Combinatorics Problems, Part 1

1. Show that $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n}{r+1}$.

2. Find a nice formulation for the sum $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$.

3. Let $S_n = \binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-2\left\lfloor \frac{n}{2} \right\rfloor}{\left\lfloor \frac{n}{2} \right\rfloor}$. Show that $S_n = F_n$.

4. Given $S = \{1, 2, \cdots, n\}$, How many unordered pairs $\{A, B\}$ are there where A and B are nonempty subsets of S with $A \cap B = \emptyset$.

5. Do there exist 10,000 10-digit numbers which are all divisible by 7 and are all rearrangements of the same digits.

6. A person has a coat of area 1 composed of five possibly overlapping patches. The area of each patch is at least $\frac{1}{2}$. Prove that there are two patches whose overlap has area of at least $\frac{1}{5}$.

7. A permutation of n elements is a one-to-one function π: $\{1, 2, \cdots, k\} \to \{1, 2, \cdots, k\}$. A permutation π of $\{1, 2, 3, \cdots, 2n\}$ has property P if $|\pi(i) - \pi(i+1)| = n$ for at least one i. Show that for any $n \in \mathbb{Z}$, there are more permutations with property P that without it.

8. Let M be a set of $n \geq 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, we perform moves where we choose two segments AB and CD that intersect and replace them by the segments AC and BD if none of them is present.

   Prove that it is impossible to perform $\frac{n^3}{4}$ moves.

Combinatorics Problem Set Part 2

1. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

   We say that a set of N flags is diverse if these flags can be arranged into an $N\times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

2. Six stacks $S_1, \dots, S_6$ of coins are standing in a row. In the beginning every stack contains a single coin. There are two types of allowed moves:

   Move 1 : If stack S_k with $1 \leq k \leq 5$ contains at least one coin, you may remove one coin from S_k and add two coins to S_{k+1}.

   Move 2 : If stack S_k with $1 \leq k \leq 4$ contains at least one coin, then you may remove one coin from S_k and exchange stacks S_{k+1} and S_{k+2}.

   Decide whether it is possible to achieve by a sequence of such moves that the first five stacks are empty, whereas the sixth stack S_6 contains exactly $2016^{2016^{2016}}$ coins.

3. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) interval of length p. In every round, player A picks some positive integer m and provides $\frac{1}{2^m}$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $\frac{k}{2^m}$ to $\frac{k+1}{2^m}$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0,1]$ is not completely blackened. Decide whether there exists a strategy for player A to win in a finite number of moves.

4. Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is colored red or blue, subject to the following condition:

   If a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$.

   Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

5. An n-term sequence $(x_1, x_2, \dots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

6. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board happy if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.