Preparation for Saudi Arabia Team 2021

June Session: Junior Balkan Mathematics Olympiad

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Lesson 3

Game Strategies

Problems:

1. On an $n \times m$ board, a coin is placed in the upper-right corner. Players take turns moving up the coin up to 3 spaces either down or to the left (but not both) and whoever reaches the lower-left corner of the board wins. Player A has the first move. Determine for which (n, m) does player A have a winning strategy and for which (n, m) does player B have a winning strategy.

2. There are n coins on a pile and two players A and B take turns taking either 1, 2 or 6 coins from the pile. The winner is the person who takes the last coin. Player A plays first. Determine for which n does player A have a winning strategy and for which n does player B have a winning strategy.

3. There are n coins in one pile and two players A and B take turns with the following game. In each move a player whose turn it is may either take a coin from a pile and discard it, or divide a pile into (at least 2) equal piles. Player A starts first. When a player takes the last coin from a pile it gone. The winner is the player who takes the last coin off the table. For which n does player A have a winning strategy and for which n does player B have a winning strategy.

4. Two players A and B place take turns to place symbols X and O, respectively, on an empty square of an infinite two-dimensional grid. Player A begins first. The first player to place n consecutive symbols horizontally or vertically wins (diagonals *do not* count). For which values of n will A have a winning strategy.

5. Alice and Bob play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Alice going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move. Prove that Bob can always win the game, no matter how Alice plays.

6. Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

7. A rook dipped in red paint is initially placed at a corner square on a m x n rectangular board. Players A and B take turns moving the same rook to a different square in the same row or same column, with player A having the first turn. As it has been dipped in red paint, the rook colours ever square it passes over red (This includes both the square it lands on as well as the squares in between). A player loses if he is forced to move the rook over a square which has already been coloured red. For each value of m and n, which player has the winning strategy?

8. A set of n points in space is given, no three of which are collinear and no four of which are coplanar (on a single plane), and each pair of points is connected by a line segment. Initially, all the line segments are colorless. A positive integer b is given and Alice and Bob play the following game. In each turn Alice colors one segment red and then Bob colors up to b segments blue. This is repeated until there are no more colorless segments left. If Alice colors a red triangle,

Alice wins. If there are no more colorless segments and Alice hasn't succeeded in coloring a red triangle, Bob wins. Neither player is allowed to color over an already colored line segment.

(a) Prove that if $b < \sqrt{2n-2} - \frac{3}{2}$, then Alice has a winning strategy.

(b) Prove that if $b \geq 2\sqrt{n}$, then Bob has a winning strategy.