Pre-Camp Problem Set Solutions

1 Algebra

1. Find all real numbers x, y and z which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.

2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive integers such that $a_1 < a_2 < \ldots < a_n$, $b_1 > b_2 > \ldots > b_n$, and $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$ is a permutation of $(1, 2, \ldots, 2n)$. Prove that

$$|a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n|$$

is a perfect square.

3. Prove that if a, b, c, d are non-negative real numbers, no two of which are zero, then

$$\frac{ab}{c^2+d^2} + \frac{ac}{b^2+d^2} + \frac{ad}{b^2+c^2} + \frac{bc}{a^2+d^2} + \frac{bd}{a^2+c^2} + \frac{cd}{a^2+b^2} \geq 3.$$

When does equality hold?

4. Prove there exists no function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all x.

2 Combinatorics

1. 100 queens are placed on a 100×100 chessboard so that no two attack each other. Prove that each of the four 50×50 corners of the board contains at least one queen.

2. Consider a planar region of area 1 which is the union of circular disks. Prove that from these disks we can select some that are mutually disjoint and have total area of at least $\frac{1}{9}$.

3. On each of 12 points around a circle we place a disk with one white side and one black side. We may perform the following move: select a black disk, and reverse its two neighbors. Find all initial configurations from which some sequence of such moves leads to a position where all disks but one are white.

4. Fix a positive integer n. Let $S_1, S_2, \ldots, S_k$ be subsets of $\{1, 2, \ldots, n\}$ such that no subset completely contains another. Prove that

$$\sum_{i=1}^{k} \frac{1}{\binom{n}{|S_i|}} \leq 1.$$

3 Geometry

1. Let A, B, C and D be points on a line in that order, and let P be a point not on the line such that $AB = BP$ and $PC = CD$. If the circumcircles of ACP and BDP intersect at P and Q, prove that Q is equidistant from A and D.

2. A trapezoid $ABCD$ with $AB||CD$ and $AD < CD$ in inscribed in a circle c. Let DP be a chord parallel to AC. The tangent to c at D meets the line AB at E, and the lines PB and DC meet at Q. Prove that $EQ = AC$.

3. Five points are given on a circle. A perpendicular is drawn through the centroid of the triangle formed by any three of them to the chord connecting the remaining two. Such a perpendicular is drawn for each triplet of points. Prove that the ten lines obtained in this way have a common point.

4. An acute-angled triangle ABC is inscribed in a circle ω. A point P is chosen inside the triangle. Line AP intersects ω at the point A_1. Line BP intersects ω at the point B_1. A line ℓ is drawn through P and intersects BC and AC at the points A_2 and B_2. Prove that the circumcircles of triangles A_1A_2C and B_1B_2C intersect again on line ℓ.

4 Number Theory

1. Natural numbers a and b are such that $\frac{a+1}{b} + \frac{b+1}{a}$ is an integer. If d is the greatest common divisor of a and b, prove that $d^2 \leq a + b$.

2. For a positive integer $n \geq 2$, consider the $n - 1$ fractions

$$\frac{2}{1}, \frac{3}{2}, \ldots, \frac{n}{n-1}$$

The product of these fractions equals n, but if you reciprocate (i.e. turn upside down) some of the fractions, the product will change. Can you make the product equal 1? Find all values of n for which this is possible and prove that you found them all.

3. Let S_0 be a finite set of positive integers. We define sets $S_1, S_2, \ldots$ of positive integers as follows: Integer a is in S_{n+1} if and only if exactly one of $a - 1$ or a is in S_n. Show that there exists infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

4. Prove that infinitely many primes divide at least one number of the form $1! + 2! + \ldots + n!$.

5 Algebra Solutions

1. If we add the three given equations, we get

$$\begin{aligned} & x^2 - 4y + 7 + y^2 - 6z + 14 + z^2 - 2x - 7 = 0 \\ \Longrightarrow \quad & (x-1)^2 + (y-2)^2 + (z-3)^2 = 0 \end{aligned}$$

   Therefore, we can only have $x = 1, y = 2, z = 3$. Checking, we see these values do indeed satisfy the given equations. *(England 2013)*

   Comment: You *have* to check your answer for questions like this (even if you do not show much work). We first showed that if (x, y, z) satisfy the three given equations, then they must be $(1, 2, 3)$. It could still be that there are no (x, y, z) which satisfy the given equations. For example, think about what happens if we replace the first equation with $x^2 - 4y + 6 = 0$ and the second equation with $y^2 - 6z + 15 = 0$.

2. Suppose $a_i > b_i$ for some i. Then $a_i > a_1, a_2, \ldots, a_{i-1}, b_i, b_{i+1}, \ldots, b_n$. Therefore, a_i is greater than n different terms from $\{1, 2, \ldots, 2n\}$, and so must be at least $n + 1$. Similarly if $b_i > a_i$, then $b_i > a_1, a_2, \ldots, a_i, b_{i+1}, b_{i+2}, \ldots, b_n$, and so must also at least $n + 1$.

   Therefore, $\{\max(a_1, b_1), \max(a_2, b_2), \ldots, \max(a_n, b_n)\}$ is a subset of $\{n+1, n+2, \ldots, 2n\}$. Since both sets are of size n, it follows that

$$\{\max(a_1, b_1), \max(a_2, b_2), \ldots, \max(a_n, b_n)\} = \{n+1, n+2, \ldots, 2n\}.$$

   This leaves $\{\min(a_1, b_1), \min(a_2, b_2), \ldots, \min(a_b, b_n)\}$ on the one side and $\{1, 2, \ldots, n\}$ on the other side, which implies

$$\{\min(a_1, b_1), \min(a_2, b_2), \ldots, \min(a_n, b_n)\} = \{1, 2, \ldots, n\}.$$

   Finally,

$$\begin{aligned} \sum_{i=1}^{n} |a_i - b_i| &= \sum_{i=1}^{n} \Big(\max(a_i, b_i) - \min(a_i, b_i) \Big) \\ &= \sum_{i=1}^{n} \max(a_i, b_i) - \sum_{i=1}^{n} \min(a_i, b_i) \\ &= (n + 1 + n + 2 + \ldots + 2n) - (1 + 2 + \ldots + n) \\ &= n^2. \end{aligned}$$

   (Mathlinks)

3. Let $u = a^2b^2 + c^2d^2, v = a^2c^2 + b^2d^2, w = a^2d^2 + b^2c^2$. Combining opposite terms and applying the AM-GM inequality[1], we have

$$\begin{aligned} \frac{ab}{c^2 + d^2} + \frac{cd}{a^2 + b^2} &= \frac{a^3b + ab^3 + c^3d + cd^3}{a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2} \\ &\geq \frac{2a^2b^2 + 2c^2d^2}{a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2} \\ &= \frac{2u}{v + w}. \end{aligned}$$

[1] For non-negative real numbers $x_1, x_2, \ldots, x_n$, the AM-GM inequality states that $\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$.

Applying similar reasoning to the other pairs of terms, we see that:

$$\frac{ab}{c^2+d^2}+\frac{ac}{b^2+d^2}+\frac{ad}{b^2+c^2}+\frac{bc}{a^2+d^2}+\frac{bd}{a^2+c^2}+\frac{cd}{a^2+b^2}\geq\frac{2u}{v+w}+\frac{2v}{w+u}+\frac{2w}{u+v}.$$

Finally, Nesbitt's inequality states that $\frac{u}{v+w}+\frac{v}{w+u}+\frac{w}{u+v}\geq\frac{3}{2}$, and so we are done. (If you have not seen Nesbitt's inequality before, you can prove it by multiplying both sides by $(u+v)(v+w)(w+u)$ and then applying AM-GM again.)

For equality to hold in the first step, we need $a^3b=ab^3$, or equivalently one of the following: $a=b, a=0, b=0$. The same holds for every pair of variables, so it must be that all variables are equal, or one of the variables is 0 and the other three are equal. It is easy to check that equality does indeed hold in both cases.

4. Let $f^{(n)}(x)$ denote the result of applying f to x exactly n times. Also, let $o(x)$ denote the smallest n such that $f^{(n)}(x)=x$(possibly infinity), and let O_n denote the set of x such that $o(x)=n$.

   Lemma: If O_n is non-empty, then it has at least n elements.

   Proof: For arbitrary x, let S_x denote the set $\{x, f(x), f^{(2)}(x), \ldots\}$. Pick $y\in S_x$, so that $y=f^{(i)}(x)$ for some i. If $o(x)=m$ for some finite m, then $f^{(m-i)}(y)=f^{(m)}(x)=x$, so $x\in S_y$. Furthermore, $f^{(m)}(y)=f^{(m+i)}(x)=f^{(i)}(f^{(m)}(x))=f^{(i)}(x)=y$, so $o(y)\leq o(x)$. Since $x\in S_y$, it follows similarly that $o(y)\leq o(x)$, and hence $o(y)$ in fact equals m. Therefore, S_x is a subset of O_m. Additionally, if $f^{(i)}(x)=f^{(j)}(x)$ for $0\leq i<j<n$, then $o(f^{(i)}(x))\leq j-i$, which we just showed is impossible. It follows that $|S_x|=m$. Taking $x\in O_n$ completes the proof of the lemma.

   Returning to the problem at hand, any f satisfying the given condition would have $f^{(2)}(x)-x=x^2-x-2=(x-2)(x+1)$. Additionally, $f^{(4)}(x)-x=x^4-4x^2-x+2=(x-2)(x+1)\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)$. Therefore, $O_4=\{\frac{-1-\sqrt{5}}{2},\frac{-1+\sqrt{5}}{2}\}$, which contradicts the lemma. *(Vietnam 1990)*

   Comment: The lemma can be strengthened slightly: If O_n is finite, then $|O_n|$ is divisible by n. This is a powerful observation in other contexts as well, especially group theory.

6 Combinatorics Solutions

1. Suppose the top-left quadrant has no queens. There are 100 queens that must fit in 100 rows, so each row must contain a queen. Since the top-left quadrant is empty, this means there must be at least 50 queens in the top-right quadrant. Similarly, by looking at columns, we see there must be at least 50 queens in the bottom-left quadrant.

   However, any square in the bottom-left or top-right quadrant lies on one of 99 different diagonals going from bottom-left to top-right. Therefore, there must be two queens on the same diagonal, which is impossible. *(Tournament of the Towns 2008)*

2. We prove by induction on n that if n disks cover an area of size k, we can choose a subset of disjoint disks that cover an area of size at least $\frac{k}{9}$.

   If $n=1$, the claim is trivial. Now let's prove the result for $n=m$, assuming the result for $n<m$. Let d be the disk with maximal radius r, let S denote the set of disks intersecting d,

and let T denote the remaining set of disks. Consider a disk $s \in S$. By assumption, it has radius at most r, and so the centers of c and s differ by distance at most $2r$. It follows that S is contained entirely inside the disk centered at c with radius $3r$. This means that the total area covered by all disks in T is at least $k - 9\pi r^2$.

By the inductive hypothesis, there exists a subset of disjoint disks in T that cover an area of size at least $\frac{k}{9} - \pi r^2$. By assumptions, these disks are all also disjoint from c, so we can add in c to get a total area of at least $\frac{k}{9}$, as required. *(Putnam 1998)*

3. Every move, we change the colour of two disks. Therefore, the parity of the number of black disks never changes, so if there are an even number of black disks to start with, the task is impossible.

   Otherwise, we claim the task is possible. We first describe how to transform the circle to have 11 black disks. Since the total number of black disks is odd, there must exist some contiguous sequence of black disks of odd length: $D_1, D_2, \ldots, D_{2n+1}$ ($n \le 4$). Apply the move to black disks $D_1, D_3, \ldots, D_{2n+1}$. When everything is done, $D_1, D_2, \ldots, D_{2n+1}$ will all still be black, and so will the disks on either side. Thus, we can keep increasing the number of black disks until all but one disk is black.

   This leaves one contiguous sequence of black disks and it is of odd length: $D_1, D_2, \ldots, D_{2n+1}$. Apply the move to black disks $D_2, D_4, \ldots, D_{2n}$. When everything is done, $D_2, D_3, \ldots, D_{2n}$ will all still be black, but D_1 and D_{2n+1} will be white. Thus, we can keep decreasing the number of black disks until only one disk is black, as required. *(Japan 1998)*

4. Pick a permutation π of $\{1, 2, \ldots, n\}$ uniformly at random. Let us say π *matches* S_i if the first $|S_i|$ elements of π are the elements of S_i in some order. Let P_i denote the probability that π matches S_i, and let P denote the probability that π matches at least one of $\{S_1, S_2, \ldots, S_k\}$.

   Note that π cannot simultaneously match both S_i and S_j for distinct i, j, because this would imply one of the two sets contained the others. Therefore, $P = P_1 + P_2 + \ldots + P_k$. Additionally, $P_i = \frac{1}{\binom{n}{|S_i|}}$ since the first $|S_i|$ elements of π are equally likely to be any of the $|S_i|$-element subsets of $\{1, 2, \ldots, 2n\}$.

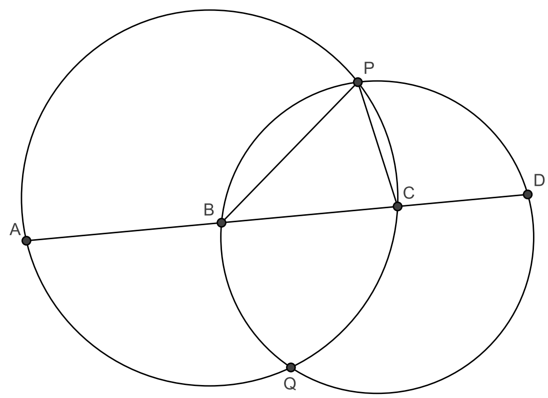
   Therefore, $\sum_{i=1}^{k} \frac{1}{\binom{n}{|S_i|}} = \sum_{i=1}^{k} P_i = P \le 1$.

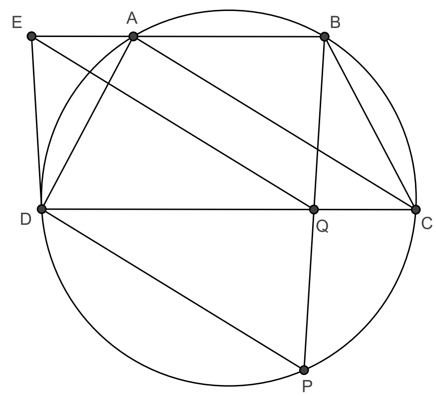
7 Geometry Solutions

1. Let O be the circumcircle of $\triangle APD$. Then $\angle POA = 2\angle PDA = \angle PCA$, so O lies on the circumcircle of $\triangle ACP$. Similarly, it lies on the circumcircle of $\triangle BDP$ so $O = Q$ and the problem is solved.

   Comment: A slicker solution is to extend PQ to hit the circumcircle of $\triangle APD$ at R, and angle chase from there. However, the solution presented here is better motivated.

2. Since AB and CD are parallel, we have $\angle DCA = \angle BAC$, which implies $DA = BC$. Additionally, we have $\angle EDA = \angle DCA = \angle PDC = \angle PBC = \angle QBC$ and $\angle DAE = \angle BCD = \angle BCQ$. Therefore $\triangle EDA$ is congruent to $\triangle QBC$ by angle-side-angle. It follows that $EA = QC$, and hence $EACQ$ is a parallelogram. This gives $EQ = AC$, as required. *(Nordic 2002)*





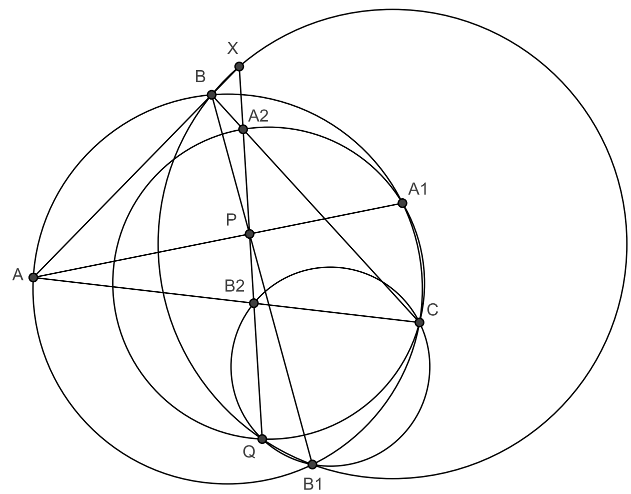
3. Let the 5 points on the circle be A, B, C, D, E. Set up vectors with the center of the circle being at 0. We claim the lines meet at $X = \frac{A+B+C+D+E}{3}$.

   Let C_{ABC} denote the centroid of $\triangle ABC$. We know C_{ABC} lies at vector $\frac{A+B+C}{3}$, so the line through C_{ABC} and X travels along the vector $\frac{D+E}{3}$. This is parallel to the line between the center of the circle and the midpoint of segment DE, and that line is orthogonal to DE because the center of a circle lies on the perpendicular bisector of any chord. It follows that $C_{ABC}X$ is also orthogonal to segment DE.

   The same argument can be applied for any triple of points, so the lines concur at X, as required.

   Comment: The main challenge with a problem like this is to figure out where X has to be in the first place. One approach is to start with just two lines. Let M be the midpoint of AB, and let C_{ABC} and C_{ABE} denote the centroids of triangles ABC and ABE. Let ℓ_C denote the line through C_{ABC} perpendicular to DE and let ℓ_E denote the line through C_{ABE} perpendicular to CD. If we perform a homothety about M with factor 3, then ℓ_C becomes the altitude from C to DE, and ℓ_E becomes the altitude from E to CD. Therefore, the point X we are looking for is mapped to the orthocenter of $\triangle CDE$ by a homothety with factor 3 about M. Now it is natural to introduce vectors since, if the vectors are centered at the circumcenter of $\triangle CDE$, then the orthocenter of $\triangle CDE$ is at precisely $C + D + E$. (Do you see why this is true? It is a consequence of the Euler line.)

4. First suppose ℓ is parallel to AB. Then $\angle A_2PA_1 = \angle BAA_1 = \angle BCA_1 = \angle A_2CA_1$, so



A_2PCA_1 is cyclic. Similarly, B_2PCB_1 is cyclic, so the two circles meet at P, which is on ℓ as required.

Otherwise, let X denote the intersection of AB and ℓ. We will assume the configuration shown in the diagram; the remaining configurations are similar[2].

Let Q be the point on ℓ so that (using directed lengths) $PQ \cdot PX = PB \cdot PB_1 = PA \cdot PA_1$. Then $XQBB_1$ and $XQAA_1$ are cyclic by Power of a Point. Therefore, $\angle B_2QB_1 = \angle XQB_1 = \angle XBB_1 = 180^\circ - \angle ABB_1 = 180^\circ - \angle ACB_1 = 180^\circ - \angle B_2CB_1$. It follows that Q lies on the circumcircle of $\triangle B_1B_2C$. A similar argument shows it also lies on the circumcircle of $\triangle A_1A_2C$, which solves the problem since Q is on ℓ. *(Mathlinks)*

8 Number Theory Solutions

1. Expanding, we have $\frac{a+1}{b} + \frac{b+1}{a} = \frac{a^2+b^2+a+b}{ab}$. We know that $d^2|ab$, so if the given quantity is an integer, then $d^2|a^2+b^2+a+b$. Since d^2 also divides a^2 and b^2, it follows that $d^2|a+b$. Therefore, $a+b$ is a positive integer that is a multiple of d^2, and hence $a+b \geq d^2$. *(Russia 1994)*

2. Suppose we reciprocate the fractions $\frac{a_1}{a_1-1}, \frac{a_2}{a_2-1}, \ldots, \frac{a_k}{a_k-1}$. Then the product becomes:

$$\begin{aligned} & \frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{n}{n-1} \cdot \left(\frac{a_1-1}{a_1}\right)^2 \cdot \left(\frac{a_2-1}{a_2}\right)^2 \cdot \ldots \cdot \left(\frac{a_k-1}{a_k}\right)^2 \\ = \quad & n \cdot \left(\frac{(a_1-1)(a_2-1)\ldots(a_k-1)}{a_1a_2\ldots a_k}\right)^2. \end{aligned}$$

If this equals 1, then $n = \left(\frac{a_1a_2\ldots a_k}{(a_1-1)(a_2-1)\ldots(a_k-1)}\right)^2$, and in particular n is a perfect square.

[2]Saying other configurations are "similar" on a contest is not usually a good idea. You should either explicitly go through every possible configuration or use something like directed angles. You can find more on directed angles here: https://sites.google.com/site/imocanada/2011-winter-camp.

Conversely, suppose $n = m^2$. If we reciprocate $\frac{2}{1}, \frac{3}{2}, \dots, \frac{m}{m-1}$, then we get:

$$\begin{aligned} & \left(\frac{m^2}{m^2-1} \cdot \frac{m^2-1}{m^2-2} \cdot \dots \cdot \frac{m+1}{m}\right) \cdot \left(\frac{m-1}{m} \cdot \frac{m-2}{m-1} \cdot \dots \cdot \frac{1}{2}\right) \\ = \; & m \cdot \frac{1}{m} \\ = \; & 1. \end{aligned}$$

Therefore, the product can be made equal to 1 if and only if n is a perfect square. *(Bay Area 2013)*

3. For each pair of integers n, i, let $s_{n,i}$ denote the number that is 1 if $i \in S_n$ and 0 otherwise. For $n \geq 0$, we are given that $s_{n+1,i} \equiv s_{n,i-1} + s_{n,i} \mod 2$.

   We are now going to work with polynomials mod 2. Specifically, given two polynomials Q and R with integer coefficients, we will write $Q \equiv R \mod 2$ if every coefficient of Q is congruent to the corresponding coefficient of R mod 2. You can add, subtract, and multiply polynomials mod 2, and they work just like integers[3].

   Now, for each n, define a polynomial $P_n(x) = \sum_i s_{n,i}x^i$. The given conditions can be rewritten to state that $P_{n+1}(x) \equiv P_n(x) \cdot (x+1) \mod 2$, and hence $P_n(x) \equiv P_0(x) \cdot (x+1)^n \mod 2$.

   Lemma: $(x+1)^{2^k} \equiv x^{2^k} + 1 \mod 2$ for all non-negative integers k.

   Proof: Using the binomial theorem, it suffices to show that $\binom{2^k}{i}$ is even for all i satisfying $0 < i < 2^k$. Recall that the largest power of 2 dividing $n!$ is exactly $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \dots$. Therefore, the largest power of 2 dividing $\binom{2^k}{i}$ is precisely

$$\begin{aligned} & \sum_{j=0}^{k} \left(\left\lfloor \frac{2^k}{2^j} \right\rfloor - \left\lfloor \frac{i}{2^j} \right\rfloor - \left\lfloor \frac{2^k - i}{2^j} \right\rfloor \right) \\ = \; & 1 + \sum_{j=0}^{k-1} \left(\left\lfloor \frac{2^k}{2^j} \right\rfloor - \left\lfloor \frac{i}{2^j} \right\rfloor - \left\lfloor \frac{2^k - i}{2^j} \right\rfloor \right) \\ \geq \; & 1 + \sum_{j=0}^{k-1} \left(\left\lfloor \frac{2^k}{2^j} \right\rfloor - \left\lfloor \frac{i + (2^k - i)}{2^j} \right\rfloor \right) \\ = \; & 1. \end{aligned}$$

   This shows $\binom{2^k}{i}$ is indeed even, which completes the proof of the lemma.

   Finally, let R denote the difference between the largest and smallest elements of S_0, and choose n to be a power of 2 larger than R. Then $P_n(x) \equiv x^n P(x) + P(x) \mod 2$ by our lemma. It follows that $S_n(x)$ consists of the integers that are in exactly one of S_0 and $\{n + a : a \in S_0\}$. Since $n > R$, these two sets are disjoint, and hence $S_n(x) = S_0 \cup \{n + a : a \in S_0\}$. *(Putnam 2000)*

[3] What does this really mean? The key property you want is this: if $Q \equiv Q' \mod 2$ and $R \equiv R' \mod 2$, then $Q + R \equiv Q' + R' \mod 2$, $Q - R \equiv Q' - R' \mod 2$, and $Q \cdot R \equiv Q' \cdot R' \mod 2$. Abstract algebra is very interested in these kinds of questions.

4. Suppose, by way of contradiction, that there are only k such primes $p_1, p_2, \ldots, p_k$.

   Let $S_n = 1!+2!+\ldots+n!$. For each n, we can write $S_n = \prod_{i=1}^{k} p_i^{e_i}$. Let $f(n)$ denote the integer i such that $p_i^{e_i}$ is maximum. For any N, the pigeonhole principle guarantees there must exist $n, m \in \{N-k, N-k+1, \ldots, N\}$ such that $n < m$ and $f(n) = f(m)$. Let $p = p_{f(n)} = p_{f(m)}$ and let e be the largest integer such that $p^e | \gcd(S_n, S_m)$. The definition of p guarantees that $p^e \geq \sqrt[k]{n!}$.

   On the other hand, $\gcd(S_n, S_m) | m! - n!$, which equals $n! \cdot \Big((n+1)(n+2)\ldots m - 1\Big)$. The largest exponent of p dividing $n!$ is

$$\begin{aligned} & \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \ldots \\ \leq & \ \frac{n}{p} + \frac{n}{p^2} + \ldots \\ \leq & \ n. \end{aligned}$$

   Therefore, $p^e \leq p^n \cdot \Big((n+1)(n+2)\ldots m - 1\Big) < p^n \cdot m^k$. Combining this with the previous observation gives us $p^n \cdot m^k > \sqrt[n]{n!}$, but we claim this is impossible for large n.

   Indeed, for large n, we have

$$\sqrt[2k]{n!} > \sqrt[2k]{\frac{n}{2}^{\frac{n}{2}}} = \frac{n}{2}^{\frac{n}{4k}} = \left(\sqrt[4k]{\frac{n}{2}} \right)^n > p^n, \text{and}$$

$$\sqrt[2k]{n!} > \sqrt[2k]{\frac{n}{2}^{\frac{n}{2}}} > \sqrt[2k]{\left(\frac{n+k}{2} \right)^{\frac{n}{4}}} \geq m^{\frac{n}{8k}} \geq m^k.$$

   Multiplying these together gives a contradiction.