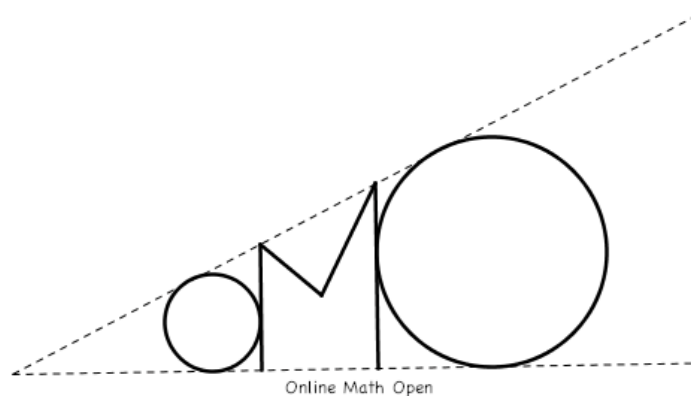


# The Online Math Open Spring Contest

## April 4 - 15, 2014



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# Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at [OnlineMathOpenTeam@gmail.com](mailto:OnlineMathOpenTeam@gmail.com).

## Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

**Only one member on each team needs to register an account on the website.** Please check the website, [http://internetolympiad.org/pages/14-omo\\_info](http://internetolympiad.org/pages/14-omo_info), for registration instructions.

*Note:* when we say “up to four”, we really do mean “up to”! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

## Contest Format and Rules

The 2014 Spring Contest will consist of 30 problems; the answer to each problem will be a nonnegative integer not exceeding  $2^{32} - 1 = 4294967295$ . The contest window will be **April 4 - 15, 2014**, from 7PM ET on the start day to 7PM ET on the end day. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. *Any other computational aids are prohibited, including scientific calculators, graphing calculators, or computer programs.* All problems on the Online Math Open are solvable without a calculator.
2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.
3. Members of different teams cannot communicate about the contest until the contest ends.
4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the “hardest” problem that a team answered correctly. Problem  $m$  is harder than problem  $n$  if fewer teams solve problem  $m$ , or if the number of solves is equal and  $m > n$ . Remaining ties will be broken by the second hardest problem solved, and so on.
5. **Participation in the Online Math Open is free.**

## Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at [http://internetolympiad.org/pages/n/omo\\_problems](http://internetolympiad.org/pages/n/omo_problems). If you have a question about problem wording, please email [OnlineMathOpenTeam@gmail.com](mailto:OnlineMathOpenTeam@gmail.com) with “Clarification” in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to [OnlineMathOpenTeam@gmail.com](mailto:OnlineMathOpenTeam@gmail.com). (Include “Protest” in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. In English class, you have discovered a mysterious phenomenon – if you spend  $n$  hours on an essay, your score on the essay will be  $100(1 - 4^{-n})$  points if  $2n$  is an integer, and 0 otherwise. For example, if you spend 30 minutes on an essay you will get a score of 50, but if you spend 35 minutes on the essay you somehow do not earn any points.

It is 4AM, your English class starts at 8:05AM the same day, and you have four essays due at the start of class. If you can only work on one essay at a time, what is the maximum possible average of your essay scores?

2. Consider two circles of radius one, and let  $O$  and  $O'$  denote their centers. Point  $M$  is selected on either circle. If  $OO' = 2014$ , what is the largest possible area of triangle  $OMO'$ ?
3. Suppose that  $m$  and  $n$  are relatively prime positive integers with  $A = \frac{m}{n}$ , where

$$A = \frac{2 + 4 + 6 + \cdots + 2014}{1 + 3 + 5 + \cdots + 2013} - \frac{1 + 3 + 5 + \cdots + 2013}{2 + 4 + 6 + \cdots + 2014}.$$

Find  $m$ . In other words, find the numerator of  $A$  when  $A$  is written as a fraction in simplest form.

4. The integers  $1, 2, \dots, n$  are written in order on a long slip of paper. The slip is then cut into five pieces, so that each piece consists of some (nonempty) consecutive set of integers. The averages of the numbers on the five slips are 1234, 345, 128, 19, and 9.5 in some order. Compute  $n$ .
5. Joe the teacher is bad at rounding. Because of this, he has come up with his own way to round grades, where a *grade* is a nonnegative decimal number with finitely many digits after the decimal point.

Given a grade with digits  $a_1a_2 \dots a_m.b_1b_2 \dots b_n$ , Joe first rounds the number to the nearest  $10^{-n+1}$ th place. He then repeats the procedure on the new number, rounding to the nearest  $10^{-n+2}$ th, then rounding the result to the nearest  $10^{-n+3}$ th, and so on, until he obtains an integer. For example, he rounds the number 2014.456 via  $2014.456 \rightarrow 2014.46 \rightarrow 2014.5 \rightarrow 2015$ .

There exists a rational number  $M$  such that a grade  $x$  gets rounded to at least 90 if and only if  $x \geq M$ . If  $M = \frac{p}{q}$  for relatively prime integers  $p$  and  $q$ , compute  $p + q$ .

6. Let  $L_n$  be the least common multiple of the integers  $1, 2, \dots, n$ . For example,  $L_{10} = 2,520$  and  $L_{30} = 2,329,089,562,800$ . Find the remainder when  $L_{31}$  is divided by 100,000.
7. How many integers  $n$  with  $10 \leq n \leq 500$  have the property that the hundreds digit of  $17n$  and  $17n + 17$  are different?
8. Let  $a_1, a_2, a_3, a_4, a_5$  be real numbers satisfying

$$\begin{aligned} 2a_1 + a_2 + a_3 + a_4 + a_5 &= 1 + \frac{1}{8}a_4 \\ 2a_2 + a_3 + a_4 + a_5 &= 2 + \frac{1}{4}a_3 \\ 2a_3 + a_4 + a_5 &= 4 + \frac{1}{2}a_2 \\ 2a_4 + a_5 &= 6 + a_1 \end{aligned}$$

Compute  $a_1 + a_2 + a_3 + a_4 + a_5$ .

9. Eighteen students participate in a team selection test with three problems, each worth up to seven points. All scores are nonnegative integers. After the competition, the results are posted by Evan in a table with 3 columns: the student's name, score, and rank (allowing ties), respectively. Here, a student's rank is one greater than the number of students with strictly higher scores (for example, if seven students score 0, 0, 7, 8, 8, 14, 21 then their ranks would be 6, 6, 5, 3, 3, 2, 1 respectively).

When Richard comes by to read the results, he accidentally reads the rank column as the score column and vice versa. Coincidentally, the results still made sense! If the scores of the students were  $x_1 \leq x_2 \leq \dots \leq x_{18}$ , determine the number of possible values of the 18-tuple  $(x_1, x_2, \dots, x_{18})$ . In other words, determine the number of possible multisets (sets with repetition) of scores.

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10. Let  $A_1A_2 \dots A_{4000}$  be a regular 4000-gon. Let  $X$  be the foot of the altitude from  $A_{1986}$  onto diagonal  $A_{1000}A_{3000}$ , and let  $Y$  be the foot of the altitude from  $A_{2014}$  onto  $A_{2000}A_{4000}$ . If  $XY = 1$ , what is the area of square  $A_{500}A_{1500}A_{2500}A_{3500}$ ?
11. Let  $X$  be a point inside convex quadrilateral  $ABCD$  with  $\angle AXB + \angle CXD = 180^\circ$ . If  $AX = 14$ ,  $BX = 11$ ,  $CX = 5$ ,  $DX = 10$ , and  $AB = CD$ , find the sum of the areas of  $\triangle AXB$  and  $\triangle CXD$ .
12. The points  $A, B, C, D, E$  lie on a line  $\ell$  in this order. Suppose  $T$  is a point not on  $\ell$  such that  $\angle BTC = \angle DTE$ , and  $\overline{AT}$  is tangent to the circumcircle of triangle  $BTE$ . If  $AB = 2$ ,  $BC = 36$ , and  $CD = 15$ , compute  $DE$ .
13. Suppose that  $g$  and  $h$  are polynomials of degree 10 with integer coefficients such that  $g(2) < h(2)$  and

$$g(x)h(x) = \sum_{k=0}^{10} \left( \binom{k+11}{k} x^{20-k} - \binom{21-k}{11} x^{k-1} + \binom{21}{11} x^{k-1} \right)$$

holds for all nonzero real numbers  $x$ . Find  $g(2)$ .

14. Let  $ABC$  be a triangle with incenter  $I$  and  $AB = 1400$ ,  $AC = 1800$ ,  $BC = 2014$ . The circle centered at  $I$  passing through  $A$  intersects line  $BC$  at two points  $X$  and  $Y$ . Compute the length  $XY$ .
15. In Prime Land, there are seven major cities, labelled  $C_0, C_1, \dots, C_6$ . For convenience, we let  $C_{n+7} = C_n$  for each  $n = 0, 1, \dots, 6$ ; i.e. we take the indices modulo 7. Al initially starts at city  $C_0$ .  
Each minute for ten minutes, Al flips a fair coin. If the coin land heads, and he is at city  $C_k$ , he moves to city  $C_{2k}$ ; otherwise he moves to city  $C_{2k+1}$ . If the probability that Al is back at city  $C_0$  after 10 moves is  $\frac{m}{1024}$ , find  $m$ .
16. Say a positive integer  $n$  is *radioactive* if one of its prime factors is strictly greater than  $\sqrt{n}$ . For example,  $2012 = 2^2 \cdot 503$ ,  $2013 = 3 \cdot 11 \cdot 61$  and  $2014 = 2 \cdot 19 \cdot 53$  are all radioactive, but  $2015 = 5 \cdot 13 \cdot 31$  is not. How many radioactive numbers have all prime factors less than 30?
17. Let  $AXYBZ$  be a convex pentagon inscribed in a circle with diameter  $\overline{AB}$ . The tangent to the circle at  $Y$  intersects lines  $BX$  and  $BZ$  at  $L$  and  $K$ , respectively. Suppose that  $\overline{AY}$  bisects  $\angle LAZ$  and  $AY = YZ$ . If the minimum possible value of

$$\frac{AK}{AX} + \left( \frac{AL}{AB} \right)^2$$

can be written as  $\frac{m}{n} + \sqrt{k}$ , where  $m, n$  and  $k$  are positive integers with  $\gcd(m, n) = 1$ , compute  $m + 10n + 100k$ .

18. Find the number of pairs  $(m, n)$  of integers with  $-2014 \leq m, n \leq 2014$  such that  $x^3 + y^3 = m + 3nxy$  has infinitely many integer solutions  $(x, y)$ .
19. Find the sum of all positive integers  $n$  such that  $\tau(n)^2 = 2n$ , where  $\tau(n)$  is the number of positive integers dividing  $n$ .
20. Let  $ABC$  be an acute triangle with circumcenter  $O$ , and select  $E$  on  $\overline{AC}$  and  $F$  on  $\overline{AB}$  so that  $\overline{BE} \perp \overline{AC}$ ,  $\overline{CF} \perp \overline{AB}$ . Suppose  $\angle EOF - \angle A = 90^\circ$  and  $\angle AOB - \angle B = 30^\circ$ . If the maximum possible measure of  $\angle C$  is  $\frac{m}{n} \cdot 180^\circ$  for some positive integers  $m$  and  $n$  with  $m < n$  and  $\gcd(m, n) = 1$ , compute  $m + n$ .
21. Let  $b = \frac{1}{2}(-1 + 3\sqrt{5})$ . Determine the number of rational numbers which can be written in the form

$$a_{2014}b^{2014} + a_{2013}b^{2013} + \dots + a_1b + a_0$$

where  $a_0, a_1, \dots, a_{2014}$  are nonnegative integers less than  $b$ .

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22. Let  $f(x)$  be a polynomial with integer coefficients such that  $f(15)f(21)f(35) - 10$  is divisible by 105. Given  $f(-34) = 2014$  and  $f(0) \geq 0$ , find the smallest possible value of  $f(0)$ .
23. Let  $\Gamma_1$  and  $\Gamma_2$  be circles in the plane with centers  $O_1$  and  $O_2$  and radii 13 and 10, respectively. Assume  $O_1O_2 = 2$ . Fix a circle  $\Omega$  with radius 2, internally tangent to  $\Gamma_1$  at  $P$  and externally tangent to  $\Gamma_2$  at  $Q$ . Let  $\omega$  be a second variable circle internally tangent to  $\Gamma_1$  at  $X$  and externally tangent to  $\Gamma_2$  at  $Y$ . Line  $PQ$  meets  $\Gamma_2$  again at  $R$ , line  $XY$  meets  $\Gamma_2$  again at  $Z$ , and lines  $PZ$  and  $XR$  meet at  $M$ . As  $\omega$  varies, the locus of point  $M$  encloses a region of area  $\frac{p}{q}\pi$ , where  $p$  and  $q$  are relatively prime positive integers. Compute  $p + q$ .
24. Let  $\mathcal{P}$  denote the set of planes in three-dimensional space with positive  $x$ ,  $y$ , and  $z$  intercepts summing to one. A point  $(x, y, z)$  with  $\min\{x, y, z\} > 0$  lies on exactly one plane in  $\mathcal{P}$ . What is the maximum possible integer value of  $(\frac{1}{4}x^2 + 2y^2 + 16z^2)^{-1}$ ?

25. If

$$\sum_{n=1}^{\infty} \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{\binom{n+100}{100}} = \frac{p}{q}$$

for relatively prime positive integers  $p, q$ , find  $p + q$ .

26. Qing initially writes the ordered pair  $(1, 0)$  on a blackboard. Each minute, if the pair  $(a, b)$  is on the board, she erases it and replaces it with one of the pairs  $(2a - b, a)$ ,  $(2a + b + 2, a)$  or  $(a + 2b + 2, b)$ . Eventually, the board reads  $(2014, k)$  for some nonnegative integer  $k$ . How many possible values of  $k$  are there?
27. A frog starts at 0 on a number line and plays a game. On each turn the frog chooses at random to jump 1 or 2 integers to the right or left. It stops moving if it lands on a nonpositive number or a number on which it has already landed. If the expected number of times it will jump is  $\frac{p}{q}$  for relatively prime positive integers  $p$  and  $q$ , find  $p + q$ .
28. In the game of Nim, players are given several piles of stones. On each turn, a player picks a nonempty pile and removes any positive integer number of stones from that pile. The player who removes the last stone wins, while the first player who cannot move loses. Alice, Bob, and Chebyshev play a 3-player version of Nim where each player wants to win but avoids losing at all costs (there is always a player who neither wins nor loses). Initially, the piles have sizes 43, 99,  $x$ ,  $y$ , where  $x$  and  $y$  are positive integers. Assuming that the first player loses when all players play optimally, compute the maximum possible value of  $xy$ .
29. Let  $ABCD$  be a tetrahedron whose six side lengths are all integers, and let  $N$  denote the sum of these side lengths. There exists a point  $P$  inside  $ABCD$  such that the feet from  $P$  onto the faces of the tetrahedron are the orthocenter of  $\triangle ABC$ , centroid of  $\triangle BCD$ , circumcenter of  $\triangle CDA$ , and orthocenter of  $\triangle DAB$ . If  $CD = 3$  and  $N < 100,000$ , determine the maximum possible value of  $N$ .
30. For a positive integer  $n$ , an  $n$ -branch  $B$  is an ordered tuple  $(S_1, S_2, \dots, S_m)$  of nonempty sets (where  $m$  is any positive integer) satisfying  $S_1 \subset S_2 \subset \cdots \subset S_m \subseteq \{1, 2, \dots, n\}$ . An integer  $x$  is said to *appear* in  $B$  if it is an element of the last set  $S_m$ . Define an  $n$ -plant to be an (unordered) set of  $n$ -branches  $\{B_1, B_2, \dots, B_k\}$ , and call it *perfect* if each of  $1, 2, \dots, n$  appears in exactly one of its branches. Let  $T_n$  be the number of distinct perfect  $n$ -plants (where  $T_0 = 1$ ), and suppose that for some positive real number  $x$  we have the convergence

$$\ln \left( \sum_{n \geq 0} T_n \cdot \frac{(\ln x)^n}{n!} \right) = \frac{6}{29}.$$

If  $x = \frac{m}{n}$  for relatively prime positive integers  $m$  and  $n$ , compute  $m + n$ .