IMO Training: Graph Theory: Solving Problems Using Dots and Lines

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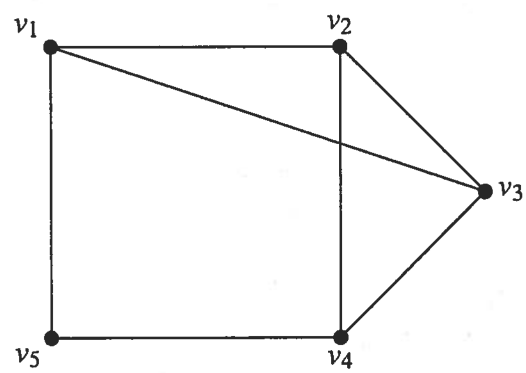
Representing objects in a mathematical problem in a graph theoretical setting can be extremely helpful to understand the underlying structures of a problem. This set of notes will present some examples of olympiad-level problems that can be solved using elementary graph theory facts.

A graph is a pair $G = (V,E)$ where V is a set of **vertices** and E is a set of **edges** with endpoints in V. The following is an example of a graph.



A graph can have finite or an infinite number of vertices and edges. In fact, there can be more than one edge joining two vertices. (In this case, it is called a **multi-edge**.)

We say that two vertices $v, w \in G$ are **adjacent** if there is an edge joining v and w. For example, in the above figure, v_1 is adjacent to v_2 but v_1 is not adjacent to v_4.

The **degree** of a vertex v is the number of vertices adjacent to v. For example, the degree of v_1 is 3 ~~2~~ and the degree of v_2 is 3.

Exercise 1: Let G be a graph with m edges and n vertices $v_1, v_2, \cdots, v_n$ with degrees $d_1, d_2, \cdots, d_n$ respectively. Prove that

$$\sum_{i=1}^{n} d_i = 2m.$$

Exercise 2: (i) A group of n people gathered and a number of handshakes took place. (No one shakes his/her own hand.) Prove that the number of people who shook an odd number of hands, is even.

(ii) Suppose each person p_i shook hands with h_i other people such that $\sum_{i=1}^{n} h_i$ is even and $1 \leq h_i < n$ for all ~~p~~ i. Is this always possible for any given such h_i?

We now describe a few common graphs.

A **complete graph** is a finite graph where every pair of vertices are adjacent.

A **path** is a finite graph where the vertices can be labelled $v_0, \cdots, v_{n-1}$ such that v_i is adjacent to v_j if and only if $|i-j| = 1$. The **length** of a path is defined to be the number of edges in the path.

$v_i \sim v_{i+1}$

A **cycle** is a finite graph where the vertices can be labelled $v_0, \cdots, v_{n-1}$ such that v_i is adjacent to v_j if and only if $|i-j| \cancel{\equiv} 1$ ~~mod n~~. for all $\{i, j\} \neq \{0, n-1\}$,

and v_0 is adj to v_{n-1}

A **connected graph** is a graph where for any two vertices $u, v \in V(G)$, there exists a path joining u and v.

Given a graph $G = (V, E)$, a **subgraph** H of G is defined to be a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap \{vw | v, w \in V(H)\}$.

A **bipartite graph** is a graph where the vertices can be partitioned into two disjoint sets A, B such that any two vertices in the same partition are not adjacent.

Exercise 3: A graph G is bipartite if and only if G does not contain an odd cycle as a subgraph.

A **tree** is a connected graph that does not contain a cycle.

Exercise 4: Let G be a connected graph with n vertices. Prove that the following statements are equivalent.

(i) G has $n-1$ edges.
(ii) G does not contain a cycle.
(iii) For every edge e in the graph, $G-\{e\}$ is not connected.
(iv) For every pair of vertices u, v in the graph, there exists a *unique* path joining u, v.

Exercise 5: Let G be any connected graph with n vertices. Prove that there exists a subgraph of G which is a tree on n vertices. (This tree is called a **spanning tree** of G.)

Exercise 6: (i) Let k be a positive integer and let G be a tree on $4k-2$ vertices such that the maximum degree of G is 4. Prove that there exist two vertex-disjoint connected subgraphs of G containing k vertices each.

(ii) Find an example of a graph on $4k-3$ vertices such that the statement in (i) is not true.

(iii) Explain why (i) holds for any connected graph G.

(Does this look familiar to you?)

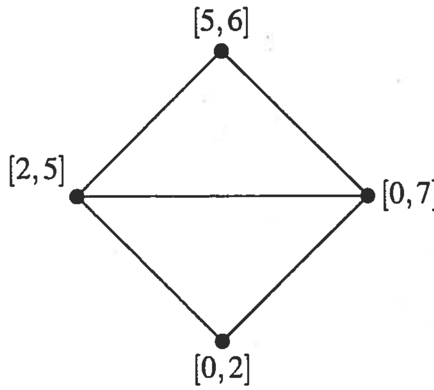
A nice part about graph theory is that you can essentially define a graph on anything. All one needs to do is define what vertices are and how edges are defined. The following question demonstrates this.

Exercise 7: A graph G is said to be an **interval graph** if every vertex is represented by a closed interval $[a,b] = \{x | a \leq x \leq b\}$ and two vertices are adjacent if and only if the intersection of their corresponding intervals is non-empty. For example, the following is an interval graph.

Prove that a cycle on n vertices where $n \geq 4$ is not an interval graph.

For a finite graph, as more edges are added to the graph, there are certain structures that are forced to appear. For example, there are n vertices in a graph. What is the most number of edges that a graph can have so that no three vertices are mutually adjacent? The area of graph theory that studies these types of problems is called **extremal graph theory**.

Exercise 8: There are n people at a party. Some pairs are friends. Some pairs are strangers.



(i) What is the maximum number of pairs that are friends such that no three people are mutually friends?

(ii) Let $r > 2$ be a positive integer. Prove that if there are at least

$$\frac{n^2}{2}(1 - \frac{1}{r-1})$$

pairs are friends, then there exists r people that are mutually friends.

Exercise 9: Let p be a prime and a be a positive integer relatively prime to p. Let G be a graph where the vertex set consists of all p-tuples of the form $(w_1, w_2, \cdots, w_p)$ such that $w_i \in \{1, 2, \cdots, a\}$ and the w_i's are not all the same. Two vertices $(w_1, w_2, \cdots, w_p)$, $(w'_1, w'_2, \cdots, w'_p)$ are adjacent if and only if $w_i = w'_{i+1}$ or $w_{i+1} = w'_i$ for all $i \in \{1, 2, \cdots, p\}$.

(i) How many vertices are in this graph?

(ii) One can show that this graph is not connected. How many vertices are there in each connected component of this graph?

(iii) What famous theorem did you just prove?