TEAM SELECTION TEST
INTRERNATIONAL MATHEMATICAL OLYMPIAD
Day 4, May 6, 2021

Problem 1. Given a positive integer k, show that there exists a prime p such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, 3, ..., k$.

Problem 2. Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac+bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Problem 3. Let p be an odd prime and let $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leq N$, denote by $r(n)$ the fraction of integers in $\{1, 2, ..., n\}$ that are red (number of red numbers divided by n). Prove that there exists a positive integer $a \in \{1, 2, ..., p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, ..., N$.