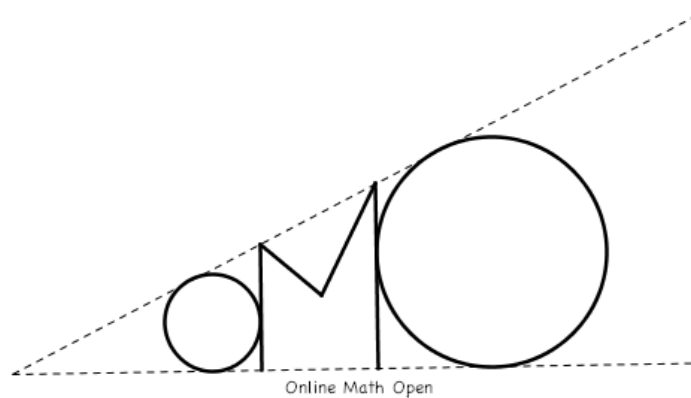


The Online Math Open Fall Contest

October 18 - 29, 2013



Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at OnlineMathOpenTeam@gmail.com.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, http://internetolympiad.org/pages/14-omo_info, for registration instructions.

Note: when we say “up to four”, we really do mean “up to”! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2013 Fall Contest will consist of **30 problems**; the answer to each problem will be a **nonnegative integer** not exceeding $2^{63} - 2 = 9223372036854775806$. The contest window will be **October 18 - 29, 2013**, from 7PM ET on the start day to 7PM ET on the end day. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.
2. Drawing aids such as graph paper, ruler, and compass are permitted. **However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.**
3. Members of different teams cannot communicate with each other about the contest while the contest is running.
4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the “hardest” problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)
5. *Participation in the Online Math Open is free.*

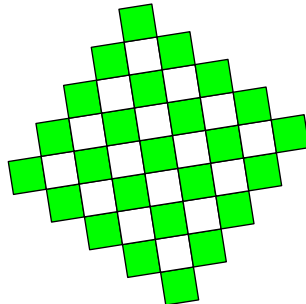
Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at http://internetolympiad.org/pages/n/omo_problems. If you have a question about problem wording, please email OnlineMathOpenTeam@gmail.com with “Clarification” in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com. (Include “Protest” in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Determine the value of $142857 + 285714 + 428571 + 571428$.
2. The figure below consists of several unit squares, M of which are white and N of which are green. Compute $100M + N$.



3. A *palindromic table* is a 3×3 array of letters such that the words in each row and column read the same forwards and backwards. An example of such a table is shown below.

O	M	O
N	M	N
O	M	O

How many palindromic tables are there that use only the letters O and M ? (The table may contain only a single letter.)

4. Suppose a_1, a_2, a_3, \dots is an increasing arithmetic progression of positive integers. Given that $a_3 = 13$, compute the maximum possible value of

$$a_{a_1} + a_{a_2} + a_{a_3} + a_{a_4} + a_{a_5}.$$

5. A wishing well is located at the point $(11, 11)$ in the xy -plane. Rachelle randomly selects an integer y from the set $\{0, 1, \dots, 10\}$. Then she randomly selects, with replacement, two integers a, b from the set $\{1, 2, \dots, 10\}$. The probability the line through $(0, y)$ and (a, b) passes through the well can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
6. Find the number of integers n with $n \geq 2$ such that the remainder when 2013 is divided by n is equal to the remainder when n is divided by 3.
7. Points M, N, P are selected on sides $\overline{AB}, \overline{AC}, \overline{BC}$, respectively, of triangle ABC . Find the area of triangle MNP given that $AM = MB = BP = 15$ and $AN = NC = CP = 25$.
8. Suppose that $x_1 < x_2 < \dots < x_n$ is a sequence of positive integers such that x_k divides x_{k+2} for each $k = 1, 2, \dots, n-2$. Given that $x_n = 1000$, what is the largest possible value of n ?
9. Let $AXYZB$ be a regular pentagon with area 5 inscribed in a circle with center O . Let Y' denote the reflection of Y over \overline{AB} and suppose C is the center of a circle passing through A, Y' and B . Compute the area of triangle ABC .
10. In convex quadrilateral $AEBC$, $\angle BEA = \angle CAE = 90^\circ$ and $AB = 15$, $BC = 14$ and $CA = 13$. Let D be the foot of the altitude from C to \overline{AB} . If ray CD meets \overline{AE} at F , compute $AE \cdot AF$.
11. Four orange lights are located at the points $(2, 0), (4, 0), (6, 0)$ and $(8, 0)$ in the xy -plane. Four yellow lights are located at the points $(1, 0), (3, 0), (5, 0), (7, 0)$. Sparky chooses one or more of the lights to turn on. In how many ways can he do this such that the collection of illuminated lights is symmetric around some line parallel to the y -axis?
12. Let a_n denote the remainder when $(n+1)^3$ is divided by n^3 ; in particular, $a_1 = 0$. Compute the remainder when $a_1 + a_2 + \dots + a_{2013}$ is divided by 1000.

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13. In the rectangular table shown below, the number 1 is written in the upper-left hand corner, and every number is the sum of the any numbers directly to its left and above. The table extends infinitely downwards and to the right.

1	1	1	1	1	...
1	2	3	4	5	...
1	3	6	10	15	...
1	4	10	20	35	...
1	5	15	35	70	...
⋮	⋮	⋮	⋮	⋮	⋮

Wanda the Worm, who is on a diet after a feast two years ago, wants to eat n numbers (not necessarily distinct in value) from the table such that the sum of the numbers is less than one million. However, she cannot eat two numbers in the same row or column (or both). What is the largest possible value of n ?

14. In the universe of Pi Zone, points are labeled with 2×2 arrays of positive reals. One can teleport from point M to point M' if M can be obtained from M' by multiplying either a row or column by some positive real. For example, one can teleport from $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ to $\begin{pmatrix} 1 & 20 \\ 3 & 40 \end{pmatrix}$ and then to $\begin{pmatrix} 1 & 20 \\ 6 & 80 \end{pmatrix}$.

A *tourist attraction* is a point where each of the entries of the associated array is either 1, 2, 4, 8 or 16. A company wishes to build a hotel on each of several points so that at least one hotel is accessible from every tourist attraction by teleporting, possibly multiple times. What is the minimum number of hotels necessary?

15. Find the positive integer n such that

$$\underbrace{f(f(\cdots f(n) \cdots))}_{2013 \text{ } f\text{'s}} = 2014^2 + 1$$

where $f(n)$ denotes the n th positive integer which is not a perfect square.

16. Al has the cards 1, 2, ..., 10 in a row in increasing order. He first chooses the cards labeled 1, 2, and 3, and rearranges them among their positions in the row in one of six ways (he can leave the positions unchanged). He then chooses the cards labeled 2, 3, and 4, and rearranges them among their positions in the row in one of six ways. (For example, his first move could have made the sequence 3, 2, 1, 4, 5, ..., and his second move could have rearranged that to 2, 4, 1, 3, 5, ...) He continues this process until he has rearranged the cards with labels 8, 9, 10. Determine the number of possible orderings of cards he can end up with.
17. Let $ABXC$ be a parallelogram. Points K, P, Q lie on \overline{BC} in this order such that $BK = \frac{1}{3}KC$ and $BP = PQ = QC = \frac{1}{3}BC$. Rays XP and XQ meet \overline{AB} and \overline{AC} at D and E , respectively. Suppose that $\overline{AK} \perp \overline{BC}$, $EK - DK = 9$ and $BC = 60$. Find $AB + AC$.
18. Given an $n \times n$ grid of dots, let $f(n)$ be the largest number of segments between adjacent dots which can be drawn such that (i) at most one segment is drawn between each pair of dots, and (ii) each dot has 1 or 3 segments coming from it. (For example, $f(4) = 16$.) Compute $f(2000)$.
19. Let $\sigma(n)$ be the number of positive divisors of n , and let $\text{rad } n$ be the product of the distinct prime divisors of n . By convention, $\text{rad } 1 = 1$. Find the greatest integer not exceeding

$$100 \left(\sum_{n=1}^{\infty} \frac{\sigma(n)\sigma(n \text{ rad } n)}{n^2 \sigma(\text{rad } n)} \right)^{\frac{1}{3}}.$$

20. A positive integer n is called *mythical* if every divisor of n is two less than a prime. Find the unique mythical number with the largest number of divisors.

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21. Let ABC be a triangle with $AB = 5$, $AC = 8$, and $BC = 7$. Let D be on side AC such that $AD = 5$ and $CD = 3$. Let I be the incenter of triangle ABC and E be the intersection of the perpendicular bisectors of \overline{ID} and \overline{BC} . Suppose $DE = \frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.
22. Find the sum of all integers m with $1 \leq m \leq 300$ such that for any integer n with $n \geq 2$, if $2013m$ divides $n^n - 1$ then $2013m$ also divides $n - 1$.
23. Let $ABCDE$ be a regular pentagon, and let F be a point on \overline{AB} with $\angle CDF = 55^\circ$. Suppose \overline{FC} and \overline{BE} meet at G , and select H on the extension of \overline{CE} past E such that $\angle DHE = \angle FDG$. Find the measure of $\angle GHD$, in degrees.
24. The real numbers $a_0, a_1, \dots, a_{2013}$ and $b_0, b_1, \dots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \dots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute

$$\sum_{k=1}^{2013} (a_k b_{k-1} - a_{k-1} b_k).$$

25. Let $ABCD$ be a quadrilateral with $AD = 20$ and $BC = 13$. The area of $\triangle ABC$ is 338 and the area of $\triangle DBC$ is 212. Compute the smallest possible perimeter of $ABCD$.
26. Let ABC be a triangle with $AB = 13$, $AC = 25$, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across $\overline{AC}, \overline{AB}$ by D, E , respectively, and let O be the circumcenter of triangle ABC . Let P be a point such that $\triangle DPO \sim \triangle PEO$, and let X and Y be the midpoints of the major and minor arcs \widehat{BC} of the circumcircle of triangle ABC . Find $PX \cdot PY$.
27. Ben has a big blackboard, initially empty, and Francisco has a fair coin. Francisco flips the coin 2013 times. On the n^{th} flip (where $n = 1, 2, \dots, 2013$), Ben does the following if the coin flips heads:
- (i) If the blackboard is empty, Ben writes n on the blackboard.
 - (ii) If the blackboard is not empty, let m denote the largest number on the blackboard. If $m^2 + 2n^2$ is divisible by 3, Ben erases m from the blackboard; otherwise, he writes the number n .

No action is taken when the coin flips tails. If probability that the blackboard is empty after all 2013 flips is $\frac{2^{u+1}}{2^k(2v+1)}$, where u, v , and k are nonnegative integers, compute k .

28. Let n denote the product of the first 2013 primes. Find the sum of all primes p with $20 \leq p \leq 150$ such that
- (i) $\frac{p+1}{2}$ is even but is not a power of 2, and
 - (ii) there exist pairwise distinct positive integers a, b, c for which

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b)$$

is divisible by p but not p^2 .

29. Kevin has 255 cookies, each labeled with a unique nonempty subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Each day, he chooses one cookie uniformly at random out of the cookies not yet eaten. Then, he eats that cookie, and all remaining cookies that are labeled with a subset of that cookie (for example, if he chooses the cookie labeled with $\{1, 2\}$, he eats that cookie as well as the cookies with $\{1\}$ and $\{2\}$). The expected value of the number of days that Kevin eats a cookie before all cookies are gone can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
30. Let $P(t) = t^3 + 27t^2 + 199t + 432$. Suppose a, b, c , and x are distinct positive reals such that $P(-a) = P(-b) = P(-c) = 0$, and

$$\sqrt{\frac{a+b+c}{x}} = \sqrt{\frac{b+c+x}{a}} + \sqrt{\frac{c+a+x}{b}} + \sqrt{\frac{a+b+x}{c}}.$$

If $x = \frac{m}{n}$ for relatively prime positive integers m and n , compute $m + n$.