Some Tricks in Synthetic Geometry

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1 Introduction

These notes outline a few basic synthetic methods to approach geometry problems on Olympiads. The solution outlines are sketches intended to emphasize motivation more than rigour. In particular, they do not deal with configuration issues and special cases. The accompanying handout includes a few geometry facts and theorems that are useful to know and also worthwhile exercises to work through an prove yourself. Some of the more useful of these facts are highlighted in Section 6.

2 Exhaust the Diagram

Sometimes it is tempting with geometry problems to immediately start guessing what magic point, line or circle to draw in the diagram leads to an elegant short solution. Before doing this, it is worthwhile to make sure that the problem actually needs something new. Oftentimes, the problem statement already has introduced all parts of the diagram that the most straightforward solution will need. In these cases, making sure you figure out everything you can with what you are given is much more productive than adding points to the diagram. Here are some things to try:

1. *Angle Chasing*: Given your knowledge of similar triangles and cyclic quadrilaterals in the diagram, find all angle relationships you can in the diagram. This is an essential step in almost all Olympiad geometry problems.

2. *Length Chasing*: Many problems can be solved by alternating between angle and length chasing – using some length relationships to find a new cyclic quadrilateral or pair of similar triangles and subsequently making use of whatever new angle relationships this yields. Here are some approaches to length chasing:

   (a) *Similar Triangles*: These arise in many different contexts. One common way is spiral similarity: If OAB and OCD are similar triangles with the same orientation, then so are OAC and OBD.

   (b) *Power of a Point*: If AB and CD meet at the point P then $PA \cdot PB = PC \cdot PD$ if and only if $ABCD$ is cyclic.

   (c) *Menelaus and Ceva*: Given a triangle ABC, let the points D, E and F be on lines AB, BC and AC, respectively. Then
$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$$
   if and only if D, E and F are collinear or CD, AE and BF are concurrent (depending on how many of D, E and F are on the sides of ABC).

3. *Work Backwards*: Assuming the result is true, what else would have to be true? Can you show any of these implications without assuming the result? Can you use any of these intermediary results to solve the problem?

Here are some examples of problems that can be solved by exhausting the diagram as given in the problem statement, without adding any drastically new points. The first two examples need no new points at all.

Example 1. *(Russia 2013) Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocentre of triangle ADE is the midpoint of BC.*

Solution. If M is the midpoint of BC, then $\angle PMB = \angle PDB = 90^\circ$ and thus $PMBD$ is cyclic. Now $\angle BDM = \angle BPM = 90^\circ - \angle CBP = \angle BAC$. Thus DM is perpendicular to AC. Similarly, EM is perpendicular to AB. □

Example 2. *(CMO 2014) The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$.*

Solution. Angle chasing gives that $\angle DPA = \angle DCB$. Now note that $QDPB$ and $RAPC$ are cyclic. Thus $\angle DPQ = \angle DBQ = \angle DBC$ and $\angle RPA = \angle DCA$. Now note that $\angle DPR = \angle DCB - \angle RPA = \angle DCB - \angle DCA = \angle ACB$. Thus $\angle QPR = \angle QPD + \angle RPD = \angle DBC + \angle ACB$ which implies the result. □

This next example succumbs easily to working backwards from the desired result.

Example 3. *(IMO 2004 #1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.*

Solution. The result is true if and only if $\angle BMR = \angle CNR$ since then the angles to segments BR and RC at the intersection of the two circles will be supplementary. This occurs if and only if $AMRN$ is cyclic. Since R lies on the bisector of $\angle BAC$, R must lie on the perpendicular bisector of MN. However, we know this is true since the angle bisector of MON is the perpendicular bisector of MN. Running this argument in reverse yields a solution. □

The next problem really illustrates the power of looking for similar triangles and stopping to think about what is already in the diagram before trying to introduce new points.

Example 4. *(ISL 2005 G3) Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that $\angle KCL$ is independent of the line g.*

Solution. Angle chasing gives that $\angle ALD = \angle KAB = \angle BAX/2$ and $\angle DAL = \angle BKA = \angle ADY/2$. Therefore triangles ADL and KBA are similar which implies that $AB/BK = DL/AD$ and therefore $DL/CD = BC/BK$. Since $\angle CDL = \angle CBK = 90^\circ - \angle ADC/2$, it follows that triangles CDL and KBC are siimilar. Now it follows that $\angle KCL = 360^\circ - \angle BCD - \angle DCL - \angle BCK = 180^\circ + \angle CDL - \angle BCD = 180^\circ - \angle BCD/2$ which is independent of g. □

These next two problems involve primarily length chasing. Both also require introducing the orthocenter of the described triangle. However, in both cases, this is just the intersection of altitudes already given in the problem statement.

Example 5. *(Russia 2005) In an acute-angled triangle ABC, AM and BN are altitudes. A point D is chosen on arc ACB of the circumcircle of the triangle. Let the lines AM and BD meet at P and the lines BN and AD meet at Q. Prove that MN bisects segment PQ.*

Solution. Assume without the loss of generality that D is on arc AC not including B. Let H be the orthocenter of ABC. Since $ADCB$ is cyclic,

$$\angle PAN = \angle DAC = \angle DBC = \angle QBM.$$

Also, it follows that

$$\angle NAH = 90^\circ - \angle ACB = \angle MBH.$$

Since $HP \perp AN$ and $HQ \perp BM$, PAN is similar to QBM and NAH is similar to MBH. Therefore

$$\frac{PM}{MH} = \frac{PM/BM}{MH/BM} = \frac{QN/AN}{NH/AN} = \frac{QN}{NH}$$

If X denotes the midpoint of PQ, then

$$\frac{PM}{MH} \cdot \frac{NH}{QN} \cdot \frac{QX}{XP} = 1$$

and by Menelaus' Theorem applied to triangle HPQ, points X, M and N are collinear. □

Example 6. *(ISL 2008 G4) In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the points A and F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.*

Solution. This problem is straightforward with power of a point and does not require introducing any new points other than the orthocenter H of ABC and foot of the perpendicular from A to BC, which are already implicitly present. Relating our goal to angles already in the diagram reduces the problem to showing that $\angle QFB = \angle PEC$. By power of a point $BQ^2 = BP^2 = BF \cdot BA$ and triangles QFB and AQB are similar. Therefore it suffices to show that $\angle PEC = \angle AQC$ which is equivalent to $AQPE$ being cyclic. By power of a point we now have

$$CP \cdot CQ = BC^2 - BP^2 = BC^2 - BF \cdot BA = BC^2 - BD \cdot BC = CD \cdot CB = CE \cdot CA$$

Therefore $AQPE$ is cyclic and we are done. □

3 Completing the Diagram

As seen in the last few examples in the previous section, it is often useful to introduce some points implicit in the problem statement, such as intersection points, triangle centers and projections. A large number of Olympiad geometry problems can be solved by (1) exhausting the diagram and (2) in this way "completing the diagram". This is a vague heuristic and can take many forms, which are impossible to characterize in a single broad stroke. Nonetheless, here is an attempt at some intuition as to when completing the diagram can be useful.

1. *The Triangle Picture*: Add in orthocenters, circumcenters, excenters, incenters, the circumcircle, midpoints of arcs, feet of altitudes, etc. depending on whether they clarify any parts of the problem statement. This is almost always a good idea to at least try.
2. *Intersecting Lines*: This can be useful, especially when the intersection is at an angle that can be calculated or has some other significance. A somewhat trivial-sounding rule of thumb is that you want to introduce intersections that add clarity rather than further complicate the diagram. Usually one or several pairs of lines will stand out as useful to intersect.
3. *Intersecting Lines with Circles*: This is often useful since circles generally give angle relationships for free.
4. *Implicit Circles*: Sometimes an angle relationship or length relationship will be best simplified when interpreted in terms of a hidden circle.
5. *Parallel and Perpendicular Lines*: Sometimes it is useful to project points onto lines, either with a perpendicular or skew projection with parallel lines. This is often to create similar triangles or measure lengths.

Another somewhat trivial-sounding rule of thumb is that a introducing a line, point or circle to a diagram is only useful if it was implicit in the problem statement or **relates two objects that were previously not relatable**. This is the entire heuristic motivation behind the "completing the transformation" tricks for finding new points that are in the next section. In the sections afterwards, we discuss more heuristics in finding the right points to add to a diagram, including phantom points and intersecting circles. This section is devoted to more generic ways to add points to a diagram, which we demonstrate through several miscellaneous examples.

Example 7. *(Russia 2015) An acute-angled ABC ($AB < AC$) is inscribed into a circle ω. Let M be the centroid of ABC, and let AH be an altitude of the triangle. The ray MH meets ω at A'. Prove that the circumcircle of the triangle $A'HB$ is tangent to AB.*

Solution. It suffices to show that $\angle BA'H = \angle B$. Since $\angle BA'H$ is an angle on the circle, it is worthwhile to see what this would imply in terms of subtended arc lengths. This motivates us to intersect the line $A'HM$ with ω at D'. We see that now our goal is to show that $AC = BD'$, or equivalently that $ABCD'$ is an isosceles trapezoid. Let D be the point such that $ABCD$ is an isosceles trapezoid. We want to show that H, M and D are collinear. Let M' be the midpoint of BC and note that $AD = HH' = 2HM$ where H' is the projection of D onto BC. Thus HD divides the segment AM' in the ratio $2 : 1$ since AD and BC are parallel. Thus HD passes through M, as desired. □

Example 8. *(ISL 1995 G1) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.*

Solution. The diagram is cluttered and we try to reduce the parts of the diagrams we need to consider. The line AM is simply the perpendicular to CP at M and DN is simply the perpendicular to BP at N. We no longer have to think about A and D in defining these lines. Now we observe that since ZP is perpendicular to BC, these lines create cyclic quadrilaterals. It seems natural to introduce their intersections with ZP. Let the perpendiculars at M and N to CP and BP intersect ZP at Q and R. We now have that $ZXMC$ and $ZYNB$ are cyclic. Power of a point yields that $PQ \cdot PZ = PM \cdot PC = PX \cdot PY = PN \cdot PB = PR \cdot PZ$. Therefore $Q = R$ and we are done. $\square$

Example 9. *(ISL 2006 G4) Let ABC be a triangle such that $\widehat{ACB} < \widehat{BAC} < \frac{\pi}{2}$. Let D be a point of $[AC]$ such that $BD = BA$. The incircle of ABC touches $[AB]$ at K and $[AC]$ at L. Let J be the center of the incircle of BCD. Prove that (KL) intersects $[AJ]$ at its middle.*

Solution. Angle chasing gives that $\angle ALK = 90^\circ - \angle A/2$ and $\angle CDJ = 90^\circ - \angle A/2$. It makes sense to try to relate these two equal angles in the diagram by trying to move one into a position so that it relates to the other. Furthermore, working on the segment AJ seems difficult as we do not know angles or lengths related to this line. Instead, we try to work on AC, where we can make use of incircle tangent length formulas. We do this by reducing the problem using non-perpendicular projections in the direction of KL onto AC. We find that this reduces the problem to a seemingly feasible alternative and also relates the equal angles originally found. Specifically, let P be the intersection of the line perpendicular to KL through J with AC. It now suffices to show that L is the midpoint of AP. Since $\angle PDJ = \angle ALK = \angle DPJ$, we have that PDJ is isosceles and if M is the midpoint of DP, then M is also the foot of the perpendicular from J onto AC. Applying incircle tangent length formulas gives that $AL = \frac{1}{2}(AB + AC - BC)$ and $AP = AD + 2AM = AD + (BD + DC - BC) = AB + AC - BC$. This implies that L is the midpoint of AP and the desired result follows. $\square$

The next example has multiple elements that are difficult to work with. Here, we follow cues presented in the diagram and obtain useful constructions (introduced points uniting more than one condition) and reduce the problem to feasible ratio calculations.

Example 10. *(ISL 1996 G3) Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\angle FHP = \angle BAC$.*

Solution. If the problem statement is true, then $\angle CHP = 180^\circ - \angle BAC$. Based on this angle relationship, intersecting HP with AB creates a cyclic quadrilateral. We reformulate the problem by defining P as the point on AC satisfying $\angle FHP = \angle BAC$ introduce this intersection point and call it D. Our goal is now to show $\angle PFO = 90^\circ$ and the two definitions are therefore equivalent. Since $CHAD$ is cyclic, we have that $\angle CDA = 180^\circ - \angle CHA = \angle CBA$. Since the line OF is difficult to deal with and angles around it have no simple formula, we try to reduce the problem to a condition relating something more directly related to P than OF. We have now that DCB is isosceles and F is the midpoint of BD. If M is the midpoint of AB, then we now note that there is a homothety sending MF to AD with center B and ratio 2. Let E be the image of O under this homothety. Note that $AE = 2OM = CH$. It now suffices to show that $\angle EDA = 90^\circ - \angle PFH$. We now try to reduce this angle condition to length conditions which will be easier to deal with since many angles in the diagram cannot be expressed simply. If G is the intersection of FP with the line through C perpendicular to CH. Since $\angle GCF = \angle EAD = 90^\circ$, it suffices to show that GCF and EAD are similar, which is equivalent to showing that

$$\frac{CH}{AD} = \frac{EA}{AD} = \frac{GC}{CF} = \frac{CP}{PA} \cdot \frac{AF}{CF}$$

Now we resort to a ratio identity for cyclic quadrilaterals. The ratio CP/PA is the ratio of the areas of triangles DCH and DAH. Therefore since $CHAD$ is cyclic, we have that

$$\frac{CP}{PA} = \frac{\sin \angle DCH \cdot CD \cdot CH}{\sin \angle DAH \cdot AD \cdot AH} = \frac{CB \cdot CH}{AD \cdot AH}$$

Therefore the desired result reduces to proving that $AH/AF = BC/CF$ which follows from the fact that AHF and CBF are similar. This completes the proof. □

4 Completing Transformations

One of the most useful techniques in synthetic geometry problems is to recognize a transformation present in a diagram, and introduce whatever points are needed to complete the set of images of points under this transformation. Often this heuristic yields the "magic point" that leads to a quick concise solution. For example, a diagram may contain a parallelogram $ABCD$ in which cases there is a translation mapping AB to DC. A diagram may contain a trapezoid $ABCD$ with $AB \| CD$ in which case there is a homothety mapping AB to CD. The transformations that most commonly appear are spiral similarities, rotations, homotheties and translations. The first few examples illustrate different ways to apply this heuristic for spiral similarities and rotations.

Example 11. *(JBMO 2002) An isosceles triangle ABC satisfies that $CA = CB$. A point P is on the circumcircle between A and B and on the opposite side of the line AB to C. If D is the foot of the perpendicular from C to PB, show that $PA + PB = 2 \cdot PD$.*

Solution. We complete the rotation with center C mapping A to B. Let the point Q be such that triangles QCB and PCA are congruent. Since $PACB$ is cyclic,

$$\angle CBQ = \angle CAP = 180^\circ - \angle CBP$$

which implies that P, B and Q are collinear. Since QCB and PCA are congruent, CPQ is isosceles and thus D is the midpoint of PQ. Therefore

$$PA + PB = PQ = 2 \cdot PD$$

□

The second example is one direction of Ptolemy's Theorem.

Example 12. *(Ptolemy's Theorem) If $ABCD$ is a cyclic quadrilateral, then*

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Here we construct similar triangles by applying a spiral similarity with center A mapping the C to D. We let the point B be mapped to P under this map, completing the transformation.

Solution. Let P be the point on BD such that $\angle APD = \angle ABC$. Note that since $\angle ADP = \angle ACB$ which implies that triangles ABC and APD are similar. This implies that triangles ADC and APB are similar. Therefore $\frac{AD}{AC} = \frac{PD}{BC}$ and $\frac{AB}{AC} = \frac{BP}{CD}$. Therefore

$$BD = BP + PD = \frac{AB \cdot CD}{AC} + \frac{AD \cdot BC}{AC}$$

which implies on multiplying up that $AB \cdot CD + AD \cdot BC = AC \cdot BD$. □

Example 13. *(ISL 2000 G6) Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.*

In this example we consider the spiral similarity with center B mapping line CX to the perpendicular bisector of AB in order to obtain the angle we want Y to have at the image Y' of C. We then show that $Y = Y'$.

Solution. Let X' and Y' be such that $AX' = BX'$, $AY' = BY'$, $\measuredangle AX'B = 2 \cdot \measuredangle BXC$ and $\measuredangle AY'B = 2 \cdot \measuredangle BCX$. We have that $AX'Y'$ and AXD are similar, and that $BX'Y'$ and BXC are similar. These similarities imply that triangles AXX' and ADY' are similar and that triangles BXX' and BCY' are similar. The ratios of similarity give that

$$DY' = \frac{AY' \cdot XX'}{AX'} = \frac{BY' \cdot XX'}{BX'} = CY'$$

Hence Y' lies on the perpendicular bisector of CD and $Y' = Y$. Thus $\measuredangle AYB = 2 \cdot \measuredangle ADX$. □

Example 14. *(IMO 1996) Let P be a point inside a triangle ABC such that*

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

Solution. Here we use spiral similarity to construct exactly the given angle condition. By the angle bisector theorem, it suffices to show that $\frac{AB}{BP} = \frac{AC}{CP}$. Let Q be such that triangles APB and ACQ are similar. It follows that APC and ABQ are similar. It follows that

$$\angle CBQ = \angle APC - \angle ABC = \angle APB - \angle ACB = \angle BCQ$$

and thus $BQ = CQ$. Ratios of similarity finish the problem

$$\frac{AB}{BP} = \frac{AQ}{CQ} = \frac{AQ}{BQ} = \frac{AC}{CP}$$

□

The next problem illustrates an often useful transformation when there is a midpoint of the side of a triangle. It is often useful to perform a 180° rotation about the midpoint to produce a parallelogram as in the example below which is from Challenging Problems in Geometry.

Example 15. *Let ABC be a given triangle and M be the midpoint of BC. If $\angle CAM = 2 \cdot \angle BAM$ and D is a point on line AM such that $\angle DBA = 90^\circ$, prove that $AD = 2 \cdot AC$.*

Solution. There is a very short trigonometric solution to this problem, but we present a synthetic one to illustrate the transformation mentioned above. Let D be such that $ABDC$ is a parallelogram. If N is the midpoint of AD, then M is the midpoint of AD. Now note

$$\angle BND = 2 \cdot \angle BAM = \angle CAM = \angle NDB$$

and thus $BD = BN$. This implies that $AC = BD = BN = \frac{1}{2}AD$. □

The next example completes another translation in the same vain as above.

Example 16. *(2013 British MO) The point P lies inside triangle ABC so that $\angle ABP = \angle PCA$. The point Q is such that $PBQC$ is a parallelogram. Prove that $\angle QAB = \angle CAP$.*

Solution. Let R be such that $RACP$ is a parallelogram. It follows that $\angle ARP = \angle PCA = \angle ABP$ which implies that $RAPB$ is cyclic. It follows that BRP and QAC are congruent and thus $\angle QAC = \angle BRP = \angle BAP$. This implies that $\angle QAB = \angle CAP$. □

This last example completes a homothety.

Example 17. *(ISL 2006 G2) Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $\frac{AK}{KB} = \frac{DL}{LC}$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concyclic.*

Solution. Since $ABCD$ is a trapezoid, there is a homothety sending AB to CD as well as one sending AB to DC. We note that the homothety sending AB to DC also sends K to L. Now we complete this homothety in the diagram. Let DA and CB intersect at T and let the homothety with center T bring P to P'. We have that K, P, Q, L and P' are collinear and $PB \| P'C$. Since $\angle DQC + \angle APB = \angle DQC + \angle DP'C = 180^\circ$, we have $DQCP'$ is cyclic. Therefore $\angle QPB = \angle QP'C = \angle QDC = 180^\circ - \angle DQC - \angle QCD = \angle QCB$. The conclusion follows. □

5 Redefining Points

In this section, we build on an idea hinted at in Example 13. Sometimes a point may be defined in a deliberately difficult way in a problem statement. This also was the case in Example 3. Often the key to the solution is to find the "useful way" to define the point and prove that this is in fact the same point. Specifically, if P is a point in the diagram that is difficult to deal with, it is often best to define P' in some other way using a property we think is true of P and then prove that $P' = P$. One thing to note is that this method requires that we have a property of P in mind. Finding out what is true of P is usually the most difficult part of problems that can be solved using this method. Sometimes working backwards is enough, but oftentimes some guesswork, intuition and wishful thinking is necessary.

Often the best conjectures are simple, such as P lies on a line in the diagram, P lies on a circle in the diagram or is concyclic with other points in the diagram, that two lines are parallel or perpendicular, or that two triangles are similar or congruent. It can be useful sometimes to try to eyeball some of these from a well-drawn diagram. Here are is an example.

Example 18. *An acute-angled triangle ABC is inscribed in a circle ω. A point P is chosen inside the triangle. Line AP intersects ω at the point A_1. Line BP intersects ω at the point B_1. A line ℓ is drawn through P and intersects BC and AC at the points A_2 and B_2. Prove that the circumcircles of triangles A_1A_2C and B_1B_2C intersect again on line ℓ.*

We want to analyze the second intersection of the circumcircles of triangles A_1A_2C and B_1B_2C. How much we can prove about this intersection Q varies greatly with how we define Q. First let's try defining Q directly as the intersection of the circumcircles of triangles A_1A_2C and B_1B_2C. From this, we know that $\angle CQB_2 = 180^\circ - \angle CB_1B_2$ and $\angle CQA_2 = 180^\circ - \angle CA_1A_2$. What we want is to show that $\angle CQB_2 + \angle CQA_2 = 180^\circ$ which now is equivalent to $\angle CB_1B_2 + \angle CA_1A_2 = 180^\circ$. However, this is not immediately true given the conditions in the problem. This doesn't seem to work. Let's try a different way of defining Q.

Solution. Define Q' as the intersection of the circumcircle of B_1PA_1 and ℓ. From cyclic quadrilaterals, we have

$$\angle B_1Q'P = \angle B_1A_1P = \angle B_1CB_2$$

which implies that Q' is on the circumcircle of B_1B_2C. By a similar argument, we have that Q' is on the circumcircle of A_1A_2C. Together these imply that $Q = Q'$. Thus Q lies on ℓ. □

A solution can also be obtained by defining Q' as the intersection of the circumcircle of B_1B_2C and ℓ. The way we define Q' above can be motivated as follows. We want to define Q' in some way and then use this way to show it lies on circles. The cleanest way to do this is to show the angle conditions for a cyclic quadrilateral. In order to get these angle conditions, one promising approach is to define Q' as the intersection of a circle with something, which in this case is ℓ.

These next examples illustrate this same method applied in more situations. Particularly in Example 19, it is hard to find a clean solution without the observations used to define P'.

Example 19. *(China 2012) In the triangle ABC, $\angle A$ is biggest. On the circumcircle of ABC, let D be the midpoint of arc ABC and E be the midpoint of arc ACB. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. Circles c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.*

If the result is true, then by the tangency conditions $\angle APB = 180^\circ - \angle BAC$ and $\angle PBA = 180^\circ - \angle APB - \angle PAB = \frac{1}{2}\angle BAC = \angle PAB$. Therefore if the problem is true, then P lies on the perpendicular bisector of AB. This gives us the hint to try defining P based on this. The method below defines P' as the intersection of c_1 and the perpendicular bisector of AB.

Solution. Let the center of c_1 be O_1 and let the center of c_2 be O_2. Since c_1 is tangent to AC, it follows that $\angle BO_1A = 2\angle BAC$. Since O_1 and E both lie on the perpendicular bisector of AB, it follows that O_1E bisects angle $\angle BO_1A$ which implies that $\angle BO_1A = \angle BAC$ and hence that $\angle BP'E = 90^\circ + \frac{1}{2}\angle BAC$. However, since P' lies on the perpencular bisector EO_1 of AB, A is the reflection of B about EO_1 and $\angle AP'E = \angle BP'E = 90^\circ + \angle BAC$. Since c_2 is tangent to AD and passes through E, it follows that $\angle AO_2E = 2\angle DAE = 180^\circ - \angle BAC$. Combining this with the angle relation above yields that P' lies on c_2. Hence P' lies on both c_1 and c_2 and $P = P'$. Therefore $\angle BAP = \frac{1}{2}\angle BO_1P = \frac{1}{2}\angle BAC$ which implies the result. □

The next example really illustrates the power of redefining a point that is difficult to work with. Here, a relatively simple restatement reduces the problem to simple angle chasing.

Example 20. *(ISL 2002 G3) The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120^\circ$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.*

Solution. We first make several preliminary observations. Since EF is the perpendicular bisector of OA, we have that $AE = OE = OA$ and therefore AOE is equilateral. Similarly, we have that AOF is equilateral which implies that $\angle EOF = 120^\circ$ and $\angle ECF = 60^\circ$. These results also imply that A

is the midpoint of arc $\widehat{EF}$ and CA bisects $\angle ECF$. After these preliminary observations, it becomes difficult to work with the point I as defined. The key here is to redefine I to be easier to work with. We now define I' to be the incenter of CEF with the goal of showing that $\angle DAO = \angle AOI'$ since this would imply that $OI' \| AD$ and therefore $I = I'$. At this point, the task becomes far more feasible than before and reduces to angle chasing. First we note that $\angle EOF = 120^\circ$ and $\angle EI'F = 90^\circ + \angle ECF/2 = 120^\circ$ which implies that $EI'OF$ is cyclic. Now we carry out our angle chasing methodically, attempting to eliminate points from consideration as we go. Note that $\angle DAO = 90^\circ - \angle AOD/2 = 90^\circ - \angle ACB/2 = 45^\circ + \angle ABC/2 = 45^\circ + \angle AFC/2$, which is enough to eliminate D and B. Now note that $\angle AOI' = \angle AOE + \angle EOI' = 60^\circ + \angle EFI = 60^\circ + \angle EFC/2$. Since $\angle AFC - \angle EFC = 30^\circ$, we have that $\angle DAO = \angle AOI'$, as desired. $\square$

A remarkably powerful way of redefining points is to try to identify them as *the intersection of a line or circle with another circle*. This yields angle information that often leads to quick solutions. To illustrate this, we outline the solution to what is possibly the hardest geometry problem on the IMO in recent memory.

Example 21. *(IMO 2011) Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.*

Exhausting the diagram yields almost nothing promising. The main issue is that we know almost nothing about the point of tangency. The key to the simplest solution to this problem is to find a way to define this supposed point of tangency. We try intersecting circumcircles in order to obtain angle information to prove that the point of intersection lies on Γ, the circumcircle of the triangle determined by the three lines and prove that the circles are tangent at this point.

Solution. Let A', B' and C' be the intersections of ℓ_b and ℓ_c, ℓ_a and ℓ_c, and ℓ_a and ℓ_b, respectively. Let P be the point of tangency between Γ and ℓ and let Q be the reflection of P through BC. Now let T be the second intersection of the circumcircles of $BB'Q$ and $CC'Q$. It can be shown that T lies on Γ and the circumcircle of $A'B'C'$ by angle chasing. Similarly, T can be shown to be a point of tangency between the circles by angle chasing. The angle chasing is made easier by first showing that AA', BB' and CC' meet at the incenter I of $A'B'C'$. $\square$

Example 22. *(CMO 2013) Let O denote the circumcenter of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.*

Here, we use the method above to define the reflection R of the point of tangency in line PQ as the intersection of triangle OBP with side BC. This construction can be motivated either by noticing this pattern in the diagram, noting that this method of intersecting circles obtains angles in exactly the way needed to prove the result, or by trying to complete the Miquel configuration.

Solution. Let the circumcircle of triangle OBP intersect side BC at the points R and B and let $\angle A$, $\angle B$ and $\angle C$ denote the angles at vertices A, B and C, respectively. Now note that since $\angle BOP = \angle B$ and $\angle COQ = \angle C$, it follows that

$$\angle POQ = 360^\circ - \angle BOP - \angle COQ - \angle BOC = 360^\circ - (180 - \angle A) - 2\angle A = 180^\circ - \angle A.$$

This implies that $APOQ$ is a cyclic quadrilateral. Since $BPOR$ is cyclic,

$$\angle QOR = 360^\circ - \angle POQ - \angle POR = 360^\circ - (180^\circ - \angle A) - (180^\circ - \angle B) = 180^\circ - \angle C.$$

This implies that $CQOR$ is a cyclic quadrilateral. Since $APOQ$ and $BPOR$ are cyclic,

$$\angle QPR = \angle QPO + \angle OPR = \angle OAQ + \angle OBR = (90^\circ - \angle B) + (90^\circ - \angle A) = \angle C.$$

Since $CQOR$ is cyclic, $\angle QRC = \angle COQ = \angle C = \angle QPR$ which implies that the circumcircle of triangle PQR is tangent to BC. Further, since $\angle PRB = \angle BOP = \angle B$,

$$\angle PRQ = 180^\circ - \angle PRB - \angle QRC = 180^\circ - \angle B - \angle C = \angle A = \angle PAQ.$$

This implies that the circumcircle of PQR is the reflection of Γ in line PQ. By symmetry in line PQ, this implies that the reflection of BC in line PQ is tangent to Γ. $\square$

6 Know the Classical Configurations

There are a lot of classical geometry configurations and miscellaneous facts that can help in math contests. Here is a selection of a few that seem to come up over and over again. Many more are included in my other handout. Some of these are difficult and worthwhile to prove on your own.

1. Given a triangle ABC, the intersections of the internal and external bisectors of $\angle BAC$ with the perpendicular bisector of ABC lie on the circumcircle of ABC.

2. Facts related to the orthocenter H of a triangle ABC with circumcircle Γ and center O:

   (a) If D is the point diametrically opposite to A on Γ and M is the midpoint of BC, then M is also the midpoint of HD.

   (b) If AH, BH and CH intersect Γ again at D, E and F, then there is a homothety centered at H sending the triangle formed by projecting H onto the sides of ABC to DEF with ratio 2.

   (c) If D and E are the intersections of AH with BC and Γ, respectively, then D is the midpoint of HE.

   (d) If M is the midpoint of BC then $AH = 2 \cdot OM$.

   (e) If BH and CH intersect AC and AB at D and E, and M is the midpoint of BC, then M is the center of the circle through B, D, E and C, and MD and ME are tangent to the circumcircle of ADE.

3. Facts related to the incenter I and excenters I_a, I_b, I_c of ABC with circumcircle Γ:

   (a) If AI intersects Γ at D then $DB = DI = DC$, D is the midpoint of II_a, and II_a is a diameter of the circle with center D which passes through B and C.

   (b) If BI and CI intersect Γ again at D and E, then I is the reflection of A in line DE and if M is the intersection of the external bisector of $\angle BAC$ with Γ, then $DMEI$ is a parallelogram.

   (c) If the incircle and A-excircle of ABC are tangent to BC at D and E, $BD = CE$.

(d) If M is the midpoint of arc BAC of Γ, then M is the midpoint of I_bI_c and the center of the circle through I_b, I_c, B and C.

4. (Symmedian) Given a triangle ABC such that M is the midpoint of BC, the symmedian from A is the line that is the reflection of AM in the bisector of angle $\angle BAC$.

   (a) If the tangents to the circumcircle Γ of ABC at B and C intersect at N, then N lies on the symmedian from A and $\angle BAM = \angle CAN$.

   (b) If the symmedian from A intersects Γ at D, then $AB/BD = AC/CD$.

5. (Apollonius Circle) Let ABC be a given triangle and let P be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line AC at Q and R, then P lies on the circle with diameter QR.

6. (Nine-Point Circle) Given a triangle ABC, let Γ denote the circle passing through the midpoints of the sides of ABC. If H is the orthocenter of ABC, then Γ passes through the midpoints of AH, BH and CH and the projections of H onto the sides of ABC.

7. (Feuerbach's Theorem) The nine-point circle is tangent to the incircle and excircles.

8. (Euler Line) If O, H and G are the circumcenter, orthocenter and centroid of a triangle ABC, then G lies on segment OH with $HG = 2 \cdot OG$.

9. (Euler's Formula) Let O, I and I_a be the circumcenter, incenter and A-excenter of a triangle ABC with circumradius R, inradius r and A-exradius r_a. Then:

   (a) $OI = \sqrt{R(R-2r)}$.

   (b) $OI_a = \sqrt{R(R+2r_a)}$.

10. Let ABC be a given triangle with incircle ω and A-excircle ω_a. If ω and ω_a are tangent to BC at M and N, then AN passes through the point diametrically opposite to M on ω and AM passes through the point diametrically opposite to N on ω_a.

11. Let ABC be a triangle with incircle ω which is tangent to BC, AC and AB at D, E and F. Let M be the midpoint of BC. The perpendicular to BC at D, the median AM and the line EF are concurrent.

12. Let ABC be a triangle with incenter I and incircle ω which is tangent to BC, AC and AB at D, E and F. The angle bisector CI intersects FE at a point T on the line adjoining the midpoints of AB and BC. It also holds that $BFTID$ is cyclic and $\angle BTC = 90°$.

13. Let ABC be a triangle with incircle ω and let D and E be the points at which ω is tangent to BC and the A-excircle is tangent to BC. Then AE passes through the point diametrically opposite to D on ω.

14. Let ABC be a triangle with A-excenter I_A and altitutde AD. Let M be the midpoint of AD and let K be the point of tangency between the incircle of ABC and BC. Then I_A, K and M are collinear.

15. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle Γ tangent to the four sides of $ABCD$ if and only if $AB + CD = AD + BC$.

16. (Simson Line) Let M, N and P be the projections of a point Q onto the sides of a triangle ABC. Then Q lies on the circumcircle of ABC if and only if M, N and P are collinear. If Q lies on the circumcircle of ABC, then the reflections of Q in the sides of ABC are collinear and pass through the orthocenter of the triangle.

17. (Butterfly Theorem) Let M be the midpoint of a chord XY of a circle Γ. The chords AB and CD pass through M. If AD and BC intersect chord XY at P and Q, then M is also the midpoint of PQ.

18. (Mixtilinear Incircles) Let ABC be a triangle with circumcircle Γ and let ω be a circle tangent internally to Γ and to AB anc AC at X and Y. Then the incenter of ABC is the midpoint of segment XY.

19. (Curvilinear Incircles) Let ABC be a triangle with circumcircle Γ and let D be a point on segment BC. Let ω be a circle tangent to Γ, DA and DC. If ω is tangent to DA and DC at F and E, then the incenter of ABC lies on FE.

20. (Pole-Polar) Let X lie on the line joining the points of tangency of the tangents from Y to a circle Ω. Then Y lies on the line joining the points of tangency of the tangents from X to Ω.

7 Problems

I have grouped the problems into three difficulty classes: A, B and C. These are loosely supposed to reflect the difficulty of Problems 1, 2 and 3 at the IMO. However, some of the harder A problems are similar in difficulty to IMO # 2's and some of the harder B problems are similar to IMO # 3's.

A1. (Japan 2012) Let ABC be a given triangle. The tangent to the circumcircle at A intersects the line BC at P. Let Q and R be the reflections of the point P across the lines AB and AC, respectively. Prove that the line BC is perpendicular to the line QR.

A2. (APMO 2007) Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

A3. (Russia 2010) Let ABC be a given triangle and let K be a point on the internal bisector of $\angle BAC$. The line CK intersects the circumcircle ω of triangle ABC at $M \neq C$. The circle Ω passes through A, touches CM at K and intersects segment AB at $P \neq A$ and ω at $Q \neq A$. Prove, that P, Q and M are collinear.

A4. (Russia 2007) A line, which passes through the incenter I of the triangle ABC, meets its sides AB and BC at the points M and N, respectively. The points K, L are chosen on the side AC such that $\angle ILA = \angle IMB$ and $\angle IKC = \angle INB$. If the triangle BMN is acute, prove that $AM + KL + CN = AC$.

A5. (Japan 2011) Let ABC be a given acute triangle and let M be the midpoint of BC. Draw the perpendicular HP from the orthocenter H of ABC to AM. Show that $AM \cdot PM = BM^2$.

A6. (CMO 1997) The point O is situated inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.

A7. (Russia 2012) The points A_1, B_1 and C_1 lie on the sides BC, CA and AB of the triangle ABC, respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let O_A, O_B and O_C be the circumcenters of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the incenter of triangle $O_AO_BO_C$ is the incenter of triangle ABC.

A8. (Russia 2012) Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the foot of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C, D lie on the same circle.

A9. (Russia 2006) Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that KL is parallel to AC. Show that the incenters of triangles ABK and CBL are equidistant from the midpoint of the arc ABC of the circumcircle of triangle ABC.

A10. (Russia 2016) The medians AM_A, BM_B and CM_C of triangle ABC intersect at M. Let Ω_A be the circle passing through the midpoint of AM and M_B and M_C. Define Ω_B and Ω_C analogously. Prove that Ω_A, Ω_B and Ω_C have a common point.

A11. (Russia 2002) Let O be the circumcenter of a triangle ABC. Points M and N are chosen on sides AB and AC, respectively, and such that $\angle MON = \angle BAC$. Prove that the perimeter of triangle AMN is not less than the length of side BC.

B1. (2007 G3) The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

B2. (Russia 2002) Diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at point O. The circumcircles of triangles AOB and COD intersect again at K. The point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if quadrilateral $BLCK$ is convex, then it has an inscribed circle.

B3. (Russia 2002) Let ABC be a given triangle. Let ℓ_a be the line parallel to the internal bisector of angle $\angle A$ passing through the point at which the excircle opposite vertex A is tangent to side BC. Define ℓ_b and ℓ_c analogously. Prove that ℓ_a, ℓ_b and ℓ_c are concurrent.

B4. (Russia 2011) The perimeter of a given triangle ABC is 4. The point X lies on ray AB and point Y lies on ray AC such that $AX = AY = 1$. If the line XY intersects segment BC at the point M, prove that the perimeter of one of the triangles ABM or ACM is 2.

B5. (Japan 2012) Let triangles PAB and PCD be such that $PA = PB$, $PC = PD$, P, A, C are collinear in that order and B, P, D are collinear in that order. The circle S_1 passes through A and C and intersects with the circle S_2 passing through B and D at distinct points X and Y. Prove that the circumcenter of the triangle PXY is the midpoint of the segment adjoining the centers of S_1 and S_2.

B6. (Japan MO 2009) Let Γ be the circumcircle of a triangle ABC. A circle with center O touches to line segment BC at P and touches the arc BC of Γ which doesn't have A at Q. If $\angle BAO = \angle CAO$, then prove that $\angle PAO = \angle QAO$.

B7. (Russia 2012) The point E is the midpoint of the segment connecting the orthocenter of the scalene triangle ABC and the point A. The incircle of triangle ABC incircle is tangent to AB and AC at points C' and B', respectively. Prove that point F, the point symmetric to point E with respect to line $B'C'$, lies on the line that passes through both the circumcenter and the incenter of triangle ABC.

B8. (1995 G8) Suppose that $ABCD$ is a cyclic quadrilateral. Let E be the intersection of AC and BD and let F be the intersection of AB and CD. Denote by H_1 and H_2 the orthocenters of triangles EAD and EBC, respectively. Prove that the points F, H_1, H_2 are collinear.

C1. (Russia 2011). Let M be the midpoint of side BC of a triangle ABC and N be the midpoint of arc BAC of the circumcircle of the triangle. Prove that the points A, N and the incenters of triangles ABM and ACM are concyclic.

C2. (Mathlinks) Let $ABCD$ be an isosceles trapezoid with AD parallel to BC. The circle ω is tangent to segments AB and AC and to the circumcircle of $ABCD$ at the point M. Let the incircle of triangle ABC be tangent to BC at P. Prove that D, P and M are collinear.

C3. (Japan 2001) Suppose that ABC and PQR are triangles such that A and P are the midpoints of QR and BC, respectively. If QR and BC are the internal bisectors of $\angle BAC$ and $\angle QPR$, respectively, prove that $AB + AC = PQ + PR$.

C4. (CGMO 2011) The A-excircle of triangle ABC is centered at I and is tangent to BC at M. The points D and E lie on rays AB and AC and satisfy that DE is parallel to BC. The incircle of triangle ADE is centered at J and tangent to DE at N. If BI and DJ intersect at F and C and EJ intersect at G, prove that the midpoint of FG lies on MN.

C5. (Russia 2006) Let ABC be an acute-angled triangle with incenter I. The lines BI and CI meet sides AC and AB at B_1 and C_1, respectively. If the line B_1C_1 meets the circumcircle of ABC at M and N, prove that the circumradius of triangle MIN is twice that of ABC.

C6. (ISL 2004 G7) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

C7. (ISL 2012 G6) Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

C8. (ISL 2002 G7) The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

C9. (RMM 2011) A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

8 Hints

If there is a theorem or fact that is useful, I have tried to indicate it with the label TL.

A1. If Q' and R' are the midpoints of PQ and PR, respectively, prove that $Q'R'$ is perpendicular to BC.

A2. Angle chase completely and look for cyclic quadrilaterals.

A3. Angle chase completely.

A4. Consider the reflection of M in line AI and the reflection of N in line CI.

A5. Consider the feet Q and R of the perpendiculars from H to AB and AC, respectively. What can be said about the relationship between MQ, MR and the circumcircle of triangle AHP.

A6. Consider the translation mapping AB to CD. Complete the picture.

A7. Consider the projections D, E and F from the incenter I of triangle ABC to sides BC, AC and AB, respectively. What can be said about the lengths DA_1, EB_1 and FC_1? Can you, from here, prove something useful about the quadrilateral AB_1C_1I? TL: lengths of the segments adjoining the vertices of a triangle to the tangency points of its incenter, the angle bisector and opposite perpendicular bisector meet on the circumcircle of a triangle.

A8. Consider the point E such that $KHBE$ is a rectangle. Notice the information we use about K to solve the problem. Is there a way to solve the problem without the point E?

A9. Let the midpoint of arc ABC be M and let the incenters of ABK and CBL be I_1 and I_2, respectively. Extend BI_1 and BI_2 to intersect the circumcircle of ABC. Can you prove that two triangles are congruent? TL: in a triangle XYZ with incenter I, the circumcenter of XYI lies on the circumcircle of XYZ.

A10. Define the intersection of two of the circles. Angle chase from here.

A11. Consider rotating triangles AMO and ANO about O. Can you create a figure which immediately implies the desired result?

B1. Consider the homothety mapping BC to DA with center P.

B2. Consider the point I such that triangles KIC and ABC are similar and equally oriented. Show that I lies on the bisectors of angles $\angle BKC$, $\angle KBL$ and $\angle KCL$. TL: the unique center of spiral similarity sending A to B and C to D is the second intersection of the circumcircles of ACP and BDP where AB and CD intersect at P, if triangles OXY and OZW are similar and equally oriented then triangles OXZ and OYW are similar and equally oriented.

B3. Prove that the lines passing through the tangency points of the incircle to the three sides and parallel to the triangle's angle bisectors are concurrent. Prove that the lines passing through the midpoints of the sides and parallel to the triangle's angle bisectors are concurrent. How does this imply the desired result? TL: the midpoint of the segment adjoining the points of tangency of the incircle and the excircle to a side of a triangle is the midpoint of that side of the triangle.

B4. Find two circles that XY is the radical axis of. Note that a circle can have radius 0. TL: radical axis theorem, the radical axis of two circles passes through the midpoints of all common tangents to the two circles.

B5. Calculate the distances from X and Y to the midpoint of the segment adjoining the centers of S_1 and S_2 in terms of the radii of the circles. Find a clean way to show that this is also the distance from P to this point. TL: Stewart's Theorem or Cosine Law.

B6. Introduce the midpoints of both arcs between B and C. What is the desired result equivalent to in terms of the quadrilateral $APOQ$?

B7. Consider the reflection A' of A in $B'C'$ and note that $A'F$ is parallel to AO.

B8. Try to overlay the similar triangles FAD and FCB and complete the picture in terms of H_1 and H_2. Is there now a complete the transformation-style argument?

C1. Reflect the incenter of ABM about the line MN and apply the trigonometric form of Ceva's Theorem. TL: trigonometric form of Ceva's Theorem or the existence of isogonal conjugates.

C2. Let I be the incenter of triangle ABC and let Q and R be the midpoints of BI and CI, respectively. Consider the second intersection of the circumcircles of triangles BQP and CRP. TL: Pascal's Theorem or inversion.

C3. Let the perpendicular bisector of QR intersect BC at D and let the perpendicular bisector of BC intersect QR at E. Let the perpendicular bisectors of BC and QR intersect each other at X. Use triangle XDE to find the ratio DP/AE in terms of the angles $\angle A$ and $\angle P$. TL: the intersection of an angle bisector of a triangle and its opposite perpendicular bisector lies on the circumcircle of the triangle.

C4. Let OB intersect O_1D at P and let OC intersect O_1E at Q. Prove that PQ is parallel to BC. Prove that if S is the internal center of homothety between the circles (O) and (O_1), then P, F and S are collinear and Q, G and S are collinear. Now use the fact that the line adjoining the midpoints of MN and OO_1 is perpendicular to BC. TL: Ceva's Theorem.

C5. Let I_B and I_C be the B and C excenters of ABC. Prove that I_BMINI_C is cyclic. TL: the nine-point circle.

C6. Prove as a lemma that given a triangle ABC, the bisector of $\angle ABC$, the line joining the midpoints of AB and AC, and the line through the points of tangency between the incircle and BC and AC are concurrent.

C7. Consider the midpoint M of arc FD on the circumcircle of BFD. Prove that M and I have the same power of a point with respect to the circumcircle of ABC. Do the same for the circumcircles of CED and AFE.

C8. Prove that KM passes through the A-excenter of ABC. Now use the fact that if X lies on the polar from Y to Ω then Y lies on the polar from Ω (pole-polar).

C9. Invert about A with power $AK \cdot AL$.