# RANDOM WALKS AND MARTINGALES

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## 1. Introduction

Random walks are an important topic in much of contemporary mathematics, from applications in statistics and computations to abstract probability theory. They are one of the simplest models we have for a large number of natural processes.

In contests, you have probably already interacted with random walks and how to study them with the idea of "state-based diagrams." At least, that's the extent to which I saw them in high school. Maybe someone's whispered the word Markov Chain to you at some point. This handout is a collection of problems involving random walks that use a number of common ideas.

## 2. Lecture notes

TBD (yes I know I know I overslept 5/7 days last week)

## 3. Exercises

**Problem 3.1** (AIME 2016)**.** Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

**Problem 3.2** (AIME 2021)**.** Let $A_1A_2A_3...A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at $A_4, A_8$, and $A_{12}$. At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**Problem 3.3.** A biased coin with probability $p$ of landing heads is tossed repeatedly until $n$ consecutive heads occurs. What is the expected number of tosses made?

**Problem 3.4** (COMC 2019)**.** Three players $A$, $B$ and $C$ sit around a circle to play a game in the order $A \to B \to C \to A \to \cdots$ . On their turn, if a player has an even number of coins, they pass half of them tothe next player and keep the other half. If they have an odd number, they discard 1 and keep the rest. For example, if players $A$, $B$ and $C$ start with $(2, 3, 1)$ coins, respectively, then they will have $(1, 4, 1)$ after $A$ moves, $(1, 2, 3)$ after $B$ moves, and $(1, 2, 2)$ after $C$ moves, etc. We call a position $(x, y, z)$ stable if it returns to the same position after every 3 moves.

What is the minimum number of coins that is needed to form a position that is neither stable nor eventually leading to $(0, 0, 0)$?

**Problem 3.5.** Let $a_1, a_2, a_3$ be real numbers and for $n \geq 4$ define $a_n$ to be the average of $a_{n-1}, a_{n-2}$ and $a_{n-3}$. Find the limit

$$\lim_{n\to\infty} a_n$$

in terms of $a_1, a_2, a_3$.

## 4. Problems

**Problem 4.1** (ISL 2001)**.** Let $T$ denote the set of all ordered triples $(p, q, r)$ of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers $p$, $q$, $r$.

**Problem 4.2** (StackExchange)**.** I flip a fair coin repeatedly and record the results. I stop as soon as the number of heads is equal to twice the number of tails (for example, I will stop after seeing HHT or THTHHH or TTTHHHHHH). What's the probability that I never stop?

**Problem 4.3.** Let $n$ be a positive integer. Let $f(n)$ be the number of sequences $(x_0, y_0), \ldots, (x_n, y_n)$ of points with integer coordinates satisfying the following conditions:

(1) $(x_0, y_0) = (0, 0)$.

(2) $|x_k - x_{k+1}| + |y_k - y_{k+1}| = 1$ for all $0 \le k < n$.

(3) For all $0 \le i, j \le n$ with $i \ne j$ we have $(x_i, y_i) \ne (x_j, y_j)$.

Prove that for all sufficiently large integers $n$ we have

$$2.4^n \le f(n) \le 2.99^n.$$

**Problem 4.4** (USA TSTST 2018)**.** Find all functions $f \colon \mathbb{Z}^2 \to [0, 1]$ such that for any integers $x$ and $y$,

$$f(x,y) = \frac{f(x-1,y) + f(x,y-1)}{2}.$$

**Problem 4.5.** A drunken person walks randomly on a tree. Each time, he chooses uniformly at random a neighbouring node and walks there. Show that wherever his starting point and goal are, the expected number of steps the person takes to reach the goal is always an integer.

**Problem 4.6.** Sauron, the king of Mordor, decided to go on a vacation and locked his Dark Tower with $n$ keys, $n \in \mathbf{N}, n \ge 2$. He distributed the keys into $n$ houses, one key per house. The houses and the Dark Tower are disposed on a circular road. Frodo wants to break into the Dark Tower, so he must collect all the keys first. He starts his journey from some of the houses. Every time he visits a house, he picks up the key (if it wasn't already taken) and tosses a fair coin to decide which of the two neighboring places to visit next. If Frodo arrives at the Dark Tower before collecting all the keys, he wouldn't enter and will be eaten by Grendel, a monster that lurks around the Tower.

Prove that the chance of Frodo not being eaten does not depend on which house he starts from.