2018 Winter Camp Mock Olympiad

1. In a group of people, some pairs are friends with each other and some are not. Every evening, one person invites all of his or her friends to a party and introduces them to each other (and they become friends). Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

2. Let G be the centroid of a triangle ABC. A line ℓ through G intersects the circumcircle of ABC at X and Y, where A and C lie on the same side of ℓ. Prove that

$$BX \cdot BY = AX \cdot AY + CX \cdot CY.$$

3. Let n be a positive integer and $a_1, a_2, \ldots, a_n$ positive integers such that

$$a_1 a_2 \cdots a_n (a_1 + a_2 + \cdots + a_n) = 100n.$$

   Find the maximal value of n for which this is possible.

4. Let $n > 1$ be an integer and let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $0 \leq x_1, x_2, \ldots, x_n \leq n$ and $x_1 x_2 \ldots x_n = (n - x_1)(n - x_2) \cdots (n - x_n)$. Find the maximum value of $x_1 + x_2 + \cdots + x_n$.

5. The squares of an infinite square grid are coloured with n^2 colours so that the squares of any $n \times n$ subgrid are of different colours. If any infinite row contains at least $n^2 - n + 1$ colours, show that there is an infinite column containing exactly n colours.