USAMO 2007 Solution Notes

Compiled by Evan Chen

22 September 2022

This is an compilation of solutions for the 2007 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k - 1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. (For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$.) Prove that for any n the sequence a_1, a_2, $\ldots$ eventually becomes constant.

2. Decide whether it possible to cover all lattice points in $\mathbb{R}^2$ by an (infinite) family of disks whose interiors are disjoint such that the radius of each disk is at least 5.

3. Let S be a set containing $n^2 + n - 1$ elements. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

4. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells (equivalently, a polyomino with n cells). A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

5. Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

6. Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω.

   Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

   $$8P_A Q_A \cdot P_B Q_B \cdot P_C Q_C \le R^3$$

   with equality if and only if triangle ABC is equilateral.

§1 USAMO 2007/1, proposed by Sam Vandervelde

Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k - 1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. (For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$.) Prove that for any n the sequence $a_1, a_2, \ldots$ eventually becomes constant.

For each k, the number

$$b_k := \frac{1}{k}(a_1 + \cdots + a_k)$$

is a nonnegative integer.

> **Claim** — The sequence (b_k) is eventually constant.

Proof. Since

$$b_{k+1} = \frac{a_1 + \cdots + a_k + a_{k+1}}{k+1} \le \frac{kb_k + k}{k+1} < b_k + 1$$

and therefore $b_{k+1} \le b_k$ for all k. Hence the sequence b_k must eventually be constant. □

This can only happen once the sequence is constant: indeed if N is an index such that $b_N = b_{N+1} = \cdots = b$ then

$$a_k = (k+1)b_{k+1} - k \cdot b_k = (k+1) \cdot b - k \cdot b = b$$

for any $k \ge N$. In other words, $a_N = a_{N+1} = \cdots = b$ as well.

§2 USAMO 2007/2, proposed by Gregory Galperin

Decide whether it possible to cover all lattice points in $\mathbb{R}^2$ by an (infinite) family of disks whose interiors are disjoint such that the radius of each disk is at least 5.

The answer is no.

Assume not. Take a disk $\odot O$ not touching any member of the family, and then enlarge it until it is maximal. Then, it must be tangent to at least three other disks, say $\odot A$, $\odot B$, $\odot C$. Suppose WLOG that $\angle AOB \leq 120°$. Denote the radii of $\odot O$, $\odot A$, $\odot B$ by r, a, b.

But the Law of Cosines gives

$$(a+b)^2 \leq (a+r)^2 + (b+r)^2 + (a+r)(b+r)$$

which rewrites as

$$12r^2 \geq (a-3r)(b-3r) \geq (5-3r)^2$$

which one can check is impossible for $r \leq 1/\sqrt{2}$. Thus $r > 1/\sqrt{2}$.

In particular $(\odot O)$ must contain a lattice point as it contains a unit square.

> **Remark.** The order of the argument here matters in subtle ways. A common approach is to try and reduce to the "optimal" case where we have three mutually tangent circles, and then apply the Descarte circle theorem. There are ways in which this approach can fail if the execution is not done with care. (In particular, one cannot simply say to reduce to this case, without some justification.)
>
> For example: it is not true that, given an infinite family of disks, we can enlarge disks until we get three mutually tangent ones. As a counterexample consider the "square grid" in which a circle is centered at $(10m, 10n)$ for each $m, n \in \mathbb{Z}$ and has radius 5. Thus it is also not possible to simply pick three nearby circles and construct a circle tangent to all three: that newly constructed circle might intersect a fourth disk not in the picture.
>
> Thus, when constructing the small disk $\odot O$ in the above solution, it seems easiest to start with a point not covered and grow $\odot O$ until it is tangent to *some* three circles, and then argue by cosine law. Otherwise it not easy to determine which three circles to start with.
>
> In all solutions it seems easier to prove that a disjoint circle of radius $1/\sqrt{2}$ exists, and then *finally* deduce it has a lattice point, rather than trying to work the lattice point into the existence proof.

§3 USAMO 2007/3, proposed by Andras Gyarfas

Let S be a set containing $n^2 + n - 1$ elements. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

We present two solutions which are really equivalent, but phrased differently. We refer to the two classes as "red" and "blue", respectively.

First solution (Grant Yu) We define a set of $n + 1$ elements to be *useful* if it has n-element subsets in each class.

Consider a **maximal collection of disjoint useful sets** and assume there are p such sets. Then, let T be the set of elements remaining (i.e. not in one of chosen useful sets).

> **Claim** — All subsets of T of size n are the same color.

Proof. Assume there was a red set R in T. Replace the elements of R one by one until we get to any other subset R' of T. At each step, because no sets of T form a useful set, the set remains red — so R' is red too. Since R' is arbitrary, this proves the claim. $\square$

We have $|T| = n^2 + n - 1 - p(n+1)$, and in particular $p < n$. WLOG all sets in T are red. We can extract another red set from each of our chosen useful sets. So we can get at least

$$p + \left\lfloor \frac{|T|}{n} \right\rfloor = p + \left\lfloor n + 1 - p - \frac{1+p}{n} \right\rfloor \geq p + (n - p) = n.$$

Second solution (by induction) We prove more strongly that:

> **Claim** — Let S be a set containing $k \cdot (n + 1) - 1$ elements. Then we can find k pairwise disjoint sets of the same color.

The proof is by induction on $k \geq 1$. The base case $k = 1$ this is immediate; $\binom{S}{n}$ is a single set.

For the inductive step, assume for contradiction the problem fails. Let T be any subset of S of size $(k-1)(n+1) - 1$. By the induction hypothesis, among the subsets of T alone, we can already find $k - 1$ pairwise disjoint sets of the same color. Now $S \setminus T$ has size $n + 1$, and so we would have to have that all $\binom{n+1}{n}$ subsets of $S \setminus T$ are the same color.

By varying T, the set $S \setminus T$ ranges over all of $\binom{S}{k+1}$. This causes all sets to be the same color, contradiction.

> **Remark.** Victor Wang writes the following:
>
> > I don't really like this problem, but I think the main motivation for generalizing the problem is that the original problem doesn't allow you to look at small cases. (Also, it's not initially clear where the $n^2 + n - 1$ comes from.) And pretty much the simplest way to get lots of similarly-flavored small cases is to start with $k = 2, 3$ in "find the smallest $N(n,k)$ such that when we partition the n-subsets of a $\geq N(n,k)$-set into 2 classes, we can find some k pairwise disjoint sets in the same class".

§4 USAMO 2007/4, proposed by Reid Barton

An animal with n cells is a connected figure consisting of n equal-sized square cells (equivalently, a polyomino with n cells). A dinosaur is an animal with at least 2007 cells. It is said to be primitive it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

In fact it's true for any tree with maximum degree ≤ 4. Here is the solution of Andrew Geng.

Let T be such a tree (a spanning tree of the dinosaur graph) which corresponds to a primitive dinosaur.

> **Claim** — There exists a vertex v such that when v is deleted, no dinosaurs result.

Proof. Assume for contradiction that all vertices are bad (leave a dinosaur when deleted). Consider two adjacent vertices v, w in T. By checking possibilities, one sees that, say, the dinosaur in $T - v$ contains w and the dinosaur of $T - w$. We can repeat in this way; since T is acyclic, this eventually becomes a contradiction. $\square$

When this vertex is deleted, we get at most 4 components, each with ≤ 2006 vertices, giving the answer of $4 \cdot 2006 + 1 = 8025$. The construction is easy (take a "cross", for example).

§5 USAMO 2007/5, proposed by Titu Andreescu

Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

We prove this by induction on n by showing that

$$\frac{X^7 + 1}{X + 1} = X^6 - X^5 + \cdots + 1$$

is never prime for $X = 7^{7^n}$, hence we gain at least two additional prime factors whenever we increase n by one.

Indeed, the quotient may be written as

$$(X + 1)^6 - 7X \cdot (X^2 + X + 1)^2$$

which becomes a difference of squares, hence composite.

§6 USAMO 2007/6, proposed by Sung-Yoon Kim

Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω.

Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \leq R^3$$

with equality if and only if triangle ABC is equilateral.

It turns out we can compute P_AQ_A explicitly. Let us invert around A with radius $s - a$ (hence fixing the incircle) and then compose this with a reflection around the angle bisector of $\angle BAC$. We denote the image of the composed map via

$$\bullet \mapsto \bullet^* \mapsto \bullet^+.$$

We overlay this inversion with the original diagram.

Let P_AQ_A meet ω_A again at P and S_A again at Q. Now observe that ω_A^* is a line parallel to S^*; that is, it is perpendicular to $\overline{PQ}$. Moreover, it is tangent to $\omega^* = \omega$.

Now upon the reflection, we find that $\omega^+ = \omega^* = \omega$, but line $\overline{PQ}$ gets mapped to the altitude from A to $\overline{BC}$, since $\overline{PQ}$ originally contained the circumcenter O (isogonal to the orthocenter). But this means that ω_A^* is none other than the $\overline{BC}$! Hence P^+ is actually the foot of the altitude from A onto $\overline{BC}$.

By similar work, we find that Q^+ is the point on $\overline{AP^+}$ such that $P^+Q^+ = 2r$.
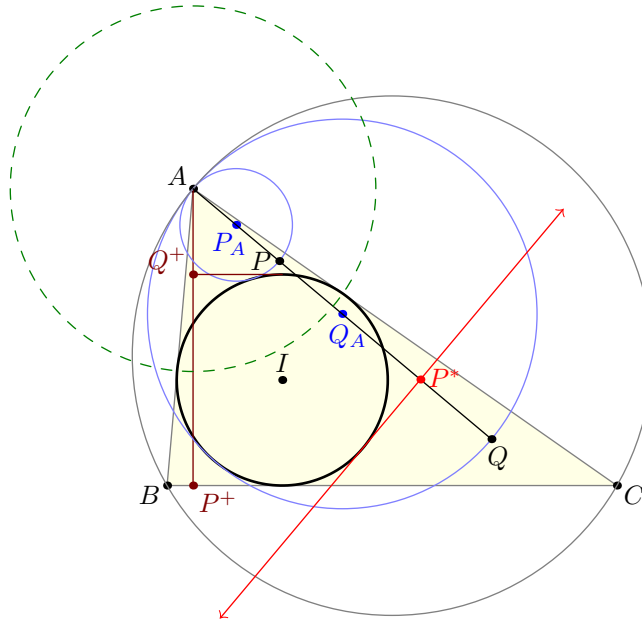
Now we can compute all the lengths directly. We have that

$$AP_A = \frac{1}{2}AP = \frac{(s-a)^2}{2AP^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a}$$

and

$$AQ_A = \frac{1}{2}AQ = \frac{(s-a)^2}{2AQ^+} = \frac{1}{2}(s-a)^2 \cdot \frac{1}{h_a - 2r}$$

where $h_a = \frac{2K}{a}$ is the length of the A-altitude, with K the area of ABC as usual. Now it follows that

$$P_A Q_A = \frac{1}{2}(s-a)^2 \left(\frac{2r}{h_a(h_a - 2r)} \right).$$

This can be simplified, as

$$h_a - 2r = \frac{2K}{a} - \frac{2K}{s} = 2K \cdot \frac{s-a}{as}.$$

Hence

$$P_A Q_A = \frac{a^2 r s(s-a)}{4K^2} = \frac{a^2(s-a)}{4K}.$$

Hence, the problem is just asking us to show that

$$a^2 b^2 c^2 (s-a)(s-b)(s-c) \le 8(RK)^3.$$

Using $abc = 4RK$ and $(s-a)(s-b)(s-c) = \frac{1}{s}K^2 = rK$, we find that this becomes

$$2(s-a)(s-b)(s-c) \le RK \iff 2r \le R$$

which follows immediately from $IO^2 = R(R-2r)$. Alternatively, one may rewrite this as Schur's Inequality in the form

$$abc \ge (-a+b+c)(a-b+c)(a+b-c).$$