Lemmas in Euclidean Geometry

Yufei Zhao

yufeiz@mit.edu

1. **Construction of the symmedian.**

   Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Show that AD coincides with a symmedian of $\triangle ABC$.

   (The *symmedian* is the reflection of the median across the angle bisector, all through the same vertex.)

2. **Diameter of the incircle.**

   Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, show that $BD = CF$.

3. **Dude, where's my spiral center?**

   Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Show that O is the center of the spiral similarity that carries AB to CD.

4. **Arc midpoints are equidistant to vertices and in/excenters**

   Let ABC be a triangle, I its incenter, and I_A, I_B, I_C its excenters. On the circumcircle of ABC, let M be the midpoint of the arc BC not containing A and let N be the midpoint of the arc BC containing A. Show that $MB = MC = MI = MI_A$ and $NB = NC = NI_B = NI_C$.

5. **I is the midpoint of the touch-chord of the mixtilinear incircles**

   Let ABC be a triangle and I its incenter. Let Γ be the circle tangent to sides AB, AC, as well as the circumcircle of ABC. Let Γ touch AB and AC at X and Y, respectively. Show that I is the midpoint of XY.

6. **More curvilinear incircles.** (A generalization of the previous lemma)

   Let ABC be a triangle, I its incenter and D a point on BC. Consider the circle that is tangent to the circumcircle of ABC but is also tangent to DC, DA at E, F respectively. Show that E, F and I are collinear.

7. **Concurrent lines from the incircle.**

   Let the incircle of ABC touch sides BC, CA, AB at D, E, F respectively. Let I be the incenter of ABC and M be the midpoint of BC. Show that the lines EF, DI and AM are concurrent.

8. **More circles around the incircle.**

   Let I be the incenter of triangle ABC, and let its incircle touch sides BC, AC, AB at D, E and F, respectively. Let line CI meet EF at T. Show T, I, D, B, F are concyclic. Also, show that $\angle BTC = 90°$ and that T lies on the line connecting the midpoints of AB and BC.

   An easy way of remembering the third part of the lemma is: for a triangle ABC, draw a midline, an angle bisector, and a touch-chord, each generated from different vertex, then the three lines are concurrent.

9. **Reflections of the orthocenter lie on the circumcircle.**

   Let H be the orthocenter of triangle ABC. Let the reflection of H across the BC be X and the reflection of H across the midpoint of BC be Y. Show that X and Y both lie on the circumcircle of ABC. Moreover, show that AY is a diameter of the circumcircle.

10. **O and H are isogonal conjugates.**

    Let ABC be a triangle, with circumcenter O, orthocenter H, and incenter I. Show AI is the angle bisector of $\angle HAO$.