Combinatorics

C1 A hundred bear-cubs picked up berries in a forest. The youngest bear-cub got one berry, the second youngest got 2 berries, the third youngest got 4 berries, and so on; the eldest cub got 2^{99} berries. They meet a fox who suggests to divide the berries 'fairly'. The fox chooses two bear-cubs and divides their berries equally between them, but if one berry is left over then the fox eats it. The fox can keep doing this as many times as it pleases (of course, this doesn't do anything meaningful if the two selected bear-cubs have the same amount of berries). What is the greatest possible number of berries that the fox can eat?

C2 For an integer $n \geq 3$, the edges of a complete graph on n vertices are coloured in one of n colours, where each colour is used at least once. Prove that there is a triangle whose edges are of three different colours.

C3 A rectangular grid is given, all of whose edges are oriented. The edges on the boundary are oriented clockwise. All the vertices in the interior of the grid have exactly two edges pointing out and two edges pointing in. Prove that there is a unit square all of whose edges are oriented clockwise.

C4 For a positive integer n, define $p(n)$ to be the number of sets of positive integers $\{x_1, x_2, \ldots, x_k\}$ such that $x_1 > x_2 > \cdots > x_k > 0$ and $n = x_1 + x_3 + x_5 + \ldots$ (i.e, the sum of all odd-indexed elements of the set). For example, $p(6) = 11$ since we have the following sets:

$$\{6\}, \{6,5\}, \{6,4\}, \{6,3\}, \{6,2\}, \{6,1\}, \{5,4,1\}, \{5,3,1\}, \{5,2,1\}, \{4,3,2\}, \{4,3,2,1\}$$

Prove that $p(n)$ is equal to the number of partitions of n for all $n \in \mathbb{N}$.

(The number of partitions of n is the number of ways of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition.)

Number Theory

N1 Find all triplets of positive integers a, b, p such that p is prime and $p^2 = a^3 + b^3$.

N2 A strictly increasing infinite sequence of natural numbers a_1, a_2, a_3, $\ldots$ is given. It is known that $a_n \leq n + 2021$ and the number $n^3 a_n - 1$ is divisible by a_{n+1} for all natural numbers n. Prove that $a_n = n$ for all natural numbers n.

N3 Define a sequence a_n such that $a_1 = 1$, $a_2 = 2$, and for all $n \geq 2$, a_{n+1} is the smallest positive integer m such that m hasn't yet occurred in the sequence and such that $\gcd(m, a_n) \neq 1$. Show that all positive integers occur in the sequence.

N4 We have functions f and g such that $f(x) = 1/x$ and $g(x) = x - 1$. Can we obtain $2^{1/3} + 1$ by applying a finite sequence of f and g to the number $2^{1/3}$?

Algebra

A1 There is a sequence $(x_1, x_2, \ldots, x_k)$ defined with $x_1 = 20$, $x_2 = 21$, and $x_{n+2} = x_n - \frac{1}{x_{n+1}}$ for $1 \le n \le k-2$. Given that $x_k = 0$, find the value of k.

A2 Solve the system of equations in real numbers:

$$(x-1)(y-1)(z-1) = xyz - 1,$$

$$(x-2)(y-2)(z-2) = xyz - 2.$$

A3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers x and y,

$$f\left(x + yf(x+y)\right) = y^2 + f(x)f(y)$$

A4 A sequence $a_1, a_2, \ldots$ is defined recursively as follows: Let $a_1 = 1, a_2 = 2, a_3 = 3$, and

$$a_{n+1} = \frac{a_n^2 + a_{n-1}^2 - 2}{a_{n-2}}$$

for $n \ge 3$. Prove that a_n is an integer for all $n \ge 1$.

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Geometry

G1 Given a square $ABCD$, let M and K be points on segments BC and CD such that $MC = KD$. Let P be the intersection of MD and BK. Prove that AP is perpendicular to MK.

G2 Let $ABCD$ is a convex quadrilateral. The internal angle bisectors of $\angle BAC$ and $\angle BDC$ meets at P. Given that $\angle APB = \angle CPD$, prove that $AB + BD = AC + CD$.

G3 In cyclic quadrilateral $ABCD$ with $AB > BC$ and $AD > DC$, define I, J as the incenters of $\triangle ABC$, $\triangle ADC$, respectively. The circle with diameter AC meets segment IB at X and the extension of JD at Y. Prove that if the four points B, I, J, D are concyclic, then X and Y are the reflections of each other across AC.

G4 In $\triangle ABC$, let D, E, and F be the feet of the altitudes from A, B, and C respectively. Let K be the orthocenter of $\triangle DEF$. Let U, V, and W be the projections of points F, D, and E on BC, CA, and AB, respectively. Assume that the circumcircles of $\triangle ABC$ and $\triangle UVW$ intersect at points M and N. Prove that $KM = KN$.