Invariants

Instructor: Dušan Djukić

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1. There are 100 pirates sitting at a round table, with the captain holding all n gold coins. At each step, if some pirates hold more than one coin, one of them will give a coin to each of his two neighbors.
   (a) If $n \geqslant 100$, prove that the pirates will never stop.
   (b) If $n < 100$, prove that they must stop at some moment.

2. The numbers $1, 2, \ldots, n$ are arranged on a circle in some order. Whenever b is the right neighbor of a and $b \leqslant a - 2$, we can switch their positions. Prove that after at most $\binom{n}{3}$ switches no more moves will be possible.

3. On a rectangular board $m \times n$ we play a game with markers having one side black and the other side white. Initially, on each of the mn squares there is a marker with the white side up, except for one marker at a corner with the black side up. A move consists of picking a black marker and turning all adjacent markers on the other side. For which m and n can we remove all markers?

4. There are $2n - 1$ light bulbs in a line. Initially the one in the middle is on and the others are off. At each step we can choose two nonadjacent bulbs that are off, under the condition that the bulbs between them must all be on, and change the state of all of them (thus, e.g. the configuration $\bullet \circ \circ \circ \bullet$ changes to $\circ \bullet \bullet \bullet \circ$). At most how many steps can we perform?

5. We have n weights with masses less than 1, arranged in a sequence. A *block* is a set of several consecutive weights. Prove that, for every k ($1 \leqslant k \leqslant n$), it is possible to partition the weights into k (disjoint) blocks so that the total masses of any two blocks differ by at most 1.