Preparation for Saudi Arabia Team 2021

June Session: Junior Balkan Mathematics Olympiad

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Lesson 1

Inequalities

Problems:

1. Let a, b, c be positive real numbers such that $abc = 1$. Prove that:
$$\frac{a^3}{(1+b)(1+c)} + \frac{b^3}{(1+a)(1+c)} + \frac{c^3}{(1+a)(1+b)} \geq \frac{3}{4}.$$
2. Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that:
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$
3. Let a, b and c be positive real numbers. Prove:
$$abc(a^2 + 2b^2)(b^2 + 2c^2)(c^2 + 2a^2) \leq (a + b + c)^3(a^2 - ab + b^2)(a^2 - ac + c^2)(b^2 - bc + c^2).$$
4. Let a, b and c be positive real numbers. Prove that
$$\frac{a}{9bc + 1} + \frac{b}{9ac + 1} + \frac{c}{9ab + 1} \geq \frac{a + b + c}{1 + (a + b + c)^2}.$$
5. Let $x, y, z \geq 1$ be real numbers such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$. Prove:
$$\sqrt{x + y + z} \geq \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$
6. Let a, b, c be positive real numbers. Prove that:
$$(a^2 + ac + c^2)\left(\frac{1}{a + b + c} + \frac{1}{a + c}\right) + b^2\left(\frac{1}{b + c} + \frac{1}{a + b}\right) > a + b + c.$$
7. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that:
$$\frac{1}{a^4 + b + c + d} + \frac{1}{a + b^4 + c + d} + \frac{1}{a + b + c^4 + d} + \frac{1}{a + b + c + d^4} \leq \frac{a + b + c + d}{4}.$$
8. Let a, b, c be positive real numbers such that $ab + bc + ac = 3$ and n a positive integer. Prove:
$$\frac{a}{\sqrt{a^n + 2n - 1}} + \frac{b}{\sqrt{b^n + 2n - 1}} + \frac{c}{\sqrt{c^n + 2n - 1}} \leq \frac{3}{\sqrt{2n}}.$$
9. Let $a \leq b \leq c \leq d$ be positive real numbers. Prove: $ab^3 + bc^3 + cd^3 + da^3 \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2$.
10. Let x, y and z be real numbers such that $x + y + z = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$. Prove that:
$$x + y + z \geq \sqrt{\frac{xy + 1}{2}} + \sqrt{\frac{yz + 1}{2}} + \sqrt{\frac{zx + 1}{2}}.$$
11. Let positive real numbers $x_1, x_2, \ldots, x_n$ satisfy:
$$\frac{1}{1 + x_1} + \frac{1}{1 + x_2} + \cdots + \frac{1}{1 + x_n} = 1.$$
Show that $x_1x_2 \ldots x_n \geq (n - 1)^n$.
12. Let $0 < x, y, z < 1$ be real numbers. Prove that: $a + b + c + 2abc > ab + bc + ac + 2\sqrt{abc}$.