Generating Functions

Wintercamp 2020

1 Motivating Example

In many problems, especially in recurrence relations, we encounter a sequence of numbers $a_0, a_1, \ldots, a_n, \ldots$, indexed by the natural numbers. It turns out to be very convenient to be able to put all these numbers together in one object, like a laundry line. We can do this by making them the coefficients of a formal power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_i x^i.$$

We will generally not worry about the convergence of this series and usually the x's are just there as place holders. However, we will encounter a couple of problems and some techniques where it does become important that we can substitute values for x, so don't completely forget about this. The idea will be that we can manipulate the power series in ways that will help us find a closed formula for the a_n as a function of n. This will usually happen by recognizing the power series as representing well-known functions that we can easily manipulate. For instance, we know that $1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$. When we assume that $|r| < 1$, this implies that

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

This is one of the basic series we will be working with and derive others from it.

Here is a first problem to warm up to some of the basic ideas.

Example 1. Let $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_0 = 2$ and $a_1 = 5$. Find a closed formula for a_n as a function of n.

2 Some Results on Infinite Series

One of the things that makes generating functions very useful, is the fact that they connect sequences to functions and one can easily multiply functions. The result for sequences may at first be surprising:

Definition 1. Given two generating functions $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, their product $A(x)B(x)$ is the generating function $C(x) = \sum_{n\geq 0} c_n x^n$ with coefficients

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{i+j=n} a_i b_j.$$

The sequence (c_n) is called the *Cauchy product* of (a_n) and (b_n).

It will often be convenient to recognize a sequence as a Cauchy product of two other sequences and use the generating functions directly to obtain a solution. One general type of situation where this occurs is:

Proposition 1. *If A is a family of sets and a_k is the number of sets of "weight" k in A and B is a family of sets and b_k is the number of sets of "weight" k is in B, then c_n is the number of pairs of a set from A and set from B of total "weight" n. We can therefore write*

$$A(x) = \sum_{a \in A} x^{|a|}, \quad B(x) = \sum_{b \in B} x^{|b|}, \quad C(x) = \sum_{c=(a,b) \in A \times B} x^{|c|}$$

where $|a|$ is the weight of a, and $|c| = |a| + |b|$.

Example 2. Find the number a_n of ways n dollars can be changed into 1 or 2 dollar coins (regardless of order). For example, when $n = 3$, there are two ways, namely three 1 dollar coins or one 1 dollar coin and one 2 dollar coin.

Example 3. (1998 IMO Shortlisted Problem) Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegtive integers such that every nonnegatve integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

Here are two other natural operations on sequences: taking partial sums gives us a new sequence for which the generating function can be expressed in terms of the original generating function. The reverse operation is that of taking finite differences.

Proposition 2. *Let $A(x) = \sum_{n \geq 0} a_n x^n$ be the generating function of (a_n) and define $s_n = a_0 + a_1 + \cdots + a_n$. Then the generating function for (s_n) is*

$$S(x) = \sum_{n \geq 0} s_n x^n = \frac{A(x)}{1-x} = A(x)(1 + x + x^2 + \cdots) = A(x) \sum_{n \geq 0} x^n.$$

Corollary 1. *Define $d_0 = a_0$, $d_n = a_n - a_{n-1}$. Then*

$$D(x) = \sum_{n \geq 0} d_n x^n = (1-x)A(x).$$

Example 4. Derivatives and antiderivatives can be very helpful in creating new generating functions.

1. Given that $A(x) = \sum_{n \geq 0} a_n x^n$, what is the generating function for $b_n = n a_n$?

2. For any polyomial $p(x)$, what is the generating function for $b_n = p(n) a_n$? In particular, what is the generating function for $b_n = p(n)$?

3. Given that $A(x) = \sum_{n\geq 0} a_n x^n$, what is the generating function for $c_n = \frac{a_n}{n+1}$?

4. Compute $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$.

Here are two other common generating functions:

1. **Generalized binomial coefficients** $(1+x)^r = \displaystyle\sum_{k\geq 0}\binom{r}{k}x^k$ for any real number r (if r is not a non-negative integer, assume that $|x| < 1$). Here $\displaystyle\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$ for $k > 0$ and $\displaystyle\binom{r}{0} = 1$.

2. **The exponential function** $e^x = \displaystyle\sum_{k\geq 0}\frac{1}{k!}x^k$

From these you can derive the following:

1. $\displaystyle\frac{1}{(1-x)^{m+1}} = \sum_{k\geq 0}\binom{k+m}{m}x^k$

2. $\displaystyle\frac{x^m}{(1-x)^{m+1}} = \sum_{k\geq 0}\binom{k}{m}x^k$

3. $\displaystyle\frac{1}{\sqrt{1-4x}} = \sum_{k\geq 0}\binom{2k}{k}x^k$

4. $\displaystyle\ln\frac{1}{1-x} = \sum_{n\geq 1}\frac{x^n}{n}$

3 Linear Recurrence Relations

Homogeneous linear recurrence relations, i.e., recurrence relations of the form $a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k}$ with initial values $a_0, a_1, \ldots, a_{k-1}$ can be solved very nicely using generating functions. We saw this in our first example. (Although you may sometimes need to use complex numbers.) If the denominator of the generating function does not have repeated factors, the general solution is of the form $a_n = c_1(\alpha_1)^n + \cdots + c_k(\alpha_k)^n$. Both the c_i and α_i can be found as we did in that example, but you can also find the α_i as roots of the characteristic polynomial:

$$x^k - b_1 x^{k-1} - b_2 x^{k-2} - \cdots - b_k = 0$$

You can then find the c_i by solving a system of linear equations to ensure that the first k values, $a_0, a_1, \ldots, a_{k-1}$ are correct.

If there is a repeated root in this polynomial (or equivalently, there is a repeated factor in the denominator of the generating function), say β has multiplicity m (this would mean that $(1-\beta x)^m$ is a factor of the denominator of the generating function), then we need to add $c_1\beta^n + c_2 n\beta^n + \cdots + c_m n^{m-1}\beta^n$ to the sum for the solution. We see that in our partial fractions approach as well, because we have to use fractions with all powers $(1-\beta x)^\ell$ with $1 \le \ell \le m$. In particular, here we see that we can just treat the generating functions as usual and we will get our solution.

Recurrence relations of the form $a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k} + f(n)$ are still linear, but not homogeneous. In this case, we need to find one special solution for this system (not thinking about the initial conditions at this point) and then the general solution can be obtained by taking the special solution and adding the general solution of the homogeneous equation $a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k}$. If you do this with the generating functions, you will get the answer in the usual way.

Here are some examples:

Example 5. Towers of Hanoi There are three pegs and there are n disks of all different sizes. At the beginning of the game these disks are all stacked on one peg in order of decreasing size, with the largest on the bottom. The goal of the game is to transfer all disks onto one of the other pegs, but at no time can we put a larger disk on top of a smaller one. The third peg can be used to take the extra steps needed to avoid this.

Let h_n be the number of moves required to transfer n disks. Obviously, $h_0 = 0$ and$h_1 = 1$, and it is not hard to check that $h_2 = 3$. It s not hard to develop a recurrence relation for h_n: if we can the top transfer $n-1$ disks to another peg, then move our largest disk to the empty peg and then move the other $n-1$ disks again, we will have them all in the right spot. So $h_n = 2h_{n-1} + 1$ for $n \ge 1$ and $h_0 = 0$. We could find a formula for h_n by iteration. You may check that $h_n = 2^{n-1} + \cdots + 2^2 + 2 + 1 = \frac{2^n-1}{2-1} = 2^n - 1$.

However, things don't always work so nicely, so it is useful to see how our more general method can work here. Let $g(x) = \sum_{n\ge 0} h_n x^n$. Then

$$\begin{aligned} g(x) &= h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots \\ 2xg(x) &= 2h_0 x + 2h_1 x^2 + \cdots + 2h_{n-1} x^n + \cdots \end{aligned}$$

and using the recurrence relation we get

$$(1-2x)g(x) = x + x^2 + \cdots + x^n + \cdots = \frac{x}{1-x}.$$

So we see that

$$g(x) = \frac{x}{(1-x)(1-2x)}$$

Using partial fractions we find

$$\begin{aligned} g(x) &= \frac{1}{1-2x} - \frac{1}{1-x} \\ &= \sum_{n\geq 0}(2x)^n - \sum_{n\geq 0} x^n \\ &= \sum_{n\geq 0}(2^n-1)x^n. \end{aligned}$$

And so we find again that $h_n = 2^n - 1$.

Below are two problems where you may have to work with particular solutions. The particular solutions will look a lot like the function $f(n)$. Here are two special cases:

- if $f(n)$ is a polynomial of degree k in n, try for your special solution to find a polynomial of degree k in n as well: find coefficients $p_0, p_1, \ldots, p_k$ such that $h_n = p_0 + p_1 n + \cdots + p_k n^k$ is a solution.
- if $f(n) = d^n$, find a coefficient p such that $h_n = pd^n$ is a solution.

Example 6. 1. Solve $h_n = 3h_{n-1} - 4n$ for $n \geq 1$ and $h_0 = 2$.

2. Solve $h_n = 2h_{n-1} + 3^n$ for $n \geq 1$ and $h_0 = 2$.

Problems 1. 1. Words of length n, using only the three letters a, b and c are to be transmitted over a communication channel subject to the condition that no work in which two a's appear consecutively is to be transmitted. Determine the number of words allowed by the communication channel.

2. Find the general solution of the recurrence relation

$$h_n - 4h_{n-1} + 4h_{n-2} = 0 \quad \text{for } (n \geq 2).$$

4 Products and Partitions

One place where products of generating functions are very useful is in counting the number of partitions of a positive integer n. A partition of n is a sequence $a_1 \leq a_2 \leq \cdots \leq a_k$ such that $\sum_{i=1}^k a_i = n$. A partition is uniquely determined by the number of 1s, the number of 2s, the number of 3s etc., so the repetition number of each of the numbers used. We can use this to build a generating function. We devote one factor to each integer:

$$\begin{aligned} (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)\cdots(1+x^k+x^{2k}+x^{3k}+\cdots)\cdots &= \prod_{k=1}^{\infty}\sum_{i=0}^{\infty} x^{ik} \\ &= \prod_{k=1}^{\infty}\frac{1}{1-x^k}. \end{aligned}$$

When we expand this product, we pick one term from each factor in all possible ways, but only a finite number of terms that are not a 1. If we take an x^4 from the first factor, an x^2 from the second factor and an x^{25} from the fifth factor, this means that we have taken four 1's, one 2, and five 5's. So we have the partition $1+1+1+1+2+5+5+5+5+5=31$. We see that the coefficient for x^k gives the number of partitions of k. Note that although this is an infinite product, in practice we only need a finite number of factorsand a finite number of terms to find a particular p_n.

Example 7. Find p_8, the number of partitions of 8.

We can further manipulate the generating function above to obtain other generating functions. Do this to solve the following problem.

Example 8. Find the generating function for partitions into all distinct parts, and the generating function for partitions into odd parts. Show that for any positive integer n, the number of partitions into distinct parts is the same as the number of partitions into odd parts.

Problems 2.

1. Find the number h_n of bags of fruit that can be made out of apples, bananas, oranges and pears, where, in each bag, the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

2. Let k_n denote the number of nonnegative integral solutions of the equation

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n$$

Find the generating function $g(x)$ for the h_n.

3. There is available an unlimited number of pennies, nickles, dimes, quarters and half-dollar pieces. Determine the generating funciton $g(x)$ for the number f_n of ways of making n cents with these pieces.

4. Let h_n denote the number of ways of dividing a convex polygonal region with $n+1$ sides into triangles be inserting diagonals that do not intersect in the interior of the region. Find both the generating function and a closed formula for the h_n. Note that these are the Catalan numbers.

5 Cyclotomic Polynomials and Roots of Unity

When working with the polynomials of the form $1+x+\cdots+x^n$ is is often useful to remember that this is a factor of $x^{n+1}-1=(x-1)(1+x+\cdots+x^n)$. Further factors are useful to recognize as well. Note that by the fact that the roots of x^n-1 are the n-th roots of unity ($e^{2k\pi i/n}$), we see that x^m-1 divides x^n-1 if and only if m divides n. For specific values of n this can lead to some unexpected relations between generating functions. Here is a nice example.

Example 9.

1. Give the generating function $\sum_{n\geq 0} p_{n,d}x^n$ for the probability $p_{n,d}$ of rolling a sum of n with d ordinary six-sided dice.

2. Use this to show that the probability distribution of rolling two normal six-sided dice is the same as the probability distribution of rolling a die with sides $1, 2, 2, 3, 3, 4$ and a die with sides $1, 3, 4, 5, 6, 8$.

Roots of unity have yet another role to play here. Let m be any positive integer and let λ be an m-th root of unity other than 1. When we consider a power series

$$B(x) = \sum_{n \geq 0} b_n x^n$$

we are sometimes interested in finding the sum of the coefficients that have an index that is a multiple of m (for instance, we may consider the sum of the even-indexed coefficients),

$$b_m + b_{2m} + b_{3m} + \cdots$$

Here is we can use the following

$$1 + \lambda^j + \lambda^{2j} + \cdots + \lambda^{(m-1)j} = \frac{1 - \lambda^{mj}}{1 - \lambda^j} = 0$$

for any j that is not divisible by m (since $\lambda^m = 1$). On the other hand, if j is a multiple of m, we see that

$$1 + \lambda^j + \lambda^{2j} + \cdots + \lambda^{(m-1)j} = m$$

So we get that

$$\frac{1}{m} \sum_{j=0}^{m-1} B(\lambda^j) = b_m + b_{2m} + b_{3m} + \cdots$$

so evaluating the generating function at this sum of m-th roots of unity allows us to filter out the sum of coefficients indexed by the multiples of m.

For odd m, we have a further useful factorization:

$$1 + t^m = (1 + t)(1 + \lambda t)(1 + \lambda^2 t) \cdots (1 + \lambda^{m-1} t)$$

since both polynomials are monic of degree m and have roots $-1/\lambda^i$ for $i = 0, 1, 2, \ldots, m - 1$.

Problems 3. 1. Can the set $\mathbb{N}$ of all positive integers be partitioned into morethan one, but still a finite number of arithmetic progressions with no two having the same common differences?

2. (IMO 1995) Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \ldots, 2p\}$ such that

   (a) A has exactly p elements, and

   (b) the sum of all the elements in A is divisible by p.

3. An $a \times b$ rectangle can be tiled by a number of $p \times 1$ and $1 \times q$ types of rectangles, where a, b, p, q are fixed positive integers. When we tile with these rectangles the orientation is fixed, so a $k \times 1$ and a $1 \times k$ rectangle are different for this purpose. Prove that a is divisible by p or b is divisible by q.

6 Further Problems

1. Let F_n be the n-th Fibonacci number. Show that
$$\sum_{n\geq 1} \frac{F_n}{2^n} = 2$$

2. Prove that for any positive integer n,
$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

3. Show that
$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

4. Prove that for any positive integer n,
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

5. (Putnam 1992, B2) For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that
$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j}.$$

6. Let $a_0 = 1$ and $a_1 = 1$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Find a formula for a_n in terms of n.

7. Let n be a positive integer. Find the number a_n of polynomials $P(x)$ with coefficients in $\{0,1,2,3\}$ such that $P(2) = n$.

8. Consider a 1-by-n chessboard, suppose that we colour each square of the chessboard with one of the two colours red and blue. Let h_n be the number of colourings in which no two squares that are coloured red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n.

9. Find a recursion for the number of ways to place flags on an n foot pole, where we have red flags that are 2 feet high, blue flags that are 1 foot high and yellow flags that are 1 foot high. The heights of the flags must add up to n. Solve the recursion.

10. Find a recurrence for the sequence (u_n) such that u_n is the number of pairs (a,b) of nonnegative numbers such that
$$2a + 5b = n$$

11. How many positive numbers are there less than $10,000,000$ whose digits sum to 10?

12. Let
$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!}.$$
Show that $a_0, a_1, a_2, \ldots$ is the unique sequence satisfying $a_0 = 1$ and
$$\sum_{k=0}^{n} a)ka_{n-k} = 1$$

13. (HMMT 2007) Let S denote the set of triples (i, j, k) of positive integers where $i+j+k = 17$. Compute
$$\sum_{(i,j,k)\in S} ijk.$$

14. Let S_n be the number of triples (a, b, c) of nonnegative integers such that $a + 2b + 3c = n$. Compute the sum
$$\sum_{n=0}^{\infty} \frac{S_n}{3^n}$$

15. (Putnam 2003, A6) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1, s_2 \in S$, $s_1 \neq s_2$ and $s_1 + s_2 = n$. Let A be the set of nonnegative integers that have an odd number of ones in their binary representation, and let B be the set of nonnegative integers that have an even number of ones in their binary representation.Show that $r_A(n) = r_B(n)$ for all n.

16. (IMO2008, problem 5) Let n and k be positive integers with $k \geq n$ and $k-n$ an even number. Let $2n$ lamps labelled $1, 2, \ldots, 2n$ be given, each of which can be either on or off. Initially all lamps are off. we consider sequences of *steps*: at each step one of hte lamps is switched (from on to off or from off to on).

    Let N be the number of such sequences consisting of k steps and resulting in the state where the lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

    Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

    Determine the ratio M/N.

17. (Leningrad Mathematical Olympiad, 1991) A finite sequence $a_1, a_2, \ldots, a_n$ is called *p-balanced* if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, \ldots, p$. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal to zero.

18. Let h_n denote the number of regions into which a convex polygonal region with $n+2$ sides is divided by its diagonals, assuming that no three diagonals have a common point. Define $h_0 = 0$. Show that

$$h_n = h_{n-1} + \binom{n+1}{3} + n, \quad (n \geq 1).$$

Then determine the generating function for h_n and obtain a formula for h_n.

19. A **partition** of a set S is a collection of non-empty subsets $A_i \subseteq S$, $1 \leq i \leq k$ (the **parts** of the partition), such that $\bigcup_{i=1}^{k} A_i = S$ and for every $i \neq j$, $A_i \cap A_j = \emptyset$. For example, a partitiion of $\{1, 2, 3, 4, 5\}$ is $\{\{1, 3\}, \{4\}, \{2, 5\}\}$.

    Suppose the integers $1, 2, 3, \ldots, n$ are arranged on a circle, in order around the circle. A partition of $\{1, 2, 3, \ldots, n\}$ is a **non-crossing partition** if it satisfies the additioinal property: if w and v are in some part A_i and x and y are in a different part A_j, then the line joining w and v does not cross the line joining x and y. The partition in the example is not a non-crossing partition: the line $1-3$ crosses the line $2-5$.

    Find the number of non-crossing partitions of $\{1, 2, 3, \ldots, n\}$.

20. Consider a set of $2n$ people sitting around a table. In how many ways can we arrange for each person to shake hands with another person at the table such that no two handshakes cross?