Miscellaneous Problems

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Today we will be doing miscellaneous problems related to combinatorics. A lot of these have very non-standard solutions and are rather difficult. The following tricks apply to pretty much all problems. If you feel that you are not getting far on a combinatorics-related problem, it is always good to try these.

- **Induction:** "Induction is awesome and should be used to its full potential" - Jacob Tsimerman, winter camp 2010. Seriously, it is awesome. For hard combinatorics problems, don't expect it to be easy application of induction. Think about how you can reduce your problem to a case for a smaller n in "a powerful way". For example, if you have $n + 1$ vertices in a graph, and you want to use the result on n vertices, take out a vertex with very specific properties and not just any vertex.
- **Extremal Principle:** Define some function f whose domain is "the configuration in your problem" (i.e. configuration of points, sequence of numbers, a graph, etc.) and the range is (usually) the positive integers. Usually it is a rather simple function, i.e. the minimum distance between two points, the largest prime dividing a number, etc. Sometimes you have to be clever and come up with a more complicated function (i.e. the sum of pairwise distances between points). Then consider a configuration for which f achieves its minimum or maximum.
- **Notice Interesting Things:** This idea applies to all olympiad problems, however more so for combinatorics problems. After playing around with the problem for some time you will hopefully come up with *useful properties* of "things" in the problem (e.g. points, edges in a graph, numbers in a sequence, differences between numbers, etc.) You can often tell if the property you found is useful or not. Sometimes it is enough to find *one thing* with a specific property. Sometimes you can remove it and use induction.
- **Reduce the Problem:** After noticing some properties in the problem, you can often reduce or transform the problem to one that is more approachable. **Important**: When you reduce the problem, you are often losing some information from the original problem. It can happen that the results in the reduced problem may not hold because you left out some conditions in the original problem. Check if the result in your reduced problem is even true.
- **Try Small Numbers and See a Pattern:** Self-explanatory.
- **Pigeon-Hole Principle:** Again, usually you have to be clever.
- **Try Lots of Things:** For combinatorics problems there is a lot of freedom and a lot of different approaches you can try. It is possible that you think your current approach is the

right one, *you feel stubborn* and keep trying to go further with this approach. If you are using it for 2 hours and having absolutely no new ideas on how to proceed, it is probably a good idea to try another approach. *Keep an open mind*, since many combinatorics problems have unexpected solutions.

Combinatorial Geometry

Some tricks relevant specifically to combinatorial geometry:

- Consider the convex hull made up of the points
- Consider the point with the smallest $x-$ or $y-$ coordinate
- Find the triangle (quadrilateral, pentagon, etc.) with the vertices being the points from your set S, so that the area of the triangle is minimal/maximal
- **Helly's Theorem:** If $X_1, X_2, ..., X_n$ are convex subsets of $\mathbb{R}^k$ so that the intersection of any $k+1$ of them is non-empty, then the intersection of all the sets is non-empty.

1. The two legs of a compass are located at two distinct lattice points in the coordinate plane drawn on an infinite sheet of paper. The distance between the two legs cannot be changed. It is allowed to fix one of the legs, and move the other leg to any other lattice point. Is it possible to switch the positions of the two legs after a finite number of steps?

2. A is a convex set in the plane (so that for any two points in A, the line segment joining the two points lies completely in A). Prove that there exists a point O in A, such that for any points X, X' on the boundary of A, such that O lies on line segment XX',

$$\frac{1}{2} \leq \frac{OX}{OX'} \leq 2$$

3. Find all sets S of finitely many points in the plane, no three of which are collinear and such that for any three points A, B, C in S, there is another point D in S such that A, B, C, D (in some order) are the vertices of a parallelogram.

4. A strip of width w is the set of all points which lie on or between two parallel lines that are a distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

5. There are two circles, each with length 1000 cm. 1000 points are marked on the first circle, and on the other circle - several arcs are marked, so that the sum of the lengths of the arcs is less than 1 cm. Prove that it is possible to lay the first circle on the second so that no marked point lies on a marked arc.

6. Find all positive integers n such that in the coordinate plane, there exists a convex n-sided polygon with all vertices having integer coordinates, and whose side-lengths are odd integers, no two of which are equal.

7. In the plane there are finitely many red and blue lines, no two of which are parallel. For every point of intersection of two lines of the same color, there exists a line of the other color passing through that point. Prove that all the lines are concurrent.

8. A convex polygon is given. Prove that there is at most one way to draw several of its diagonals in such a way, that no two diagonals intersect each other and as a result the polygon is partitioned into acute triangles.

9. There are n lines in the plane, all passing through a point O. For any two lines, there is a third line which bisects one of the pairs of vertical angles formed by the two lines. Prove that the n lines divide the $360°$ angle at O into equal angles.

10. A finite collection of squares has total area 4. Show that they can be arranged to cover a square of side 1.

11. Several identical paper squares of n different colors are lying on a rectangular table, with sides of the squares parallel to the sides of the table. Among any n squares of pairwise distinct colors it is possible to find 2 which can be pinned to the table using one pin. Prove that all squares of a certain color can be pinned to the table using $2n - 2$ pins.

12. There are $N \geq 3$ points in the plane. Among the pairwise distances between these points there are at most n different distances. Prove that $N \leq (n + 1)^2$.

Processes

Some tricks relevant specifically to processes:

- Find an invariant - a quantity that does not change; or a half-invariant - a quantity that does not increase/decrease.
- Use discrete continuity - if a quantity changes by -1, 0, or 1 each time; and it is equal to a and b at two different times, then for any integer between a and b, the quantity will be equal to that integer at some point
- Group things (i.e. moves made, part of the configuration, etc.)

1. A number is written in each of the squares of an $m \times n$ grid. It is allowed to switch the sign of all numbers from one row or one column. Prove that eventually it is possible to get a grid, in which the sums of the numbers in every column and every row are non-negative.

2. Ivan has a 52-card deck. He draws the cards from the deck one by one, without putting them back in the deck. Every time before drawing a card he guesses the suit of the card he will draw. He decides to always guess the suit that occurs most frequently in the remaining deck (if there are several such suits, he chooses any one of them). Prove that he will guess the right suit at least 13 times.

3. Two distinct positive integers a, b are written on the board. The smaller of them is erased and instead of it the number $\frac{ab}{|a-b|}$ is written. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

4. A checker is placed in each of the unit squares of a $n \times n$ square, which is part of an infinite chessboard. A *move* is the process of selecting two checkers located in unit squares sharing a side, and using one of the checkers to jump over the other checker into an empty square adjacent by a side to the square in which the other checker is located. The checker that has been jumped over is removed from the board. After several moves it will be impossible to make a move. Prove that this will happen after at least $\lfloor \frac{n^2}{3} \rfloor$ moves.

5. There are 2000 distinct points, every two of which are connected by a line segment. Danny and Cynthia take turns erasing line segments, so that Danny is allowed to erase only one line segment per turn, and Cynthia is allowed to erase two or three line segments per turn. The person after whose move there is a point not connected to any other points loses. Who will win in this game?

6. A $n \times n$ grid is given, $n-1$ squares of which contain a one, and the rest of the squares contain a zero. It is allowed to select a square, subtract 1 from the number in that square, and add 1 to all numbers in the same row and column as this square. Is it possible to get a grid where all numbers in the squares are equal?

7. An infinite strip of paper is given, divided into unit squares, numbered by the integers from left to right (like a number line). Several stones lie in some of the squares, and a square can have more than 1 stone in it. It is allowed to make the following moves:
(a) Remove a stone from each of the squares n and $n+1$ and place one stone into square $n+2$.
(b) Remove two stones from square n and place one stone into each of the squares $n+1, n-2$.
Prove that eventually it is impossible to make any more moves. Also, prove that the final configuration of the stones is always the same regardless of the order in which the moves were made.

8. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Graphs

Some results relevant specifically to graphs:

- The sum of the degrees of the vertices in a graph is even (a degree is the number of edges exiting the graph).
- If a graph with n vertices has no cycles, but there is a path connecting any two vertices, the graph has $n-1$ edges.
- The set of vertices in a graph can be partitioned into two sets A and B such that no two vertices in A are connected by an edge, and no two vertices in B are connected by an edge iff there are no odd cycles in this graph.
- **Hall's Theorem**: In a graph with vertices partitioned into two sets A and B, it is possible to match every vertex in A with a unique vertex in B iff:
  For any set of vertices X in A, the set $C(X)$ of all vertices in B, that are connected by an edge with some vertex in X, is at least as large as X (i.e. $|C(X)| \geq |X|$).

Some useful tricks:

- Take a vertex A and consider the disjoint sets $A_1, A_2, ..., A_n$ so that all vertices in A_1 are connected by an edge to A; all vertices in A_{i+1} are connected by an edge to some vertex in A_i, and not connected by an edge to any of the vertices in $\{A\}, A_1, A_2, ...,$ or A_{i-1}.
- Look at specific parts of the graph satisfying various properties
- Color vertices or edges in the graph
- Define your graph, vertices, and edges in a non-standard way
- Create an algorithm which will produce the desired configuration

1. In a country there are several cities and several roads. Every road connects exactly 2 cities. Out of every city exit at least 3 roads. Prove that there is a cycle, the number of cities in which is not divisible by 3.

2. The inhabitants of a village start getting sick with the flu. One day in the morning some of them ate too much ice cream and got sick; and after that day the only way a healthy person would get sick is if they visited a sick friend. Every person in the village is sick for exactly 1 day, and the next day he is immune to the flu virus - he cannot get sick that day. Despite the pandemic every day a healthy person visits all his sick friends. After the pandemic started, nobody got vaccines.

Prove that:
(a) If some people got a vaccine before the first day when the pandemic started, and were immune to the flu on the first day, the pandemic can last forever.
(b) If nobody was immune to the flu on the first day, eventually the pandemic will end.

3. In a certain group of 12 people, among any 9 people, there are 5 who know each other. Show that in this group there are 6 people who know each other.

4. In a country with 2010 cities, there are several two-way roads. Every road connects exactly 2 cities. It is possible to travel from any city to any city along the roads. Furthermore, it is possible to do this even if any one of the roads is closed. Two construction companies A and B are playing a game. On every turn a construction company selects, if possible, one of the roads and enforces one-way traffic on that road (if there is already one-way traffic traffic on the road, this road is not allowed to be selected). A company loses its license if after its move it is impossible to travel from some city to some other city. Company A goes first. Can one of the companies guarantee that the other company will lose its license?

5. In a country with 2010 cities, there are several roads. Every road connects exactly 2 cities. Through every city there are at most n different non-self-intersecting cycles of odd length. Prove that the cities can be divided into $n+2$ groups, so that any two cities from two different groups are not connected by a road.

6. There are 100 representatives from 25 countries seating at a round table, 4 representatives from each country. Prove that it is possible to divide the representatives into 4 groups with one representative from each group so that no two representatives from the same group are sitting side by side at the table.

7. An $m \times n$ rectangular board is given where m, n are odd integers. The board is covered with 2×1 dominoes, so that no two dominoes overlap and only the bottom left square of the board is empty (i.e. not covered by any dominoes). At any point in time it is allowed to slide a domino so that it covers an empty square and still stays on the board, and none of the other dominoes are moved during this process. As a result, a new square becomes empty. Prove that after several such moves it is possible to make any corner square on the board empty.

8. $2n+3$ players participate in a chess tournament. Every two play exactly one game. The schedule is set so that no two games are played at the same time, and each player, after playing game, is free for at least n next (consecutive) games. Prove that one of the players who plays in the first game will also play in the last game.

Solutions
Combinatorial Geometry

1. No. Call a lattice point even if the sum of its coordinates is even, and call it odd otherwise. Call one of the legs first, and other one second. Then every time a leg is moved from an even point to an even point, or from an odd point to an odd point. If in the beginning one of the legs is at an even point, and the other - at an odd point, the legs cannot switch. If both legs are at even points or both at odd points, consider a new coordinate system on the sheet of paper, with unit length twice as large as unit length in the original coordinate system, and so that the two legs are located at lattice points of the new coordinate system.
(Russian Math Olympiad 1998)

2. For each point P on the boundary of A, consider the homothety with centre P and factor $\frac{2}{3}$, which carries the set A to a set A_P lying inside A. For any three points K, L, M on the boundary of A, the centroid of $\triangle KLM$ lies in each of the sets A_K, A_L, A_M so the three sets have non-empty intersection. By Helly's theorem all sets have non-empty intersection; take a point O in this intersection.
Consider any two points X, X' on the boundary of A such that O lie on XX'. By construction $O \in A_X, O \in A_{X'}$ so $\frac{XX'}{3} \leq OX, OX' \leq \frac{2XX'}{3}$ and the result follows.
(AOPS)

3. We claim that any such set contains at most 4 points. Assume S has more than 4 points. Take A, B, C such that $\triangle ABC$ has the largest possible area over all choices of A, B, C. There is a fourth point D such that $ABCD$ is a parallelogram. It is clear that if a point X is outside $ABCD$ then one of the triangles ABX, BCX, CDX, ADX has area greater than the area of triangle ABC, BCD, CDA, DAB respectively. Each of these triangles has area equal to the area of ABC. Hence there are no points outside of $ABCD$.
Consider a point E inside $ABCD$. There is a point F such that A, B, E, F are vertices of a parallelogram. It is clear that F cannot be inside $ABCD$. This gives a contradiction.
(USA TST 2005)

4. Let A, B be points in S such that AB is maximal. Consider a point C in S such that the distance from C to AB is maximal. It suffices to show the distance from C to AB is at most 1, since then we can cover S by a strip of length 2 such that line AB passes is the middle line in the strip.
We can assume A, B, C are not collinear (otherwise we are done). The triangle ABC can be covered by a strip of length 1, hence one of its altitudes has length at most 1. Since AB is the longest side in the triangle, the altitude to it is the shortest among the three altitudes, so it has length at most 1. Therefore the distance from C to AB is at most 1.
(Balkan Math Olympiad 2010)

5. Place the first circle on the second. Then place a painter at a fixed point on the first circle. Rotate the first circle in one direction, and make the painter paint the point on the second circle, under which the painter is located, whenever some marked point lies on a marked arc. It suffices to show that after one full revolution of the first circle, some point on the second arc is not painted. The resulting painting of the second circle will be the same to the one if the first circle is rotated 1000 times, and on the ith rotation the painter paints the point under him whenever the ith marked point lies on a marked arc. During each revolution less than 1 cm is painted on the second circle, hence after 1000 revolutions less than 1000 cm will be painted on the second circle. The result

follows.
(A Russian Book with Olympiad Problems)

6. If (x_i, y_i) and (x_{i+1}, y_{i+1}) are the coordinates of two consecutive vertices in the polygon, then $x_i + y_i \neq x_{i+1} + y_{i+1} \bmod 2$, since the length of the segment joining the two vertices is an odd integer. Therefore n cannot be odd (otherwise we will have $x_1 + y_1 \equiv x_n + y_n \bmod 2$ which is impossible.
We will show any even $n \geq 4$ works. For every n we will construct a polygon with n vertices $X_1, ..., X_n$ satisfying the conditions of the problem, and furthermore the side $X_i X_{n+1-i}$ is parallel to the $x-$axis.
We proceed by induction. For $n = 4$ take the vertices of the polygon to have coordinates $(0,0), (69,12), (100,12), (109,0)$. Assume we have the polygon with n vertices, we need to add two new vertices A, B. Notice that we can shift all vertices $X_{\frac{n}{2}+1}, X_{\frac{n}{2}+2}, ..., X_n$ in the positive $x-$direction by an arbitrary number of units m. The resulting polygon with the new vertices $X_1, X_2, ..., X_n$ still satisfies the conditions of the problem. Hence it suffices to prove that there exists a trapezoid $ABCD$ with points A, B on the $x-$axis and $CD || AB$, so that all its sides are pairwise distinct odd integers, which are arbitrarily large, and the angles $\angle CAB, \angle DBA$ are arbitrarily small; since then we can shift vertices $X_{\frac{n}{2}+1}, X_{\frac{n}{2}+2}, ..., X_n$, and then let $X_{\frac{n}{2}} X_{\frac{n}{2}+1}$ be side AB and CD be above AB so that C, D are the two new vertices.
Construction of $ABCD$ is not hard. Let x be the height of the trapezoid. We want to find integers y, z and s, t such that $x^2 = z^2 - y^2, x^2 = s^2 - t^2$ so that AC, BD have lengths y, s; and $\tan(\angle CAB), \tan(\angle DBA)$ are $\frac{x}{y}, \frac{x}{t}$; we want these to be arbitrarily small. Take $x = 6k + 3, y = 18k^2 + 18k + 4, z = 18k^2 + 18k + 5, s = 6k^2 + 6k, t = 6k^2 + 6k + 3$. Then since we can take k arbitrarily large, the integers x, y, z, s, t satisfy the conditions we want.
(Chinese TST Quiz 6, 2008; most of the solution is from Adrei Frimu's post on AOPS)

7. Assume the contrary. Notice that for any point of intersection of two lines, there is a red line passing through it.
Consider a line l and points A and B on l, which are points of intersections of l with two red lines m and n respectively. Choose A and B such that AB is maximal. Let C be the point of intersection of m and n. There is a blue line passing through C and intersecting l at a point D; D must be a point on line segment AB (since AB is maximal).
Look at all configurations of 4 lines l', m', n', p' that look like the configuration of l, m, n, p; with lines l', p' being of one color, m', n' of the other color; m', l', n' concurrent and the point of intersection of p' with l' is between the points of intersection of m', n' with l. Find the configuration for which the area of the triangle formed by lines l', m', n' is minimal. Let C', A', B' be the points of intersection of $m', n'; m', l'; l', n'$ respectively. Through the point of intersection of p', l' there is another line q' of the same color as m' and n'. It will intersect the line segment $C'A'$ or $C'B'$; wolog it is line segment $C'A'$. Then lines m', p', l', q' form a configuration with a triangle of smaller area. This is a contradiction.
(Russian Math Olympiad 2002)

8. Call such drawing of diagonals a *triangulation*, and call a triangle *special* if two of its sides coincide with two of the sides of the polygon. Let the number of sides of the polygon be n. The sum of the angles in the polygon is $(n-2)\pi$. The sum of the angles in a triangle is π so there are $(n-2)$ triangles in the triangulation. Hence there are at least 2 distinct special triangles.
Furthermore, there are at most 3 acute angles in any convex polygon, since otherwise the sum of

the angles in the polygon is less than $(n-2)\pi$.
Consider two distinct triangulations T_1, T_2. There are two triangles t_1, t_2 that are special in triangulation T_1. Each of them has an angle that coincides with an angle of the polygon; let the two angles be α_1, α_2. Similarly define two special triangles s_1, s_2 in triangulation T_2, and the two angles β_1, β_2. Since angles α_i, β_i are all acute, it follows that two of them are the same; wolog α_1, β_1 are the same. Then triangles t_1, s_1 are the same. Cut them off and consider the remaining convex polygon with $n-1$ sides. Finish the problem with induction.
(Russian Math Olympiad 2003)

9. Let the sides of the squares be equal to a_i for $i = 1, 2, ..., N$(N is the number of squares).
If some $a_k > 1$ then kth square will cover the unit square. Assume $a_i < 1$ for all i. Each a_i must belong to some interval $[2^{-k_i}, 2^{-k_i+1})$ where k_i are positive integers for all i. Let us decrease every ith square to the square with side length $b_i = \frac{1}{2^{k_i}}$. Its area will decrease by at most 4 times, because $1 \leq \frac{a_i}{b_i} < 2$. Therefore the area of the new squares will be greater than 1.
Let us prove we can tile the unit square with the new squares. Divide the unit square into 4 squares with side length $\frac{1}{2}$. First place the squares with side $\frac{1}{2}$(if they exist). On the non-tiled squares with side $\frac{1}{2}$(if they exist) place the squares with side $\frac{1}{4}$(if they exist), by dividing each non-tiled square with side $\frac{1}{2}$ into 4 equal squares. We will continue this procedure for $k = 3, 4, ...$ by placing squares with side length $\frac{1}{2^k}$ on the on-tiled squares with side length $\frac{1}{2^{k-1}}$, each time dividing them into 4 equal squares.
Because the sum of areas of squares is greater than 1, then eventually we will cover the unit square. Increasing the ith square with side b_i to the square with side a_i, we will get the tiling of the unit square with the given squares.
(Russian Math Olympiad 1979)

10. Call the $2n$ angles into which the lines divide the angle around O - *basic*, and two lines that form a basic angle - *adjacent*. Since no line passes inside a basic angle, then for any two adjacent lines, there is a third line bisecting the other pair of vertical angles (complementary with the basic angles). If this line is rotated by $90°$ around O, it will bisect the simple angles. There are as many "bisecting" lines, as all the lines, hence if the whole figure is rotated by $90°$ around O, each of the rotated n lines will be bisect two vertical simple angles.
Let α be the largest vertical simple angle, and β, θ be two adjacent simple angles, the angle bisectors of which become after the $90°$ degree rotation the sides of α. Then $\alpha = \frac{\beta+\theta}{2}$ hence $\alpha = \beta = \theta$. Rotating by $90°$ in opposite direction the angles β, θ, we see that the two angles adjacent to α are equal to α. Hence all simple angles are equal.
(Russian Math Olympiad 2003)

11. We use induction on the number of colors, n. Base case will be done for $n = 2$. Let S be the left-most square. If it is of color 1, all squares of color 2 have a point in common with it, hence each of the squares of color 2 contains one of the right vertices of S; hence all squares of color 2 can be pinned using 2 pins at those two vertices.
Assume the result holds for n colors; we need to prove it for $n+1$ colors. Consider all squares and find the left-most square S; wolog it has color $n+1$. All squares intersecting S contain one of its right vertices, hence they can all be pinned using 2 pins. Remove from the table all squares of color $n+1$ and all squares intersecting S. Among the remaining squares, each square has one of n colors. Among any n of these squares of pairwise distinct colors, there are two that intersect; or else add in square S and we get $n+1$ squares of pairwise distinct colors, no two of which intersect.

Hence there is a color i, such that all the remaining squares of color i can be pinned using $2n-2$ pins. The squares of color i that are not pinned all intersect square S and can be pinned using 2 pins at the two right vertices of S.
(Russian Math Olympiad 2000)

12. Let $A_1, A_2, ..., A_N$ be the points, and each of the distances A_iA_j is equal to one of the numbers $r_1, r_2, ..., r_n$. Then for each i, all of the points except A_i lie on one of the circles $\omega(A_i, r_1), \omega(A_i, r_2), ..., \omega(A_i, r_n)$ where $\omega(O, r)$ denotes the circle with centre O and radius r.
Introduce a system of coordinates so that the coordinates axes are not parallel to any of the lines A_iA_j. Wolog A_1 has the smallest $x-$coordinate among the points. Among the lines A_1A_i find the one with the largest absolute value of the slope; wolog it is A_1A_2. Then all points $A_3, A_4, ..., A_N$ lie in the same half-plane π with respect to line A_1A_2.
Each of the points $A_3, A_4, ..., A_N$ is the intersection of circles $\omega(A_1, r_k), \omega(A_2, r_l)$ for some $k, l \in \{1, 2, ..., n\}$. Each of the n^2 pairs of these circles has at most one point of intersection in π. Hence among $N-2$ points $A_3, A_4, ..., A_N$ at most n^2 are distinct. Hence $N-2 \leq n^2$ so $N \leq n^2+2 = (n+1)^2$.
(Russian Math Olympiad 2004)

Processes

1. Among all grids we can get (there are finitely many of them) take the one with the largest possible sum of all numbers. Assume the sum of the numbers in one of the rows in this grid is negative. If we switch the sign of the numbers in the grid, we get a grid with a larger sum of numbers. The result follows.
(Russian Math Olympiad 1961)

2. Call a characteristic of a deck the number of cards of the most frequently occurring suit. Every time the characteristic stays the same or decreases by 1. If it decreases by 1, Igor guessed the right suit. In the beginning the characteristic is 13, in the end it is 0. Hence it decreased by 1 exactly 13 times.
(Russian Math Olympiad 1998)

3. Let us write numbers on a second board. Every time numbers x, y appear on the first board, we will write down numbers $\frac{ab}{x}, \frac{ab}{y}$ on the second board. Then when numbers x, y are on the first board, $x < y$, and the operation is performed, then on the other board the pair $(\frac{ab}{x}, \frac{ab}{y})$ will be replaced by $(\frac{ab}{y}, \frac{ab}{x} - \frac{ab}{y})$ (verify this). This is just Euclid's algorithm, so eventually both numbers on the second board will be equal to $\gcd(a, b)$. At that point both numbers on the first board will be equal.
(Russian Math Olympiad 1998)

4. Call a move *internal* if the checker jumps into the $n \times n$ square S, and *external* if the checker jumps outside of S. Assume we got to a position from which it is impossible to make any moves, and k internal and l external moves have been made.
There are at least $\lfloor \frac{n^2}{2} \rfloor$ empty squares in S (or else two checkers are in adjacent squares); an internal move increases the number of empty squares in S by at most 1; an external move increases this number by at most 2. Hence

$$2l + k \geq \lfloor \frac{n^2}{2} \rfloor \tag{1}$$

Assume n is even. Divide S into $\frac{n^2}{4}$ 2×2 squares. In every such square there were at least 2 moves which involved the checkers from this square (either the checker was used to jump in a move, or it was jumped over). In every internal move, the checkers used were from at most 2 squares; in every external moves, the checkers used were from at most 1 square; hence

$$2k + l \geq 2\frac{n^2}{4} \tag{2}$$

Adding (1) and (2) the result follows for all even n.
For odd $n = 2m + 1$ consider the cross formed by taking the third column from the left and third row from the top. Divide the cross into one unit square and $2m$ dominoes made up of two squares. Divide the remaining part into m^2 2×2 squares. Every internal move uses at checkers from at most 2 figures among the figures we selected; and every external move uses at most one such figure. Hence

$$2k + l \geq 2m^2 + 2m \tag{3}$$

Then add (1) and (2) and get the result for all odd n (you might need to consider some cases of n mod 3).
(Russian Math Olympiad 1999)

5. Cynthia wins. Let us divide the points into 4 equal groups A, B, C, D and index the points from 1 to 500 in each group. We will prove Cynthia can always make a move so that after her move the number of line segments exiting from the points in each group is the same.

If Danny erases a line segment connecting two points from the same group, i.e. A_iA_j, Cynthia will erase B_iB_j, C_iC_j, D_iD_j. If Danny erases a line segment connecting two points from different groups and having different indices, i.e. A_iB_j Cynthia will erase A_jB_i, C_iD_j, D_iC_j.

If Danny erases a line segment connecting two points from different groups but with different indices, i.e. A_kB_k Cynthia does the following. There are 6 line segments connecting pairs of points from A_k, B_k, C_k, D_k. Cynthia can always erase two other line segments among these 6 so that the remaining three have an endpoint in common. For example she can erse A_kC_k, B_kC_k so the remaining three segments are A_kD_k, B_kD_k, C_kD_k. If Danny ever erases one of these line segments, there must exist $l \neq k$ such that A_lD_k, B_lD_k or C_lD_k is not yet erased (otherwise no line segments are exiting from D_k). Then the same thing will hold for B_k, A_k, C_k so Cynthia can erase the other two segments from A_kD_k, B_kD_k, C_kD_k and not lose.

(Russian Math Olympiad 2000)

6. No. We first prove there is a 2×2 square in the original grid with three zeroes and one 1. This follows from the fact that there is a row containing only zeroes, hence there is a row containing only zeroes adjacent to a row containing a 1.

Consider any 2×2 square K in the grid; let a, b, c, d be the numbers in top left, top right, bottom left, bottom right corners of this square, respectively. After the operation is performed let these numbers be a', b', c', d' respectively. Let $S = (a + d) - (b + c); S' = (a' + d') - (b' + c')$. We will show $D \equiv D' \bmod 3$.

Call the square on which the operation is performed cool. If the cool square is outside K, it is easy to verify $D = D'$.

If the cool square is inside K, wolog it is in the top left corner of K; then after operation is performed $a' = a - 1, b' = b + 1, c' = c + 1, d' = d$ hence $D = D' - 3$. Hence $D \equiv D' \bmod 3$. The result follows.

(Russian Math Olympiad 1998)

7. Let r be the positive root of the equation $x^2 - x - 1 = 0$. For any configuration A of the stones, let a_i be the number of stones in square i and let $w(A) = \sum a_ir^i$. Then $w(A)$ stays constant (verify this yourself).

We now show the process cannot repeat forever by induction on n, the total number of stones. For sufficiently large M, $r^M > w(A)$ since $r > 1$. Hence no stones can ever be in a square with index greater than M. Hence eventually a stone will appear in a square and will not be moved from there anymore. Throw out this stone when this happens. Now use the induction hypothesis.

Assume it is possible to get two distinct *final* configurations A, B with a_i, b_i respectively being the number of stones in square i; so that for some j, $a_j \neq b_j$. Assume $w(A) = w(B)$. Choose the largest k for which $a_k \neq b_k$.; wolog $a_k = 0, b_k = 1$. Throw out stones on squares $k+1, k+2, ...$ in both configurations A, B (they are identical in A and B). For the remaining configurations A', B' we have

$$w(B') \geq r^k = \frac{r^{k-1}}{1 - r^{-2}} = r^{k-1} + r^{k-3} + ... > w(A')$$

since in configuration A' it is impossible to make any more moves, hence no two stones in A' are in one square, and no two are in adjacent squares. Hence $w(A) \neq w(B)$ and the result follows.

(Russian Math Olympiad 1997)

8. Label the two rooms X and Y. Let $2n$ be the size of the largest clique. Place all students in one such clique C in X and everyone else in Y. Let $s(X), s(Y)$ be the sizes of the largest cliques in rooms X and Y respectively. Move students from X to Y one by one. Every time a student is moved, $d = s(X) - s(Y)$ decreases by 1 or 2. Hence at one point it will become 0 or -1.
It suffices to consider the case $d = -1$. Then $s(X) = k, s(Y) = k+1$ for some positive integer k. If some student from C is in Y and is not in one of the cliques in Y of size $k+1$, move this student to X so that $d(X) = d(Y) = k+1$. Hence we can assume each clique in Y of size $k+1$ contains $2n-k$ students from C. Then in each such clique there are $2(k-n)+1 \geq 1$ students no in C. Move these students one by one to X. We will show every time the size of the largest clique in X is k. Assume not, then the last student that has been moved is friends with all students in C, which contradicts the fact that $2n$ is the size of the largest clique. Hence at some point $d(X) = d(Y) = k$ and we are done.
(IMO 2007/3; proposed by Russia. Surprising, eh?)

Graphs

1. Assume there is a graph G in which the length of every cycle is divisible by 3. Take such graph with the smallest possible number of vertices, and look at any cycle C in this graph containing the cities $A_1, A_2, ..., A_{3k}$. Assume there is a path connecting cities A_m, A_n which does not use any of the edges in C. The union of this path with C gives two cycles C_1, C_2 each of which has length (i.e. number of edges in it) divisible by 3. Hence the length of the path is also divisible by 3. Then for any vertex B not in C, it cannot be connected by an edge with two cities from C.
Replace the cycle C by a vertex A. (!) Connect it by an edge with those vertices that were previously connected by an edge with some vertex in C. It is clear the new graph G' has less vertices, there are at least 3 edges coming out of every vertex, and every cycle in the new graph has length which is divisible by 3. This contradicts the choice of G as the smallest possible graph.

2. (a) The pandemic will last forever if for example on the first day one person is sick, one is healthy, and one is immune, and every two people are friends.
(b) Let us prove nobody can get sick twice. Divide the people into groups $G_1, G_2, ...$ so that G_1 is the group of all people who got sick on day 1; G_2 contains all people who are not in G_1 and are friends with somebody from G_1, G_3 contains all people who are not in G_1 or G_2 and are friends with somebody from G_2, etc. If two people are friends, the indices of the groups they belong to differ by 1. By induction it is easy to prove that on day i only people from G_i are sick, and only people from G_{i-1} are immune to the flu.
(Russian Math Olympiad 1980)

3. Look at the graph where vertices are people, two vertices are connected by an edge if the two corresponding people *do not* know each other. (!) If this graph has no odd cycles, we can split it into two groups; in each group no two vertices are connected, so every two people know each other. We can then find 6 people who know each other.
Assume there is an odd cycle. Consider the minimal odd cycle of length k, we have 5 cases:
Call a person *good* if he is in the cycle and *bad* if he is not in the cycle.
$\underline{k = 3}$: If among the bad people every two know each other we are done. Otherwise we can find two that don't know each other. For the other 7 bad people, among any four, we can find three who know each other. (Take those four, the 3 good people, and the two who don't know each other). Then any 2 edges in the subgraph on these 7 people have a vertex in common. Furthermore, this vertex must be the same for any 2 edges. Removing this vertex gives 6 vertices no two of which are connected by an edge, giving the required 6 people.
$\underline{k = 5}$: For the 7 bad people, among any four, we can find three who know each other. (Take those four and the five good people). This case is done in the same way as the previous one.
$\underline{k = 7}$: For any two bad people, if we take them and the 7 good people, we get 9. Hence any 2 bad people do not know each other. If there is some good person who does not know any bad person, we are done. Otherwise there is a bad person A and good people B, C, so that AB and AC are edges in the graph. By choice of minimal cycle the only way this can happen is if as we go along the cycle from B to C, we visit only one more vertex D, so that BD, DC are edges. But then D does not know any bad people, since if we remove D from the cycle and insert A, we get a cycle of length 7, and no two people in the complement of this cycle know each other.
$\underline{k = 9}$: Impossible, as if we take the 9 good people, among any 5 we can find two who do not know each other.
$\underline{k = 11}$: The bad person A does not know at most 2 good people (otherwise there is an odd cycle

with smaller length). Let the two good people be B, C. Arguing in the same way as when $k = 7$ we get that BD, DC are edges in the cycle. Now take person A, D, and every second person as we go along the cycle starting from D, until we have 6 people in total. No two of them will know each other.

4. No. Consider the graph where vertices are cities and edges are roads. At any point in the game choose any edge that is not yet oriented. We will show that it is possible to orient it in a way so that it is still possible to get from any vertex to any other vertex.
Let the ends of the edge be x and y, and call the edge xy. If there is a path P from x to y (or from y to x) without using the edge xy, we direct the edge xy from y to x (or from x to y, respectively). Then in any path between any two cities that used the edge xy in direction from x to y, the edge xy can be replaced by P, and everything is fine.
Assume now there is no path P from x to y or from y to x that does not use edge xy. Let S be the set of all vertices to which one can get from x without using edge xy, and T be the set of vertices to which one can get from y, without using edge xy. For any vertex in S, there is a path from it to vertex y. By the assumption, this path must use edge xy. Therefore for any vertex in S, there is a path from it to x that does not use edge xy. Then for any two vertices in S, it is possible to get from one to the other without using edge xy.
Similarly for any two vertices in T, it is possible to get from one to the other without using edge xy.
Since in the original graph it is possible to get from any vertex to any other vertex, then every vertex belongs to exactly one of S and T. If edge xy is removed, it is possible to get from any vertex to any other vertex so there is a vertex p in S and vertex q in T so that pq is an edge, wolog directed from p to q. But there is a path from x to p, and from q to y not using edge xy, so there is a path from x to y not using edge xy, contradicting our assumption.

5. We use induction on the number of cities. When there is 1 city, the result is obvious.
Now, assume the result holds for k cities and we want to prove it for $k + 1$ cities. Look at a graph G with $k+1$ cities satisfying the conditions of the problem and remove city X and all roads coming out of it. By the induction assumption, the cities in the remaining graph G' can be colored in $n+2$ colors, so that no two cities of the same color are connected by a road.
For every $m = 2, 3, ..., n + 2$ consider the graph G'_m made up only of cities of colors 1 and m and the roads between these cities. This graph is bipartite hence has no odd cycles. Let G_m be the graph obtained by adding to G'_m the city X and all roads connecting it with the cities in G'_m.
If for some m, the graph G_m has no odd cycle through X, then it has no odd cycles at all. Hence it is bipartite and we can recolor the cities in G_m in colors 1 and m so that no two cities of the same color are connected by a road. Now add back all the remaining cities and roads. The resulting colored graph G will also satisfy the condition that no two cities of the same color are connected by a road, and we are done.
Otherwise for every $m = 2, 3, ..., n + 2$ the graph G_m has no odd cycle C_m through X. This cycle must pass through some vertex of color m. Therefore all cycles $C_2, C_3, ..., C_{n+2}$ are pairwise distinct. We get $n + 1$ odd cycles through X, a contradiction.

6. *Lemma:* We are given $2n$ people from n countries, two from each country. Assume it is possible to divide the people into n pairwise disjoint pairs, so that these pairs are the only pairs of friends among then $2n$ people. Then it is possible to divide the people into 2 groups, each containing one person from each country, so that no two people from the same group are friends.

Proof: Number the countries from 1 to n; call two people from the same country *comrades*. We will construct two groups A, B. Place one person from country 1 in group A and in group B place his comrade; place his friend in group A; wolog that friend is from country i; in group B place his comrade; place his friend in group A, etc. This process stops when the next person to be placed has already been placed; this person must be the first person from country 1. He is placed in group A which is exactly what we want.
If the process stops and there are still people remaining, we do the same thing with the remaining people. The lemma is proved.
Let us now solve the problem. Consider 4 people in a country X. Create new countries X', X'' and place 2 of the people from those 4 in country X' and the other 2 in country X''. Divide the people around the table into 50 pairs, so that in each pair people are sitting side by side; call them friends. Apply the lemma; we can divide the people into 2 groups, each with 50 people, and in each group no two people are friends and they are all from different "new" countries. In each group, for every person there is at most one other person in that group sitting beside him. Hence we can again split the people in the group into pairs of neighbors and apply the lemma again (now using the original countries) to get 4 groups, so that in each group no two people sit side by side.
(Russian Math Olympiad 2003)

7. Number the rows and columns from top to bottom and from left to right, respectively. Call a square located in an odd-numbered row and odd-numbered column *special*. Notice that an empty square is always special.
If a special square A is covered by a domino, then one of the short sides of the domino is adjacent to another special square B. Draw an arrow from the centre of square A to the centre of square B. Notice if B is empty, then by sliding the domino we can move the empty square to A.
Draw these arrows for all special squares. If there is a directed path along the arrows from a special square A to the empty square, it is possible to move the empty square to A.
A path along these arrows either ends in an empty square, or is a cycle. Let us show that if it is a cycle, then it bounds a polygon with an odd number of squares inside. Look at any such cycle. Consider a new square grid with vertical and horizontal lines passing through the centres of the special squares. The cycle will bound a polygon made up of squares of side length 2 in the new grid. We now use induction on the number of these squares. For one square the result is obvious. A polygon made up of k squares is obtained by adding a 2×2 square on the boundary of a polygon made up of $k-1$ squares; it is easy to check then the number of unit squares inside the cycle increases by 2 or 4.
Any polygon bounded by a cycle must contain an even number of unit squares, since it can be completely tiled by dominoes. Hence all paths along the arrows end in the empty square. The corner squares are special, and the result follows.
(Russian Math Olympiad 1998)

8. For every player call a break the number of games between two consecutive matches (including the second match). All breaks are at least $n+1$. Number the games in order of their occurrence. Consider $n+3$ consecutive games $g_1, g_2, ..., g_{n+3}$ with $2n+6$ players in them. We claim at most 3 players could participate in two of these games. This is clear since they would have to play games $(g_1, g_{n+2}), (g_2, g_{n+3}), (g_1, g_{n+3})$ or else two of them play each other twice. There are $2n+3$ players in total, hence the breaks of the players of g_1 are $n+1$ and $n+2$. Hence every break is equal to $n+1$ or $n+2$.
It suffices to find a player all whose breaks were equal to $n+2$, since the total number of games is

$\frac{(2n+3)(2n+2)}{2} = (n+2)(2n+1)+1$ he will be the one playing in the first and the last games. Assume the contrary. Find the player X who was the last to have a break of $n+1$ games. Assume he played a player Z in game c at the end of this break. Let a be the number of the last game Z played, right after which he had a break of $n+1$ games. Then between games $a+n+1$ and c, Z had all breaks of length $n+2$; so $c = a+(n+1)+k(n+2)$. All the breaks that X had before game c were $n+2$. If he had at least k such breaks, then he had to play game a with Z. Otherwise he had at most $k-1$ such breaks, and his first game was at least $c-(n+1)-(k-1)(n+2) = a+(n+2) \geq n+3$. Hence at most $2n+2$ games played in the first $n+2$ games, so the games 1 and $n+2$ were played by the same two players, a contradiction.
(Russian Math Olympiad 2008)

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