Combinatorics - Grids and chessboards
Canada IMO Camp, Winter 2020

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1 Introduction

The following problems involve grids or chessboards.

2 Important techniques

Theorem 2.1. *Given a large enough $n \times n$ grid, every colouring of the unit-squares using k colours will contain a rectangle (with grid corners) where all corners are the same colour.*

This proof is called a **product colouring argument**. It will give us a horrible bound of $n \geq k^{k+1} + 1$. [1]

Proof. In a given column with $k + 1$ rows, the pigeonhole principle tells us that two of the squares must be the same colour. There are k^{k+1} ways to colour a $(k + 1) \times 1$ grid. So with $k^{k+1} + 1$ columns, by the pigeonhole principle we must have two identically coloured columns.

So in a $(k^{k+1} + 1) \times (k + 1)$ grid, every colouring using k colours will have two identically coloured columns with $k + 1$ rows. So both columns will have a pair of matching colours in identical rows. That is our desired rectangle. □

By repeating this again, we can get that for a large enough n, every $n \times n \times n$ box will contain a rectangular box where all corners are the same colour.

[1] You can read about the rectangle-free 4 colouring of a 17×17 grid here: `https://mathlesstraveled.com/2012/02/09/17x17-4-coloring-with-no-monochromatic-rectangles/`

Corollary 2.2 (Gallai)**.** *For a fixed number of colours k, there is an n (a function of k) such that every colouring of an $n \times n$ grid using k colours contains a square all of whose corners are the same.*

The bounds for the previous theorem are very bad.

Example 2.3. By introducing tilings or grid colourings into problems we can often find invariants in the problem. For example, the checkerboard tiling can be useful, or more generally the colouring $\chi(i, j) = (i + j) \mod n$ for some relevant n (like 4).

This shows that the five unique tetris pieces cannot form a rectangle with area 20, since such a rectangle would contain an equal amount of red and black squares. However, all tetris pieces contain an equal amount of red and black squares *except* the T piece.

By introducing the checkerboard colouring, combined with the adjacent-edge graph (every unit square is a node, place an edge on unit squares that share an edge) we see that this is a bipartite graph whose bipartition is given by the checkerboard colouring. You can then use facts about bipartite graphs, such as they have no odd cycles.

The colouring should be chosen to help you identify a mathematical invariant in the problem. This can be useful when the question describes a procedure or game or movement of a piece on the board.

Example 2.4. In problems that involve conditions on grids with labels like "every column and every row has some property", it can be helpful to consider the bipartite graph associated to the grid as follows: Let the vertices be $V = R \cup C$ where R is the collection of rows, C is the collection of columns. For the edges, we have some choice, but it's common to place an edge between a row r_i and a column c_j if the label $a_{i,j}$ has some relevant property.

Doing this allows you to use facts about bipartite graphs (no odd cycles), and potentially use Hall's Marriage theorem.

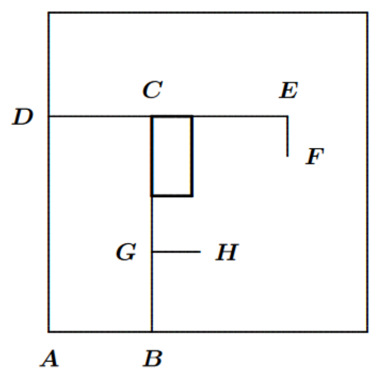
3 Worked Problem

Theorem 3.1 (Vietnam 2007 Shortlist)**.** *A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that one of the rectangles has no point on the boundary of the square.*

This solution appears in Crux, Feb 2012, solved by Oliver Geupe.

Proof. Suppose we have a counterexample with a minimal n.

Consider the rectangle $ABCD$ that contains the corner A of the unit square. By assumption, one of BC or CD must have an extension beyond C. WLOG, assume CD extends until EF, which is parallel to BC.



Now the edge BC cannot be shared with another rectangle R, otherwise it would violated the minimality of the counterexample (by combining R and $ABCD$). So there is a point G on the line BC such that GH is the edge of a rectangle parallel to CD.

Thus there must be a rectangle R containing the point C that is contained in the rectangle with sides CE and CG. This R is completely in the interior of the unit square, a contradiction. □

4 Problems

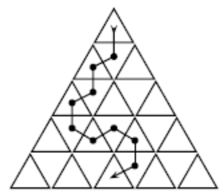
4.1 Warm-up Problems

These questions are below the level of the IMO, but are meant to get you thinking about grids.

1. [India 2006] Some 46 squares are randomly chosen from a 9×9 chess board and colored in red. Show that there exists a 2×2 block of 4 squares of which at least three are colored in red.

2. [CMO 2005] Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the

bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



3. [USAMO 1998] A 98×98 chess board has the squares colored alternately black and white in the usual way. A move consists of selecting a rectangular subset of the squares (with boundary parallel to the sides of the board) and changing their color. What is the smallest number of moves required to make all the squares black?

   `https://mks.mff.cuni.cz/kalva/usa/usoln/usol984.html`

4. [Czech and Slovak Olympiad 2007] A stone is placed in a square of a chessboard with n rows and n columns. We can **alternately** undertake two operations:

   (a) move the stone to a square that shares a common side with the square in which it stands;

   (b) move it to a square sharing only one common vertex with the square in which it stands.

   In addition, we are required that the first step must be (b). Find all integers n such that the stone can go through a certain path visiting every square exactly once. (This path need not start at a corner of the chessboard.)

4.2 USAMO Problems

1. [USAMO 2000] How many squares of a 1000×1000 chessboard can be chosen, so that we cannot find three chosen squares with two in the same row and two in the same column?

   `https://mks.mff.cuni.cz/kalva/usa/usoln/usol004.html`

2. [USAMO 1999] Certain squares of an $n \times n$ board are colored black and the rest white. Every white square shares a side with a black

square. Every pair of black squares can be joined by chain of black squares, so that consecutive members of the chain share a side. Show that there are at least $\frac{n^2-2}{3}$ black squares.

`https://mks.mff.cuni.cz/kalva/usa/usoln/usol991.html`

4.3 Other Problems

1. Let $f(n)$ be the number of ways to colour the unit squares of a $2 \times n$ grid using black and white such that there are no black squares that are diagonally adjacent. Note $f(1) = 4$ and $f(2) = 9$. Find $f(n)$.

2. [Iran 2003] Assume that we have a $n \times n$ table and we fill it with the numbers $1, \ldots, n$ such that each number appears exactly n times. Prove that there exist a row or column such with at least $\sqrt{n}$ different numbers.

3. [Russian 2002] Can the cells of a 2002×2002 table be filled with the numbers from 1 to 2002^2 (one per cell) so that for any cell we can find three numbers a, b, c in the union of the same row and column with $a = bc$?

4. [Russian 2002] Eight rooks are placed on an 8×8 chessboard, so that there is just one rook in each row and column. Show that we can find four rooks, A, B, C, D, so that the distance between the centers of the squares containing A and B equals the distance between the centers of the squares containing C and D.

5. [China 1999] A $4 \times 4 \times 4$ cube is composed of 64 unit cubes. The faces of 16 unit cubes are to be coloured red. A colouring is called interesting if there is exactly 1 red unit cube in every $1 \times 1 \times 4$ rectangular box composed of 4 unit cubes. Determine the number of interesting colourings.

6. [ITAMO 2009] A natural number k is said to be n-squared if by colouring the squares of a $2n \times k$ chessboard, in any manner, with n different colours, we can find 4 separate unit squares of the same colour, the centers of which are vertices of a rectangle having sides parallel to the sides of the board. Determine, as a function of n, the smallest natural number k that is n-squared.

4.4 IMO Problems

1. [IMO Shortlist 1999] Let n be an even positive integer. We say that two different cells of a $n \times n$ board are neighboring if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that any cell (marked or not marked) has a marked neighboring cell.

2. [IMO Shortlist 2005] Consider a $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called adjacent if they have a common edge, and a path is a sequence of unit squares in which any two consecutive squares are adjacent. Two paths are called non-intersecting if they don't share any common squares.

   Each unit square of the rectangular board can be colored black or white. We speak of a coloring of the board if all its mn unit squares are colored.

   Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

   Prove that $N^2 \geq M \cdot 2^{mn}$.

3. [IMO 2010] Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

   Type 1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

   Type 2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

   Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

4. [IMO Shortlist 1998] A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each nonintegral number x in the array can be changed into either $\lceil x \rceil$ or $\lfloor x \rfloor$ so that the row-sums and column-sums remain unchanged. (Note that $\lceil x \rceil$ is the least integer greater than or equal to x, while $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

5. [IMO Shortlist 2009] For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of a chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

6. [IMO Shortlist 2009] On a 999×999 board a limp rook can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A non-intersecting route of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called cyclic, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

   How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?