Similarity

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1. Let $ABCD$ be a convex quadrilateral. Show that

$$AC^2 \cdot BD^2 = AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cos(A + C).$$

2. (IMO Shortlist 1998) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

Prove that

$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

3. A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Show that

$$BI^2 + \frac{AI \cdot BI \cdot CI}{DI} = AB \cdot BC.$$

4. (Turkey 1998) Let ABC be a triangle. Suppose that the circle through C tangent to AB at A and the circle through B tangent to AC at A have different radii, and let D be their second intersection. Let E be the point on the ray AB such that $AB = BE$. Let F be the second intersection of the rat CA with the circle through A, D, E. Prove that $AF = AC$.

5. A circle with center O passes through the vertices A and C of triangle ABC and intersects segments AB and BC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersects at exactly two distinct points B and M. Prove that $\angle OMB = 90^\circ$.

6. Circles ω_1 and ω_2 meet at points O and M. Circle ω, centered at O, meet circles ω_1 and ω_2 in four distinct points A, B, C and D, such that $ABCD$ is a convex quadrilateral. Lines AB and CD meet at N_1. Lines AD and BC meet at N_2. Prove that $N_1N_2 \perp MO$.

$AC \in \omega_1$
$B, D \in \omega_2$

7. (Crux) Points O and H are the circumcenter and orthocenter of acute triangle ABC, respectively. The perpendicular bisector of segment AH meets sides AB and AC at D and E, respectively. Prove that $\angle DOA = \angle EOA$.

8. (IMO Shortlist 2000) Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, and let X be a point inside $ABCD$ such that $\angle ADX = \angle BCX < 90^\circ$ and $\angle DAX = \angle CBX < 90$. If the perpendicular bisectors of segments AB and CD intersect at Y, prove that $\angle AYB = 2\angle ADX$.