York Math Camp
January 2008

Polynomial Practice Problems

1. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_0 a_n \neq 0$, and suppose that p has n real roots (counting multiplicities). Prove that the number of sign changes in the sequence $(a_0, a_1, \ldots, a_n)$ is precisely the number of positive roots of p.

2. Let $q(x)$ be a polynomial with real coefficients, all of whose roots have negative real part. Prove that the nonzero coefficients of q all have the same sign.

3. Let $q(x)$ be a polynomial with real coefficients. Prove that the number of real roots of q is congruent modulo 2 to the degree of q.

4. Determine the number of distinct, real roots of the polynomial $x^5 + 2x^4 - 5x^3 + 8x^2 - 7x - 3$.

5. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where $a_n \neq 0$. If $p(\alpha) = 0$, show that $|\alpha| < 1 + |a_{n-1}/a_n| + \cdots + |a_0/a_n|$.

6. Let $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$. If $p(\alpha) = 0$, show that $|\alpha| < 1 + \max\{|a_k| : 0 \leq k \leq n-1\}$.

7. Show that if $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$ then $a = b = c = 0$.

8. (a) Prove that for each $n \in \mathbb{N}$ there is a polynomial T_n with integer coefficients and the leading coefficient 2^{n-1} such that $T_n(\cos x) = \cos nx$ for all x.

   (b) Prove that the polynomials T_n satisfy $T_{m+n} + T_{m-n} = 2T_m T_n$ for all $m, n \in \mathbb{N}, mn$.

   (c) Prove that the polynomial U_n given by $U_n(2x) = 2T_n(x)$ also has integer coefficients and satisfies $U_n(x + x^{-1}) = x^n + x^{-n}$.
   The polynomials Tn(x) are known as the Chebyshev polynomials.

9. Prove that if $\cos(p\pi/q) = a$ is a rational number for some $p, q \in \mathbb{Z}$, then $a \in \{0, \pm\frac{1}{2}, \pm 1\}$.

10. For what real values of a does there exist a rational function $f(x)$ that satisfies $f(x^2) = f(x)^2 - a$? (A rational function is a quotient of two polynomials.)

11. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every positive integer n.

12. Find all polynomials P satisfying $P(x^2 + 1) = P(x)^2 + 1$ for all x.

13. A sequence of integers $(a_n)_{n=1}^{\infty}$ has the property that $m - n | a_m - a_n$ for any distinct $m, n \in \mathbb{N}$. Suppose that there is a polynomial $P(x)$ such that $|a_n| < P(n)$ for all n. Show that there exists a polynomial $Q(x)$ such that $a_n = Q(n)$ for all n.

14. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a natural number. Consider the polynomial $Q(x) = P(P(...P(P(x))...))$, where P is applied k times. Prove that there exist at most n integers t such that $Q(t) = t$.

15. Show that $n^4 - 20n^2 + 4$ is composite when n is any integer.

16. Determine the triples of integers (x, y, z) satisfying the equation $x^3 + y^3 + z^3 = (x + y + z)^3$.

17. Prove that every polynomial over the complex numbers has a nonzero polynomial multiple whose exponents are all divisible by 10^9.

18. Let r and s be the roots of $x^2 - (a + d)x + (ad - bc) = 0$. Prove that r^3 and s^3 are the roots of $y^2 - (a^3 + d^3 + 3abc + 3bcd)y + (ad - bc)^3 = 0$.

19. Show that there is only one natural number n such that $2^8 + 2^{11} + 2^n$ is a perfect square.