Number theory

August 31, 2020

Problem 1. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2}\geq\frac{a+b+c}{2}$$

Problem 2. *It is known that positive integers a and b have different parity. Moreover, A and B are disjoint sets with $A \cup B = \mathbb{N}$. Prove that either A or B contains two numbers, difference of which is either a or b.*

Problem 3. *Let $n \geq 2$. Prove that there exist n distinct positive integers, so that sum of each two divides their product.*

Problem 4. *Let $n > 0$ and $k > 2$. Prove that $2^k - 1$ does not divide $2^n + 1$.*

Problem 5. *Prove that if m, n are distinct positive integers, then $2^{2^m} + 1$ and $2^{2^n} + 1$ are co-prime.*

Problem 6. *Find all positive numbers, which are co-prime with all the members of sequence $a_n = 2 \cdot 3^n + 4^n + 12^n - 1$.*

Problem 7. *a) There are $2n+2$ points on the plane, so that no three of them are collinear. Prove that there exists a line, which contains 2 of these points, and divides plane into two parts containing exactly n of these points.*
b) There are $2n+3$ points on the plane, so that no three of them are collinear, and no four of which do not lie on the same circle. Prove that there exists a circle, which contains 3 of these points, and divides plane into two parts containing exactly n of these points. c)

There are $4n$ red points on the plane, so that no three of them are collinear. Prove that it is possible to form n quadrilaterals (not necessarily convex), which do not intersect and have as their vertices the red points.