The Online Math Open Fall Contest
December 4–15, 2020

Online Math Open

Acknowledgments

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Contestants may compete in teams of up to four people, but no contestant may belong to more than one team. Starting Fall of 2019, we no longer require that all contestants must not have graduated from high school. Teams need not remain the same between the Fall and Spring contests, and contestants are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.

Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The Fall 2020 Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7:01PM ET on the start day to **7:01PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs, are prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams may not communicate with each other about the contest while the contest is running.

4. Anything that may be considered offensive or inappropriate is not allowed as a team name.

5. Team member names must be the actual team members' names. In particular, names of other people (including celebrities) or phrases that do not represent actual people are not allowed. This includes names of mathematical objects and multiple names concatenated into one name entry.

6. At our discretion, we may rename teams or team members that do not satisfy these rules.

7. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

8. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.

Contest Information

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com` (include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. A circle with radius r has area 505. Compute the area of a circle with diameter $2r$.

2. For any positive integer x, let $f(x) = x^x$. Suppose that n is a positive integer such that there exists a positive integer m with $m \neq 1$ such that $f(f(f(m))) = m^{m^{n+2020}}$. Compute the smallest possible value of n.

3. Compute the number of ways to write the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 in the cells of a 3 by 3 grid such that
   - each cell has exactly one number,
   - each number goes in exactly one cell,
   - the numbers in each row are increasing from left to right,
   - the numbers in each column are increasing from top to bottom, and
   - the numbers in the diagonal from the upper-right corner cell to the lower-left corner cell are increasing from upper-right to lower-left.

4. An alien from the planet OMO Centauri writes the first ten prime numbers in arbitrary order as U, W, XW, ZZ, V, Y, ZV, ZW, ZY, and X. Each letter represents a nonzero digit. Each letter represents the same digit everywhere it appears, and different letters represent different digits. Also, the alien is using a base other than base ten. The alien writes another number as UZWX. Compute this number (expressed in base ten, with the usual, human digits).

5. Compute the number of ordered triples of integers (a, b, c) between 1 and 12, inclusive, such that, if
$$q = a + \frac{1}{b} - \frac{1}{b + \frac{1}{c}},$$
then q is a positive rational number and, when q is written in lowest terms, the numerator is divisible by 13.

6. Let x, y, and z be nonnegative real numbers with $x + y + z = 120$. Compute the largest possible value of the median of the three numbers $2x + y, 2y + z$, and $2z + x$.

7. On a 9×9 square lake composed of unit squares, there is a 2×4 rectangular iceberg also composed of unit squares (it could be in either orientation; that is, it could be 4×2 as well). The sides of the iceberg are parallel to the sides of the lake. Also, the iceberg is invisible. Lily is trying to sink the iceberg by firing missiles through the lake. Each missile fires through a row or column, destroying anything that lies in its row or column. In particular, if Lily hits the iceberg with any missile, she succeeds. Lily has bought n missiles and will fire all n of them at once. Let N be the smallest possible value of n such that Lily can guarantee that she hits the iceberg. Let M be the number of ways for Lily to fire N missiles and guarantee that she hits the iceberg. Compute $100M + N$.

8. Let λ be a real number. Suppose that if $ABCD$ is any convex cyclic quadrilateral such that $AC = 4$, $BD = 5$, and $\overline{AB} \perp \overline{CD}$, then the area of $ABCD$ is at least λ. Then the greatest possible value of λ is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

9. Hong and Song each have a shuffled deck of eight cards, four red and four black. Every turn, each player places down the two topmost cards of their decks. A player can thus play one of three pairs: two black cards, two red cards, or one of each color. The probability that Hong and Song play exactly the same pairs as each other for all four turns is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

10. Let w, x, y, and z be nonzero complex numbers, and let n be a positive integer. Suppose that the following conditions hold:
    - $\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$,

- $wx + wy + wz + xy + xz + yz = 14,$
- $(w+x)^3 + (w+y)^3 + (w+z)^3 + (x+y)^3 + (x+z)^3 + (y+z)^3 = 2160,$ and
- $w + x + y + z + i\sqrt{n} \in \mathbb{R}.$

Compute n.

11. Let ABC be a triangle such that $AB = 5$, $AC = 8$, and $\angle BAC = 60^\circ$. Let P be a point inside the triangle such that $\angle APB = \angle BPC = \angle CPA$. Lines BP and AC intersect at E, and lines CP and AB intersect at F. The circumcircles of triangles BPF and CPE intersect at points P and $Q \neq P$. Then $QE + QF = \frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

12. At a party, there are 100 cats. Each pair of cats flips a coin, and they shake paws if and only if the coin comes up heads. It is known that exactly 4900 pairs of cats shook paws. After the party, each cat is independently assigned a "happiness index" uniformly at random in the interval $[0, 1]$. We say a cat is *practical* if it has a happiness index that is strictly greater than the index of every cat with which it shook paws. The expected value of the number of practical cats is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

13. Let a, b, c, x, y, and z be positive integers such that
$$\frac{a^2 - 2}{x} = \frac{b^2 - 37}{y} = \frac{c^2 - 41}{z} = a + b + c.$$
Let $S = a + b + c + x + y + z$. Compute the sum of all possible values of S.

14. Let $BCB'C'$ be a rectangle, let M be the midpoint of $B'C'$, and let A be a point on the circumcircle of the rectangle. Let triangle ABC have orthocenter H, and let T be the foot of the perpendicular from H to line AM. Suppose that $AM = 2$, $[ABC] = 2020$, and $BC = 10$. Then $AT = \frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

15. Let m and n be positive integers such that $\gcd(m, n) = 1$ and
$$\sum_{k=0}^{2020} (-1)^k \binom{2020}{k} \cos(2020 \cos^{-1}(\tfrac{k}{2020})) = \frac{m}{n}.$$
Suppose n is written as the product of a collection of (not necessarily distinct) prime numbers. Compute the sum of the members of this collection. (For example, if it were true that $n = 12 = 2 \times 2 \times 3$, then the answer would be $2 + 2 + 3 = 7$.)

16. For a positive integer n, we will say that a sequence $a_1, a_2, \ldots a_n$ where $a_i \in \{1, 2, \ldots, n\}$ for all i is *n-highly divisible* if, for every positive integer d that divides n and every nonnegative integer k less than $\frac{n}{d}$ we have that
$$d \,\Bigg|\, \sum_{i=kd+1}^{(k+1)d} a_i.$$
Let $\chi(n)$ be the probability that a sequence $a_1, a_2, \ldots, a_n$ where a_i is chosen randomly from $\{1, 2, \ldots n\}$ independently for all i is n-highly divisible. Suppose that n is a positive integer such that there exists a positive integer m not divisible by 3 such that $3^{40}\chi(n) = \frac{1}{m}$. Compute the sum of all possible values of n.

17. Let ABC be a triangle with $AB = 11, BC = 12$, and $CA = 13$, let M and N be the midpoints of sides CA and AB, respectively, and let the incircle touch sides CA and AB at points X and Y, respectively. Suppose that R, S, and T are the midpoints of line segments MN, BX, and CY, respectively. Then $\sin \angle SRT = \frac{k\sqrt{m}}{n}$, where k, m, and n are positive integers such that $\gcd(k, n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

18. The people in an infinitely long line are numbered $1, 2, 3, \ldots$. Then, each person says either "Karl" or "Lark" independently and at random. Let S be the set of all positive integers i such that people i, $i+1$, and $i+2$ all say "Karl," and define $X = \sum_{i \in S} 2^{-i}$. Then the expected value of X^2 is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

19. Compute the smallest positive integer M such that there exists a positive integer n such that

    - M is the sum of the squares of some n consecutive positive integers, and
    - $2M$ is the sum of the squares of some $2n$ consecutive positive integers.

20. Given a string of at least one character in which each character is either A or B, Kathryn is allowed to make these moves:

    - she can choose an appearance of A, erase it, and replace it with BB, or
    - she can choose an appearance of B, erase it, and replace it with AA.

    Kathryn starts with the string A. Let a_n be the number of strings of length n that Kathryn can reach using a sequence of zero or more moves. (For example, $a_1 = 1$, as the only string of length 1 that Kathryn can reach is A.) Then $\sum_{n=1}^{\infty} \frac{a_n}{5^n} = \frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute $100m + n$.

21. Among all ellipses with center at the origin, exactly one such ellipse is tangent to the graph of the curve $x^3 - 6x^2y + 3xy^2 + y^3 + 9x^2 - 9xy + 9y^2 = 0$ at three distinct points. The area of this ellipse is $\frac{k\pi\sqrt{m}}{n}$, where k, m, and n are positive integers such that $\gcd(k, n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

22. Three points P_1, P_2, and P_3 and three lines ℓ_1, ℓ_2, and ℓ_3 lie in the plane such that none of the three points lie on any of the three lines. For (not necessarily distinct) integers i and j between 1 and 3 inclusive, we call a line ℓ (i, j)-*good* if the reflection of P_i across ℓ lies on ℓ_j, and call it *excellent* if there are two distinct pairs (i_1, j_1) and (i_2, j_2) for which it is good. Suppose that exactly N excellent lines exist. Compute the largest possible value of N.

23. For a positive integer $k > 1$ with $\gcd(k, 2020) = 1$, we say a positive integer N is *k-bad* if there do not exist nonnegative integers x and y with $N = 2020x + ky$. Suppose k is a positive integer with $k > 1$ and $\gcd(k, 2020) = 1$ such that the following property holds: if m and n are positive integers with $m + n = 2019(k - 1)$ and $m \geq n$ and m is k-bad, then n is k-bad. Compute the sum of all possible values of k.

24. In graph theory, a *triangle* is a set of three vertices, every two of which are connected by an edge. For an integer $n \geq 3$, if a graph on n vertices does not contain two triangles that do not share any vertices, let $f(n)$ be the maximum number of triangles it can contain. Compute $f(3) + f(4) + \cdots + f(100)$.

25. Let n be a positive integer with exactly twelve positive divisors $1 = d_1 < \cdots < d_{12} = n$. We say n is *trite* if

    $$5 + d_6(d_6 + d_4) = d_7 d_4.$$

    Compute the sum of the two smallest trite positive integers.

26. The bivariate functions $f_0, f_1, f_2, f_3, \ldots$ are sequentially defined by the relations $f_0(x, y) = 0$ and $f_{n+1}(x, y) = \big|x + |y + f_n(x, y)|\big|$ for all integers $n \geq 0$. For independently and randomly selected values $x_0, y_0 \in [-2, 2]$, let p_n be the probability that $f_n(x_0, y_0) < 1$. Let a, b, c, and d be positive integers such that the limit of the sequence $p_1, p_3, p_5, p_7, \ldots$ is $\frac{\pi^2 + a}{b}$ and the limit of the sequence $p_0, p_2, p_4, p_6, p_8, \ldots$ is $\frac{\pi^2 + c}{d}$. Compute $1000a + 100b + 10c + d$.

27. Let ABC be a scalene, non-right triangle. Let ω be the incircle and let γ be the nine-point circle (the circle through the feet of the altitudes) of $\triangle ABC$, with centers I and N, respectively. Suppose ω and γ are tangent at a point F. Let D be the foot of the perpendicular from A to line BC and let M be the midpoint of side $\overline{BC}$. The common tangent to ω and γ at F intersects lines AB and AC at points P and Q, respectively. Let lines DP and DQ intersect γ at points $P_1 \neq D$ and $Q_1 \neq D$, respectively. Suppose that point Z lies on line P_1Q_1 such that $\angle MFZ = 90^\circ$ and $MZ \perp DF$. Suppose that γ has radius 11 and ω has radius 5. Then $DI = \frac{k\sqrt{m}}{n}$, where k, m, and n are positive integers such that $\gcd(k, n) = 1$ and m is not divisible by the square of any prime. Compute $100k + 10m + n$.

28. Julia bakes a cake in the shape of a unit square. Each minute, Julia makes two cuts through the cake as follows:

    - she picks a **square** piece $\mathcal{S}$ of the cake with no cuts through its interior; then
    - she slices the entire cake along the two lines parallel to the sides of the cake passing through the center of $\mathcal{S}$.

    She does not move any pieces of cake during this process. After eight minutes, she has a grid of $9^2 = 81$ pieces of cake. (The pieces can be various sizes.) Compute the number of distinct grids that she could have ended up with. Two grids are the same if they have the same set of cuts; in particular, two grids that differ by a rotation or reflection are distinct.

29. Let ABC be a scalene triangle. Let $I_0 = A$ and, for every positive integer t, let I_t be the incenter of triangle $I_{t-1}BC$. Suppose that the points $I_0, I_1, I_2, \ldots$ all lie on some hyperbola $\mathcal{H}$ whose asymptotes are lines ℓ_1 and ℓ_2. Let the line through A perpendicular to line BC intersect ℓ_1 and ℓ_2 at points P and Q respectively. Suppose that $AC^2 = \frac{12}{7}AB^2 + 1$. Then the smallest possible value of the area of quadrilateral $BPCQ$ is $\frac{j\sqrt{k}+l\sqrt{m}}{n}$ for positive integers j, k, l, m, and n such that $\gcd(j, l, n) = 1$, both k and m are squarefree, and $j > l$. Compute $10000j + 1000k + 100l + 10m + n$.

30. Suppose that F is a field* with exactly 5^{14} elements. We say that a function $f : F \to F$ is *happy*, if, for all $x, y \in F$,
$$(f(x+y) + f(x))\,(f(x-y) + f(x)) = f(y^2) - f(x^2).$$
    Compute the number of elements z of F such that there exist distinct happy functions h_1 and h_2 such that $h_1(z) = h_2(z)$.

*A *field* is a set S, together with functions $+ : S \times S \to S$ and $\cdot : S \times S \to S$, called *addition* and *multiplication*, respectively, (where we write $+(x, y) = x + y$ and $\cdot(x, y) = xy$) such that there are elements 0 and 1 in S with $0 \neq 1$ such that the following conditions hold:

- $(a + b) + c = a + (b + c)$ for all $a, b, c \in S$,
- $a + b = b + a$ for all $a, b \in S$,
- $(ab)c = a(bc)$ for all $a, b, c \in S$,
- $ab = ba$ for all $a, b \in S$,
- $a + 0 = 0 + a = a$ for all $a \in S$,
- $a1 = 1a = a$ for all $a \in S$,
- $a(b + c) = ab + ac$ for all $a, b, c \in S$,
- $(a + b)c = ac + bc$ for all $a, b, c \in S$,
- for all $a \in S$, there exists a unique element $-a \in S$ such that $a + (-a) = (-a) + a = 0$, and
- for all $a \in S$ with $a \neq 0$, there exists a unique element $a^{-1} \in S$ such that $aa^{-1} = a^{-1}a = 1$.

Note that "addition," "multiplication," "0," and "1" are allowed to be totally different from what they normally mean. Also, we let $a - b = a + (-b)$ and $a^2 = aa$ for all $a, b \in S$.