May Online Camp 2021

Number Theory – Level L3

Problems

Problem 1. A positive integer is called *nice* if it can be represented as a sum of two squares of non-negative integers. Prove that any positive integer is the difference of two nice numbers.

Problem 2. Let p_i for $i = 1, 2, \ldots, k$ be a sequence of consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 3 \ldots$). Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_k$. Prove that in a set $\{1, 2, \ldots, N\}$ there are exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

Problem 3. Find all sets of positive integers $\{x_1, x_2, \ldots, x_{20}\}$ such that

$$x_{i+2}^2 = \text{lcm}(x_{i+1}, x_i) + \text{lcm}(x_i, x_{i-1})$$

for $i = 1, 2, \ldots, 20$ where $x_0 = x_{20}$, $x_{21} = x_1$, $x_{22} = x_2$.

Problem 4. Let $n > 1$ be odd integer. Consider numbers n, $n+1$, $n+2$, $\ldots$, $2n-1$ written on the blackboard. Prove that we can erase one number, such that the sum of all numbers will be not divided any number on the blackboard.

Problem 5. Let $n > 20$ and $k > 1$ be integers such that k^2 divides n. Prove that there exist positive integers a, b, c, such that

$$n = ab + bc + ca.$$

Problem 6. Let a, $b > 1$ be integers such that $a^2 + b$, and $a + b^2$ are primes. Prove $\gcd(ab + 1, a + b) = 1$.

Problem 7. Let p, q be primes such that $p < q < 2p$. Prove that there are two consecutive positive integers, such that largest prime divisor of first number is p, and the largest prime divisor of second number is q.

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that

$$a = \frac{x+y}{z} \quad \text{and} \quad b = \frac{xy}{z}.$$

Problem 9. We say that a positive integer is an almost square, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.

Problem 10. It is known that a cells square can be cut into n equal figures of k cells. Prove that it is possible to cut it into k equal figures of n cells.

Problem 11. Prove that any rational number may be written as

$$\frac{a^2+b^3}{c^5+d^7},$$

where a, b, c, d are positive integers.

Problem 12. Let n be a positive integer. Prove that there exists positive integers a and b, such that

$$a^2+a+1 = (n^2+n+1)(b^2+b+1).$$

Problem 13. Let a, b, c be positive integers. Prove that there is a positive integer n such that

$$(a^2+n)(b^2+n)(c^2+n)$$

is a perfect square.

Problem 14. Let a, b, z be positive integers such that $ab = z^2+1$. Prove that there are positive integers such x, y such that

$$\frac{a}{b} = \frac{x^2+1}{y^2+1}.$$

Problem 15. Prove that there are infinitely many pairwise distinct positive integers a, b, c and d such that $a^2+2cd+b^2$ and $c^2+2ab+d^2$ are squares.

Problem 16. Let $a, b, c \in \mathbb{N}$ with $\gcd(a^2-1, b^2-1, c^2-1) = 1$. Prove that,

$$\gcd(ab+c, bc+a, ca+b) = \gcd(a, b, c).$$

Problem 17. Define the sequence $a_1, a_2, a_3, \ldots$ by

$$a_1 = 1, \quad a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}.$$

Does the sequence contain infinitely many multiples of 7?