42. Let x and y be real numbers such that

$$x^3 + y^3 + (x+y)^3 + 30xy = 2000.$$

Prove that $x + y = 10$.

$$2000 = x^3 + y^3 + (x+y)^3 + 30xy + 3xy(x+y) - 3xy(x+y)$$

$$2000 = (x+y)^3 + (x+y)^3 + 30xy - 3xy(x+y)$$

$$2000 = 2(x+y)^3 + 3xy(10 - x - y)$$

$$2(10^3 - (x+y)^3) - 3xy(10 - (x+y)) = 0$$

$$(10 - (x+y))(2x^2 + 2y^2 + xy - 20(x+y-10)) = 0$$

Case 1: $x + y = 10$

which is what we want

Case 2: $2x^2 + (y-20)x + (2y^2 - 20y + 200) = 0$

$$\Delta = (y-2)^2 - 4 \cdot 2(2y^2 - 20y + 200)$$

$$= -15y^2 - 200y - 1200$$

$$= -(15y^2 + 200y + 1200)$$

$$= -(5y^2 + 10(y+10)^2 + 200) < 0$$

$\Rightarrow$ no real solutions for x

$\square$

Another way:

$$(x+y)^3 - 1000 + x^3 + y^3 + (-10)^3 + 30xy = 0$$

$$(x+y-10)\left((x+y)^2 - 10(x+y) + 100\right) + (x+y-10)\left(x^2+y^2+10^2-10x-10y-xy\right)$$

$$(x+y-10)\left(2x^2+2y^2+xy-20(x+y-10)\right) = 0$$

Case 1: $x+y=10$

which is what we want

Case 2:

$$\begin{cases} \dfrac{x^2+y^2}{2} \geq -xy \quad \forall\ x,y \in \mathbb{R} \\ \dfrac{3}{2}x^2 + \dfrac{200}{3} \geq 20x \\ \dfrac{3}{2}y^2 + \dfrac{200}{3} \geq 20y \end{cases}$$

$$2x^2 + 2y^2 + \frac{400}{3} \geq -xy + 20x + 20y$$

$$20x^2 + 2y^2 + xy - 20(x+y-10) \geq 200 - \frac{400}{3} > 0$$

$$\Longrightarrow \Leftarrow !!$$

45. The real numbers a, b, c, d, e, and f satisfy the conditions

$$a+b+c+d+e+f=10$$

and

$$(a-1)^2+(b-1)^2+(c-1)^2+(d-1)^2+(e-1)^2+(f-1)^2=6.$$

Determine the greatest possible value of f.

let $u, v, w, x, y, z = (a-1), (b-1), \dots, (f-1)$

$$\begin{cases} u+v+w+x+y+z=4 \\ u^2+v^2+w^2+x^2+y^2+z^2=6 \end{cases}$$

We want to maximize z

$$(u+v+w+x+y)=4-z$$
$$(u^2+v^2+w^2+x^2+y^2)=6-z^2$$

$$(u^2+v^2+w^2+x^2+y^2)(1+1+1+1+1) \geq (u+v+w+x+y)^2$$

$$5(6-z^2) \geq (4-z)^2$$

$$30-5z^2 \geq 16-8z+z^2$$

$$0 \geq 6z^2-8z-14$$

$$0 \geq 3z^2-4z-7$$

$$0 \geq (3z-7)(z+1)$$

$$\Rightarrow \quad -1 \leq z \leq \frac{7}{3} \Rightarrow \max z \leq \frac{7}{3}$$

To give example $\frac{u}{1}=\frac{v}{1}=\frac{w}{1}=\frac{x}{1}=\frac{y}{1}$, $z=\frac{7}{3}$

$$\Rightarrow u=v=w=x=y=\frac{1}{3}, \quad z=\frac{7}{3} \Rightarrow \max f=\frac{10}{3}$$

7. Let x, y, z be real numbers. Prove that

$$(x^2+y^2+z^2)^2 + xyz(x+y+z) \geq (xy+yz+zx)^2 + (x^2y^2+y^2z^2+z^2x^2).$$

$$(a+b+c)^2 \geqslant 3(ab+bc+ca) \quad \text{(AM-GM)}$$

$$\Rightarrow (x^2+y^2+z^2)^2 \geqslant 3(x^2y^2+y^2z^2+z^2x^2) \quad (1)$$

Also

$$xy^2z + x^2yz + xyz^2$$

$$= (xy)(yz) + (xy)(xz) + (xz)(yz)$$

$$a^2+b^2+c^2 \geqslant ab+bc+ca \quad \text{(AM-GM)}$$

$$\Rightarrow (xy)^2 + (yz)^2 + (zx)^2 \geqslant xy^2z + x^2yz + xyz^2$$

$$\Rightarrow x^2y^2 + y^2z^2 + z^2x^2 \geqslant xyz(x+y+z) \quad (2)$$

$$(x^2+y^2+z^2)^2 + xyz(x+y+z) \geqslant \underline{\underline{3}}(x^2y^2+y^2z^2+z^2x^2) + xyz(x+y+z)$$

$$\geqslant 2(x^2y^2+y^2z^2+z^2x^2) + 2\,xyz(x+y+z)$$

$$= x^2y^2+y^2z^2+z^2x^2 + (x^2y^2+y^2z^2+z^2x^2 + 2\,xyz(x+y+z))$$

$$(x^2y^2+y^2z^2+z^2x^2) + (xy+yz+zx)^2 \quad \square$$

7. Let x, y, z be real numbers. Prove that

$$(x^2+y^2+z^2)^2+xyz(x+y+z) \geq (xy+yz+zx)^2+(x^2y^2+y^2z^2+z^2x^2).$$

$$\sum x^4 + 2\sum x^2y^2 + \sum x^2yz \geqslant \sum x^2y^2 + 2\sum x^2yz + \sum x^2y^2$$

$$\Leftrightarrow \quad \sum x^4 \geqslant \sum x^2yz$$

$$\Leftrightarrow \quad x^4+y^4+z^4 \geqslant x^2yz+xy^2z+xyz^4$$

which is true because

$$\begin{cases} x^4+x^4+y^4+z^4 \geqslant 4\,x^2yz \\ y^4+y^4+x^4+z^4 \geqslant 4\,xy^2z \\ z^4+z^4+y^4+x^4 \geqslant 4xyz^2 \end{cases}$$

$$x^4+y^4+z^4 \geqslant x^2yz+xy^2z+xyz^4$$

$\square$

another proof for $x^4+y^4+z^4 \geqslant x^2yz+xy^2z+xyz^4$

$$\Rightarrow \quad x^4+y^4+z^4 \geqslant x^2y^2+y^2z^2+z^2x^2$$
$$\geqslant (xy)(yz)+(yz)(zx)+(zx)(xy)$$
$$= x^2yz+xy^2z+xyz^4$$

$\square$

52. (a) Show that $x^4 - x^3 - x + 1 \geq 0$ for all real numbers x.

(b) Find all real numbers x_1, x_2, x_3 that satisfy $x_1 + x_2 + x_3 = 3$ and $x_1^3 + x_2^3 + x_3^3 = x_1^4 + x_2^4 + x_3^4$.

(a) Note that 1 is a root for $x^4 - x^3 - x + 1$

so $x^4 - x^3 - x + 1 = (x-1)x^3 - (x-1)$

$= (x-1)(x^3-1)$

$= (x-1)^2(x^2+x+1)$

However, $x^2+x+1 = (x+\frac{1}{2})^2 + \frac{3}{4} > 0$

or $\frac{1}{2}(x+1)^2 + \frac{x^2}{2} + \frac{1}{2} > 0$

$\Rightarrow$ $x^4 - x^3 - x + 1 \geq 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ equality holds when $x = 1$

(b) From part (a)

$$x_i^4 \geq x_i^3 + x_i - 1$$

$\Rightarrow$ $x_1^4 + x_2^4 + x_3^4 \geq x_1^3 + x_2^3 + x_3^3 + x_1 + x_2 + x_3 - 3$

$\Rightarrow$ $x_1 + x_2 + x_3 \leq 3$

but $x_1 + x_2 + x_3 = 3 \Rightarrow$ equality case holds

$\Rightarrow$ $x_i = 1 \quad \forall i \in \{1, 2, 3\}$