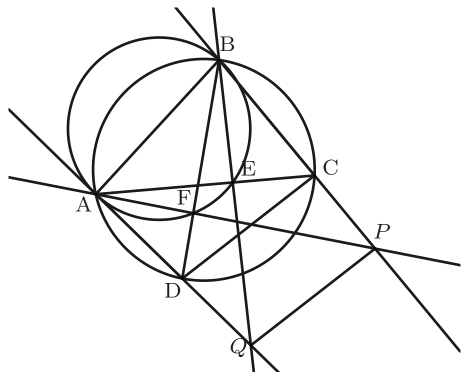
2019 Winter Camp Mock Olympiad Solutions

1. We have

$$\angle QAP = \angle QAC - \angle PAC = \angle DAC - \angle FAE = \angle DBC - \angle FBE = \angle QBP,$$

whence $QABP$ is cyclic. In particular, $\angle AQP = 180° - \angle ABP = 180° - \angle ABC = \angle ADC$, whence CD and PQ are parallel.



2. Let x denote the largest even number such that both x and $x+2$ divide $4k$. Since $2, 4 \mid 4k$, such an x exists. If $x+1 \mid 4k$, then since $x+1$ is odd, we have $2x+2 \mid 4k$. But one of $x, x+2$ is equivalent to 2 modulo 4, whence at least one of $2x, 2(x+2) = 2x+4$ divides $4k$. In particular, either $2x, 2x+2$ or $2x+2, 2x+4$ divide $4k$, contradicting the definition of x. Therefore our assumption is wrong, and $x+1$ is not a divisor of $4k$. If $d_i = x+2$, then $d_{i-1} = x$ necessarily, and the statement has been proven.

3. We prove the statement by induction on $x_1 + x_2 + \cdots + x_n$. Let $S = \{x_1, x_2, \ldots, x_n\}$ and suppose that S contains an element with at least two digits. Let $\overline{xa}$ be the largest element of S where a is a single digit. The given condition implies $\overline{x} \notin S$. Let S' be formed by removing any of $\overline{x0}, \overline{x1}, \ldots, \overline{x9}$ from S and adding in $\overline{x}$. Since $\overline{xa} \in S$, no prefix of $\overline{x}$ can be in S and S' is also a set satisfying the condition with a lower sum of elements since $\overline{xa} > \overline{x}$. Observe that

$$\sum_{y \in S} \frac{1}{y} - \sum_{y \in S'} \frac{1}{y} \leq \sum_{b=0}^{9} \frac{1}{10x+b} - \frac{1}{x} < \sum_{b=0}^{9} \frac{1}{10x} - \frac{1}{x} = 0$$

Therefore by the induction hypothesis applied to S', we have that

$$\sum_{y \in S} \frac{1}{y} < \sum_{y \in S'} \frac{1}{y} < 3$$

completing the inductive step. It suffices to show that the statement is true for all sets S all of whose elements have one digit. This follows from the fact that

$$1 + \frac{1}{2} + \cdots + \frac{1}{9} = 2.8289\ldots < 3$$

which completes the proof.

4. The task is to prove that for any connected finite graph G with chromatic number $\chi(G) \geq n+1$, there are $n(n-1)/2$ edges that can be removed from G without disconnecting it. We prove this by induction on the number of vertices in G. The claim trivially holds if G contains one vertex. Now consider an arbitrary graph G with $\chi(G) = n+1$ and let v be the leaf of a spanning tree of G. This implies that $G' = G - \{v\}$ is connected. If G' satisfies that $\chi(G') \leq n-1$, then colouring v with a new colour implies that $\chi(G) \leq n$, which is a contradiction. If $\chi(G') = n+1$ then the induction hypothesis implies that $n(n-1)/2$ edges can be removed from G' without disconnecting it. Adding v and any edge incident to v to G' implies the same is true for G. Now consider the case where $\chi(G') = n$ and fix an n-colouring of G'. Since G is not n-colourable, v must be adjacent to at least one vertex of each colour in this colouring of G' since otherwise v could be assigned one of the first n colours. Thus v has at least n neighbours. Applying the induction hypothesis to G' guarantees $(n-1)(n-2)/2$ edges can be removed without disconnecting G'. Then adding in v along with a single edge from v to a vertex in G' ensures the overall graph is connected. The total number of edges removed from G is at least $(n-1)(n-2)/2 + (n-1) = n(n-1)/2$, completing the induction.

5. Assume for contradiction that there are k primes $p_1, p_2, \ldots, p_k$ such that all of the prime divisors of $a_i + a_j$ are contained in $\{p_1, p_2, \ldots, p_k\}$. Let $v_p(n)$ denote the largest power of p dividing n. We now form k graphs $G_1, G_2, \ldots, G_k$. Let i and j be joined by an edge in G_t if
$$v_{p_t}(a_i + a_j) \geq \min\{v_{p_t}(a_i), v_{p_t}(a_j)\} + 1$$
if $p_t \neq 2$ and if $v_{p_t}(a_i + a_j) \geq \min\{v_{p_t}(a_i), v_{p_t}(a_j)\} + 2$ if $p_t = 2$. We first observe that every pair of $i, j \in \{1, 2, \ldots, 2^k + 1\}$ are joined by an edge in at least one G_t. Suppose not and $v_{p_t}(a_i + a_j) \leq \min\{v_{p_t}(a_i), v_{p_t}(a_j)\}$ for each t with $p_t \neq 2$ and $v_{p_t}(a_i + a_j) \leq \min\{v_{p_t}(a_i), v_{p_t}(a_j)\} + 1$ if $p_t = 2$. Since $p_1, p_2, \ldots, p_k$ are the only divisors of $a_i + a_j$, we have
$$a_i + a_j \leq 2\min\{a_i, a_j\}$$
which is a contradiction since a_i and a_j are distinct. We now will show that each G_t is bipartite. First, if a_i and a_j are joined by an edge in G_t, then we must have $v_{p_t}(a_i) = v_{p_t}(a_j) := N$. Furthermore, we have $p_t^{1+v_2(p_t)} \mid \frac{a_i}{p_t^N} + \frac{a_j}{p_t^N}$. If G_t is not bipartite, then there exists an odd cycle in G_t, say $a_1, a_2, \ldots, a_m, a_1$ with m odd. Therefore we have
$$\frac{a_1}{p_t^N} \equiv -\frac{a_2}{p_t^N} \equiv \frac{a_3}{p_t^N} \equiv \cdots \equiv \frac{a_m}{p_t^N} \equiv -\frac{a_1}{p_t^N} \pmod{p_t^{1+v_2(p_t)}},$$
and so $2a_1 \equiv 0 \pmod{p_t^{N+1+v_2(p_t)}}$. But this implies that $v_{p_t}(a_1) \geq N+1$, contradiction. Therefore G_t is bipartite.
Now, we have k graphs $G_1, \ldots, G_k$ which are all bipartite and have the same vertex set. The bipartiteness divides each graph into two classes, and so a vertex can fall in 2^k distinct classes over all graphs. Since we start with $2^k + 1$ vertices, there exists a pair of vertices i, j which are in the same class in each graph. But then there is no edge between i and j on all of $G_1, \ldots, G_k$, a contradiction, and the problem statement follows.