0.1 Bounding in Number Theory

Problem 0.1.1. *Let n be a positive integer. Is it possible to express n^2+3n+3 into the form ab with a and b being positive integers, and such that the difference between a and b is smaller than $2\sqrt{n+1}$?*

Problem 0.1.2. *We have $n > 2$ non-zero integers such that each one of them is divisible by the sum of the other $n-1$ numbers. Prove that the sum of all the given numbers is zero.*

Problem 0.1.3. *We call a set of integers special if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \ldots, 4n\}$ cannot be partitioned into n disjoint special sets.*

Problem 0.1.4. *Two infinite arithmetic sequences with positive integers are given:*

$$a_1 < a_2 < a_3 < \cdots ; b_1 < b_2 < b_3 < \cdots$$

It is known that there are infinitely many pairs of positive integers (i, j) for which $i \leq j \leq i + 2021$ and a_i divides b_j. Prove that for every positive integer i there exists a positive integer j such that a_i divides b_j.

Problem 0.1.5. *Does there exist a nonnegative integer a for which the equation*

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?

Problem 0.1.6. *Let $a_1 = 2$ and, for every positive integer n, let a_{n+1} be the smallest integer strictly greater than a_n that has more positive divisors than a_n. Prove that $2a_{n+1} = 3a_n$ only for finitely many indicies n.*

Problem 0.1.7. *Positive integers $a_1, a_2, \ldots, a_{2020}$ are given. For $n \geq 2021$, a_n is the smallest number different from $a_1, a_2, \ldots, a_{n-1}$ which doesn't divide $a_{n-2020}...a_{n-2}a_{n-1}$. Prove that every number large enough appears in the sequence.*

Problem 0.1.8. *Find all permutations $(a_1, a_2, ..., a_{2021})$ of $(1, 2, ..., 2021)$, such that for every two positive integers m and n with difference bigger than 20^{21}, the following inequality holds: $GCD(m+1, n+a_1) + GCD(m+2, n+a_2) + ... + GCD(m+2021, n+a_{2021}) < 2|m-n|$.*

Problem 0.1.9. *Let S be a set of positive integers such that for every $a, b \in S$, there always exists $c \in S$ such that c^2 divides $a(a+b)$. Show that there exists an $a \in S$ such that a divides every element of S.*

Problem 0.1.10. *For each positive integer k, denote by $\tau(k)$ the number of all positive divisors of k, including 1 and k. Let a and b be positive integers such that $\tau(\tau(an)) = \tau(\tau(bn))$ for all positive integers n. Prove that $a = b$.*

Problem 0.1.11. *Find all infinite sequences of positive integers $\{a_n\}_{n\geq 1}$ satisfying the following condition : there exists a positive constant c such that $\gcd(a_m + n, a_n + m) > c(m+n)$ holds for all pairs of positive integers (m, n).*