1 The basics

1.1 Homogenous Coordinates and Projective transformations

For our purposes, we want to view the projective plane as the usual Euclidean plane plus points at infinity. In particular, for every direction we have a point at infinity, and all the points at infinity lie on a line at infinity. From this point of view, a family of parallel lines will all be concurrent "at infinity", and we call a family of lines passing through a point a "pencil" of lines. Algebraically speaking, the projective plane is represented by homogenous coordinates $[X, Y, Z]$; these coordinates are 'immune' to scaling. The usual Euclidean plane sits inside of it by the collection of points $[x, y, 1]$, and the points at infininity are precisely those which have Z-coordinate 0.

Just like how in Euclidean geometry, the automorphisms consist of the usual translation, scaling, rotation and reflection transformations, we have a class of automorphisms on the projective plane called "Projective Transformations". Without going into too much detail, they're the collection of invertible three-by-three matrices. For our purposes, we collect the following facts:

Theorem 1.1. *Any four points that are not collinear can be mapped to any other four points by a unique projective transformation. Furthermore, projective transformations are one-to-one and preserve incidence.*

Let us give some examples of how this is useful.

Proposition 1.2. *Let ABC be a triangle.*

(i) Suppose DEF is another triangle. Then the statements, "AD, BE, CF are concurrent" (perspective from a point) and "$AB \cap DE, AC \cap DF, BC \cap EF$ are collinear" (perspective from a line), are equivalent

(ii) Let P be a point, and let $P_1P_2P_3$ be the cevian triangle of P in ABC. Let $Q_1Q_2Q_3$ be the cevian triangle of a point Q in $P_1P_2P_3$. Then ABC and $Q_1Q_2Q_3$ are perspective from a point.

The above examples highlight an important "geo-sense" that one should develop, and that is when a problem *feels projective*. In general, being able to reducing olympiad problems into more manageable questions about incidence, for which projective geometry is great for, is a good strategy for tackling the tough ones.

1.2 Cross ratio and Harmonic points

Given four collinear points A, B, C, D, the cross-ratio is defined to be the ratio:

$$(A, B; C, D) := \frac{AC \cdot BD}{AD \cdot BC}.$$

In the case that one of these points is infinity, say B, then we declare that the ratio $\frac{BD}{BC} := 1$. As an applicaiton of sine-law, we can obtain that if a, b, c, d are four concurrent lines passing through A, B, C, D then:

$$(A, B; C, D) = \frac{\sin(a, c) \cdot \sin(b, d)}{\sin(a, d) \cdot \sin(b, c)}.$$

Theorem 1.3. *Cross-ratio is preserved under arbitrary projective transformations!*

Due to the results just mentioned, it is comvenient to define the cross ratio of concurrent lines too, via $(a, b; c, d) := (A, B; C; D)$, which we we've shown is independent of the choice of line. (As an exercise, show that if A, B, C, D lie on a circle, then the cross-ratio of these points can be defined as the ratio of the lengths as-well.) This particular viewpoint then motivates the definition of a *perspectivity*, and it gives us a technique to "cross-ratio bash". Let us see it in action with a proof of Menelaus' Theorem.

When the cross-ratio is equal to -1, we call the points A, B, C, D to be harmonic. These are particularly useful, as we will see later on. For the moment, convince yourself that if D, E are the feet of an internal and external angle bisector on BC, then $(D, E; B, C) = -1$, and that if M is the midpoint of BC and ∞_{BC} is the point at infinity on BC, then $(\infty_{BC}, M; B, C) = -1$.
Also, convince yourself of the following:

Theorem 1.4. *Let $ABCD$ be and quadrilateral. Let $E := AB \cap CD, F := BC \cap DA$, $X := AC \cap EF$ and $Y := BD \cap EF$. Then:*

$$(E, F; X, Y) = -1.$$

1.3 Conics

A conic section, from the viewpoint of projective geometry, is the image of a circle under a projective transformation. In particular, parabolas are tangent to the line a infinity, and hyperbolas hits the line at infinity at two distinct points, while ellipses pass through no points at infinity (and are affinely equivalent to circles). As a consequence of the theory we managed to build up, we have the following:

Theorem 1.5. *Given four points A, B, C, D, the locus of points E such that $(EA, EB; EC, ED)$ is constant is a conic.*

This theorem comes in handy quite often. In particular, as lines are degerate conics, it can be useful in proving collinearity results. This will be seen in the problems at the end.

We turn out attention to harmonic quadrilaterals on circles:

Proposition 1.6. *Consider a point P and circle $\mathcal{C}$ with tangents from P being A, B and a line through P meeting $\mathcal{C}$ at C, D. The angle bisectors of $\angle CAD, \angle CBD$ meet on CD at some point E. (using ratios obtained by harmonic quadrilateral). By angle chasing, and using the fact that B is a symmedian of $\triangle CAD$, it is not too hard to obtain that $E \in (P, PA = PB)$ (the circle centered at P with radius $PA = PB$).*

Proposition 1.7. *(Butterfly Theorem) Let M be the midpoint of a chord XY, and let AB and CD be the chords passing through M. Let $E := AD \cap XY$ and $F = BC \cap XY$. Prove that $EM = MF$.*

Proposition 1.8. *Let ABC be a triangle with H its orthocenter. The circle with diameter AC cuts the circumcircle of triangle ABH at K. Prove that the point of intersection of the lines CK and BH is the midpoint of the segment BH.*

2 Transformations

Fix a circle $\mathcal{C}$ and centre O. The polar map with respect to $\mathcal{C}$ interchanges points and lines as follows:

(i) If P is a point other than O, the *pole* of P is the line p through P' perpenndicular to PP', there P' is the inverse of P through $\mathcal{C}$.

(ii) If p is a line not passing through O, the polar of p is the inverse through $\mathcal{C}$ of the foot of the perpendicular from O to p.

(iii) If P is the point at infinity, the pole of P is the line through O perpendicular to the direction of P

(iv) If P is O, the pole of P is the line at infinity, and vice versa

Theorem 2.1. *The polar map satisfies the following properties:*

1. *Every point is the polar of its pole, and ever line is the pole of its polar*
2. *The polar of the line through the points A and B is the intersection of the poles a and b*
3. *Three points are collinear if and only if their poles are concurrent.*

As a consequence, we have the duality principle, which says that a theorem of projective geometry remains true if points and lines are interchanged. For example, Deargue's theorem is self-dual, and proving one direction automatically proves the other. Pascal's theorem and Brianchon's theorem also are dual to eachother, so it suffices to prove one of them.

Proposition 2.2. *(Brokard's Theorem) The points A, B, C, D lie in this order on a circle $\mathcal{C}$ with center O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of PQR. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to $\mathcal{C}$.*

3 Some tricky problems

Proposition 3.1. *Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another*

Proof. Let P', Q' be the inverses for P, Q wrt $\odot ABCD$. Let $L = \odot OP'Q' \cap \odot OPQ$. Note that $ML \equiv MR \perp OL$ since M is midpoint of PQ and R is orthocentre. Let $N = P'Q' \cap PQ \cap OL$. Note the polar of N is therefore KL. It follows NK is tangent to $\odot PQK, \odot ABK$ so done ☐

Proposition 3.2. *Given a cyclic quadrilateral $ABCD$, let L_a lie in the interrior of BCD and such that distancs of this triangle are proportional to the lengths of the corresponding sides. The points L_b, L_c, L_d are defined analogously. Given that the quadrilateral $L_aL_bL_cL-d$ is cyclic, prove that the quadrilateral $ABCD$ has two parallel sides.*

(Hint: Recall one classificaiton of symmedians).

Proposition 3.3. *Let $\triangle ABC$ be a triangle with incenter I and circumcenter O. Let P, Q, R be midpoints of arcs BAC, CBA, ACB in (O) respectively. Let $X = PI \cap BC$, $Y = QI \cap AC$, $Z = RI \cap AB$. Prove that AX, BY, CZ, OI are concurrent.*

Lemma 3.4. *Let the tangency of A mixtilinear incircle $(\mathcal{A})$ with AC be M and tangency of B mixtilinear incircle $(\mathcal{B})$ with BC be N. Then, $MN \parallel AB$.*

Proof. Consider the tangency of both mixtilinear incircles to AB, name them M', N' in obvious way. We know I is the midpoint of $M'M, N'N$, hence it follows $MN \parallel AB$. ☐

Proof. IX, IY meets $\odot ABC$ at X', Y' respectively. Note that $X' \in \mathcal{A}, Y' \in \mathcal{B}$. Let AI, BI meet $\odot ABC$ at A', B' respectively. We have:

$$B(I, X'; A, X) = B'(B', X'; A, C) = A'(A', Y', B, C) = A(I, Y', B, Y)$$

This means $I, AX' \cap BY', AX \cap BY$ are collinear. We know $AX' \cap BY' \in OI \implies AX \cap BY \in OI$ ☐

Proposition 3.5. *In a convex quadrilateral $ABCD$. We have $AB = AD$, $CB = CD$, M is the intersection of AC and BD. Through M draw two lines. One line intersect AB and CD at P and Q. The other one intersect BC and AD at R and S. PR meets BD at G and SQ meets BD at H. If M is the midpoint of BD, prove: $GM = HM$.*

Proof. As we move F on AC, the map $AE \longmapsto AG$ that this movement induces from the pencil of lines through A to itself is a projective map. When $F = C, F = A$, and $F = AC \cap BD$, the lines AE, AG are symmetric wrt AC, which means that our projective transformation coincides with the reflection in AC (of the lines through A, remember) in three different positions. This actually means that the two transformations coincide, and we're done. $\square$

Proposition 3.6. *Let ABC be a triangle and P, Q be isogonal conjugates. Suppose that D and E are the altitudes from P and Q to BC, and let AP and AQ cut the circumcircle at S, T. Prove that $SD \cap TE \in PQ$.*

Proposition 3.7. *Given a $\triangle ABC$ with orthocenter H. Let D be the point such that $ABCD$ is a parallelogram. Let $P \equiv CH \cap BD$, $Q \equiv BH \cap AD$, $R \equiv AC \cap PQ$. Prove that $\angle CBD = \angle RBA$.*

Proof. Let $\mathcal{H}$ be the conic through $ABCDH$ (a hyperbola) and let M be the midpoint $\overline{AD}$. Let $K = BR \cap AD$. Since P, Q, R are collinear, the converse of Pascal's Theorem applied to $CHBBDA$ implies that BR is tangent to $\mathcal{H}$. Thus,

$$\frac{KA}{KD} = (A, D; K, M) \stackrel{B}{=} (A, D; B, C) = \frac{AB^2}{BD^2},$$

so K lies on the B-symmedian of $\triangle ABD$, and the result follows. $\square$

4 Problems

The problems will be arranged in order of difficulty later on.

Proposition 4.1. *Let $ABCD$ be a quadrilateral inscribed in a circle k. AC and BD meet at E. The rays CB and DA meet at F Prove that the line throguh the incenters of ABE and ABF and the line through the incenters of CDE and CDF meet a point lying on circle k.*

Proposition 4.2. *Let ABC be a non-equilateral triangle and let M_a, M_b, M_c be the midpoints of the sides of BC, CA, AB respectively. Let S be a point lying on the EUler line. Denote by X, Y, Z the second intersections of M_AS, M_bS, M_cS with the nine-point circle. Prove that AX, BY, CZ are concurrent.*

For this one, try proving the following first:

Theorem 4.3. *Let ABC be a triangle, Let $P_aP_bP_c$ and $Q_aQ_bQ_c$ be the circumcevian triangles of P and Q respectively. Prove that ABC and the triangle determined by the lines P_aQ_a, P_bQ_b, P_cQ_c are perspective.*

Proposition 4.4. *Consider two circles $\mathcal{C}_1, \mathcal{C}_2$. that intersect at A, B and let their centers be O_1, O_2. Take their common tangents, and denote them ℓ_1, ℓ_2. Suppose $\ell_1 \cap \mathcal{C}_1, \mathcal{C}_2 = X_0, Y_0$. Take a point on ℓ_1, call it P, and consider a line ℓ through P. Let ℓ cut $\mathcal{C}_1$ at X_1, X_2 and $\mathcal{C}_2$ at Y_1, Y_2. Let $X_0X_1 \cap Y_0Y_1 = Q, X_0X_2 \cap Y_0Y_2 = R, X_0X_1 \cap Y_0Y_2 = S, X_0X_2 \cap Y_0Y_1 = T$.*

Claim: Q, R, S, T lie on the same, fixed circle that passes through A, B.

Proposition 4.5. *Let ABC be a triangle with circumcentre O. Let P lie inside the triangle ABC with $\angle PAB\angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on the line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.*

Proposition 4.6. *Let ABD be a triangle with I its incetner. Q be the point at which the incricel touches the line AC and Q the midpoint of AC and K the orthocentre of BIC. Prove that the line KQ is perpendicular to IE.*

Proposition 4.7. *Let ABC have orthocentre H and DEF be the orthic triangle. Let I be the midpoint of BC and suppose IH cuts EF at K, and KD cuts $\mathcal{C}$ (the circumcircle) at T. Prove that AT is a bisector of angle ETF.*

Proposition 4.8. *Let O be the circumcentre and H the orthocentre of a triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\angle FHP = \angle BAC$.*

Proposition 4.9. *Let ABC be a triangle with $AB < AC$. Suppose D and P are feet of the internal and external angle bisectors of $\angle BAC$. Let M be the midpoint of segment BC and $\mathcal{C}$ the ciircumcircle of APD. Suppose Q is on the minor arc AD of $\mathcal{C}$ such that MQ is tangent to $\mathcal{C}$. Suppose that QB meets $\mathcal{C}$ again at R, and the line throught R perpendicular to BC meets PQ at S. Prove that SD is tangent to the circumcircle of QDM.*

Proposition 4.10. *Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.*

Proposition 4.11. *Let M be midpoint of angle bisector AD of triangle ABC. Circle k_1 with diameter AC cuts BM at E, and circle k_2 with diameter AB cuts CM at F.Prove that B, E, F and C are concyclic.*

Proposition 4.12. *An acute triangle ABC with $AB \neq AC$ is given. Let V and D be the feet of the altitude and angle bisector from A, and let E and F be the intersection points of the circumcircle of $\triangle AVD$ with sides AC and AB, respectively. Prove that AD, BE and CF have a common point.*

For the next one, I encourage to think about Apolonius circles:

Proposition 4.13. *Let X be a point inside triangle ABC such that $XA.BC = XB.AC = XC.AC$. Let I_1, I_2, I_3 be the incenters of XBC, XCA, XAB. Prove that AI_1, BI_2, CI_3 are concurrent.*

Proposition 4.14. *Let the quadrilateral $ABCD$ be inscribed in a circle with center O. Suppose that $\angle B$ and $\angle C$ are obtuse, and let lines AB and CD intersect at E. Let P and R be the foot of the perpendiculars dropped from E to the lines BC and AD respectively. Extend EP to hit AD at Q, and let ER hit BC at S. Finally, let K be the midpoint of QS. Prove that E, K, O are collinear.*