Buffet Solutions

Winter Camp 2017

Algebra

1. From $\frac{a_1^2}{a_1-1} > S = a_1 + a_2 + a_3$, we have

$$S > a_1 S - a_1^2 = a_1(a_2 + a_3)$$

Symmetric inequalities in a_2 and a_3 now yield

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_1+a_3} + \frac{a_3}{a_1+a_2} > \frac{a_1}{S} + \frac{a_2}{S} + \frac{a_3}{S} = 1$$

2.

3. First note that $f(1) = f(1)^2$ and thus $f(1) = 1$. We now show by induction on k that if n is a positive integer with $n \leq 2^k$ then $f(n) \leq 2^k$. The claim is true for $k = 0$. Now note that if $2^k < n \leq 2^{k+1}$ then

$$f(n) \leq 2 \max \left\{ f(n-2^k), f(2^k) \right\} \leq 2^{k+1}$$

since $n - 2^k \leq 2^k$, completing the induction. Now $f(n) \leq 2^{\lceil \log_2 n \rceil} \leq 2n$. Also observe that for any 2^k nonnegative real numbers $a_1, a_2, \dots, a_{2^k}$, we have that

$$f(a_1 + a_2 + \cdots + a_{2^k}) \leq 2^k \left(f(a_1) + f(a_2) + \cdots + f(a_{2^k}) \right)$$

This follows by induction on k by applying $f(a+b) \leq 2 \max\{f(a), f(b)\} \leq 2f(a) + 2f(b)$ to $a = a_1 + \cdots + a_{2^{k-1}}$ and $b = a_{2^{k-1}+1} + \cdots + a_{2^k}$. Now note that

$$\begin{aligned}
f(a+b)^{2^k-1} = f\left((a+b)^{2^k-1}\right) &= f\left(\sum_{j=0}^{2^k-1} \binom{2^k-1}{j} a^j b^{2^k-1-j} \right) \\
&\leq 2^k \sum_{j=0}^{2^k-1} f\left(\binom{2^k-1}{j} a^j b^{2^k-1-j} \right) \\
&\leq 2^k \sum_{j=0}^{2^k-1} f\left(\binom{2^k-1}{j} \right) f(a)^j f(b)^{2^k-1-j} \\
&\leq 2^{k+1} \sum_{j=0}^{2^k-1} \binom{2^k-1}{j} f(a)^j f(b)^{2^k-1-j} \\
&= 2^{k+1} \cdot (f(a) + f(b))^{2^k-1}
\end{aligned}$$

Therefore for all k we have that $f(a+b) \leq 2^{\frac{k+1}{2^k-1}} \cdot (f(a)+f(b))$. As k gets arbitrarily large, $\frac{k+1}{2^k-1}$ tends to zero and thus it must hold that $f(a+b) \leq f(a)+f(b)$.

Combinatorics

1. Assume for contradiction that after cancelling the $N-1$ flights, the cities can be divided into two sets A and B such that there are no roads between A and B. Note that each city is paired with a unique other city joined by a flight offered by c for each company c. Since it was possible to travel between any cities before any flights were cancelled, there is a company c_1 such that there was originally a flight f between A and B offered by c_1. Thus A consists of pairs of cities joined by a flight offered by c_1 along with one city on f. Thus $|A|$ is odd. However, if c_2 is the company without a cancelled flight, A consists of pairs of cities joined by flights offered by c_2. This implies that $|A|$ is even, which is a contradiction. Therefore it is still possible to travel between any two cities.

2. Note that $p(1) \geq 2$ since any x with one digit is rational. We now will prove the result by induction on k. Assume that $p(k-1) \geq k$ while $p(k) < k+1$ for some $k \geq 2$. Each distinct digit sequence of length $k-1$ in x leads to at least one sequence of length k by extending by one digit on the right. Thus we must have that $p(k) = p(k-1) = k$ and each digit sequence of length k extends uniquely to a sequence of length $k-1$. Therefore each digit of x is uniquely determined by the $k-1$ preceding digits. By pigeonhole principle, some sequence of length $k-1$ appears twice in x and thus the digits of x are identical starting at two distinct points. This implies that x is periodic and hence rational, which is a contradiction.

3.

Number Theory

1. If there is some $b = (b_1, b_2, \ldots, b_m)$ such that $\gcd(a_1+b_1, a_2+b_2, \ldots, a_m+b_m) \geq n^m$, it must follow that $a_1+b_1 = \cdots = a_m+b_m$ since $a_i+b_i < n^m+n < 2n^m$ for each i. Changing one b_i by 1 yields a gcd of 1, implying the result. Thus there is no such b. Now note that since there are n^m possible b, pigeonhole implies that there are two $b = (b_1, b_2, \ldots, b_m)$ and $b' = (b'_1, b'_2, \ldots, b'_m)$ such that

$$\gcd(a_1+b_1, a_2+b_2, \ldots, a_m+b_m) = \gcd(a_1+b'_1, a_2+b'_2, \ldots, a_m+b'_m) = g$$

Thus $g | a_1+b_1$ and $g | a_1+b'_1$ which implies that g divides $|b_1-b'_1|$ and hence $g < n$, as desired.

2. Assume for contradiction that $r > n-2$. Note that p_k divides

$$S = p_2p_3 \ldots p_n + p_1p_3 \cdots p_n + \cdots + p_1p_2 \cdots p_{n-1} - r$$

for each k and therefore S is divisible by $p_1p_2 \cdots p_k$. Also observe that $p_k > r$ and hence $p_k \geq n$ for each k. Therefore

$$\frac{S}{p_1p_2 \cdots p_k} < \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} \leq 1$$

which is a contradiction since the left hand side is a positive integer.

3. We first recall the lifting the exponent lemma: Let p be an odd prime, and a, b distinct positive integers coprime to p such that $p \mid a-b$. Then $v_p(a^n - b^n) = v_p(a-b) + v_p(n)$ for positive integers n. For $p = 2$, we have for odd distinct positive integers such that $4 \mid a-b$, then $v_2(a^n - b^n) = v_2(a-b) + v_2(n)$. In any case, for any prime p and distinct positive integers a, b both coprime to p with $p \mid a-b$, we have

$$v_p(a^n - b^n) \leq v_p(a^{2n} - b^{2n}) = v_p(a^2 - b^2) + v_p(n)$$

Let $P = \{p_1, p_2, \dots, p_r\}$ be the primes less than 10^{2016} which do not divide a, and assume that there are infinitely many n such that all prime divisors of $a^m - 1$ are in P, say $n_1 < n_2 < \cdots$. Let e_i be the order of a modulo p_i, so that $p_i \mid a^n - 1$ if and only if $e_i \mid n$. Let $f_i = v_{p_i}(a^{2e_i} - 1)$, and then we have

$$v_{p_i}(a^n - 1) \leq v_{p_i}(a^{2e_i} - 1) + v_{p_i}\left(\frac{n}{e_i}\right) \leq f_i + v_{p_i}(n)$$

Thus

$$a^{n_j} - 1 = \prod_{i=1}^{r} p_i^{v_{p_i}(a^{n_j} - 1)} \leq \prod_{i=1}^{r} p_i^{f_i + v_{p_i}(n_j)} = n_j \prod_{i=1}^{r} p_i^{f_i}$$

where $\prod_{i=1}^{r} p_i^{f_i} = C$ is a constant. Thus for all j, $a^{n_j} - 1 \leq C n_j$, but this is false for sufficiently large n_j.

Geometry

1. Let B' be on ray AC and such that $AB' = AB$ and let C' be on ray AB with $AC' = AC$. Note that both BB' and CC' are perpendicular to AL since ABB' and ACC' are isosceles. Since MD, BB' and CC' are parallel and MD passes through the midpoint of BC, it follows that MD is the midline of trapezoid $BB'CC'$ and M is the midpoint of BC'. Thus $AD = \frac{1}{2}(AB + AC)$ and since $MC = \frac{1}{2}BC$, the result follows.

2. Let B_0 be the midpoint of arc AC. Note that AIA_0, BIB_0 and CIC_0 are each collinear triples of points. We now have that $\angle C_0BI = \angle C_0BB_0 = \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90^\circ - \angle A$. Furthermore, $\angle BC_0I = \angle BC_0C = \angle A$. Thus C_0BI is isosceles with $C_0B = C_0I$. Similarly, we have $A_0B = A_0I$. Thus I is the reflection of B over A_0C_0. Let D and E be the points at which the tangents from B to S_1 and S_2 other than BA and BC touch these circles. We have that $\angle DBE = 2\angle C_0BA + \angle B + 2\angle CBA_0 = \angle C + \angle B + \angle A = 180^\circ$. Thus a common tangent to S_1 and S_2 passes through A. Reflecting about A_0C_0 gives that I also lies on a common tangent to S_1 and S_2.

3.