE), (DF)$ yields that $XD \perp EF$. Letting H' be the orthocenter of DEF, we equivalently have X, D, H' are collinear and similarly for Y, Z.

Now let r_A be the length of the D-excircle, centered at A. Let the D-excircle meet EF at point A_1. Let the line through D parallel to EF meet XY at D_1. I claim XDD_1, AA_1F are congruent. Indeed, it's easy to verify they are homothetic and therefore similar, so it's enough to show $DD_1 = A_1F = s - e$. Let the line through D parallel to XY meet EF at D_2 and let XY meet EF at D_3. By the excenter version of the Iran Incenter Lemma, we know lineP_CQ_C contains the F-excircle's tangency points with FE, FD, so D_3 must be this tangency point with FE. Since $DD_1D_3D_2$ is a parallelogram, we therefore have $DD_1 = FD_3 - FD_2 = s - FD = s - e$ as desired, proving the congruency and therefore proving $XD = AA_1 = r_A$.

Now let X' lie on XD with $AX' \parallel EF$. Let the incircle of DEF meet EF at A_2 and let A_3 be the midpoint of EF. Then clearly $(AX', AA_3; AA_2, AA_1)$ is harmonic. Let $DH' \cap EF = A_4$, let the midpoint of DA_4 be A_5, and let $A_6 = AA_3 \cap XD$. Projecting the above bundle onto line DH', and noting that A_5, A_2, A are well-known to be collinear (often cited as the Midpoint of Altitude Lemma), it follows that $(X', A_6; A_5, \infty_{XH'})$ is harmonic so $A_5A_6 = A_5X'$. It then follows that $DA_6 = X'A_4 = AA_1 = r_A$, hence $A_6 = X$ and A, A_3, X are collinear. Similarly we can define and see B, B_3, Y and C, C_3, Z are collinear.

Now we go back to the original problem. First, I claim that the polar of N in $\mathcal{H}$ always passes through H'. First, since $\mathcal{H}$ is a rectangular hyperbola, it's well-known it passes through H. Now by Desargues' Involution Theorem, the two intersections of any conic through A, B, C, H with line NH' must determine a fixed involution. Since N, H' are isogonal conjugates in triangle DEF, when $\mathcal{H}$ degenerates as $HA \cup BC$, this involution coincides with harmonic conjugation in NH', since $(DA, DB; DN, DH')$ is harmonic. Similarly, when $\mathcal{H}$ degenerates as $HB \cup CA, HC \cup B$, the involution coincides with harmonic conjugation in NH', hence this conjugation determines the involution completely. Therefore, if NH' meets $\mathcal{H}$ at R, S, then $(R, S; N, H')$ is harmonic, implying H' always lies on the polar of N.

Now I claim H' lies on OK. First, we angle-chase to see $\angle YXH' = \angle(AB, DH') = 90^\circ - \angle(EF, AB) = 0.5\angle F = \angle XYH'$, so $H'X = H'Y$ and similar arguments give H' is the circumcenter of XYZ.Therefore, since we observed earlier that ABC, XYZ are homothetic, this homothety also sends O to H'. The homothety has center $AX \cap BY \cap CZ$, and we noted earlier that $AX \equiv AA_3$. Since AA_3 is a median in $\triangle AEF$ and $BCEF$ is cyclic, it is a symmedian in $\triangle ABC$, so $K \in AA_3$ and similar arguments show K is the center of homothety, so $K \in OH'$ as claimed.

Now, let the tangents to $\mathcal{H}$ at N meet $\mathcal{H}$ at R', S', so $R'S'$ is the polar of N. We know H', R', S' are collinear. If $H' \neq R', S'$, then H', P are both intersections of $R'S'$ with OK, contradiction unless $R'S' \equiv OK$. In this case, the isogonal conjugate of $\mathcal{H}$ in $\triangle ABC$ must be line GH, where G is the centroid. It follows that H lies on the isogonal conjugate, so O lies on $\mathcal{H}$ after taking conjugates again. In that case, since $(O, H; N, \infty_{OH})$ is harmonic and O, H lie on $\mathcal{H}$, we know ∞_{OH} lies on the polar of N. But the polar of N is OK, so this implies O, K, H are collinear. Since ABC is scalene, taking conjugates yields that H, G, O lie on some non-degenerate conic, a contradiction to Bezout's Theorem. Therefore the above case is impossible, so $H' \in \{R', S'\}$.

The above case shows R', S' cannot both lie on OK, so it follows that $P = H' \in OK$. Now since DN, DH' are isogonal in $\angle EDF$, it follows that the reflection of N over BC lies on line $H'D$, so $Q = AK \cap DH'$, and hence $Q = X$ from earlier. Then $PQ = H'X = DX + DH' = AA_1 + DH' = AA_1 + 2NA_3$. Defining H_1 to be the projection of H on EF, from N being the midpoint of OH we deduce $AA_1 + 2NA_3 = AA_1 + OA_1 + HH_1 = AO + HH_1 = R + r$, where r is the inradius of DEF. We easily compute $r = DH \cos A = 2R \cos A \cos B \cos C$, so we're left with finding $R(1 + 2\cos A \cos B \cos C)$. To compute $\cos A$, we use the given values and the identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$, noting $\cos A > 0$ since ABC is acute to find $\cos A = -\frac{1}{12} + \frac{\sqrt{30}}{6}$. Now plugging everything in yields $PQ = 71 + 2\sqrt{30}$, for a final answer of 7132. $\square$