CYCLES IN GRAPHS

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1. Introduction

Graphs encode structure in a variety of mathematical contexts. In olympiads, this inclusion is reversed, and the solution to many problems phrased as graph theory (or airports, highways, parties, etc.) is obtained by figuring out what structure the graph is describing. One of the most important structures is the cycle. We'll see how to exploit this simple notion by finding cycles and decomposing into cycles to solve problems.

2. Examples

We'll start with a classical result as a warm-up.

Lecture Problem 2.1 (Euler's Theorem)
A connected graph has an Euler cycle if and only if all vertices have even degree.

(An Euler cycle is a cycle that uses every edge exactly once.)

Proof: Induct on the number of edges. The base case $n = 3$ is easy enough.

Start on some vertex. Keep on walking along edges that haven't been walked before. This process can only end in a cycle. Remove this cycle and apply the induction hypothesis to find another cycle. Those two Euler cycles must coincide at some vertex by connectivity. Start at that vertex and walk along both Euler cycles in turn to find the big Euler tour. ■

Another warm-up. This is a very useful characterization of bipartite graphs, and is very much in the spirit of many problems in this handout.

Proposition 2.2
Prove that a graph is bipartite if and only if all of its cycles have even length.

Thanks to CMS and the Winter camp for inviting me back! You can reach me at `edgar.wang@yale.edu` if you have questions/need sources or hints.

Proof: One direction is very easy: If a graph is bipartite, then there can clearly be no odd cycles, as then we cannot split the vertices in that odd circle into two parts. Thus, bipartite graphs have all cycles of even length.

Now assume the graph has no odd cycles. We'll algorithmically split the vertices of the graph into two groups. Pick an arbitrary, call it v, vertex and colour it in white. Now, let's consider every single path out of this vertex at once! Colour every neighbour of v and colour it in black. Now, we iterate this by taking every black vertex as another starting point, and colour every one of its neighbours that isn't already coloured in white. This process continues, and we follow every path at once, until the whole graph is coloured.

To prove that this process is well-defined, we need to show that we'll never colour a vertex in both black and white. Assume for the sake of contradiction that there is a vertex that is coloured in both black and white. Then, we can find two paths from v to that vertex, one traversing an even number of edges (the path leading to a white colour) and another with an odd number of edges (the one leading to a black colour). However, traversing the white path and then the black path backwards gives a cycle of odd length starting and ending at v, contradiction!

Thus, the colouring is well-defined, and the graph is bipartite, where the two parts are the black and white vertices. ■

We see that there is some type of global/local principle going on, where we are rewarded for both considering all possible cycles, and for computing a cycle in particular. We'll see this played out in three elegant problems.

The first way to find cycles is numerically. We can often formulate an algebraic condition that guarantees the existence of a cycle.

The next problem is extremely instructive. It's a very common way to disguise an easy problem about bipartite graphs as a problem about grids. The solution shows one way to see the equivalence between those two objects.

Lecture Problem 2.3

Suppose $2n$ points of an $n \times n$ grid are marked. Prove that there exists a $k > 1$ and $2k$ distinct marked points $a_1, \ldots, a_{2k}$ such that, for all i, a_{2i-1} and a_{2i} are in the same row, while a_{2i} and a_{2i+1} are in the same column.

Proof: There is a very natural graph-theoretic interpretation of grids. Set up a bipartite graph as such: let the two parts be A and B, where the vertices in A correspond to the columns of the grid, and the vertices in B correspond to the rows. Then, draw an edge between $a \in A$ and $b \in B$ if and only if the square at the intersection of the column represented by a and the row represented by b is marked.

What about the sequence of points we want to find? The graph theoretic formulation lets us translate it into something more manageable: such a sequence would give us an alternating sequnce of row, column, row, column, etc. which is exactly a cycle on the bipartite graph!

We can now rephrase this problem graph-theoretically: Given a bipartite graph on $2n$ vertices with $2n$ edges, can we find a cycle? This problem is actually pretty weak: for any graph (not just bipartite!) with m vertices and at least m edges, we can find a cycle. The proof of this *very important* fact is left as an exercise in induction. ■

The second way we'll see to find a cycle is by tracing it out. There is often a clever way to force ourselves into finding a cycle.

Lecture Problem 2.4

Consider an $n \times n$ square grid, where n is an odd integer, where every square except the bottom-left corner is covered by a 1×2 domino. We can perform the following operation: If a domino's short side is adjacent to an empty square, we may slide it across the empty square. Prove that we can perform a sequence of moves after which the top-right corner will be empty.

Proof: Drawing many examples for small n leads to the key structure rather quickly. We'll define it formally.

Consider the following graph: Label the squares on the grid with coordinates $\{(a, b), 1 \leq a, b \leq n\}$, so that the bottom-left corner is $(1, 1)$ and the top right is (n, n). Note that our empty square is on the graph, so is its final destination. Draw a vertex on every square both of whose coordinates are *odd*. Given the initial tiling, draw an edge between two vertices if and only if one of the vertices is covered by a domino whose short side is adjacent to the other vertex.

Let's now walk along some path starting at any vertex. It can either end on the empty square, or result in a cycle.

We'll show that we'll never get a cycle. This is motivated primarily by drawing a bunch of examples. Assume for the sake of contradiction that there is some cycle. We can prove by induction that this cycle bounds inside it (i.e. completely contains) an odd number of squares.

However, any shape bounded completely by our cycle is necessarily tiled by dominoes! (Exercise: why?) Therefore, it must contain an even number of squares, contradiction!

Thus, every single vertex is connected to the empty square (i.e. the graph is simply connected). It's easy, by pushing dominoes along the tree connecting $(1, 1)$ and (n, n), then to move the empty square to the top-right corner. ■

It's often helpful to consider decomposition of a graph into many smaller cycles. In that case, we can focus on each cycle independently.

Lecture Problem 2.5 (USA TSTST 2018)

In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

Proof: Let G denote the graph.

The condition about exactly two roads in a and exactly two roads out suggest that we can find many cycles in this graph. However, the same condition makes it hard for us to directly act on the edges.

The trick is to build a graph on top of this graph! Consider the bipartite graph Γ with parts A, B, both corresponding to all the vertices in G. Draw an edge between $a \in A, b \in B$ if and only if there is an oriented edge from the vertex represented by a to the vertex represented by b. Note that every vertex has degree 2.

Let's see what information this encodes: Consider $a_1, b_1, a_2, b_2, \ldots, a_1$ such that $a_1 b_1 a_2 b_2 \ldots a_1$ is a cycle in Γ. What happens if we delete the edge corresponding to $a_1 b_1$ in G? We must keep the edge corresponding to $a_2 b_1$. Similarly, we must delete the edge $a_2 b_2$, and so on alternating delete/keep, until we loop all the way around the cycle. Conversely, if we keep $a_1 b_1$, we must delete $a_2 b_1$, and alternate so on.

Now we note that, since every vertex has degree 2, which is even, we can partition Γ into a disjoint collection of such cycles. In fact, we have exactly one cycle for each connected component.

Thus, every valid configuration on G leads to one of two possible states on each cycle of Γ. Conversely, it's not hard to check that every combination of either state on each cycle yields a valid configuration of G.

Thus, there are 2^k configurations of G, where k is the number of connected components of Γ. ■

3. Problems

The problems are *very* roughly arranged in order of difficulty. I tried to make it so that the problems here all use the ideas in the lectures to a certain extent, and that no "crazy ideas" or "very unorthodox" constructions are necessary, to make a point that looking for simple structure is often your best bet!

Problem 3.1. If a graph has $2k$ vertices of odd degree, then its edges can be partitioned into k trails. (A trail is a sequence of distinct adjacent edges, whose vertices may repeat.)

Problem 3.2 (Hall's Marriage Lemma)**.** Let G be a finite bipartite graph with bipartite sets X and Y. An X-perfect matching (also called: X-saturating matching) is a matching which covers every vertex in X.

For a subset W of X, let $N_G(W)$ denote the neighbourhood of W in G, i.e. the set of all vertices in Y adjacent to some element of W. There is an $X-$perfect matching if and only if for every subset W of X:

$$|W| \leq |N_G(W)|.$$

(The full proof of this result is actually quite involved, but I put it this early because it's a very important fact that appears all over the place!)

Problem 3.3 (De Brujin's Theorem)**.** Let n be a positive integer. Each vertex of a 2^n-gon is labeled 0 or 1. There are 2^n sequences obtained by starting at some vertex and reading the first n labels encountered clockwise. Show that there exists a labelling such that these 2^n sequences are all distinct.

Problem 3.4 (Putnam 2014)**.** The mn squares of a $m \times n$ rectangular grid are filled with rational numbers. Among the absolute values of these rational numbers, there are $m + n$ distinct primes. Prove that not every row is a multiple of the first row.

(Note: A row with entries $(a_1, a_2, \ldots, a_n)$ is a *multiple* of a row with entries $(b_1, b_2, \ldots, b_m)$ if and only if either $a_i/b_i = k$ or $a_i = b_i = 0$ for each i for some constant k.)

Problem 3.5. In an $m \times n$ rectangular grid, every unit edge is oriented such that the arrows on the boundary are oriented clockwise, and there are two edges pointing in and two edges pointing away from each vertex in the interior of the grid. Prove that there is some unit square such that its four edges are oriented clockwise.

Problem 3.6 (IMO 1998)**.** A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each nonintegral number x in the array can be changed into either $\lceil x \rceil$ or $\lfloor x \rfloor$ so that the row-sums and column-sums remain unchanged. (Note that $\lceil x \rceil$ is the least integer greater than or equal to x, while $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

Problem 3.7 (Bulgaria TST 2004)**.** On an $n \times n$ table real numbers are put in the unit squares such that no two rows are identically filled. Prove that one can remove a column of the table such that the new table has no two rows identically filled.

Problem 3.8. The edges of a complete graph on $2^n + 1$ vertices are colored using n colours. Prove that we can find a monochromatic cycle of odd length.

Problem 3.9. Two people are playing a very silly game: They each have in their hands a card with a positive integer that only the other player can see. The goal of the game is to have both players guess what card they have. They are only allowed to say one of two sentences: "I don't know what my card is" and "I know what my card is."

This game is obviously very silly. All of a sudden, an angel descends from the skies and says: "Behold! One of you has a card with N written on it, and the other has $N+1$ for some positive integer N" and leaves.

Is the game still as silly as it was before? Can they both win, now?

Problem 3.10 (Ukraine TST)**.** There are $2n$ students at a math contest. Each of them submits to the jury a probelm. At the end, the jury gives each student one of the problems submitted. Say that the contest is fair if there exists n students who receive their problems from the other n participants. Prove that the number of distributions of problems that result in a fair contest is a perfect square.

Problem 3.11 (Kasteleyn Weightings)**.** Given a planar bipartite graph, prove that one can assign each edge with a weight from $\{-1, 1\}$ such that the products of weights on each face with 2 (mod 4) edges is 1 and the product of weights on each face with 0 (mod 4) edges is -1.

Problem 3.12 (ISL 2006)**.** Let n be an even positive integer. Show that there is a permutation $(x_1, x_2, \dots, x_n)$ of $(1, 2, \dots, n)$ such that, for every $i \in \{1, 2, \dots, n\}$, the number x_{i+1} is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$.

Problem 3.13 (IMO 1986)**.** Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

Problem 3.14 (Follow-up to De Brujin)**.** How many different ways are there to label the 2^n−gon as described in the De Brujin's Theorem? Assume rotations count as distinct labellings.

Problem 3.15 (EGMO 2016)**.** Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are related to each other if they are in either the same row or in the same column. No cell is related to itself. Some cells are coloured blue, such that every cell is related to at lest two blue cells. Determine the minimum number of blue cells.

Problem 3.16 (David Arthur, via Matt Brennan)**.** An *arrowgram* is a finite rectangular grid with an arrow drawn on every square pointing in one of the eight compass directions such that:

(1) Every arrow points to an adjacent square (i.e. no arrow points off the edge of the grid), and

(2) No two arrows point to the same square.

Two arrowgrams A and B are said to be similar if they are on equally-sized grids, and for every square, the arrow in it points in the same direction or in opposite directions. For which integers N does there exist an arrowgram that is equivalent to exactly N other arrowgrams?

Problem 3.17 (CMO 2019)**.** A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.

Problem 3.18 (USA TST 2011)**.** In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

Problem 3.19 (RMM 2017)**.** Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Problem 3.20 (IMO 2020)**.** There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is coloured in one of n colours and there are four pebbles of each colour. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

(1) The total weights of both piles are the same.

(2) Each pile contains two pebbles of each colour.