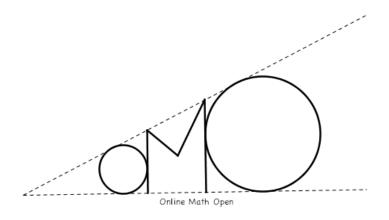
# The Online Math Open Fall Contest Official Solutions November $4-15,\,2016$



# Acknowledgements

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# Late Late May 12 May 1

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1. Kevin is in first grade, so his teacher asks him to calculate  $20 + 1 \cdot 6 + k$ , where k is a real number revealed to Kevin. However, since Kevin is rude to his Aunt Sally, he instead calculates  $(20+1) \cdot (6+k)$ . Surprisingly, Kevin gets the correct answer! Assuming Kevin did his computations correctly, what was his answer?

Proposed by James Lin.

Answer. 21

**Solution.** Equating the given expression and Kevin's incorrect expression gives  $26 + k = 21(6 + k) = 126 + 21k \implies k = -5$ , so hence 26 + k = 21.

2. Yang has a standard 6-sided die, a standard 8-sided die, and a standard 10-sided die. He tosses these three dice simultaneously. The probability that the three numbers that show up form the side lengths of a right triangle can be expressed as  $\frac{m}{n}$ , for relatively prime positive integers m and n. Find 100m+n. Proposed by Yannick Yao.

**Answer.** 1180

**Solution.** Notice that the only ways for the three rolls to form a right triangle are by getting 3-4-5 and 6-8-10, in some order. It is not difficult to see that there are 3!=6 ways to get 3-4-5 and 1 way to get 6-8-10, so the desired probability is  $\frac{6+1}{6\cdot8\cdot10}=\frac{7}{480}$ , so our answer is 1180.

3. In a rectangle ABCD, let M and N be the midpoints of sides BC and CD, respectively, such that AM is perpendicular to MN. Given that the length of AN is 60, the area of rectangle ABCD is  $m\sqrt{n}$  for positive integers m and n such that n is not divisible by the square of any prime. Compute 100m + n. Proposed by Yannick Yao.

**Answer.** 160002.

**Solution.** Notice that  $\triangle ABM$  is similar to  $\triangle MCN$ , so  $\frac{AB}{BM} = \frac{CM}{CN}$ . Since BM = CM and AB = 2CN, it follows that  $BC = \sqrt{2}AB$  and therefore  $AD = 2\sqrt{2}DN$ . Hence,  $\frac{AN}{DN} = \sqrt{\left(\frac{AD}{DN}\right)^2 + \left(\frac{DN}{DN}\right)^2} = \sqrt{\left(2\sqrt{2}\right)^2 + 1^2} = 3$ . Thus,  $DN = 60/3 = 20 \Rightarrow CD = 40$  and the area of ABCD is  $40 \cdot 40\sqrt{2} = 1600\sqrt{2}$ , giving an answer of 160002.

4. Let  $G = 10^{10^{100}}$  (a.k.a. a googolplex). Then

$$\log_{\left(\log_{\left(\log_{10}G\right)}G\right)}G$$

can be expressed in the form  $\frac{m}{n}$  for relatively prime positive integers m and n. Determine the sum of the digits of m + n.

Proposed by Yannick Yao.

Answer. 18

**Solution.** We compute  $\log_{10} G = 10^{100}$ , and

$$\log_{(\log_{10} G)} G = \frac{\log_{10} G}{\log_{10} (\log_{10} G)} = \frac{10^{100}}{100} = 10^{98};$$

$$\log_{\left(\log_{(\log_{10}G)}G\right)}G = \frac{\log_{10}G}{\log_{10}(\log_{(\log_{10}G)}G)} = \frac{10^{100}}{98} = \frac{5\cdot 10^{99}}{49}.$$

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Therefore  $m + n = 5 \cdot 10^{99} + 49$ , so the sum of the digits of m + n is 18.

5. Jay notices that there are n primes that form an arithmetic sequence with common difference 12. What is the maximum possible value for n?

Proposed by James Lin.

Answer. 5

**Solution.** Assume there exist 5 prime numbers that form an arithmetic sequence with common difference 12, denoted by p, p+12, p+24, p+36, p+48. Notice that these 5 primes have 5 different residues modulo 5, hence one of them is divisible 5. Therefore p=5. It follows that any arithmetic progression that does not contain 5 must have at most four primes. Otherwise, if the sequence contains a 5, it can have at most 5 terms: 5, 17, 29, 41, 53, because the potential next term is 65. Thus the answer is 5.  $\square$ 

6. For a positive integer n, define n? =  $1^n \cdot 2^{n-1} \cdot 3^{n-2} \cdots (n-1)^2 \cdot n^1$ . Find the positive integer k for which 7?9? = 5?k?.

Proposed by Tristan Shin.

**Answer.** 10.

**Solution.** Notice that  $\frac{n?}{(n-1)?}=n!$ . Thus, we can see that  $n?=1!2!\cdots n!$ . Because of this, we have that  $\frac{k?}{9?}=\frac{7?}{5?}=7!6!$ . Noting that  $10\cdot 9\cdot 8=6! \implies 10!=7!6!$ , it follows that k?=9?10!=10? and k=10.

7. The 2016 players in the Gensokyo Tennis Club are playing Up and Down the River. The players first randomly form 1008 pairs, and each pair is assigned to a tennis court (The courts are numbered from 1 to 1008). Every day, the two players on the same court play a match against each other to determine a winner and a loser. For  $2 \le i \le 1008$ , the winner on court i will move to court i-1 the next day (and the winner on court 1 does not move). Likewise, for  $1 \le j \le 1007$ , the loser on court j will move to court j+1 the next day (and the loser on court 1008 does not move). On Day 1, Reimu is playing on court 123 and Marisa is playing on court 876. Find the smallest positive integer value of n for which it is possible that Reimu and Marisa play one another on Day n.

Proposed by Yannick Yao.

**Answer.** 500

**Solution.** If both Reimu and Marisa (or neither of them) change courts each day, then the difference between their court numbers will increase by 2, decrease by 2, or stay constant. If one of them doesn't change courts (because she either won on court 1 or lost on court 1008), then the difference changes by 1. Since 876 - 123 = 753 is odd, this means that they will never play each other unless one of them gets to court 1 and 1008 and stays for an odd number of rounds. Since Reimu is closer to court 1 than Marisa to court 1008, it can be seen that the fastest way for Reimu and Marisa to play one another is to have Reimu rise to court 1 (Reaching there on Day 123) and stay for 1 day (So she loses on Day 124) before losing to meet Marisa, who is rising the whole time. At Day 124, Marisa is on court 876 - 123 = 753. Then it takes (753 - 1)/2 = 376 days for them to meet each other on court 377 on Day 124 + 376 = 500. Thus 500 is our answer.

8. For a positive integer n, define the nth triangular number  $T_n$  to be  $\frac{n(n+1)}{2}$ , and define the nth square number  $S_n$  to be  $n^2$ . Find the value of

$$\sqrt{S_{62} + T_{63} \sqrt{S_{61} + T_{62} \sqrt{\cdots \sqrt{S_2 + T_3 \sqrt{S_1 + T_2}}}}}.$$

Proposed by Yannick Yao.

**Answer.** 1954

**Solution.** For each positive integer n, let  $K_n = \sqrt{S_n + T_{n+1} \sqrt{S_{n-1} + T_n \sqrt{\cdots \sqrt{S_1 + T_2}}}}$ . We claim that  $K_n = T_n + 1$  for all n.

We proceed by induction. For the base case of n=',  $K_1 = \sqrt{S_1 + T_2} = \sqrt{1+3} = 2 = T_1 + 1$ . Now assume that the claim is true for n = t, then it suffices to show that  $K_{t+1} = \sqrt{S_{t+1} + T_{t+2}(T_t + 1)} = T_{t+1} + 1$ . Note that

$$(T_{t+1}+1)^2 - S_{t+1} = (T_{t+1}+1)^2 - (t+1)^2 = (T_{t+1}+1+(t+1))(T_{t+1}+1-(t+1)) = (T_{t+1}+(t+2))(T_{t+1}-t) = T_{t+2}(T_t+1),$$

so hence  $T_{t+1} + 1 = \sqrt{S_{t+1} + T_{t+2}(T_t + 1)}$ , thus finishing the inductive step and proving the claim.

Now, 
$$K_{62} = T_{62} + 1 = \frac{62 \cdot 63}{2} + 1 = 1954$$
.

9. In quadrilateral ABCD, AB = 7, BC = 24, CD = 15, DA = 20, and AC = 25. Let segments AC and BD intersect at E. What is the length of EC?

Proposed by James Lin.

Answer. 18.

**Solution.** Note that 
$$\angle ABC = \angle ADC = 90^\circ$$
, so  $ABCD$  is a cyclic quadrilateral. Then,  $\triangle BEC \sim \triangle AED$  so  $\frac{BE}{AE} = \frac{BC}{AD} = \frac{6}{5}$  and  $\triangle AEB \sim \triangle DEC$  so  $\frac{CE}{BE} = \frac{CD}{AB} = \frac{15}{7}$ . Hence,  $\frac{CE}{AE} = \frac{BE}{AE} \cdot \frac{CE}{BE} = \frac{6}{5} \cdot \frac{15}{7} = \frac{18}{7}$ . Since  $AE + EC = 25$ , it follows that  $EC = 18$ .

- 10. Let  $a_1 < a_2 < a_3 < a_4$  be positive integers such that the following conditions hold:
  - $gcd(a_i, a_j) > 1$  holds for all integers  $1 \le i < j \le 4$ .
  - $gcd(a_i, a_j, a_k) = 1$  holds for all integers  $1 \le i < j < k \le 4$ .

Find the smallest possible value of  $a_4$ .

Proposed by James Lin.

**Answer**. 231

**Solution.** Every two integers must share at least one prime factor, but this prime factor cannot divide a third number to comply with the second condition. Therefore there must be at least six different primes  $p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}$  such that  $p_{1,2}p_{1,3}p_{1,4} \mid a_1, p_{1,2}p_{2,3}p_{2,4} \mid a_2, p_{1,3}p_{2,3}p_{3,4} \mid a_3, p_{1,4}p_{2,4}p_{3,4} \mid a_4$ . In order to minimize  $a_4$ , we may assume that  $\{p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}\} = \{2, 3, 5, 7, 11, 13\}$  and  $a_1 = p_{1,2}p_{1,3}p_{1,4}$ ;  $a_2 = p_{1,2}p_{2,3}p_{2,4}$ ;  $a_3 = p_{1,3}p_{2,3}p_{3,4}$ ;  $a_4 = p_{1,4}p_{2,4}p_{3,4}$ . We will ignore the condition that  $a_1 < a_2 < a_3 < a_4$  and instead find  $\max(a_1, a_2, a_3, a_4)$ . Assume that  $p_{3,4} = 13$ . If  $11 \in \{p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}\}$ , then  $\max(a_3, a_4) \ge 2 \cdot 11 \cdot 13 = 286$ . Otherwise  $p_{1,2} = 11$ , and as  $p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}$  are again symmetric, we can assume that  $p_{2,4} = 7$ . Hence, we have  $a_1 = 11p_{1,3}p_{1,4}, a_2 = 77p_{2,3}, a_3 = 13p_{1,3}p_{2,3}, a_4 = 91p_{1,4}$ , so since  $\max(p_{1,4}, p_{2,3}) \ge 3$ , we have that  $\max(a_3, a_4) \ge 77 \cdot 3 = 231$ . This maximum value of 231 can be achieved by letting  $p_{2,3} = 3, p_{1,4} = 2, p_{1,3} = 5$ , so that  $a_1 = 110, a_2 = 231, a_3 = 195, a_4 = 182$ .

11. Let f be a random permutation on  $\{1, 2, ..., 100\}$  satisfying f(1) > f(4) and f(9) > f(16). The probability that f(1) > f(16) > f(25) can be written as  $\frac{m}{n}$  where m and n are relatively prime positive integers. Compute 100m + n.

Note: In other words, f is a function such that  $\{f(1), f(2), \dots, f(100)\}$  is a permutation of  $\{1, 2, \dots, 100\}$ . Proposed by Evan Chen.

**Answer.** 730

**Solution.** There are  $\frac{5!}{2! \cdot 2!} = 30$  ways to order f(1), f(4), f(9), f(16), f(25) such that the given condition is true.

Now, we will count the number of these orderings of f(1), f(4), f(9), f(16), f(25) such that the second condition is true. First we count the number of conditions such that f(1) > f(16). Consider f(1), f(4), f(9), f(16), and note that f(1), f(9) > f(16). If f(4) > f(16), there are 3 ways to order f(1), f(4), f(9) since f(1) > f(4) and there is 1 way to place f(25). If f(16) > f(4), then there are 2 ways to order f(1) and f(9), and there are 2 ways to place f(25). Hence, of the 30 ways to order the five numbers,  $3 + 2 \cdot 2 = 7$  of them are valid, so the desired probability is  $\frac{7}{30}$ , and our answer is 730.

Alternate Solution. consider the ordering of f(1), f(9), f(16), f(25) first. If it's f(9) > f(1) > f(16) > f(25), then there are 3 ways to place f(4). If it's f(1) > f(9) > f(16) > f(25), then there are 4 ways to place f(4), this also gives 3+4=7 valid orders. Hence, the probability is  $\frac{7}{30}$ .

12. For each positive integer  $n \ge 2$ , define k(n) to be the largest integer m such that  $(n!)^m$  divides 2016!. What is the minimum possible value of n + k(n)?

Proposed by Tristan Shin.

Answer. 89

**Solution.** Let a = 2016.

Let m be a positive integer. First, note that  $m |x| \leq |mx|$  for all positive real numbers x. In particular,

$$m\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \leq \sum_{k=1}^{\infty} \left\lfloor \frac{mn}{p^k} \right\rfloor$$

for all primes p and positive integers m and n. But by Legendre's Formula, this is just  $mv_p(n!) \le v_p((mn)!)$ . We can then see that  $(n!)^m \mid (nm)!$  for all positive integers m and n.

Now, this means that  $(n!)^{\left\lfloor \frac{a}{n} \right\rfloor} \mid (n \lfloor \frac{a}{n} \rfloor)! \mid a!$ , as  $n \lfloor \frac{a}{n} \rfloor \leq a$ . This gives that  $k(n) \geq \lfloor \frac{a}{n} \rfloor > \frac{a}{n} - 1$ . But then  $n + k(n) > n + \frac{a}{n} - 1 \geq 2\sqrt{a} - 1 \approx 88.8$  by AM-GM. Thus,  $n + k(n) \geq 89$ , as it is an integer.

Next, we will that  $k(47) \le 42$ . This is obvious, as  $(47!)^{43}$  cannot divide 2016! because  $v_{47}(2016!) = \left|\frac{2016}{47}\right| = 42$ .

But then  $89 \le 47 + k (47) \le 89$ , so hence 89 = 47 + k (47). Thus, the answer is 89.

- 13. Let  $A_1B_1C_1$  be a triangle with  $A_1B_1 = 16$ ,  $B_1C_1 = 14$ , and  $C_1A_1 = 10$ . Given a positive integer i and a triangle  $A_iB_iC_i$  with circumcenter  $O_i$ , define triangle  $A_{i+1}B_{i+1}C_{i+1}$  in the following way:
  - (a)  $A_{i+1}$  is on side  $B_iC_i$  such that  $C_iA_{i+1} = 2B_iA_{i+1}$ .
  - (b)  $B_{i+1} \neq C_i$  is the intersection of line  $A_i C_i$  with the circumcircle of  $O_i A_{i+1} C_i$ .
  - (c)  $C_{i+1} \neq B_i$  is the intersection of line  $A_i B_i$  with the circumcircle of  $O_i A_{i+1} B_i$ .

Find

$$\left(\sum_{i=1}^{\infty} [A_i B_i C_i]\right)^2.$$

Note: [K] denotes the area of K.

Proposed by Yang Liu.

**Answer.** 10800

**Solution.** Note that for all integers  $i \geq 1$ ,  $A_i B_{i+1} O_i C_{i+1}$  is a cyclic quadrilateral by Miquel's Theo- $\angle B_i A_i O_i = \angle B_i A_i C_i. \text{ Similarly, we can show } \angle C_{i+1} B_{i+1} A_{i+1} = \angle C_i B_i A_i, \text{ so } \triangle A_{i+1} B_{i+1} C_{i+1} \sim \triangle A_i B_i C_i. \text{ Hence, it follows that } \frac{[A_{i+1} B_{i+1} C_{i+1}]}{[A_i B_i C_i]} = \frac{[A_2 B_2 C_2]}{[A_1 B_1 C_1]}, \text{ so it suffices to calculate } \frac{[A_2 B_2 C_2]}{[A_1 B_1 C_1]}.$ 

Let  $R_1$  and  $R_2$  denote the circumradii of  $A_1B_1C_1$  and  $A_2B_2C_2$ , respectively. Note that  $\angle C_2A_2O_1 + \angle A_2C_2B_2 = \angle B_1A_1O_1 + \angle A_1C_1B_1 = 90^\circ$ , so  $C_2O_1$  is perpendicular to  $A_2B_2$ . Similarly,  $B_2O_1$  is perpendicular to  $C_2A_2$ , so  $O_1$  is the orthocenter of  $A_2B_2C_2$ . Note that it's well-known by the Law of Cosines that  $\angle B_1A_1C_1 = 60^\circ \implies O_2B_1C_1 = 30^\circ$ . Also,  $R_1 = \frac{B_1C_1}{2\sin B_1A_1C_1} = \frac{14}{\sqrt{3}}$ ,  $B_1A_2 = \frac{14}{3}$ ,

and  $[A_1B_1C_1] = \frac{1}{2} \cdot A_1B_1 \cdot A_1C_1 \cdot \sin B_1A_1C_1 = 40\sqrt{3}$ . By the Law of Cosines on  $\triangle A_2B_1O_1$ ,  $A_2B_1 = \sqrt{\left(\frac{14}{\sqrt{3}}\right)^2 + \left(\frac{14}{3}\right)^2 - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{14}{\sqrt{3}} \cdot \frac{14}{3}} = \frac{14}{3}$ . Since  $O_1$  is the orthocenter of  $A_2B_2C_2$ , it's well-known

that  $\frac{14}{3} = A_2 O_1 = 2R_2 \cos B_2 A_2 C_2 = 2R_2 \cos B_1 A_1 C_1 = R_2$ . Then,  $\frac{[A_2 B_2 C_2]}{[A_1 B_1 C_1]} = \left(\frac{R_2}{R_1}\right)^2 = \frac{1}{3}$  and

$$\left(\sum_{i=1}^{\infty} [A_i B_i C_i]\right)^2 = \left(\frac{[A_1 B_1 C_1]}{1 - \frac{1}{3}}\right)^2 = \left(60\sqrt{3}\right)^2 = 10800.$$

Alternate Solution: As before, note that  $\triangle A_{i+1}B_{i+1}C_{i+1} \sim \triangle A_iB_iC_i$ , so we only need to compute  $\frac{[A_2B_2C_2]}{[A_1B_1C_1]}$ . Let M denote the midpoint of  $B_1C_1$ . It can be computed that  $A_2M=\frac{7}{3}$  and

 $O_1M = A_1O_1 \sin O_1A_1M = \frac{7}{\sqrt{3}}$ , so  $\angle A_2O_1M = 30^\circ$  and  $90^\circ = \angle A_2O_1C_1 = \angle A_2B_2C_1$ . Then since

 $\sin B_1 C_1 A_1 = \frac{B_1 A_1 \sin B_1 A_1 C_1}{B_1 C_1} = \frac{16 \cdot \frac{\sqrt{3}}{2}}{14} = \frac{4\sqrt{3}}{7}$ , it follows that  $A_2 B_2 = A_2 C_1 \sin B_1 C_1 A_1 = \frac{16 \cdot \frac{\sqrt{3}}{2}}{14} = \frac{4\sqrt{3}}{7}$ 

 $\frac{28}{3} \cdot \frac{4\sqrt{3}}{7} = \frac{16}{\sqrt{3}}.$  It follows that  $\frac{[A_2B_2C_2]}{[A_1B_1C_1]} = \left(\frac{A_2B_2}{A_1B_1}\right)^2 = \frac{1}{3}.$  Proceed to do the calculation as in the

- 14. In Yang's number theory class, Michael K, Michael M, and Michael R take a series of tests. Afterwards, Yang makes the following observations about the test scores:
  - Michael K had an average test score of 90, Michael M had an average test score of 91, and Michael R had an average test score of 92.
  - Michael K took more tests than Michael M, who in turn took more tests than Michael R.
  - Michael M got a higher total test score than Michael R, who in turn got a higher total test score than Michael K. (The total test score is the sum of the test scores over all tests)

What is the least number of tests that Michael K, Michael M, and Michael R could have taken combined?

Proposed by James Lin.

**Answer.** | 413 |

**Solution.** Say that Michael K took x tests, Michael M took y tests, and Michael R took z tests, so we need to minimize x+y+z given that x>y>z and 91y>92z>90x. Note that  $y\leq x-1\implies 90x<$  $91y \le 91(x-1) \implies 91 < x$ . If z = x-2, then  $90x < 92z = 92(x-2) \implies x < 92$ , contradicting 91 < x. Hence,  $z \le x - 3$ .

If z = x - 3, then  $92z > 90x = 90(z + 3) \implies z > 135$ . If y - z = 1, it follows that 92z < 91y = 1 $91(z+1) \implies z < 91$ , a contradiction. Hence, y-z=2, so  $x+y+z \ge 139+138+136=413$ .

Now, assume that z = x - k for an integer  $k \ge 3$ . Then,  $92z > 90x = 90(z + k) \implies z > 45k$ . Then,  $x + y + z > 3z > 135k \ge 540$ . It follows that the minimum possible value of x + y + z is 413.

15. Two bored millionaires, Bilion and Trilion, decide to play a game. They each have a sufficient supply of \$1,\$2,\$5, and \$10 bills. Starting with Bilion, they take turns putting one of the bills they have into a pile. The game ends when the bills in the pile total exactly \$1,000,000, and whoever makes the last move wins the \$1,000,000 in the pile (if the pile is worth more than \$1,000,000 after a move, then the person who made the last move loses instead, and the other person wins the amount of cash in the pile). Assuming optimal play, how many dollars will the winning player gain?

Proposed by Yannick Yao.

**Answer.** 333333

16. For her zeroth project at Magic School, Emilia needs to grow six perfectly-shaped apple trees. First she plants six tree saplings at the end of Day 0. On each day afterwards, Emilia attempts to use her magic to turn each sapling into a perfectly-shaped apple tree, and for each sapling she succeeds in turning it into a perfectly-shaped apple tree that day with a probability of  $\frac{1}{2}$ . (Once a sapling is turned into a perfectly-shaped apple tree, it will stay a perfectly-shaped apple tree.) The expected number of days it will take Emilia to obtain six perfectly-shaped apple trees is  $\frac{m}{n}$  for relatively prime positive integers m and n. Find 100m + n.

Proposed by Yannick Yao.

**Answer.** 789953

**Solution.** Let N be the number of days that Emilia took to grow six perfectly-shaped apple trees. We can see that

$$\mathbb{E}(N) = 1 \cdot P(N = 1) + 2 \cdot P(N = 2) + 3 \cdot P(N = 3) + \dots$$
  
=  $P(N \ge 1) + P(N \ge 2) + P(N \ge 3) + \dots$ 

which means that it suffices to sum the probability that the first day is needed, the second day is needed, etc.

For each integer k, the probability that Emilia does *not* need the k-th day is the probability that all 6 saplings are grown into perfectly-shaped apple trees in the first k-1 days. By complementary counting, we can find  $P(N \ge k) = 1 - \left(1 - \left(\frac{1}{2}\right)^{k-1}\right)^6$ .

In order to sum the probabilities over all positive integers k, we rewrite the probability as

$$P(N \geq k) = 6(\frac{1}{2})^{k-1} - 15(\frac{1}{4})^{k-1} + 20(\frac{1}{8})^{k-1} - 15(\frac{1}{16})^{k-1} + 6(\frac{1}{32})^{k-1} - (\frac{1}{64})^{k-1}$$

and sum each of the six terms term across all k since each of them is a geometric series. We can see that the sum is 6(2/1) - 15(4/3) + 20(8/7) - 15(16/15) + 6(32/31) - (64/63) = 7880/1953, and the answer is 789953.

- 17. Let n be a positive integer. S is a set of points such that the points in S are arranged in a regular 2016-simplex grid, with an edge of the simplex having n points in S. (For example, the 2-dimensional analog would have  $\frac{n(n+1)}{2}$  points arranged in an equilateral triangle grid). Each point in S is labeled with a real number such that the following conditions hold:
  - Not all the points in S are labeled with 0.
  - If  $\ell$  is a line that is parallel to an edge of the simplex and that passes through at least one point in S, then the labels of all the points in S that are on  $\ell$  add to 0.
  - The labels of the points in S are symmetric along any such line  $\ell$ .

Find the smallest positive integer n such that this is possible.

Note: A regular 2016-simplex has 2017 vertices in 2016-dimensional space such that the distances between every pair of vertices are equal.

Proposed by James Lin.

**Answer.** 4066273

**Solution.** We interpret S as a polynomial  $P(x_1, x_2, \ldots, x_{2017})$  expressed in Chinese Dumbass Notation. That is, if the vertices of the 2016-simplex are  $V_1, V_2, \ldots, V_{2017}$ , then we associate  $V_i$  with the monomial  $x_i^{n-1}$ , and for all points of the form  $W = \frac{1}{n-1} \left( c_1 V_1 + c_2 V_2 + \cdots + c_{2017} V_{2017} \right)$  for non-negative integers  $c_1, c_2, \ldots, c_{2017}$  summing to n-1 (Where the points are treated as vectors), we associate W with the monomial  $x_1^{c_1} x_2^{c_2} \ldots x_{2017}^{c_{2017}}$ . It's clear that the set of all possible tuples  $(c_1, c_2, \ldots, c_{2017})$  corresponds exactly with the points in S. Then, if W is labeled with w, we let  $P = \sum_{W \in S} w x_1^{c_1} x_2^{c_2} \ldots x_{2017}^{c_{2017}}$ . The first condition tells us that  $P \neq 0$ . Let T be the set of all

lines  $\ell$  parallel to  $V_1V_2$  and passing through at least one point S. Then, applying the second condition over all lines in T tells us that  $P(x_1,x_1,x_3,\ldots,x_{2017})=0$ , and applying the third condition over all lines in T tells us that  $P(x_1,x_2,x_3,\ldots,x_{2017})=P(x_2,x_1x_3,\ldots,x_{2017})$ . It's well-known that this occurs if and only if  $(x_1-x_2)^2|P(x_1,x_2,\ldots,x_{2017})$ , so by symmetry, it follows that  $\prod_{1\leq i < j \leq 2017} (x_i-x_j)^2|P(x_1,x_2,\ldots,x_{2017}), \text{ so } n-1=\deg P \geq 2\binom{2017}{2}.$  Equality can be achieved simply

by letting 
$$P(x_1, x_2, \dots, x_{2017}) = \prod_{1 \le i < j \le 2017} (x_i - x_j)^2$$
, so  $n = 2\binom{2017}{2} + 1 = 4066273$ .

18. Find the smallest positive integer k such that there exist positive integers M, O > 1 satisfying

$$(O \cdot M \cdot O)^k = (O \cdot M) \cdot \underbrace{(N \cdot O \cdot M) \cdot (N \cdot O \cdot M) \cdot \dots \cdot (N \cdot O \cdot M)}_{2016 \ (N \cdot O \cdot M)s},$$

where  $N = O^M$ .

Note: This is edited from the previous text, which did not clarify that NOM represented  $N \cdot O \cdot M$ , for example.

Proposed by Yannick Yao and James Lin.

**Answer.** 2823

**Solution.** Rewrite as  $O^{2k}M^k = O^{2017+2016M}M^{2017}$ . Note that if  $k \leq 2016$ , then  $O^{4032}M^{2016} \geq O^{2k}M^k = O^{2017+2016M}M^{2017}$ , a contradiction since M>1. If k=2017, then we need  $O^{4034}M^{2017}=O^{2017+2016M}M^{2017}$ , which has no integer solution for M since O>1. Hence, k>2017, so rewrite the equation as  $M^{k-2017}=O^{2017+2016M-2k}$ . Note that  $\log_M O$  is a rational number.

If M=2, then we have  $2^{k-2017}=O^{6049-2k}$ , so we have  $6049-2k\mid k-2017\mid 2k-4034\implies 6049-2k\mid 2015=5\cdot 13\cdot 31$ . Since k>2017, 6049-2k<2015, so hence we have  $6049-2k\le 403=13\cdot 31\implies k\ge 2823$ . k=2823 is achieved when M=2 and O=4.

Now, assume for the sake of contradiction that some k < 2823 gives a solution for M and O. We must have  $M \ge 3$ . If  $M \le O$ , then  $k - 2017 \ge 2017 + 2016M - 2k \ge 2017 + 2016 \cdot 3 - 2k \implies 3k \ge 10082$ , which contradicts k < 2823.

Hence, M > O. Let  $M = x^y$ , where  $x \ge 2$  is not the perfect power of any integer (Other than itself) and  $y \ge 2$ , since otherwise there is no choice for O. Then,  $O \ge x$ , so

$$x^{y(k-2017)} = M^{k-2017} = O^{2017+2016M-2k} \ge x^{2017+2016x^y-2k}$$

$$\implies y(k-2017) \ge 2017 + 2016x^y - 2k \ge 2017 + 2016 \cdot 2^y - 2k$$

$$\implies (y+2)(k-2017) \ge 2016 \cdot 2^y - 2017$$

$$\implies 806 > k - 2017 \ge \frac{2016 \cdot 2^y - 2017}{y+2}.$$

We will now show by induction on y that for  $y \geq 2$ , we have  $806 \leq \frac{2016 \cdot 2^y - 2017}{y+2}$ , which will show that k < 2823 is impossible. This is clearly true for y = 2, now assume it is true for y = z for some integer  $z \geq 2$ . We will show it holds for y = z + 1. Note that  $806 \leq \frac{2016 \cdot 2^z - 2017}{z+2} = \frac{2(2016 \cdot 2^z - 2017)}{2(z+2)} \leq \frac{2016 \cdot 2^{z+1} - 2017}{z+3}$ , so the induction is complete.

19. Let S be the set of all polynomials Q(x, y, z) with coefficients in  $\{0, 1\}$  such that there exists a homogeneous polynomial P(x, y, z) of degree 2016 with integer coefficients and a polynomial R(x, y, z) with integer coefficients so that

$$P(x, y, z)Q(x, y, z) = P(yz, zx, xy) + 2R(x, y, z)$$

and P(1,1,1) is odd. Determine the size of S.

Note: A homogeneous polynomial of degree d consists solely of terms of degree d.

Proposed by Vincent Huang.

**Answer.** 509545

**Solution.** We work entirely in  $\mathbb{F}_2[X,Y,Z]$ . First it's clear that deg Q=2016 by examining degrees in the relation.

Now, notice that P(x, y, z)|P(yz, zx, xy). Therefore, we know that  $P(yz, zx, xy)|P(zx \cdot xy, yz \cdot xy, yz \cdot zx) = P(xyz \cdot x, xyz \cdot y, xyz \cdot z) = (xyz)^{2016}P(x, y, z)$ , where the last equality holds because P is homogeneous with degree 2016.

Then it's clear that  $Q(x, y, z)|(xyz)^{2016}$ , implying that Q(x, y, z) is a monomial of the form  $x^ay^bz^c$  with a + b + c = 2016.

Now suppose P has any term  $x^dy^ez^f$ . Then P(yz,zx,xy) contains the term  $x^{d+a}y^{e+b}z^{f+c}$ . But this term can only come from multiplying  $(yz)^{n-d-a}(zx)^{n-b-e}(xy)^{n-c-f}$  which means that P(x,y,z) contains a term of the form  $x^{n-d-a}y^{n-b-e}z^{n-c-f}$ .

So we have a bijection in the nonzero terms of P which is  $(d, e, f) \iff (n - d - a, n - b - e, n - c - f)$ . But since  $P(1, 1, 1) \neq 0 \pmod{2}$  by the given condition, we must have that some (d, e, f) is equal to (n - d - a, n - b - e, n - c - f), otherwise the total number of terms would be even.

Then we deduce n=2d+a=2e+b=2f+c. It's evident that  $a,b,c\equiv 2016 \mod 2$  with a+b+c=2016. Clearly all such polynomials in Q will work as we can just take  $P=x^{0.5(n-a)}y^{0.5(n-b)}z^{0.5(n-c)}$ .

To evaluate the size of Q, note there's a bijection between the working triples (a, b, c) and the solutions to x + y + z = 1008, which evaluates to  $\binom{1008+2}{2} = 509545$ .

20. For a positive integer k, define the sequence  $\{a_n\}_{n\geq 0}$  such that  $a_0=1$  and for all positive integers n,  $a_n$  is the smallest positive integer greater than  $a_{n-1}$  for which  $a_n\equiv ka_{n-1}\pmod{2017}$ . What is the number of positive integers  $1\leq k\leq 2016$  for which  $a_{2016}=1+\binom{2017}{2}$ ?

Proposed by James Lin.

**Answer.** 1953

**Solution.** Let p=2017 and  $M=\mathbb{E}(a_{n+1}-a_n)$ , where n ranges from 0 to p-2, inclusive. We need  $M=\frac{p}{2}$  for  $a_{2016}=1+\binom{2017}{2}$ . Note that for  $k=1,\ M=p$ , so we can assume that k>1. If  $k^{n+1}-k^n\equiv i\pmod{p}$ , for  $1\leq i\leq p-1$ , then  $i\equiv k^n(k-1)\pmod{p}$ .

Let  $ord_p(k)=d$ . We will show that  $M=\frac{p}{2}$  if and only if d is even. Note that M is also  $\mathbb{E}(a_{n+1}-a_n)$ , where n ranges from 0 to d-1, inclusive. If d=2g is even, then note that  $k^{-g}\equiv -1\pmod{p}$  so  $k^j\equiv -k^{j+g}\neq 0\pmod{p}$  for all nonnegative integers j. Since  $k-1\neq 0\pmod{p}$ ,  $k^j(k-1)=-k^{j+g}(k-1)\neq 0\pmod{p}$ . Then, it's clear that  $M=\frac{p}{2}$  by ranging over all  $0\leq j\leq g-1$ . If d is

odd, then note that it is impossible for  $M = \frac{p}{2}$ , as then it would require  $a_d - a_0 = \frac{dp}{2}$ , which is not a positive integer.

Let q be the largest odd factor of p-1. It's well-known that the number of residues with order d modulo p, where d|p-1, is  $\phi(d)$ , so our answer is  $(p-1)-\sum_{d|q}\phi(d)=(p-1)-q$ , so letting p=2017

gives q = 63, and our answer is 1953.

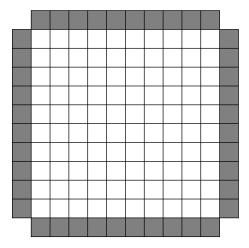
21. Mark the Martian and Bark the Bartian live on planet Blok, in the year 2019. Mark and Bark decide to play a game on a  $10 \times 10$  grid of cells. First, Mark randomly generates a subset S of  $\{1, 2, ..., 2019\}$  with |S| = 100. Then, Bark writes each of the 100 integers in a different cell of the  $10 \times 10$  grid. Afterwards, Bark constructs a solid out of this grid in the following way: for each grid cell, if the number written on it is n, then she stacks  $n \times 1 \times 1 \times 1$  blocks on top of one other in that cell. Let B be the largest possible surface area of the resulting solid, including the bottom of the solid, over all possible ways Bark could have inserted the 100 integers into the grid of cells. Find the expected value of B over all possible sets S Mark could have generated.

Proposed by Yang Liu.

**Answer.** 234040

**Solution.** Let the numbers Mark generate be  $s_1 < s_2 < \cdots < s_{100}$  be the number that Mark chooses. Note that any other number  $s \in \{1, 2, \dots, 2019\} \setminus S$  if equally likely to be in the open intervals  $(0, s_1), (s_1, s_2), \dots (s_{99}, s_{100}), (s_{100}, 2020)$ , so it follows that  $\mathbb{E}(s_k) = \frac{2019 - 100}{101} \cdot k + k = 20k$ .

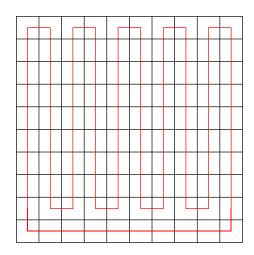
Now, fix  $s_1, s_2, \ldots s_{100}$ , and randomly place them in the grid cells. Draw *empty* cells next to the edges of the grid (they are shaded below), and define *total* cells to be the union of grid cells and empty cells. Say that two total cells are *adjacent* if they share a side. Define a grid cell to be an *edge* cell if it is adjacent to an empty cell.



Label the rows and columns of the grid by 1, 2, ..., 10, and let the cell in the *i*th row and *j* th column be denoted by  $c_{i,j}$ , with a total of  $t_{i,j}$  blocks on it. Furthermore, denote  $t_{i,j} = 0$  when either i = 0, i = 11, j = 0, or j = 11. Let  $a_{i,j}$  for  $1 \le i, j \le 10$  denote the size of the set  $\{t: t \in \{t_{i-1,j}, t_{i,j-1}, t_{i+1,j}, t_{i,j+1}\}$  and  $t < t_{i,j}\}$ . That is, the stack on a grid cell  $c_{i,j}$  is taller than  $a_{i,j}$  of the stack on its 4 adjacent total cells (Where the empty cells are stacks of 0 blocks). Note that the top and bottom of the stacks on grid cells contribute 200 to the surface area of Bark's solid; now, it remains to calculate the lateral surface area of the solid. Note that the region between any two stacks on two adjacent total cells of size x and y contribute |x - y| to the surface area, so it becomes clear that the lateral surface area of the solid is simply  $\sum_{1 \le i,j \le 10} (2a_{i,j} - 4)t_{i,j}$ . Since the  $t_{i,j}$ 's are fixed, it

that the lateral surface area of the solid is simply  $\sum_{1 \leq i,j \leq 10} (2a_{i,j}-4)t_{i,j}$ . Since the  $t_{i,j}$ 's are fixed, it suffices to maximize  $\sum_{1 \leq i,j \leq 10} a_{i,j}t_{i,j}$ . Write the set  $\{a_{i,j}: 1 \leq i,j \leq 10\}$  as  $\{b_1,b_2,\dots b_{100}\}$ , so we need to maximize  $M = \sum_{1 \leq i \leq 100} b_i s_{101-i}$ .

Define a sequence  $c_i$  by  $c_i=4$  for  $1 \le i \le 50$ ,  $c_i=2$  for  $i=51 \le i \le 52$ ,  $c_i=1$  for  $53 \le i \le 68$ , and  $c_i=0$  for  $69 \le i \le 100$ . We will show that the sequence  $c_i$  majorizes the sequence  $b_i$ , or that  $c_1+c_2+\cdots+c_i \ge b_1+b_2+\cdots+b_i$  for  $1 \le i \le 100$ , where equality holds for i=100. Since the sequence  $\{s_{101-i}\}$  is increasing, it's clear that M is maximized when  $b_i=c_i$  for all i. Let  $y_i=b_1+b_2+\cdots+b_i$  and  $z_i=c_1+c_2+\cdots+c_i$  for  $1 \le i \le 100$ . Note that  $y_{100}=220=z_{100}$  since there are 220 pairs of adjacent total cells, and it follows that  $y_i \le z_i=220$  for  $68 \le i \le 99$ . Furthermore,  $0 \le b_i \le 4$  for all i, so  $y_i \le z_i=4i$  for  $1 \le i \le 50$  certainly holds.



Next, we will show that  $y_{51} \leq z_{51} = 202$ . If  $b_{50} < 4$ , then it's clear that we are done. Otherwise, assume  $b_{50} = 4$ . We can find a Hamiltonian cycle on the grid cells, as labeled in red above. Note that  $a_{i,j} = 4$  cannot hold for consecutive grid cells along the Hamiltonian cycle, since  $b_{50} = 4$ , it follows that in fact every other cell along the Hamiltonian cycle satisfies  $a_{i,j} = 4$ . This means that a the cells satisfying  $a_{i,j} = 4$  are the black squares on a  $10 \times 10$  chessboard, but it's then clear that  $b_{51} \leq 2$  since any white square on a chessboard neighbors at least two black squares.

Now, it remains to prove that  $b_i \le c_i = 152 + i$  for  $52 \le i \le 67$ . Note that we can tile the  $8 \times 8$  grid cells in rows 2-9 and columns 2-9 with 32 dominoes, and  $a_{i,j} \ge 1$  for at least one grid cell in each domino, and  $a_{i,j} \ge 1$  for all 36 edge grid cells, so  $b_{68} \ge 1$ . It's then clear that that  $b_{i+1} + b_{i+2} + \cdots + b_{68} \ge 68 - i$  for  $52 \le i \le 67$ , so we are done.

Hence, our answer is maximized when  $b_i = 4$  for  $1 \le i \le 50$ ,  $b_i = 2$  for  $51 \le i \le 52$ ,  $b_i = 1$  for  $53 \le i \le 68$ , and  $b_i = 0$  for  $69 \le i \le 100$ . This means that by linearity of expectation, our answer is

$$200 + \mathbb{E}(4 \cdot (s_{51} + s_{52} + \dots + s_{100}) - 2 \cdot (s_{33} + s_{34} + \dots + s_{48}) - 4 \cdot (s_1 + s_2 + \dots + s_{32})))$$

$$= 200 + (4 \cdot (\mathbb{E}(s_{51}) + \mathbb{E}(s_{52}) + \dots + \mathbb{E}(s_{100})) - 2 \cdot (\mathbb{E}(s_{33}) + \mathbb{E}(s_{34}) + \dots + \mathbb{E}(s_{48})) - 4 \cdot (\mathbb{E}(s_1) + \mathbb{E}(s_2) + \dots + \mathbb{E}(s_{32}))$$

$$= 200 + 302000 - 25920 - 42240$$

$$= 234040$$

22. Let ABC be a triangle with AB = 3 and AC = 4. It is given that there does not exist a point D, different from A and not lying on line BC, such that the Euler line of ABC coincides with the Euler line of DBC. The square of the product of all possible lengths of BC can be expressed in the form  $m + n\sqrt{p}$ , where m, n, and p are positive integers and p is not divisible by the square of any prime. Find 100m + 10n + p.

Note: For this problem, consider every line passing through the center of an equilateral triangle to be an Euler line of the equilateral triangle. Hence, if D is chosen such that DBC is an equilateral triangle and the Euler line of ABC passes through the center of DBC, then consider the Euler line of ABC to coincide with "the" Euler line of DBC.

Proposed by Michael Ren.

**Answer.** 10782

**Solution.** Suppose that ABC and DBC have the same Euler line  $\ell$ . Then, note that the circumcenters of ABC and DBC must lie on both the perpendicular bisector of BC and  $\ell$ . Because  $AB \neq AC$ ,  $\ell$  is not the perpendicular bisector of BC, so the two lines do not coincide. This means that ABC and DBC have a common circumcenter, O.

Let M be the midpoint of BC. Note that a homothety centered at M with ratio 3 takes  $\ell$  to AD because the centroids of ABC and DBC lie on  $\ell$ . Hence, D does not exist if and only if the parallel to  $\ell$  through A does not intersect the circumcircle of ABC at a point different from A, B, and C. We either have that  $\ell \parallel AB$ ,  $\ell \parallel AC$ , or  $\ell$  is parallel to the tangent to the circumcircle of ABC at A.

Let 
$$x = BC$$
. By the Law of Cosines,  $\cos A = \frac{25-x^2}{24}$ ,  $\cos B = \frac{x^2-7}{6x}$ , and  $\cos C = \frac{x^2+7}{8x}$ .

Let H and O respectively be the orthocenter and circumcenter of ABC. It is well-known that the heights from H and O to AB have lengths  $2R\cos A\cos B$  and  $R\cos C$ , respectively. If  $OH \parallel AB$ , then we must have  $2\cos A\cos B = \cos C$ , or  $2\cdot \frac{25-x^2}{24}\cdot \frac{x^2-7}{6x} = \frac{x^2+7}{8x} \Longrightarrow (25-x^2)(x^2-7) = 9(x^2+7) \Longrightarrow x^4-23x^2+238=0$ , which has no real solutions.

Similarly, if  $OH \parallel AC$ , then we must have  $2\cos A\cos C = \cos B$ , or  $2 \cdot \frac{25-x^2}{24} \cdot \frac{x^2+7}{8x} = \frac{x^2-7}{6x} \implies (25-x^2)(x^2+7) = 16(x^2-7) \implies x^4-2x^2+287 \implies x^2=1\pm 12\sqrt{2}$ , so  $x^2=1+12\sqrt{2}$  is the only solution for this case.

Finally, if  $\ell$  is parallel to the tangent to the circumcircle of ABC at A, then note that  $\angle AOG = 90^\circ$ , where G is the centroid of ABC. Let R be the circumradius of ABC. It is well known that  $AG^2 = \frac{-a^2 + 2b^2 + 2c^2}{9}$  and  $OG^2 = R^2 - \frac{a^2 + b^2 + c^2}{9}$ . Hence, by the Pythagorean theorem we have that  $AO^2 + OG^2 = AG^2 \implies R^2 + R^2 - \frac{a^2 + b^2 + c^2}{9} = \frac{-a^2 + 2b^2 + 2c^2}{9} \implies b^2 + c^2 = 6R^2 \implies R^2 = \frac{25}{6}$ . We just want to find the product of the possible lengths of the third side of a triangle with sides 4 and 3 and a fixed circumradius R. Suppose that those sides correspond to an inscribed arc of measure x and y respectively. Then, note that the third side can correspond to an inscribed arc of measure x-y or x+y. Then, their product is  $2R\sin(x+y)2R\sin(x-y) = 4R^2\sin(x+y)\sin(x-y) = 2R^2(\cos(2x)-\cos(2y)) = 4R^2(\sin^2 x - \sin^2 y) = 4^2 - 3^2 = 7$ .

The product of the squares of all solutions is  $7^2(1+12\sqrt{2})=49+588\sqrt{2}$ , so the answer is 10782.

- 23. Let  $\mathbb{N}$  denote the set of positive integers. Let  $f: \mathbb{N} \to \mathbb{N}$  be a function such that the following conditions hold:
  - For any  $n \in \mathbb{N}$ , we have  $f(n)|n^{2016}$ .
  - For any  $a, b, c \in \mathbb{N}$  satisfying  $a^2 + b^2 = c^2$ , we have f(a)f(b) = f(c).

Over all possible functions f, determine the number of distinct values that can be achieved by f(2014) + f(2) - f(2016).

Proposed by Vincent Huang.

**Answer.** 2035153.

**Solution.** Let P(x, y, z) denote the assertion that  $x^2 + y^2 = z^2$  and f(x)f(y) = f(z). Let a denote an odd positive integer and let k denote a positive integer. Note that f(1) = 1, and if a > 1,

$$P\left(a, \frac{a^2 - 1}{2}, \frac{a^2 + 1}{2}\right) \implies f(a) \mid f\left(\frac{a^2 + 1}{2}\right) \mid \left(\frac{a^2 + 1}{2}\right)^{2016}$$

$$\implies f(a) \mid \gcd\left(a^{2016}, \left(\frac{a^2 + 1}{2}\right)^{2016}\right) = \gcd\left(a, \frac{a^2 + 1}{2}\right)^{2016} = 1,$$

so f(x) = 1 for all odd integers x. Then,

$$P(2^k a, 2^{2k-2}a^2 - 1, 2^{2k-2}a^2 + 1) \implies f(2^k a) \mid f(2^{2k-2}a^2 + 1).$$

If k > 1, then  $f(2^{2k-2}a^2 + 1) = 1$  since  $2^{2k-2}a^2 + 1$  is odd, so  $f(2^ka) = 1$ , so f(x) = 1 for all integers x divisible by 4.

If 
$$k = 1$$
, then  $f(2a) \mid f(a^2 + 1) \mid (a^2 + 1)^{2016} \implies f(2a) \mid \gcd((2a)^{2016}, (a^2 + 1)^{2016}) = \gcd(2a, a^2 + 1)^{2016} = 2^{2016}$ .

Hence,  $f(2a) \in \{2^0, 2^1, 2^2, \dots, 2^{2016}\}$  for all odd a. Note that f(2) can be any of these 2017 values without affecting any of the other values of f since 2 is not a part of any Pythagorean triple. For all  $0 \le i \le 2016$ , there exists a function f such that  $f(2014) = 2^i$ , namely  $f(2a) = 2^i$  for all odd a > 1. Thus, it suffices to find the number of values that can be achieved by  $2^x + 2^y - 1$  for integers  $0 \le x, y \le 2016$ , which is just  $\binom{2018}{2} = 2035153$ .

24. Let P(x,y) be a polynomial such that  $\deg_x(P), \deg_y(P) \leq 2020$  and

$$P(i,j) = \binom{i+j}{i}$$

over all 2021<sup>2</sup> ordered pairs (i, j) with  $0 \le i, j \le 2020$ . Find the remainder when P(4040, 4040) is divided by 2017.

Note:  $\deg_x(P)$  is the highest exponent of x in a nonzero term of P(x,y).  $\deg_y(P)$  is defined similarly. Proposed by Michael Ren.

**Answer.** 1555

**Solution.** Let n = 2020. Note that the polynomial is unique by solving a linear system of  $(n+1)^2$  linearly independent equations in  $(n+1)^2$  variables. We claim that

$$P(x,y) = {x \choose 0} {y \choose 0} + {x \choose 1} {y \choose 1} + {x \choose 2} {y \choose 2} + \dots + {x \choose n} {y \choose n}$$

satisfies the condition.

Indeed, note that

$$P(i,j) = \sum_{k=0}^{n} \binom{i}{k} \binom{j}{k} = \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k}$$

because  $\binom{i}{k} = 0$  for k > i and  $\binom{j}{k} = 0$  for k > j.

Without loss of generality, assume that  $i \leq j$ . Then, we have that

$$P(i,j) = \sum_{k=0}^{i} \binom{i}{k} \binom{j}{k} = \sum_{k=0}^{i} \binom{i}{k-i} \binom{j}{k} = \binom{i+j}{i}$$

by Vandermonde's identity, as desired.

Hence, the answer is

$$P(2n,2n) = \sum_{k=0}^{n} {2n \choose k}^2 = \sum_{k=0}^{n-1} {2n \choose k} {2n \choose 2n-k} + {2n \choose n}^2 = \frac{\sum_{k=0}^{2n} {2n \choose k} {2n \choose 2n-k} + {2n \choose n}^2}{2} = \frac{{4n \choose 2n} + {2n \choose 2n}^2}{2}$$

by Vandermonde's identity again.

Now, by Lucas's Theorem, the answer is  $\frac{\binom{8080}{4040} + \binom{4040}{2020}^2}{2} \equiv \frac{\binom{4}{2}\binom{12}{6} + \left[\binom{2}{1}\binom{6}{3}\right]^2}{2} = 3572 \equiv 1555 \pmod{2017}$ 

25. Let  $X_1X_2X_3$  be a triangle with  $X_1X_2=4, X_2X_3=5, X_3X_1=7$ , and centroid G. For all integers  $n\geq 3$ , define the set  $S_n$  to be the set of  $n^2$  ordered pairs (i,j) such that  $1\leq i\leq n$  and  $1\leq j\leq n$ . Then, for each integer  $n\geq 3$ , when given the points  $X_1,X_2,\ldots,X_n$ , randomly choose an element  $(i,j)\in S_n$  and define  $X_{n+1}$  to be the midpoint of  $X_i$  and  $X_j$ . The value of

$$\sum_{i=0}^{\infty} \left( \mathbb{E}\left[ X_{i+4} G^2 \right] \left( \frac{3}{4} \right)^i \right)$$

can be expressed in the form  $p + q \ln 2 + r \ln 3$  for rational numbers p, q, r. Let  $|p| + |q| + |r| = \frac{m}{n}$  for relatively prime positive integers m and n. Compute 100m + n.

Note:  $\mathbb{E}(x)$  denotes the expected value of x.

Proposed by Yang Liu.

**Answer.** 932821

**Solution.** Without loss of generality, let the centroid G be at the origin, so G = 0 as a vector. Let  $\mathbb{E}_n$  denote the expectation over all the choices of  $(i,j) \in S_m$  for  $3 \leq m \leq n-1$ . In other words, the points  $X_1, \ldots, X_n$  have been chosen. Define

$$a_n = \mathbb{E}_n[\mathbb{E}_{1 \le i \le n}[X_i^2]]$$
 and  $b_n = \mathbb{E}_n[\mathbb{E}_{1 \le i \le n}[X_i]^2].$ 

A direct computation shows that  $a_3 = 10$  and  $b_3 = 0$ . We now compute a recursion for  $a_n$  and  $b_n$ .

Note that

$$\mathbb{E}_{n+1}[X_{n+1}^2] = \mathbb{E}_n \left[ \mathbb{E}_{(i,j) \in S_n} \left[ \left( \frac{X_i + X_j}{2} \right)^2 \right] \right] = \frac{1}{2} a_n + \frac{1}{2} b_n.$$

Therefore,

$$a_{n+1} = \mathbb{E}_{n+1}[\mathbb{E}_{1 \le i \le n+1}[X_i^2]] = \frac{na_n + \mathbb{E}_{n+1}[X_{n+1}^2]}{n+1} = \frac{(2n+1)a_n + b_n}{2n+2}.$$

We can do similarly computations for  $b_n$ . We can compute

$$\mathbb{E}_{n+1}[\mathbb{E}_{1 \le i \le n+1}[X_i]^2] = \mathbb{E}_n \left[ \mathbb{E}_{(i,j) \in S_n} \left[ \left( \frac{X_1 + \dots + X_n + X_{n+1}}{n+1} \right)^2 \right] \right].$$

If we let  $T_n = X_1 + \dots X_n$ , then the previous expression equals

$$\frac{1}{(n+1)^2}\mathbb{E}_n\left[\mathbb{E}_{(i,j)\in S_n}\left[T_n^2+2T_nX_{n+1}+X_{n+1}^2\right]\right] = \frac{1}{(n+1)^2}\mathbb{E}_n\left[\frac{n+2}{n}T_n^2+X_{n+1}^2\right] = \frac{(2n^2+4n+1)b_n+a_n}{2(n+1)^2}$$

as  $\mathbb{E}_n[X_{n+1}^2] = \frac{1}{2}a_n + \frac{1}{2}b_n$  as above and  $\mathbb{E}_n[T_n^2] = n^2b_n$  by definition

Our next claim is the explicit formulas for  $b_{n+1} - b_n$  and  $a_{n+1} - a_n$  for  $n \ge 3$ . The formulas are

$$a_{n+1} - a_n = -\frac{96}{7} \frac{1}{4^{n+1} \cdot n} {2(n+1) \choose n+1}$$
 and  $b_{n+1} - b_n = \frac{96}{7} \frac{1}{4^{n+1} \cdot n(n+1)} {2(n+1) \choose n+1}$ .

Though the proof is messy, one can verify this by induction. The proof is omitted. Now, define  $s_n = \mathbb{E}_{n+1}[X_{n+1}^2] = \frac{1}{2}(a_n + b_n)$  for  $n \geq 3$ , and otherwise,  $s_n = 0$ . In particular,  $s_3 = \frac{1}{2}a_3 = 5$ . By the above,

$$s_{n+1} - s_n = -\frac{48}{7} \frac{1}{4^{n+1} \cdot (n+1)} {2(n+1) \choose n+1}.$$

Our final step will be to compute the generating function

$$\sum_{n\geq 0} s_{n+3} x^n = (1-x)^{-1} \left( 5 + \sum_{n\geq 1} (s_{n+3} - s_{n+2}) x^n \right) = (1-x)^{-1} \left( 5 - \frac{48}{7} \sum_{n\geq 1} \frac{1}{4^{n+3}(n+3)} {2(n+3) \choose n+3} x^n \right).$$

Let's only deal with the innermost sum for now. Note that

$$\sum_{n\geq 1} \frac{1}{4^{n+3}(n+3)} \binom{2(n+3)}{n+3} x^n = x^{-3} \int \sum_{n\geq 1} \frac{1}{4^{n+3}} \binom{2(n+3)}{n+3} x^{n+2} dx,$$

where the  $\int$  denotes an antiderivative (we will find the correct constant term later). Only dealing with the antiderivative right now, and remembering that  $\sum_{n>0} \frac{1}{4^n} {2n \choose n} x^n = \frac{1}{\sqrt{1-x}}$ ,

$$\begin{split} \int \sum_{n \geq 1} \frac{1}{4^{n+3}} \binom{2(n+3)}{n+3} x^{n+2} dx &= \int \frac{1}{x} \left( \frac{1}{\sqrt{1-x}} - 1 - \frac{1}{2}x - \frac{3}{8}x^2 - \frac{5}{16}x^3 \right) dx \\ &= C - 2\ln(1+\sqrt{1-x}) - \frac{1}{2}x - \frac{3}{16}x^2 - \frac{5}{48}x^3 \end{split}$$

for some constant C. Note that substituting x=0 should give a result of 0, so  $C=2\ln 2$ . Substituting everything back into the original expression,

$$\sum_{n>0} s_{n+3} x^n = (1-x)^{-1} \left( 5 - \frac{48}{7} \left( \frac{2\ln 2 - 2\ln(1+\sqrt{1-x}) - \frac{1}{2}x - \frac{3}{16}x^2 - \frac{5}{48}x^3}{x^3} \right) \right).$$

Substituting  $x = \frac{3}{4}$  gives us the final answer

$$\frac{1136}{21} - \frac{16384}{63} \ln 2 + \frac{8192}{63} \ln 3 \implies |p| + |q| + |r| = \frac{9328}{21} \implies 100m + n = 932821.$$

26. Let ABC be a triangle with BC = 9, CA = 8, and AB = 10. Let the incenter and incircle of ABC be I and  $\gamma$ , respectively, and let N be the midpoint of major arc BC of the circumcircle of ABC. Line NI meets the circumcircle of ABC a second time at P. Let the line through I perpendicular to AI meet segments AB, AC, and AP at  $C_1$ ,  $B_1$ , and Q, respectively. Let  $B_2$  lie on segment CQ such that line  $B_1B_2$  is tangent to  $\gamma$ , and let  $C_2$  lie on segment BQ such that line  $C_1C_2$  tangent to  $\gamma$ . The length of  $B_2C_2$  can be expressed in the form  $\frac{m}{n}$  for relatively prime positive integers m and n. Determine 100m + n.

Proposed by Vincent Huang.

**Answer.** 72163

**Solution.** It is well known that  $(B_1C_1P)$  is the A-mixtillinear circle  $\omega$  of  $\triangle ABC$ , i.e.  $\omega$  is tangent to AB, AC, and (ABC).

**Lemma 1:**  $BC_1 : B_1C = BP : CP$ .

**Proof:** Let BP, CP meet  $\omega$  at  $B_3, C_3$ . A homothety centered at P takes  $\omega$  to (ABC) and  $B_3C_3$  to BC, so  $B_3C_3||BC$ . Then  $BC_1^2: B_1C^2 = BB_3 \cdot BP: CC_3 \cdot CP = BP^2: CP^2$ , hence  $BC_1: B_1C = BP: CP\blacksquare$ .

Now let NI meet BC at R, so that by the Angle Bisector Theorem on  $\angle BPC$ , BR : CR = BP : CP. Combining this with lemma 1, we know by the converse of Ceva that AR,  $BB_1$ ,  $CC_1$  concur.

Let Q' be the projection of R onto  $B_1C_1$  and let  $B_1C_1$ , BC meet at S. Due to the concurrency (S,R;B,C) is harmonic, hence since  $\angle SQ'R=90^\circ$  it's well-known that Q'R,Q'S must bisect  $\angle BQ'C$ . Then  $\angle CQ'B_1=\angle BQ'C_1$ ,  $\angle BC_1Q'=\angle CB_1Q'$ , so  $\triangle BC_1Q'\sim\triangle CB_1Q'$ , thus  $BQ':CQ'=BC_1:CB_1=BP:CP$ .

It's clear that  $C_1Q: B_1Q = AC_1\sin QAC_1AB_1\sin B_1AQ = \sin BAP: \sin CAP = BP: CP$ , hence Q = Q'. Define  $X = BQ \cap AC, Y = CQ \cap AB$ . Then since  $QY, QC_2$  are reflections across  $B_1C_1$ , and it's easy to see  $C_1C_2$ , AB are reflections, we see that  $C_2, Y$  are reflections across  $B_1C_1$ , and similarly  $X, B_2$  are reflections so  $B_2C_2 = XY$ .

It remains to compute some lengths. It's known that  $\omega$  is the image of  $\gamma$  under a homothety centered at A with factor  $\frac{1}{\cos^2 0.5A}$ , so  $AB_1 = AC_1 = \frac{(s-a)}{\cos^2 0.5A} = \frac{b+c-a}{1+\cos A} = \frac{2bc}{a+b+c}$ , and thus  $BC_1 = \frac{c(a+c-b)}{a+b+c}$ ,  $CB_1 = \frac{b(a+b-c)}{a+b+c}$ .

Now from similar triangles YBQ, XCQ we know that  $BY: CX = BC_1: B_1C = c(a+c-b): b(a+b-c)$ . From  $\angle XBY = \angle XCY$  we know BCXY is cyclic, hence  $AX \cdot AC = AY \cdot AB$ . Let k be the ratio of similarity between  $\triangle AXY, \triangle ABC$  so that BY = c - bk, CX = b - ck. Then using the ratio from before, we can solve to deduce  $k = \frac{2bc}{b^2+c^2+ab+ac} \implies B_2C_2 = XY = ka = \frac{2abc}{b^2+c^2+ab+ac}$  which is  $\frac{720}{163}$ , yielding an answer of 72163

27. Compute the number of monic polynomials q(x) with integer coefficients of degree 12 such that there exists an integer polynomial p(x) satisfying  $q(x)p(x) = q(x^2)$ .

Proposed by Yang Liu.

**Answer.** 569

**Solution.** Notice that it is necessary and sufficient to have  $q(x)|q(x^2)$  due to polynomial long division. Now, if r is a root of q(x) then we easily find that  $r^2$  is as well. Thus  $r^{2^n}$  is a root of q for all positive integers n, so since  $\{r^{2^n}: n \in \mathbb{N}\}$  is a finite set so there are positive integers  $m \neq n$  with  $r^{2^m} = r^{2^n}$ . Therefore each root of q(x) is either zero or a root of unity.

Now let

$$q(x) = x^n \prod_{m=1}^{\infty} \Phi_m(x)^{e_m},$$

where n and the  $e_i$  are nonnegative integers with

$$n + \sum_{m=1}^{\infty} e_m \phi(m) = 12,$$

defining  $\Phi_m$  to be the *m*th cyclotomic polynomial.

Utilizing  $\Phi_n(x^2) = \Phi_n(x)\Phi_{2n}(x)$  when n is odd and  $\Phi_n(x^2) = \Phi_{2n}(x)$  when n is even, we find that  $q(x)|q(x^2)$  becomes

$$x^{n} \prod_{m=1}^{\infty} \Phi_{m}(x)^{e_{m}} \left| x^{2n} \prod_{k=1}^{\infty} (\Phi_{2k-1}(x) \Phi_{4k-2}(x))^{e_{2k-1}} \prod_{k=1}^{\infty} \Phi_{4k}(x)^{e_{2k}} \right|.$$

It follows that  $q(x)|q(x^2)$  if and only if  $e_n \ge e_{2n}$  for all integers n.

Therefore, if t is the largest non-negative integer for which  $x^t|q(x)$ , then we can break  $q(x)/x^t$  uniquely up into products of the form

$$\kappa_{2^{v}m}(x) = \Phi_m(x)\Phi_{2m}(x)\cdots\Phi_{2^{v}m}(x),$$

where m is odd. For instance,

$$\Phi_3(x)^5 \Phi_6(x)^3 \Phi_{12}(x)^2 \Phi_{24}(x) = \kappa_{24}(x) \kappa_{12}(x) \kappa_6(x) \kappa_3(x)^2.$$

Let  $k_{2^v m} = \deg \kappa_{2^v m}(x)$ . Note that  $k_{2^v m} = 2^v \phi(m)$  if m is odd and v is a nonnegative integer, and that this degree must be at most 12.

The only odd powers of primes n with  $\phi(n) \leq 12$  are n = 3, 9, 5, 7, 11, 13, with  $\phi(3) = 2, \phi(9) = 6, \phi(5) = 4, \phi(7) = 6, \phi(11) = 10, \phi(13) = 12$ , hence it follows from  $\phi(mn) = \phi(m)\phi(n)$  for relatively prime m, n, the only other odd n with  $\phi(n) \leq 12$  are n = 1, 15, 21 with  $\phi(1) = 1, \phi(15) = 8, \phi(21) = 12$ . Hence,  $k_1 = 1, k_2 = k_3 = 2, k_4 = k_5 = k_6 = 4, k_7 = k_9 = 6, k_8 = k_{10} = k_{12} = k_{15} = 8, k_{11} = 10$ , and  $k_{13} = k_{14} = k_{18} = k_{21} = 12$ . It then becomes clear that the number of desired q(x) is just the coefficient of  $x^{12}$  in

$$\frac{1}{(1-x)^2(1-x^2)^2(1-x^4)^3(1-x^6)^2(1-x^8)^4(1-x^{10})(1-x^{12})^4},$$

where the extra factor of 1-x in the denominator accounts for the factor of  $x^t$  in q(x). Now, it simply remains the above expression modulo  $x^{13}$ :

$$\frac{1}{(1-x)^2(1-x^2)^2(1-x^4)^3(1-x^6)^2(1-x^8)^4(1-x^{10})(1-x^{12})^4} \\
\equiv \frac{(1+x^{10})(1+4x^{12})}{(1-x)^2(1-x^2)^2(1-x^4)^3(1-x^6)^2(1-x^8)^4} \pmod{x^{13}} \\
\equiv \frac{(1+4x^8)(1+x^{10}+4x^{12})}{(1-x)^2(1-x^2)^2(1-x^4)^3(1-x^6)^2} \pmod{x^{13}} \\
\equiv \frac{(1+2x^6+3x^{12})(1+4x^8+x^{10}+4x^{12})}{(1-x)^2(1-x^2)^2(1-x^4)^3} \pmod{x^{13}} \\
\equiv \frac{(1+3x^4+6x^8+10x^{12})(1+2x^6+4x^8+x^{10}+7x^{12})}{(1-x)^2(1-x^2)^2} \pmod{x^{13}} \\
\equiv \frac{(1+2x^2+3x^4+4x^6+5x^8+6x^{10}+7x^{12})(1+3x^4+2x^6+10x^8+7x^{10}+29x^{12})}{(1-x)^2} \pmod{x^{13}} \\
\equiv \frac{1+2x^2+6x^4+12x^6+28x^8+51x^{10}+103x^{12}}{(1-x)^2} \pmod{x^{13}}.$$

We wish to find the coefficient of  $x^{12}$  in the final expression. Since the numerator is an even polynomial, we just need to find

$$[x^{12}](1+3x^2+5x^4+7x^6+9x^8+11x^{10}+13x^{12})(1+2x^2+6x^4+12x^6+28x^8+51x^{10}+103x^{12})\\ =1\cdot 103+3\cdot 51+5\cdot 28+7\cdot 12+9\cdot 6+11\cdot 2+13\cdot 1\\ =569.$$

28. Let ABC be a triangle with AB = 34, BC = 25, and CA = 39. Let O, H, and  $\omega$  be the circumcenter, orthocenter, and circumcircle of  $\triangle ABC$ , respectively. Let line AH meet  $\omega$  a second time at  $A_1$  and let the reflection of H over the perpendicular bisector of BC be  $H_1$ . Suppose the line through O perpendicular to  $A_1O$  meets  $\omega$  at two points Q and R with Q on minor arc AC and R on minor arc AB. Denote  $\mathcal{H}$  as the hyperbola passing through  $A, B, C, H, H_1$ , and suppose HO meets  $\mathcal{H}$  again at P. Let X, Y be points with  $XH \parallel AR \parallel YP, XP \parallel AQ \parallel YH$ . Let  $P_1, P_2$  be points on the tangent to  $\mathcal{H}$  at P with  $P_1 \parallel P_2$  and let  $P_3, P_4$  be points on the tangent to  $P_4$  at  $P_4$  with  $P_4$  and  $P_4$  meet at  $P_4$  meet at  $P_4$  and  $P_4$  meet at  $P_4$  meet at  $P_4$  and  $P_4$  meet at  $P_4$  and  $P_4$  meet at  $P_4$  meet at P

Proposed by Vincent Huang.

**Answer**. 43040

**Solution.** Let D be on  $\omega$  with AD||BC.

**Lemma 1:** Given any points A', B', C', a hyperbola  $\mathcal{H}'$  through A', B', C' is rectangular (has perpendicular asymptotes) if and only if  $\mathcal{H}'$  passes through the orthocenter H' of  $\triangle A'B'C'$ .

**Proof:** This is well-known, see Theorems 1 and 2 here: http://www.artofproblemsolving.com/community/c2927h1273728\_rectangular\_circumhyperbolas. □

By Lemma 1 on  $\triangle ABC$ ,  $\mathcal{H}$  is rectangular. By the converse of lemma 1 on  $\triangle BH_1C$ , since D is the orthocenter of  $\triangle DH_1C$ , we know  $\mathcal{H}$  passes through D. Therefore the isogonal conjugate of  $\mathcal{H}$  in  $\triangle ABC$  is line OD' where D' is the isogonal conjugate of D (so it is at infinity). It's clear from the isogonality of AD, AD' that OD' is parallel to the A-tangent in  $\omega$ . Then if OD' meets  $\omega$  at Q', R' with Q' on minor arc AB and R' on minor arc AC, we know that  $Q'R' = OD' \perp AO$ . Hence Q'R', QR are symmetric in the perpendicular bisector of BC, meaning that AR, AR' are isogonal, as are AQ, AQ' in  $\angle BAC$ . Since Q', R' are the isogonal conjugates of the points at infinity lying on  $\mathcal{H}$ , it follows that AQ, AR are parallel to the asymptotes of  $\mathcal{H}$ .

Next let  $U = \infty_{AR}$ ,  $V = \infty_{AQ}$ . Define N' as the center of  $\mathcal{H}$ . Let  $P'_1 = N'U \cap PP$ ,  $P'_2 = N'V \cap PP$ ,  $P'_3 = N'V \cap HH$ ,  $P'_4 = N'U \cap HH$ . Let  $Z = PP \cap HH$ .

**Lemma 2:**  $P'_1, X, P'_3$  are collinear, as are  $P'_2, Y, P'_4$ .

**Proof:** By Newton's Theorem on quadrilateral  $N'P_1'ZP_3'$  with inscribed conic  $\mathcal{H}$  we deduce that  $N'Z, P_1'P_3', HU, PV$  concur at X. Similarly,  $N'Z, P_2'P_4', HV, PU$  concur at Y. We also know that since N'Z passes through X, Y that N'XYZ is collinear.  $\square$ 

**Lemma 3:**  $P_1'P_3'||HP||P_2'P_4'$ .

**Proof:** First, note that the polar of  $P_1'$  in  $\mathcal{H}$  is PU, and similarly the polar of  $P_3'$  in  $\mathcal{H}$  is HV, hence the pole of  $P_1'P_3'$  is just  $PU \cap HV = Y$ . Meanwhile, the pole of UV is N', and the pole of PH is Z, Since Y, N', Z are collinear from lemma 2, we know that  $UV, PH, P_1'P_3'$  concur at infinity, hence  $P_1'P_3'||PH$ , and similarly  $P_2'P_4'||HP$ , as desired.  $\square$ 

By Lemmas 2 and 3, we know that  $P'_1 = P_1$ ,  $P'_2 = P_2$ ,  $P'_3 = P_3$ ,  $P'_4 = P_4$ . Hence  $P_1P_4$ ,  $P_2P_3$  meet at N = N', the center of  $\mathcal{H}$ .

Let  $N_1$  be the midpoint of DH. Let  $H_B, H_C$  be the orthocenters of  $\triangle ADC, \triangle ADB$ . It's well-known that  $N_1$  is the midpoint of  $AH_1, DH, BH_B, CH_C$ , and we know by lemma 1 on  $\triangle ADC, \triangle ADB$  that  $H_B, H_C \in \mathcal{H}$ , hence the reflection of  $\mathcal{H}$  about  $N_1$  has at least eight points in common with  $\mathcal{H}$ , so they are the same hyperbola. Hence  $N = N_1$  is the center of the hyperbola.

Now by homothety centered at D with ratio 2, if  $A_2$  is the foot of the altitude from A to BC, we know  $NO=0.5HA_1=HA_2$ . But  $HA_2=2R\cos B\cos C=2\cdot\frac{1105}{56}\cdot\frac{13}{85}\cdot\frac{33}{65}=\frac{429}{140}$ , hence the answer is 43040.

- 29. Let n be a positive integer. Yang the Saltant Sanguivorous Shearling is on the side of a very steep mountain that is embedded in the coordinate plane. There is a blood river along the line y = x, which Yang may reach but is not permitted to go above (i.e. Yang is allowed to be located at (2016, 2015) and (2016, 2016), but not at (2016, 2017)). Yang is currently located at (0,0) and wishes to reach (n,0). Yang is permitted only to make the following moves:
  - Yang may spring, which consists of going from a point (x, y) to the point (x, y + 1).
  - Yang may stroll, which consists of going from a point (x, y) to the point (x + 1, y).
  - Yang may sink, which consists of going from a point (x,y) to the point (x,y-1).

In addition, whenever Yang does a sink, he breaks his tiny little legs and may no longer do a spring at any time afterwards. Yang also expends a lot of energy doing a spring and gets bloodthirsty, so he must visit the blood river at least once afterwards to quench his bloodthirst. (So Yang may still spring while bloodthirsty, but he may not finish his journey while bloodthirsty.) Let there be  $a_n$  different ways for which Yang can reach (n,0), given that Yang is permitted to pass by (n,0) in the middle of his journey. Find the 2016th smallest positive integer n for which  $a_n \equiv 1 \pmod{5}$ .

Proposed by James Lin.

**Answer.** 475756

**Solution.** Let  $C_k = \frac{1}{k+1} \binom{2k}{k}$  denote the kth Catalan number. If (k,k) is the last point on Yang's path where he is at the River, then there are  $C_k$  ways for Yang to reach (k,k), and  $\binom{n}{k}$  ways for Yang to go

from (k,k) to (n,0). Hence,  $a_n = \sum_{i=0}^n C_i \binom{n}{i}$ . We will now use generating functions, with the aid of two well-known lemmas:

$$\bullet \sum_{n=i}^{\infty} \binom{n}{i} x^n = \frac{x^i}{(1-x)^{i+1}}.$$

$$\bullet \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Now, notice that  $a_n$  is the coefficient of  $x^n$  in

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n} C_i \binom{n}{i} x^n$$

$$= \sum_{i=0}^{\infty} C_i \sum_{n=i}^{\infty} \binom{n}{i} x^n$$

$$= \sum_{i=0}^{\infty} C_i \left( \frac{x^i}{(1-x)^{i+1}} \right)$$

$$= \frac{1}{1-x} \sum_{i=0}^{\infty} C_i \left( \frac{x}{1-x} \right)^i$$

$$= \frac{1}{1-x} \cdot \frac{1-\sqrt{1-\frac{4x}{1-x}}}{\frac{2x}{1-x}}$$

$$= \frac{1-\sqrt{\frac{1-5x}{1-x}}}{2x}$$

Note that  $a_n \equiv 1 \pmod{5}$  exactly when the coefficient of  $x^{n+1}$  in  $\sqrt{\frac{1-5x}{1-x}}$  is 3 (mod 5), where we note that all coefficients of this expression are integers since the  $a_i$ 's are integers. Note that  $\sqrt{1-5x} \equiv 1 \pmod{5}$ , so it remains to compute when  $N = [x^{n+1}]\sqrt{\frac{1}{1-x}} = \frac{\binom{2n+2}{4^{n+1}}}{4^{n+1}}$  is 3 (mod 5).

Write the base 5 representation of n+1 as  $b_k b_{k-1} \dots b_0$  and the base 5 representation of 2n+2 as  $c_l c_{l-1} \dots c_0$ . If there is a 3 or 4 among  $b_k, b_{k-1}, \dots, b_0$ 's, then consider  $b_i$ , the rightmost 3 or 4 in the the representation. Then, since there is no carryover when doubling any of  $b_{i-1}, b_{i-2}, \dots, b_0$ , it follows that  $c_i \equiv 2b_i \pmod{5}$ . But then  $c_i < b_i$  whether  $b_i = 3$  or  $b_i = 4$ , so by Lucas's Theorem,  $5 \binom{c_i}{b_i} \binom{2n+2}{n+1}$ , so  $N \neq 3 \pmod{5}$  in this case.

Otherwise, k = l and  $b_k, b_{k-1}, \ldots, b_0 \in \{0, 1, 2\}$ . Note that  $\binom{0}{0} \equiv 1 \pmod{5}$ ,  $\binom{2}{1} \equiv 2 \pmod{5}$ , and  $\binom{4}{2} \equiv 1 \pmod{5}$ . Hence, if there are t 1's in the base 5-representation of n+1, then  $\binom{2n+2}{n+1} \equiv 2^t \pmod{5}$ . Furthermore, note that  $n+1 \equiv t \pmod{2}$ , so  $4^{n+1} \equiv 4^t \pmod{5}$  and thus  $N \equiv 3^t \pmod{5}$ . Hence,  $a_n \equiv 1 \pmod{5}$  if and only if n+1 has only 0,1,2 in its base 5 representation, where the number of 1's are 1 (mod 4).

It can easily be computed that the 2016th smallest integer is  $n+1=110211012_5, n=110211011_5=475756.$ 

30. Let  $P_1(x), P_2(x), \ldots, P_n(x)$  be monic, non-constant polynomials with integer coefficients and let Q(x) be a polynomial with integer coefficients such that

$$x^{2^{2016}} + x + 1 = P_1(x)P_2(x)\dots P_n(x) + 2Q(x).$$

Suppose that the maximum possible value of 2016n can be written in the form  $2^{b_1} + 2^{b_2} + \cdots + 2^{b_k}$  for nonnegative integers  $b_1 < b_2 < \cdots < b_k$ . Find the value of  $b_1 + b_2 + \cdots + b_k$ .

Proposed by Michael Ren.

**Answer.** 3977

**Solution.** Let k = 2016. Working in  $\mathbb{F}_2$ , we want to find the number of irreducible factors of  $x^{2^k} + x + 1$ . First, we claim that  $x^{2^m} + x + 1 \mid x^{2^k} + x + 1$  if and only if  $\frac{k}{m}$  is an odd integer. Note that  $(x^{2^m} + x + 1)^{2^i} = x^{2^{m+i}} + x^{2^i} + 1$  in  $\mathbb{F}_2$ , so  $x^{2^m} + x + 1 \mid x^{2^{m+i}} + x^{2^i} + 1$  for any positive integer i. Now,  $x^{2^m} + x + 1 \mid x^{2^k} + x^{2^{k-m}} + 1$ , so  $x^{2^m} + x + 1 \mid x^{2^{k-m}} + x$ . Also,  $x^{2^m} + x + 1 \mid x^{2^{k-m}} + x^{2^{k-2m}} + 1$ ,

so  $x^{2^m} + x + 1 \mid x^{2^{k-2m}} + x + 1$ . Hence, we can reduce  $k \mod 2m$ . Clearly, if  $k \equiv m \pmod{2m}$  then the divisibility is true, so the if direction is proven. For the only if direction, assume that  $0 \le k < 2m$ . Clearly,  $k \ge m$  or the degrees don't work out. But then, we can reduce to  $x^{2^{k-m}} + x$ , so if  $k \ne m$  then we obtain another contradiction with degrees.

Now, we claim that all irreducible factors of  $x^{2^k}+x+1$  either are of degree 2k or divide  $x^{2^m}+x+1$  for some m < k. Suppose that z is a root of an irreducible factor of  $x^{2^k}+x+1$  that does not divide  $x^{2^m}+x+1$  for any m < k. Then, by the Frobenius endomorphism,  $z^{2^k}=z+1$  is also a root of  $x^{2^k}+x+1$ , so  $z=(z+1)^{2^k}=z^{2^{2^k}}$ , so z is an element of  $\mathbb{F}_{2^{2k}}$ . Since  $z^{2^k}=z+1 \neq z$ , z is not an element of  $\mathbb{F}_{2^k}$ . Suppose that z is an element of  $\mathbb{F}_{2^m}$  for some  $2m \mid 2k$  and  $\frac{k}{m}$  is an odd positive integer greater than 1 since z is not an element of  $\mathbb{F}_{2^k}$ . However, this means that  $z+1=z^{2^k}=(z^{2^{2^m}})^{2^{\frac{k-m}{2m}}}\cdot z^{2^m}=z^{2^m}$ , so z is a root of  $x^{2^m}+x+1$ , a contradiction.

To finish, we define the sequence  $a_1, a_2, \ldots$  as  $\sum_{\substack{n+d \ 2d} \in \mathbb{N}} a_d = 2^n$ . Note that we wish to compute  $2016\left(\frac{1}{2}\sum_{\substack{k+d \ 2d} \in \mathbb{N}} \frac{a_d}{d}\right)$ . Note that  $a_{32} = 2^{32}, \ a_{96} = 2^{96} - 2^{32}, \ a_{224} = 2^{224} - 2^{32}, \ a_{288} = 2^{288} - 2^{96}, \ a_{672} = 2^{672} - 2^{224} - 2^{96} + 2^{32}, \ \text{and} \ a_{2016} = 2^{2016} - 2^{672} - 2^{288} + 2^{96}.$  The desired sum then becomes  $2016\left(\frac{1}{2}\left(\frac{2^{2016}}{2016} + \frac{2^{672}}{1008} + \frac{2^{288}}{336} + \frac{2^{224}}{336} + \frac{2^{96}}{168} + \frac{2^{32}}{56}\right)\right) = 2^{2015} + 2^{672} + 2^{289} + 2^{288} + 2^{225} + 2^{224} + 2^{98} + 2^{97} + 2^{36} + 2^{33}$  so the answer is 2015 + 672 + 289 + 288 + 225 + 224 + 98 + 97 + 36 + 33 = 3977.

Note: In general, the sum is  $\frac{1}{2} \sum_{\frac{n}{d} \in 2\mathbb{Z}+1} \frac{2^d}{d} \frac{\varphi(\frac{n}{d})}{\frac{n}{d}} = \frac{1}{2n} \sum_{\frac{n}{d} \in 2\mathbb{N}-1} 2^d \varphi(\frac{n}{d}) = \frac{1}{2n} \left( \sum_{d|n, \ d \text{ odd}} \varphi(d) 2^{\frac{n}{d}} \right)$ .