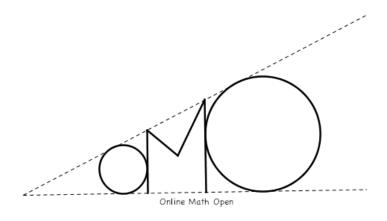
The Online Math Open Fall Contest Official Solutions October 18 - 29, 2013



Acknowledgements

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1. Determine the value of 142857 + 285714 + 428571 + 571428.

Proposed by Ray Li.

Answer. 1428570

Solution. Let N = 142857. Because

$$142857 = N$$

285714 = 2N

428571 = 3N

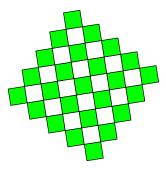
571428 = 4N

the answer is 10N = 1428570.

Comment. For the curious: The cyclic shifting of the digits in the multiples of N is a consequence of the expansion $\frac{1}{7} = 0.\overline{142857}$, and the fact that 10 is the primitive root modulo 7.

See also USAMO 2013 #5, which is solved by basically the same idea.

2. The figure below consists of several unit squares, M of which are white and N of which are green. Compute 100M + N.



Proposed by Evan Chen.

Answer. 1625

Solution. It turns out that the white squares form a 4×4 grid, while the green squares form a 5×5 grid. (Rotate the grid!) So, the answer is $4^2 \cdot 100 + 5^2 = 1625$.

3. A palindromic table is a 3×3 array of letters such that the words in each row and column read the same forwards and backwards. An example of such a table is shown below.

$$egin{array}{cccc} O & M & O \\ N & M & N \\ O & M & O \\ \end{array}$$

How many palindromic tables are there that use only the letters O and M? (The table may contain only a single letter.)

Proposed by Evan Chen.

Answer. 16

Solution. It's not hard to check the table must be of the form

 $\begin{array}{cccc} A & B & A \\ C & X & C \\ A & B & A \end{array}$

where A, B, C, X are letters, not necessarily distinct. This follows by looking at each pair of letters that must be the same for the table to be palindromic.

Then, the answer is $2^4 = 16$.

4. Suppose a_1, a_2, a_3, \ldots is an increasing arithmetic progression of positive integers. Given that $a_3 = 13$, compute the maximum possible value of

$$a_{a_1} + a_{a_2} + a_{a_3} + a_{a_4} + a_{a_5}$$
.

Proposed by Evan Chen.

Answer. 365

Solution. The sum in question is equal to $5a_{a_3} = 5a_{13}$. Now, because $a_1 \ge 1$, the common difference is at most $\frac{1}{2}(13-1) = 6$. Then, $a_{13} \le 1+6\cdot 12 = 73$, so $5a_{13} \le 365$. Equality occurs when $a_n = 6n-5$ for all n.

5. A wishing well is located at the point (11,11) in the xy-plane. Rachelle randomly selects an integer y from the set $\{0,1,\ldots,10\}$. Then she randomly selects, with replacement, two integers a,b from the set $\{1,2,\ldots,10\}$. The probability the line through (0,y) and (a,b) passes through the well can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute m+n.

Proposed by Evan Chen.

Answer. 111

Solution. Note that the number 11 is prime. The key observation is that if $y \neq 0$, then no lattice point on the line from (0, y) to (11, 11) can have an x-coordinate in $\{1, 2, ..., 10\}$. Therefore, we must have y = 0. This occurs with probability $\frac{1}{11}$.

In that case, the probability that (a,b) lies on the line is just the probability that a=b, or $\frac{1}{10}$.

So,
$$\frac{m}{n} = \frac{1}{11} \cdot \frac{1}{10}$$
, and the answer is $1 + 110 = 111$.

6. Find the number of integers n with $n \ge 2$ such that the remainder when 2013 is divided by n is equal to the remainder when n is divided by 3.

Proposed by Michael Kural.

Answer. 6

Solution. Note that the remainder when dividing by 3 is either 0, 1, or 2. If both remainders are 0, then $3 \mid n$ and $n \mid 2013 = 3 \cdot 11 \cdot 61$, so 4 possibilities for n are $3, 3 \cdot 11, 3 \cdot 61$, and $3 \cdot 11 \cdot 61$. If both remainders are 1, then $n \mid 2012 = 2^2 \cdot 503$, and $n \equiv 1 \pmod{3}$, so we can check to see that $n = 2^2$ and $n = 2 \cdot 503$ are the only solutions in this case. If both remainders are 2, then $n \mid 2011$ and $n \equiv 2 \pmod{3}$. As 2011 is prime and $n \geq 2$, the only possibility is n = 2011, but $2011 \not\equiv 2 \pmod{3}$, so there are no solutions in this case. Thus the answer is 4 + 2 = 6.

7. Points M, N, P are selected on sides $\overline{AB}, \overline{AC}, \overline{BC}, \overline{BC}, \overline{BC}$ respectively, of triangle ABC. Find the area of triangle MNP given that AM = MB = BP = 15 and AN = NC = CP = 25.

Proposed by Evan Chen.

Answer. 150

Solution. Remark that AB = 30, BC = 40 and CA = 50. Then, note that M and N are the midpoints of \overline{AC} and \overline{BC} . This causes the area of triangle MNP to be one quarter of the area of ABC; after all, one can "slide" P to the midpoint of \overline{AB} without affecting the area! Hence the answer is $\frac{1}{4}(600) = 150$.

8. Suppose that $x_1 < x_2 < \cdots < x_n$ is a sequence of positive integers such that x_k divides x_{k+2} for each $k = 1, 2, \dots, n-2$. Given that $x_n = 1000$, what is the largest possible value of n?

Proposed by Evan Chen.

Answer. 13

Solution. First, consider $1000 = 2^3 \cdot 5^3$. Among the sequence of numbers x_n, x_{n-2}, \ldots we must reduce the number of prime factors by one. This process is forced to terminate in at most six steps, so we have $n-12 \le 1$. Hence, $n \le 13$.

Now we need to show n = 13 is actually achievable. Using the "chain" idea above, it is natural to write

$$x_{13} = 1000$$

$$x_{11} = 200$$

$$x_9 = 40$$

$$x_7 = 20$$

$$x_5 = 10$$

$$x_3 = 5$$

$$x_1 = 1$$

From here, we can weave in another chain by

$$x_{12} = 240$$

$$x_{10} = 48$$

$$x_8 = 24$$

$$x_6 = 12$$

$$x_4 = 6$$

$$x_2 = 2$$

This shows that n = 13 is indeed achievable.

9. Let AXYZB be a regular pentagon with area 5 inscribed in a circle with center O. Let Y' denote the reflection of Y over \overline{AB} and suppose C is the center of a circle passing through A, Y' and B. Compute the area of triangle ABC.

Proposed by Evan Chen.

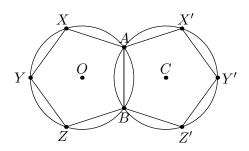
Answer. 1

Solution. Instead of reflecting just the point Y, reflect the entire pentagon! Then we see that C is the center of pentagon AX'Y'Z'B – in other words, C is the reflection of O across \overline{AB} .

It then follows that the area of $\triangle CAB$ is equal to the area of triangle OAB, which is one-fifth of the total area. Hence, $\frac{1}{5} \cdot 5 = 1$.

10. In convex quadrilateral AEBC, $\angle BEA = \angle CAE = 90^{\circ}$ and AB = 15, BC = 14 and CA = 13. Let D be the foot of the altitude from C to \overline{AB} . If ray CD meets \overline{AE} at F, compute $AE \cdot AF$.

Proposed by David Stoner.



Problem 9: The pentagon.

Answer. 99

Solution. We know $\overline{CA} \perp \overline{AE}$ and $\overline{AE} \perp \overline{BE}$. Note that $\angle ACF = \angle BAE = 90^{\circ} - A$, so $\triangle CAF \sim \triangle AEB$, and $AE \cdot AF = CA \cdot BE$. Let the foot from B to \overline{AC} be G, and note that BE = AG. But we can easily compute

$$BE = \frac{13^2 + 15^2 - 14^2}{2 \cdot 13} = \frac{99}{13}$$

So $AE \cdot AF = BE \cdot CA = 99$.

11. Four orange lights are located at the points (2,0), (4,0), (6,0) and (8,0) in the xy-plane. Four yellow lights are located at the points (1,0), (3,0), (5,0), (7,0). Sparky chooses one or more of the lights to turn on. In how many ways can he do this such that the collection of illuminated lights is symmetric around some line parallel to the y-axis?

Proposed by Evan Chen.

Answer. [52]

Solution. Consider a line of symmetry. The alternating colors of the lights forces the line to pass through one of the lights. Let's assume that the line of symmetry is x = k for some k.

When k = 1, the only possibility is a single illuminated light (1, 0).

When k = 2, we may choose whether (2,0) is on, and whether the lights (1,0) or (3,0) are both on. However, we cannot have all lights off; in other words, we need to discard the empty set. Hence there are $2^2 - 1 = 3$ possibilities.

Continuing, we get 7, 15, 15, 7, 3, 1 possibilities for $k = 3, 4, \dots, 8$. Hence the answer is 1 + 3 + 7 + 15 + 15 + 7 + 3 + 1 = 52.

12. Let a_n denote the remainder when $(n+1)^3$ is divided by n^3 ; in particular, $a_1 = 0$. Compute the remainder when $a_1 + a_2 + \cdots + a_{2013}$ is divided by 1000.

Proposed by Evan Chen.

Answer. [693]

Solution. Remark that for any integer n with $n \ge 4$, we have $n^3 \le (n+1)^3 \le 2n^3$. Thus, we have $a_n = (n+1)^3 - n^3$ for all integers n with $n \ge 4$. In that case, we discover a "telescoping" sequence

$$a_4 + a_5 + \dots + a_{2013} = 2014^3 - 4^3 \equiv 14^3 - 4^3 \equiv 680 \pmod{1000}.$$

Then, we just compute $a_1 = 0$, $a_2 = 3$, and $a_3 = 10$. Hence, the answer is 680 + 0 + 3 + 10 = 693. \square

13. In the rectangular table shown below, the number 1 is written in the upper-left hand corner, and every number is the sum of the any numbers directly to its left and above. The table extends infinitely downwards and to the right.

Wanda the Worm, who is on a diet after a feast two years ago, wants to eat n numbers (not necessarily distinct in value) from the table such that the sum of the numbers is less than one million. However, she cannot eat two numbers in the same row or column (or both). What is the largest possible value of n?

Proposed by Evan Chen.

Answer. 20.

Solution. We claim optimal diet for any fixed n is to eat the n numbers from the (n-1)th row of Pascal's triangle.

The main idea of the proof is as follows: suppose Wanda eats two numbers x and y where x is above and to the left of y. The associated rows/columns determine a rectangle with four vertices, two of which are x and y. Then, it's not too hard to see that by replacing x, y with the numbers at the other two vertices, we get a smaller sum. (This follows a "smoothing"-type idea.) So, in an optimal situation, the numbers from a "diagonal" moving from bottom-left to upper-right.

The smallest possible sum of such a configuration is the numbers $\binom{n-1}{0},\ldots,\binom{n-1}{n}$, which occupies the n smallest rows and n smallest columns. The minimal sum is therefore 2^{n-1} . So we want $2^{n-1} < 10^6$ or $n \le 20$.

14. In the universe of Pi Zone, points are labeled with 2×2 arrays of positive reals. One can teleport from point M to point M' if M can be obtained from M' by multiplying either a row or column by some positive real. For example, one can teleport from $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ to $\begin{pmatrix} 1 & 20 \\ 3 & 40 \end{pmatrix}$ and then to $\begin{pmatrix} 1 & 20 \\ 6 & 80 \end{pmatrix}$.

A tourist attraction is a point where each of the entries of the associated array is either 1, 2, 4, 8 or 16. A company wishes to build a hotel on each of several points so that at least one hotel is accessible from every tourist attraction by teleporting, possibly multiple times. What is the minimum number of hotels necessary?

Proposed by Michael Kural.

Answer. 17

Solution. For an array $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define its *height* to be $\frac{ad}{bc}$. It is not hard to see that one can only teleport between two points of the same height. One can also check that this is a complete invariant: that is, if two points have the same height, then it is possible to teleport from one to another.

So, the answer is just the number of possible heights of tourist attractions. This can be any of the numbers $2^{-8}, 2^{-7}, \dots, 2^{8}$, of which there are 17.

Comment. Trivia: the problem is really equivalent to finding the equivalence classes of a binary relation \sim on the set of 2×2 matrices, where \sim is the transitive closure of the teleportation operation.

15. Find the positive integer n such that

$$\underbrace{f(f(\cdots f(n)\cdots))}_{2013 \text{ f's}}(n)\cdots)) = 2014^2 + 1$$

where f(n) denotes the nth positive integer which is not a perfect square.

Proposed by David Stoner.

Answer. 1015057

Solution. We claim that

$$f^{-1}(k^2+1) = k^2 - k + 1.$$

Indeed, there are k positive squares at most $k^2 + 1$, so $k^2 + 1$ is the $k^2 - k + 1$ th non-square. Now because

$$(k-1)^2 < k^2 - k + 1 < k^2$$

for k > 1, we similarly obtain

$$f^{-1}(k^2 - k + 1) = k^2 - k + 1 - (k - 1) = (k - 1)^2 + 1$$

Repeatedly using these two formulas, it is now easy to obtain $f^{-2013}(2014^2 + 1) = 1008^2 - 1008 + 1 = 1015057$.

16. Al has the cards 1, 2, ..., 10 in a row in increasing order. He first chooses the cards labeled 1, 2, and 3, and rearranges them among their positions in the row in one of six ways (he can leave the positions unchanged). He then chooses the cards labeled 2, 3, and 4, and rearranges them among their positions in the row in one of six ways. (For example, his first move could have made the sequence 3, 2, 1, 4, 5, ..., and his second move could have rearranged that to 2, 4, 1, 3, 5,) He continues this process until he has rearranged the cards with labels 8, 9, 10. Determine the number of possible orderings of cards he can end up with.

Proposed by Ray Li.

Answer. 13122

Solution. This is recursive. After the first move, which chooses one of three positions for card 1, it reduces to the same problem for 9 cards. Note that the ordering of cards 2 and 3 doesn't matter because all possible second moves are independent of their order. For example, from $1, 2, 3, 4, \ldots$ and $1, 3, 2, 4, \ldots$, we have the same set of possible resulting sequences.

Thus, if a_n is the answer for n cards, we derive a recursion $a_n = 3a_{n-1}$ for every integer n with $n \ge 4$. Because $a_3 = 6$, we compute $a_{10} = 3^7 \cdot 6 = 13122$.

17. Let ABXC be a parallelogram. Points K, P, Q lie on \overline{BC} in this order such that $BK = \frac{1}{3}KC$ and $BP = PQ = QC = \frac{1}{3}BC$. Rays XP and XQ meet \overline{AB} and \overline{AC} at D and E, respectively. Suppose that $\overline{AK} \perp \overline{BC}$, EK - DK = 9 and BC = 60. Find AB + AC.

Proposed by Evan Chen.

Answer. 100

Solution. P and Q are the centroids of $\triangle ABX$ and $\triangle ACX$, so it follows that D and E are the midpoints. Let M be the midpoint of BC. Then DKEM is an isosceles trapezoid, so

$$9 = EK - DK = DM - EM = \frac{1}{2}(AC - AB) \implies AC - AB = 18.$$

6

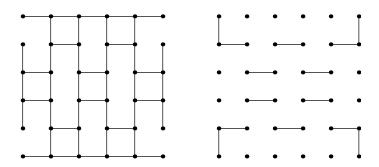
Now BK = 15, CK = 45, so $AK^2 = AB^2 - 15^2 = AC^2 - 45^2$. In other words,

$$AC^2 - AB^2 = 45^2 - 15^2 = 30 \cdot 60 \implies AC + AB = \frac{30 \cdot 60}{18} = 100.$$

18. Given an $n \times n$ grid of dots, let f(n) be the largest number of segments between adjacent dots which can be drawn such that (i) at most one segment is drawn between each pair of dots, and (ii) each dot has 1 or 3 segments coming from it. (For example, f(4) = 16.) Compute f(2000).

Proposed by David Stoner.

Answer. 5999992



Problem 18: Complements in a square grid when n = 6.

Solution. Consider the "complement" of such a drawing; i.e. the set of segments which is not chosen. Then we require that in the complement,

- Each of the corner dots, which has two neighbors, has degree 2-1=1,
- Each of the side dots, which has three neighbors, has degree 3-3=0 or 3-1=2, and
- Each of the interior dots, which has four neighbors, has degree 4-3=1 or 4-1=3.

We wish to minimize the number of edges in the complement.

It is not hard to see the construction in the figure is now minimal. We just handle the corners first, and then connect the rest of the points in the interior to handle the condition the degree needs to be at least one. This will yield $2 \cdot 4 + \frac{(n-2)^2 - 4}{2}$ segments (when n is even). This corresponds to

$$2n(n-1) - \left(8 + \frac{(n-2)^2 - 4}{2}\right) = \frac{3}{2}n^2 - 8$$

segments in the actual construction. When n = 2000, we obtain 5999992.

19. Let $\sigma(n)$ be the number of positive divisors of n, and let rad n be the product of the distinct prime divisors of n. By convention, rad 1 = 1. Find the greatest integer not exceeding

$$100 \left(\sum_{n=1}^{\infty} \frac{\sigma(n) \sigma(n \operatorname{rad} n)}{n^2 \sigma(\operatorname{rad} n)} \right)^{\frac{1}{3}}.$$

Proposed by Michael Kural.

Answer. 164

Solution. Let $f(n) = \frac{\sigma(n)\sigma(n\operatorname{rad} n)}{n^2\sigma(\operatorname{rad} n)}$. Note that f(n) is multiplicative, so the desired sum is simply

$$\sum_{n=1}^{\infty} f(n) = \prod_{p \text{ prime } k=0}^{\infty} f(p^k)$$

Now for $k \ge 1$, $\sigma(p^k) = k + 1$, and rad $p^k = p$, so $\sigma(p^k \operatorname{rad} p^k) = k + 2$. $\sigma(\operatorname{rad} p^k)$ is simply 2, so

$$f(p^k) = \frac{(k+1)(k+2)}{2p^{2k}} = {k+2 \choose 2} \frac{1}{p^{2k}}$$

Note that f(1) = 1, so this formula holds for all $k \ge 0$. Then

$$\sum_{k\geq 0} f(p^k) = \sum_{k\geq 0} {k+2 \choose 2} \frac{1}{p^{2k}} = \left(\sum_{k\geq 0} \frac{1}{p^{2k}}\right)^3$$

Then

$$\prod_{p \text{ prime}} \sum_{k=0}^{\infty} f(p^k) = \prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{1}{p^{2k}} \right)^3 = \left(\prod_{p \text{ prime}} \left(\sum_{k \geq 0} \frac{1}{p^{2k}} \right) \right)^3$$

But again by multiplicativity

$$\prod_{p \text{ prime}} \left(\sum_{k \ge 0} \frac{1}{p^{2k}} \right) = \sum_{n \ge 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Thus

$$100 \left(\sum_{n=1}^{\infty} \frac{\sigma(n)\sigma(n \operatorname{rad} n)}{n^2 \sigma(\operatorname{rad} n)} \right)^{\frac{1}{3}} = \frac{100\pi^2}{6} \approx 164.5$$

So the answer is 164.

20. A positive integer n is called *mythical* if every divisor of n is two less than a prime. Find the unique mythical number with the largest number of divisors.

Proposed by Evan Chen.

Answer. 135

Solution. Consider n mythical and observe that $2 \nmid n$. Now, remark that there cannot be two distinct primes $p, q \neq 3$ dividing n. Otherwise, p+2, q+2, and pq+2 must both be prime. Then we must have $p, q \equiv 2 \pmod{3}$ but then $pq+2 \equiv 0 \pmod{3}$, which is impossible. Similar work also allows us to prove that $p^2 \nmid n$ for $p \neq 3$.

Hence, we will write $n = 3^k p$ for some prime p. (The case $n = 3^k$ has five divisors at 81 and this is maximal here.) If p = 5, then it is easy to check that $3^3 \cdot 5 = 135$ is the best possible (since $3^4 \cdot 5 + 2 = 11 \cdot 37$). Now, we will prove that if p > 5 then k < 3. Indeed, if $k \ge 3$, then

$$3^0p + 2 \equiv p + 2 \pmod{5}$$

$$3^1p+2\equiv 3p+2\pmod 5$$

$$3^2p + 2 \equiv 4p + 2 \pmod{5}$$

$$3^3p + 2 \equiv 2p + 2 \pmod{5}$$

must all be prime. But unless $p \equiv 0 \pmod 5$, at least one of these must be 0 (mod 5), which is a contradiction. (This isn't a miracle – it occurs because 3 is a primitive root modulo 5, so all nonzero residues appear as a coefficient.)

So the optimal case is $4 \cdot 2 = 8$ factors, achieved at $3^3 \cdot 5 = 135$.

21. Let ABC be a triangle with AB = 5, AC = 8, and BC = 7. Let D be on side AC such that AD = 5 and CD = 3. Let I be the incenter of triangle ABC and E be the intersection of the perpendicular bisectors of \overline{ID} and \overline{BC} . Suppose $DE = \frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find a + b + c.

Proposed by Ray Li.

Answer. 13

Solution. We claim that BIDC is a cyclic quadrilateral. First, note that as

$$7^2 = 8^2 + 5^2 - 2 \cdot 5 \cdot 8 \cdot \frac{1}{2}$$

it follows that $\cos \angle A = \frac{1}{2}$ and $\angle A = 60^\circ$. Then $\angle BIC = 90^\circ + \frac{\angle A}{2} = 120^\circ$. Also, since AB = AD and $\angle BAD = 60^\circ$, triangle BAD is equilateral, so $\angle BDC = 180^\circ - \angle ADB = 120^\circ$. Thus BIDC is cyclic, and E is its circumcenter. Thus DE is the circumradius of triangle DBC, but this is easy to compute with the law of sines:

$$DE = \frac{1}{2} \cdot \frac{BC}{\sin \angle BDC} = \frac{7}{2\sin 120^{\circ}} = \frac{7\sqrt{3}}{3}$$

So the answer is 7 + 3 + 3 = 13.

22. Find the sum of all integers m with $1 \le m \le 300$ such that for any integer n with $n \ge 2$, if 2013m divides $n^n - 1$ then 2013m also divides n - 1.

Proposed by Evan Chen.

Answer. 4360

Solution. Call an integer M stable if $n^n \equiv 1 \pmod M$ implies $n \equiv 1 \pmod M$. We claim that M is stable if for every prime $p \mid M$, we have $q \mid M$ for each prime factor q of p-1. Suppose $n^n \equiv 1 \pmod M$. It suffices to show that $n \equiv 1 \pmod p^k$ for each $p^k \mid M$, (where p is a prime). Let $u \mid \varphi(p^k)$ be the order of n modulo p^k . Because $n^n \equiv 1 \pmod M \implies (n,M) = 1$, we find that $(n,\varphi(p^k)) = 1$. But $u \mid n$ as well. This forces u = 1, which is the desired.

Let M=2013m. First, we claim that M must be even. Otherwise, take n=M-1. Then, we claim that 5 must divide M. Otherwise, take $n\equiv 0\pmod 5$, $n\equiv 3\pmod 11$, and $n\equiv 1\pmod n$ modulo any other primes powers dividing M.

Now for $m=10,20,\ldots,300$, it is easy to check by the condition that M is stable by our condition above – unless m=290. It turns out that m=290 is not stable; simply select $n\equiv 0\pmod 7$, $n\equiv 2^4\pmod 29$, and $n\equiv 1\pmod 10\cdot 2013$. It is not hard to check that $n^n-1\equiv 1\pmod 29\cdot 10\cdot 2013$ and yet $n\not\equiv 1\pmod 29$, as desired.

So, the answer is $10 + 20 + 30 + \cdots + 280 + 300 = 4360$.

In fact, the converse to the stability lemma is true as well. We can generate the necessary counterexamples using primitive roots. \Box

23. Let \overline{ABCDE} be a regular pentagon, and let F be a point on \overline{AB} with $\angle CDF = 55^{\circ}$. Suppose \overline{FC} and \overline{BE} meet at G, and select H on the extension of \overline{CE} past E such that $\angle DHE = \angle FDG$. Find the measure of $\angle GHD$, in degrees.

Proposed by David Stoner.

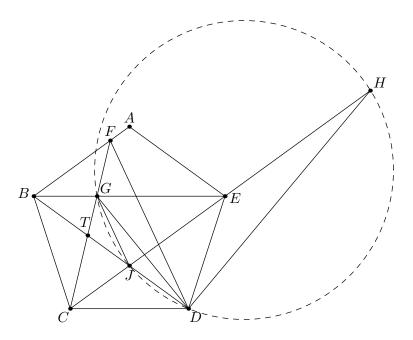
Answer. 19

Solution. Let J be the intersection of diagonals \overline{BD} and \overline{CE} , and note that ABEJ is a rhombus. First, we claim that $\overline{JG} \parallel \overline{DF}$. Let \overline{CF} and \overline{BD} meet at T. Observe that

$$\angle TCJ = \angle TFB$$
, $\angle BTG = \angle CTJ$

and

$$\angle GBT = \angle FBG = \angle JCD = \angle JDC = 36^{\circ}.$$



Problem 23: A pentagon! Point F is not drawn to scale.

Applying the law of sines now yields TG: GF = TJ: JD readily, proving the claim.

Then

$$\angle JGD = \angle FDG = \angle EHD = \angle JHD$$

so J, D, G, H are concyclic. Now we deduce

$$\angle GHD = 180^{\circ} - \angle GJD = \angle JDF = \angle CDF - 36^{\circ} = 19^{\circ}.$$

24. The real numbers $a_0, a_1, \ldots, a_{2013}$ and $b_0, b_1, \ldots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \ldots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute

$$\sum_{k=1}^{2013} \left(a_k b_{k-1} - a_{k-1} b_k \right).$$

Proposed by Evan Chen.

Answer. 671

Solution. Let n = 2013 for brevity. Then compute

$$\sum_{k=1}^{n} (a_k b_{k-1} - a_{k-1} b_k) = \sum_{k=1}^{n} ((a_k - a_{k-1}) (b_k + b_{k-1}) - (a_k b_k - a_{k-1} b_{k-1}))$$

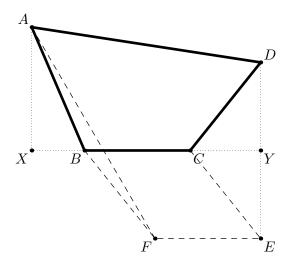
$$= \sum_{k=1}^{n} (a_k - a_{k-1}) (b_k + b_{k-1}) + \sum_{k=1}^{n} (a_k b_k - a_{k-1} b_{k-1})$$

$$= \frac{1}{63 \cdot 96} \sum_{k=1}^{n} (2k+2) + (a_n b_n - a_0 b_0)$$

$$= \frac{1}{63 \cdot 96} \left(n \cdot \frac{4 + (2n+2)}{2} \right)$$

$$= \frac{2013 \cdot 2016}{63 \cdot 96}$$

$$= 671.$$



Problem 25: It's the burning tent problem!

25. Let ABCD be a quadrilateral with AD=20 and BC=13. The area of $\triangle ABC$ is 338 and the area of $\triangle DBC$ is 212. Compute the smallest possible perimeter of ABCD.

Proposed by Evan Chen.

Answer. 118

Solution. Let X and Y be the feet of the altitudes from A and D to BC. Compute $AX = \frac{676}{13}$, $DY = \frac{424}{13}$ and finally $XY = \frac{64}{13}$.

The intuition now is to let BC slide along line XY, reducing this to the river problem. Let E be the reflection of D over Y and F be the point such that BCEF is a parallelogram. Then it's easy to compute

$$AF^2 = (XY - BC)^2 + (AX + DY)^2 = \left(\frac{105}{13}\right)^2 + \left(\frac{1100}{13}\right)^2 = 85^2.$$

Then $AB + CD = AB + BE \le AF = 85$, and the answer is 85 + 20 + 13 = 118.

26. Let ABC be a triangle with AB = 13, AC = 25, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across $\overline{AC}, \overline{AB}$ by D, E, respectively, and let O be the circumcenter of triangle ABC. Let P be a point such that $\triangle DPO \sim \triangle PEO$, and let X and Y be the midpoints of the major and minor arcs \overline{BC} of the circumcircle of triangle ABC. Find $PX \cdot PY$.

Proposed by Michael Kural.

Answer. 274

Solution. Let the complex coordinates of A, B, C, D, E be a, b, c, d, e. Without loss of generality assume that O is at the origin. Note that $\triangle BOC \sim \triangle BAD$, so

$$\frac{d-a}{a-b} = \frac{c-0}{0-b}$$

implying

$$d = a + (b - a)\frac{c}{b} = a + c - \frac{ac}{b}$$

and similarly, $e=a+b-\frac{ab}{c}$. Now note that $p=\sqrt{de}$, as $\frac{p}{d}=\frac{e}{p}$. Also, $X,Y=\pm\sqrt{bc}$ for a similar resason. Then

$$PX \cdot PY = \left| \sqrt{de} - \sqrt{bc} \right| \left| \sqrt{de} + \sqrt{bc} \right|$$

$$= \left| \left(a + c - \frac{ac}{b} \right) \left(a + b - \frac{ab}{c} \right) - bc \right|$$

$$= \left| -\frac{a^2c}{b} - \frac{a^2b}{c} + 2a^2 \right|$$

$$= \left| -\frac{a^2(b-c)^2}{bc} \right|.$$

Now as we assumed O, the circumcenter of ABC, was at the origin, we have |a| = |b| = |c|, so $\left| \frac{a^2}{bc} \right| = 1$. Thus

$$PX \cdot PY = \left| \frac{a^2}{bc} \right| |(b-c)^2| = |b-c|^2 = BC^2$$

But now it is easy to compute

$$BC^{2} = AB^{2} + AC^{2} - 2 \cdot AB \cdot AC \cos A = 13^{2} + 25^{2} - 2 \cdot 13 \cdot 25 \cdot \frac{4}{5} = 274.$$

- 27. Ben has a big blackboard, initially empty, and Francisco has a fair coin. Francisco flips the coin 2013 times. On the $n^{\rm th}$ flip (where $n=1,2,\ldots,2013$), Ben does the following if the coin flips heads:
 - (i) If the blackboard is empty, Ben writes n on the blackboard.
 - (ii) If the blackboard is not empty, let m denote the largest number on the blackboard. If $m^2 + 2n^2$ is divisible by 3, Ben erases m from the blackboard; otherwise, he writes the number n.

No action is taken when the coin flips tails. If probability that the blackboard is empty after all 2013 flips is $\frac{2u+1}{2^k(2v+1)}$, where u, v, and k are nonnegative integers, compute k.

Proposed by Evan Chen.

Answer. 1336

Solution. Consider non-commutative variables s,t which satisfy $s^2=t^2=1$. Then the problem is equivalent to computing $\frac{1}{2^{2013}}M$, where M is the number of choices of $i_1,i_2,\ldots,i_{2013}\in\{0,1\}^{2013}$ which satisfy

$$s^{i_1}s^{i_2}t^{i_3}s^{i_4}ts^{i_5}t^{i_6}s^{i_7}$$
 $s^{i_{2011}}s^{i_{2012}}t^{i_{2013}} = 1$

Because $s^a s^b$ has the same possible outcomes as s^i , we have $M=2^{671}N$ where N is the number of choices of $j_1, j_2, \ldots, j_{1342}$ which satisfy

$$s^{j_1}t^{j_2}s^{j_3}t^{j_4}\dots s^{j_{1341}}t^{j_{1342}}=1.$$

Applying Proposition 5.2 from this RSI 2013 paper directly gives $N = \binom{1341}{671}$. (A general closed form is given in the paper, and the proof is just a long induction.) A standard computation shows that 2^6 is the largest power of 2 which divides N. So, the answer is 2013 - 671 - 6 = 1336.

Comment. This is essentially the two variable version of 2012-2013 Winter OMO Problem 50, but the connection is not immediately obvious. Refer to official solution 2 for more details.

One may also consider the slightly easier variant of this problem with the second condition reversed: that is, Ben erases m precisely when $m^2 + 2n^2$ is not divisible by 3.

28. Let n denote the product of the first 2013 primes. Find the sum of all primes p with $20 \le p \le 150$ such that

- (i) $\frac{p+1}{2}$ is even but is not a power of 2, and
- (ii) there exist pairwise distinct positive integers a, b, c for which

$$a^{n}(a-b)(a-c) + b^{n}(b-c)(b-a) + c^{n}(c-a)(c-b)$$

is divisible by p but not p^2 .

Proposed by Evan Chen.

Answer. 431

Solution. First, observe that because of condition (i), we have that $p(p-1) \mid n$ for each suitable prime. This will allow us to apply Fermat's Little Theorem later.

Let $N = a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b)$. Suppose that we indeed have $p^1 \parallel N$, (where $p^k \parallel n$ means $p^k \mid n$ but $p^{k+1} \nmid n$). First, we claim that we cannot have $a \equiv 0 \pmod{p}$. Otherwise, because $n \geq 2$ this would imply $a^n(a-b)(a-c) \equiv 0 \pmod{p^2}$ and hence we obtain

$$N \equiv b^{n}(b-c)(b-a) + c^{n}(c-a)(c-b) \pmod{p^{2}}$$

= $(b-c)[b^{n}(b-a) - c^{n}(c-a)]$

This is clearly fatal if $c \equiv 0 \pmod{p}$ as well, so consider the case where $b, c \not\equiv 0 \pmod{p}$. Observe that $b^n(b-a) - c^n(c-a) \equiv b - c \pmod{p}$. Hence, either both or neither of the terms above are divisible by p^2 . So we need only consider the case where $a, b, c \not\equiv 0 \pmod{p}$.

In that case, $a^n \equiv b^n \equiv c^n \equiv 1 \pmod{p}$. Assume without loss of generality that $b - c \not\equiv 0 \pmod{p}$; otherwise $a - b \equiv b - c \equiv c - a \equiv 0 \pmod{p}$ gives $p^2 \mid N$. Now we compute

$$2N \equiv 2(a-b)(a-c) + 2(b-c)(b-a) + 2(c-a)(c-b) \pmod{p^2}$$

$$\equiv (a-b)^2 + (b-c)^2 + (c-a)^2 \pmod{p^2}$$

$$= (a-b)^2 + (b-c)^2 + [(a-b) + (b-c)]^2$$

$$= 2 \left[(a-b)^2 + (a-b)(b-c) + (b-c)^2 \right]$$

$$\implies N \equiv (a-b)^2 + (a-b)(b-c) + (b-c)^2$$

Let x = a - b and $y = b - c \not\equiv 0 \pmod{p}$.

$$N \equiv x^2 + xy + y^2 \pmod{p^2}$$

At this point we drop down to modulo p and find

$$0 \equiv x^2 + xy + y^2 \pmod{p}$$

$$\implies -3 \equiv \left(2\frac{x}{y} + 1\right)^2 \pmod{p}$$

Quadratic reciprocity now implies $\left(\frac{-3}{p}\right) = 1 \implies p \equiv 1 \pmod{3}$. Hence, in the original criterion the only possibilities are $p \in \{43, 67, 79, 103, 139\}$.

The sum of these is 431. To construct them, just pick y = 1 and 0 < x < p-1 such that $\left(2\frac{x}{y} + 1\right)^2 \equiv -3$ (mod p) (again possible by quadratic reciprocity). Then $x^2 + xy + y^2 < (x+y)^2 < p^2$. So we simply reconstruct a suitable a, b, c from the x and y.

29. Kevin has 255 cookies, each labeled with a unique nonempty subset of {1,2,3,4,5,6,7,8}. Each day, he chooses one cookie uniformly at random out of the cookies not yet eaten. Then, he eats that cookie, and all remaining cookies that are labeled with a subset of that cookie (for example, if he chooses the

cookie labeled with $\{1, 2\}$, he eats that cookie as well as the cookies with $\{1\}$ and $\{2\}$). The expected value of the number of days that Kevin eats a cookie before all cookies are gone can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m+n.

Proposed by Ray Li.

Answer. 213

Solution. The proof is by linearity of expectation. Let n = 8.

We claim the probability that a cookie with a subset S is ever chosen is $\frac{1}{2^{n-|S|}}$. Note that S is uneaten at a given time if and only if all of its supersets are also uneaten at that time. Because each of S and its supersets is equally likely to be eaten at any such point in time, the probability is just

$$\frac{1}{2^{\#(\text{supersets})}} = \frac{1}{2^{n-|S|}}$$

as claimed.

But the number of days Kevin takes equals the number of cookies chosen, or the sum of the indicator function [cookie S chosen] over all nonempty subsets S. Therefore the expected number of days taken simply equals

$$\sum_{S\subseteq\{1,2,\dots,8\}} \frac{1}{2^{n-|S|}} = \sum_{k=1}^{8} \frac{\text{\#sets with size } k}{2^{n-k}}$$

$$= \frac{\binom{n}{n}}{2^{0}} + \frac{\binom{n}{n-1}}{2^{1}} + \dots + \frac{\binom{n}{0}}{2^{n}} - \frac{\binom{n}{0}}{2^{n}}$$

$$= \frac{\binom{n}{n} \cdot 2^{n} + \binom{n}{n-1} 2^{n-1} + \dots + \binom{n}{0} - \binom{n}{0}}{2^{n}}$$

$$= \frac{(2+1)^{n} - 1}{2^{n}}$$

$$= \frac{3^{8} - 1}{2^{8}}$$

$$= \frac{205}{2^{6}}$$

where we have subtracted off the last term since the empty set is not among the cookies. Hence, the answer is 205 + 8 = 213.

30. Let $P(t) = t^3 + 27t^2 + 199t + 432$. Suppose a, b, c, and x are distinct positive reals such that P(-a) = P(-b) = P(-c) = 0, and

$$\sqrt{\frac{a+b+c}{x}} = \sqrt{\frac{b+c+x}{a}} + \sqrt{\frac{c+a+x}{b}} + \sqrt{\frac{a+b+x}{c}}.$$

If $x = \frac{m}{n}$ for relatively prime positive integers m and n, compute m + n.

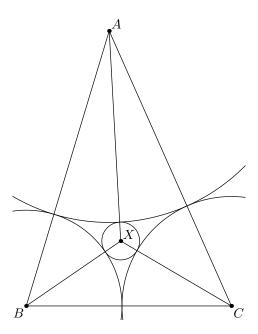
Proposed by Evan Chen.

Answer. [847]

Solution. First note that x is unique, because if we multiply both sides of the equation by x, we get a constant on the LHS and an increasing function f(x) on the RHS (such that f(0) = 0 and $\lim_{x\to\infty} f(x) = \infty$). (*)

Construct three mutually tangent circles centered at A, B, C with radii a, b, c, respectively, and consider a fourth circle centered at X with radius y externally tangent to (A), (B), (C). By Descartes's theorem on mutually tangent circles,

$$y^{-1} = a^{-1} + b^{-1} + c^{-1} + 2\sqrt{a^{-1}b^{-1} + b^{-1}c^{-1} + c^{-1}a^{-1}} = \frac{199}{432} + 2\sqrt{\frac{27}{432}} = \frac{199}{432} + \frac{1}{2}.$$



Problem 30: Actually a geometry problem.

On the other hand, by Heron's formula on the equation [ABC] = [AXB] + [BXC] + [CXA], we find

$$\sqrt{abc(a+b+c)} = \sqrt{ybc(b+c+y)} + \sqrt{yca(c+a+y)} + \sqrt{yab(a+b+y)},$$

whence $f(y) = f(x) \implies y = x$ by (*).

Finally,
$$x = \frac{1}{\frac{199}{432} + \frac{1}{2}} = \frac{432}{415}$$
, yielding an answer of $432 + 415 = 847$.

Comment. This problem is more or less equivalent to 2011 ELMO Shortlist A5, which was inspired by a failed approach on 2003 IMO Shortlist G7.