SAMC 2013

SAUDI ARABIA MATHEMATICAL COMPETITIONS

مسابقات الرّياضيات للمملكة العربيّة السّعوديّة

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Introduction

This booklet contains problems used in the training and selection of the Saudi team for the International Mathematics Olympiad. The training was supported by The Ministry of Education, which commissioned **Mawhiba**, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. **Mawhiba** is an independent establishment, presided by the King of Saudi Arabia, with the Minister of Education as Vice President. We thank King Saud University for giving the opportunity to trainers to contribute in the training of the Saudi Team.

The Saudi team had four main training camps during the academic year 2012-2013 beside the full-time training period that started on April 1, 2013. The team participated in the Asian Pacific Olympiad in March 12, 2013, and in the Gulf Mathematical Olympiad, which was held in Qatar in the period April 1-5, 2013.

It is our pleasure to share these training and selection problems with other IMO teams, hoping it will contribute to a future cooperation.

Dr. Fawzi A. Al-Thukair
Leader of the Saudi Math Olympiad Team

مقدّمة

يحوي هذا الكتيّب على مسائل التّدريب والاختيار للفريق السّعودي للاولمبياد الدّولي للرّياضيات والّذي يعقد في شهر يوليو ٢٠١٣ في جمهوريّة كولومبيا. لقد كان التّدريب مدعوما من وزارة التّربية والتّعليم في المملكة العربيّة السّعوديّة والّتي كلّفت مؤسّسة الملك عبدالعزيز ورجاله للموهبة والابداع " موهبة " للقيام بالمهمّة. تعتبر مؤسّسة " موهبة " مؤسّسة مستقلّة يرأسها الملك عبدالله بن عبدالعزيز رعاه الله وينوب عنه سمو وزير التّربية والتّعليم. وتجدر الاشارة إلى دعم جامعة الملك سعود الّتي أتاحت الفرصة لبعض الاساتذة للقيام بتدريب الفريق.

تمّ تنظيم أربع ملتقيات تدريبيّة للفريق السّعودي خلال العام الدّراسي ٢٠١٢ - ٢٠١٣، بالإضافة إلى فترة التّفريغ الكلّي للتّدريب الّتي بدأج في ١ - ٤ - ٢٠١٣. كما شارك الفريق في اولمبياد الرّياضيات لدول آسيا والباسيفيك $APMO$ في ١٢ مارس ٢٠١٣، واولمبياد دول الخليج GMO الّذي عقد في دولة قطر في الفترة ١ - ٥ أبريل ٢٠١٣.

يسعدني أن أهدي لكم مسائل هذا الكتيّب، راجيا أن تكون بداية تعاون بيننا في مجال الاولمبياد وغيره من حقول المنافسة العلميّة.

د. فوزي بن أحمد الذكير
رئيس الفريق السّعودي للاولمبياد الدّولي.

Acknowledgement

Problems for the 2013 KSA MO contests were chosen by Abdullah Alghamdi, Abdulaziz Bin Obaid, Safwat Eltanany, Zuming Feng (co-chair), Yunhao Fu, Abdallah Laradji, Ian Le, Carlos Shine, Malik Talbi (co-chair), and Pin Yu. Many thanks to all of these persons, especially those who contributed additional solutions that were helpful in grading these tests and construction of this booklet.

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Part I

Problems

Chapter 1

Preselection tests for the full-time training

The students who attended the full-time training that started on April 1, 2013, were chosen according to their combined performance in this set of tests during the October camp at Riyadh. These tests are prepared by Abdullah Alghamdi, Abdulaziz Bin Obaid, Safwat Eltanany, Abdallah Laradji, and Malik Talbi.

1.1 Day I - October 16, 2013

Allowed time: 3 hours

1. Let $-1 \leq x, y \leq 1$. Prove the inequality

$$2\sqrt{(1-x^2)(1-y^2)} \leq 2(1-x)(1-y)+1.$$

2. Let x, y be two non-negative integers. Prove that 47 divides $3^x - 2^y$ if and only if 23 divides $4x + y$.

3. Ten students take a test consisting of 4 different papers in Algebra, Geometry, Number Theory and Combinatorics. First, the proctor distributes randomly the Algebra paper to each student. Then the remaining papers are distributed one at a time in the following order: Geometry, Number Theory, Combinatorics in such a way that no student receives a paper before he finishes the previous one. In how many ways can the proctor distribute the test papers given that a student may for example finish the Number Theory paper before

another student receives the Geometry paper, and that he receives the Combinatorics paper after that the same other student receives the Combinatorics papers.

4. ABC is a triangle, G its centroid and A', B', C' the midpoints of its sides BC, CA, AB, respectively. Prove that if the quadrilateral $AC'GB'$ is cyclic then
$$AB \cdot CC' = AC \cdot BB'.$$

1.2 Day II - October 19, 2013

Allowed time: 3 hours

1. Prove that if a is an integer relatively prime with 35 then
$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \mod 35.$$

2. The quadratic equation $ax^2 + bx + c = 0$ has its roots in the interval $[0, 1]$. Find the maximum of
$$\frac{(a-b)(2a-b)}{a(a-b+c)}.$$

3. The positive integer a is relatively prime with 10. Prove that for any positive integer n, there exists a power of a whose last n digits are $\underbrace{0\cdots0}_{n-1}1$.

4. ΔABC is a triangle and I_b, I_c its excenters opposite to B, C. Prove that ΔABC is right at A if and only if its area is equal to $\frac{1}{2}AI_b \cdot AI_c$.

1.3 Day III - October 21, 2013

Allowed time: 3 hours

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(f(x)) = 4x + 1$ for all real number x. Prove that the equation $f(x) = x$ has a unique solution.

2. Let $a_1, a_2, \ldots, a_9$ be integers. Prove that if 19 divides $a_1^9 + a_2^9 + \cdots + a_9^9$ then 19 divides the product $a_1 a_2 \cdots a_9$.

3. The points of the plan have been colored by 2013 different colors. We say that a triangle ΔABC has the color X if its three vertices A, B, C has the color X. Prove that there are infinitely many triangles with the same color and the same area.

4. ΔABC is a triangle with $AB < BC$, $\mathcal{C}$ its circumcircle, K the midpoint of the minor arc $\widehat{CA}$ of the circle $\mathcal{C}$ and T a point on $\mathcal{C}$ such that KT is perpendicular to BC. If A', B' are the intouch points of the incircle of ΔABC with the sides BC, AC, prove that the lines $AT, BK, A'B'$ are concurrent.

1.4 Day IV - October 23, 2013

Allowed time: 3 hours

1. Let $a_1, a_2, a_3, ...$ be a sequence of real numbers which satisfy the relation

$$a_{n+1} = \sqrt{a_n^2 + 1}.$$

Suppose that there exists a positive integer n_0 such that $a_{2n_0} = 3a_{n_0}$. Find the value of a_{46}.

2. Let x, y be two integers. Prove that if 2013 divides $x^{1433} + y^{1433}$ then 2013 divides $x^7 + y^7$.

3. How many permutations $(s_1, s_2, \cdots, s_n)$ of $(1, 2, \ldots, n)$ are there satisfying the condition $s_i > s_j$ for all $i \geq j + 3$ when $n = 5$ and when $n = 7$?

4. ΔABC is a triangle, M the midpoint of BC, D the projection of M on AC and E the midppoint of MD. Prove that the lines AE, BD are orthogonal if and only if $AB = AC$.

Chapter 2

Selection tests for the Gulf Mathematical Olympiad 2013

The KSA 2013 Gulf MO team members were chosen according to their combined performance in this set of tests during the January camp at Riyadh. These tests are prepared by Zuming Feng, Ian Le, Carlos Shine, Malik Talbi, and Pin Yu.

Members of the KSA 2013 Gulf MO team were Sameh Zawawi, Mahdi Alshaikh, Alzubair Habibullah, Ibraheem Khan, Salman Tawfik, Abdallah Alnashwan. The total team scores was 162 out of 240, and was ranked 1st place among the 7 participating teams. The team's individual performances were as follows:

Sameh Zawawi	GOLD Medallist
Mahdi Alshaikh	GOLD Medallist
Alzubair Habibullah	SILVER Medallist
Ibraheem Khan	SILVER Medallist
Salman Tawfik	SILVER Medallist
Abdallah Alnashwan	SILVER Medallist

2.1 Day I - January 22, 2013

Allowed time: 4 hours and half

1. Tarik wants to choose some distinct numbers from the set $S = \{2, \ldots, 111\}$ in such a way that each of the chosen numbers cannot be written as the product of two other distinct chosen numbers. What is the maximum number of numbers Tarik can choose?

2. For positive real numbers a, b and c, prove that

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}.$$

3. Define a *regular n-pointed star* to be a union of n lines segments P_1P_2, P_2P_3, $\ldots$, P_nP_1 such that

   - the points $P_1, P_2, \ldots, P_n$ are coplanar and no three of them are collinear;
   - each of the n line segments intersects at least one of the other line segments at a point other than an endpoint;
   - all of the angles at $P_1, P_2, \ldots, P_n$ are congruent;
   - all of the n line segments P_1P_2, P_2P_3, $\ldots$, P_nP_1 are congruent; and
   - the path $P_1P_2 \ldots P_nP_1$ turns counterclockwise at an angle less than $180°$ at each vertex.

   There are no regular 3-pointed, 4-pointed, or 6-pointed stars. All regular 5-pointed star are similar, but there are two non-similar regular 7-pointed stars. Find all possible values of n such that there are exactly 29 non-similar regular n-pointed stars.

4. In acute triangle ABC, points D and E are the feet of the perpendiculars from A to BC and B to CA, respectively. Segment AD is a diameter of circle ω. Circle ω intersects sides AC and AB at F and G (other than A), respectively. Segment BE intersects segments GD and GF at X and Y respectively. Ray DY intersects side AB at Z. Prove that lines XZ and BC are perpendicular.

2.2 Day II - January 26, 2013

Allowed time: 4 hours and half

1. An acute triangle ABC is inscribed in circle ω centered at O. Line BO and side AC meet at B_1. Line CO and side AB meet at C_1. Line B_1C_1 meets circle ω at P and Q. If $AP = AQ$, prove that $AB = AC$.

2. Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + p$ be a polynomial of integer coefficients where p is a prime number. Assume that
$$p > \sum_{i=1}^{n} |a_i|.$$
Prove that $f(X)$ is irreducible.

3. Find the largest integer k such that k divides $n^{55} - n$ for all integer n.

4. Let $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for all positive integer n, be the Fibonacci sequence. Prove that for any positive integer m there exist infinitely many positive integers n such that
$$F_n + 2 \equiv F_{n+1} + 1 \equiv F_{n+2} \mod m.$$

2.3 Day III - January 29, 2013

Allowed time: 3 hours and half

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy
$$\begin{aligned} f\left(\tfrac{\sqrt{3}}{3}x\right) &= \sqrt{3}f(x) - \tfrac{2\sqrt{3}}{3}x, \\ f(x)f(y) &= f(xy) + f\left(\tfrac{x}{y}\right), \end{aligned}$$
for all $x, y \in \mathbb{R}$, with $y \neq 0$.

2. Find all values of n for which there exists a convex cyclic non-regular polygon with n vertices such that the measures of all its internal angles are equal.

3. ABC is a triangle, H its orthocenter, I its incenter, O its circumcenter and ω its circumcircle. Line CI intersects circle ω at point D different from C. Assume that $AB = ID$ and $AH = OH$. Find the angles of triangle ABC.

4. Find all pairs of positive integers (a, b) such that $a^2 + b^2$ divides both $a^3 + 1$ and $b^3 + 1$.

Chapter 3

Selection tests for the Balkan Mathematical Olympiad 2013

The KSA 2013 Balkan MO team members were chosen according to their combined performance in this set of tests during the March-April camp at Riyadh. These tests are prepared by Zuming Feng, Yunhao Fu, and Malik Talbi.

Members of the KSA 2013 Balkan MO team were Abdallah Alnashwan, Ali Alnasser, Mahdi Alshaikh, Alzubair Habibullah, Ibraheem Khan, and Sameh Zawawi. However, the team could not take part to the competition this year because of its schedule.

Members of the KSA 2012 Balkan MO team were Saleh Algamdi, Abdulrahman Alharbi, Doha Aljeddawi, Hasan Eid, Husain Eid, Alyazeed Basuni. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Abdul Aziz bin Obaid (Mawhiba, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Adel Mohammad Alghadir, and Mrs. Abeer Kawther as observers of the KSA delegation. The total team scores was 131 out of 240, and was ranked 14th place among the 22 participating teams. The team's individual performances were as follows:

Alyazeed Basuni	SILVER Medallist
Saleh Algamdi	BRONZE Medallist
Abdulrahman Alharbi	BRONZE Medallist
Doha Aljeddawi	BRONZE Medallist
Hasan Eid	BRONZE Medallist
Husain Eid	

3.1 Day I - April 7, 2013

Time allowed: 5 hours

1. The set G is defined by the points (x, y) with integer coordinates, $1 \leq x \leq 5$ and $1 \leq y \leq 5$. Determine the number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ such that for $1 \leq i \leq 5$, $P_i = (x_i, i)$ is in G and $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_4| = |x_4 - x_5| = 1$.

2. For positive integers a and b, $\gcd(a, b)$ denote their greatest common divisor and $\operatorname{lcm}(a, b)$ their least common multiple. Determine the number of ordered pairs (a, b) of positive integers satisfying the equation

$$ab + 63 = 20 \operatorname{lcm}(a, b) + 12 \gcd(a, b).$$

3. Solve the following equation where x is a real number:

$$\lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 = 0.$$

4. $ABCDEF$ is an equiangular hexagon of perimeter 21. Given that $AB = 3$, $CD = 4$, and $EF = 5$, compute the area of hexagon $ABCDEF$.

5. Let k be a real number such that the product of real roots of the equation

$$X^4 + 2X^3 + (2 + 2k)X^2 + (1 + 2k)X + 2k = 0$$

is -2013. Find the sum of the squares of these real roots.

6. Let ABC be a triangle with incenter I, and let D, E, F be the midpoints of sides BC, CA, AB, respectively. Lines BI and DE meet at P, and lines CI and DF meet at Q. Line PQ meets sides AB and AC at T and S, respectively. Prove that $AS = AT$.

7. Ayman wants to color the cells of a 50×50 chessboard into black and white so that each 2×3 or 3×2 rectangle contains an even number of white cells. Determine the number of ways Ayman can color the chessboard.

8. Prove that the ratio

$$\frac{1^1 + 3^3 + 5^5 + \cdots + (2^{2013} - 1)^{(2^{2013}-1)}}{2^{2013}}$$

is an odd integer.

3.2 Day II - April 9, 2013

4 Time allowed: 5 hours

1. In triangle ABC, $AB = AC = 3$ and $\angle A = 90°$. Let M be the midpoint of side BC. Points D and E lie on sides AC and AB respectively such that $AD > AE$ and $ADME$ is a cyclic quadrilateral. Given that triangle EMD has area 2, find the length of segment CD.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy for all $x, y \in \mathbb{R}$ the relation
$$f(f(f(x) + y) + y) = x + y + f(y).$$

3. Find all positive integers x, y, z such that
$$2^x + 21^y = z^2.$$

4. Ten students are standing in a line. A teacher wants to place a hat on each student. He has two colors of hats, red and white, and he has 10 hats of each color. Determine the number of ways in which the teacher can place hats such that among any set of consecutive students, the number of students with red hats and the number of students with blue hats differ by at most 2.

5. We call a positive integer *good* if it doesn't have a zero digit and the sum of the squares of its digits is a perfect square. For example, 122 and 34 are good and 304 and 12 are not not good. Prove that there exists a n-digit good number for every positive integer n.

6. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that
$$a\sqrt{b^2 + c^2 + bc} + b\sqrt{c^2 + a^2 + ca} + c\sqrt{a^2 + b^2 + ab} \geq \sqrt{3}.$$

7. The excircle ω_B of triangle ABC opposite B touches side AC, rays BA and BC at B_1, C_1 and A_1, respectively. Point D lies on major arc $\widehat{A_1C_1}$ of ω_B. Rays DA_1 and C_1B_1 meet at E. Lines AB_1 and BE meet at F. Prove that line FD is tangent to ω_B (at D).

8. A social club has 101 members, each of whom is fluent in the same 50 languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

3.3 Day III - April 14, 2013

Time allowed: 4 hours

1. $ABCD$ is a cyclic quadrilateral and ω its circumcircle. The perpendicular line to AC at D intersects AC at E and ω at F. Denote by ℓ the perpendicular line to BC at F. The perpendicular line to ℓ at A intersects ℓ at G and ω at H. Line GE intersects FH at I and CD at J. Prove that points C, F, I, and J are concyclic.

2. Define Fibonacci sequence $\{F\}_{n=0}^{\infty}$ as $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for every integer $n > 1$. Determine all quadruples (a, b, c, n) of positive integers with $a < b < c$ such that each of $a, b, c, a + n, b + n, c + 2n$ is a term of the Fibonacci sequence.

3. Let T be a real number satisfying the property: For any nonnegative real numbers a, b, c, d, e with their sum equal to 1, it is possible to arrange them around a circle such that the products of any two neighboring numbers are no greater than T. Determine the minimum value of T.

4. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function which satisfies for all integer $n \geq 0$:

$$(a) \quad f(2n+1)^2 - f(2n)^2 = 6f(n) + 1, \qquad (b) \quad f(2n) \geq f(n);$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers. Solve the equation $f(n) = 1000$.

3.4 Day IV - April 16, 2013

Time allowed: 1 hour 30 minutes

1. $ABCD$ is a cyclic quadrilateral such that $AB = BC = CA$. Diagonals AC and BD intersect at E. Given that $BE = 19$ and $ED = 6$, find the possible values of AD.

2. The base-7 representation of number n is $\overline{abc}_{(7)}$, and the base-9 representation of number n is $\overline{cba}_{(9)}$. What is the decimal (base-10) representation of n?

3. Find the area of the set of points of the plane whose coordinates (x, y) satisfy

$$x^2 + y^2 \leq 4|x| + 4|y|.$$

4. Find all positive integers $n < 589$ for which 589 divides $n^2 + n + 1$.

Chapter 4

Selection tests for the International Mathematical Olympiad 2013

The KSA 2013 IMO team members were chosen according to their combined performance in this set of tests during the May camp at Riyadh. These tests are prepared by Zuming Feng, Carlos Shine, and Malik Talbi.

Members of the KSA 2013 IMO team were Alyazeed Basyoni, Sameh Zawawi, Ibraheem Khan, Abdulrahman Alharbi, Ali Alnasser, Alzubair Habibullah. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Mansour Almoaigel, and Dr. Abdulrahman Albarak (Ministry of Education) as observers of the KSA delegation.

Members of the KSA 2012 IMO team were Alyazeed Basuni, Husain Eid, Saleh Algamdi, Abdulrahman Alharbi, Wael Alsaeed, and Hasan Eid. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Palmer Mebane, Mansour Almoaigel, and Dr. Abdulrahman Albarak (Ministry of Education) as observers of the KSA delegation. The total team scores was 105 out of 252, and was ranked 29th place among the 100 participating teams. The team's individual performances were as follows:

Alyazeed Basuni	SILVER Medallist
Husain Eid	SILVER Medallist
Saleh Algamdi	BRONZE Medallist
Abdulrahman Alharbi	BRONZE Medallist
Wael Alsaeed	BRONZE Medallist
Hasan Eid	

4.1 Day I - May 28, 2013

4 hours 30 minutes

1. Triangle ABC is inscribed in circle ω. Point P lies inside triangle ABC. Lines AP, BP and CP intersect ω again at points A_1, B_1 and C_1 (other than A, B, C), respectively. The tangent lines to ω at A_1 and B_1 intersect at C_2. The tangent lines to ω at B_1 and C_1 intersect at A_2. The tangent lines to ω at C_1 and A_1 intersect at B_2. Prove that the lines AA_2, BB_2 and CC_2 are concurrent.

2. Let $S = \{0, 1, 2, 3, \ldots\}$ be the set of the non-negative integers. Find all strictly increasing functions $f\colon S \to S$ such that $n + f(f(n)) \leq 2f(n)$ for every n in S.

3. A Saudi company has two offices. One office is located in Riyadh and the other in Jeddah. To insure the connection between the two offices, the company has designated from each office a number of correspondents so that

   (a) each pair of correspondents from the same office share exactly one common correspondent from the other office.

   (b) there are at least 10 correspondents from Riyadh.

   (c) Zayd, one of the correspondents from Jeddah, is in contact with exactly 8 correspondents from Riyadh.

   What is the minimum number of correspondents from Jeddah who are in contact with the correspondent Amr from Riyadh?

4. Determine whether it is possible to place the integers $1, 2, \ldots, 2012$ in a circle in such a way that the 2012 products of adjacent pairs of numbers leave pairwise distinct remainders when divided by 2013.

4.2 Day II - May 29, 2013

4 hours 30 minutes - 4 problems

1. Find the maximum and the minimum values of

$$S = (1 - x_1)(1 - y_1) + (1 - x_2)(1 - y_2)$$

for real numbers x_1, x_2, y_1, y_2 with $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 2013$.

2. Let ABC be an acute triangle, and let AA_1, BB_1, and CC_1 be its altitudes. Segments AA_1 and B_1C_1 meet at point K. The perpendicular bisector of segment A_1K intersects sides AB and AC at L and M, respectively. Prove that points A, A_1, L, and M lie on a circle.

3. For a positive integer n, we consider all its divisors (including 1 and itself). Suppose that $p\%$ of these divisors have their unit digit equal to 3 (For example $n = 117$, has six divisors, namely 1,3,9,13,39,117. Two of these divisors, namely 3 and 13, have unit digits equal to 3. Hence for $n = 117$, $p = 33.33\cdots$). Find, when n is any positive integer, the maximum possible value of p.

4. Determine if there exists an infinite sequence of positive integers

$$a_1, a_2, a_3, \ldots$$

such that

(i) each positive integer occurs exactly once in the sequence, and

(ii) each positive integer occurs exactly once in the sequence $|a_1 - a_2|$, $|a_2 - a_3|$, $\ldots$, $|a_k - a_{k+1}|$, $\ldots$

4.3 Day III - May 30, 2013

4 hours 30 minutes

1. Adel draws an $m \times n$ grid of dots on the coordinate plane, at the points of integer coordinates (a, b) where $1 \le a \le m$ and $1 \le b \le n$. He proceeds to draw a closed path along k of these dots, $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, such that (a_i, b_i) and (a_{i+1}, b_{i+1}) (where $(a_{k+1}, b_{k+1}) = (a_1, b_1)$) are 1 unit apart for each $1 \le i \le k$. Adel makes sure his path does not cross itself, that is, the k dots are distinct. Find, with proof, the maximum possible value of k in terms of m and n.

2. Given an integer $n \geq 2$, determine the number of ordered n-tuples of integers $(a_1, a_2, \ldots, a_n)$ such that

   (a) $a_1 + a_2 + \cdots + a_n \geq n^2$; and

   (b) $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$.

3. Let ABC be an acute triangle, M be the midpoint of BC and P be a point on line segment AM. Lines BP and CP meet the circumcircle of ABC again at X and Y, respectively, and sides AC at D and AB at E, respectively. Prove that the circumcircles of AXD and AYE have a common point $T \neq A$ on line AM.

4. Find all polynomials $p(x)$ with integer coefficients such that for each positive integer n, the number $2^n - 1$ is divisible by $p(n)$.

Part II

Problems in arabic

اختبارات ترشيح الطّلاب للتّدريب المكثّف 2013

مسائل اليوم الأوّل ـ 30 ذوالقعدة 1433 هـ

1 ـ لتكن $-1 \leq x, y \leq 1$. أثبت المتباينة
$$2\sqrt{(1-x^2)(1-y^2)} \leq 2(1-x)(1-y)+1.$$

2 ـ ليكن x, y عددين صحيحين غير سالبين. أثبت أنّ المقدار $3^x - 2^y$ يقبل القسمة على 47 إذا وفقط إذا كان المقدار $4x+y$ يقبل القسمة على 23 .

3 ـ يشارك عشر طلّاب في اختبار في أربع مواد مختلفة: جبر، وهندسة، ونظريّة أعداد، وتركيبات. طبعت مسائل كلّ مادة في ورقة مستقلّة. يبدأ المراقب بتوزيع مسائل الجبر على الطلّاب بترتيب عشوائي. كلّما ينتهي طالب من حلّ مسائل الجبر يعطيه المراقب مسائل الهندسة، وإذا انتهى من حلّ مسائل الهندسة يعطيه مسائل نظريّة الأعداد ثمّ في الأخير يعطيه مسائل التركيبات بهذا الترتيب. بكم طريقة يمكن للمراقب توزيع المسائل على الطلّاب، علما بأنّه يمكن لطالب أن يحصل مثلا على مسائل نظرية الأعداد قبل أن يحصل زميله على مسائل الهندسة، وأن يحصل على مسائل التركيبات بعد أن يحصل نفس الزميل على مسائل التركيبات.

4 ـ ΔABC مثلّث، و G مركزه المتوسّط و A', B', C' منتصفات أضلاعه BC, CA, AB على الترتيب. أثبت أنّه إذا كان الرباعي $AC'GB'$ دائريّا فإنّ لدينا العلاقة: $AB \cdot CC' = AC \cdot BB'$.

مسائل اليوم الثّاني ـ 3 ذوالحجّة 1433 هـ

1 ـ إذا كان a عددا صحيحا أوّليّا نسبيّا مع 35 ، فأثبت أنّ
$$(a^4-1)(a^4+15a^2+1) \equiv 0 \mod 35.$$

2 ـ لتكن $ax^2+bx+c=0$ معادلة تربيعيّة جذراها على الفترة $[0,1]$. جد القيمة العظمى للمقدار
$$\frac{(a-b)(2a-b)}{a(a-b+c)}.$$

3 ـ ليكن a عددا صحيحا موجبا أوّليّا نسبيّا مع 10 . أثبت أنّه لكلّ عدد صحيح موجب n ، توجد قوّة للعدد a تنتهي خاناتها الـ n بـ $\cdots 00\cdots 01$.

4 ـ ΔABC مثلّث، و I_b, I_c مركزا الدائرتيه الخارجيّتين المقابلتين للرأسين B, C . أثبت أنّ المثلّث ΔABC قائم عند الرأس A إذا وفقط إذا كانت قيمة مساحته تساوي $\frac{1}{2}AI_b \cdot AI_c$.

مسائل اليوم الثّالث ـ 5 ذوالحجّة 1433 هـ

1 ـ لتكن $f:\mathbb{R}\to\mathbb{R}$ دالة تحقّق $f(f(x))=4x+1$ لكلّ x عدد حقيقي. أثبت أنّ المعادلة $f(x)=x$ لها حلّ وحيد.

2 ـ لتكن $a_1, a_2, \ldots, a_9$ أعدادا صحيحة. أثبت أنّه إذا كان المقدار $a_1^9+a_2^9+\cdots+a_9^9$ يقبل القسمة على 19 فإنّ حاصل الضرب $a_1a_2\cdots a_9$ يقبل القسمة على 19 .

3 ـ تمّ تلوين نقط المستوي بـ 2013 لونا مختلفا. نقول إنّ المثلّث ΔABC له لون كذا إذا كانت رؤوسه الثلاثة A, B, C بهذا اللّون. أثبت أنّه يوجد عدد غير منته من المثلّثات لها نفس اللّون ونفس المساحة.

4 ـ ΔABC مثلّث يحقّق $AB<BC$ ، و $\mathcal{C}$ دائرته المحيطة، و K نقطة منتصف القوس $\widehat{CA}$ من الدائرة $\mathcal{C}$ ، و T نقطة على $\mathcal{C}$ بحيث KT عمودي على BC . إذا كانت

A', B' نقطتي تماس الدائرة الداخليّة للمثلّث ΔABC مع الضلعين BC, AC على الترتيب، فأثبت أنّ المستقيمات $AT, BK, A'B'$ تلتقي في نقطة واحدة.

مسائل اليوم الرّابع ـ 7 ذوالحجّة 1433 هـ

1 ـ لتكن $a_1, a_2, a_3, ...$ متتالية أعداد حقيقيّة تحقّق العلاقة

$$a_{n+1} = \sqrt{a_n^2 + 1}.$$

لنفرض وجود عدد صحيح موجب n_0 بحيث $a_{2n_0} = 3a_{n_0}$. أوجد قيمة a_{46} .

2 ـ ليكن x, y عددين صحيحين. أثبت أنّ المقدار $x^7 + y^7$ يقبل القسمة على 2013 إذا كان المقدار $x^{1433} + y^{1433}$ يقبل القسمة على 2013 .

3 ـ كم توجد من تبديلات $(s_1, s_2, \cdots, s_n)$ لـ $(1, 2, \ldots, n)$ تحقّق الشرط $s_i > s_j$ لكلّ $i \geq j + 3$ عندما $n = 5$ وعندما $n = 7$ ؟

4 ـ ΔABC مثلّث، و M نقطة المنتصف للضلع BC ، و D مسقط النقطة M على الضلع AC ، و E نقطة المنتصف للضلع MD . أثبت أنّ المستقيمين AE, BD عموديّان إذا وفقط إذا كان $AB = AC$.

اختبارات ترشيح الفريق السّعودي لأولمبياد الرياضيات لدول الخليج 2013

مسائل اليوم الأوّل ـ 10 ربيع الأوّل 1434 هـ

1– يريد طارق أن يختار بعض الأعداد المختلفة من المجموعة $S = \{1, 2, \ldots, 111\}$ بحيث لا يمكن كتابة أيّ عدد مختار كحاصل ضرب عددين مختارين آخرين مختلفين. ما هو أكبر كمّ من الأعداد يمكن لطارق أن يختارها؟

2– لكلّ الأعداد الحقيقيّة الموجبة a و b و c ، أثبت المتباينة

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3}.$$

3– نعرّف **النجمة المنتظمة ذات** n **من الرؤوس** على أنّها اتّحاد n من القطع المستقيمة $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n, P_nP_1$ تحقّق

- النقط $P_1, P_2, \ldots, P_n$ تقع في نفس المستوى ولا توجد ثلاثة نقط منها على استقامة واحدة؛
- كلّ قطعة مستقيمة تقطع على الأقل قطعة مستقيمة أخرى في غير الرؤوس؛
- كلّ الزوايا عند الرؤوس $P_1, P_2, \ldots, P_n$ متطابقة؛
- كلّ القطع المستقيمة $P_1P_2, P_2P_3, \ldots, P_nP_1$ متطابقة؛ وأخيرا
- الخطّ المضلّع $P_1P_2\ldots P_nP_1$ يستدير باتّجاه معاكس لعقارب السّاعة بزاوية أقل من $180°$ عند كلّ رأس.

لاتوجد نجوم منتظمة ذات 3 ، أو 4 ، أو 6 رؤوس. كلّ النجوم المنتظمة ذات 5 رؤوس متشابهة فيما بينها، وتوجد نجمتان منتظمتان غير متشابهتين ذات 7 رؤوس. جد كلّ قيم n الممكنة بحيث توجد بالضبط 29 نجمة منتظمة غير متشابهة ذات n رأس.

4– في مثلّث حاد ABC ، النقطتان D و E قدما الارتفاعين من A على BC ، ومن B على CA ، على الترتيب. القطعة المستقيمة AD قطر لدائرة ω . الدائرة ω تقطع الضلعين

AC و AB عند نقطتين F و G ، مختلفتين عن A ، على الترتيب. القطعة المستقيمة BE تقطع القطعتين المستقيمتين GD و GF عند X و Y على الترتيب. الشعاع DY يقطع الضلع AB عند Z . أثبت أنّ المستقيم XZ عموديّ على BC .

مسائل اليوم الثّاني ـ 14 ربيع الأوّل 1434 هـ

1– ABC مثلّث حاد، و ω الدّائرة المارة برؤوسه، و O مركز الدّائرة ω . يتقاطع المستقيم BO مع الضلع AC عند B_1 . يتقاطع المستقيم CO مع الضلع AB عند C_1 . المستقيم B_1C_1 يقطع الدّائرة ω عند P و Q . إذا كان $AP = AQ$ ، فأثبت أنّ $AB = AC$.

2– لتكن $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + p$ كثيرة حدود معاملاتها أعداد صحيحة و p عدد أوّليّ. لنفرض أنّ

$$p > \sum_{i=1}^{n} |a_i|.$$

أثبت أنّ $f(X)$ غير قابلة للتّحليل (للاختزال).

3– جد أكبر عدد صحيح k بحيث k يقسم $n^{55} - n$ لكلّ عدد صحيح n .

4– لتكن $F_0 = 0$ ، و $F_1 = 1$ ، و $F_{n+1} = F_n + F_{n-1}$ ، لكلّ عدد صحيح موجب n ، متتابعة فيبوناتشي. أثبت أنّه لكلّ عدد صحيح m يوجد عدد غير منته من الأعداد الصّحيحة الموجبة n الّتي تحقّق

$$F_n + 2 \equiv F_{n+1} + 1 \equiv F_{n+2} \mod m.$$

مسائل اليوم الثّالث ـ 17 ربيع الأوّل 1434 هـ

1– جد كلّ الدّوال $f:\mathbb{R}\to\mathbb{R}$ الّتي تحقّق

$$\begin{aligned} f\left(\tfrac{\sqrt{3}}{3}x\right) &= \sqrt{3}f(x)-\tfrac{2\sqrt{3}}{3}x, \\ f(x)f(y) &= f(xy)+f\left(\tfrac{x}{y}\right), \end{aligned}$$

لكلّ $x,y\in\mathbb{R}$ ، حيث $y\neq 0$.

2– جد كلّ قيم n الّتي لأجلها يوجد مضلّع دائريّ محدّب غير منتظم له n رأس وقيم كلّ زواياه الدّاخليّة متساوية.

3– ABC مثلّث، و H ملتقى ارتفاعاته، و I مركزه الدّاخليّ، و O مركزه المحيط، و ω دائرته المحيطة (الّتي تمرّ برؤوسه). المستقيم CI يقطع الدائرة ω عند نقطة D مختلفة عن C . لنفرض أنّ $AB=ID$ و $AH=OH$. جد زوايا المثلّث ABC .

4– جد كلّ أزواج الأعداد الصحيحة الموجبة (a,b) الّتي لأجلها يقسم المقدار a^2+b^2 المقدارين a^3+1 و b^3+1 .

اختبارات ترشيح الفريق السّعودي لأولمبياد الرياضيات لدول البلقان 2013

مسائل اليوم الأوّل ـ 26 جمادى الأولى 1434 هـ

1– لتكن G مجموعة النقط (x,y) ذات الاحداثيّات الصّحيحة $1 \le x \le 5$ و $1 \le y \le 5$. حدّد عدد متتابعات النقط الخمس (P_1,P_2,P_3,P_4,P_5) الّتي تحقّق لكلّ $1 \le i \le 5$ ، $P_i=(x_i,i)$ في G و $|x_1-x_2|=|x_2-x_3|=|x_3-x_4|=|x_4-x_5|=1$.

2– لكلّ الأعداد الصّحيحة a و b ، يرمز $\gcd(a,b)$ لأكبر قاسم مشترك بينهما ويرمز $lcm(a,b)$ لأصغر مضاعف مشترك بينهما. حدّد عدد الأزواج المرتّبة (a,b) من الأعداد الصحيحة الموجبة الّتي تحقّق المعادلة

$$ab+63=20\ lcm(a,b)+12\gcd(a,b).$$

3– حلّ المعادلة التّالية حيث x عدد حقيقي

$$\lfloor x^2\rfloor-10\lfloor x\rfloor+24=0.$$

4– السّداسي $ABCDEF$ متطابق الزّوايا ومحيطه يساوي 21 . إذا علمت أنّ $AB=3$ ، و $CD=4$ ، و $EF=5$ ، فاحسب مساحة السّداسي $ABCDEF$.

5– ليكن k عددا حقيقيّا بحيث يساوي حاصل ضرب الجذور الحقيقيّة للمعادلة

$$X^4+2X^3+(2+2k)X^2+(1+2k)X+2k=0$$

2013– . احسب حاصل جمع مربّعات هذه الحلول.

6– ABC مثلّث و I مركز دائرته المحاطة بأضلاعه (الدّاخليّة)، و D ، E ، F نقط منتصف أضلاعه BC ، CA ، AB ، على التّرتيب. يتقاطع المستقيمان BI و DE عند P ، ويتقاطع المستقيمان CI و DF عند Q . المستقيم PQ يقطع الضلعين AB و AC عند T و S ، على التّرتيب. أثبت أنّ $AS=AT$.

7– يريد أيمن أن يلوّن خانات طاولة 50×50 شطرنج باللّونين الأبيض والأسود بحيث يشتمل كلّ مستطيل من القياس 3×2 أو 2×3 على عدد زوجي من الخانات البيضاء. حدّد عدد الطرق الّتي يمكن لأيمن أن يلوّن بها طاولة الشطرنج.

8– أثبت أنّ حاصل القسمة

$$\frac{1^1 + 3^3 + 5^5 + \cdots + (2^{2013} - 1)^{(2^{2013}-1)}}{2^{2013}}$$

عدد فردي.

مسائل اليوم الثّاني ـ 28 جمادى الأولى 1434 هـ

1– في المثلّث ABC ، $AB = AC = 3$ و $\angle A = 90^\circ$. لتكن M نقطة منتصف الضّلع BC . النقطتان D و E تقعان على الضّلعين AC و AB على التّرتيب بحيث يكون $AD > AE$ ويكون $ADME$ رباعيّا دائريّا. إذا علمت أنّ مساحة المثلّث EMD تساوي 2 ، جد طول القطعة المستقيمة CD ؟

2– جد كلّ الدّوال $f : \mathbb{R} \to \mathbb{R}$ الّتي تحقّق لكلّ $x, y \in \mathbb{R}$ العلاقة

$$f(f(f(x) + y) + y) = x + y + f(y).$$

3– جد كلّ الأعداد الصّحيحة الموجبة x, y, z الّتي تحقّق

$$2^x + 21^y = z^2.$$

4– هناك عشر طلّاب متصافّون في خط مستقيم. أراد مدرّس أن يضع غطرة على رأس كلّ طالب. لديه لونان من الغطرات، لون أحمر ولون أبيض، ولديه عشر غطرات من كلّ لون. جد عدد الطّرق الّتي تمكّن المدرّس من وضع الغطرات بحيث إذا اعتبرنا أي مجموعة طلّاب متتاليّين، الفارق بين عدد الطّلاب الّذين على رأسهم غطرة حمراء وعدد الطّلاب الّذين على رأسهم غطرة بيضاء لا يتجاوز 2 .

5– نقول لعدد صحيح موجب إنّه طيّب إذا كانت خاناته مختلفة عن صفر ومجموع

مربّعات خاناته مربّع كامل. فمثلا، 122 و 34 عددان طيّبان، في حين أنّ 304 و 12 عددان غير طيّبين. أثبت أنّه يوجد عدد طيّب مكوّن من n خانة لكلّ عدد صحيح موجب n .

6– لتكن a, b, c أعدادا حقيقيّة موجبة تحقّق $ab + bc + ca = 1$. أثبت أنّ
$$a\sqrt{b^2 + c^2 + bc} + b\sqrt{c^2 + a^2 + ca} + c\sqrt{a^2 + b^2 + ab} \geq \sqrt{3}.$$

7– الدّائرة ω_B المحاطة خارج المثلّث ABC مقابلة الرّأس B ، تمسّ الضلع AC ، والشعاعين BA و BC عند B_1 ، و C_1 ، و A_1 ، على التّرتيب. النقطة D تقع على القوس الكبير $\widehat{A_1C_1}$ لـ ω_B . الشّعاعان DA_1 و C_1B_1 يلتقيان عند E . المستقيمان AB_1 و BE يلتقيان عند F . أثبت أنّ المستقيم FD مماس لـ ω_B (عند D).

8– يضمّ نادي اجتماعي 101 عضوا كلّهم يتقنون نفس الـ 50 لغة. اختار كلّ زوج من الأعضاء لغة واحدة يتحدّثان بها فيما بينهما. لنفرض عدم وجود ثلاث أعضاء يتحدّثون فيما بينهم بلغة واحدة. ليكن A عدد المجموعات المكوّنة من ثلاث أعضاء يتكلّمون فيما بينهم بثلاث لغات مختلفة. جد أكبر قيمة ممكنة لـ A .

مسائل اليوم الثّالث ـ 4 جمادى الآخرة 1434 هـ

1– $ABCD$ رباعي دائري و ω الدائرة المحيطة برؤوسه. المستقيم العمودي على AC والمار بـ D يقطع AC عند E ويقطع ω عند F . ليكن ℓ المستقيم العمودي على BC والمار بـ F . المستقيم العمودي على ℓ والمار بـ A يقطع ℓ عند G ويقطع ω عند H . المستقيم GE يقطع FH عند I ويقطع CD عند J . أثبت أنّ النّقط C ، و F ، و I ، و J على دائرة واحدة.

2– نعرّف متتابعة فيبوناتشي $\{F\}_{n=0}^{\infty}$ بـ $F_0 = 0$ ، و $F_1 = 1$ ، و $F_{n+1} = F_n + F_{n-1}$ لكلّ عدد صحيح $n > 1$. حدد كلّ رباعيّات (a, b, c, n) الأعداد الصّحيحة الموجبة الّتي تحقّق $a < b < c$ ويكون فيها كلّ من a ، b ، c ، $a + n$ ، $b + n$ ، و $c + 2n$ عنصرا من عناصر متتابعة فيبوناتشي؟

3– ليكن T عددا حقيقيّا يحقّق الخاصيّة التّالية: لكلّ أعداد حقيقيّة غير سالبة a ، b ، c ، d ، e حاصل جمعها 1 ، يمكن ترتيبها على دائرة بحيث لا يكبر حاصل ضرب كلّ عددين متتاليين العدد T . حدّد أصغر قيمة ممكنة لـ T .

4– لتكن $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ دالة تحقّق لكلّ عدد صحيح $n \geq 0$:

$$(a) \quad f(2n+1)^2 - f(2n)^2 = 6f(n) + 1, \qquad (b) \quad f(2n) \geq f(n);$$

حيث يرمز $\mathbb{Z}_{\geq 0}$ لمجموعة الأعداد الصحيحة غير السّالبة. حلّ المعادلة $f(n) = 1000$.

مسائل اليوم الرّابع ـ 6 جمادى الآخرة 1434 هـ

1– $ABCD$ رباعي دائري يحقّق $AB = BC = CA$. القطران AC و BD يلتقيان عند E . إذا علمت أنّ $BE = 19$ و $ED = 6$ ،فجد كلّ القيم الممكنة لـ AD .

2– يكتب العدد n في القاعدة 7 على شكل $\overline{abc}_{(7)}$ ، ويكتب في القاعدة 9 على شكل $\overline{cba}_{(9)}$. ما هي كتابة العدد n في القاعدة 10 ؟

3– جد مساحة مجموعة نقط المستوي الّتي تحقّق إحداثيّاتها (x, y) المتباينة

$$x^2 + y^2 \leq 4|x| + 4|y|.$$

4– جد كلّ الأعداد الصحيحة الموجبة $n < 589$ الّتي تجعل 589 يقسم المقدار $n^2 + n + 1$.

اختبارات ترشيح الفريق السّعودي لأولمبياد الرياضيات الدّولي 2013

مسائل اليوم الأوّل ـ 18 رجب 1434 هـ

1– ليكن ABC مثلّثا محاطا بالدّائرة ω . النقطة P تقع داخل المثلّث ABC . المستقيمات AP و BP و CP تقطع الدّائرة ω مرّة ثانية عند النقط A_1 و B_1 و C_1 ، على التّرتيب، مختلفة عن A و B و C . المستقيمان الماسّان للدّائرة ω عند النقطتين A_1 و B_1 يتقاطعان عند النقطة C_2 . المستقيمان الماسّان للدّائرة ω عند النقطتين B_1 و C_1 يتقاطعان عند النقطة A_2 . المستقيمان الماسّان للدّائرة ω عند النقطتين C_1 و A_1 يتقاطعان عند النقطة B_2 . أثبت أنّ المستقيمات AA_2 و BB_2 و CC_2 تلتقي عند نقطة واحدة.

2– لتكن $S = \{0, 1, 2, 3, \ldots\}$ مجموعة الأعداد الصّحيحة غير السّالبة. جد كلّ الدّوال $f\colon S \to S$ المتزايدة قطعا والّتي تحقّق $n + f(f(n)) \leq 2f(n)$ لكلّ عدد n في S .

3– لدى شركة سعوديّة مكتبان، أحدهما في الرّياض والآخر في جدّه. حتّى يتمّ التّنسيق بين المكتبين، قامت الشّركة بتعيين في كلّ مكتب عددا من المراسلين بحيث يتحقّق ما يلي

ا ـ كلّ مراسلين اثنين من نفس المكتب يشتركان في مراسل واحد فقط من المكتب الثّاني.

ب ـ هناك على الأقل 10 مراسلون في الرّياض.

ج ـ أحد المراسلين في جدّة، واسمه زيد، على اتّصال مع 8 مراسلين فقط من الرّياض.

ما هو أصغر عدد ممكن من المراسلين في جدّه الّذين هم على اتّصال مع المراسل عمرو من الرّياض؟

4– حدّد ما إذا كان ممكنا إعادة ترتيب الأعداد الصّحيحة $1, 2, \ldots, 2012$ على دائرة بحيث تشكّل بواقي القسمة على 2013 لكلّ عددين متجاورين، وعدد هذه البواقي 2012 ، أعدادا مختلفة فيما بينها.

مسائل اليوم الثّاني ـ 19 رجب 1434 هـ

1– جد أكبر وأصغر قيمة للمقدار

$$S = (1 - x_1)(1 - y_1) + (1 - x_2)(1 - y_2)$$

لمّا تحقّق الأعداد الحقيقيّة x_1 ، x_2 ، y_1 ، y_2 العلاقة $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 2013$.

2– ليكن ABC مثلّثا حادّا و AA_1 و BB_1 و CC_1 ارتفاعاته. تلتقي القطعتان المستقيمتان AA_1 و B_1C_1 عند النقطة K . يقطع المنصّف العموديّ على القطعة المستقيمة A_1K الضلعين AB و AC عند النقطتين L و M ، على التّرتيب. أثبت أنّ النقط A و A_1 و L و M تقع على دائرة واحدة.

3– نعتبر لكلّ عدد صحيح n قائمة قواسمه (بما فيهم 1 وذات العدد). لنفرض أنّ خانات الآحاد لـ $p\%$ من هذه القواسم تساوي 3 . فمثلا، إذا كان $n = 117$ ، فإنّ لديه 6 قواسم ، وهي 1 ، 3 ، 9 ، 13 ، 39 ، 117 . اثنان من هذه القواسم ، وهي 3 و 13 ، خانتا آحاديها تساوي 3 . إذن لدينا $p = 33.33\cdots$ لمّا $n = 117$. جد أكبر قيمة ممكنة لـ p إذا كان n أي عدد صحيح موجب.

4– حدّد هل توجد متسلسلة غير متهية من الأعداد الصّحيحة الموجبة

$$a_1, a_2, a_3, \ldots$$

الّتي تحقّق

ا ـ كلّ عدد صحيح موجب يظهر مرّة واحدة فقط في المتسلسلة،

ب ـ كلّ عدد صحيح موجب يظهر مرّة واحدة فقط في المتسلسلة $|a_1 - a_2|$ ، $|a_2 - a_3|$ ، ... ، $|a_k - a_{k+1}|$ ، ...

مسائل اليوم الثّالث ـ 20 رجب 1434 هـ

1– رسم عادل شبكة $m \times n$ رؤوس على المستوي. إحداثيّات هذه الرؤوس أعداد صحيحة (a,b) تحقّق $1 \le a \le m$ و $1 \le b \le n$. قام عادل برسم مسار مغلق يمرّ بـ k من هذه الرؤوس، (a_1,b_1) ، (a_2,b_2) ، ... ، (a_k,b_k) ، بحيث يساوي طول المسافة بين (a_i,b_i) و (a_{i+1},b_{i+1}) واحدا لكلّ $1 \le i \le k$ (حيث $(a_{k+1},b_{k+1}) = (a_1,b_1)$). تأكّد عادل أنّ مساره غير متداخل، أي أنّ كلّ رؤوسه الـ k مختلفة فيما بينها. احسب بالإثبات أكبر قيمة ممكنة لـ k بدلالة m و n .

2– ليكن n عددا صحيحا يحقّق $n \ge 2$. حدّد عدد المجموعات المرتّبة $(a_1,a_2,\ldots,a_n)$ من الأعداد الصحيحة الّتي تحقّق

ا ـ $a_1 + a_2 + \cdots + a_n \ge n^2$ ؛

ب ـ و $a_1^2 + a_2^2 + \cdots + a_n^2 \le n^3 + 1$.

3– ليكن ABC مثلّثا حادّا، و M نقطة المنتصف للضلع BC ، و P نقطة على القطعة المستقيمة AM . المستقيمان BP و CP يقطعان الدائرة المحيطة بالمثلّث ABC مرّة ثانية عند X و Y ، على التّرتيب، ويقطعان، الأوّل منهما الضلع AC عند D والآخر الضّلع AB عند E. أثبت انّ الدّائرتين المحيطتين بالمثلّثين AXD و AYE تتقاطعان مرّة ثانية عند نقطة $T \neq A$ على المستقيم AM .

4– جد كلّ كثيرات الحدود $p(x)$ ذوات المعاملات الصحيحة الّتي تجعل لكلّ عدد صحيح موجب n ، المقدار $2^n - 1$ يقبل القسمة على $p(n)$.

Part III

Solutions

Chapter 5

Solutions to the preselection tests for the full-time training

5.1 Solutions to the selection test of day I

1. *First solution.* Because $-1 \leq x, y \leq 1$, we have $(1-x^2), (1-y^2) \geq 0$. Therefore, by applying AM-GM, we obtain

$$2\sqrt{(1-x^2)(1-y^2)} \leq (1-x^2) + (1-y^2).$$

So, it remains to prove that

$$(1-x^2) + (1-y^2) \leq 2(1-x)(1-y) + 1$$

which is equivalent to

$$x^2 + y^2 - 2x - 2y + 2xy + 1 \geq 0.$$

But

$$x^2 + y^2 - 2x - 2y + 2xy + 1 = (x+y-1)^2.$$

This ends the proof.
The equality holds when $1-x^2 = 1-y^2$ and $x+y-1 = 0$. This is equivalent to $x = y = \frac{1}{2}$.
Second solution. Because $-1 \leq x, y \leq 1$, there exist $-\frac{\pi}{2} \leq \theta, \phi \leq \frac{\pi}{2}$ such that $x = \sin\theta$ and $y = \sin\phi$. The inequality to prove becomes

$$2\cos\theta\cos\phi \leq 2(1-\sin\theta)(1-\sin\phi) + 1,$$

or, equivalently

$$2[\cos(\theta+\phi)+\sin\theta+\sin\phi]\le 3.$$

Using the relations

$$\cos(\theta+\phi)=1-2\sin^2\frac{\theta+\phi}{2}\qquad\text{and}\qquad\sin\theta+\sin\phi=2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

the inequality simplifies to

$$\cos^2\frac{\theta-\phi}{2}\le 1+\left(2\sin\frac{\theta+\phi}{2}-\cos\frac{\theta-\phi}{2}\right)^2,$$

which is satisfied.
The equality holds when $\theta=\phi=\frac{\pi}{6}$, which means $x=y=\frac{1}{2}$.

2. Because 2 and 47 are relatively prime numbers, 47 divides 3^x-2^y if and only if 47 divides $2^{4x}(3^x-2^y)$. But

$$2^{4x}(3^x-2^y)=48^x-2^{4x+y}\equiv 1-2^{4x+y}\mod 47.$$

Therefore 47 divides 3^x-2^y if and only if $2^{4x+y}\equiv 1 \mod 47$.
On the other hand, we have $2^{23}\equiv 49^{23}\equiv 7^{46}\equiv 1 \mod 47$. We deduce that the prime number 23 is the order of 2 modulo 47. This implies that $2^{4x+y}\equiv 1$ mod 47 if and only if 23 divides $4x+y$.

3. First, since the proctor distribute randomly the Algebra paper to each student, he has 10! ways to do it depending on how he orders the students.
For the other three papers, we order the students from 1 to 30, each student receiving three positions corresponding to the three papers. Since the first position a student receives corresponds to the Geometry paper, the second to the Number Theory paper and the third to the Combinatorics paper, the problem is equivalent to count the number of partitions of the set $\{1,2,\ldots,30\}$ into 10 subsets, each of 3 elements. This is equal to

$$\binom{30}{3,\ldots,3}=\frac{30!}{3!^{10}}.$$

Therefore, the number of ways the proctor can distribute the test papers is

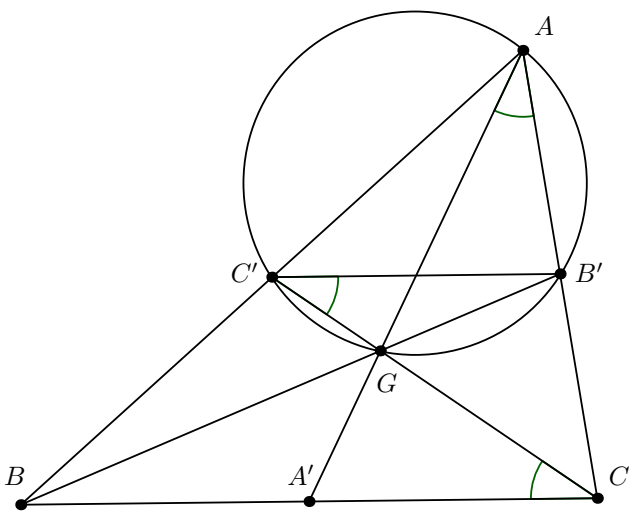
$$\frac{10!30!}{3!^{10}}.$$

4. *Remark.* This problem has a converse:

$$\text{If } AB \cdot CC' = AC \cdot BB', \text{ then either } AB = AC \text{ or } AC'GB' \text{ is cyclic.}$$

The third solution contains a proof for the coverse.
First solution. Because $AC'GB'$ is cyclic, we have $\measuredangle GAB' = \measuredangle GC'B'$. Because $B'C'$ is parallel to BC, we have $\measuredangle CC'B' = \measuredangle C'CB$. We deduce that

$$\sin \measuredangle A'AC = \sin \measuredangle C'CB.$$



But triangles $AA'C, BCC'$ have the same area which is half of the area of triangle ABC. We deduce that

$$\frac{1}{2}AA' \cdot AC \sin \measuredangle A'AC = \frac{1}{2}CC' \cdot BC \sin \measuredangle C'CB,$$

and therefore

$$\frac{AC}{CC'} = \frac{BC}{AA'}.$$

We proceed in a similar way, by considering triangles ABA' and BCB', obtaining

$$\frac{AB}{BB'} = \frac{BC}{AA'},$$

and deduce the relation

$$AB \cdot CC' = AC \cdot BB'.$$

Second solution. Lets us consider the power of the point B with respect to the circumcircle of the cyclic quadrilateral $AC'GB'$. We have

$$\mathcal{P}(B) = \overline{BC'} \cdot \overline{BA} = \overline{BG} \cdot \overline{BB'}.$$

But $\overline{BC'} = \frac{1}{2}\overline{BA}$ and $\overline{BG} = \frac{2}{3}\overline{BB'}$. We deduce that

$$BB' = \frac{\sqrt{3}}{2}AB.$$

Similarly, by considering the power of the point C with respect to the circumcircle of the cyclic quadrilateral $AC'GB'$ we obtain

$$CC' = \frac{\sqrt{3}}{2}AC.$$

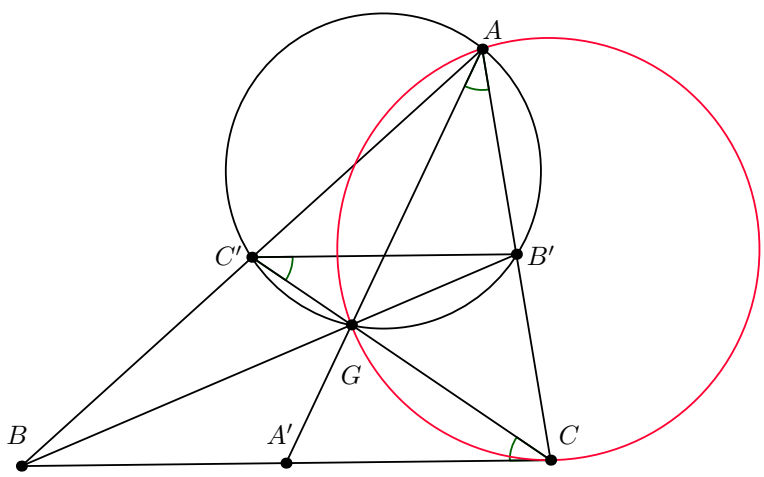
From these two relations we deduce easily that

$$AB \cdot CC' = AC \cdot BB'.$$

Third solution (Contains also a proof for the converse). We can see from the first solution that the quadrilateral $AC'GB'$ is cyclic if and only

$$\measuredangle GAC = \measuredangle GCB.$$

This is equivalent to saying that the line BC is tangent to the circumcircle of triangle AGC.



This property is equivalent to saying that $A'C^2 = \overline{A'G} \cdot \overline{A'A}$ by using the power of the point A' with respect to this circumcircle. But

$$A'C = \frac{1}{2}BC, \quad \overline{A'G} = \frac{1}{3}\overline{A'A} \quad \text{and} \quad AA' = \sqrt{\frac{2AB^2 + 2AC^2 - BC^2}{4}}.$$

We deduce that $AC'GB'$ is cyclic if and only if

$$2BC^2 = AB^2 + AC^2.$$

Now, using the formulas for the medians

$$BB' = \sqrt{\frac{2BC^2 + 2AB^2 - CA^2}{4}} \quad \text{and} \quad CC' = \sqrt{\frac{2CA^2 + 2BC^2 - AB^2}{4}},$$

we deduce that the relation

$$AB \cdot CC' = AC \cdot BB'$$

is equivalent to

$$AB^2 \cdot (2CA^2 + 2BC^2 - AB^2) = AC^2 \cdot (2BC^2 + 2AB^2 - CA^2),$$

which is equivalent to

$$(AB^2 - AC^2) \cdot (2BC^2 - AB^2 - AC^2) = 0,$$

which is equivalent to either $AB = AC$ or $AC'GB'$ is cyclic.

5.2 Solutions to the selection test of day II

1. If a is relatively prime with 35 then it is relatively prime with both $5, 7$. Since a is relatively prime with 5 then, by Fermat, $a^4 \equiv 1 \mod 5$ which implies

$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \mod 5.$$

Since a is relatively prime with 7 then $a^6 \equiv 1 \mod 7$. Hence

$$\begin{aligned}(a^4 - 1)(a^4 + 15a^2 + 1) &\equiv (a^2 + 1)(a^2 - 1)(a^4 + a^2 + 1) \\ &\equiv (a^2 + 1)(a^6 - 1) \equiv 0 \qquad \mod 7.\end{aligned}$$

But $5, 7$ are relatively prime and therefore

$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \mod 35.$$

2. Let u, v be the roots of the quadratic equation $ax^2 + bx + c = 0$ such that $0 \le u \le v \le 1$. We have the relations $b = -a(u+v)$ and $c = uv$. Therefore

$$\begin{aligned}\frac{(a-b)(2a-b)}{a(a-b+c)} &= \frac{(1+u+v)(2+u+v)}{1+u+v+uv} \\ &= 2 + \frac{u}{1+v} + \frac{v}{1+u} \\ &\le 2 + \frac{u}{1+u} + \frac{1}{1+u} = 3.\end{aligned}$$

Clearly, when $u = v = 1$, the equality holds. Thus, 3 is the maximum.

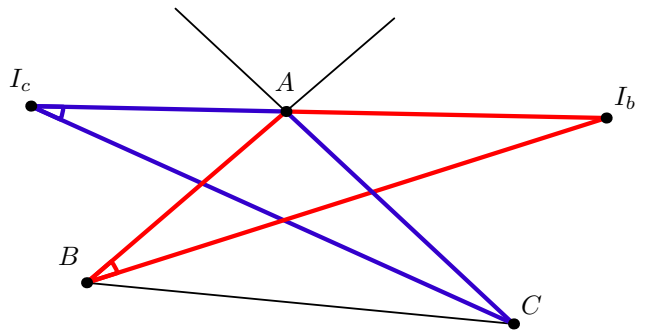
3. This is equivalent to prove that there exists a positive integer k such that 10^n divides $a^k - 1$.
First solution. Consider the remainders of the division of the $10^n + 1$ powers $a^1, a^2, \ldots, a^{10^n+1}$ of a by 10^n. Since there are at most 10^n possible reminders, by the pigeonhole principle there exist at least two powers a^i, a^j, $i < j$, having the same remainder. Therefore 10^n divides $a^j - a^i = a^i(a^{j-i} - 1)$. But 10^n is relatively prime with a^i. This proves that 10^n divides $a^{j-i} - 1$.
Second solution. Since a and 10^n are relatively prime, by Euler's theorem

$$a^{\phi(10^n)} = a^{4\cdot 10^{n-1}} \equiv 1 \mod 10^n.$$

Therefore, 10^n divides $a^{4\cdot 10^{n-1}} - 1$.

4. *First solution.* Let ΔABC be any triangle. Considering triangle AI_cC, we have

$$\begin{aligned}\measuredangle CI_cA &= 180^\circ - (\measuredangle ACI_c + \measuredangle I_cAC) \\ &= 180^\circ - (\tfrac{1}{2}\measuredangle ACB + \measuredangle BAC + \tfrac{1}{2}(180^\circ - \measuredangle BAC)) \\ &= 90^\circ - \tfrac{1}{2}(\measuredangle ACB + \measuredangle BAC) \\ &= \measuredangle I_bBA.\end{aligned}$$



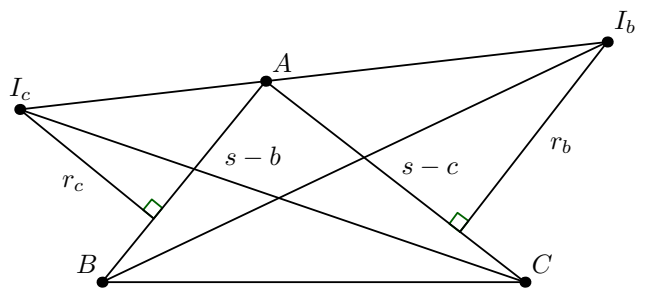
On the other hand, $\measuredangle I_cAC = \measuredangle BAI_b$. We deduce that triangles ABI_b, AI_cC are similar and therefore

$$\frac{AI_c}{AB} = \frac{AC}{AI_b} \quad \Leftrightarrow \quad AI_b \cdot AI_c = AB \cdot AC.$$

Thus, the area of triangle ABC is equal to $\frac{1}{2}AI_b \cdot AI_c \sin\measuredangle BAC$. Hence, triangle ΔABC is right at A if and only if its area is equal to $\frac{1}{2}AI_b \cdot AI_c$.

Second solution. Let ΔABC be any triangle. Denote a, b, c the lengths of the opposite sides to the vertices A, B, C respectively, s the semiperimeter, r the inradius, r_b, r_c the exradius opposite to the vertices B, C respectively. We have

$$AI_b^2 = r_b^2 + (s-c)^2 \quad \text{and} \quad AI_c^2 = r_c^2 + (s-b)^2.$$



But the area of the triangle ΔABC is equal to

$$K = rs = r_b(s-b) = r_c(s-c) = \sqrt{s(s-a)(s-b)(s-c)},$$

and that $r_br_c = s(s-a)$ (which follows from the above formulas for the area). Hence

$$AI_b^2 \cdot AI_c^2 = (r_b^2 + (s-c)^2)(r_c^2 + (s-b)^2) = s^2(s-a)^2 + 2K^2 + (s-b)^2(s-c)^2.$$

Therefore, $K = \frac{1}{2}AI_b \cdot AI_c$ is equivalent to

$$2K^2 = s^2(s-a)^2 + (s-b)^2(s-c)^2.$$

But $2K^2 = 2s(s-a)(s-b)(s-c)$. Hence, this is equivalent to

$$((s-b)(s-c) - s(s-a))^2 = 0,$$

which simplifies to

$$a^2 = b^2 + c^2.$$

5.3 Solutions to the selection test of day III

1. Let x_0 be a a solution of the equation $f(x) = x$. We have

$$x_0 = f(x_0) = f(f(x_0)) = 4x_0 + 1.$$

Therefore, $x_0 = -\frac{1}{3}$. This proves the uniqueness of the solution.
On the other hand,

$$f\left(f\left(-\frac{1}{3}\right)\right) = 4\left(-\frac{1}{3}\right) + 1 = -\frac{1}{3}.$$

We deduce that

$$f\left(-\frac{1}{3}\right) = f\left(f\left(f\left(-\frac{1}{3}\right)\right)\right) = 4f\left(-\frac{1}{3}\right) + 1,$$

and therefore,

$$f\left(-\frac{1}{3}\right) = -\frac{1}{3}.$$

This prove the existence of the solution.

Remark. There are infinitely many functions f satisfying the relation $f(f(x)) = 4x + 1$, for all $x \in \mathbb{R}$. For example, the function given by

$$f(x) = 2\left(x + \frac{1}{3}\right) - \frac{1}{3}, \quad \text{for all } x \in \mathbb{R},$$

or the function given by

$$f(x) = -2\left(x + \frac{1}{3}\right) - \frac{1}{3}, \quad \text{for all } x \in \mathbb{R},$$

or the function given by

$$f(x) = \begin{cases} 2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \in \mathbb{Q} \\ -2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \notin \mathbb{Q} \end{cases},$$

all satisfy the relation. More generaly, if $A \subseteq \mathbb{R}$ has the property: $ra \in A$ whenever $r \in \mathbb{Q},\ a \in A$, the function given by

$$f(x) = \begin{cases} 2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \in A \\ -2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \notin A \end{cases},$$

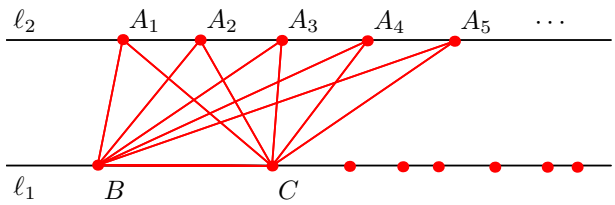
satisfies the relation.

2. Assume that 19 does not divide the product $a_1a_2\cdots a_9$. This means that $a_1, a_2, \ldots, a_9$ are relatively prime with 19. Using Fermat,

$$a_1^{18} \equiv a_2^{18} \equiv \cdots a_9^{18} \equiv 1 \mod 19.$$

But $a_i^{18} \equiv 1 \mod 19$, is equivalent to $(a_i^9 - 1)(a_i^9 + 1) \equiv 0 \mod 19$. Since 19 is a prime number, this implies that $a_i^9 \equiv \pm 1 \mod 19$, for $i = 1, \ldots, 9$, and therefore $a_1^9 + a_2^9 + \cdots + a_9^9 \equiv \pm k \mod 19$ for some odd number k between 1 and 9. This is a contradiction. Thus 19 divides the product $a_1a_2\cdots a_9$.

3. Consider 2014 parallel lines. Each line contains infinitely many points. Since the number of the colors is finite, by the pigeonhole principle, there exist on each line infinitely many points of the same color. Choose for each line one color for which there exist infinitely many points. Since there are 2013 colors and 2014 lines, by the pigeonhole principle there exist at least two lines ℓ_1, ℓ_2 for which the same color $\mathcal{C}$ have been choosen. Choose two points B, C from the first line ℓ_1 of this color $\mathcal{C}$. Choose infinitely many points $A_1, A_2, \ldots$ from the second line ℓ_2 of this color $\mathcal{C}$. Triangles $A_1BC, A_2BC, \ldots$ are all of the same color $\mathcal{C}$ and have the same area since ℓ_1, ℓ_2 are parallel.



4. Let E be the intersection point of BK with AT. The problem is equivalent to prove that points A', E, B' are collinear.

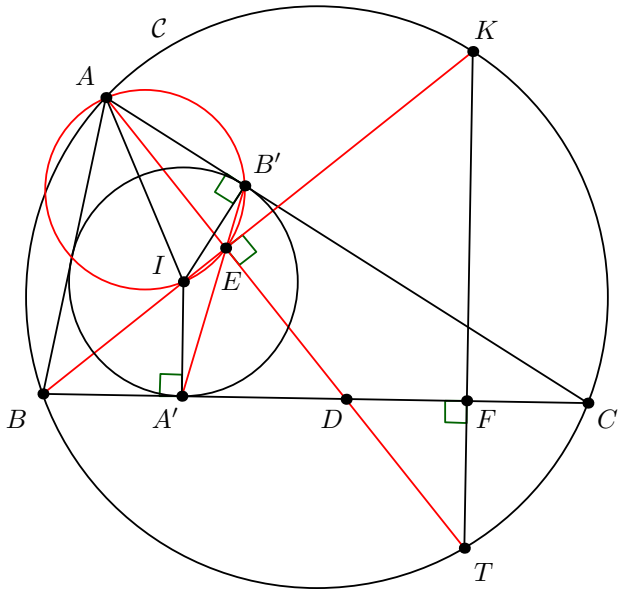
First solution. Let F be the intersection point of BC with KT. Since K is the midpoint of $\widehat{CA}$, by expressing the angles in terms of arc lengths we obtain

$$\measuredangle KEA = \frac{1}{2}\left(\widehat{KA} + \widehat{BT}\right) = \frac{1}{2}\left(\widehat{CK} + \widehat{BT}\right) = \measuredangle CFK = 90^\circ.$$

But B' is an intouch point. This implies that

$$\measuredangle AB'I = 90^\circ = \measuredangle AEI,$$

and the points A, I, E, B' are concyclic.



We deduce from this that

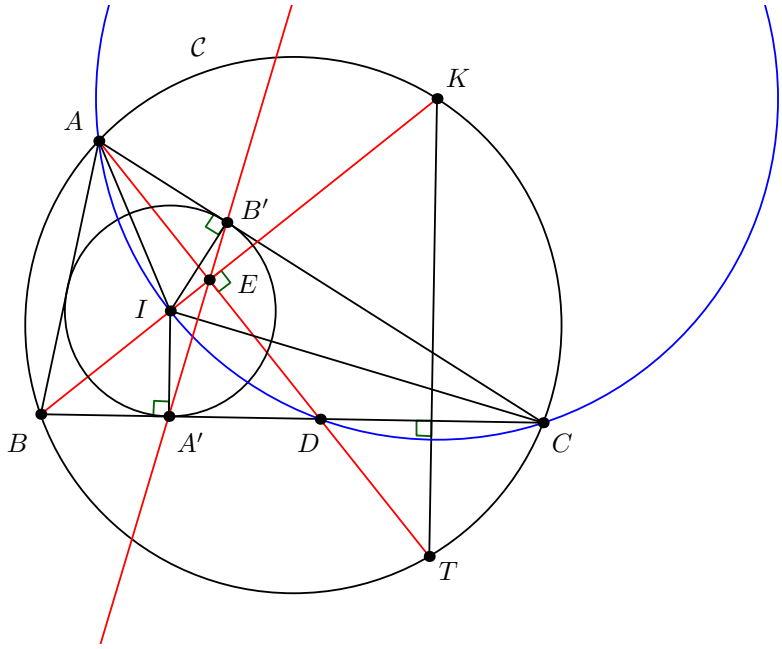
$$\measuredangle IB'E = \measuredangle IAE = 90^\circ - \measuredangle EIA = 90^\circ - \frac{1}{2}\measuredangle BAC - \frac{1}{2}\measuredangle CBA = \frac{1}{2}\measuredangle ACB.$$

On the other hand, triangle $IA'B'$ is isosceles ($IA' = IB'$). Therefore,

$$\measuredangle IB'A' = \frac{1}{2}(180^\circ - \measuredangle A'IB') = \frac{1}{2}\measuredangle ACB = \measuredangle IB'E,$$

since $\measuredangle CA'I = 90^\circ$. This proves that the points A', E, B' are collinear.

Second solution. We prove as in the first solution that $\measuredangle KEA = 90^\circ$. Let D be the intersection point of AT with BC. The pedal triangle of I with respect to triangle ADC is $A'B'E$. Proving that points A', B', E are collinear is equivalent to proving that I is on the circumcircle of triangle ADC and therefore, the line $A'B'E$ will be the Simson line of point I with respect to triangle ADC.



We have

$$\measuredangle CDA = \measuredangle DBA + \measuredangle BAD = \measuredangle CBA + 90^\circ - \frac{1}{2}\measuredangle CBA = 90^\circ + \frac{1}{2}\measuredangle CBA.$$

On the other hand

$$\measuredangle CIA = 180^\circ - \frac{1}{2}\measuredangle BAC - \frac{1}{2}\measuredangle ACB = 90^\circ + \frac{1}{2}\measuredangle CBA = \measuredangle CDA.$$

This proves that I is on the circumcircle of triangle ADC and thus A', E, B' are collinear.

5.4 Solutions to the selection test of day IV

1. We have $a_{n+1}^2 - a_n^2 = 1$ for all natural number n. Using a telescopic sum we obtain

$$n_0 = \sum_{k=n_0}^{2n_0-1} (a_{k+1}^2 - a_k^2) = a_{2n_0}^2 - a_{n_0}^2 = 9a_{n_0}^2 - a_{n_0}^2 = 8a_{n_0}^2.$$

Hence

$$a_{n_0}^2 = \frac{n_0}{8}.$$

On the other hand,

$$n_0 - 1 = \sum_{k=1}^{n_0-1} (a_{k+1}^2 - a_k^2) = a_{n_0}^2 - a_1^2 = \frac{n_0}{8} - a_1^2,$$

or equivalently

$$a_1^2 = \frac{8 - 7n_0}{8}.$$

Since $a_1^2 \geq 0$, we deduce that $n_0 = 1$ and $a_1^2 = \frac{1}{8}$.

Therefore

$$45 = \sum_{k=1}^{45} (a_{k+1}^2 - a_k^2) = a_{46}^2 - a_1^2 = a_{46}^2 - \frac{1}{8},$$

which leads to

$$a_{46} = \frac{19\sqrt{2}}{4}.$$

2. Since $2013 = 3 \times 11 \times 61$, we will prove that for $p = 3, 11, 61$, if p divides $x^{1433} + y^{1433}$ then p divides $x^7 + y^7$.
   Let $p = 3, 11, 61$, and assume that p divides $x^{1433} + y^{1433}$.
   If p divides x, then it divides x^7 and x^{1433}. But p divides $x^{1433} + y^{1433}$. Then it divides y^{1433}. Since p is a prime number, we deduce that it divides y and y^7. Therefore, it divides $x^7 + y^7$. In a similar way we prove that if p divides y, then it divides $x^7 + y^7$.
   Assume now that p is relatively prime with x, y. Then, using Fermat,

$$x^{p-1} \equiv y^{p-1} \equiv 1 \mod p.$$

   But $p - 1$ divides 1440 for $p = 3, 11, 61$. We deduce that

$$x^{1440} \equiv y^{1440} \equiv 1 \mod p.$$

   Hence

$$x^7 + y^7 \equiv x^7 y^{1440} + y^7 x^{1440} \equiv x^7 y^7 (y^{1433} + x^{1433}) \equiv 0 \mod p.$$

   This proves that p divides $x^7 + y^7$.

3. Let N_n be the number of permutations satisfying these conditions.

For $n = 5$, the conditions are $s_4 > s_1$ and $s_5 > s_2, s_1$. Among the 5! permutations of $(1, \cdots, 5)$, half of them satisfy the condition $s_4 > s_1$. Among these permutations, half of them satisfy also the condition $s_5 > s_2$. Therefore, there are 30 permutations satisfying both conditions $s_4 > s_1$ and $s_5 > s_2$.
To compute N_5, it is easier to substract from 30 the number of permutations which do not satisfy the condition $s_5 > s_1$. These permutations satisfy the condition $s_4 > s_1 > s_5 > s_2$ with no condition on s_3. Since there are 5 possibilities for s_3, there are 5 such permutations and

$$N_5 = 30 - 5 = 25.$$

For $n = 7$. There are more conditions and the method used for $n = 5$ becomes complicated. That is why, we will use a different method based on an inductive relation for N_n.
For $n = 1, 2, 3$ there are no conditions on the permutations. So

$$N_1 = 1! = 1, \quad N_2 = 2! = 2, \quad N_3 = 3! = 6.$$

For $n = 4$, there is only one condition: $s_4 > s_1$. This gives,

$$N_4 = \frac{1}{2}4! = 12.$$

For $n \geq 5$, we have the conditions $s_n \geq s_1, s_2, \ldots, s_{n-3}$. This means that at most there are only s_{n-1}, s_{n-2} at most which can be greater than s_n. Therefore $s_n \in \{n, n-1, n-2\}$.

(a) When $s_n = n$, there are as many permutations as for $n-1$, that is N_{n-1}.

(b) When $s_n = n - 1$, because of the conditions $s_n \geq s_1, s_2, \ldots, s_{n-3}$, we have $n \in \{s_{n-1}, s_{n-2}\}$.
   - If $s_{n-1} = n$, there are as many permutations as for $n - 2$, that is N_{n-2}.
   - If $s_{n-2} = n$, because of the conditions $s_{n-1} \geq s_1, s_2, \ldots, s_{n-4}$ we have $s_{n-1} \in \{n-2, n-3\}$.
     i. If $s_{n-1} = n - 2$, there are as many permutations as for $n - 3$, that is N_{n-3}.
     ii. If $s_{n-1} = n - 3$ then $s_{n-3} = n - 2$ and there are as many permutations as for $n - 4$, that is N_{n-4}.

(c) When $s_n = n - 2$, because of the conditions $s_n \geq s_1, s_2, \ldots, s_{n-3}$, we have either $s_{n-1} = n,\ s_{n-2} = n - 1$ or $s_{n-1} = n - 1,\ s_{n-2} = n$. In each case there are as many permutations as for $n - 3$, that is N_{n-3}.

Hence, we obtain the inductive relation

$$N_n = N_{n-1} + N_{n-2} + 3N_{n-3} + N_{n-4}.$$

Therefore

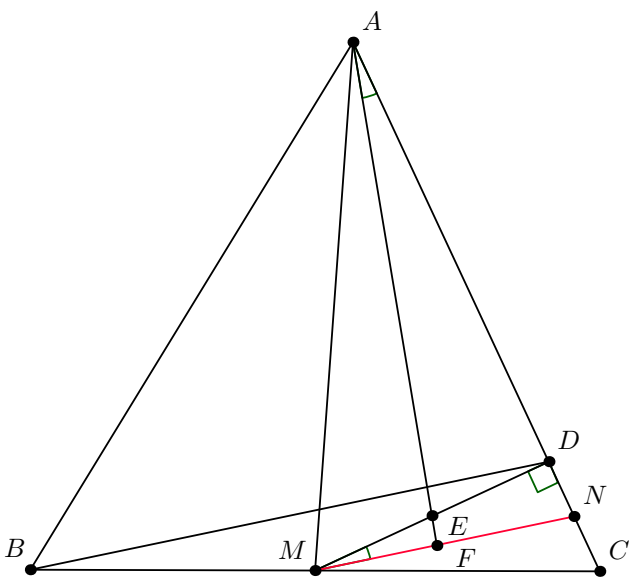
$$N_5 = 12 + 6 + 3 \times 2 + 1 = 25,$$

$$N_6 = 25 + 12 + 3 \times 6 + 2 = 57,$$

$$N_7 = 57 + 25 + 3 \times 12 + 6 = 124.$$

Remark. If the condition in this problem is replaced by $s_i > s_j$ for all $i \geq j+2$, N_n will be given by the Fibonacci sequence.

4. *First solution.* Let N be the midpoint of CD and F the intersection point of lines AE and MN. Since M is the midpoint of BC, the segment MN is parallel to BD.



Therefore, AE, BD are perpendicular if and only if AF, MN are perpendicular.
Equivalently, triangles AFN and MDN are similar since they already share the same angle at N.
Since AFN and ADE are similar, then triangles ADE and MDN are similar, which is equivalent to the fact that triangles ADM and MDC are similar

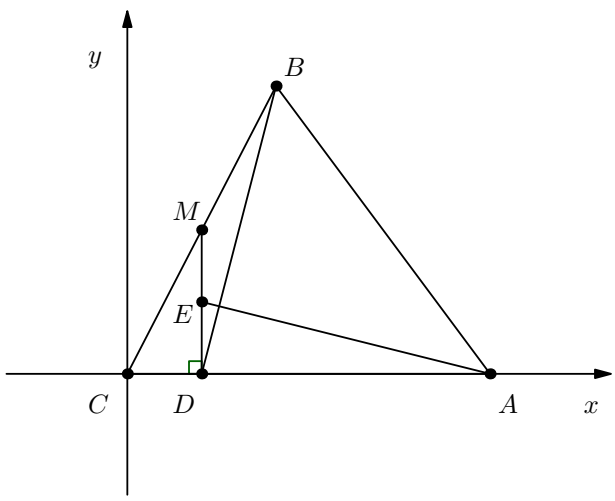
since E is the midpoint of DM and N is the midpoint of DC.
In conclusion, AE and BD are perpendicular if and only if the median AM of triangle ABC is also an altitude. That is $AB = AC$.

Second solution. By coordinates or equivalently by complex numbers. Fix the origin at C, and the $x-$ axis to be CA. The affixes of the points are

$$C(0), \qquad A(a) \qquad \text{and} \qquad B(b+ic),$$

where a, b, c are positive real numbers. Hence, the affixes of the other points are

$$M\left(\frac{b}{2}+i\frac{c}{2}\right), \qquad D\left(\frac{b}{2}\right) \qquad \text{and} \qquad E\left(\frac{b}{2}+i\frac{c}{4}\right).$$



Therefore, the affixes of the vectors are

$$\overrightarrow{DB}\left(\frac{b}{2}+ic\right) \qquad \text{and} \qquad \overrightarrow{EA}\left(a-\frac{b}{2}-i\frac{c}{4}\right).$$

On the other hand, the lengths of the sides are equal to

$$AB = \sqrt{(a-b)^2+c^2} \qquad \text{and} \qquad AC = a.$$

The segments AE, BD are perpendicular if and only if

$$Re\left(\left(\frac{b}{2}+ic\right)\left(a-\frac{b}{2}+i\frac{c}{4}\right)\right) = 0,$$

which is equivalent to $2ab - b^2 - c^2 = 0$, or, after a straight computation, to $AB = AC$.

Third solution. By using scalar products of vectors. Because M is the midpoint of BC and E the midpoint of MD, we have

$$\overrightarrow{DM} = \frac{1}{2}(\overrightarrow{DB} + \overrightarrow{DC}), \text{ or, equivalently } \overrightarrow{BD} = 2\overrightarrow{MD} + \overrightarrow{DC},$$

and

$$\overrightarrow{AE} = \overrightarrow{AM} + \overrightarrow{ME} = \frac{1}{2}(\overrightarrow{AM} + \overrightarrow{AD}).$$

Because MD and AC are perpendicular, we have

$$\overrightarrow{AD} \cdot \overrightarrow{MD} = \overrightarrow{ME} \cdot \overrightarrow{DC} = 0.$$

Therefore

$$\begin{aligned}
\overrightarrow{AE} \cdot \overrightarrow{BD} &= 2\overrightarrow{AE} \cdot \overrightarrow{MD} + \overrightarrow{AE} \cdot \overrightarrow{DC} \\
&= (\overrightarrow{AM} + \overrightarrow{AD}) \cdot \overrightarrow{MD} + (\overrightarrow{AM} + \overrightarrow{ME}) \cdot \overrightarrow{DC} \\
&= \overrightarrow{AM} \cdot \overrightarrow{MD} + \overrightarrow{AM} \cdot \overrightarrow{DC} = \overrightarrow{AM} \cdot \overrightarrow{MC} \\
&= \tfrac{1}{2}\overrightarrow{AM} \cdot \overrightarrow{BC}.
\end{aligned}$$

Hence, AE and BD are perpendicular if and only if the median AM of the triangle ABC is an altitude. This is equivalent to $AB = AC$.

Chapter 6

Solutions to the selection tests for the Gulf Mathematical Olympiad 2013

6.1 Solutions to the selection test of day I

- **Problem 1.** First, we see that it is possible for Tarik to choose the 101 numbers 11,12,. . . ,111, since the product $11 \times 12 > 111$.

  Assume that Tarik has choosen k numbers and let d be the smallest among these numbers. If $d \geq 11$, then clearly, $k \leq 101$.

  If $2 \leq d \leq 6$, from each of the 9 sets $\{9, 9d\}; \{10, 10d\}; \ldots; \{17, 17d\}$, Tarik can choose at most one number. Because $9d > 17$, these sets are pairwise disjoint. Because $17d \leq 102$, there are at least 9 numbers between 9 and 102 that Tarik could not choose. Therefore $k \leq 101$.

  If $3 \leq d \leq 10$, from each of the sets $\{d+1, d(d+1)\}$; $\{d+2, d(d+2)\}$; $\ldots$; $\{11, 11d\}$, Tarik can choose at most one element. Because $d(d+1) \geq 12$, these sets are pairwise disjoint. Because $11d < 111$, There are at least $11 - d$ numbers from these sets that Tarik could not choose. But Tarik didn't choose the numbers $2, \ldots, d-1$. Therefore, Tarik didn't choose at least $11-d+d-2 = 9$ numbers. Hence $k \leq 101$.

Therefore, the maximum number of numbers Tarik can choose is 101.

- **Problem 2.** We have

$$\begin{aligned}
&\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\\
&\qquad=\frac{a^4}{a^3+a^2b+ab^2}+\frac{b^4}{b^3+b^2c+bc^2}+\frac{c^4}{c^3+c^2a+ca^2}\\
&\qquad\geq\frac{(a^2+b^2+c^2)^2}{a^3+ab^2+ac^2+ba^2+b^3+bc^2+ca^2+cb^2+c^3},
\end{aligned}$$

by applying Cauchy-Schwarz inequality.

On the other hand

$$\begin{aligned}
\frac{(a^2+b^2+c^2)^2}{a^3+ab^2+ac^2+ba^2+b^3+bc^2+ca^2+cb^2+c^3} &= \frac{a^2+b^2+c^2}{a+b+c}\\
&\geq \frac{a+b+c}{3},
\end{aligned}$$

by applying again Cauchy-Schwarz inequality.

The equality holds when $a=b=c$.

- **Problem 3.** Because all angles are congruent and all line segments are congruent, the regular n-pointed star is cyclic. Indeed, consider any four consecutive vertices $P_i, P_{i+1}, P_{i+2}, P_{i+3}$, for some $i \in \{1,2,\ldots,n\}$. They form an isosceles trapezoid. So they are cocyclic. By induction, all vertices $P_1, P_2, \ldots, P_n$ are on the same circle. Let O be the center of this circle.

  Because all the line segments are congruent and the path turns counter-clockwise, there exists a positive real number $0^\circ < \theta < 180^\circ$ such that $\angle P_iOP_{i+1} = \theta$ for all $i = 1,2,\ldots,n$. Because $P_{n+1} = P_1$, there exists a positive integer k such that $n\theta = 360k^\circ$. But $\theta < 180^\circ$. Therefore $k < \frac{n}{2}$. Because each of the n line segments intersects at least one of the other line segments at a point other than an endpoint, $k > 1$. Because all the vertices $P_1, P_2, \ldots, P_n$, are different, the integers k and n have no common divisors.

Conversely, for a given circumradius, if $1 < k < \frac{n}{2}$ is an integer relatively prime to n, there exists a unique regular n-pointed star of angle $\theta = \frac{360k}{n}^\circ$. Hence, the number of non-similar regular n-pointed stars is the number of such integers k, that is $\frac{\phi(n)}{2} - 1$, where ϕ is the Euler totient function. Therefore, it remains to solve the equation $\phi(n) = 60$.

There are three cases. The first case is when there exists a prime p divisor of n such that 3×5 divides $p-1$. Here, the only possibilities are $p = 61$ or 31.

If $p = 61$, there are 2 solutions for n. Either $n = 61$ or $n = 2 \times 61 = 122$.

If $p = 31$, there are 3 solutions for n. Either $n = 3 \times 31 = 93$, or $n = 2 \times 3 \times 31 = 186$, or $n = 2^2 \times 31 = 124$.

The second case is when there exist two prime numbers p and q such that 3 divides $p - 1$ and 5 divides $q - 1$. In this case, the only possibility is $p = 7$ and $q = 11$. This gives only two solutions for n. Either $n = 7 \times 11 = 77$ or $n = 2 \times 7 \times 11 = 154$.

The third case is when there exists an odd prime number p such that p^2 divides n. Then $p = 3$ or 5. If $n = 5^2 m$ and m is relatively prime to 5, we obtain $\phi(m) = 3$ which is impossible since 3 is an odd number. If $n = 3^2 m$ and m is relatively prime to 3, we obtain $\phi(m) = 10$. Therefore, $m = 11$ or 22. Hence $n = 99$ or 198.

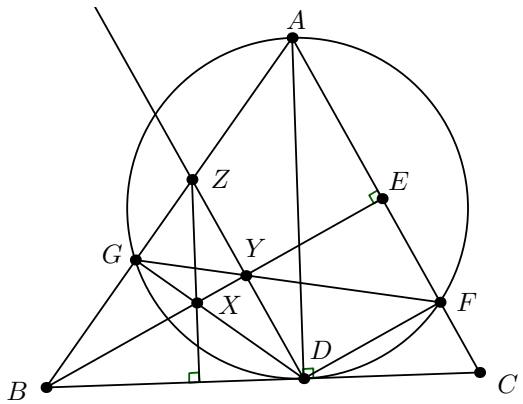
In conclusion, all possible values for n are 61,77,93,99,122,124,154,186 and 198.

- **Problem 4.** We present for this problem two solutions.

First solution. We have $\angle AFG = \angle ADG$ since $AGDF$ is cyclic. On the other hand $\angle DGA = \angle AFD = 90^\circ$, since AD is a diameter. We deduce that triangles AGD and ADB are similar, and therefore $\angle AFG = \angle CBA$.

Because $\angle AEB = \angle ADB = 90^\circ$, quadrilateral $ABDE$ is cyclic. Therefore $\angle DEB = \angle CBA = \angle EFY$. But $\angle EFD = \angle YEF = 90^\circ$. We deduce that $DFEY$ is a rectangle and therefore BY is an altitude in triangle BDZ. Line

segment DG is also an altitude in triangle BDZ which intersects BY at X. We deduce that ZX and BC are perpendicular.

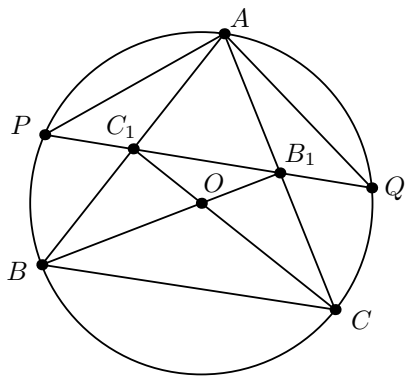


Second solution. Because $\angle ADB = \angle AEB = 90°$, the quadrilateral $ABDE$ is cyclic. Therefore, the projections of the point D on the lines AB, BE, and EA are collinear (Simson line). But G and F are the projections of D on lines AB and EA, respectively, since AD is a diameter of the circle ω. Then the projection of D on BE is Y, the intersection point of BE with GF. Therefore, BY is perpendicular to DZ. Hence, in triangle BDZ, line segments BY and DG are altitudes and intersect at X. Thus ZX is also an altitude and therefore ZX and BC are perpendicular.

6.2 Solutions to the selection test of day II

- **Problem 1.** Assume $AP = AQ$. Because $OP = OQ$, the line AO is perpendicular to PQ. But since $\angle C_1AO = 90° - \angle ACB$ then $\angle B_1C_1A = \angle ACB$, and therefore, quadrilateral BCB_1C_1 is cyclic.

  Using the power of the point O with respect to the circumcircle of BCB_1C_1 we get
  $$OB_1 = \frac{OB_1 \cdot OB}{R} = \frac{OC_1 \cdot OC}{R} = OC_1,$$
  where R is the circumradius of triangle ABC. Therefore, $BB_1 = CC_1$, that is the minors $\widehat{BB_1}$ and $\widehat{CC_1}$ of the circumcircle of BCB_1C_1 have the same length. This is equivalent to saying that $\angle CBC_1 = \angle B_1CB$, and therefore $AB = AC$.



- **Problem 2.** Assume that there exist two non constant polynomials $g(X)$ and $h(X)$ with integer coefficients such that $f(X) = g(X)h(X)$. Because, $p = g(0)h(0)$ is prime, we can assume that $|g(0)| = 1$. Because the modulus of the product of the complex roots of $g(X)$ is equal to 1, at least one of these roots, say ω_0, has modulus less than or equal to 1. But $f(\omega_0) = 0$. We deduce that

$$\begin{aligned} p &= |a_n\omega_0^n + a_{n-1}\omega_0^{n-1} + \cdots + a_1\omega_0| \\ &\leq |a_n| \cdot |\omega_0|^n + |a_{n-1}| \cdot |\omega_0|^{n-1} + \cdots + |a_1| \cdot |\omega_0| \\ &\leq |a_n| + |a_{n-1}| + \cdots + |a_1|, \end{aligned}$$

  which is a contradiction. Therefore, $f(X)$ is irreducible.

- **Problem 3.** Let p be a prime divisor of $n^{55} - n$ for all integer n. Whenever n is not divisible with p, we have

$$n^{54} \equiv 1 \mod p.$$

  In this case, the order of n modulo p divides 54. But there exists an integer n of order $p-1$ modulo p. We deduce that $p-1$ divides 54. But the only primes p such that $p-1$ divides 54 are $p = 2, 3, 7$ and 19. Conversely, all these four primes $2, 3, 7$ and 19 divide $n^{55}-n$ for all integer n by Fermat's little theorem.

  Notice that for $p(p^{55} - 1)$ is not divisible by p^2 for all prime numbers p since $p^{54} - 1$ is relatively prime with p. Therefore, the greatest integer k which divides $n^{55} - n$ for all integer n is $2 \times 3 \times 7 \times 19 = 798$.

- **Problem 4.** Let m be a positive integer and consider the infinite set of pairs (F_k, F_{k+1}), for $k \in \mathbb{N}$. By the pigeonhole principle, there exists a pair (a, b) of

integers $0 \le a, b \le m-1$ and an infinite sequence of integers $0 < k_1 < k_2 < \cdots$ such that

$$(F_{k_i}, F_{k_i+1}) \equiv (a, b) \mod m, \quad \text{for all} \quad i \ge 1.$$

Therefore

$$(F_{k_i-1}, F_{k_i}) = (F_{k_i+1} - F_{k_i}, F_{k_i}) \equiv (b-a, a) \mod m, \quad \text{for all} \quad i \ge 1.$$

We keep descending in this way until we get

$$(F_{k_i-k_1+2}, F_{k_i-k_1+3}) \equiv (F_2, F_3) \equiv (1, 2) \mod m, \quad \text{for all} \quad i \ge 1.$$

Let $n_i = k_i - k_1 + 2$, for all $i \ge 1$. Clearly, the infinite sequence $2 = n_1 < n_2 < \cdots$ is increasing and we have

$$F_{n_i} + 2 \equiv F_2 + 2 \equiv 3 \mod m, \qquad F_{n_i+1} + 1 \equiv F_3 + 1 \equiv 3 \mod m,$$

$$\text{and} \quad F_{n_i+2} \equiv F_{n_i+1} + F_{n_i} \equiv F_3 + F_2 \equiv 3 \mod m.$$

6.3 Solutions to the selection test of day III

- **Problem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the two functional equations.

  Because $f\left(\frac{\sqrt{3}}{3}x\right) - \sqrt{3}f(1) = -\frac{2\sqrt{3}}{3} \neq 0$, either $f(1) \neq 0$ or $f\left(\frac{\sqrt{3}}{3}x\right) \neq 0$.

  Fix $x_0 \in \mathbb{R}$ with $f(x_0) \neq 0$ and let $y \in \mathbb{R}$ with $y \neq 0$. We have

  $$f(x_0)f\left(\frac{1}{y}\right) = f\left(\frac{x_0}{y}\right) + f(x_0 y) = f(x_0)f(y).$$

  Therefore, $f\left(\frac{1}{y}\right) = f(y)$ for all $y \neq 0$.

Let $x \neq 0$. We have

$$\begin{aligned} f\left(\frac{\sqrt{3}}{3}x\right) &= \sqrt{3}f(x) - \frac{2\sqrt{3}}{3}x \\ &= \sqrt{3}f\left(\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{3}x}\right) - \frac{2\sqrt{3}}{3}x \\ &= \sqrt{3}\left(\sqrt{3}f\left(\frac{1}{\frac{\sqrt{3}}{3}x}\right) - \frac{2}{x}\right) - \frac{2\sqrt{3}}{3}x \\ &= 3f\left(\frac{\sqrt{3}}{3}x\right) - \frac{2\sqrt{3}}{x} - \frac{2\sqrt{3}}{3}x. \end{aligned}$$

We deduce that

$$f\left(\frac{\sqrt{3}}{3}x\right) = \frac{\sqrt{3}}{x} + \frac{\sqrt{3}}{3}x,$$

and therefore

$$f(x) = x + \frac{1}{x},$$

for all $x \neq 0$.

For $x = 0$ and $y = 2$, we have from the second functional equation

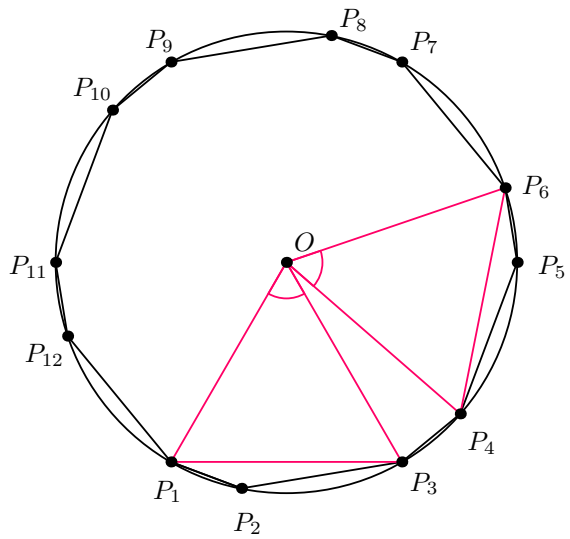
$$\frac{5}{2}f(0) = f(0) + f(0),$$

which implies that $f(0) = 0$. Hence

$$f(x) = \begin{cases} x + \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, it is easy to check that this function satisfies the two functional equations.

- **Problem 2.** Let $P_1P_2 \cdots P_n$ be a convex cyclic non-regular polygon with the measures of all its internal angles are equal and let O be its circumcenter. Because, all the angles are equal, all the arcs $\widehat{P_iP_{i+2}}$, for $i = 1, \ldots, n$, have the same length. Hence, $\angle P_iOP_{i+2} = \frac{4\pi}{n}$, for all $i = 1, \ldots, n$, since the polygon is convex.



Let $\theta = \angle P_1OP_2$. We have

$$\angle P_{2i-1}OP_{2i} = \angle P_1OP_2 + \angle P_2OP_{2i} - \angle P_1OP_{2i-1} = \angle P_1OP_2 = \theta.$$

If n is an odd integer,

$$\theta = \angle P_nOP_1 = \frac{n+1}{2}\angle P_nOP2 = \frac{n+1}{2} \cdot \frac{4\pi}{n} = \frac{2\pi}{n} = \angle P_1OP_2 \mod 2\pi.$$

This means that the polygon is regular, which contradicts the hypothesis.

If n is even, any value of θ with $0 < \theta < \frac{4\pi}{n}$ and $\theta \neq \frac{2\pi}{n}$ defines a unique non-regular such a polygon.

Therefore, the possible values of n are all even positive integer $n \geq 4$.

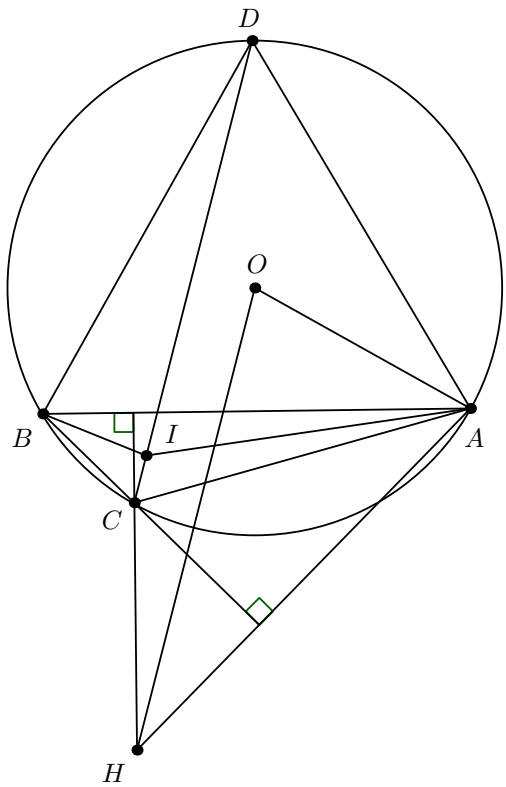
- **Problem 3.** We already know that $DA = DB = DI$. Because $DI = AB$, triangle ADB is equilateral. But $\angle ACB + \angle BDA = 180°$, we deduce that $\angle ACB = 120°$.

  Because $\angle BOA = 2\angle BDA = 120°$, we deduce that $\angle OAB = 30°$. On the other hand, we have $\angle CAH = 90° - \angle CBA - \angle BAC = 30°$. By applying cosine law in the triangle AOH we obtain

$$OH^2 = AH^2 + R^2 - 2R \cdot AH \cos(\angle BAC + 60°),$$

where R is the circumradius of triangle ABC. This is equivalent to

$$2AH\cos(\angle BAC + 60^\circ) = R.$$



Because HC is perpendicular to AB and BC is perpendicular to AH, we have $\angle AHC = \angle CBA$. On the other hand, we have $\angle HCA = 180^\circ - \angle AHC - 30^\circ = 90^\circ + \angle BAC$. We deduce, by applying sine law to the triangle ACH that

$$\frac{AH}{\sin(90^\circ + \angle BAC)} = \frac{AC}{\sin\angle CBA} = 2R.$$

Therefore, $AH = 2R\cos\angle BAC$. By plugging this into our previous relation we obtain

$$2\cos\angle BAC \cdot \cos(\angle BAC + 60^\circ) = \frac{1}{2}.$$

This is equivalent to

$$\cos(2\angle BAC + 60^\circ) + \cos 60^\circ = \frac{1}{2},$$

and therefore, $\angle BAC = 15°$. Thus

$$\angle BAC = 15°, \quad \angle CBA = 45°, \quad \text{and } \angle ACB = 120°.$$

- **Problem 4.** We have

$$0 \equiv (a^3+1) - (b^3+1) \equiv (a-b)(a^2+ab+b^2) \equiv (a-b)ab \mod (a^2+b^2).$$

Let d be a common divisor of a and a^2+b^2. Then d divides a^3+1 and a^3, so it divides 1. Hence a and a^2+b^2 are coprime. In a similar way b and a^2+b^2 are coprime. Thus $a-b \equiv 0 \mod (a^2+b^2)$.

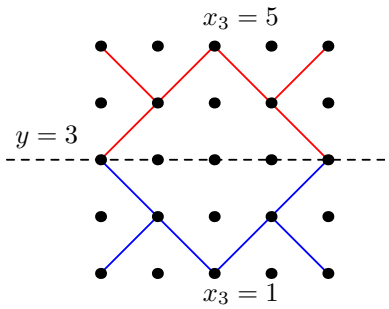
If $a \neq b$ then $a^2+b^2 \leq |a-b| \leq (a-b)^2 < a^2+b^2$, since $ab \geq 1$, which is a contradiction. Hence $a = b = 1$, since a and a^2+b^2 are coprime.

Chapter 7

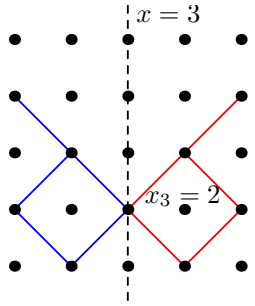
Solutions to the selection tests for the Balkan Mathematical Olympiad 2013

7.1 Solutions to the selection test of day I

- **Problem 1.** Let us count the number of five-point sequences according to the position of point P_3.



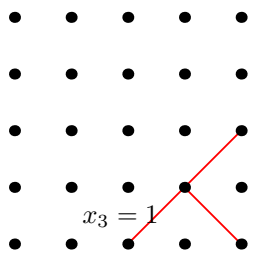
Because the horizontal line of equation $y = 3$ is a symmetry axis for this figure, the number of sequences with $x_3 = i$ is equal to the number of sequences with $x_3 = 6 - i$, for $i = 1, 2$.



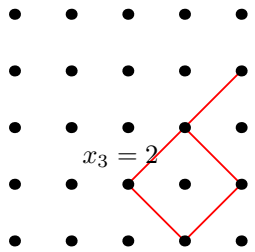
Similarly, the vertical line of equation $x = 3$ is a symmetry axis for this figure, so that the number N_i of three-point sequences (P_1, P_2, P_3) ending at P_3 with $x_3 = i$ is equal to the number of three-point sequences (P_3, P_4, P_5) starting at P_3, with $x_3 = i$, for $i = 1, 2, 3$. Therefore, the number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ with $x_3 = i$ is N_i^2. Hence, the total number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ is

$$2(N_1^2 + N_2^2) + N_3^2.$$

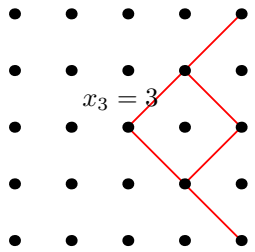
It is easy to see that $N_1 = 2$, since the only possibilites for x_4, x_5 are $x_4 = 2$ and $x_5 = 1$ or 3.



It is also easy to see that $N_2 = 3$, since $x_4 = 1$ or 3 and $x_5 = 2$ in the first case and 2 or 4 in the second case.



Finaly, $N_3 = 4$, since $x_4 = 2$ or 4, and $x_5 = 1$ or 3 in the first case, and 3 or 5 in the second case.



Hence, the number of sequences is $2\left(2^2 + 3^2\right) + 4^2 = 42$.

- **Problem 2.** Let $d = \gcd(a, b)$ and $a = da'$ and $b = db'$ with a', b' two relatively prime positive integers. The equation becomes

$$a'b'd^2 + 63 = 20a'b'd + 12d.$$

Therefore, d divides 63 and we have

$$a'b'd + \frac{63}{d} = 20a'b' + 12.$$

If $5 < d < 20$, then $a'b'd + \frac{63}{d} < 20a'b' + 12$, and this is impossible. Hence, $d = 1, 3, 21$, or 63.

1. If $d = 1$, the equation is equivalent to $51 = 19a'b'$, which is impossible since 19 does not divide 51.
2. If $d = 3$, the equation is equivalent to $9 = 17a'b'$, which is impossible since 17 does not divide 9.
3. If $d = 21$, the equation is equivalent to $a'b' = 9$. Because a', b' are relatively prime positive integers, either $a' = 1, b' = 9$ or $a' = 9, b' = 1$. This leads to two solutions $(a, b) = (21, 189)$ and $(a, b) = (189, 21)$.
4. If $d = 63$, the equation is equivalent to $43a'b' = 11$, which is impossible since 43 does not divide 11.

Therefore, the equation has two solutions $(a, b) = (21, 189)$ and $(a, b) = (189, 21)$.

- **Problem 3.** Notice first that if $\lfloor x \rfloor \leq 0$ then

$$\lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 \geq 24 > 0.$$

This means that if x is a solution to the equation then $\lfloor x \rfloor \geq 1$.

Let x be a solution to the equation, $m = \lfloor x \rfloor \geq 1$, and $r = x - \lfloor x \rfloor \geq 0$. We have

$$0 = \lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 = \lfloor r(r+2m) \rfloor + (m-4)(m-6).$$

Because $r(r+2m) \geq 0$, we deduce that $(m-4)(m-6) \leq 0$, which is equivalent to $m = 4, 5$, or 6.

1. If $m = 4$, the equation becomes $\lfloor r(r+8) \rfloor = 0$. This is equivalent to $(r+4)^2 - 17 < 0$ and $(r+4)^2 - 16 \geq 0$, and its solutions are $0 \leq r < \sqrt{17} - 4$. This means that the solutions to the equation in this case are $4 \leq x < \sqrt{17}$.

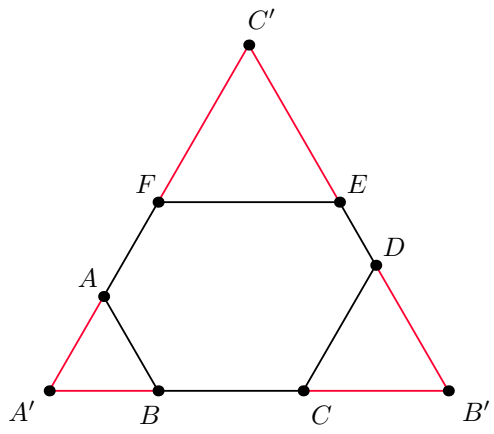
2. If $m = 5$, the equation becomes $\lfloor r(r+10) \rfloor = 1$. This is equivalent to $(r+5)^2 - 27 < 0$ and $(r+5)^2 - 26 \geq 0$, and its solutions are $\sqrt{26} - 5 \leq r < \sqrt{27} - 5$. This means that the solutions to the equation in this case are $\sqrt{26} \leq x < \sqrt{27}$.

3. If $m = 6$, the equation becomes $\lfloor r(r+12) \rfloor = 0$. This is equivalent to $(r+6)^2 - 37 < 0$ and $(r+6)^2 - 36 \geq 0$, and its solutions are $0 \leq r < \sqrt{37} - 6$. This means that the solutions to the equation in this case are $6 \leq x < \sqrt{37}$.

Thus, the set of solutions to this equation is

$$\left[4, \sqrt{17}\right) \cup \left[\sqrt{26}, \sqrt{27}\right) \cup \left[6, \sqrt{37}\right).$$

- **Problem 4.** We extend sides FA and BC to intersect at A', and sides BC and DE to intersect at B', and sides DE and FA to intersect at C'.



Triangles $A'B'C'$, $A'BA$, $B'DC$, and $C'FE$ are equilateral with side lengths 11,3,4, and 5 respectively. Therefore, the area of hexagon $ABCDEF$ is

$$\frac{\sqrt{3}}{4} \cdot 11^2 - \frac{\sqrt{3}}{4} \cdot 3^2 - \frac{\sqrt{3}}{4} \cdot 4^2 - \frac{\sqrt{3}}{4} \cdot 5^2 = \frac{71\sqrt{3}}{4}.$$

- **Problem 5.** Notice first that

$$X^4 + 2X^3 + (2+2k)X^2 + (1+2k)X + 2k = (X^2 + X + 1)(X^2 + X + 2k).$$

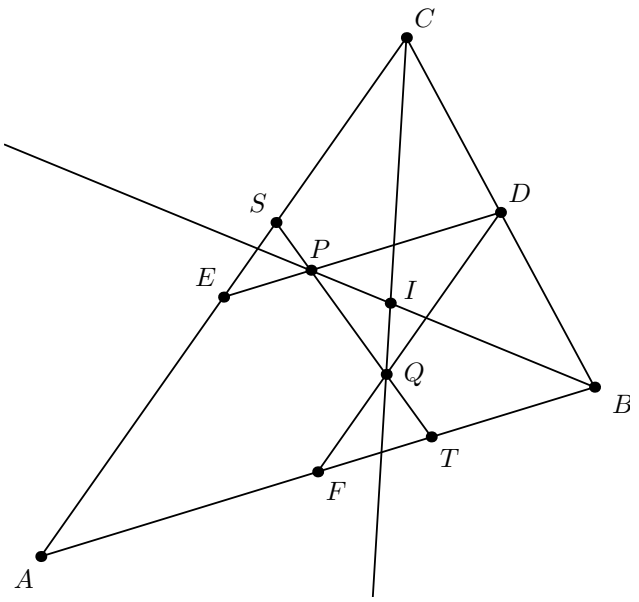
Because the factor $X^2 + X + 1$ has no real roots, we deduce from Vieta relations that $r_1 + r_2 = -1$ and $r_1 r_2 = 2k = -2013$, where r_1, r_2 are the real roots of the equation $X^4 + 2X^3 + (2+2k)X^2 + (1+2k)X + 2k = 0$. Therefore,

$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = 1 + 2 \times 2013 = 4027.$$

- **Problem 6.** Because sides of triangles ABC and DEF are parallel (homothetic triangles), we have

$$\begin{aligned} \angle BPD &= 180^\circ - \angle EDF - \angle FDB - \tfrac{1}{2}\angle CBA \\ &= 180^\circ - \angle BAC - \angle ACB - \tfrac{1}{2}\angle CBA \\ &= \tfrac{1}{2}\angle CBA = \angle DBP. \end{aligned}$$

We deduce that triangle DPB is isosceles and therefore $DP = DB$.



Similarly, we prove that $DQ = DC$. But $DB = DC$. We conclude that triangle DPQ is isosceles, and therefore, its bisector at D is perpendicular to PQ.

But the bisector of triangle ABC at A is parallel to the bisector of triangle DEF at E since the two triangles are homothetic. We deduce that the bisector of triangle ATS at A is perpendicular to ST and therefore $AS = AT$.

- **Problem 7.** Let us associate a 1 to each cell with a black color and a 0 to each cell with a white color. The condition is equivalent to the sum of numbers in each 2×3 rectangle and in each 3×2 rectangle is even. Let $a_{i,j}$ be this number at the cell in the ith row and j^{th} column, for $1 \leq i, j \leq 50$.

  Consider, for a fixed pair i, j, with $1 \leq i \leq 48$ and $1 \leq j \leq 47$, the 3×4 rectangle:

$a_{i,j}$	$a_{i,j+1}$	$a_{i,j+2}$	$a_{i,j+3}$
$a_{i+1,j}$	$a_{i+1,j+1}$	$a_{i+1,j+2}$	$a_{i+1,j+3}$
$a_{i+2,j}$	$a_{i+2,j+1}$	$a_{i+2,j+2}$	$a_{i+2,j+3}$

  By applying the condition to the two 2×3 rectangles which contain cells from the second and the third rows, we get

$$a_{i+1,j} + a_{i+1,j+1} + a_{i+1,j+2} + a_{i+2,j} + a_{i+2,j+1} + a_{i+2,j+2} \equiv 0 \mod 2,$$

and

$$a_{i+1,j+1} + a_{i+1,j+2} + a_{i+1,j+3} + a_{i+2,j+1} + a_{i+2,j+2} + a_{i+2,j+3} \equiv 0 \mod 2.$$

By applying the condition to all the 3×2 rectangles, we get

$$a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} + a_{i+2,j} + a_{i+2,j+1} \equiv 0 \mod 2,$$

$$a_{i,j+1} + a_{i,j+2} + a_{i+1,j+1} + a_{i+1,j+2} + a_{i+2,j+1} + a_{i+2,j+2} \equiv 0 \mod 2,$$

and

$$a_{i,j+2} + a_{i,j+3} + a_{i+1,j+2} + a_{i+1,j+3} + a_{i+2,j+2} + a_{i+2,j+3} \equiv 0 \mod 2.$$

By adding these 5 relations and cancelling all even numbers we get

$$a_{i,j} + a_{i,j+3} \equiv 0 \mod 2.$$

This proves that

$$a_{i,j+3} = a_{i,j}.$$

We prove in a similar way that

$$a_{i+3,j} = a_{i,j}.$$

Therefore, it is enough to know the numbers in the 3×3 rectangle

$a_{1,1}$	$a_{1,2}$	$a_{1,3}$
$a_{2,1}$	$a_{2,2}$	$a_{2,3}$
$a_{3,1}$	$a_{3,2}$	$a_{3,3}$

to deduce, by periodicity, the numbers in all the other cells.

Applying the condition to the first 2×3 rectangle, we will get

$$a_{2,3} \equiv a_{1,1} + a_{1,2} + a_{1,3} + a_{2,1} + a_{2,2} \mod 2$$

Applying the condition to the second 2×3 rectangle and to the second 3×2 rectangle and adding the two relations, we will get

$$a_{3,1} \equiv a_{2,1} + a_{1,2} + a_{1,3} \mod 2$$

We obtain in a similar way

$$a_{3,2} \equiv a_{2,2} + a_{1,3} + a_{1,1} \mod 2$$

$$a_{3,3} \equiv a_{2,3} + a_{1,1} + a_{1,2} \mod 2$$

Therefore, it is enough to know the 5 numbers

$$a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2},$$

to deduce all the numbers in the cells of the 50×50 chessborad.

Conversely, choose a value in $\{0, 1\}$ for each number $a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}$, and deduce the values in all the other cells of the chessboard. It is easy to check that the condition on the two 2×3 rectangles and the two 3×2 rectangles in the 3×3 rectangle above is satisfied. We deduce, by periodicity, that it is satisfied in all the chessboard. Hence there are 2^5 ways to color the chessboard.

- **Problem 8.** We prove by induction on $n \geq 2$ that

$$T_n = \frac{1^1 + 3^3 + 5^5 + \cdots + (2^n - 1)^{(2^n - 1)}}{2^n}$$

is an odd integer.

For $n = 2$, $T_2 = \frac{1^1 + 3^3}{2^2} = 7$ is an odd integer.

Assume that T_n is an odd integer. We have

$$\begin{aligned} T_{n+1} &= \frac{1^1 + 3^3 + 5^5 + \cdots + \left(2^{n+1} - 1\right)^{\left(2^{n+1} - 1\right)}}{2^{n+1}} \\ &= T_n + \frac{\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k - 1)^{2k-1}\right)}{2^{n+1}}. \end{aligned}$$

Therefore, it remains to prove that

$$\frac{\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k - 1)^{2k-1}\right)}{2^{n+1}}$$

is an even integer, that is

$$\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k - 1)^{2k-1}\right) \equiv 0 \mod 2^{n+2}.$$

To prove this, we will use the fact that if l is an odd integer, then $l^{2^n} \equiv 1 \mod 2^{n+2}$. This is because each factor in

$$l^{2^n} - 1 = \left(l^{2^{n-1}} + 1\right)\left(l^{2^{n-2}} + 1\right) \cdots \left(l^2 + 1\right)(l+1)(l-1)$$

is even and $(l+1)(l-1)$ is divisible by 8.

Because $n \geq 2$, we have for $1 \leq k \leq 2^{n-1}$

$$\begin{aligned}
(2^n + 2k - 1)^{2^n + 2k - 1} &- (2k-1)^{2k-1} \\
&\equiv (2^n + 2k - 1)^{2k-1} - (2k-1)^{2k-1} \\
&\equiv \sum_{i=0}^{2k-1} \binom{2k-1}{i} 2^{in} (2k-1)^{2k-1-i} - (2k-1)^{2k-1} \\
&\equiv (2k-1) \cdot 2^n \cdot (2k-1)^{2k-2} \\
&\equiv 2^n (2k-1)^{2k-1} \qquad \mod 2^{n+2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k-1)^{2k-1}\right) \\
&\qquad \equiv 2^n \sum_{k=1}^{2^{n-1}} (2k-1)^{2k-1} \equiv 2^{2n} T_n \equiv 0 \qquad \mod 2^{n+2}.
\end{aligned}$$

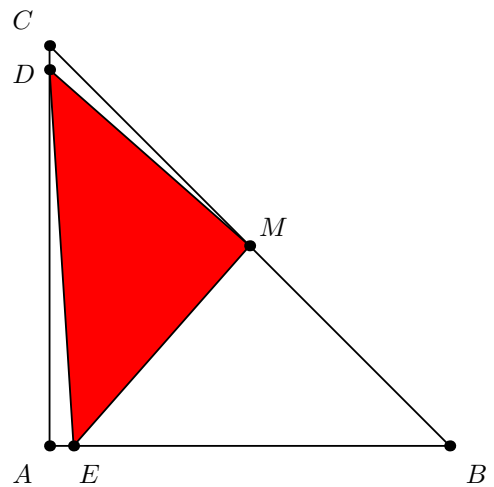
7.2 Solutions to the selection test of day II

- **Problem 1.** Because $AEMD$ is cyclic, we have $\angle BEM = \angle ADM$. But $\angle MBE = \angle MAD = 45°$ and $BM = AM$. We deduce that triangles BME and AMD are congruent and therefore $ME = MD$.

  Because $AEMD$ is cyclic, $\angle DME = 90°$. Therefore, the area of triangle EMD is

$$2 = \frac{1}{2} ME \cdot MD = \frac{1}{2} MD^2,$$

and hence, $MD = ME = 2$.



We have from the cosine law in triangle AMD

$$4 = MD^2 = AD^2 + AM^2 - 2AD \cdot AM \cos 45^\circ = AD^2 + \frac{9}{2} - 3AD.$$

Because AE satisfies the same equation as AD above, and $AE < AD$, we deduce that $AD = \frac{3+\sqrt{7}}{2}$ and therefore

$$CD = \frac{3-\sqrt{7}}{2}.$$

- **Problem 2.** Plug in $y = 0$. The functionnal equation becomes

$$f(f(f(x))) = x + f(0),$$

for all $x \in \mathbb{R}$. Since the map $x \mapsto x + f(0)$ is bijective, then so is f.

Plug in $y = -x$. The functionnal equation becomes

$$f(f(f(x) - x) - x) = f(-x),$$

for all $x \in \mathbb{R}$. By injectivity of f, we can cancel f in both sides and obtain

$$f(f(x) - x) = 0,$$

for all $x \in \mathbb{R}$. By surjectivity of f there exists a real number a such that $f(a) = 0$. Again by cancelling f from both sides we obtain

$$f(x) = x + a,$$

for all $x \in \mathbb{R}$. But $0 = f(a) = 2a$. We deduce that

$$f(x) = x,$$

for all $x \in \mathbb{R}$.

Conversely, we check easily that this function is a solution to the problem.

- **Problem 3.** Let us consider the two possible cases for the integer x.

If x is an odd positive integer, we have

$$z^2 \equiv 2^x + 21^y \equiv 2 + 0 \equiv 2 \mod 3.$$

This is impossible since a perfect square is congruent to either 0 or 1 modulo 3.

If x is an even positive integer, write $x = 2x_0$ for a positive integer x_0. We have

$$21^y = z^2 - 2^x = (z - 2^{x_0})(z + 2^{x_0}).$$

Assume that one of the two primes 3 or 7 divides both factors $z - 2^{x_0}$ and $z + 2^{x_0}$. This implies that this prime divides their sum $2z$ and therefore z. But $2^x = 21^y - z^2$ is neither divisible by 3 nor by 7. This proves that the factors $z - 2^{x_0}$ and $z + 2^{x_0}$ are relatively prime. Because $z - 2^{x_0} < z + 2^{x_0}$, we have two possibilities

a. First possibility is when $z - 2^{x_0} = 1$ and $z + 2^{x_0} = 21^y$. In this case $2^{x_0+1} + 1 = 21^y$. Because y is positive, we have

$$2^{x_0+1} \equiv 6 \mod 7.$$

But this is impossible since 2^{x_0+1} is congruent to either $1, 2$ or 4 modulo 7.

b. Second possibility is when $z - 2^{x_0} = 3^y$ and $z + 2^{x_0} = 7^y$. This is equivalent to $z = 2^{x_0} + 3^y$ and $2^{x_0+1} = 7^y - 3^y$. If $y = 1$, then $x = 2$ and $z = 5$ gives a solution for the problem. If $y \geq 2$, because $7^y - 3^y \equiv 0$

mod 8 only for even number, y must be even. Write $y = 2y_0$ for a positive integer y_0. We have $2^{x_0+1} = (7^{y_0} - 3^{y_0})(7^{y_0} + 3^{y_0})$. But $7^{y_0} + 3^{y_0}$ is greater than 4 and is congruent to 2 modulo 4. This means that it is not a power of 2, which is a contradiction.

Hence, the only solution is $(x, y, z) = (2, 1, 5)$.

- **Problem 4.** Notice first that there are no 3 consecutive students with hats of the same color.

  If there are two consecutive students with hats of the same color, the next two consecutive students with hats of the same color must be different from the first color. Moreover, the number of students standing between this two pairs of consecutive students with hats of the same colors, must be even since they alternate between the colors of their hats.

  We conclude from this that there are three possible cases:

  1. First case, when there are no consecutive students with hats of the same color. In this case, the students must alternate the color of their hats and there are precisely 2 possible ways depending on the color of the hat of the first student.
  2. Second case, when the position of the first student followed by another student with hat of the same color is odd. In this case, each nonempty subset of the set of pairs $\{(1, 2); (3, 4); (5, 6); (7, 8); (9, 10)\}$ determine the positions of pairs of students with hat of the same color. Because the color of hats of the first pair in this subset determines the color of hats of all the other students, there are $2 \cdot (2^5 - 1)$ ways in which the teacher can place the hats.
  3. Third case, when the position of the first student followed by another student with hat of the same color is even. In this case, each nonempty subset of the set of pairs $\{(2, 3); (4, 5); (6, 7); (8, 9)\}$ determine the positions of pairs of students with hat of the same color. Because the color of hats of the first pair in this subset determines the color of hats of all the other students, there are $2 \cdot (2^4 - 1)$ ways in which the teacher can place the hats.

  Therefore, the number of ways in which the teacher can place hats is

$$2 + 2 \cdot (2^5 - 1) + 2 \cdot (2^4 - 1) = 94.$$

- **Problem 5.** We prove this by induction on n.

  For $n = 1, 2, 3$, the numbers 1,34, and 122 are good numbers and all their odd digits are less than 5.

  Assume that a_n is an n-digit good number and all its odd digits are less than 5.

  If a_n has an even digit $2m$, remove this digit and replace it by 4 digits $mmmm$ to obtain an $n+3$-digit number. The sum of squares of the digits of this new number is equal to the sum of squares of the digits of a_n and is therefore a perfect square. Notice that all the odd digits of this new $n+3$-digit good number are less than 5. For example, replace the 4 in the 2-digit good number 34 by 2222 to obtain the 5-digit good number 32222.

  If a_n has no even digit, all its digits are less than 5. We can multiply all its digits by 2 to obtain a new n-digit good number. Choose one of the new even digits $2m$ remove it and replace it by four digits $mmmm$ to obtain a new $n+3$-digit good number and its odd digits m are less than 5. For example, for the 4-digit good number 3333 multiply all its digit by 2 to obtain the 4-digit good number 6666 and then replace one of the 6 by 3333 to obtain the 7-digit good number 3333666.

  This proves by induction that for any positive integer n, there exists an n-digit good number.

- **Problem 6.** We present for this problem two solutions, one using classical inequalities and the second using geometry.

  First solution. Notice that, by using AM-GM inequality, we have

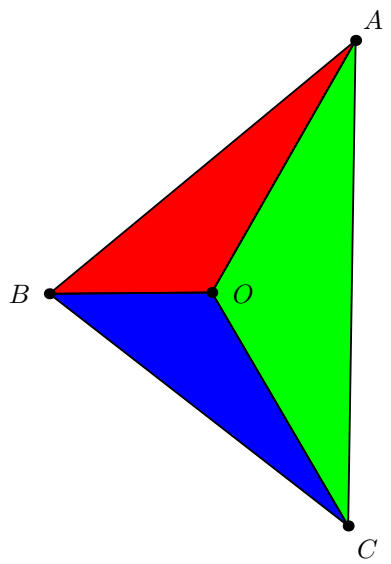
$$a\sqrt{b^2+c^2+bc} = a\sqrt{(b+c)^2-bc} \geq a\sqrt{(b+c)^2-\left(\frac{b+c}{2}\right)^2} = \frac{\sqrt{3}}{2}a(b+c).$$

  Applying the same inequality for the two other terms, we deduce that

$$\begin{aligned} a\sqrt{b^2+c^2+bc} \;+\; & b\sqrt{c^2+a^2+ca} \;+\; c\sqrt{a^2+b^2+ab} \\ & \geq \frac{\sqrt{3}}{2}a(b+c)+\frac{\sqrt{3}}{2}b(c+a)+\frac{\sqrt{3}}{2}c(a+b) \\ & \geq \sqrt{3}(ab+bc+ca) \\ & \geq \sqrt{3}. \end{aligned}$$

The equality holds when $a=b=c=\frac{\sqrt{3}}{3}$.

Second solution. Consider four points, O, A, B, and C, in the plane such that $OA=a, OB=b$, and $OC=c$, and $\angle AOB=\angle BOC=\angle COA=120°$.



The area of triangle ABC is

$$[ABC]=\frac{1}{2}(OA\cdot OB+OB\cdot OC+OC\cdot OA)\sin 120°=\frac{\sqrt{3}}{4}(ab+bc+ca)=\frac{\sqrt{3}}{4}.$$

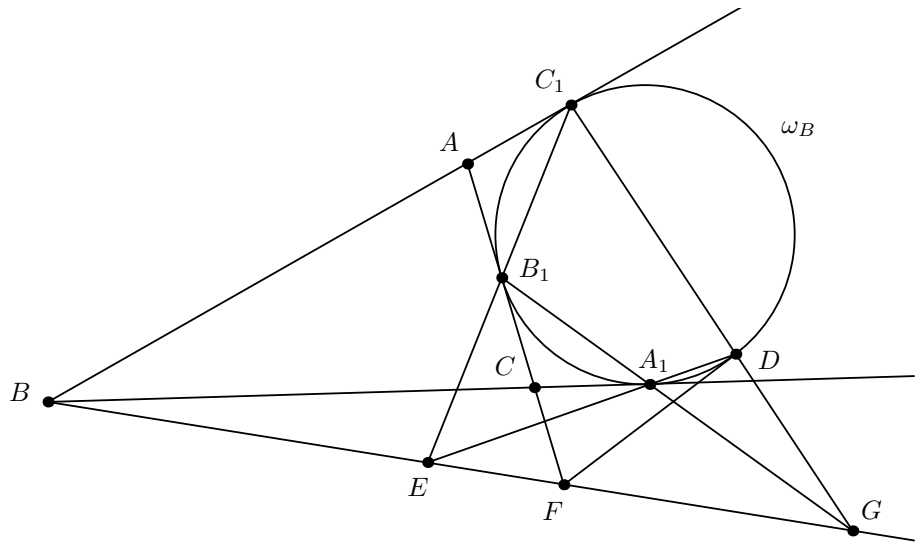
On the other hand, we have

$$a\sqrt{b^2+c^2+bc}=OA\cdot BC\geq 2([AOB]+[COA]).$$

Applying the same inequality for the two other terms, we obtain

$$\begin{aligned}
&a\sqrt{b^2+c^2+bc}+\ b\sqrt{c^2+a^2+ca}\ +\ c\sqrt{a^2+b^2+ab}\\
&\quad\geq 2([AOB]+[COA])+2([BOC]+[AOB])+2([COA]+[BOC])\\
&\quad\geq 4[ABC]\\
&\quad\geq \sqrt{3}.
\end{aligned}$$

- **Problem 7.** Let F' be the intersection point of the tangents to circle ω_B at B_1 and D.



Let G be the intersection point of lines B_1A_1 and DC_1.

Consider the six cyclic points $C_1, C_1, B_1, A_1, A_1, D$. Because the tangent lines to ω_B at C_1 and A_1 intersect at B, lines C_1B_1 and A_1D intersect at E, and lines B_1A_1 and DC_1 intersect at G, from Pascal theorem, the three points B, E, and G are collinear.

Consider the six cyclic points C_1, B_1, B_1, A_1, D, D. Because lines C_1B_1 and A_1D intersect at E, the tangent lines to ω_B at B_1 and D intersect at F', and

lines B_1A_1 and DC_1 intersect at G, from Pascal theorem, the three points E, F', and G are collinear.

We deduce that B, E, and F' are collinear and therefore $F' = F$. This proves that FD is tangent to ω_B at D.

- **Problem 8.** Let B be the number of three-member subsets such that one of the members uses the same language to talk to the two other members and the two other members talk to each other using a different language. Because there are no three-members who use the same language to talk to each other, $A + B$ is the number of all three-member subsets, that is $\binom{101}{3}$. To maximize A, is equivalent to minimize B.

  Let X be a member, and $x_1, x_2, \ldots, x_{50}$ the number of members with whom X talks in the 50 different languages $l_1, l_2, \ldots, l_{50}$, respectively. We have

$$x_1 + x_2 + \cdots + x_{50} = 100.$$

  There are

$$B_X = \binom{x_1}{2} + \binom{x_2}{2} + \cdots + \binom{x_{50}}{2}$$

  three-member subsets each containing X with two other members to whom X talks using the same language. We have, from Cauchy-Schwarz inequality,

$$B_X = \sum_{i=1}^{50} \frac{x_i(x_i - 1)}{2} = \frac{1}{2}\sum_{i=1}^{50} x_i^2 - 50 \geq \frac{\left(\sum_{i=1}^{50} x_i\right)^2}{2 \times 50} - 50 = 50.$$

  Therefore,

$$B \geq 101 \times 50,$$

  and hence,

$$A \leq \binom{101}{3} - 101 \times 50 = 161600.$$

  The equality is satisfied when each member X uses each language to talk exactly with two other members, that is

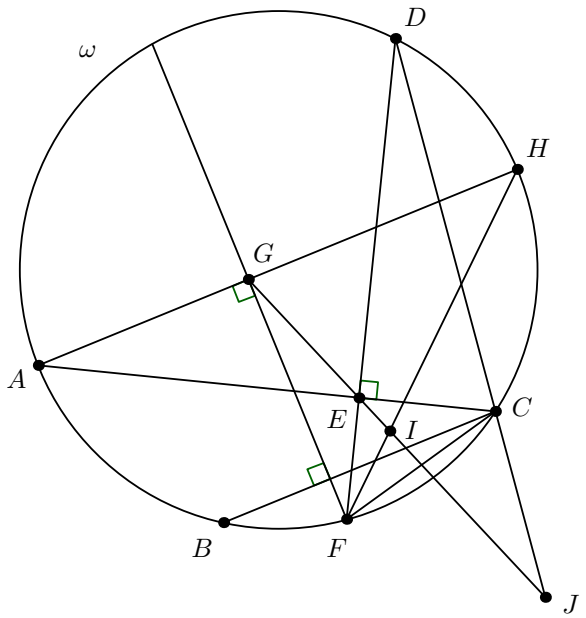
$$x_1 = x_2 = \cdots = x_{50} = 2.$$

  Assume that $X_1, X_2, \ldots, X_{101}$ are the 101 members, and each member X_i, for $i = 1, \ldots, 101$, talks with the members X_{i+j} and X_{i-j} using the language

l_j, for $j = 1, \ldots, 50$ (here, the indices are modulo 101). Because $50 = \frac{101-1}{2}$, each member will use each language to talk exactly with two other members and he will not use two different languages to talk to same members. This proves that the maximum possible value of A is 161600.

7.3 Solutions to the selection test of day III

- **Problem 1.** To prove that C, F, I, and J are concyclic, it is equivalent to prove that $\angle CFI = \angle CJI$ or $\angle CFI + \angle CJI = 180°$, depending on the configuration. We will present here the proof for one configuration. The proof for the other configuration is similar.



Because $AFCH$ is cyclic, we have $\angle CFI = \angle CAH$.

Because $\angle FEA = \angle FGA = 90°$, the quadrilateral $AFEG$ is cyclic, and therefore $\angle CAH = \angle EFG$.

It remains to prove that $\angle EFG = \angle DJG$, which is equivalent to proving that quadrilateral $DGFJ$ is cyclic.

But $\angle EGF = \angle EAF$ since $AFEG$ is cyclic. On the other hand, because $AFCD$ is cyclic, we deduce that $\angle EAF = \angle CDF$. Therefore, $\angle EGF = \angle CDF$, which proves that $DGFJ$ is cyclic.

- **Problem 2.** Let (a, b, c, n) be a quadruplet of positive integers with $a < b < c$ such that each of $a, b, c, a+n, b+n, c+2n$ is a term of the Fibonacci sequence, and let

$$b + n = F_k,$$

for some positive integer k. Because $b < b + n$ and $a + n < b + n$, we have $\max\{b, a + n\} \leq F_{k-1}$.

Assume $\min\{b, a+n\} \leq F_{k-2}$. We have $a+n+b \leq F_{k-1}+F_{k-2} = F_k = b+n$, which is impossible since $a > 0$. Therefore,

$$a + n = b = F_{k-1},$$

$$n = (b + n) - b = F_k - F_{k-1} = F_{k-2},$$

and

$$a = (a + n) - n = F_{k-1} - F_{k-2} = F_{k-3}.$$

Let $c + 2n = F_m$. We have $F_k \leq c \leq F_{m-1}$ and therefore $F_{m-2} \leq 2n = 2F_{k-2} \leq F_k$.

If $F_{m-2} = F_k$ then $F_{k-2} = F_{k-1} = 1$ and $a = F_{k-3} = 0$ which is impossible. Therefore

$$c = b + n = F_{m-1} = F_k,$$

and

$$c + 2n = F_m = F_{k+1}.$$

But

$$2n = (c + 2n) - c = F_{k+1} - F_k = F_{k-1} = a + n.$$

We deduce that

$$F_{k-3} = a = n = F_{k-2} = 1.$$

Hence, $k = 4$, $(a, b, c, n) = (1, 2, 3, 1)$ and we check easily that

$$a = F_1, b = a + n = F_3, c = b + n = F_4 \text{ and } c + 2n = F_5.$$

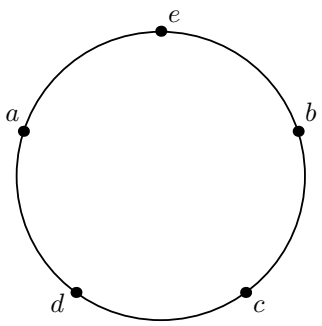
- **Problem 3.** Assume, without loss of generality, that

$$0 \le a \le b \le c \le d \le e.$$

Because

$$ea \le \max\{eb, cd\} \le ec \le ed,$$

to get the smallest possible maximum, the best arrangment arround the circle is



In this case, the maximum is given by

$$\max\{eb, cd\}.$$

We have

$$eb \le e\frac{b+c+d}{3} \le \frac{e(1-e)}{3} \le \frac{1}{12},$$

and the equality holds when $a = 0, b = c = d = \frac{1}{6}$, and $e = \frac{1}{2}$.

We have, on the other hand,

$$cd = \sqrt[3]{c^2d^2(cd)} \le \left(\sqrt[3]{cde}\right)^2 \le \left(\frac{c+d+e}{3}\right)^2 \le \frac{1}{9},$$

and the equality holds when $a = b = 0$, and $c = d = e = \frac{1}{3}$.

Therefore, the minimum possible value of T is $\frac{1}{9}$.

- **Problem 4.** Let n be a nonnegative integer. We have

$$f(2n)^2 < f(2n)^2+6f(n)+1 = f(2n+1)^2 < f(2n)^2+6f(2n)+9 = (f(2n)+3)^2.$$

Therefore,

$$f(2n) < f(2n+1) < f(2n) + 3.$$

Assume that $f(2n+1) = f(2n) + 2$. In this case

$$6f(n) + 1 = f(2n+1)^2 - f(2n)^2 = 4f(2n) + 4.$$

This is impossible since the left hand side is odd while the right hand side is even. Therefore $f(2n+1) = f(2n) + 1$.

On the other hand,

$$6f(n) + 1 = f(2n+1)^2 - f(2n)^2 = 2f(2n) + 1.$$

We deduce that $f(2n) = 3f(n)$, and $f(0) = 0$.

Now, let $n \geq 0$ and write $n = \overline{a_1 a_2 \cdots a_k}_{(2)}$ in basis 2. We prove by induction on k that $f(n) = \overline{a_1 a_2 \cdots a_k}_{(3)}$ in basis 3.

For $k = 1$, we have

$$f(\overline{0}_{(2)}) = f(0) = 0 = \overline{0}_{(3)} \quad \text{and} \quad f(\overline{1}_{(2)}) = f(1) = f(0) + 1 = 1 = \overline{1}_{(3)}.$$

Assume this true for k. We have

$$\begin{aligned} f(\overline{a_1 a_2 \cdots a_k a_{k+1}}_{(2)}) &= f(2\overline{a_1 a_2 \cdots a_k}_{(2)} + a_{k+1}) \\ &= f(2\overline{a_1 a_2 \cdots a_k}_{(2)}) + a_{k+1} \\ &= 3f(\overline{a_1 a_2 \cdots a_k}_{(2)}) + a_{k+1} \\ &= 3\overline{a_1 a_2 \cdots a_k}_{(3)} + a_{k+1} \\ &= \overline{a_1 a_2 \cdots a_k a_{k+1}}_{(3)}. \end{aligned}$$

This completes the induction.

Applying this to 1000, we have

$$1000 = \overline{1101001}_{(3)} = f\left(\overline{1101001}_{(2)}\right) = f(105).$$

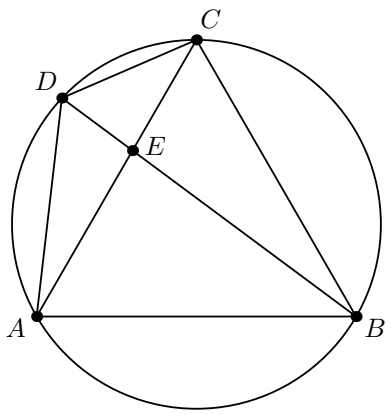
7.4 Solutions to the selection test of day IV

- **Problem 1.** Applying Ptolemy relation to the cyclic quadrilateral $ABCD$, we get

$$AB \cdot CD + BC \cdot DA = AC \cdot BD,$$

which simplifies to

$$CD + DA = 25.$$



Let $a = AB = BC = CA$ and $x = AE$.

We have, from similarity of triangles AED and BEC, that

$$\frac{DA}{a} = \frac{x}{19}.$$

We have, from similarity of triangles CED and BEA, that

$$\frac{CD}{a} = \frac{a-x}{19}.$$

We deduce that

$$\frac{a}{19} = \frac{x}{19} + \frac{a-x}{19} = \frac{DA}{a} + \frac{CD}{a} = \frac{25}{a},$$

and therefore $a = 5\sqrt{19}$.

Applying sine law on the circumcircle of $ABCD$, we obtain

$$\frac{\sin\angle BAD}{25} = \frac{\sin\angle BAC}{5\sqrt{19}} = \frac{\sqrt{57}}{190}.$$

Hence $\sin\angle BAD = \frac{5\sqrt{57}}{38}$, that is

$$\cos\angle BAD = \pm\frac{\sqrt{38^2 - 25\times 57}}{38} = \pm\frac{\sqrt{19}}{38}.$$

Applying cosine law to triangle ABD, we obtain

$$25^2 = 25\times 19 + AD^2 \pm 2\times 5\sqrt{19}\times AD\times\frac{\sqrt{19}}{38},$$

and therefore, $AD = 10$ or 15.

- **Problem 2.** We have $n = c + 7b + 49a = a + 9b + 81c$. This implies that $20(2c - a) = (4a - b)$. Because $0 \le a, b, c \le 6$, either $4a - b = 2c - a = 0$ or $4a - b = 20$ and $2c - a = 1$.

  1. If $4a - b = 2c - a = 0$, then $b = 8c \le 6$ and therefore $a = b = c = 0$, that is $n = 0$.
  2. If $4a - b = 20$ and $2c - a = 1$, then $a = 5$, or 6. If $a = 5$ then $b = 0$ and $c = 3$ and $n = 248$. If $a = 6$, then $2c = 7$, which is impossible.

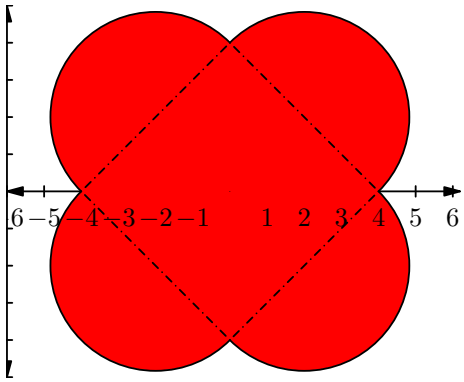
  Therefore, $n = 0$ or 248.

- **Problem 3.** Notice that, if (x, y) is a solution of the inequality, then $(-x, y), (x, -y)$, and $(-x, -y)$ are all solutions of the inequality. Therfore, we can assume $x, y \ge 0$ and deduce the other solutions by symmetries with respect to $x-$axis and $y-$axis.

  Assume $x, y \ge 0$. The inequality is equivalent to

  $$(x-2)^2 + (y-2)^2 \le 8.$$

  The points whose coordinates satisfy this inequality are precisely the points in the intersection of the first quadrant with the disk of center $(2, 2)$ and radius $\sqrt{8}$. By applying the above symmetries we obtain the following surface of points whose coordinates satisfy the inequality.



Its area, is the area of a square of side length $4\sqrt{2}$ and four half disks of radius $2\sqrt{2}$, that is $32 + 16\pi$.

- **Problem 4.** Because $589 = 19 \times 31$, we will find all positive integers $n < 589$ such that both 19 and 31 divide $n^2 + n + 1$.

  Let n be such an integer. We have

  $$0 \equiv n^2 + n + 1 \equiv n^2 + 20n + 1 \equiv (n+10)^2 - 2^2 \equiv (n+8)(n+12) \mod 19.$$

  Hence, 19 divides $n^2 + n + 1$ if and only if $n \equiv 7$ or $11 \mod 19$.

  On the other hand, we have

  $$0 \equiv n^2 + n + 1 \equiv n^2 + 32n + 1 \equiv (n+16)^2 - 10^2 \equiv (n+26)(n+6) \mod 31.$$

  Hence, 31 divides $n^2 + n + 1$ if and only if $n \equiv 5$ or $25 \mod 31$.

  Using the fact that $8 \times 31 \equiv 1 \mod 19$ and $18 \times 19 \equiv 1 \mod 31$, we consider the following four cases:

  1. If $n \equiv 7 \mod 19$ and $n \equiv 5 \mod 31$. There exists an integer k such that

     $$n = 7 \times 8 \times 31 + 5 \times 18 \times 19 + k \times 19 \times 31 = 501 + 589(k+5).$$

     But $0 < n < 589$. We deduce that $n = 501$.

2. If $n \equiv 7 \mod 19$ and $n \equiv 25 \mod 31$. There exists an integer k such that

$$n = 7 \times 8 \times 31 + 25 \times 18 \times 19 + k \times 19 \times 31 = 273 + 589(k + 17).$$

But $0 < n < 589$. We deduce that $n = 273$.

3. If $n \equiv 11 \mod 19$ and $n \equiv 5 \mod 31$. There exists an integer k such that

$$n = 11 \times 8 \times 31 + 5 \times 18 \times 19 + k \times 19 \times 31 = 315 + 589(k + 7).$$

But $0 < n < 589$. We deduce that $n = 315$.

4. If $n \equiv 11 \mod 19$ and $n \equiv 25 \mod 31$. There exists an integer k such that

$$n = 11 \times 8 \times 31 + 25 \times 18 \times 19 + k \times 19 \times 31 = 87 + 589(k + 19).$$

But $0 < n < 589$. We deduce that $n = 87$.

Therefore, the possible values for n are 87,273,315, and 501.

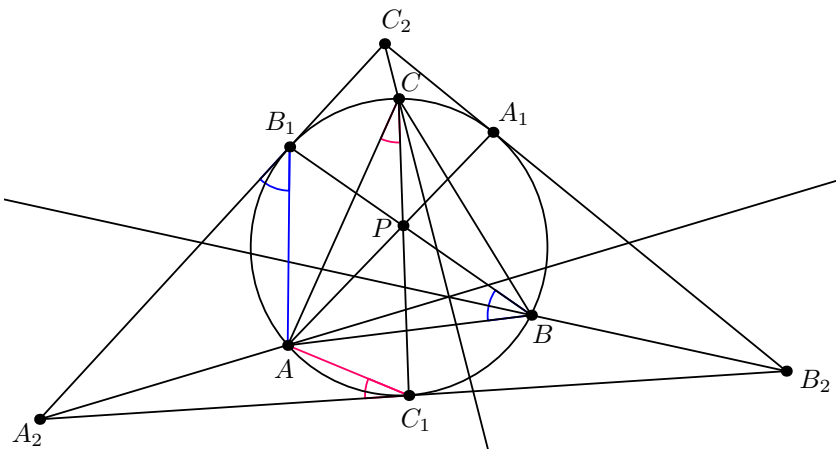
Chapter 8

Solutions to the selection tests for the International Mathematical Olympiad 2013

8.1 Solutions to the selection test of day I

- **Problem 1.** Let R be the circumradius of triangle ABC. We have, using sine law,

$$C_1A = 2R\sin\angle ACP \qquad \text{and} \qquad AB_1 = 2R\sin\angle PBA.$$



On the other hand, because lines A_2C_1 and A_2B_1 are tangent to ω, we have

$$\angle AC_1A_2 = \angle ACP \qquad \text{and} \qquad \angle A_2B_1A = \angle PBA.$$

Applying sine law to triangles AA_2C_1 and AB_1A_2, we obtain

$$\frac{\sin\angle AA_2C_1}{AC_1} = \frac{\sin\angle AC_1A_2}{AA_2} \qquad \text{and} \qquad \frac{\sin\angle AA_2B_1}{AB_1} = \frac{\sin\angle A_2B_1A}{AA_2}.$$

Therefore, we have

$$\frac{\sin\angle C_1A_2A}{\sin\angle AA_2B_1} = \frac{AC_1\cdot\sin\angle AC_1A_2}{AB_1\cdot\sin\angle A_2B_1A} = \frac{\sin^2\angle ACP}{\sin^2\angle PBA}.$$

We have, in a similar way

$$\frac{\sin\angle A_1B_2B}{\sin\angle BB_2C_1} = \frac{\sin^2\angle BAP}{\sin^2\angle PCB} \qquad \text{and} \qquad \frac{\sin\angle B_1C_2C}{\sin\angle CC_2A_1} = \frac{\sin^2\angle CBP}{\sin^2\angle PAC}.$$

But, because AP, BP and CP are concurrent, we have by trigonometric Ceva

$$\frac{\sin\angle BAP}{\sin\angle PAC}\cdot\frac{\sin\angle CBP}{\sin\angle PBA}\cdot\frac{\sin\angle ACP}{\sin\angle PCB} = 1.$$

We deduce that

$$\frac{\sin\angle C_1A_2A}{\sin\angle AA_2B_1}\cdot\frac{\sin\angle A_1B_2B}{\sin\angle BB_2C_1}\cdot\frac{\sin\angle B_1C_2C}{\sin\angle CC_2A_1} = 1.$$

This proves, by using the reciprocal of trigonometric Ceva, that lines AA_2, BB_2 and CC_2 are concurrent.

- **Problem 2.** Since f is strictly increasing, we have $f(n) \geq n$ for all n in S.

  Assume that there exists an integer n in S such that $f(n) > n$. Let n_0 be the smallest such n and write $f(n_0) = n_0 + k_0$, for some positive integer $k_0 \geq 1$. Again, since f is strictly increasing, we have $f(n) \geq n + k_0$, for all integers $n \geq n_0$.

Because $k_0 \geq 1$, we have

$$f(n_0 + k_0) \geq n_0 + 2k_0.$$

On the other hand, we have

$$f(n_0 + k_0) = f(f(n_0)) \leq 2f(n_0) - n_0 = n_0 + 2k_0.$$

We deduce that

$$f(n_0 + k_0) = n_0 + 2k_0.$$

Because $f(n_0) = n_0 + k_0$, $f(n_0 + k_0) = (n_0 + k_0) + k_0$ and f is strictly increasing, we deduce that $f(n) = n + k_0$, for all $n_0 \leq n \leq n_0 + k_0$.

We prove by induction on m that for all integer n such that

$$n_0 + mk_0 \leq n \leq n_0 + (m+1)k_0,$$

we have $f(n) = n + k_0$. Hence

$$f(n) = \begin{cases} n + k_0 & \text{if } n \geq n_0, \\ n & \text{otherwise.} \end{cases}$$

Conversely, if f is such a function and $n \geq n_0$ then

$$n + f(f(n)) = n + f(n + k_0) = 2n + 2k_0 = 2f(n).$$

And if $f(n) = n$, the inequality is clearly satisfied.

Therefore, the solutions to this functional inequality are the functions that can be written as

$$f(n) = \begin{cases} n + k_0 & \text{if } n \geq n_0, \\ n & \text{otherwise,} \end{cases}$$

for some $n_0 \in S$ and some nonnegative integer $k_0 \geq 0$.

- **Problem 3.** Assume that we have two correspondents, R from Riyadh and J from Jeddah, who are not connected to each other. Let $\mathcal{J}_R$ be the set of correspondents from Jeddah connected to R and $\mathcal{R}_J$ the set of correspondents from Riyadh connected to J. Define the function

$$f : \mathcal{J}_R \longrightarrow \mathcal{R}_J$$

in the following way:

For each correspondent $J' \in \mathcal{J}_R$, since the correspondents J and J' are different, they share a unique correspondent in $\mathcal{R}_J$. Let $f(J')$ be this unique correspondent. So f is well-defined.

Assume that there exist two correspondents $J_1, J_2 \in \mathcal{J}_R$ such that $f(J_1) = f(J_2) = R' \in \mathcal{R}_J$. This means that the correspondent R' from Riyadh shares with the correspondent R two correspondents J_1, J_2 from Jeddah. Hence $J_1 = J_2$ and the map f is injective.

Consider a correspondent $R' \in \mathcal{R}_J$ and let J' be the unique correspondent in $\mathcal{J}_R$ that R' shares with R. Then R' is the unique correspondent in $\mathcal{R}_J$ that J' shares with J. Thus $R' = f(J')$ and f is surjective. This proves that R and J have the same number of correspondents from the other city.

We deduce from this that if the correspondent Amr from Riyadh is not connected to the correspondent Zayd from Jeddah, the correspondent Amr has exactly eight correspondents from Jeddah.

Assume now that the correspondent Amr is connected to the correspondent Zayd from Jeddah. Since there are at least ten correspondents in Riyadh and Zayd is connected only with eight, let R_1, R_2 be two correspondents from Riyadh who are not connected to Zayd. Let J_1 be the unique correspondent that Amr and R_1 share. Let J_2 be the unique correspondent that R_1 shares with R_2. Since R_1 is not in contact with Zayd, he has eight correspondents from Jeddah. So there exists at least six correspondents in contact with R_1 who are not in contact neither with Amr nor with R_2. Let J be one of these correspondents.

The correspondent R_2 from Riyadh has exactly eight correspondents from Jeddah since he is not connected to the correspondent Zayd. The correspondent J from Jeddah has exactly eight correspondents from Riyadh since he is not connected to the correspondent R_2 from Riyadh. The correspondent Amr from Riyadh has exactly eight correspondents from Jeddah since he is not connected to the correspondent J from Jeddah.

This proves that in all cases the correspondent Amr from Riyadh has exactly eight correspondents from Jeddah.

- **Problem 4.** Assume that it is possible to place the integers $1, 2, \ldots, 2012$ in a circle in such a way that the 2012 products of adjacent pairs of numbers leave pairwise distinct remainders when divided by 2013. Let $a_1, a_2, \ldots, a_{2012}$ be such a reordering of the integers $1, 2, \ldots, 2012$ on the circle.

  Because $2013 = 3 \times 11 \times 61$, a number is a multiple of 3 or 11 or 61 if and only if its remainder when divided by 2013 is a multiple of 3 or 11 or 61.

  By the pigeonhole principle, there are at least two adjacent numbers in the list a_1, a_2, $\ldots$, a_{2012} which are not multiple of 3. Make this list starting from these two adjacent numbers and consider the list $b_1, b_2, \ldots, b_{2012}$ of their products, where $b_i = a_i \cdot a_{i+1}$, for $i = 1, \ldots, 2012$ with $a_{2013} = a_1$. Consider $a_{i_1}, a_{i_2}, \ldots, a_{i_{670}}$, all the multiples of 3 with $2 < i_1 < i_2 < \cdots < i_{670}$. It is clear that $b_{i_1}, b_{i_2}, \ldots, b_{i_{670}}$ are all multiples of 3 and that $b_{i_1-1}, b_{i_2-1}, \ldots, b_{i_{670}-1}$ are also all multiples of 3. So their reminders when divided by 2013 are all multiples of 3. But there are only 671 different multiples of 3 between 0 and 2012 included. Therefore $i_{j+1} = i_j + 1$ for all $j = 1, \ldots, 669$. This means that the multiples of 3 in the list a_1, a_2, $\ldots$, a_{2012} form a block and are not separated by any non multiple of 3.

  In a similar way we prove that multiples of 11 form a block in this list and that multiples of 61 form also a block in this list. But since there are common multiples of 3 and 11, their blocks must be connected to each other. For the same reason the 3 blocks must be connected by pairs to each others.

  But, in each block, there are numbers which are in none of the two other blocks, for example there are multiples of 3 which are neither multiples of 11 nor multiples of 61 like $3, 6, 9$, and the same thing happens for 11 and 61. So, these numbers are in the middle of each of the blocks and make the blocks intersecting only in their sides. There are also numbers which are not multiples of none of these 3 numbers, like $1, 2, 4$. So, these numbers will prevent two of the three blocks to intersect, and here is the contradiction.

  Hence, it is not possible to place the integers $1, 2, \ldots, 2012$ in a circle under the given condition.

8.2 Solutions to the selection test of day II

- **Problem 1.** We present for this problem two solutions, one using trigonometric functions and the other using classical inequalities.

First solution. The condition $x_1^2 + x_2^2 = y_1^2 + y_2^2 = c^2$, here $c^2 = 2013$, is equivalent to saying that there exist $\alpha, \beta \in \mathbb{R}$ such that

$$x_1 = c\cos\alpha, \quad x_2 = c\sin\alpha, \quad y_1 = c\cos\beta, \quad \text{and} \quad y_2 = c\sin\beta.$$

Therefore

$$\begin{aligned} S &= (1-x_1)(1-y_1) + (1-x_2)(1-y_2) = 2 - (x_1+x_2+y_1+y_2) + x_1y_1 + x_2y_2 \\ &= 2 - c(\cos\alpha + \sin\alpha + \cos\beta + \sin\beta) + c^2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2 - \sqrt{2}c\left(\sin\left(\alpha + \frac{\pi}{4}\right) + \sin\left(\beta + \frac{\pi}{4}\right)\right) + c^2\cos(\alpha-\beta) \\ &= \left(2 - c^2\right) - 2\sqrt{2}c\sin\left(\frac{\alpha+\beta}{2} + \frac{\pi}{4}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2c^2\cos^2\left(\frac{\alpha-\beta}{2}\right) \\ &= \left(2 - c^2\right) - 2\sqrt{2}cst + 2c^2t^2, \end{aligned}$$

where

$$s = \sin\left(\frac{\alpha+\beta}{2} + \frac{\pi}{4}\right) \quad \text{and} \quad t = \cos\left(\frac{\alpha-\beta}{2}\right),$$

are two independent variables, since $\alpha+\beta$ and $\alpha-\beta$ are independent, taking all the real values between -1 and 1 included.

Hence, the maximum of S is equal to $2 + c^2 + 2\sqrt{2}c = 2015 + 2\sqrt{4026}$, and is reached when $s = -t = \pm 1$. That is precisely when $x_1 = x_2 = y_1 = y_2 = -\frac{\sqrt{4026}}{2}$.

For the minimum, notice that

$$S = \left(2 - c^2\right) - s^2 + (\sqrt{2}ct - s)^2$$

Therefore, the minimum of S is equal to $1 - c^2 = -2012$, and is reached when $s = \pm 1$ and $t = \frac{\sqrt{2}s}{2c} = \frac{\pm\sqrt{4026}}{4026}$. That is precisely when

$$x_1 = y_2 = \frac{1+\sqrt{4025}}{2} \quad \text{and} \quad x_2 = y_1 = \frac{1-\sqrt{4025}}{2},$$

or vice-versa.

Second solution. For the maximum, we have by Cauchy-Schwartz inequality

$$\begin{aligned} S &= 2 - (x_1 + x_2 + y_1 + y_2) + x_1y_1 + x_2y_2 \\ &\leq 2 + \sqrt{4 \cdot (x_1^2 + x_2^2 + y_1^2 + y_2^2)} + \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} \\ &\leq 2015 + 2\sqrt{4026}, \end{aligned}$$

and the equality holds when $x_1 = x_2 = y_1 = y_2 = -\frac{\sqrt{4026}}{2}$.

For the minimum, we have

$$\begin{aligned} S &= 2 - (x_1 + y_1) - (x_2 + y_2) + \frac{(x_1 + y_1)^2}{2} + \frac{(x_2 + y_2)^2}{2} - c^2 \\ &= (1 - c^2) + \frac{(x_1 + y_1 - 1)^2}{2} + \frac{(x_2 + y_2 - 1)^2}{2} \\ &\geq 1 - c^2 = -2012, \end{aligned}$$

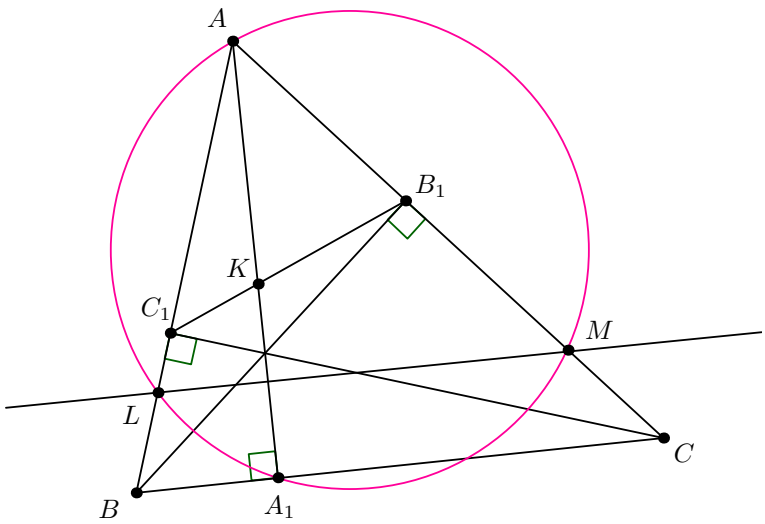
where $c^2 = 2013$.

Thus the minimum of S is -2012 and is reached when $y_1 = 1 - x_1$, $y_2 = 1 - x_2$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 2013$. This is equivalent to

$$x_1 = y_2 = \frac{1 + \sqrt{4025}}{2} \quad \text{and} \quad x_2 = y_1 = \frac{1 - \sqrt{4025}}{2},$$

or vice-versa.

- **Problem 2.** Because LM is parallel to BC, the problem is equivalent to proving that $\angle AA_1L = \angle AML = \angle ACB$. We present two solutions:



First solution. Let L_1 be the point on AB such that $\angle AA_1L_1 = \angle ACB$ and let us prove that $L_1 = L$.

Quadrilateral BCB_1C_1 is cyclic since $\angle BC_1C = \angle BB_1C = 90°$. Therefore, $\angle B_1C_1A = \angle ACB = \angle AA_1L_1$. We deduce that the quadrilateral $A_1KC_1L_1$ is cyclic. Hence, $\angle L_1KA_1 = \angle L_1C_1A_1$.

Quadrilateral AC_1A_1C is cyclic since $\angle CA_1A = \angle CC_1A = 90°$. We deduce that $\angle L_1C_1A_1 = \angle ACB$. Thus, $\angle L_1KA_1 = \angle AA_1L_1$. This proves that $L_1A_1 = L_1K$, that is $L_1 = L$, the intersection point of the perpendicular bisector of KA_1 with AB.

Second solution. Since $LK = LA_1$, the problem is equivalent to proving that $\angle LKA_1 = \angle ACB$. But $\angle BHA_1 = 90° - \angle A_1BH = \angle ACB$, where H is the orthocenter of triangle ABC. Therefore, the problem is equivalent to proving that LK and BH are parallel.

We know that $\frac{AL}{LB} = \frac{AN}{NA_1}$, where N is the midpoint of KA_1, since LM and BC are parallel. It remains to prove that $\frac{AK}{KH} = \frac{AN}{NA_1}$.

To simplify the notations let us write $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$.

We know that $\angle KC_1A = \gamma$, $\angle HC_1K = 90^\circ - \gamma$, $\angle C_1AK = 90^\circ - \beta$ and $\angle KHC_1 = \beta$. We deduce by applying sine laws on triangles AC_1K and C_1HK that

$$\frac{AK}{KH} = \frac{\sin\angle KC_1A \cdot \sin\angle KHC_1}{\sin\angle HC_1K \cdot \sin\angle C_1AK} = \tan\beta \cdot \tan\gamma.$$

On the other hand, we have $\frac{AN}{NA_1} = \frac{AK}{NA_1} + 1 = 2\frac{AK}{KA_1} + 1$. We also know that $\angle A_1C_1K = 180^\circ - 2\gamma$ and $\angle KA_1C_1 = 90^\circ - \alpha$. We deduce by applying sine laws on triangles AC_1K and C_1A_1K that

$$\frac{AK}{KA_1} = \frac{\sin\angle KC_1A \cdot \sin\angle KA_1C_1}{\sin\angle A_1C_1K \cdot \sin\angle C_1AK} = \frac{\sin\gamma \cdot \cos\alpha}{\sin 2\gamma \cdot \cos\beta} = \frac{\cos\alpha}{2\cos\beta \cdot \cos\gamma}.$$

We deduce that

$$\frac{AN}{NA_1} = \frac{\cos\alpha}{\cos\beta \cdot \cos\gamma} + 1 = \frac{\cos\beta \cdot \cos\gamma - \cos(\beta+\gamma)}{\cos\beta \cdot \cos\gamma} = \tan\beta \cdot \tan\gamma = \frac{AK}{KH}.$$

- **Problem 3.** Let n be a positive integer. Consider D_1, the set of all divisors of n with unit digit 3, and D_2 the set of all the other divisors of n.

If the unit digit of n is different from 9, consider the map $\delta : D_1 \longrightarrow D_2$ defined by $\delta(d) = \frac{n}{d}$ for all $d \in D_1$. If $d \equiv 3 \mod 10$ and $\delta(d) \equiv 3 \mod 10$ for some $d \in D_1$, then $n = d \cdot \delta(d) \equiv 9 \mod 10$ which is a contradiction. Then the map δ is well defined. Clearly, the map δ is an injective map. Therefore the cardinality of D_1 is less than or equal to the cardinality of D_2. Hence $p \leq 50$.

If the unit digit of n is equal to 9, the integer n is neither divisible by 2 nor by 5. This means that all prime divisors of n have unit digit equal to 1, 3, 5, or 9.

If all prime divisors of n have unit digits equal to 1 or 9, all divisors of n have unit digits equal to 1 or 9. Therefore $p = 0$.

If the integer n has a prime divisor p with unit digit 3 or 7, consider the map $\delta : D_1 \longrightarrow D_2$ defined for $d \in D_1$ by $\delta(d) = \frac{d}{p}$ if p divides d and by $\delta(d) = pd$ otherwise. We can see easily in both cases, that the unit digit of $\delta(d)$ is equal to 1 or 9. Hence δ is well defined.

Assume that there exist $d_1, d_2 \in D_1$ such that $\delta(d_1) = \delta(d_2)$. It is clear that if both d_1, d_2 are divisible by p or both d_1, d_2 are not divisible by p, we have $d_1 = d_2$. Assume that d_1 is divisble by p and d_2 is not divisible by p. We have $\frac{d_1}{p} = d_2 p$, which is equivalent to $d_1 = p^2 d_2$. Because $d_1 \equiv d_2 \equiv 3 \mod 10$, we deduce that $p^2 \equiv 1 \mod 10$, which contradicts the fact that $p^2 \equiv -1 \mod 10$ since $p \equiv 3, 7 \mod 10$. Hence δ is injective and therefore the cardinality of D_1 is less than or equal to the cardinality of D_2. Thus $p \leq 50$.

Notice that for $n = 3$, its divisors are $1, 3$ and $p = 50$. This proves that the maximum possible value of p is 50.

- **Problem 4.** We will construct such a sequence by induction:

Define $a_1 = 1$ and $a_2 = 2$. In this case we have $b_1 = |a_1 - a_2| = 1$.

Assume that $a_1, a_2, \ldots, a_{2n}$ are defined such that there is no positive integer which occurs at least twice neither in the finite sequence $a_1, a_2, \ldots, a_{2n}$ nor in the finite sequence $b_1 = |a_1 - a_2|, b_2 = |a_2 - a_3|, \ldots, b_{2n-1} = |a_{2n-1} - a_{2n}|$.

Let M_n be the maximum element of the set of integers $\{a_1, a_2, \ldots, a_{2n}\}$, c_n the maximum of the integers k such that $\{1, 2, \ldots, k\} \subseteq \{a_1, a_2, \ldots, a_{2n}\}$, and d_n the maximum of the integers k such that $\{1, 2, \ldots, k\} \subseteq \{b_1, b_2, \ldots, b_{2n-1}\}$. Notice that for all $1 \leq i \leq 2n - 1$, we have

$$b_i = |a_i - a_{i+1}| \leq \max_{1 \leq j \leq 2n} a_j - \min_{1 \leq j \leq 2n} a_j = M_n - 1.$$

If $c_n < d_n$ define $a_{2n+1} = 2M_n + c_n + 1$ and $a_{2n+2} = c_n + 1$. If $c_n \geq d_n$ define $a_{2n+1} = 2M_n + 1$ and $a_{2n+2} = 2M_n + d_n + 2$.

It is clear that none of the two possible values for a_{2n+1} and for a_{2n+2} occur in the finite sequence $a_1, a_2, \ldots, a_{2n}$.

In the first case, we have

$$b_{2n} = 2M_n + c_n + 1 - a_{2n} \geq M_n + c_n + 1 > M_n - 1 \geq b_i,$$

and

$$b_{2n+1} = 2M_n > M_n - 1 \geq b_i,$$

for all $1 \leq i \leq 2n-1$, and $b_{2n+1} \neq b_{2n}$ since $a_{2n} \neq c_n + 1$.

In the second case, we have

$$b_{2n} = 2M_n + 1 - a_{2n} \geq M_n + 1 > M_n - 1 \geq b_i,$$

and

$$b_{2n+1} = d_n + 1 \neq b_i,$$

for all $1 \leq i \leq 2n-1$, and $b_{2n+1} = d_n + 1 \leq M_n < M_n + 1 \leq b_{2n}$.

Therefore, there is no positive integer which occurs at least twice in the finite sequence $a_1, a_2, \ldots, a_{2n+2}$ or in the finite sequence $b_1, b_2, \ldots, b_{2n+1}$.

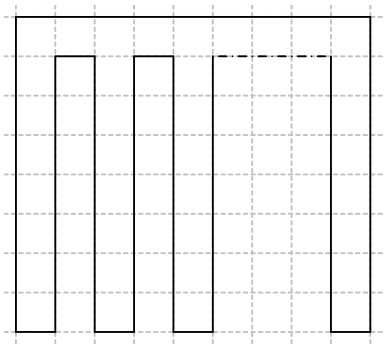
This proves that there is no positive integer which occurs at least twice in the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ or in the infinite sequence $b_1, b_2, \ldots, b_n, \ldots$.

Notice moreover, that in the first case $c_{n+1} \geq c_n + 1$ and in the second case $d_{n+1} \geq d_n + 1$. Therefore, if $e_n = \min\{c_n, d_n\}$ then $e_{n+2} \geq e_n + 1$. But $c_1 = 2$ and $d_1 = 1$. We deduce that $e_{2n} \geq n$ for all integer n. Therefore, any positive integer n occurs in both finite sequences $a_1, a_2, \ldots, a_{4n}$ and $b_1, b_2, \ldots, b_{4n-1}$.

This proves that any positive integer n occurs exactly once in each infinite sequence $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$.

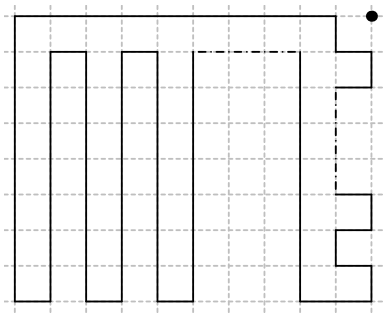
8.3 Solutions to the selection test of day III

- **Problem 1.** If m is even, Adel can draw the following closed path which passes through all the dots of his grid. Therefore, the maximum possible value of k is mn.



If n is even, Adel can draw a similar closed path obtained by symmetry with respect to the first diagonal. Therefore, the maximum possible value of k is mn.

If mn is odd, Adel can draw the following closed path which passes through $mn - 1$ dots of his grid:



It remains to prove that this is the maximal possible value of k. For this, assign black color to all dots of the grid of coordinates (a, b) with $a + b$ odd and white color to all dots with $a + b$ even. Any consecutive dots in a path $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ that Adel can draw have different colors. Therefore, the colors of the first and last dots in Adel's path describe the parity of the length of the path. Since the path of Adel is closed, it will start and end with the same color and therefore, its length is even. Therefore k is even. This proves that the maximum possible value of k is $mn - 1$.

- **Problem 2.** We present for this problem two solutions.

First solution. Notice that we have

$$\begin{aligned}(a_1-n)^2+\cdots+(a_n-n)^2 &= (a_1^2+\cdots+a_n^2)-2n(a_1+\cdots+a_n)+n^3\\ &\le n^3+1-2n^3+n^3=1.\end{aligned}$$

Therefore, there are two cases:

The first case is when $a_1=a_2=\cdots=a_n=n$. This is a solution since it satisfies both inequalities.

The second case is when there exists $1\le i_0\le n$ such that $|a_{i_0}-n|=1$ and $a_i=n$ for all $1\le i\le n$ with $i\ne i_0$.

If $a_{i_0}=n-1$, the first inequality becomes $n^2-1=a_1+\cdots+a_n\ge n^2$ which is impossible.

If $a_{i_0}=n+1$, the second inequality becomes $n^3+2n+1=a_1^2+\cdots+a_n^2\le n^3+1$ which is also impossible.

Hence, the only solution to both inequalities is given by $a_1=a_2=\cdots=a_n=n$.

Second solution. We have by Cauchy-Schwartz inequality

$$(n^2+1)^2>n^4+n\ge n(a_1^2+\cdots+a_n^2)\ge(a_1+\cdots+a_n)^2\ge n^4.$$

We deduce that

$$a_1+\cdots+a_n=n^2,$$

and

$$n^3+1\ge a_1^2+\cdots+a_n^2\ge n^3,$$

and the right hand side equality occurs if and only if $a_1=\cdots=a_n=n$.

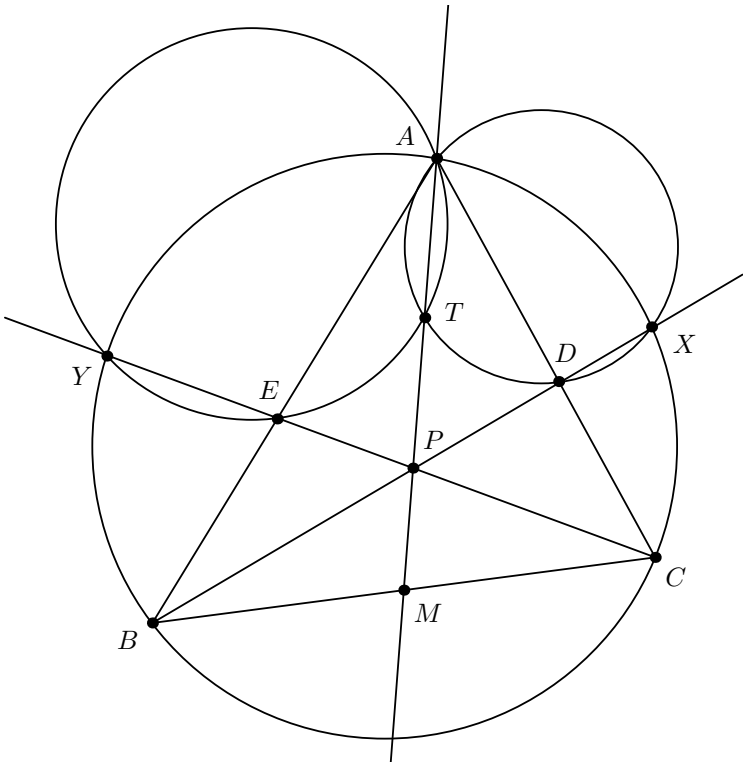
Now, assume, looking for a contradiction, that $a_1^2+\cdots+a_n^2=n^3+1$. Since a number and all his powers have the same parity, we deduce that

$$n\equiv n^2\equiv a_1+\cdots+a_n\equiv a_1^2+\cdots+a_n^2\equiv n^3+1\equiv n+1 \mod 2,$$

which is impossible.

Hence, the only solution to both inequalities is given by $a_1=a_2=\cdots=a_n=n$.

- **Problem 3.** By Applying Ceva to the concurrent cevians AM, BD and CE, we obtain
$$\frac{AE}{EB} = \frac{AD}{DC} \cdot \frac{CM}{MB} = \frac{AD}{DC}.$$
We deduce from Thales theorem that segments ED and BC are parallel.



Since quadrilateral $BCXY$ is cyclic, we have $\angle BXY = \angle BCY = \angle CED$. We deduce that quadrilateral $EDXY$ is cyclic and therefore
$$PD \cdot PX = PE \cdot PY.$$
Hence, point P lies on the radical axis of circumcircles of triangles AXD and AYE which passes trough A.

If these two circles are tangent to AP at A then
$$\angle MAC = \angle AXB = \angle ACB, \quad \text{and} \quad \angle BAM = \angle CYA = \angle CBA,$$

which implies that triangle ABC is a right triangle at A and this is a contradiction.

We conclude that the two circles intersect in a second point $T \neq A$ on line AM.

- **Problem 4.** Suppose there is some value of n such that $p(n) \neq \pm 1$. Let q be a prime divisor of $p(n)$. Because $q = (n+q) - q$ divides $p(n+q) - p(n)$, we deduce that q divides $p(n+q)$. Therefore, q divides both $2^n - 1$ and $2^{n+q} - 1$, which implies in particular that q is an odd prime. We have

$$1 \equiv 2^{n+q} \equiv 2^n \cdot 2^q \equiv 2^q \equiv 2 \mod q,$$

by Fermat's little theorem. This is a contradiction.
Hence, for all values of n, $p(n) = \pm 1$. Since p is a polynomial and takes infinitely many times the same value, either 1 or -1, it is constant. This proves that the only polynomials with integer coefficients which have the required property are $p(X) = 1$ and $p(X) = -1$.

SAMC 2014

SAUDI ARABIA MATHEMATICAL COMPETITIONS

مسابقات الرّياضيات للمملكة العربيّة السّعوديّة

Malik Talbi

Riyadh, June 2014

مقدّمة

يحوي هذا الكتيّب على مسائل التّدريب والاختيار للفريق السّعودي للاولمبياد الدّولي للرّياضيات والّذي يعقد في شهر يوليو ٢٠١٤ في جنوب افريقيا. لقد كان التّدريب مدعوما من وزارة التّربية والتّعليم في المملكة العربيّة السّعوديّة والّتي كلّفت مؤسّسة الملك عبدالعزيز ورجاله للموهبة والابداع " موهبة " للقيام بالمهمّة. تعتبر مؤسّسة " موهبة " مؤسّسة مستقلّة يرأسها الملك عبدالّله بن عبدالعزيز رعاه الّله وينوب عنه سمو وزير التّربية والتّعليم. وتجدر الاشارة إلى دعم جامعة الملك سعود الّتي أتاحت الفرصة لبعض الاساتذة للقيام بتدريب الفريق. كما لا بدّ أن نذكر بالشّكر والعرفان لجامعة الملك عبدالله للعلوم والتّقنية ما قامت به من استضافة كريمة لتدريب الطّلبة في المرحلة الاخيرة، وهي مرحلة التّفريغ الّتي تسبق اولمبياد الدّولي.

تمّ تنظيم أربع ملتقيات تدريبيّة للفريق السّعودي خلال العام الدّراسي ٢٠١٣ - ٢٠١٤، بالإضافة إلى فترة التّفريغ الكلّي للتّدريب الّتي بدأت في الاسبوع الثّاني من شهر أبريل. كما شارك الفريق في اولمبياد الرّياضيات لدول آسيا والباسيفيك $APMO$ في مارس ٢٠١٤، واولمبياد دول الخليج GMO الّذي عقد في سلطنة عمان في الاسموع الأوّل من أبريل ٢٠١٤. كما شارك في أولمبياد البلقان في بداية شهر مايو ٢٠١٤.

يسعدني أن أهدي لكم مسائل هذا الكتيّب، راجيا أن تكون بداية تعاون بيننا في مجال الاولمبياد وغيره من حقول المنافسة العلميّة.

د. فوزي بن أحمد الذكير
رئيس الفريق السّعودي للاولمبياد الدّولي.

Introduction

This booklet contains problems used in the training and selection of the Saudi team for the International Mathematics Olympiad. The training was supported by The Ministry of Education, which commissioned *Mawhiba*, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. *Mawhiba* is an independent establishment, presided by the King of Saudi Arabia, with the Minister of Education as Vice President. We thank King Saud University for giving the opportunity to trainers to contribute in the training of the Saudi Team. We certainly thank King abdullah University for Science and Technology KAUST, for allowing us to use their campus facilities to train our students during the final stage of our program.

The Saudi team had four main training camps during the academic year 2013-2014 beside the full-time training period that started in the second week of April, 2014. The team participated in the Asian Pacific Olympiad in March 2014, the Gulf Mathematical Olympiad, which was held in Oman in April, and in the Balkan Olympiad BMO in Bulgaria in the first week of May.

It is our pleasure to share these training and selection problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Fawzi A. Al-Thukair
Leader of the Saudi Math Olympiad Team

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Part I

Problems in Arabic

اختبارات ترشيح الطّلاب للتّدريب المكثّف 2014

مسائل اليوم الأوّل ـ 28 ذوالقعدة 1434 هـ

1 ـ لتكن $a_1, a_2, \ldots, a_{2n}$ أعدادا حقيقيّة موجبة تحقّق $a_i + a_{n+i} = 1$ ، لكلّ $i = 1, \cdots, n$. أثبت وجود عددين صحيحين مختلفين $1 \leq j, k \leq 2n$ يحقّقان

$$\sqrt{a_j^2 - a_k^2} < \frac{1}{\sqrt{n} + \sqrt{n-1}}.$$

2 ـ لتكن D نقطة المنتصف للضّلع BC للمثلّث ABC و E نقطة المنتصف للمتوسّط AD . المستقيم BE يقطع الضّلع CA عند F . أثبت أنّ مساحة الرّباعي $CDEF$ تساوي $\frac{5}{12}$ مساحة المثلّث ABC .

3 ـ جد كلّ الأعداد الصّحيحة الموجبة n الّتي تجعل المقدار $1 - 5^n + 5^{2n+1}$ مربّعا كاملا.

4 ـ يريد ماجد أن يلوّن خانات طاولة $n \times n$ شطرنج باللّونين الأبيض والأسود بحيث يشتمل كلّ مربّع من القياس 2×2 على خانتين ملوّنتين بالأبيض وخانتين ملوّنتين بالأسود. حدّد عدد الطرق الّتي يمكن لماجد أن يلوّن بها طاولة الشطرنج.

مسائل اليوم الثّاني ـ 1 ذوالحجّة 1434 هـ

1– أثبت أنّ 2014 يقسم $53n^{55} - 57n^{53} + 4n$ ، لكلّ عدد صحيح n .

2– لتكن a_1, a_2, a_3, a_4, a_5 أعدادا حقيقيّة لا تساوي صفرا. أثبت أنّ لكثيرة الحدود

$$P(X) = \prod_{k=0}^{4} \left(a_{k+1}X^4 + a_{k+2}X^3 + a_{k+3}X^2 + a_{k+4}X + a_{k+5}\right),$$

حيث $a_{5+i} = a_i$ لمّا $i = 1, 2, 3, 4$ ، جذرا ذا جزء حقيقيّ سالب.

3– تمّ ترتيب الـ 2013 عددا

$$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \ldots, \frac{1}{2013 \times 2014}$$

على دائرة بطريقة عشوائيّة.

ا. أثبت أنّ هناك عشرة أعداد متتالية على الدائرة حاصل جمعها أقل من $\frac{1}{4000}$.

ب. أثبت أنّ هناك عشرة أعداد متتالية على الدائرة حاصل جمعها أقل من $\frac{1}{10000}$.

4– ليكن ABC مثلّثا حادّا يحقّق $\angle A < \angle B \leq \angle C$ ، و O مركز دائرته المحيطة برؤوسه. يقطع المنصّف العموديّ على الضّلع AB الضّلع AC عند D . يقطع المنصّف العموديّ على الضّلع AC الضّلع AB عند E . اعط قيمة زوايا المثلّث DEO بدلالة زوايا المثلّث ABC .

مسائل اليوم الثّالث ـ 4 ذوالحجّة 1434 هـ

1– شارك 14 طالبا في اختبار 3 ساعات يشمل 15 مسألة قصيرة. كلّ طالب حلّ عددا مختلفا من المسائل، وكلّ مسألة حلّها عدد مختلف من الطّلاب. أثبت أنّه يوجد طالب حلّ 5 مسائل فقط.

2– ليكن x و y عددين حقيقين موجبين. جد أصغر قيمة للمقدار

$$x^2 + xy + \frac{y^2}{2} + \frac{2^6}{x+y} + \frac{3^4}{x^3}.$$

3– ليكن ABC مثلّثا و I مركزه الدّاخلي. يقطع المستقيم AI الضّلع BC عند D والمنصّف العمودي على BC عند E . ليكن J المركز الدّاخلي للمثلّث CDE . أثبت أنّ المثلّث CIJ متطابق الضّلعين.

4– أثبت أنّه يوجد عدد صحيح موجب n يجعل آخر خانات n^3 تنتهي بـ $201320132013\square\cdots$.

مسائل اليوم الرّابع ـ 6 ذوالحجّة 1434 هـ

1– ليكن p عددا أوّليّا و $n \geq 2$ عددا صحيحا، بحيث $p \mid (n^6 - 1)$. برهن أنّ $n > \sqrt{p} - 1$.

2– ليكن $x \geq 0$ عددا حقيقيّا. برهن أنّ

$$\frac{(x^2+1)^6}{2^7} + \frac{1}{2} \geq x^5 - x^3 + x.$$

3– تلعب فاطمة وأسماء اللّعبة التّالية. تبدأ فاطمة باختيار 2013 من الأعداد المختلفة هي a_1 ، a_2 ، ... ، a_{2013} . تحاول أسماء أن تعرف قيمة كلّ واحد من الأعداد a_1 ، a_2 ، ... ، a_{2013} . في كلّ مرّة، تختار أسماء $1 \leq i < j \leq 2013$ وتسأل فاطمة " ما هي المجموعة $\{a_i, a_j\}$ ؟ " (فمثلا، إذا سألت أسماء عن المجموعة $\{a_1, a_2\}$ ، وكان $a_1 = 17$ و $a_2 = 13$ ، أجابتها فاطمة بـ $\{13, 17\}$). ما هو أقل عدد من الأسئلة الّتي على أسماء أن تسألها كي تعرف قيم جميع الأعداد a_1 ، a_2 ، ... ، a_{2013} .

4– ليكن $\triangle ABC$ مثلّثا حاد الزّوايا فيه $\widehat{A} > \widehat{B} \geq \widehat{C}$. النقط D ، E ، F هي نقط تماس الدّائرة الدّاخليّة للمثلّث مع الأضلاع BC ، CA ، AB ، على التّرتيب. لتكن J نقطة على (BD) ، K نقطة على (DC) ، L نقطة على (EC) ، M نقطة على (FB) ، بحيث

$$AF = FM = JD = DK = LE = EA.$$

لتكن النقطة P نقطة تقاطع AJ مع KM والنّقطة Q نقطة تقاطع AK مع JL . أثبت أنّ $PJKQ$ رباعيّ دائريّ.

اختبارات ترشيح الطّلاب لأولمبياد الرّياضيّات للدّول الأعضاء في مكتب التّربية العربي لدول الخليج 2014

مسائل اليوم الأوّل ـ 21 ربيع الأوّل 1435 هـ

الوقت المتاح: 3 ساعات

1– ليكن ABC مثلّثا يحقّق $\angle A < \angle B \leq \angle C$ ، و M و N نقطتي منتصف الضّلعين CA و AB ، على التّرتيب، و P و Q مسقطي B و C على المتوسّطين CN و BM ، على التّرتيب. أثبت أنّ الرّباعيّ $MNPQ$ دائريّ.

2– ليكن $p \geq 2$ عددا أوّليّا و $\frac{a_p}{b_p} = 1 + \frac{1}{2} + \cdots + \frac{1}{p^2-1}$ ، حيث a_p و b_p عددان صحيحان موجبان أوّليّان نسبيّا. احسب $\gcd(p, b_p)$.

3– قسّم تركي مربّعا إلى عدد منته من المستطيلات البيضاء والخضراء، بحيث تكون أضلاع كلّ مستطيل موازية لأضلاع المربّع. كتب تركي داخل كلّ مستطيل أبيض حاصل قسمة طول ضلعه الأفقيّ على طول ضلعه العموديّ، وكتب داخل كلّ مستطيل أخضر حاصل قسمة طول ضلعه العموديّ على طول ضلعه الأفقيّ. في الأخير قام تركي بحساب S ، حاصل جمع هذه الأعداد داخل المستطيلات. إذا كان مجموع مساحات المستطيلات البيضاء يساوي مجموع مساحات المستطيلات الخضراء، فحسب أقلّ قيمة ممكنة لـ S .

4– ليكن ABC مثلّثا، و D منتصف الضّلع BC ، و E نقطة تقاطع منصّف الزّاوية $\angle BAC$ مع الضّلع BC. يقطع المنصّف العموديّ على AE منصّفي الزّاويتين $\angle CBA$ و $\angle CDA$ عند M و N ، على التّرتيب. يتقاطع منصّفا الزّاويتين $\angle CBA$ و $\angle CDA$ عند P . أثبت أنّ النّقط A ، M ، N ، P على دائرة واحدة.

مسائل اليوم الثّاني ـ 24 ربيع الأوّل 1435 هـ

الوقت المتاح: 3 ساعات

1– لتكن A, B, C ثلاثة نقط على استقامة واحدة بهذا التّرتيب، و $\mathcal{C}$ دائرة تمرّ بـ B و C ، و l خطّ مستقيم مختلف عن BC ويمرّ بـ A ويقطع $\mathcal{C}$ عند M و N . منصّفا الزّاويتين $\angle CMB$ و $\angle CNB$ يقطعان BC عند P و Q . أثبت أنّ $AP \cdot AQ = AB \cdot AC$.

2– ليكن p عددا أوّليّا. أثبت أنّه يوجد عدد غير منته من الأعداد الصّحيحة الموجبة n تجعل p يقسم المقدار

$$1^n + 2^n + \cdots + (p+1)^n.$$

3– ليكن $ABCDE$ خماسيّا دائريّا بحيث إذا قطع القطران AC و AD القطر BE عند P و Q ، على التّرتيب، فإنّ $BP \cdot QE = PQ^2$. أثبت أنّ $BC \cdot DE = CD \cdot PQ$.

4– لتكن $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ أعدادا حقيقيّة. أثبت أنّ

$$a_1a_2(a_1 - a_2) + a_2a_3(a_2 - a_3) + \cdots + a_{n-1}a_n(a_{n-1} - a_n) \geq a_1a_n(a_1 - a_n).$$

مسائل اليوم الثّالث ـ 27 ربيع الأوّل 1435 هـ

الوقت المتاح: 3 ساعات

1– جد كلّ الثّلاثيّات المرتّبة (a,b,c) للأعداد الصّحيحة الموجبة الّتي تحقّق

$$5^a + 3^b - 2^c = 32.$$

2– لتكن المجموعة $S = \{(a,b) \mid a,b = 1,2,3,4,\ and\ a \neq b\}$ ، واعتبر كلّ كثيرات الحدود $p(X,Y)$ غير المطابقة لصفر، ذوات المعاملات الصّحيحة، الّتي تحقّق $p(a,b) = 0$ لكلّ عنصر (a,b) في S .

ا. ما هي أقل درجة ممكنة لكثيرات الحدود $p(X,Y)$ هذه؟

ب. حدّد كلّ كثيرات الحدود $p(X,Y)$ هذه الّتي لها أقل درجة ممكنة.

3– ليكن ABC مثلّثا، و I مركزه الدّاخليّ، و ω دائرة مركزها I . تقع النّقط A', B', C' على ω بحيث تجعل الأشعّة IA', IB', IC' الّتي تنطلق من I ، تقطع الأضلاع BC, CA, AB عموديّا، على التّرتيب. أثبت أنّ المستقيمات AA', BB', CC' تتقاطع في نقطة واحدة.

4– لتكن X مجموعة أعداد نسبيّة تحقّق الشّرطين التّاليين:

ا. المجموعة X تشتمل على الأقل على عنصرين،

ب. لكلّ x, y في X ، إذا كان $x \neq y$ فإنّه يوجد z في X يجعل $\left|\frac{x-z}{y-z}\right| = 2$ أو $\left|\frac{y-z}{x-z}\right| = 2$.

أثبت أنّ المجموعة X تشتمل على عدد غير منته من العناصر.

اختبارات ترشيح الطّلاب
لأولمبياد الرّياضيّات لدول البلقان 2014

مسائل اليوم الأوّل ـ 24 جمادى الأولى 1435 هـ

الوقت المتاح: 4 ساعات

1– نسمّي قاسم فعلي موجب للعدد n كلّ قاسم موجب للعدد n مختلف عن n . لكلّ عدد صحيح موجب $n \geq 2$ ، نعرّف $f(n)$ على أنّه العدد الّذي يساوي واحدا زائدا أكبر قاسم فعليّ للعدد n. حدّد كلّ الأعد الصّحيحة الموجبة n الّتي تحقّق $f(f(n)) = 2$.

2– أثبت أنّه من بين أي 16 مكعّبات كاملة، يوجد مكعّبان يقبل حاصل طرحهما القسمة على 91.

3– ليكن $n \geq 2$ عددا صحيحا موجبا، ولنكتب

$$\frac{1}{n} = 0.a_1a_2\ldots.$$

إذا كان $n = a_1 + a_2 + \cdots$ حاصل جمع الخانات، فحدّد كلّ القيم الممكنة للعدد n.

4– ليكن ABC مثلّثا يحقّق $\angle B \leq \angle C$ ، و I مركز دائرته الدّاخليّة، و D نقطة تقاطع المستقيم AI مع الضّلع BC. لتكن M و N نقطتين على الضّلعين BA و CA ، على التّرتيب، تحقّقان $BM = BD$ و $CN = CD$. الدّائرة المحيطة بالمثلّث CMN تقطع المستقيم BC مرّة ثانية عند P. أثبت أنّ الرّباعيّ $DIMP$ دائريّ.

5– جد كلّ الأعداد الصّحيحة الموجبة n الّتي تحقّق

$$3^n + 4^n + \cdots + (n+2)^n = (n+3)^n.$$

مسائل اليوم الثّاني ـ 29 جمادى الأولى 1435 هـ

الوقت المتاح: 4 ساعات

1– جد القيمة الصّغرى للمقدار $\sum_{k=0}^{40}\left(x+\frac{k}{2}\right)^2$ حيث x عدد حقيقيّ.

2– لتكن $\mathbb{N}$ مجموعة الأعداد الصّحيحة الموجبة، و S مجموعة بحيث توجد دالة $f:\mathbb{N}\to S$ تحقّق الخاصّيّة التّالية: إذا كان x و y عددين صحيحين موجبين حاصل طرحهما عدد أوّليّ، فإنّ $f(x)\neq f(y)$. جد أقل عدد ممكن من العناصر في S.

3– ليكن $ABCD$ متوازي الأضلاع. يقطع المستقيم ℓ المستقيمات AB و BC و CD و DA عند أربع نقط مختلفة E و F و G و H ، على التّرتيب. تتقاطع الدّائرتان المحيطتان بالمثلّثين AEF و AGH مرّة ثانية عند P. تتقاطع الدّائرتان المحيطتان بالمثلّثين CEF و CGH مرّة ثانية عند Q . أثبت أنّ المستقيم PQ ينصّف القطر BD.

4– ليكن n عددا صحيحا أكبر من 2. لدينا n نقطة مختلفة على المستوى، لا توجد ثلاثة منها على استقامة واحدة. أثبت أنّه يمكننا تسمية النقط $P_1,P_2,\ldots,P_n$ بحيث لا يقطع ضلع من أضلاع المضلّع $P_1P_2\ldots P_n$ ضلعا آخر في غير رأس (لا يشترط في المضلّع أن يكون محدّبا).

5– ليكن $n>3$ عددا صحيحا فرديّا موجبا لا يقبل القسمة على 3 . حدّد إذا كان بالإمكان كتابة أعداد على جدول قياسه $n\times n$ بحيث:

ا. أعداد كلّ صفّ هي إعادة ترتيب للأعداد $0,1,\ldots,n-1$ ، وأعداد كلّ عمود هي إعادة ترتيب للأعداد $0,1,\ldots,n-1$ ؛

ب. الجدول غير متناظر تماما، بمعنى أنّه إذا كان $1\le i<j\le n$ و $1\le i'<j'\le n$ ، و $(i,j)\neq(i',j')$ فإنّ $(a_{i,j},a_{j,i})\neq(a_{i',j'},a_{j',i'})$ ، حيث يرمز $a_{i,j}$ للعدد المكتوب في الصّفّ i والعمود j.

مسائل اليوم الثّالث ـ 1 جمادى الآخرة 1435 هـ

الوقت المتاح: 4 ساعات

1– جد كلّ الدّوال $f:\mathbb{N}\to(0,\infty)$ الّتي تحقّق $f(4)=4$ والعلاقة

$$\frac{1}{f(1)f(2)}+\frac{1}{f(2)f(3)}+\cdots+\frac{1}{f(n)f(n+1)}=\frac{f(n)}{f(n+1)},\quad \forall n\in\mathbb{N},$$

حيث يرمز $\mathbb{N}=\{1,2,\ldots\}$ لمجموعة الأعداد الصّحيحة الموجبة.

2– تتقاطع دائرتان ω_1 و ω_2 عند نقطتين P و Q . القطعتان المستقيمتان AC و BD وتران في ω_1 و ω_2 على التّرتيب، بحيث تقطع القطعة المستقيمة AB الشّعاع CD في P . يقطع الشّعاع BD القطعة المستقيمة AC في X . تقع النقطة Y على ω_1 بحيث $PY \parallel BD$. تقع النقطة Z على ω_2 بحيث $PZ \parallel AC$. أثبت أنّ النّقط Q,X,Y,Z على استقامة واحدة.

3– ليكن a,b عددين حقيقيّين غير سالبين و n عددا صحيحا موجبا. أثبت أنّ

$$\left(1-2^{-n}\right)\left|a^{2^n}-b^{2^n}\right|\geq\sqrt{ab}\left|a^{2^n-1}-b^{2^n-1}\right|.$$

4– لتكن $f:\mathbb{N}\to\mathbb{N}$ دالة متباينة تحقّق $f(1)=2$ ، $f(2)=4$ و

$$f(f(m)+f(n))=f(f(m))+f(n)$$

لكلّ $m,n\in\mathbb{N}$. أثبت أنّ $f(n)=n+2$ لكلّ $n\geq 2$.

5– ليكن ABC مثلّثا، تمرّ الدّائرة Ω بالرّأسين B و C ، تمسّ الدّائرة ω الدّائرة Ω دخليّا وكذا الضّلعين AB و AC عند النّقط T و P و Q ، على التّرتيب، لتكن M نقطة منتصف القوس $\widehat{BC}$ (الّذي يشمل T) للدّائرة Ω . أثبت أنّ المستقيمات PQ و BC و MT تلتقي عند نقطة واحدة.

اختبارات ترشيح الطّلاب لأولمبياد الرّياضيّات الدّولي

مسائل اليوم الأوّل ـ 24 رجب 1435 هـ

الوقت المتاح: 4 ساعات ونصف

1– يلعب طارق وسلطان اللّعبة التّالية. يفكّر طارق في عدد أكبر من 100 ، ثمّ يقترح سلطان عددا أكبر من 1 . إذا كان عدد طارق يقبل القسمة على عدد سلطان فإنّ سلطان سيفوز، وإلّا يطرح طارق عدد سلطان من عدده ليحصل على عدد جديد، ثمّ يقترح سلطان عددا آخرا مختلفا عن الأعداد الّتي اقترحها من قبل، وهكذا. إذا صار عدد طارق سالبا فإنّ سلطان سيخسر. هل لسلطان استراتجيّة للفوز؟

2– نعرّف حجر الدومينو على أنّه زوج مرتّب من عددين صحيحين موجبين مختلفين. نعرّف المتتالية الفعليّة للدّومينو على أنّها قائمة أحجار دومينو مختلفة مرتّبة الواحدة تلوى الأخرى بحيث يتساوى العدد الأوّل لأيّ حجر دومينو مختلّف عن حجر الدّومينو الأوّل في القائمة مع العدد الثّاني لحجر الدّومينو الّذي يسبقه مباشرة، وبحيث لا يظهر فيها حجرا الدومينو (i,j) و (j,i) معا، لكلّ i و j . لتكن D_n مجموعة أحجار الدّومينو المكوّنة من أعداد أقل من أو تساوي n . جد أكبر طول ممكن لمتتالية فعليّة للدّومينو أحجارها كلّها في D_n .

3– ليكن ABC مثلّثا، و P و M و N نقط على BC و AB و AC ، على التّرتيب، بحيث لايتوازى MN و BC ويكون $AMPN$ متوازي الأضلاع. يقطع المستقيم MN الدّائرة المحيطة بالمثلّث ABC عند R و S . أثبت أنّ الدّائرة المحيطة بالمثلّث RPS مماسّة للضّلع BC .

4– جد كلّ الدّوال $f\colon \mathbb{N} \to \mathbb{N}$ الّتي تحقّق

$$f(n+1) > \frac{f(n) + f(f(n))}{2}$$

لكلّ $n \in \mathbb{N}$ ، حيث $\mathbb{N}$ هي مجموعة الأعداد الصّحيحة الموجبة.

مسائل اليوم الثّاني ـ 25 رجب 1435 هـ

الوقت المتاح: 4 ساعات ونصف

1– لتكن Γ دائرة مركزها O و AE قطر فيها. تقع النقطة D على القطعة المستقيمة OE والنقطة B هي منتصف أحد القوسين $\widehat{AE}$ في الدّائرة Γ. رسمت النقطة C بحيث يشكّل $ABCD$ متوازي الأضلاع. يلتقي المستقيمان EB و CD عند F. يقطع المستقيم OF القوس الصّغير $\widehat{EB}$ في I. أثبت أنّ EI منصّف الزّاوية $\angle BEC$.

2– لتكن S مجموعة خمسة أعداد حقيقيّة موجبة كلّما اخترنا منها ثلاثة أعداد مختلفة a, b, c ، كان المقدار $ab + bc + ca$ عددا نسبيّا. أثبت أنّه لكلّ عددين a و b في S ، المقدار $\frac{a}{b}$ عدد نسبيّ.

3– أثبت أنّه يمكننا تعبئة جدول $n \times n$ بأعداد صحيحة غير سالبة (ليست شرطا مختلفة) بحيث يكون كلّ حاصل جمع أعداد في عمود أو في صفّ كلّها مربّعات كاملة مختلفة فيما بينها.

4– يلعب أوس لعبة الورق باثنين وخمسين ورقة من لونين، في كلّ لون ستّ وعشرون ورقة. يوزّع أوس الأوراق على صفّ، وكلّما تتالت ورقتان من نفس اللّون، يمكنه إزالتهما. يفوز أوس في اللّعبة إذا استطاع أن يزيل كلّ الأوراق. إذا بدأ أوس اللّعبة بأوراق مرتّبة بطريقة عشوائيّة، ما هي احتمالية أن يفوز في اللّعبة؟

مسائل اليوم الثّالث ـ 28 رجب 1435 هـ

الوقت المتاح: 4 ساعات ونصف

1– نقول لعدد صحيح إنّه **مثالي** إذا كان ضعفه يساوي حاصل جمع قواسمه الموجبة، فمثلا العدد 28 مثاليّ لكون $2 \cdot 28 = 1 + 2 + 4 + 7 + 14 + 28$.

ا. نقول لعدد صحيح إنّه خال من المربّعات إذا كان لا يقبل القسمة على مربّع عدد أوّلي. جد كلّ الأعداد المثاليّة الخالية من المربّعات،

ب. أثبت أنّه لا يوجد مربّع كامل مثاليّ.

2– حدّد كلّ الدّوال $f : [0, \infty) \to \mathbb{R}$ الّتي تحقّق $f(0) = 0$ ، ولكلّ $x > 0$ العلاقة

$$f(x) = 1 + 5f\left(\left\lfloor \frac{x}{2} \right\rfloor\right) - 6f\left(\left\lfloor \frac{x}{4} \right\rfloor\right)$$

3– هناك 2015 قطعة نقود على الطّاولة. لكلّ $i = 1, 2, \ldots, 2015$ على التّرتيب، يجب قلب تماما i قطعة نقود. أثبت أنّه يمكننا دائما، بعد إنهاء هذه العمليّة، إمّا أن نضع كلّ قطع النّقود على وجهها، أو أن نضع كلّ قطع النّقود على ظهرها، وأنّه إذا أمكننا الحصول على إحدى الحالتين فلا يمكننا الحصول على الحالة الأخرى.

4– تتقاطع دائرتان ω_1 و ω_2 ، مركزهما O_1 و O_2 على التّرتيب، عند نقطتين A و B . لتكن X و Y نقطتين على ω_1 . يقطع المستقيمان XA و YA الدّائرة ω_2 عند Z و W ، على التّرتيب، بحيث تقع النّقطة A بين X و Z وبين Y و W . لتكن M نقطة منتصف O_1O_2 ، و S نقطة منتصف XA ، و T نقطة منتصف WA . أثبت أنّ $MS = MT$ إذا وفقط إذا كانت النّقط X ، Y ، Z ، W على دائرة.

مسائل اليوم الرّابع ـ 29 رجب 1435 هـ

الوقت المتاح: 4 ساعات ونصف

1– لتكن a_1 ، ... ، a_n متتالية غير متزايدة من الأعداد الحقيقيّة الموجبة. أثبت أنّ
$$\sqrt{a_1^2+a_2^2+\cdots+a_n^2} \le a_1+\frac{a_2}{\sqrt{2}+1}+\cdots+\frac{a_n}{\sqrt{n}+\sqrt{n-1}}.$$
وادرس حالة التّساوي.

2– يلعب في دورة كلّ لاعب مرّة واحدة مع كلّ واحد من اللّاعبين الآخرين. تعطى في كلّ مباراة 1 نقطة للفائز، و 0 نقطة للخاسر، و $\frac{1}{2}$ نقطة لكلّ من المتعادلين. لوحظ في آخر الدّورة أنّ كلّ لاعب، حصل على نصف رصيده تماما في مباراته ضد آخر عشر لاعبين أصحاب أقل رصيد (بما في ذلك آخر عشر لاعبين، حصل كلّ واحد منهم على نصف رصيده في مباراته ضدّ بقيّة اللّاعبين التّسع). كم كان عدد اللّاعبين في هذه الدّورة؟

3– لدينا شبكة مربّعات منتظمة، وتمّ وضع حجرين على رأسين من الشّبكة. يمكننا في كلّ خطوة تحريك أحد الحجرين إلى رأس آخر وإبقاء الحجر الثّاني ثابتا، بحيث لا تتغيّر المسافة بين الحجرين. هل يمكننا بعد عدد منته من الخطوات أن نبدّل بين موقعي الحجرين.

4– تقع النّقط A_1 و B_1 و C_1 على الأضلع BC و AC و AB للمثلّث ABC ، على التّرتيب، بحيث تحقّق $AB_1-AC_1=CA_1-CB_1=BC_1-BA_1$. لتكن I_A و I_B و I_C المراكز الدّاخليّة للمثلّثات AB_1C_1 و A_1BC_1 و A_1B_1C ، على التّرتيب. أثبت أنّ مركز الدّائرة المحيطة بالمثلّث $I_AI_BI_C$ ، هو المركز الدّاخليّ للمثلّث ABC .

Part II

Problems in English

Chapter 1

Preselection tests for the full-time training

The students who attended the full-time training that started on April 13, 2014, were chosen according to their combined performance in this set of tests during the October camp at Riyadh. These tests are prepared by Yunhao Fu and Malik Talbi.

1.1 Day I - October 4, 2013

Allowed time: 3 hours

1. Let $a_1, a_2, \ldots, a_{2n}$ be positive real numbers such that $a_i + a_{n+i} = 1$, for all $i = 1, \cdots, n$. Prove that there exist two different integers $1 \leq j, k \leq 2n$ for which
$$\sqrt{a_j^2 - a_k^2} < \frac{1}{\sqrt{n} + \sqrt{n-1}}.$$

2. Let D be the midpoint of side BC of triangle ABC and E the midpoint of median AD. Line BE intersects side CA at F. Prove that the area of quadrilateral $CDEF$ is $\frac{5}{12}$ the area of triangle ABC.

3. Find all positive integers n for which $1 - 5^n + 5^{2n+1}$ is a perfect square.

4. Majid wants to color the cells of an $n \times n$ chessboard into white and black so that each 2×2 subsquare contains two white cells and two black cells. In how many ways can Majid color this $n \times n$ chessboard?

1.2 Day II - October 6, 2013

Allowed time: 3 hours

1. Prove that 2014 divides $53n^{55} - 57n^{53} + 4n$ for all integer n.

2. Let a_1, a_2, a_3, a_4, a_5 be nonzero real numbers. Prove that the polynomial

$$P(X) = \prod_{k=0}^{4} \left(a_{k+1}X^4 + a_{k+2}X^3 + a_{k+3}X^2 + a_{k+4}X + a_{k+5}\right),$$

   where $a_{5+i} = a_i$ for $i = 1, 2, 3, 4$, has a root with negative real part.

3. The 2013 numbers

$$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \ldots, \frac{1}{2013 \times 2014}$$

   are arranged randomly on a circle.

   (a) Prove that there exist ten consecutive numbers on the circle whose sum is less than $\frac{1}{4000}$.

   (b) Prove that there exist ten consecutive numbers on the circle whose sum is less than $\frac{1}{10000}$.

4. Let ABC be an acute triangle with $\angle A < \angle B \leq \angle C$, and O its circumcenter. The perpendicular bisector of side AB intersects side AC at D. The perpendicular bisector of side AC intersects side AB at E. Express the angles of triangle DEO in terms of the angles of triangle ABC.

1.3 Day III - October 9, 2013

Allowed time: 3 hours

1. There are 14 students who have particiated to a 3 hour test consisting on 15 short problems. Each student has solved a different number of problems and each problem has been solved by a different number of students. Prove that there exists a student who has solved exactly 5 problems.

2. Let x, y be positive real numbers. Find the minimum of

$$x^2 + xy + \frac{y^2}{2} + \frac{2^6}{x+y} + \frac{3^4}{x^3}.$$

3. Let ABC be a triangle and I its incenter. The line AI intersects the side BC at D and the perpendicular bisector of BC at E. Let J be the incenter of triangle CDE. Prove that triangle CIJ is isosceles.

4. Prove that there exists a positive integer n such that the last digits of n^3 are $\cdots\square 201320132013$.

1.4 Day IV - October 11, 2013

Allowed time: 3 hours

1. Let p be a prime number and $n \geq 2$ a positive integer, such that $p \mid (n^6 - 1)$. Prove that $n > \sqrt{p} - 1$.

2. Given $x \geq 0$, prove that

$$\frac{(x^2+1)^6}{2^7} + \frac{1}{2} \geq x^5 - x^3 + x.$$

3. Fatima and Asma are playing the following game. First, Fatima chooses 2013 pairwise different numbers, called a_1, a_2, ..., a_{2013}. Then, Asma tries to know the value of each number a_1, a_2, ..., a_{2013}. At each time, Asma chooses $1 \leq i < j \leq 2013$ and asks Fatima "*What is the set* $\{a_i, a_j\}$*?*"(For example, if Asma asks what is the set $\{a_1, a_2\}$, and $a_1 = 17$ and $a_2 = 13$, Fatima will answer $\{13, 17\}$). Find the least number of questions Asma needs to ask, to know the value of all the numbers a_1, a_2, ..., a_{2013}.

4. Let $\triangle ABC$ be an acute triangle, with $\widehat{A} > \widehat{B} \geq \widehat{C}$. Let D, E and F be the tangency points between the incircle of triangle and sides BC, CA, AB, respectively. Let J be a point on (BD), K a point on (DC), L a point on (EC) and M a point on (FB), such that

$$AF = FM = JD = DK = LE = EA.$$

Let P be the intersection point between AJ and KM and let Q be the intersection point between AK and JL. Prove that $PJKQ$ is cyclic.

Chapter 2

Team selection tests for the Gulf Mathematical Olympiad 2014

The KSA 2014 Gulf MO team members were chosen according to their combined performance in this set of tests during the January camp at Riyadh. These tests are prepared by Yunhao Fu and Malik Talbi.

Members of the KSA 2014 Gulf MO team were Ibraheem Khan, Alzubair Habibullah, Mahdi Al-Shaikh Saleh, Shaden Al-Shammari, Ammar Alqatari, Majid Almarhumi. The total team scores was 219 out of 240, and was ranked 1st place among the 7 participating teams. The team's individual performances were as follows:

Shaden Al-Shammari	GOLD Medallist
Alzubair Habibullah	GOLD Medallist
Ibraheem Khan	GOLD Medallist
Mahdi Al-Shaikh Saleh	GOLD Medallist
Majid Almarhumi	SILVER Medallist
Ammar Alqatari	SILVER Medallist

2.1 Day I - January 22, 2014

Allowed time: 3 hours

1. Let ABC be a triangle with $\angle A < \angle B \leq \angle C$, M and N the midpoints of sides CA and AB, respectively, and P and Q the projections of B and C on the medians CN and BM, respectively. Prove that the quadrilateral $MNPQ$ is cyclic.

2. Let $p \geq 2$ be a prime number and $\dfrac{a_p}{b_p} = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{p^2 - 1}$, where a_p and b_p are two relatively prime positive integers. Compute $\gcd(p, b_p)$.

3. Turki has divided a square into finitely many white and green rectangles, each with sides parallel to the sides of the square. Within each white rectangle, he writes down its width divided by its height. Within each green rectangle, he writes down its height divided by its width. Finally, he calculates S, the sum of these numbers. If the total area of white rectangles equals the total area of green rectangles, determine the minimum possible value of S.

4. Let ABC be a triangle, D the midpoint of side BC and E the intersection point of the bisector of angle $\angle BAC$ with side BC. The perpendicular bisector of AE intersects the bisectors of angles $\angle CBA$ and $\angle CDA$ at M and N, respectively. The bisectors of angles $\angle CBA$ and $\angle CDA$ intersect at P. Prove that points A, M, N, P are concyclic.

2.2 Day II - January 25, 2014

Allowed time: 3 hours

1. Let A, B, C be colinear points in this order, $\mathcal{C}$ an arbitrary circle passing through B and C, and l an arbitrary line different from BC, passing through A and intersecting $\mathcal{C}$ at M and N. The bisectors of the angles $\angle CMB$ and $\angle CNB$ intersect BC at P and Q. Prove that $AP \cdot AQ = AB \cdot AC$.

2. Let p be a prime number. Prove that there exist infinitely many positive integers n such that p divides
$$1^n + 2^n + \cdots + (p+1)^n.$$

3. Let $ABCDE$ be a cyclic pentagon such that the diagonals AC and AD intersect BE at P and Q, respectively, with $BP \cdot QE = PQ^2$. Prove that $BC \cdot DE = CD \cdot PQ$.

4. Let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ be real numbers. Prove that
$$a_1a_2(a_1 - a_2) + a_2a_3(a_2 - a_3) + \cdots + a_{n-1}a_n(a_{n-1} - a_n) \geq a_1a_n(a_1 - a_n).$$

2.3 Day III - January 28, 2014

Allowed time: 3 hours

1. Find all ordered triples (a, b, c) of positive integers which satisfy

$$5^a + 3^b - 2^c = 32.$$

2. Let $S = \{(a, b) \mid a, b = 1, 2, 3, 4, \text{ and } a \neq b\}$, and consider all nonzero polynomials $p(X, Y)$ with integer coefficients such that $p(a, b) = 0$ for every element (a, b) in S.

   (a) What is the minimal degree of such polynomial $p(X, Y)$?

   (b) Determine all such polynomials $p(X, Y)$ with minimal degree.

3. Let ABC be a triangle, I its incenter, and ω a circle of center I. Points A', B', C' are on ω such that rays IA', IB', IC' starting from I intersect perpendicularly sides BC, CA, AB, respectively. Prove that lines AA', BB', CC' are concurrent.

4. Let X be a set of rational numbers satisfying the following two conditions:

   (a) The set X contains at least two elements,

   (b) For any x, y in X, if $x \neq y$ then there exists z in X such that either $\left|\dfrac{x-z}{y-z}\right| = 2$ or $\left|\dfrac{y-z}{x-z}\right| = 2$.

   Prove that X contains infinitely many elements.

Chapter 3

Team selection tests for the Balkan Mathematical Olympiad 2014

The KSA 2014 Balkan MO team members were chosen according to their combined performance in this set of tests during the March camp at Riyadh. These tests are prepared by Ivan Borsenco, Zuming Feng, and Malik Talbi.

Members of the KSA 2014 Balkan MO team were Alzubair Habibullah, Ibraheem Khan, Ammar Alqatari, Mahdi Al-Shaikh Saleh, Shaden Al-Shammari, Omar Alrabiah. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Malik Talbi (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Tarek S. Shehata (Mawhiba, Riyadh) as observer A of the KSA delegation. The total team scores was 100 out of 240, and was ranked 14th place among the 20 participating teams. The team's individual performances were as follows:

Alzubair Habibullah	BRONZE Medallist
Omar Alrabiah	BRONZE Medallist
Ammar Alqatari	Honorable Mention
Mahdi Al-Shaikh Saleh	
Ibraheem Khan	
Shaden Al-Shammari	

3.1 Day I - March 25, 2014

Allowed time: 4 hours

1. A positive proper divisor is a positive divisor of a number, excluding itself. For positive integers $n \geq 2$, let $f(n)$ denote the number that is one more than the largest proper divisor of n. Determine all positive integers n such that $f(f(n)) = 2$.

2. Prove that among any 16 perfect cubes we can always find two cubes whose difference is divisible by 91.

3. Let $n \geq 2$ be a positive integer, and write in a digit form
$$\frac{1}{n} = 0.a_1a_2\ldots.$$
Suppose that $n = a_1 + a_2 + \cdots$. Determine all possible values of n.

4. Let ABC be a triangle with $\angle B \leq \angle C$, I its incenter and D the intersection point of line AI with side BC. Let M and N be points on sides BA and CA, respectively, such that $BM = BD$ and $CN = CD$. The circumcircle of triangle CMN intersects again line BC at P. Prove that quadrilateral $DIMP$ is cyclic.

5. Find all positive integers n such that
$$3^n + 4^n + \cdots + (n+2)^n = (n+3)^n.$$

3.2 Day II - March 30, 2014

Allowed time: 4 hours

1. Find the minimum of $\sum_{k=0}^{40} \left(x + \frac{k}{2}\right)^2$ where x is a real number.

2. Let $\mathbb{N}$ denote the set of positive integers, and let S be a set. There exists a function $f : \mathbb{N} \to S$ such that if x and y are a pair of positive integers with their difference being a prime number, then $f(x) \neq f(y)$. Determine the minimum number of elements in S.

3. Let $ABCD$ be a parallelogram. A line ℓ intersects lines AB, BC, CD, DA at four different points E, F, G, H, respectively. The circumcircles of triangles AEF and AGH intersect again at P. The circumcircles of triangles CEF and CGH intersect again at Q. Prove that the line PQ bisects the diagonal BD.

4. Let n be an integer greater than 2. Consider a set of n different points, with no three collinear, in the plane. Prove that we can label the points $P_1, P_2, \ldots, P_n$ such that $P_1P_2 \ldots P_n$ is not a self-intersecting polygon. (*A polygon is self-intersecting if one of its side intersects the interior of another side. The polygon is not necessarily convex*)

5. Let $n > 3$ be an odd positive integer not divisible by 3. Determine if it is possible to form an $n \times n$ array of numbers such that

   (a) the set of the numbers in each row is a permutation of $0, 1, \ldots, n-1$; the set of the numbers in each column is a permutation of $0, 1, \ldots, n-1$;

   (b) the board is *totally non-symmetric*: for $1 \leq i < j \leq n$ and $1 \leq i' < j' \leq n$, if $(i, j) \neq (i', j')$ then $(a_{i,j}, a_{j,i}) \neq (a_{i',j'}, a_{j',i'})$, where $a_{i,j}$ denotes the entry in the ith row and j^{th} column.

3.3 Day III - April 1, 2014

Allowed time: 4 hours

1. Find all functions $f : \mathbb{N} \to (0, \infty)$ such that $f(4) = 4$ and

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)}, \quad \forall n \in \mathbb{N},$$

where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers.

2. Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.

3. Let a, b be two nonnegative real numbers and n a positive integer. Prove that

$$\left(1 - 2^{-n}\right) \left|a^{2^n} - b^{2^n}\right| \geq \sqrt{ab} \left|a^{2^n - 1} - b^{2^n - 1}\right|.$$

4. Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function such that $f(1) = 2$, $f(2) = 4$ and

$$f(f(m) + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}$. Prove that $f(n) = n + 2$ for all $n \geq 2$.

5. Let ABC be a triangle. Circle Ω passes through points B and C. Circle ω is tangent internally to Ω and also to sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc $\widehat{BC}$ (containing T) of Ω. Prove that lines PQ, BC, and MT are concurrent.

Chapter 4

Team Selection tests for the International Mathematical Olympiad 2014

The KSA 2014 IMO team members were chosen according to their combined performance in this set of tests during the April-May camp at KAUST University in Thuwal. These tests are prepared by Ivan Borsenco, Zuming Feng and Carlos Shine.

Members of the KSA 2014 IMO team were Ibraheem Khan, Omar Alrabiah, Alzubair Habibullah, Mahdi Al-Shaikh Saleh, Shaden Al-Shammari, Salman Saleh. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulrahman Alguwaizani (MAWHIBA, Riyadh), Mansour Almoaigel, Dr. Abdulrahman Albarak (Ministry of Education), Dr. Adel Alkeaid (MAWHIBA, Riyadh) and Sultan Albarakati (MAWHIBA, Riyadh) as observers of the KSA delegation.

Members of the KSA 2013 IMO team were Alyazeed Basuni, Ibraheem Khan, Abdulrahman Alharbi, Sameh Zawawi, Alzubair Habibullah, and Ali Alnasser. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Mansour Almoaigel, and Dr. Abdulrahman Albarak (Ministry of Education) as observers of the KSA dele-

gation. The total team scores was 84 out of 252, and was ranked 42nd place among the 97 participating teams. The team's individual performances were as follows:

Alyazeed Basuni	BRONZE Medallist
Ibraheem Khan	BRONZE Medallist
Abdulrahman Alharbi	BRONZE Medallist
Sameh Zawawi	BRONZE Medallist
Alzubair Habibullah	
Ali Alnasser	

4.1 Day I - May 23, 2014

Allowed time: 4 hours and half

1. Tarik and Sultan are playing the following game. Tarik thinks of a number that is greater than 100. Then Sultan is telling a number greater than 1. If Tarik's number is divisible by Sultan's number, Sultan wins, otherwise Tarik subtracts Sultan's number from his number and Sultan tells his next number. Sultan is forbidden to repeat his numbers. If Tarik's number becomes negative, Sultan loses. Does Sultan have a winning strategy?

2. Define a *domino* to be an ordered pair of *distinct* positive integers. A *proper sequence* of dominoes is a list of distinct dominoes in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not *both* appear for any i and j. Let D_n be the set of all dominoes whose coordinates are no larger than n. Find the length of the longest proper sequence of dominoes that can be formed using the dominoes of D_n.

3. Let ABC be a triangle and let P be a point on BC. Points M and N lie on AB and AC, respectively such that MN is not parallel to BC and $AMPN$ is a parallelogram. Line MN meets the circumcircle of ABC at R and S. Prove that the circumcircle of triangle RPS is tangent to BC.

4. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(n+1) > \frac{f(n) + f(f(n))}{2}$$

for all $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of strictly positive integers.

4.2 Day II - May 24, 2014

Allowed time: 4 hours and half

1. Let Γ be a circle with center O and AE be a diameter. Point D lies on segment OE and point B is the midpoint of one of the arcs $\widehat{AE}$ of Γ. Construct point C such that $ABCD$ is a parallelogram. Lines EB and CD meet at F. Line OF meets the minor arc $\widehat{EB}$ at I. Prove that EI bisects $\angle BEC$.

2. Let S be a set of positive real numbers with five elements such that for any distinct a, b, c in S, the number $ab + bc + ca$ is rational. Prove that for any a and b in S, $\frac{a}{b}$ is a rational number.

3. Show that it is possible to write a $n \times n$ array of non-negative numbers (not necessarily distinct) such that the sums of entries on each row and each column are pairwise distinct perfect squares.

4. Aws plays a solitaire game on a fifty-two card deck: whenever two cards of the same color are adjacent, he can remove them. Aws wins the game if he removes all the cards. If Aws starts with the cards in a random order, what is the probability for him to win?

4.3 Day III - May 27, 2014

Allowed time: 4 hours and half

1. A *perfect number* is an integer that equals half the sum of its positive divisors. For example, because $2 \cdot 28 = 1 + 2 + 4 + 7 + 14 + 28$, 28 is a perfect number.

   (a) A *square-free* integer is an integer not divisible by a square of any prime number. Find all square-free integers that are perfect numbers.

   (b) Prove that no perfect square is a perfect number.

2. Determine all functions $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ and

$$f(x) = 1 + 5f\left(\left\lfloor \frac{x}{2} \right\rfloor\right) - 6f\left(\left\lfloor \frac{x}{4} \right\rfloor\right)$$

   for all $x > 0$.

3. There are 2015 coins on a table. For $i = 1, 2, \ldots, 2015$ in succession, one must turn over exactly i coins. Prove that it is always possible either to make all of the coins face up or to make all of the coins face down, but not both.

4. Two circles ω_1 and ω_2, with center O_1 and O_2 respectively, meet at points A and B. Let X and Y be points on ω_1. Lines XA and YA meet ω_2 at Z and W, respectively, such that A lies between X and Z and between Y and W. Let M be the midpoint of O_1O_2, S be the midpoint of XA and T be the midpoint of WA. Prove that $MS = MT$ if and only if X, Y, Z and W are concyclic.

4.4 Day IV - May 28, 2014

Allowed time: 4 hours and half

1. Let $a_1, \ldots, a_n$ be a non increasing sequence of positive real numbers. Prove that
$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \leq a_1 + \frac{a_2}{\sqrt{2}+1} + \cdots + \frac{a_n}{\sqrt{n}+\sqrt{n-1}}.$$
When does equality hold?

2. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of his points against the other nine of the ten). What was the total number of players in the tournament?

3. We are given a lattice and two pebbles A and B that are placed at two lattice points. At each step we are allowed to relocate one of the pebbles to another lattice point with the condition that the distance between pebbles is preserved. Is it possible after finite number of steps to switch positions of the pebbles?

4. Points A_1, B_1, C_1 lie on the sides BC, AC and AB of a triangle ABC, respectively, such that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incenters of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcenter of triangle $I_AI_BI_C$, is the incenter of triangle ABC.

Chapter 5

Authors of the problems

Among the problems used to select our students and presented in this booklet, we believe that the following are original and we would like to thank their authors:

1.1.1 Suggested by Yunhao Fu & Malik Talbi,

1.1.2 Suggested by Malik Talbi,

1.1.4 Suggested by Malik Talbi,

1.2.2 Suggested by Yunhao Fu,

1.2.3 Suggested by Yunhao Fu,

1.2.4 Suggested by Malik Talbi,

1.3.1 Suggested by Yunhao Fu,

1.3.3 Suggested by Malik Talbi,

1.3.4 Suggested by Yunhao Fu,

1.4.2 Suggested by Yunhao Fu,

2.1.1 Suggested by Malik Talbi,

2.1.4 Suggested by Malik Talbi,

2.2.4 Suggested by Yunhao Fu,

2.3.3 Suggested by Malik Talbi,

2.3.4 Suggested by Yunhao Fu,

3.1.4 Suggested by Malik Talbi,

3.2.3 Suggested by Malik Talbi,

3.3.3 Suggested by Malik Talbi,

Part III

Solutions

Chapter 1

Solutions to the preselection tests for the full-time training

1.1 Solutions to the preselection test of day I

1. Let $a_1, a_2, \ldots, a_{2n}$ be positive real numbers such that $a_i + a_{n+i} = 1$, for all $i = 1, \cdots, n$. Prove that there exist two different integers $1 \leq j, k \leq 2n$ for which
$$\sqrt{a_j^2 - a_k^2} < \frac{1}{\sqrt{n} + \sqrt{n-1}}.$$

Solution. Assume, without loss of generality, that
$$0 < a_1 \leq a_2 \leq \cdots \leq a_n \leq \frac{1}{2} \leq a_{2n} \leq a_{2n-1} \leq \cdots \leq a_{n+1} < 1,$$
and let
$$K = \min\left\{\left|a_i^2 - a_j^2\right|, \text{ with } i, j = 1, \ldots, 2n, \text{ and } i \neq j\right\}.$$
We have
$$K \leq a_{2n}^2 - a_n^2 = (a_{2n} + a_n)(a_{2n} - a_n) = 1 - 2a_n,$$
which means that
$$\frac{1-K}{2} \geq a_n.$$
We deduce that
$$\frac{(1-K)^2}{4} \geq a_n^2 > a_n^2 - a_1^2 = \sum_{i=1}^{n-1}\left(a_{i+1}^2 - a_i^2\right) \geq (n-1)K.$$

This implies that

$$K^2 - 2(2n-1)K + 1 > 0.$$

Because $K < 1$, we deduce that

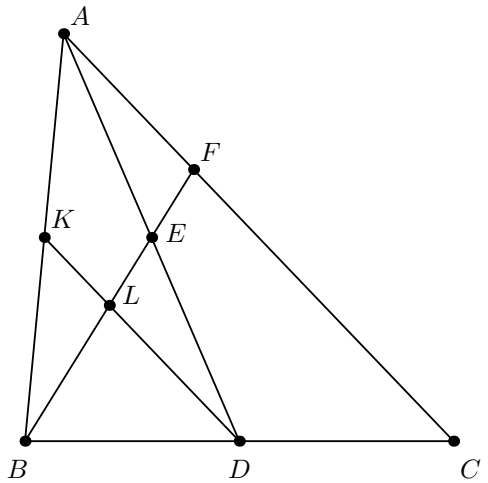
$$K < 2n - 1 - \sqrt{(2n-1)^2 - 1} = \left(\sqrt{n} - \sqrt{n-1}\right)^2,$$

and therefore

$$\sqrt{K} < \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n} + \sqrt{n-1}}.$$

2. Let D be the midpoint of side BC of triangle ABC and E the midpoint of median AD. Line BE intersects side CA at F. Prove that the area of quadrilateral $CDEF$ is $\frac{5}{12}$ the area of triangle ABC.

Solution 1. Let K be the midpoint of AB and L the intersection point of DK with BE.



Because D and K are midpoints of BC and BA, respectively, segment DK is parallel to side AB and we have

$$\frac{EF}{LE} = \frac{EA}{DE} = 1.$$

Because E and K are midpoints of AD and AB, respectively, the point L is the centroid of triangle ABD. Therefore

$$\frac{LE}{BE} = \frac{1}{3}.$$

We deduce that the ratio of the areas of the triangles

$$\frac{[AEF]}{[ABE]} = \frac{EF}{BE} = \frac{1}{3}.$$

Therefore

$$\frac{[AEF]}{[ABC]} = \frac{[AEF]}{[ABE]} \cdot \frac{[ABE]}{[ABD]} \cdot \frac{[ABD]}{[ABC]} = \frac{1}{12},$$

since D and E are midpoints of BC and AD, respectively. We deduce that

$$\frac{[CFED]}{[ABC]} = \frac{[ADC] - [AEF]}{[ABC]} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}.$$

Solution 2. Let

$$x = \frac{[CFD]}{[ABC]} \quad \text{and} \quad y = \frac{[DFE]}{[ABC]}$$

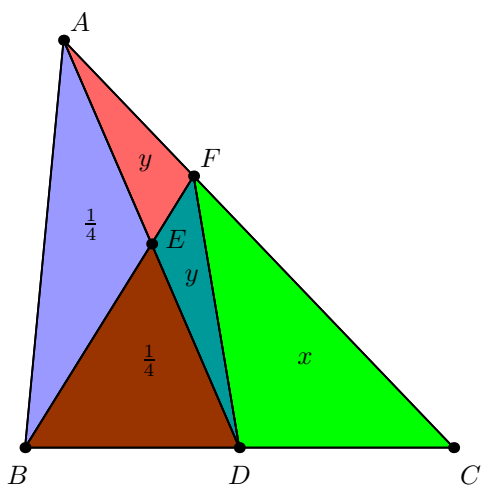
be the ratios of the areas.

Because E is the midpoint of AD, we have

$$[ABE] = [BDE] = \frac{1}{2}[ABD] \quad \text{and} \quad [AEF] = [DFE].$$

Because D is the midpoint of BC, we have

$$[ABD] = [ADC] = \frac{1}{2}[ABC] \quad \text{and} \quad [FBD] = [FDC].$$



We deduce that

$$x + 2y = \frac{1}{2} \quad \text{and} \quad x = y + \frac{1}{4},$$

and therefore

$$x = \frac{1}{3} \quad \text{and} \quad y = \frac{1}{12}.$$

This proves that

$$\frac{[CFED]}{[ABC]} = x + y = \frac{5}{12}.$$

3. Find all positive integers n for which $1 - 5^n + 5^{2n+1}$ is a perfect square.

Solution. Assume that $1 - 5^n + 5^{2n+1} = m^2$, for a positive integer m. We have

$$5^n \left(5^{n+1} - 1\right) = (m-1)(m+1).$$

Because $(m+1) - (m-1) = 2$, the number 5 cannot devide both $m-1$ and $m+1$. Therefore, we have two cases:

First case when 5 divides $m-1$. In this case, there exists a positive integer k such that $m - 1 = 5^n k$ and $5^{n+1} - 1 = (m+1)k = (5^n k + 2)\, k$. We, therefore, obtain the equation

$$5^n \left(5 - k^2\right) = 2k + 1.$$

For $k = 1$, this equation has no solution. For $k = 2$, $n = 1$ is a solution for the equation. For $k \geq 3$ this equation has no solution since its left hand side is negative.

Second case when 5 divides $m+1$. In this case, there exists a positive integer k such that $m+1 = 5^n k$ and $5^{n+1} - 1 = (m-1)k = (5^n k - 2)\, k$. We, therefore, obtain the equation

$$5^n \left(k^2 - 5\right) = 2k - 1.$$

This equation has no solution since for $k = 1, 2$ or 3 the left hand side is either negative or even, and for $k \geq 4$ we have

$$5^n \left(k^2 - 5\right) > 5(k+2)(k-3) \geq 5k + 10 > 2k - 1.$$

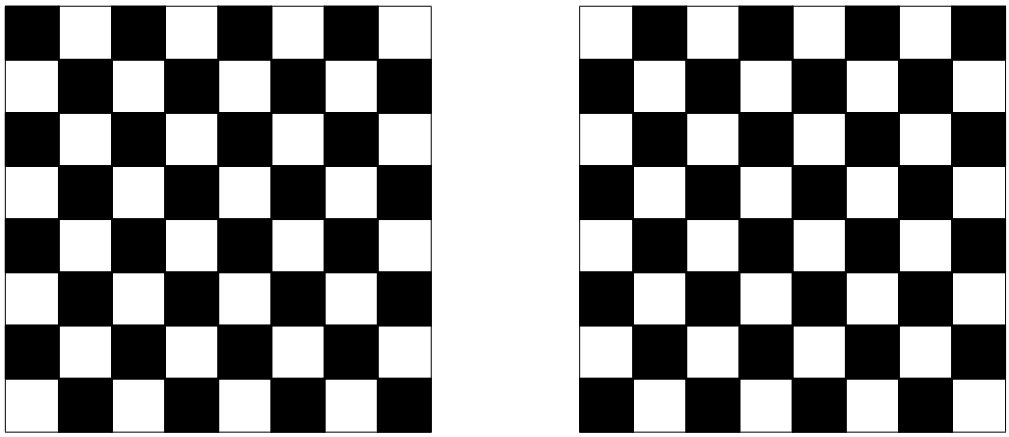
Therefore, the unique solution to this problem is $n = 1$ since we have

$$1 - 5 + 5^3 = 11^2.$$

4. Majid wants to color the cells of an $n \times n$ chessboard into white and black so that each 2×2 subsquare contains two white cells and two black cells. In how many ways can Majid color this $n \times n$ chessboard?

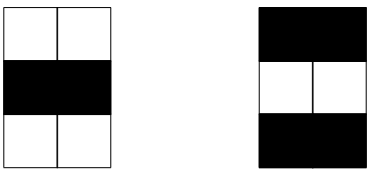
Solution. We have three possible cases.

The first case is when there are no two adjacent cells with the same color. In this case there are exactly two possible ways to color this $n \times n$ chessboard depending on the color of the first cell on the top left of the chessboard:



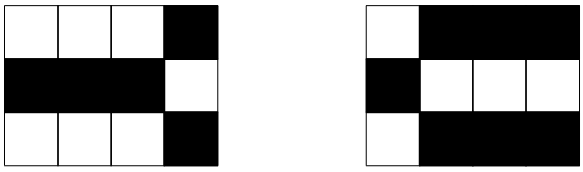
The second case is when there are two adjacent cells with the same color on the same row.

In this case the two adjacent cells in the previous row and the two adjacent cells in the next row must be of the other color.

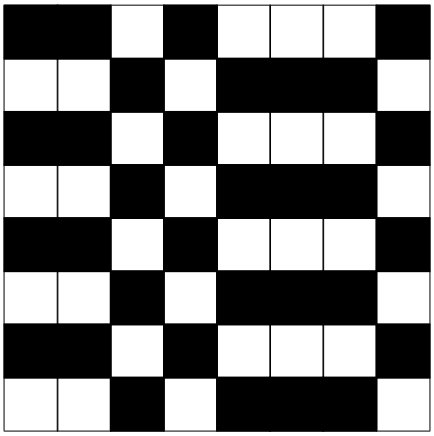


If the previous cell or the next cell in this row is of the same color, the adjacent cells in the previous or the next row must be of the other color. If

the previous cell or the next cell in this row is of the other color, the adjacent cells in the previous or the next row must be of this color.



Therefore, once fixing the colors of the cells of this row, the colors of the cells of the other rows are fixed by just alternating verticaly the colors of adjacent cells.



Hence, in this case any coloring of the first row with at least two adjacent cells with the same color will fix the coloring of the cells of the hole chessboard. There are precisely $2^n - 2$ possible such coloring corresponding to the number of subsets of $\{1, 2, \ldots, n\}$ (the positions of the black cells in the row) minus the two alternating coloring of the first case.

The third case is when there are two adjacent cells with the same color on the same column. This case is similar to the previous one and in the same way we obtain $2^n - 2$ possible such coloring.

Notice that the second and the third cases cannot occur at the same time. Therefore, there are precisely $2 + 2\,(2^n - 2) = 2^{n+1} - 2$ possible coloring of the $n \times n$ chessboard.

1.2 Solutions to the preselection test of day II

1. Prove that 2014 divides $53n^{55} - 57n^{53} + 4n$ for all integer n.

   Solution. Notice first that $2014 = 2 \times 19 \times 53$. Applying Fermat' little theorem for the prime numbers $2, 19, 53$ we obtain:

$$53n^{55} - 57n^{53} + 4n \equiv n - n + 0 \equiv 0 \mod 2,$$

$$53n^{55} - 57n^{53} + 4n \equiv 15n^{17}\left(n^{19}\right)^2 - 0 + 4n \equiv 15n^{19} + 4n \equiv 19n \equiv 0 \mod 19,$$

$$\text{and } \ 53n^{55} - 57n^{53} + 4n \equiv 0 - 4n + 4n \equiv 0 \mod 53,$$

   for all integer n. Therefore 2014 divides $53n^{55} - 57n^{53} + 4n$ for all integer n.

2. Let a_1, a_2, a_3, a_4, a_5 be nonzero real numbers. Prove that the polynomial

$$P(X) = \prod_{k=0}^{4} \left(a_{k+1}X^4 + a_{k+2}X^3 + a_{k+3}X^2 + a_{k+4}X + a_{k+5}\right),$$

   where $a_{5+i} = a_i$ for $i = 1, 2, 3, 4$, has a root with negative real part.

   Solution. Assume, to the contrary, that all roots of the polynomial $P(X)$ have nonegative real parts. We deduce that the real parts of the sums of the roots of its factors

$$a_kX^4 + a_{k+1}X^3 + a_{k+2}X^2 + a_{k+3}X + a_{k+4},$$

   for $k = 1, 2, 3, 4$, are nonegative. Therefore, by Vieta' relations, we have

$$Re\left(-\frac{a_{k+1}}{a_k}\right) \geq 0,$$

   for $k = 1, 2, 3, 4$. Hence

$$-1 = Re\prod_{k=1}^{4}\left(-\frac{a_{k+1}}{a_k}\right) = \prod_{k=1}^{4} Re\left(-\frac{a_{k+1}}{a_k}\right) \geq 0,$$

   which is a contradiction.

   This proves that the polynomial $P(X)$ has a root with negative real part.

3. The 2013 numbers

$$\frac{1}{1\times 2}, \frac{1}{2\times 3}, \frac{1}{3\times 4}, \ldots, \frac{1}{2013\times 2014}$$

are arranged randomly on a circle.

(a) Prove that there exist ten consecutive numbers on the circle whose sum is less than $\frac{1}{4000}$.

(b) Prove that there exist ten consecutive numbers on the circle whose sum is less than $\frac{1}{10000}$.

Solution.

(a) Consider 201 disjoint blocks $B_1, B_2, \ldots, B_{201}$ consisting each on 10 numbers, consecutive on the circle. By the pigeonhole principle, there exists a block B_k, for some $1 \le k \le 201$, not containing any of the 200 numbers

$$\frac{1}{1\times 2}, \frac{1}{2\times 3}, \ldots, \frac{1}{200\times 201}.$$

The sum of the ten consecutive numbers in this block B_k is less than or equal to

$$\begin{aligned}
&\frac{1}{201\times 202} + \frac{1}{202\times 203} + \cdots + \frac{1}{210\times 211} \\
&\qquad = \left(\frac{1}{201} - \frac{1}{202}\right) + \left(\frac{1}{202} - \frac{1}{203}\right) + \cdots + \left(\frac{1}{210} - \frac{1}{211}\right) \\
&\qquad = \frac{1}{201} - \frac{1}{211} = \frac{10}{201\times 211} < \frac{1}{4000}.
\end{aligned}$$

(b) Again, consider the 201 disjoint blocks $B_1, B_2, \ldots, B_{201}$ like in (a). By the pigeonhole principle, there exists at least 101 blocks $B_{k_1}, B_{k_2}, \ldots, B_{k_{201}}$, not containing any of the 100 numbers

$$\frac{1}{1\times 2}, \frac{1}{2\times 3}, \ldots, \frac{1}{100\times 101}.$$

The sum of the 1010 numbers in these 101 blocks $B_{k_1}, B_{k_2}, \ldots, B_{k_{201}}$ is

less than or equal to

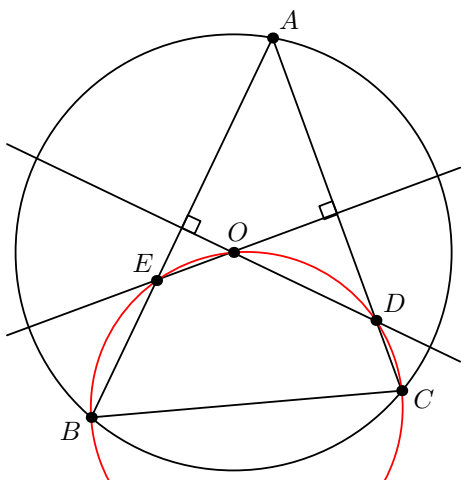
$$\begin{aligned}\sum_{i=1}^{201}\left(\sum_{b\in B_{k_i}} b\right) &\le \frac{1}{101\times 102}+\frac{1}{102\times 103}+\cdots+\frac{1}{210\times 211}\\ &\le \left(\frac{1}{101}-\frac{1}{102}\right)+\left(\frac{1}{102}-\frac{1}{103}\right)+\cdots+\left(\frac{1}{210}-\frac{1}{211}\right)\\ &< \frac{1}{101}.\end{aligned}$$

Therefore, there exists at least one block $B_{k_{i_0}}$ among these 101 blocks with the sum of its ten consecutive numbers is less than

$$\frac{1}{101\times 101}<\frac{1}{10000}.$$

4. Let ABC be an acute triangle with $\angle A<\angle B\le\angle C$, and O its circumcenter. The perpendicular bisector of side AB intersects side AC at D. The perpendicular bisector of side AC intersects side AB at E. Express the angles of triangle DEO in terms of the angles of triangle ABC.

Solution. Let M be the midpoint of AC. Because $\measuredangle COM=\frac{1}{2}\measuredangle COA=\measuredangle CBE$, the quadrilateral $BCOE$ is cyclic.



Let N be the midpoint of AB. Because $\measuredangle NOB=\frac{1}{2}\measuredangle AOB=\measuredangle DCB$, the quadrilateral $BCDO$ is cyclic. Therefore, the pentagon $BCDOE$ is cyclic

and we have:

$$\measuredangle ODE = \measuredangle OBE = 90^\circ - \measuredangle ACB, \qquad \measuredangle DEO = \measuredangle DCO = 90^\circ - \measuredangle CBA,$$

and

$$\measuredangle EOD = 180^\circ - \measuredangle ODE - \measuredangle DEO = \measuredangle CBA + \measuredangle ACB.$$

1.3 Solutions to the preselection test of day III

1. There are 14 students who have particiated to a 3 hour test consisting on 15 short problems. Each student has solved a different number of problems and each problem has been solved by a different number of students. Prove that there exists a student who has solved exactly 5 problems.

   Solution. Because there are 14 students, the possible numbers of students solving a problem are $0, 1, \ldots, 14$. Because there are 15 problems and each problem has been solved by a different number of students, there is a problem which has been solved by all the 14 students and another problem which has not been solved by any of the 14 students.

   If we cancel both problems, the one solved by all the students and the one not solved by any of the students, we still have that each of the 14 students has solved a different number of the 13 remaining problems. But from these 13 remaining problems, the possible number of problems that a student has solved are $0, 1, \ldots, 13$. Again, because there are 14 students and each student has solved a different number of problems, there is a student who has solved exactly 4 problems from these 13 remaining problems. But this student has also solved the canceled problem solved by all the students and has not solved the canceled problem which has not been solved by any of the students. Therefore, this student has solved exactly 5 problems.

2. Let x, y be positive real numbers. Find the minimum of

$$x^2 + xy + \frac{y^2}{2} + \frac{2^6}{x+y} + \frac{3^4}{x^3}.$$

   Solution. We have

$$x^2 + xy + \frac{y^2}{2} + \frac{2^6}{x+y} + \frac{3^4}{x^3} = \frac{x^2}{2} + \frac{(x+y)^2}{2} + \frac{2^6}{x+y} + \frac{3^4}{x^3}.$$

By applying AM-GM inequality we have

$$\frac{x^2}{2}+\frac{3^4}{x^3}=\frac{x^2}{6}+\frac{x^2}{6}+\frac{x^2}{6}+\frac{3^4}{2x^3}+\frac{3^4}{2x^3}\geq 5\sqrt[5]{\left(\frac{x^2}{6}\right)^3\left(\frac{3^4}{2x^3}\right)^2}=\frac{15}{2},$$

and the equality holds when $\dfrac{x^2}{6}=\dfrac{3^4}{2x^3}$. This is when $x=3$.

By applying AM-GM inequality we have

$$\frac{(x+y)^2}{2}+\frac{2^6}{x+y}=\frac{(x+y)^2}{2}+\frac{2^5}{x+y}+\frac{2^5}{x+y}\geq 3\sqrt[3]{\frac{(x+y)^2}{2}\left(\frac{2^5}{x+y}\right)^2}=24,$$

and the equality holds when $\dfrac{(x+y)^2}{2}=\dfrac{2^5}{x+y}$. This is when $x+y=4$.

Therefore, the minimum of

$$x^2+xy+\frac{y^2}{2}+\frac{2^6}{x+y}+\frac{3^4}{x^3}$$

is

$$\frac{15}{2}+24=\frac{63}{2},$$

and it is reached when $x=3$ and $y=1$.

3. Let ABC be a triangle and I its incenter. The line AI intersects the side BC at D and the perpendicular bisector of BC at E. Let J be the incenter of triangle CDE. Prove that triangle CIJ is isosceles.

Solutions. It is well known that the bisector of the angle at vertex A and the perpendicular bisector of side BC intersect on the circumcircle of triangle ABC. We give here two solutions.

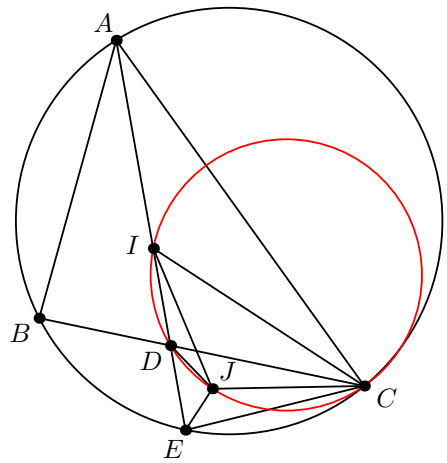
Solution 1. We have

$$\begin{aligned}\measuredangle JDI &= 180^\circ-\measuredangle EDJ=180^\circ-\frac{1}{2}\measuredangle ADB=180^\circ-\frac{1}{2}\left(\measuredangle DAC+\measuredangle ACB\right)\\ &=180^\circ-\left(\frac{1}{4}\measuredangle BAC+\frac{1}{2}\measuredangle ACB\right).\end{aligned}$$

On the other hand

$$\measuredangle ICJ = \frac{1}{2}\measuredangle ACB + \frac{1}{2}\measuredangle BCE = \frac{1}{2}\measuredangle ACB + \frac{1}{2}\measuredangle BAE = \frac{1}{2}\measuredangle ACB + \frac{1}{4}\measuredangle BAC.$$

Therefore, quadrilateral $CIDJ$ is cyclic.



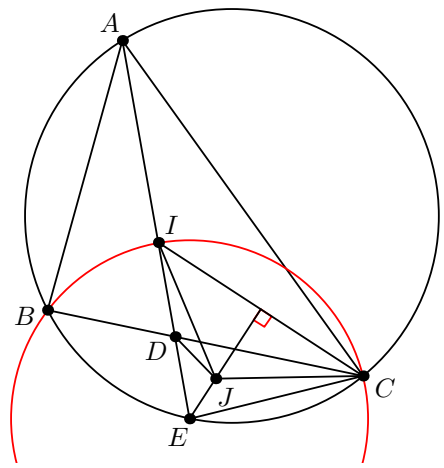
We deduce that

$$\measuredangle JIC = \measuredangle JDC = \measuredangle EDJ = \measuredangle ICJ,$$

and triangle CIJ is isosceles.

Solution 2. It is well known that $EC = EI$. This is because

$$\measuredangle ICE = \measuredangle ICD + \measuredangle DCE = \measuredangle ACI + \measuredangle BAE = \measuredangle ACI + \measuredangle IAC = \measuredangle EIC.$$



Therefore, the bisector of angle $\angle CEI$ is the perpendicular bisector of segment CI. But J is on this bisector. Hence, $JC = JI$ and triangle CIJ is isosceles.

4. Prove that there exists a positive integer n such that the last digits of n^3 are $\cdots\square 201320132013$.

Solution. Notice that $a = 201320132013$ and 10^{12} are coprime. By Euler's theorem

$$a^{\phi\left(10^{12}\right)} \equiv 1 \mod \left(10^{12}\right).$$

Let $b = a^{\phi\left(10^{12}\right)-1}$. We have $ab \equiv 1 \mod \left(10^{12}\right)$. Because b and 10^{12} are coprime, again by Euler's theorem

$$a \equiv ab^{\phi\left(10^{12}\right)} \equiv b^{\phi\left(10^{12}\right)-1} \mod \left(10^{12}\right).$$

But

$$\phi\left(10^{12}\right) - 1 = 2^{13} \times 5^{11} - 1 \equiv (-1)^{13} \times (-1)^{11} - 1 \equiv 0 \mod (3).$$

Therefore, there exists a positive integer m such that $\phi\left(10^{12}\right) - 1 = 3m$.

Let $n = b^m$. We have

$$n^3 = b^{\phi\left(10^{12}\right)-1} \equiv 201320132013 \mod \left(10^{12}\right).$$

1.4 Solutions to the preselection test of day IV

1. Let p be a prime number and $n \geq 2$ a positive integer, such that $p \mid (n^6 - 1)$. Prove that $n > \sqrt{p} - 1$.

Solution. Because p is prime and divides

$$n^6 - 1 = (n-1)(n+1)(n^2 - n + 1)(n^2 + n + 1),$$

it divides at least one of these positive factors. The prime number p is therefore less or equal to at least one of these factors. Because

$$n - 1 < n + 1 \leq (n-1)n + 1 = n^2 - n + 1 < n^2 + n + 1 < (n+1)^2,$$

we have $p < (n+1)^2$ and therefore $\sqrt{p} - 1 < n$.

2. Given $x \geq 0$, prove that

$$\frac{(x^2+1)^6}{2^7} + \frac{1}{2} \geq x^5 - x^3 + x.$$

Solution. Because $x^2 + 1 \geq 2x$, we have

$$\frac{(x^2+1)^7}{2^7} \geq \frac{(2x)^7}{2^7} = x^7, \quad \text{and} \quad \frac{x^2+1}{2} \geq x,$$

and deduce

$$(x^2+1)\cdot\left(\frac{(x^2+1)^6}{2^7} + \frac{1}{2}\right) = \frac{(x^2+1)^7}{2^7} + \frac{x^2+1}{2} \geq x^7 + x = x(x^2+1)(x^4-x^2+1).$$

Therefore

$$\frac{(x^2+1)^6}{2^7} + \frac{1}{2} \geq x^5 - x^3 + x,$$

and the equality holds when $x = 1$.

3. Fatima and Asma are playing the following game. First, Fatima chooses 2013 pairwise different numbers, called a_1, a_2, ..., a_{2013}. Then, Asma tries to know the value of each number a_1, a_2, ..., a_{2013}. At each time, Asma chooses $1 \leq i < j \leq 2013$ and asks Fatima "*What is the set* $\{a_i, a_j\}$*?*"(For example, if Asma asks what is the set $\{a_1, a_2\}$, and $a_1 = 17$ and $a_2 = 13$, Fatima will answer $\{13, 17\}$). Find the least number of questions Asma needs to ask, to know the value of all the numbers a_1, a_2, ..., a_{2013}.

Solution. Let n be the number of different questions Asma asks to know the values of each number a_1, a_2, ..., a_{2013}. Let us make the following remarks:

(a) Because Asma needs to know the value of each number a_i, each number a_i must appear at least in one of the sets that Asma chooses to ask about.

(b) If Asma chooses a set $\{a_{i_1}, a_{i_2}\}$, she needs to choose another set that contains only one of the numbers a_{i_1}, a_{i_2}, so that she can distinguish between the values of a_{i_1} and a_{i_2}.

(c) If Asma asks about the sets $\{a_{i_1}, a_{i_2}\}$ and $\{a_{i_2}, a_{i_3}\}$, for different indices i_1, i_2, i_3, she will deduce the value of each number $a_{i_1}, a_{i_2}, a_{i_3}$, by taking the intersection of the two sets and their differences.

(d) If for some $k \geq 3$, Asma asks about the sets $\{a_{i_1}, a_{i_2}\}$, $\{a_{i_2}, a_{i_3}\}$, $\ldots$, $\{a_{i_{k-1}}, a_{i_k}\}$, for different indices $i_1, i_2, \ldots, i_k$, she will deduce the value of each number $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ and does not need to ask after this about the set $\{a_{i_k}, a_{i_1}\}$ since she already knows the answer.

From now on, we assume, without loss of generality, that Asma never asks about a set that she already knows.

According to these remarks, we can make a partition of the set $\{a_1, a_2, \ldots, a_{2013}\}$ into non empty disjoint subsets $A_1, A_2, \ldots, A_m$ such that two numbers a_i, a_j are in the same subset A_k if and only if there exist sets $\{a_{i_1}, a_{i_2}\}$, $\{a_{i_2}, a_{i_3}\}$, $\ldots$, $\{a_{i_{l-1}}, a_{i_l}\}$ that Asma has asked about each one of them, and such that $a_{i_1} = a_i$, $a_{i_l} = a_j$. Notice that if $a_{i_1}, a_{i_2}, \ldots, a_{i_l}$ are different, because of remark (d) they are unique.

For each A_i, there are exactly $\#A_i - 1$ sets in A_i that Asma has asked about. To see this, fix $a_{i_0} \in A_i$, and for each $a_j \in A_i$ with $a_j \neq a_{i_0}$, associate the unique set $\{a_{i_{l-1}}, a_{i_l}\}$ in the previous paragraph when $a_{i_1} = a_{i_0}$, $a_{i_l} = a_j$ and $a_{i_1}, a_{i_2}, \ldots, a_{i_l}$ are different. It is not hard to see from remark (d) that this map is a bijection between $A_i \setminus \{a_{i_0}\}$ and the set of subsets of A_i in the questions of Asma.

Because each set in the questions of Asma is contained exactly in one of the subsets $A_1, A_2, \ldots, A_m$, we have

$$n = \sum_{i=1}^{m} (\#A_i - 1) = 2013 - m.$$

Because each set A_i must contain at least 3 elements (remarks (a) and (b)),

$$m \leq \left\lfloor \frac{2013}{3} \right\rfloor = 671.$$

This proves that $n \geq 1342$.

Conversely, if Asma asks about the 1342 sets

$$\{a_{3i+1}, a_{3i+2}\}, \{a_{3i+2}, a_{3i+3}\}, \quad \text{for} \quad i = 0, 1, \ldots, 670,$$

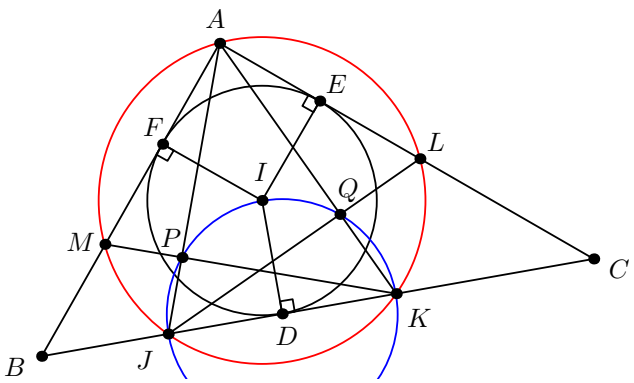
she will know the values of all the numbers a_1, a_2, $\ldots$, a_{2013}.

4. Let $\triangle ABC$ be an acute triangle, with $\widehat{A} > \widehat{B} \geq \widehat{C}$. Let D, E and F be the tangency points between the incircle of triangle and sides BC, CA, AB, respectively. Let J be a point on (BD), K a point on (DC), L a point on (EC) and M a point on (FB), such that

$$AF = FM = JD = DK = LE = EA.$$

Let P be the intersection point between AJ and KM and let Q be the intersection point between AK and JL. Prove that $PJKQ$ is cyclic.

Solution. Let I be the incenter of triangle ABC. Because $ID = IE = IF$, $AF = FM = JD = DK = LE = EA$, and the angles at D, E, and F are right, we have by pythagoras $IA = IM = IJ = IK = IL$. Therefore, the pentagon $AMJKL$ is cyclic.



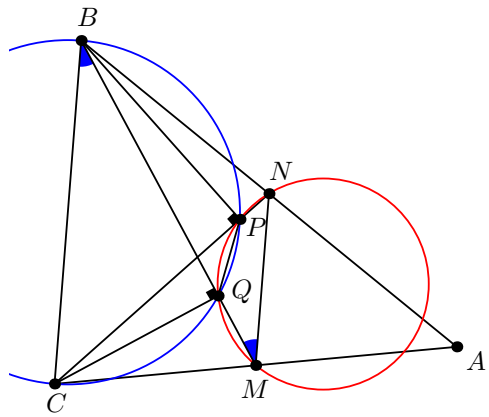
On the other hand, $AM = 2AF = 2EA = LA$. We deduce for the circumcircle of the pentagon $AMJKL$ that $\widehat{AM} = \widehat{LA}$. This implies that $\measuredangle QKP = \measuredangle QJP$, and therefore the quadrilateral $PJKQ$ is cyclic.

Chapter 2

Solutions to the team selection tests for the Gulf Mathematical Olympiads

2.1 Solutions to the team selection test of day I

1. Let ABC be a triangle with $\angle A < \angle B \leq \angle C$, M and N the midpoints of sides CA and AB, respectively, and P and Q the projections of B and C on the medians CN and BM, respectively. Prove that the quadrilateral $MNPQ$ is cyclic.



Solution. Because $\angle BPC = \angle BQC = 90°$, quadrilateral $BCQP$ is cyclic and therefore

$$\angle CPQ = \angle CBQ.$$

Because M and N are midpoints of sides AC and AB, segment MN is parallel to side BC. Hence $\angle NMQ = \angle CBQ$.

We deduce that $\angle NMQ = \angle CPQ$. This proves that quadrilateral $MNPQ$ is cyclic.

2. Let $p \geq 2$ be a prime number and $\dfrac{a_p}{b_p} = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{p^2-1}$, where a_p and b_p are two relatively prime positive integers. Compute $\gcd(p, b_p)$.

Solutions. For $p = 2$ we have $\dfrac{a_2}{b_2} = \dfrac{11}{6}$, and therefore $\gcd(2, b_2) = 2$.

For p an odd prime we propose two solutions to this problem:

First solution. We have

$$\frac{a_p}{b_p} = \frac{\displaystyle\sum_{k=1}^{p^2-1} \frac{(p^2-1)!}{k}}{(p^2-1)!}.$$

Because the only multiples of p appearing in $1 \times 2 \times \cdots \times (p^2-1)$ are $p, 2p, \ldots, (p-1)p$, and p is a prime number, p^{p-1} is the largest power of p dividing $(p^2-1)!$.

For $1 \leq k \leq p^2-1$, if k is not a multiple of p then p^{p-1} divides $\frac{(p^2-1)!}{k}$. If k is a multiple of p then p^{p-2} is the largest power of p dividing $\frac{(p^2-1)!}{k}$. Summing up the terms where k is a multiple of p we obtain

$$\sum_{j=1}^{p-1} \frac{(p^2-1)!}{jp} = \frac{(p^2-1)!}{p!} \sum_{j=1}^{p-1} \frac{(p-1)!}{j}.$$

Notice that all these fractions are integers. Using the fact that for each integer $1 \leq j \leq p-1$ there is a unique inverse integer $1 \leq \psi(j) \leq p-1$ modular p (i.e. $j\psi(j) \equiv 1 \bmod p$) and that this inverse map $j \to \psi(j)$ is a permutation of $\{1, 2, \ldots, p-1\}$, we get

$$\sum_{j=1}^{p-1} \frac{(p-1)!}{j} \equiv \sum_{j=1}^{p-1} \frac{(p-1)!}{j} j\psi(j) \equiv (p-1)! \sum_{j=1}^{p-1} \psi(j)$$

$$\equiv (p-1)! \sum_{j=1}^{p-1} j \equiv (p-1)! \frac{p(p-1)}{2} \equiv 0, \qquad (\text{mod } p).$$

This proves that p^{p-1} divides $\sum_{j=1}^{p-1} \frac{(p^2-1)!}{jp}$ and so it divides $\sum_{k=1}^{p^2-1} \frac{(p^2-1)!}{k}$. We deduce that p^{p-1} cancels from the denominator and therefore

$$gcd(p, b_p) = 1.$$

Second solution. Because p is odd, we have

$$\frac{a_p}{b_p} = \sum_{k=1}^{\frac{p^2-1}{2}} \left(\frac{1}{k} + \frac{1}{p^2-k} \right) = \sum_{k=1}^{\frac{p^2-1}{2}} \frac{p^2}{k(p^2-k)}.$$

For $1 \leq k \leq \frac{p^2-1}{2}$, if k is not a multiple of p then the denominator $k(p^2-k)$ is not divisible by p. If k is a multiple of p then

$$\frac{p^2}{k(p^2-k)} = \frac{1}{j(p-j)},$$

where $k = jp$ for some integer $1 \leq j \leq \frac{p-1}{2}$. But the denominator $j(p-j)$ is not divisible by p. Therefore $\frac{a_p}{b_p}$ is a sum of fractions where none of the denominators is divisible by p and, since $\frac{a_p}{b_p}$ is irreducible, then $\gcd(p, b_p) = 1$.

Remark. In fact, for $p \geq 5$ one can prove that p divides a_p.

Indeed, following the second solution, it is enough to prove that

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{(p-1)!}{j(p-j)} \equiv 0 \ \ (\text{mod } p).$$

Because $\psi(p-j) = p - \psi(j)$, for $1 \le j \le \frac{p-1}{2}$, we have

$$\begin{aligned}
2\sum_{j=1}^{\frac{p-1}{2}} \frac{(p-1)!}{j(p-j)} &\equiv 2\sum_{j=1}^{\frac{p-1}{2}} (p-1)!\psi(j)\psi(p-j) \equiv -2(p-1)!\sum_{j=1}^{\frac{p-1}{2}} \psi(j)^2 \\
&\equiv -(p-1)!\sum_{j=1}^{p-1} \psi(j)^2 \equiv -(p-1)!\sum_{j=1}^{p-1} j^2 \\
&\equiv -(p-1)!\frac{(p-1)p(2p-1)}{6} \equiv 0 \pmod{p}.
\end{aligned}$$

3. Turki has divided a square into finitely many white and green rectangles, each with sides parallel to the sides of the square. Within each white rectangle, he writes down its width divided by its height. Within each green rectangle, he writes down its height divided by its width. Finally, he calculates S, the sum of these numbers. If the total area of white rectangles equals the total area of green rectangles, determine the minimum possible value of S.

Solution. Let s be the sidelength of the original square, m and n the numbers of white and green rectangles $W_1, W_2, \ldots, W_m$ and $G_1, G_2, \ldots, G_n$, respectively, a_i and b_i the width and heigth of the white rectangle W_i, respectively, for $i = 1, 2, \ldots, m$, and c_j and d_j the width and heigth of the green rectangle G_j, respectively, for $j = 1, 2, \ldots, n$.

The total area of the white rectangles and the total area of the green rectangles satisfy

$$a_1b_1 + a_2b_2 + \cdots + a_mb_m = c_1d_1 + c_2d_2 + \cdots + c_nd_n = \frac{s^2}{2}.$$

We have

$$\begin{aligned}
S &= \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_m}{b_m} + \frac{d_1}{c_1} + \frac{d_2}{c_2} + \cdots + \frac{d_n}{c_n} \\
&= \frac{a_1^2}{a_1b_1} + \frac{a_2^2}{a_2b_2} + \cdots + \frac{a_m^2}{a_mb_m} + \frac{d_1^2}{c_1d_1} + \frac{d_2^2}{c_2d_2} + \cdots + \frac{d_n^2}{c_nd_n} \\
&\ge \frac{(a_1 + a_2 + \cdots + a_m)^2}{\frac{s^2}{2}} + \frac{(d_1 + d_2 + \cdots + d_n)^2}{\frac{s^2}{2}},
\end{aligned}$$

by Cauchy-Schwarz inequality.

Assume that $a_1 + a_2 + \cdots + a_m < \frac{s}{2}$. Because $b_i \leq s$, for $i = 1, 2, \ldots, m$, we have

$$\frac{s^2}{2} = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m \leq (a_1 + a_2 + \cdots + a_m)s < \frac{s^2}{2},$$

which is a contradiction. Therefore $a_1 + a_2 + \cdots + a_m \geq \frac{s}{2}$ and, similarly, $d_1 + d_2 + \cdots + d_n \geq \frac{s}{2}$.

Now assume that $a_1 + a_2 + \cdots + a_m < s$. There exists a vertical line L crossing the original square and not intersecting any of the white rectangles. Therefore, the sum of the heigths of the green rectangles intersecting L is greater than or equal to s. Hence, either $a_1 + a_2 + \cdots + a_m \geq s$ or $d_1 + d_2 + \cdots + d_n \geq s$.

We deduce that

$$S \geq \frac{2}{s^2}\left(s^2 + \left(\frac{s}{2}\right)^2\right) = \frac{5}{2}.$$

If Turki divides the square into two horizontal rectangles of the same area then $S = \frac{2}{1} + \frac{1}{2} = \frac{5}{2}$. Thus the minimum possible value of S is $\frac{5}{2}$.

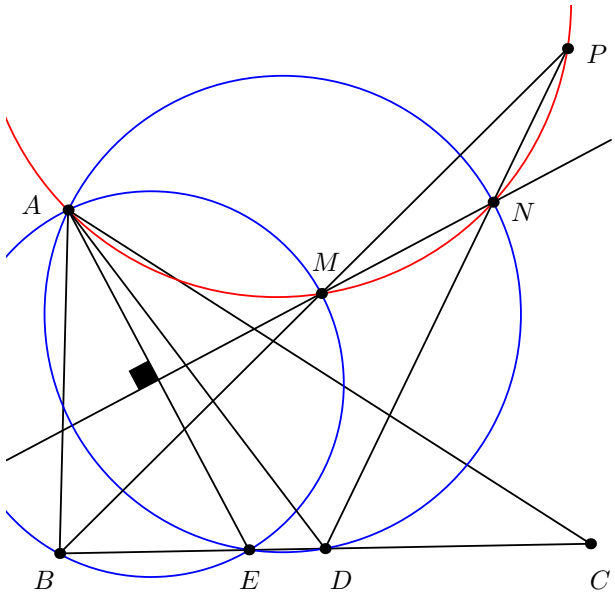
4. Let ABC be a triangle, D the midpoint of side BC and E the intersection point of the bisector of angle $\angle BAC$ with side BC. The perpendicular bisector of AE intersects the bisectors of angles $\angle CBA$ and $\angle CDA$ at M and N, respectively. The bisectors of angles $\angle CBA$ and $\angle CDA$ intersect at P. Prove that points A, M, N, P are concyclic.

Solution. The bisector of angle $\angle EBA$ and the perpendicular bisector of side EA, both bisect the arc $\widehat{EA}$ of the circumcircle of triangle ABE opposite to the vertex B. Therefore quadrilateral $ABEM$ is cyclic and hence

$$\angle EAM = \angle EBM.$$

The segment defined by the two intersection points of the perpendicular bisector of AE with the circumcircle of triangle AED is a diameter and intersects the bisector of the angle $\angle ADE$ on this circumcircle. But the bisectors of angles $\angle CDA$ and $\angle ADE$ are perpendicular. Therefore quadrilateral $AEDN$ is cyclic and hence

$$\angle CDN = \angle EAN.$$



We deduce that

$$\angle MAN = \angle EAN - \angle EAM = \angle CDN - \angle EBM = \angle MPN,$$

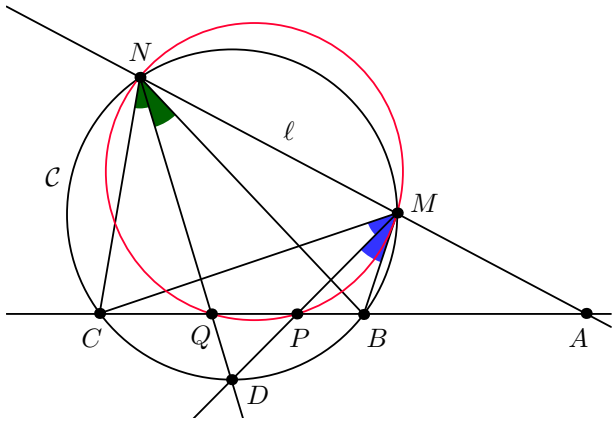
from triangle BDP. This proves that points A, M, N, P are concyclic.

Remark: This problem remains true if E is any point on segment BD different from D.

2.2 Solutions to the team selection test of day II

1. Let A, B, C be colinear points in this order, $\mathcal{C}$ an arbitrary circle passing through B and C, and l an arbitrary line different from BC, passing through A and intersecting $\mathcal{C}$ at M and N. The bisectors of the angles $\angle CMB$ and $\angle CNB$ intersect BC at P and Q. Prove that $AP \cdot AQ = AB \cdot AC$.

Solution. Let D be the midpoint of the arc $\widehat{BC}$ of the circle $\mathcal{C}$ opposite to both M, N. Because $\widehat{CD} = \widehat{DB}$ the bisectors of angles $\angle CMB$ and $\angle CNB$, both, intersect the arc $\widehat{CB}$ at D.



Therefore, we have

$$\angle NQC = \frac{1}{2}\left(\widehat{NC} + \widehat{DB}\right) = \frac{1}{2}\left(\widehat{NC} + \widehat{CD}\right) = \frac{1}{2}\widehat{ND} = \angle NMP.$$

This proves that quadrilateral $MNQP$ is cyclic. But quadrilateral $MNCB$ is also cyclic and lines MN, BC intersect at A, hence by the power of the point A with respect to these two circles we have

$$AP \cdot AQ = AM \cdot AN = AB \cdot AC.$$

2. Let p be a prime number. Prove that there exist infinitely many positive integers n such that p divides

$$1^n + 2^n + \cdots + (p+1)^n.$$

Solution. Let k be a positive integer. Using Fermat's little theorem we have

$$1^{k(p-1)}+2^{k(p-1)}+\cdots+(p+1)^{k(p-1)} \equiv \underbrace{1^k + 1^k + \cdots + 1^k}_{p-1 \text{ times}} + 0^k + 1^k \equiv 0 \pmod{p}.$$

Therefore, for $n = k(p-1)$, and $k = 1, 2, 3, \ldots$, the prime number p divides

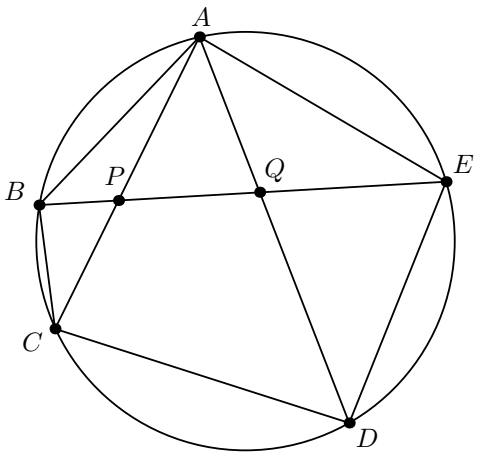
$$1^n + 2^n + \cdots + (p+1)^n.$$

3. Let $ABCDE$ be a cyclic pentagon such that the diagonals AC and AD intersect BE at P and Q, respectively, with $BP \cdot QE = PQ^2$. Prove that

$BC \cdot DE = CD \cdot PQ.$

Solution. Because triangles PBC and PAE are similar, and triangles QED and QAB are similar, we have

$$\frac{BC}{AE} = \frac{PB}{PA}, \quad \text{and} \quad \frac{DE}{BA} = \frac{EQ}{AQ}.$$



Therefore

$$\frac{BC \cdot DE}{AE \cdot BA} = \frac{PB \cdot EQ}{PA \cdot AQ} = \frac{PQ^2}{PA \cdot AQ}. \qquad (*)$$

On the other hand, applying sine law on triangles ABQ and APQ, we have

$$\frac{AB}{AQ} = \frac{\sin \angle AQB}{\sin \angle EBA}, \quad \text{and} \quad \frac{PQ}{AP} = \frac{\sin \angle PAQ}{\sin \angle AQP}.$$

Finally, by applying sine law on the circumcircle of the pentagon $ABCDE$, we get

$$\frac{AB}{AQ} \cdot \frac{PQ}{AP} = \frac{\sin \angle AQB}{\sin \angle EBA} \cdot \frac{\sin \angle PAQ}{\sin \angle AQP} = \frac{\sin \angle CAD}{\sin \angle EBA} = \frac{CD}{EA}. \qquad (**)$$

Combining the two relations $(*)$ and $(**)$, we obtain after cancellation

$$BC \cdot DE = CD \cdot PQ.$$

4. Let $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ be real numbers. Prove that

$$a_1a_2(a_1 - a_2) + a_2a_3(a_2 - a_3) + \cdots + a_{n-1}a_n(a_{n-1} - a_n) \geq a_1a_n(a_1 - a_n).$$

First solution. Assume that there exists $1 \leq i_0 < n$ such that $a_{i_0} = a_{i_0+1}$. The sum of the two terms in the left hand side $a_{i_0-1}a_{i_0}(a_{i_0-1} - a_{i_0}) + a_{i_0}a_{i_0+1}(a_{i_0} - a_{i_0+1})$ is equal to $a_{i_0-1}a_{i_0+1}(a_{i_0-1} - a_{i_0-1})$. This means that we can skip i_0 in the left hand side. This means that it is enough to prove this inequality when $a_1 > a_2 > \cdots > a_n > 0$. In this case we have

$$\begin{aligned}
&a_1a_2(a_1 - a_2) + a_2a_3(a_2 - a_3) + \cdots + a_{n-1}a_n(a_{n-1} - a_n) \\
&\quad = \frac{(a_1 - a_2)^2}{\frac{1}{a_1} - \frac{1}{a_2}} + \frac{(a_2 - a_3)^2}{\frac{1}{a_2} - \frac{1}{a_3}} + \cdots + \frac{(a_{n-1} - a_n)^2}{\frac{1}{a_{n-1}} - \frac{1}{a_n}} \\
&\quad \geq \frac{((a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n))^2}{\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \cdots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)} = a_1a_n(a_1 - a_n),
\end{aligned}$$

by Cauchy-Schwarz inequality.

The equality holds here when

$$\frac{a_1 - a_2}{\frac{1}{a_1} - \frac{1}{a_2}} = \frac{a_2 - a_3}{\frac{1}{a_2} - \frac{1}{a_3}} = \cdots = \frac{a_{n-1} - a_n}{\frac{1}{a_{n-1}} - \frac{1}{a_n}},$$

which is equivalent to

$$a_1a_2 = a_2a_3 = \cdots = a_{n-1}a_n.$$

This implies that $n = 2$, otherwise $a_1 = a_3$ which is rejected. Therefore, in the general case, the equality holds when at least $n-2$ of the $n-1$ inequalities $a_1 \geq a_2 \geq \cdots \geq a_n$ are in fact equalities.

Second solution. We prove this inequality by induction on $n \geq 2$. For $n = 2$, clearly both sides are the same. Assume this inequality true for n. Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq a_{n+1} > 0$. We have

$$\begin{aligned}
&a_1a_2(a_1 - a_2) + \cdots + a_{n-1}a_n(a_{n-1} - a_n) + a_na_{n+1}(a_n - a_{n+1}) \\
&\qquad \geq a_1a_n(a_1 - a_n) + a_na_{n+1}(a_n - a_{n+1}) \geq a_1a_{n+1}(a_1 - a_{n+1}),
\end{aligned}$$

since

$$a_1a_n(a_1 - a_n) + a_na_{n+1}(a_n - a_{n+1}) - a_1a_{n+1}(a_1 - a_{n+1})$$

$$= (a_1 - a_n)(a_n - a_{n+1})(a_1 - a_{n+1}) \geq 0.$$

The equality holds here when $a_1 = a_n$ or when $a_n = a_{n+1}$ and there is equality in the hypothesis of induction. Therefore, in the general case, the equality holds when at least $n-2$ of the $n-1$ inequalities $a_1 \geq a_2 \geq \cdots \geq a_n$ are in fact equalities.

2.3 Solutions to the team selection test of day III

1. Find all ordered triples (a, b, c) of positive integers which satisfy

$$5^a + 3^b - 2^c = 32.$$

Solution. By considering this equation modular 3, we get

$$2^a - 2^c \equiv 2 \pmod 3.$$

This occurs only when a is even and c is odd.

If $c = 1$, the equation becomes $5^a + 3^b = 34$, which is equivalent to $a = 2$ and $b = 1$.

If $c \neq 1$, then $c \geq 3$, since c is odd. By considering this equation modular 8, and using the fact that a is even, we get

$$1 + 3^b \equiv 0 \pmod 8,$$

which can never occur since 3^b is congruent to either 1 or 3 modular 8.

Hence, the unique solution to this equation is $(a, b, c) = (2, 1, 1)$.

2. Let $S = \{(a, b) \mid a, b = 1, 2, 3, 4, \text{ and } a \neq b\}$, and consider all nonzero polynomials $p(X, Y)$ with integer coefficients such that $p(a, b) = 0$ for every element (a, b) in S.

   (a) What is the minimal degree of such polynomial $p(X, Y)$?

(b) Determine all such polynomials $p(X, Y)$ with minimal degree.

Solution.

(a) Let $p(X, Y)$ be such a polynomial. There are two possible cases:

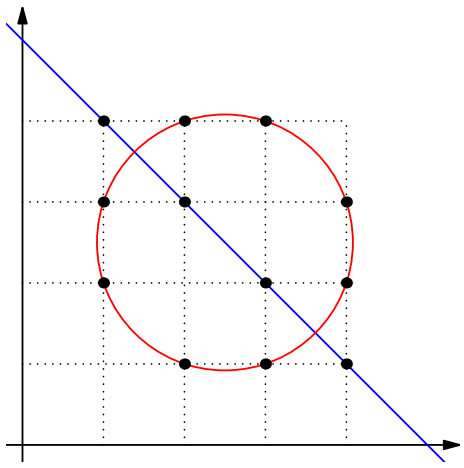
Case 1. If $p(1, Y) = p(2, Y) = p(3, Y) = p(4, Y) = 0$. In this case

$$(X - 1)(X - 2)(X - 3)(X - 4)$$

is a factor of $p(X, Y)$ and therefore $\deg p(X, Y) \geq 4$.
Case 2. If there exists $a_0 \in \{1, 2, 3, 4\}$ such that $p(a_0, Y) \neq 0$. Because $p(a_0, b) = 0$ for $b \in \{1, 2, 3, 4\} \setminus \{a_0\}$, the nonzero polynomial $p(a_0, Y)$ has 3 different roots and therefore $\deg p(X, Y) \geq \deg p(a_0, Y) \geq 3$.

Hence $\deg p(X, Y) \geq 3$.

To construct such a polynomial of degree 3, consider, as shown in the figure, the line of equation $x + y - 5 = 0$ and the circle of equation $(x - \frac{5}{2})^2 + (y - \frac{5}{2})^2 - \frac{5}{2} = 0$.



The graph obtained by union of this line and this circle passes by all the points of S. Therefore, the polynomial

$$p_0(X, Y) = (X + Y - 5)(X^2 + Y^2 - 5X - 5Y + 10)$$

satisfies the condition $p_0(a, b) = 0$ for all $(a, b) \in S$ and is of degree 3. Hence the minimal degree of such polynomials $p(X, Y)$ is 3.

(b) Let $p(X,Y)$ be a nonzero polynomial of degree 3 with integer coefficients such that $p(a,b)=0$ for every element (a,b) in S. From (a), there exists $a_0 \in \{1,2,3,4\}$ such that $p(a_0,Y) \neq 0$. Let $p_0(X,Y) = (X+Y-5)((2X-5)^2+(2Y-5)^2-10)$ and define the polynomial

$$q(X,Y) = p_0(a_0,a_0)p(X,Y) - p(a_0,a_0)p_0(X,Y).$$

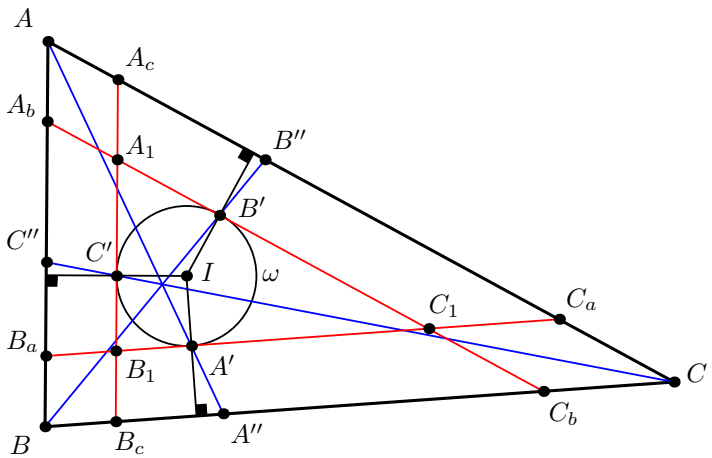
Polynomial $q(X,Y)$ is of degree at most 3. But polynomial $q(a_0,Y)$ has at least $1,2,3,4$ as roots. Therefore $q(a_0,Y)=0$, which means that $q(X,Y)=(X-a_0)q_1(X,Y)$ for some polynomial $q_1(X,Y)$ of degree at most 2. But for any $a \in \{1,2,3,4\} \setminus \{a_0\}$, $q_1(a,Y)$ has at least 3 roots, which implies that $q_1(a,Y)=0$. Hence $(X-1)(X-2)(X-3)(X-4)$ divides $q(X,Y)$ of degree 3. Therefore $q(X,Y)=0$. Because $p_0(X,Y)$ is a unitary polynomial, we deduce that

$$p(X,Y) = kp_0(X,Y), \quad \text{for some } k \in \mathbb{Z}.$$

3. Let ABC be a triangle, I its incenter, and ω a circle of center I. Points A', B', C' are on ω such that rays IA', IB', IC' starting from I intersect perpendicularly sides BC, CA, AB, respectively. Prove that lines AA', BB', CC' are concurrent.

First solution. Define A'', B'', C'' to be the intersection points of the lines AA', BB', CC' with the sides BC, CA, AB, respectively, B_a, C_a the intersection points of the tangent line to ω at A' with the lines AB, AC, respectively, C_b, A_b the intersection points of the tangent line to ω at B' with the lines BC, BA, respectively, A_c, B_c the intersection points of the tangent line to ω at C' with the lines CA, CB, respectively, A_1, B_1, C_1 the intersection points of the three tangent lines to ω at A', B', C', as shown in the figure, r the radius of ω and r_0 the inradius of triangle ABC.

The distance from A_1 to each line AB, AC is equal to $|r-r_0|$. Therefore, A_1 is on the bisector of angle $\angle BAC$. But $AA_bA_1A_c$ is a parallelogram. We deduce that $AA_bA_1A_c$ is a rhombus and therefore $A_1A_b = A_1A_c$. On the other hand, we have $A_1B' = A_1C'$ as tangents to ω from A_1. We deduce that $A_bB' = A_cC'$. We deduce in a similar way that $B_cC' = B_aA'$ and $C_aA' = C_bB'$.



Because lines B_aC_a and BC are parallel, we have from Thales $\dfrac{BA''}{A''C} = \dfrac{B_aA'}{A'C_a}$. We have similarly $\dfrac{CB''}{B''A} = \dfrac{C_bB'}{B'A_b}$, and $\dfrac{AC''}{C''B} = \dfrac{A_cC'}{C'B_c}$. Therefore

$$\frac{BA''}{A''C} \cdot \frac{CB''}{B''A} \cdot \frac{AC''}{C''B} = \frac{B_aA'}{A'C_a} \cdot \frac{C_bB'}{B'A_b} \cdot \frac{A_cC'}{C'B_c} = 1,$$

and by Ceva's theorem, lines AA', BB', CC' are concurrent.

Second solution. Because IA' and IC' are prependicular to BC and AB, respectively, at the intouch points of the incircle of ABC, and $IA' = IC'$, we have $\angle CBA' = \angle C'BA$ and $\angle A'BA = \angle CBC'$ as oriented angles. We have in a similar way $\angle BAC' = \angle B'AC$, $\angle C'AC = \angle BAB'$, $\angle ACB' = \angle A'CB$ and $\angle B'CB = \angle ACA'$ as oriented angles.

Because AA', BA', CA' intersect in a point A', we have from the trigonometric form of Ceva's theorem

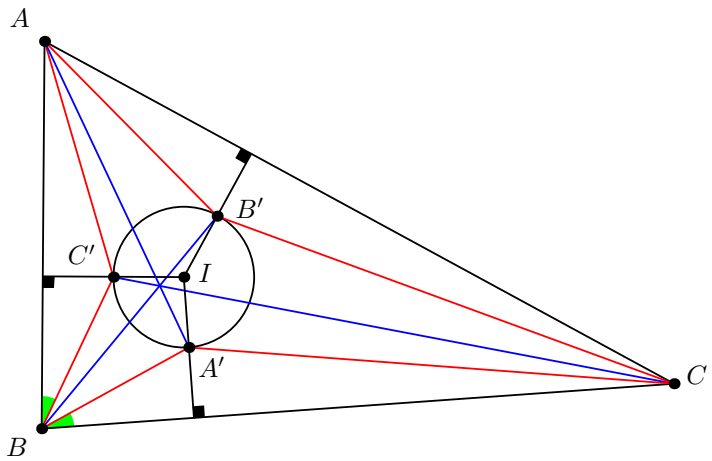
$$\frac{\sin\angle BAA'}{\sin\angle A'AC} = \frac{\sin\angle A'BA}{\sin\angle CBA'} \cdot \frac{\sin\angle A'CB}{\sin\angle ACA'}.$$

Similarly, we have

$$\frac{\sin\angle CBB'}{\sin\angle B'BA} = \frac{\sin\angle B'CB}{\sin\angle ACB'} \cdot \frac{\sin\angle B'AC}{\sin\angle BAB'},$$

and

$$\frac{\sin\angle ACC'}{\sin\angle C'CB} = \frac{\sin\angle C'AC}{\sin\angle BAC'} \cdot \frac{\sin\angle C'BA}{\sin\angle CBC'}.$$



After multiplication and cancellation, we obtain

$$\frac{\sin\angle BAA'}{\sin\angle A'AC}\cdot\frac{\sin\angle CBB'}{\sin\angle B'BA}\cdot\frac{\sin\angle ACC'}{\sin\angle C'CB}=1.$$

We deduce from the trigonometric form of Ceva's theorem that AA', BB', CC' are concurrent.

4. Let X be a set of rational numbers satisfying the following two conditions:

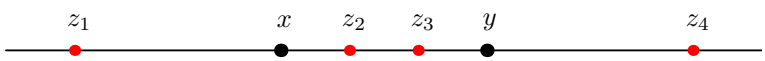
   (a) The set X contains at least two elements,

   (b) For any x, y in X, if $x \neq y$ then there exists z in X such that either $\left|\frac{x-z}{y-z}\right| = 2$ or $\left|\frac{y-z}{x-z}\right| = 2$.

   Prove that X contains infinitely many elements.

Solution. The condition (b) tells that if $x \neq y$ are in X, then at least one of the four numbers

$$z_1 = x-(y-x),\quad z_2 = x+\frac{1}{3}(y-x),\quad z_3 = x+\frac{2}{3}(y-x),\quad z_4 = x+2(y-x),$$



is in X.

Assume that X is finite and let x_0 be the smallest element of X. By condition (a), the set $X \setminus \{x_0\}$ is nonempty. Consider a map

$$f : X \setminus \{x_0\} \to X \setminus \{x_0\}$$

which satisfies, for $y \in X \setminus \{x_0\}$, the condition

$$f(y) \in \left\{x_0 + \frac{1}{3}(y - x_0), x_0 + \frac{2}{3}(y - x_0), x_0 + 2(y - x_0)\right\} \cap X.$$

Notice that $f(y)$ is defined for all $y \in X \setminus \{x_0\}$ by condition (b) since $x_0 - (y - x_0) < x_0$ is not in X.

Now, fix an element $y_0 \in X \setminus \{x_0\}$ and define by induction $a_0 = y_0 - x_0$ and $a_{n+1} = f(x_0 + a_n) - x_0$ for all $n \in \mathbb{N}_0$. Clearly, the sequence (a_n) is well defined and positive and we have, for each $n \in \mathbb{N}_0$, either

$$a_{n+1} = \frac{1}{3}a_n, \quad \text{or} \quad a_{n+1} = \frac{2}{3}a_n, \quad \text{or} \quad a_{n+1} = 2a_n.$$

Therefore, for each positive integers m, n, there exist non-negative integers p, q with $p + q > 0$ such that

$$a_{m+n} = \frac{2^p}{3^q}a_m.$$

Because $X \setminus \{x_0\}$ is finite, there exist two positive integers m_0, n_0 such that $a_{m_0+n_0} = a_{m_0}$. Hence the corresponding two non-negative integers p_0, q_0 with $p_0 + q_0 > 0$ satisfy

$$1 = \frac{2^{p_o}}{3^{q_0}},$$

which is impossible.

This proves that X contains infinitely many elements.

Chapter 3

Solutions to the team selection tests for the Balkan Mathematical Olympiads

3.1 Solutions to the team selection test of day I

1. A positive proper divisor is a positive divisor of a number, excluding itself. For positive integers $n \geq 2$, let $f(n)$ denote the number that is one more than the largest proper divisor of n. Determine all positive integers n such that $f(f(n)) = 2$.

   Solution. Let $n \geq 2$ such that $f(f(n)) = 2$. The largest proper divisor of $f(n)$ is 1 if and only if $f(n)$ is a prime number.

   If $f(n) = 2$, this is equivalent to n being a prime number.

   If $f(n) = p \geq 3$, then $p-1$, the largest proper divisor of n is an even number. This means that n is even and therefore, its largest proper divisor is $\frac{n}{2}$. This is equivalent to $n = 2(p-1)$.

   Hence, the positive integers n such that $f(f(n)) = 2$ are prime numbers or numbers $n = 2(p-1)$ for any prime number p.

2. Prove that among any 16 perfect cubes we can always find two cubes whose difference is divisible by 91.

Solution. Notice first that any perfect cube is congruent to either 0, 1 or 6 modulo 7 and it is congruent to either 0, 1, 5, 8, or 12 modulo 13. Because 7 and 13 are relatively prime numbers, by the Chinese Reminder theorem, a perfect cube is congruent to precisely one of $15 = 3 \times 5$ different residues modulo $91 = 7 \times 13$. Therefore, by the Pigeonhole principle, we can always find among any 16 perfect cubes, two cubes which are congruent modulo 91.

3. Let $n \geq 2$ be a positive integer, and write in a digit form

$$\frac{1}{n} = 0.a_1a_2\ldots.$$

Suppose that $n = a_1 + a_2 + \cdots$. Determine all possible values of n.

Solution. Let $n \geq 2$ be such an integer. Because $n = a_1 + a_2 + \cdots$, there exists a positive integer $k > 0$ such that $a_k \neq 0$ and $a_i = 0$ for all integers $i > k$.
Because $\dfrac{10^k}{n} = \overline{a_1a_2\cdots a_k}$ is an integer and $\dfrac{10^{k-1}}{n} = \overline{a_1a_2\cdots a_{k-1}, a_k}$ is not an integer, there exists two non-negative integers $a, b \geq 0$ such that $n = 2^a \cdot 5^b$, with $\max\{a, b\} = k$. We have three cases:

(a) If $k = a = b$, then $n = 10^k \geq 10$ and $a_1 + a_2 + \cdots + a_k = 1 \neq n$, contradiction.

(b) If $k = b > a$, then

$$5^k \leq 2^a \cdot 5^k = n = a_1 + a_2 + \cdots + a_k \leq 9k.$$

For $k = 2$, we have $5^2 = 25 > 18 = 9 \times 2$. If $k \geq 2$ and $5^k > 9k$, then $5^{k+1} > 5 \times 9k = 9(k + 4k) > 9(k + 1)$. This implies that $k = 1$ and therefore $a = 0$, that is $n = 5$. But $\frac{1}{n} = 0.2$ and $n \neq 2$, contradiction.

(c) If $k = a > b$, then

$$2^k \leq 2^k \cdot 5^b = n = a_1 + a_2 + \cdots + a_k \leq 9k.$$

For $k = 6$, we have $2^6 = 64 > 54 = 9 \times 6$. If $k \geq 6$ and $2^k > 9k$, then $2^{k+1} = 2^k + 2^k > 9k + 2^6 > 9(k + 1)$. This implies that $k \leq 5$.
Now, assume that $b \geq 1$. We have

$$10 \times 2^{k-1} \leq n = a_1 + a_2 + \cdots + a_k \leq 9k.$$

For $k = 1$, this is not satisfied. Assume that $10 \times 2^{k-1} > 9k$. We have $10 \times 2^k > 9k + 10 \times 2^{k-1} \geq 9k + 10 > 9(k+1)$, contradiction.
Hence, $b = 0$ and $n = 2^k$.
By checking for $k = 1, 2, 3, 4, 5$ we find that only $n = 2^3$ has this property.

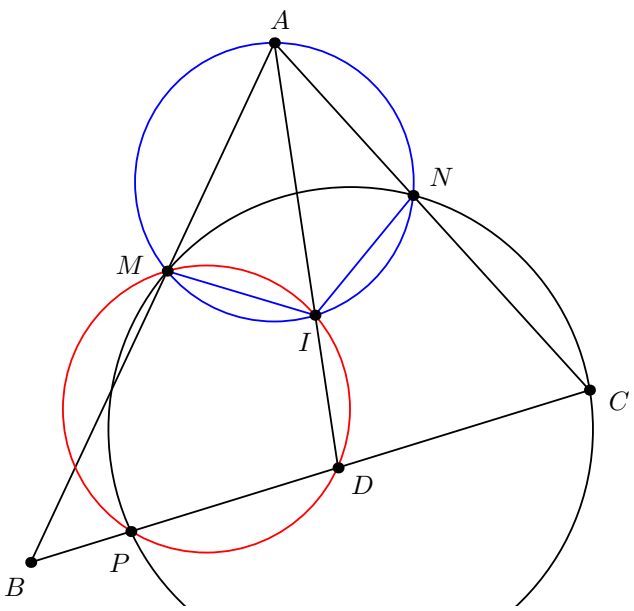
Hence $n = 8$.

4. Let ABC be a triangle with $\angle B \leq \angle C$, I its incenter and D the intersection point of line AI with side BC. Let M and N be points on sides BA and CA, respectively, such that $BM = BD$ and $CN = CD$. The circumcircle of triangle CMN intersects again line BC at P. Prove that quadrilateral $DIMP$ is cyclic.

Solution. By the bisector theorem we have

$$\frac{MB}{AB} = \frac{DB}{AB} = \frac{DC}{AC} = \frac{NC}{AC}.$$

We deduce that segments MN and BC are parallel.



Because $CNMP$ is a cyclic trapezoid, it is isosceles. Hence

$$\angle DPM = \angle ACB.$$

Because $BD = BM$ and I is on the bisector of $\angle DBM$, the quadrilateral $BDIM$ is a kite. We deduce that

$$\begin{aligned}\angle MID &= 360^\circ - \angle DBM - 2\angle ADB \\ &= 360^\circ - \angle CBA - 2\left(\angle ACB + \tfrac{1}{2}\angle BAC\right) \\ &= 180^\circ - \angle ACB = 180^\circ - \angle DPM.\end{aligned}$$

This proves that quadrilateral $DIMP$ is cyclic.

5. Find all positive integers n such that

$$3^n + 4^n + \cdots + (n+2)^n = (n+3)^n.$$

Solution. Notice first that

$$3^2 + 4^2 = 5^2, \quad \text{and} \quad 3^3 + 4^3 + 5^3 = 6^3.$$

However

$$3^4 + 4^4 + 5^4 + 6^4 = 2258 < 2401 = 7^4.$$

Assume that

$$3^n + 4^n + \cdots + (n+2)^n < (n+3)^n$$

for some $n \geq 4$. We have

$$\begin{aligned}3^{n+1} + 4^{n+1} + \quad & \cdots + (n+2)^{n+1} + (n+3)^{n+1} \\ & < (n+2)\left(3^n + 4^n + \cdots + (n+2)^n\right) + (n+3)^{n+1} \\ & < (n+2)(n+3)^n + (n+3)^{n+1} < 2(n+3)^{n+1}.\end{aligned}$$

For $n = 4$, we have $(n+2)(n+3)^n + (n+3)^{n+1} = 31213 < 32768 = (n+4)^{n+1}$. For $n \geq 5$, we have

$$\begin{aligned}\left(\frac{n+4}{n+3}\right)^{n+1} &= \left(1 + \frac{1}{n+3}\right)^{n+1} > 1 + \frac{n+1}{n+3} + \frac{n(n+1)}{2(n+3)^2} \\ &> 2 + \frac{(n-5)(n+2)+2}{2(n+3)^2} > 2.\end{aligned}$$

Therefore, $3^{n+1} + 4^{n+1} + \cdots + (n+2)^{n+1} + (n+3)^{n+1} < (n+4)^{n+1}$. Hence, we have proved by induction that $3^n + 4^n + \cdots + (n+2)^n < (n+3)^n$ for all $n \geq 4$.

3.2 Solutions to the team selection test of day II

1. Find the minimum of $\sum_{k=0}^{40}\left(x+\frac{k}{2}\right)^2$ where x is a real number.

   Solution. We have

$$\begin{aligned}\sum_{k=0}^{40}\left(x+\frac{k}{2}\right)^2 &= (x+10)^2+\sum_{k=1}^{20}\left(\left((x+10)+\frac{k}{2}\right)^2+\left((x+10)-\frac{k}{2}\right)^2\right)\\ &= (x+10)^2+2\sum_{k=1}^{20}\left((x+10)^2+\left(\frac{k}{2}\right)^2\right)\\ &= 41(x+10)^2+\frac{1}{2}\sum_{k=1}^{20}k^2 = 41(x+10)^2+\frac{20\cdot 21\cdot 41}{2\cdot 6}\\ &= 41(x+10)^2+1435.\end{aligned}$$

   Therefore, the minimum value is 1435, the current year in the islamic calendar, which is obtained when $x=-10$.

2. Let $\mathbb{N}$ denote the set of positive integers, and let S be a set. There exists a function $f:\mathbb{N}\to S$ such that if x and y are a pair of positive integers with their difference being a prime number, then $f(x)\neq f(y)$. Determine the minimum number of elements in S.

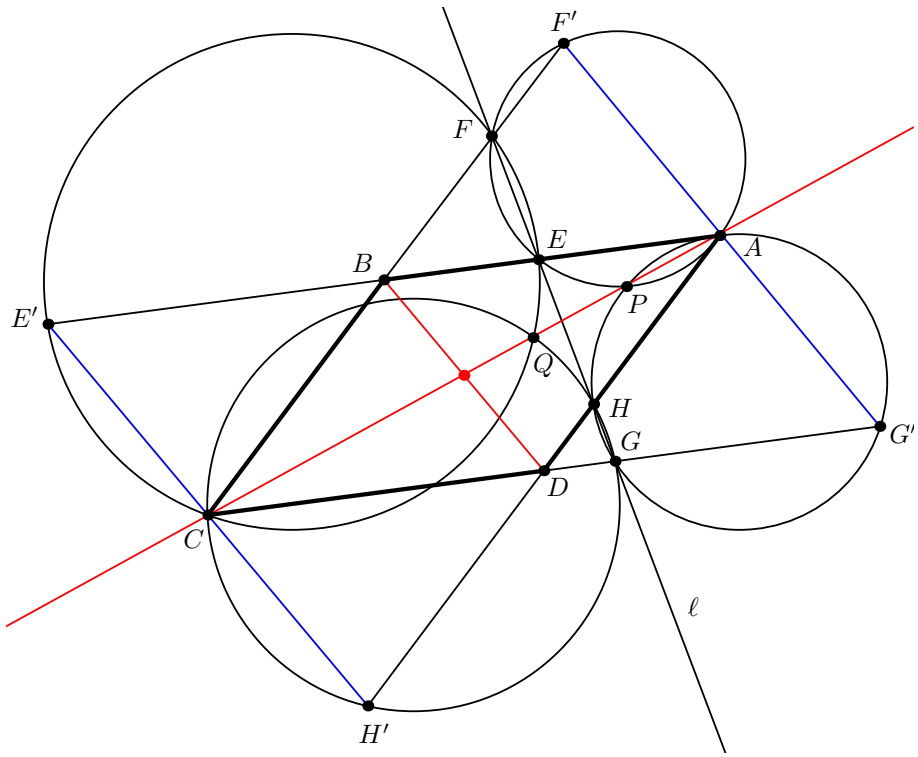
   Solution. Let $f:\mathbb{N}\to S$ be such a function. Because the difference of any two numbers in $\{1,3,6,8\}$ is a prime number, the cardinality of $\{f(1),f(3),f(6),f(8)\}$ is 4. Hence, the minimum number of elements in S is greater than or equal to 4.

   Now, consider the function $f:\mathbb{N}\to\{0,1,2,3\}$ that associates to each $n\in\mathbb{N}$ its remainder when divided by 4. This function satisfies the condition of the problem since when $f(n_1)=f(n_2)$, for $n_1,n_2\in\mathbb{N}$, then n_1-n_2 is not a prime number since it is a multiple of 4. Therefore, the minimum number of elements in S is 4.

3. Let $ABCD$ be a parallelogram. A line ℓ intersects lines AB,BC,CD,DA at four different points E,F,G,H, respectively. The circumcircles of triangles

AEF and AGH intersect again at P. The circumcircles of triangles CEF and CGH intersect again at Q. Prove that the line PQ bisects the diagonal BD.

Solution. Let F' be the second intersection point of line BC with the circumcircle of triangle AEF and G' the second intersection point of line CD with the circumcircle of triangle AGH.



Because $AF'FE$ is cyclic, we have $\angle F'AB = \angle CFG$. Because $AHGG'$ is cyclic, we have $\angle DAG' = \angle FGC$. Because $ABCD$ is a parallelogram, we have $\angle BAD = \angle FCG$. We deduce that

$$\angle F'AB + \angle BAD + \angle DAG' = \angle CFG + \angle FCG + \angle FGC = 180^\circ.$$

Hence, points F', A, G' are collinear.

Because $AF'FE$ is cyclic and lines AB and CD are parallel, we have

$$\angle CF'G' = \angle FEB = \angle FGC.$$

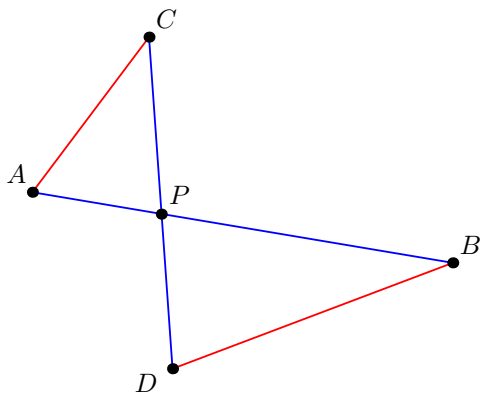
This proves that quadrilateral $FGG'F'$ is cyclic and therefore $CG \cdot CG' = CF \cdot CF'$. This proves that C is on the radical axis of the circumcircles of triangles AFE and AFG. Hence, points A, P, C are collinear.

We prove in a similar way that points A, Q, C are collinear. Thus, line PQ is equal to line AC which bisects the diagonal BD of the parallelogram $ABCD$.

4. Let n be an integer greater than 2. Consider a set of n different points, with no three collinear, in the plane. Prove that we can label the points $P_1, P_2, \ldots, P_n$ such that $P_1P_2 \ldots P_n$ is not a self-intersecting polygon. (*A polygon is self-intersecting if one of its side intersects the interior of another side. The polygon is not necessarily convex*)

First solution. We start first by proving the following easy lemma:

> If A, B, C, D are four points in the plane no three of them are collinear and if segments AB and CD intersect, than $AB + CD > AC + BD$.



Indeed, let P be the intersection point of AB with CD. We have, by triangular inequality,

$$AB + CD = AP + PC + BP + PD > AC + BD.$$

Now consider a set of n different points, with no three collinear, in the plane. Let $\mathcal{P}$ be the set of all possible labelling $P_1, P_2, \ldots, P_n$ of these points. Define

for each labelling $P_1, P_2, \ldots, P_n$,

$$S(P_1, P_2, \ldots, P_n) = P_1P_2 + P_2P_3 + \cdots + P_{n-1}P_n + P_nP_1.$$

Because the set $\mathcal{P}$ is finite, there exists a labelling $P_1, P_2, \ldots, P_n$ for which $S(P_1, P_2, \ldots, P_n)$ is minimum. For this labelling, $P_1P_2 \ldots P_n$ is not a self-intersecting polygon.

Indeed, assume that there exist four different points $P_i, P_{i+1}, P_j, P_{j+1}$, with $1 \leq i < j \leq n$ and $P_{n+1} = P_1$, such that segments P_iP_{i+1} and P_jP_{j+1} intersect. By the lemma, we have

$$S(P_1, P_2, \ldots, P_n) > S(P_1, \ldots, P_i, P_j, P_{j-1}, \ldots, P_{i+1}, P_{j+1}, P_{j+2}, \ldots, P_n),$$

which contradicts the minimality of $S(P_1, P_2, \ldots, P_n)$.

Second solution. We fix for the plane an oriented Cartesian coordinate system and order its points by ordering their coordinates with the colexicographic order relation. This is defined by

$$(x_1, y_1) <_c (x_2, y_2) \text{ if } y_1 < y_2, \text{ or if } y_1 = y_2 \text{ and } x_1 < x_2,$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

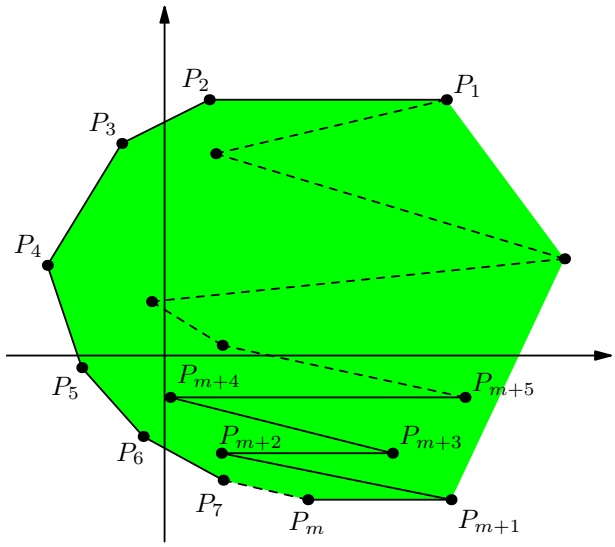
Clearly, if two points A, B of the plane satisfy $A <_c B$ and C is on the interior of the segment AB, then $A <_c C <_c B$. Therefore, if $A <_c B <_c C <_c D$ are four points in the plane, the segments AB and CD do not intersect.

Now consider a set X of n different points, with no three collinear, in the plane. Consider the convex hull of X. This is the intersection of all half planes containing the points of X. Clearly, if A, B, C, D are four points in X and segment AB lies on the border of the convex hull of X, then segments AB and CD do not intersect.

Notice that both maximal and minimal points of X for the colexicographic order are on the border of the convex hull.

Start from the maximal point and label it P_1. Then move on the border of the convex hull in the counterclockwise direction and label the consecutive

points in this motion $P_2, P_3, \ldots$, respectively, until you arrive to the minimal point labeled P_m. The remaining points will be labeled according to their colexicographic order so that $P_m <_c P_{m+1} <_c P_{m+2} <_c \cdots <_c P_n <_c P_1$, as shown in the figure.



For this labeling, we deduce from the previous remarks that $P_1P_2 \ldots P_n$ is not a self-intersecting polygon.

5. Let $n > 3$ be an odd positive integer not divisible by 3. Determine if it is possible to form an $n \times n$ array of numbers such that

   (a) the set of the numbers in each row is a permutation of $0, 1, \ldots, n-1$; the set of the numbers in each column is a permutation of $0, 1, \ldots, n-1$;

   (b) the board is *totally non-symmetric*: for $1 \leq i < j \leq n$ and $1 \leq i' < j' \leq n$, if $(i, j) \neq (i', j')$ then $(a_{i,j}, a_{j,i}) \neq (a_{i',j'}, a_{j',i'})$, where $a_{i,j}$ denotes the entry in the ith row and j^{th} column.

Solution. For $1 \leq i, j \leq n$, choose $a_{i,j}$ equal the remainder of $3i + j$ when divided by n. Clearly, $0 \leq a_{i,j} \leq n - 1$, for $1 \leq i, j \leq n$.

If we fix a row $1 \leq i \leq n$, $a_{i,j_1} = a_{i,j_2}$, for $1 \leq j_1, j_2 \leq n$, implies $3i + j_1 \equiv 3i + j_2 \pmod{n}$, that is $j_1 \equiv j_2 \pmod{n}$ and therefore $j_1 = j_2$.

Hence, $a_{i,1}, \ldots, a_{i,n}$ is a permutation of $0, 1, \ldots, n-1$.

If we fix a column $1 \le j \le n$, $a_{i_1,j} = a_{i_2,j}$, for $1 \le i_1, i_2 \le n$, implies $3i_1 + j \equiv 3i_2 + j \pmod{n}$, that is $3i_1 \equiv 3i_2 \pmod{n}$. But n is not divisible by 3. Therefore $i_1 = i_2$. This proves that $a_{1,j}, \ldots, a_{n,j}$ is a permutation of $0, 1, \ldots, n-1$.

Now, let $1 \le i < j \le n$ and $1 \le i' < j' \le n$ such that $(a_{i,j}, a_{j,i}) = (a_{i',j'}, a_{j',i'})$ this implies that $3i+j \equiv 3i'+j' \pmod{n}$ and $3j+i \equiv 3j'+i' \pmod{n}$. This is equivalent to $8i \equiv 8i' \pmod{n}$ and $8j \equiv 8j' \pmod{n}$. But n is an odd number. Therefore, $(i, j) = (i', j')$. Hence, the board is totally non-symmetric.

3.3 Solutions to the team selection test of day III

1. Find all functions $f : \mathbb{N} \to (0, \infty)$ such that $f(4) = 4$ and

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)}, \quad \forall n \in \mathbb{N},$$

where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers.

Solution. Taking $n = 1$, we obtain $\dfrac{1}{f(1)} = f(1)$. Because $f(1) > 0$, we deduce that $f(1) = 1$.

Taking $n = 2$, we obtain $f(3) + 1 = f(2)^2$, and taking $n = 3$ we obtain $4f(3) + 4 + f(2) = f(2)f(3)^2$. Because $f(2) \neq 0$, this is equivalent to

$$f(3) + 1 = f(2)^2, \quad \text{and} \quad 4f(2) + 1 = f(3)^2.$$

We deduce that $4f(2) + 1 = \left(f(2)^2 - 1\right)^2$, and therefore

$$f(2)\left(f(2) - 2\right)\left(f(2)^2 + 2f(2) + 2\right) = 0.$$

But $f(2) \neq 0$. Hence $f(2) = 2$ and $f(3) = 3$.

Now assume, for all $1 \le k \le n$, that $f(k) = k$. We have

$$\frac{n-1}{n} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{n}{f(n+1)} - \frac{1}{n \cdot f(n+1)} = \frac{n^2 - 1}{nf(n+1)},$$

and therefore $f(n+1) = n+1$.

This proves that $f(n) = n$, for all $n \in \mathbb{N}$.

2. Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.

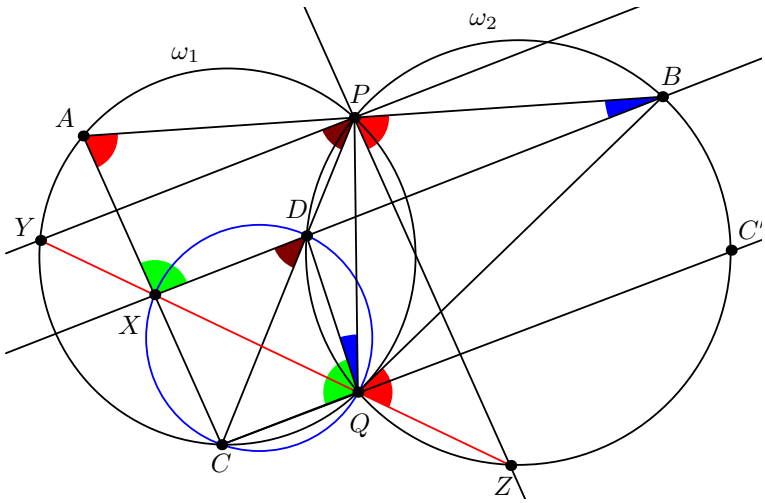
Solution. Because quadrilateral $BPDQ$ is cyclic, we have $\angle PQD = \angle PBD$. Because quadrilateral $ACQP$ is cyclic, we have $\angle PQC = 180° - \angle CAP$. We deduce that

$$\begin{aligned}\angle DQC &= \angle PQC - \angle PQD = 180° - \angle CAP - \angle PBD \\ &= 180° - \angle XAB - \angle ABX = \angle BXA = \angle DXA.\end{aligned}$$

This proves that quadrilateral $CQDX$ is cyclic and therefore

$$\angle XQC = \angle XDC.$$

Because PA is parallel to DX, we have $\angle YPC = \angle XDC$. Because $CQPY$ is cyclic, we have $\angle YPC = \angle YQC$. Therefore, $\angle YQC = \angle XQC$, which means that points Q, X, Y are collinear.



Because quadrilateral $BPQZ$ is cyclic, we have $\angle ZQB = \angle ZPB$. Because lines PZ and AC are parallel, we have $\angle ZPB = \angle CAP$.

Let C' be the second intersection point of line CQ with circle ω_1. Because quadrilateral $ACQP$ is cyclic, we have $\angle C'QP = \angle CAP$. Therefore, $\angle ZQB = \angle C'QP$. We deduce, by cyclicity of quadrilaterals $BPDQ$ and $CQDX$, that

$$\angle ZQC' = \angle BQP = \angle BDP = \angle XDC = \angle XQC,$$

which means that points Q, X, Z are collinear.

3. Let a, b be two nonnegative real numbers and n a positive integer. Prove that

$$\left(1 - 2^{-n}\right)\left|a^{2^n} - b^{2^n}\right| \geq \sqrt{ab}\left|a^{2^n-1} - b^{2^n-1}\right|.$$

First Solution. If $a = b$, the inequality is satisfied. If $a \neq b$, we can assume without loss of generality that $a > b$. In this case, the inequality can be written

$$(2^n - 1)\left(a^{2^n} - b^{2^n}\right) \geq 2^n\sqrt{ab}\left(a^{2^n-1} - b^{2^n-1}\right),$$

or equivalently

$$2^n\left(\sqrt{a} - \sqrt{b}\right)\left(a^{2^n-\frac{1}{2}} + b^{2^n-\frac{1}{2}}\right) \geq a^{2^n} - b^{2^n}.$$

By dividing in both sides by $\sqrt{a} - \sqrt{b}$, the inequality becomes

$$2^n\left(A^{2^{n+1}-1} + B^{2^{n+1}-1}\right) \geq (A+B)\left(A^2 + B^2\right)\left(A^4 + B^4\right)\cdots\left(A^{2^n} + B^{2^n}\right),$$

where $A = \sqrt{a}$ and $B = \sqrt{b}$. We prove this last inequality by induction on $n \geq 1$.

For $n = 1$, we have

$$\begin{aligned}(A+B)(A^2+B^2) &= A^3 + B^3 + A^2B + AB^2 \\ &\leq A^3 + B^3 + \frac{2A^3 + B^3}{3} + \frac{A^3 + 2B^3}{3} \\ &\leq 2(A^3 + B^3),\end{aligned}$$

by AM-GM inequality.

Assume the inequality true for n. We have

$$\begin{aligned}
&\left(A^{2^{n+1}-1}+B^{2^{n+1}-1}\right)\left(A^{2^{n+1}}+B^{2^{n+1}}\right)\\
&\qquad = A^{2^{n+2}-1}+B^{2^{n+2}-1}+A^{2^{n+1}-1}B^{2^{n+1}}+A^{2^{n+1}}B^{2^{n+1}-1}\\
&\qquad \le A^{2^{n+2}-1}+B^{2^{n+2}-1}+\frac{(2^{n+1}-1)A^{2^{n+2}-1}+2^{n+1}B^{2^{n+2}-1}}{2^{n+2}-1}\\
&\qquad\qquad +\frac{2^{n+1}A^{2^{n+2}-1}+(2^{n+1}-1)B^{2^{n+2}-1}}{2^{n+2}-1}\\
&\qquad \le 2\left(A^{2^{n+2}-1}+B^{2^{n+2}-1}\right),
\end{aligned}$$

by AM-GM inequality. We deduce the inequality for $n+1$.

Second Solution. Consider the polynomial

$$P(X)=\left(1-2^{-n}\right)\left(X^{2^{n+1}}-1\right)-X\left(X^{2^{n+1}-2}-1\right).$$

Its derivative polynomial is

$$P'(X)=\left(\left(2^{n+1}-2\right)X-(2^{n+1}-1)\right)X^{2^{n+1}-2}+1.$$

Therefore, for all $x\ge 1$, $P'(x)\ge P'(1)=0$. Hence, the polynomial function P is increasing on $[1,\infty[$.

Let a,b be two positive real numbers with $a\ge b$. Because $\sqrt{\frac{a}{b}}\ge 1$, we have

$$\begin{aligned}
b^{2^n}P\left(\sqrt{\frac{a}{b}}\right) &= \left(1-2^{-n}\right)b^{2^n}\left(\left(\frac{a}{b}\right)^{2^n}-1\right)-b^{2^n}\sqrt{\frac{a}{b}}\left(\left(\frac{a}{b}\right)^{2^n-1}-1\right)\\
&= \left(1-2^{-n}\right)\left(a^{2^n}-b^{2^n}\right)-\sqrt{ab}\left(a^{2^n-1}-b^{2^n-1}\right)\ge b^{2^n}P(1)=0,
\end{aligned}$$

which proves the inequality.

4. Let $f:\mathbb{N}\to\mathbb{N}$ be an injective function such that $f(1)=2$, $f(2)=4$ and

$$f(f(m)+f(n))=f(f(m))+f(n)$$

for all $m,n\in\mathbb{N}$. Prove that $f(n)=n+2$ for all $n\ge 2$.

Solution. Taking $n = 1$, we obtain $f(f(m)+2) = f(f(m))+2$, for all $m \in \mathbb{N}$. Taking $m = 1$, we obtain $f(2 + f(n)) = 4 + f(n)$, for all $n \in \mathbb{N}$. Therefore

$$f(f(n)) = f(n) + 2, \quad \text{for all } n \in \mathbb{N}.$$

Because $f(2) = f(f(1)) = f(1) + 2 = 4$, we prove by induction that

$$f(n) = n + 2$$

for all positive even integer $n \geq 2$.

Because the set $\{f(1)\} \cup \{f(2), f(4), f(6), \ldots\}$ contains all even positive integers and because f is an injective function, the image by f of any odd number $n \geq 3$ is an odd number.

Let

$$k = \min \left\{ f(n) \;\middle|\; f(n) \text{ is an odd integer} \right\},$$

and let $f(n_0) = k$ for some odd positive number $n_0 \geq 3$. We prove by induction that

$$f(n) = n + 2, \quad \text{for any odd number } n \geq k.$$

Indeed, $f(k) = f(f(n_0)) = f(n_0) + 2 = k + 2$. Assume that $f(n) = n + 2$, for some odd number $n \geq k$. We deduce that $f(n + 2) = f(f(n)) = f(n) + 2 = (n + 2) + 2$.

Clearly, $n_0 < k$, otherwise $k = f(n_0) = n_0 + 2 \geq k + 2$, which is impossible. Therefore,

$$\{f(n_0), f(k), f(k + 2), f(k + 4), \ldots\} = \{k, k + 2, k + 4, k + 6, \ldots\}$$

covers all possible odd values taken by f. Because, f is injective, and $f(3), f(5) \geq k$ are odd integers, we deduce that $n_0 = 3$, $k = 5$ and

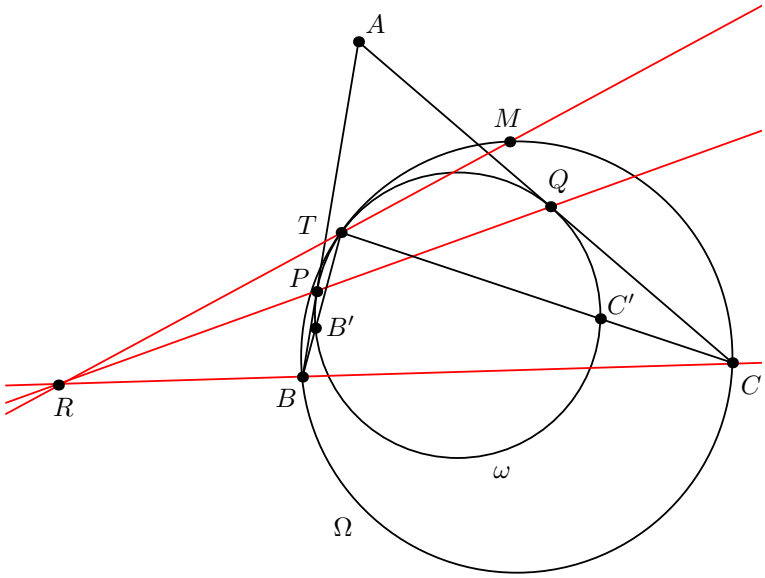
$$f(n) = n + 2$$

for all positive odd integer $n \geq 2$.

5. Let ABC be a triangle. Circle Ω passes through points B and C. Circle ω is tangent internally to Ω and also to sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc $\widehat{BC}$ (containing T) of Ω. Prove that

lines PQ, BC, and MT are concurrent.

Solution. Let R be the intersection point of lines PQ and BC, and S the intersection point of lines MT and BC.



Because circle ω is tangent to both sides AB and AC at P and Q, we have $AP = AQ$. Applying Menelaus' theorem to triangle ABC and line PQ we obtain

$$\frac{\overline{BR}}{\overline{RC}} = -\frac{BP}{PA} \cdot \frac{AQ}{QC} = -\frac{BP}{QC}.$$

Because M is the midpoint of arc $\widehat{BC}$ of circle Ω containing T, line MT is the external bisector of angle $\angle BTC$. We deduce that

$$\frac{\overline{BS}}{\overline{SC}} = -\frac{TB}{TC}.$$

To prove that $S = R$, we have to prove that $\frac{\overline{BS}}{\overline{SC}} = \frac{\overline{BR}}{\overline{RC}}$, which is equivalent to proving that

$$\frac{BP}{QC} = \frac{TB}{TC}.$$

Let B' and C' be the second intersection points of lines TB and TC with circle ω, respectively. Because circles Ω and ω are tangent at T, segments

BC and $B'C'$ are parallel, and therefore

$$\frac{TB}{TC} = \frac{BB'}{CC'}.$$

Because lines AB and AC are tangent to ω at P and Q, respectively, we have from the power of points B and C with respect to circle ω

$$\left(\frac{BP}{CQ}\right)^2 = \frac{BT \cdot BB'}{CT \cdot CC'} = \left(\frac{BT}{CT}\right)^2.$$

Chapter 4

Solutions to the team selection tests for the International Mathematical Olympiads

The author would like to thank the students Ibraheem Khan, Omar Alrabiah, Alzubair Habibullah, Mahdi Al-Shaikh Saleh, Shaden Al-Shammari, Salman Saleh, and Alhamza Al-Nufayli for providing him with solutions of problems in this chapter.

4.1 Solutions to the team selection test of day I

1. Tarik and Sultan are playing the following game. Tarik thinks of a number that is greater than 100. Then Sultan is telling a number greater than 1. If Tarik's number is divisible by Sultan's number, Sultan wins, otherwise Tarik subtracts Sultan's number from his number and Sultan tells his next number. Sultan is forbidden to repeat his numbers. If Tarik's number becomes negative, Sultan loses. Does Sultan have a winning strategy?

   Solution. Yes, Sultan has winning strategies. Here are two examples:

   First winning strategy. Sultan plays the following numbers in order: 2, 3, 4, 6, 8, 20, 24 and he wins. Indeed, let $n \geq 100$ be Tarek's number and assume that Sultan will not win before playing the last number 24.

   - n is not divisible by 2, so it is congruent to one of the twelve numbers

$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23$, in modular 24.

- $n-2$ is not divisible by 3, so it is congruent to one of the eight numbers $1, 5, 7, 11, 13, 17, 19, 23$, in modular 24.
- $n-5 = (n-2)-3$ is not divisible by 4, so it is congruent to one of the four numbers $2, 10, 14, 22$, in modular 24.
- $n-9 = (n-5)-4$ is not divisible by 6, so it is congruent to one of the two numbers $10, 22$, in modular 24.
- $n-15 = (n-9)-6$ is not divisible by 8, so it is congruent to 4 in modular 24.
- $n-23 = (n-15)-8$ is not divisible by 20 and is congruent to 20 in modular 24.
- Finally, Sultan ends up with $n-43 \equiv 0 \bmod 24$, which is still a positive number divisible by 24, and Sultan wins.

Second winning strategy. Sultan plays the following numbers in order: 2, 3, 4, 6, 16, 12 and he wins. Indeed, let $n \geq 100$ be Tarek's number and assume that Sultan will not win before playing the last number 12.

- n is not divisible by 2, so it is congruent to one of the six numbers $1, 3, 5, 7, 9, 11$, in modular 12.
- $n-2$ is not divisible by 3, so it is congruent to one of the four numbers $1, 5, 7, 11$, in modular 12.
- $n-5 = (n-2)-3$ is not divisible by 4, so it is congruent to one of the two numbers $2, 10$, in modular 12.
- $n-9 = (n-5)-4$ is not divisible by 6, so it is congruent to 10 in modular 12.
- $n-15 = (n-9)-6$ is not divisible by 16 and is congruent to 4 in modular 12.
- Finally, Sultan ends up with $n-31 \equiv 0 \bmod 12$, which is still a positive number divisible by 12, and Sultan wins.

2. Define a *domino* to be an ordered pair of *distinct* positive integers. A *proper sequence* of dominoes is a list of distinct dominoes in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not *both* appear for any i and j. Let D_n be the set of all dominoes whose coordinates are no larger than n. Find the length of the longest proper sequence of dominoes that can be formed using the dominoes of D_n.

Solution. Consider the complete graph $\mathcal{G}_n$, over the set of vertices $\{1, 2, \ldots, n\}$. To each domino (i, j) in D_n, we associate the edge joining the two vertices i, j. In this way, to each edge in $\mathcal{G}_n$ we have associated exactly two dominoes (i, j) and (j, i), for some $1 \leq i \neq j \leq n$. A proper sequence of dominoes corresponds exactly to a path in $\mathcal{G}_n$ which visits every edge at most once. The problem asks for the length of the longest such path. We will use and prove the following:

> If a finite graph $\mathcal{G}$ is connected and each of its vertices has an even degree, there exists a closed path which visits all its edges exactly once. This closed path is called Eulerian.

We will prove this fact by induction on the number of edges of $\mathcal{G}$.

If $\mathcal{G}$ has 3 edges, then it is a triangle and the statment is obvious. Assume there exists a closed Eulerian path in any connected graph with all vertices of even degree having less than or equal to m edges.

Let $\mathcal{G}$ be a connected graph with $m + 1$ edges and all its vertices are of even degree. Let $\mathcal{P}$ be a longest path visiting each edge of $\mathcal{G}$ at most once.

Assume that $\mathcal{P}$ starts and ends at different vertices v, v'. Path $\mathcal{P}$ visits an odd number of edges incident to v. But the degree of v is even. Then, there exists an edge e incident to v which is not visited by $\mathcal{P}$ and so, we can extend $\mathcal{P}$ with e. This contradicts the maximality of $\mathcal{P}$. Hence path $\mathcal{P}$ starts and ends at the same vertex.

Assume that there exists an edge e not visited by $\mathcal{P}$ and let v be a vertex incident to e and v_1 a vertex visited by $\mathcal{P}$. Because $\mathcal{G}$ is connected, there exists a path from v to v_1. Replace v_1 by the first vertex in this connecting path visited by $\mathcal{P}$. Let $\mathcal{G}'$ be the graph obtained by deleting from the graph $\mathcal{G}$ all edges visited by $\mathcal{P}$ and taking the connected component of v_1. Graph $\mathcal{G}'$ contains edge e and has less than m edges. Each vertex v' of $\mathcal{G}'$ has even degree since path $\mathcal{P}$ has visited precisely an even number of edges incident to v'. Hence, there exists a closed Eulerian path $\mathcal{P}'$ in $\mathcal{G}'$. By considering in $\mathcal{G}$ the path starting at v_1 and walking along all the path $\mathcal{P}$ and then along all the path $\mathcal{P}'$ we obtain a longer path in $\mathcal{G}$ visiting each edge at most once. This contradicts the maximality of $\mathcal{P}$. This proves that edge e does not exist

and that $\mathcal{P}$ is Eulerian.

Back to our problem, we will consider two cases:

If n is an odd number, the degree of each vertex of the complete graph $\mathcal{G}_n$ is $n-1$, even, and $\mathcal{G}_n$ is connected. So there exists a path visiting all the edges of $\mathcal{G}_n$ exactly once. Hence the longest proper sequence of dominoes that can be formed using the dominoes of D_n uses all dominoes. It has length $\binom{n}{2} = \dfrac{n(n-1)}{2}$.

If n is an even number, the degree of each vertex of the complete graph $\mathcal{G}_n$ is odd. Let $\mathcal{P}$ be a longest path visiting each edge at most once. For each vertex v different from the starting and the ending vertices of $\mathcal{P}$, path $\mathcal{P}$ visites an even number of edges incident to v. So there exists at least one edge incident to v not visited by $\mathcal{P}$. Because, each edge is incident to two vertices, there exists at least $\frac{n-2}{2}$ edges not visited by $\mathcal{P}$. Hence the length of $\mathcal{P}$ is less than or equal to $\binom{n}{2} - \dfrac{n-2}{2} = \dfrac{n^2-2n+2}{2}$.

Now, consider the graph $\mathcal{G}'_n$ obtained from $\mathcal{G}_n$ by deleting the edges between 1 and 3, 2 and 3, and $2i-1$ and $2i$, for $2 \leq i \leq \frac{n}{2}$. Graph $\mathcal{G}'_n$ is connected and all its vertices have even degree. So there exists an Eulerian closed path $\mathcal{P}'$ starting and ending at 3. Extending path $\mathcal{P}'$ in $\mathcal{G}_n$ at the begining with the edge between 1 and 3 and at the end with the edge between 2 and 3, we obtain a path $\mathcal{P}$ of length $\frac{n^2-2n+2}{2}$ visiting each edge at most once. Hence the longest proper sequence of dominoes that can be formed using the dominoes of D_n has length $\dfrac{n^2-2n+2}{2}$.

3. Let ABC be a triangle and let P be a point on BC. Points M and N lie on AB and AC, respectively such that MN is not parallel to BC and $AMPN$ is a parallelogram. Line MN meets the circumcircle of ABC at R and S. Prove that the circumcircle of triangle RPS is tangent to BC.

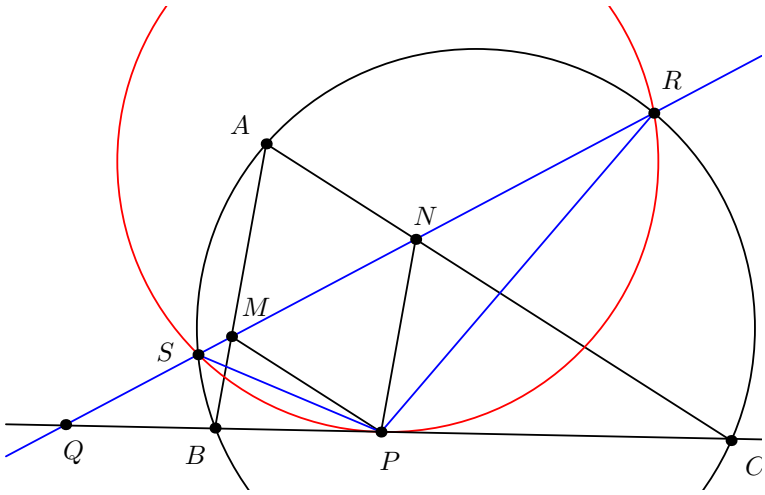
Solution. Let Q be the intersection point of lines BC and MN.

Because BM and NP are parallel, and AN and MP are parallel, we have

$$\frac{QP}{QB} = \frac{NP}{MB} = \frac{AM}{MB} = \frac{CP}{PB},$$

and

$$\frac{QC}{QP} = \frac{CN}{PM} = \frac{CN}{NA} = \frac{CP}{PB}.$$



We deduce that

$$QP^2 = QB \cdot QC.$$

But from the power of point Q with respect to the circumcircle of ABC, we have $QB \cdot QC = QR \cdot QS$. We deduce that

$$QP^2 = QR \cdot QS,$$

that is line BC is tangent to the circumcircle of triangle PRS.

4. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(n+1) > \frac{f(n) + f(f(n))}{2}$$

for all $n \in \mathbb{N}$, were $\mathbb{N}$ is the set of strictly positive integers.

Solution. It is clear that $f(n) \geq 1$ for all $n \geq 1$. Assume that $f(n) \geq m$ for all $n \geq m$, for some $m \geq 1$. Let $n \geq m+1$. Because $n-1 \geq m$, we have $f(n-1) \geq m$ and therefore $f(f(n-1)) \geq m$. We deduce that $f(n) > \frac{f(n)+f(f(n))}{2} \geq m$. Hence, $f(n) \geq m+1$. In particular, $f(n) \geq n$ for all $n \in \mathbb{N}$.

Assume there exists m for which $f(m) \geq m+2$. Then $f(f(m)) \geq f(m+2) \geq m+2$, and $f(m+1) > \frac{f(m)+f(f(m))}{2} \geq m+2$. This gives $f(m+1) \geq (m+1)+2$. It follows by induction that $f(n) \geq n+2$ for all $n \geq m$. Therefore, for all $n \geq m$, $f(n+1) - f(n) > f(f(n)) - f(n+1) \geq f(n+2) - f(n+1)$, which means that the sequence $f(n+1) - f(n)$ becomes decreasing from m and this contradicts the fact $f(n) \geq n$ for all $n \in \mathbb{N}$. We deduce that

$$n \leq f(n) \leq n+1 \quad \text{for all} \quad n \in \mathbb{N}.$$

It is clear that the identity function is a solution. Assume that there exists $m \in \mathbb{N}$ such that $f(m) = m+1$. We have $m+2 \geq f(m+1) > f(m) + f(f(m)) - f(m+1) = m+1$. This proves that $f(m+1) = (m+1)+1$ and by induction $f(n) = n+1$, for all $n \geq m$. After checking, we conclude that the solutions are the identity function, $f(n) = n+1$ for all $n \in \mathbb{N}$, and all the functions which can be written

$$f(n) = \begin{cases} n & \text{if } n < m, \\ n+1 & \text{if } n \geq m, \end{cases}$$

for some $m > 1$.

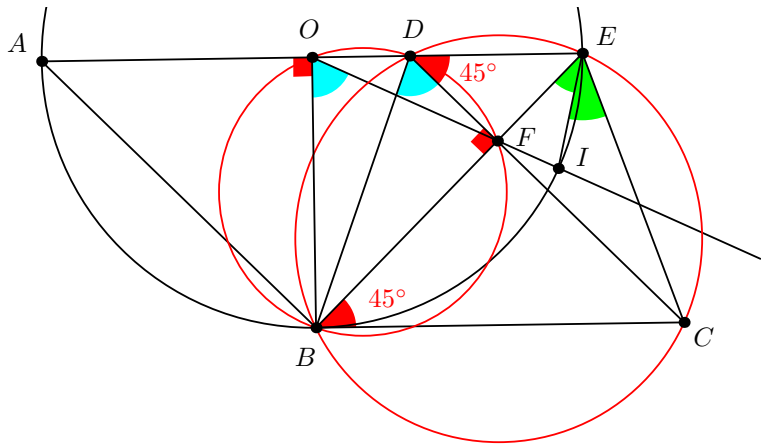
4.2 Solutions to the team selection test of day II

1. Let Γ be a circle with center O and AE be a diameter. Point D lies on segment OE and point B is the midpoint of one of the arcs $\widehat{AE}$ of Γ. Construct point C such that $ABCD$ is a parallelogram. Lines EB and CD meet at F. Line OF meets the minor arc $\widehat{EB}$ at I. Prove that EI bisects $\angle BEC$.

   Solution. Because DC and AB are parallel, $\angle CDE = \angle BAO = 45°$. Because BC and AE are parallel, $\angle CBE = \angle OEB = 45°$. Hence quadrilateral $BCED$ is cyclic and therefore $\angle BEC = \angle BDC$.

On the other hand, $\angle DFB = \angle DEF + \angle FDE = 45° + 45° = 90° = \angle AOB$. We deduce that quadrilateral $BFDO$ is cyclic and therefore $\angle BDC = \angle BOI$.

But $\angle BOI = 2\angle BFI$ since both angles intercept the same arc $\widehat{BI}$. We conclude that $\angle BEC = 2\angle BEI$, which means that EI bisects $\angle BEC$.



2. Let S be a set of positive real numbers with five elements such that for any distinct a, b, c in S, the number $ab + bc + ca$ is rational. Prove that for any a and b in S, $\frac{a}{b}$ is a rational number.

Solution. Let a, b, c be three distinct elements in S.

If we denote by $\mathcal{S}_i$, the set of subsets of S of i elements, $i = 2, 3$, we notice that

$$\sum_{\{x,y\}\in\mathcal{S}_2} xy = \frac{1}{3} \sum_{\{x,y,z\}\in\mathcal{S}_3} (xy + yz + zx) \in \mathbb{Q}.$$

If we denote by $\mathcal{S}_{x,i}$, the set of subsets of $S \setminus \{x\}$ of i elements, $i = 2, 3$, for $x \in S$, we notice that

$$ac = \sum_{\{x,y\}\in\mathcal{S}_2} xy - \frac{1}{2}\left(\sum_{\{x,y,z\}\in\mathcal{S}_{a,3}} (xy + yz + zx) + \sum_{\{x,y,z\}\in\mathcal{S}_{c,3}} (xy + yz + zx) \right).$$

Hence $ac \in \mathbb{Q}$. Similarly $bc \in \mathbb{Q}$. We deduce that $\dfrac{a}{b} = \dfrac{ac}{bc} \in \mathbb{Q}$.

3. Show that it is possible to write a $n \times n$ array of non-negative numbers (not necessarily distinct) such that the sums of entries on each row and each column are pairwise distinct perfect squares.

Solutions. We give for this problem two solutions:

First solution suggested by the student Salman Saleh. We will construct an $n \times n$ array by induction on $n \geq 2$.

For $n = 2$, consider the following example:

0	9
0	16

Assume there exists an $n \times n$ array

$a_{1,1}$	$a_{1,2}$	$\dots$	$a_{1,n}$
$a_{2,1}$	$a_{2,2}$	$\dots$	$a_{2,n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$a_{n,1}$	$a_{n,2}$	$\dots$	$a_{n,n}$

such that the sums of entries on each row and each column are pairwise distinct perfect squares and such that the sum of the last column is greater than all the other sums. Let

$$r_i^2 = \sum_{j=1}^{n} a_{i,j}, \quad \text{for} \quad i = 1, 2, \dots, n,$$

and

$$c_j^2 = \sum_{i=1}^{n} a_{i,j}, \quad \text{for} \quad j = 1, 2, \dots, n,$$

be the distinct perfect squares. Notice that

$$\sum_{i=1}^{n} r_i^2 = \sum_{j=1}^{n} c_j^2.$$

Consider the following $(n+1)\times(n+1)$ array

$4a_{1,1}$	$4a_{2,1}$	$\dots$	$4a_{n,1}$	$12c_1^2$
$4a_{1,2}$	$4a_{2,2}$	$\dots$	$4a_{n,2}$	$12c_2^2$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$4a_{1,n-1}$	$4a_{2,n-1}$	$\dots$	$4a_{n,n-1}$	$12c_{n-1}^2$
$4a_{1,n}$	$4a_{2,n}$	$\dots$	$4a_{n,n}$	$4(25n^2-1)c_n^2$
$12r_1^2$	$12r_2^2$	$\dots$	$12r_n^2$	x_n

where

$$x_n = \left((25n^2-4)c_n^2-1\right)^2 - 12\sum_{i=1}^{n} r_i^2.$$

Notice that

$$x_n > \left((25n^2-4)c_n^2-1\right)^2 - 12nc_n^2 > (21n^2-4)c_n^2 > 0.$$

The sums of entries of each row and each columns are

$$(4c_1)^2, (4c_2)^2, \dots, (4c_{n-1})^2, (4r_1)^2, (4r_2)^2, \dots, (4r_n)^2,$$

which are all distinct and less than the three other sums

$$(10nc_n)^2 < \left((25n^2-4)c_n^2-1\right)^2 < \left((25n^2-4)c_n^2+1\right)^2,$$

and clearly, the sum of the last column is the greatest perfect square.

Second solution suggested by the student Alhamza Alnufayli. If $n=2k\geq 2$, we consider the following $n\times n$ array:

$48 \cdot 2^{2(k-1)}$	0	$\cdots$	0	0	$\cdots$	0	$33 \cdot 2^{2(k-1)}$
0	$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\iddots$	$\iddots$	0
$\vdots$	$\ddots$	$48 \cdot 2^2$	0	0	$33 \cdot 2^2$	$\iddots$	$\vdots$
0	$\ldots$	0	48	33	0	$\ldots$	0
0	$\ldots$	0	1	3	0	$\ldots$	0
$\vdots$	$\iddots$	2^2	0	0	$3 \cdot 2^2$	$\ddots$	$\vdots$
0	$\iddots$	$\iddots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	0
$2^{2(k-1)}$	0	$\cdots$	0	0	$\cdots$	0	$3 \cdot 2^{2(k-1)}$

The sums of entries of each row and each column are

$$\left(9 \cdot 2^{k-1}\right)^2, \ldots, (9 \cdot 2)^2, (9 \cdot 1)^2, (2 \cdot 1)^2, (2 \cdot 2)^2, \ldots, \left(2 \cdot 2^{k-1}\right)^2,$$
$$\left(7 \cdot 2^{k-1}\right)^2, \ldots, (7 \cdot 2)^2, (7 \cdot 1)^2, (6 \cdot 1)^2, (6 \cdot 2)^2, \ldots, \left(6 \cdot 2^{k-1}\right)^2,$$

which are all distinct perfect squares.

If $n = 2k + 1 \geq 2$, we consider the following $n \times n$ array:

$48 \cdot 2^{2(k-1)}$	0	$\cdots$	0	0	0	$\cdots$	0	$33 \cdot 2^{2(k-1)}$
0	$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\iddots$	0
$\vdots$	$\ddots$	$48 \cdot 2^2$	0	0	0	$33 \cdot 2^2$	$\iddots$	$\vdots$
0	$\ldots$	0	48	360	33	0	$\ldots$	0
0	$\ldots$	$\ldots$	0	13^2	0	$\ldots$	$\ldots$	0
0	$\ldots$	0	1	0	3	0	$\ldots$	0
$\vdots$	$\iddots$	2^2	0	0	0	$3 \cdot 2^2$	$\ddots$	$\vdots$
0	$\iddots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	0
$2^{2(k-1)}$	0	$\cdots$	0	0	0	$\cdots$	0	$3 \cdot 2^{2(k-1)}$

The sums of entries on each row and each column are

$$\left(9\cdot 2^{k-1}\right)^2,\ldots,(9\cdot 2)^2,21^2,13^2,(2\cdot 1)^2,(2\cdot 2)^2,\ldots,\left(2\cdot 2^{k-1}\right)^2,$$
$$\left(7\cdot 2^{k-1}\right)^2,\ldots,(7\cdot 2)^2,(7\cdot 1)^2,23^2,(6\cdot 1)^2,(6\cdot 2)^2,\ldots,\left(6\cdot 2^{k-1}\right)^2,$$

which are all distinct perfect squares.

4. Aws plays a solitaire game on a fifty-two card deck: whenever two cards of the same color are adjacent, he can remove them. Aws wins the game if he removes all the cards. If Aws starts with the cards in a random order, what is the probability for him to win?

Solution. Let us consider the positions $1,2,\ldots,52$, from left to right, of the fifty-two cards at the begining of the game. Whenever Aws removes two adjacent cards, we substract 2 to the positions of all the cards on the right of these adjacent removed cards.

We notice that, at each operation, if a card is not removed, even if its position changes, it will keep the same parity. We notice also that whenever two adjacent cards of the same color are removed, the positions of these two cards have different parity.

Define R to be the number of red cards with even positions at the begining substracted from the number of red cards with odd positions at the begining. Clearly, R is an invariant under Aws' operations. Hence, if at the begining, $R\neq 0$, Aws can never remove all his cards.

Now assume that $R=0$ and that Aws has started playing. If there are no more red cards, then Aws is kept with only black cards, so he can keep removing adjacent cards until he ends up with no card. If there is at least one red card, because $R=0$, there is at least one red card with even position e_1 and one red card with odd position o_1. Assume, without loss of generality that $e_1<o_1$.

Consider $o=\min\mathcal{O}$, where $\mathcal{O}$ is the set of odd positions o' of red cards with $o'>e_1$. Number o exists since $o_1\in\mathcal{O}$. Consider $e=\max\mathcal{E}$, where $\mathcal{E}$ is the set of even positions e' of red cards with $o>e'$. Number e exists since $e_1\in\mathcal{E}$. If $o=e+1$, then there are two adjacent red cards that Aws can remove. If

$o > e+1$ then $o \geq e+3$ and the positions between e and o are all occupied by adjacent black cards. In this case, Aws can remove adjacent black cards. This proves that if $R = 0$, Aws can keep removing cards until he removes all cards.

Hence, Aws can win if and only if $R = 0$. In this situation, there must be 13 red cards with even positions and 13 red cards with odd positions. There are precisely $\binom{26}{13}^2 \cdot 26!^2$ possible such starting positions and the probability for Aws to win is

$$\frac{\binom{26}{13}^2}{\binom{52}{26}}.$$

4.3 Solutions to the team selection test of day III

1. A *perfect number* is an integer that equals half the sum of its positive divisors. For example, because $2 \cdot 28 = 1+2+4+7+14+28$, 28 is a perfect number.

   (a) A *square-free* integer is an integer not divisible by a square of any prime number. Find all square-free integers that are perfect numbers.

   (b) Prove that no perfect square is a perfect number.

Solution. Recall that if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, then the sum of the positive divisors of n is

$$\sigma(n) = \prod_{i=1}^{k} \left(\sum_{j=0}^{\alpha_i} p_i^j \right).$$

(a) Let n be a square-free perfect number. In this case, $n = p_1 p_2 \cdots p_k$, for some prime numbers $p_1 < p_2 < \ldots < p_k$, and

$$2p_1 p_2 \cdots p_k = (p_1+1)(p_2+1)\cdots(p_k+1).$$

If $p_1 \neq 2$ then $p_1+1, p_2+1, \ldots, p_k+1$ are all even numbers while $p_1 p_2 \cdots p_k$ is an odd number. We deduce that $k = 1$ and $2p_1 = p_1 + 1$

which is impossible.

If $p_1 = 2$ then $p_1 + 1 = 3 = p_2$ and if $k > 2$, we have

$$p_3 p_4 \cdots p_k = (p_3 + 1)(p_4 + 1) \cdots (p_k + 1),$$

which is impossible since both sides have different parity. Hence $k = 2$ and $n = 6$.

This proves that the only square-free perfect number is 6.

(b) Let n be a perfect square. There exist prime numbers $p_1 < p_2 < \ldots < p_k$ and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that

$$n = p_1^{2\alpha_1} p_2^{2\alpha_2} \cdots p_k^{2\alpha_k}.$$

Because $\sum_{j=0}^{2\alpha_i} p_i^j$ is an odd number, for $i = 1, 2, \ldots, k$, the sum $\sigma(n)$ of the positive divisors of n is odd while $2n$ is even. Hence n cannot be a perfect number.

2. Determine all functions $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ and

$$f(x) = 1 + 5f\left(\left\lfloor \frac{x}{2} \right\rfloor\right) - 6f\left(\left\lfloor \frac{x}{4} \right\rfloor\right)$$

for all $x > 0$.

Solution. Let $x \geq 0$. If $x \in (0, 2)$ then $f(x) = 1 + 5f(0) - 6f(0) = 1$. If $x \in [2, 4)$ then $f(x) = 1 + 5f(1) - 6f(0) = 6 = a_1$. If $x \in [4, 8)$ then $\lfloor \frac{x}{2} \rfloor \in [2, 4)$ and $\lfloor \frac{x}{4} \rfloor \in [1, 2)$, and therefore $f(x) = 1 + 5 \cdot 6 - 6 \cdot 1 = 25 = a_2$.

Assume for $n \geq 1$, that function f is constant on $[2^n, 2^{n+1})$ taking a value a_n, and constant on $[2^{n+1}, 2^{n+2})$ taking a value a_{n+1}, and let $x \in [2^{n+2}, 2^{n+3})$. Because $\lfloor \frac{x}{2} \rfloor \in [2^{n+1}, 2^{n+2})$ and $\lfloor \frac{x}{4} \rfloor \in [2^n, 2^{n+1})$, we deduce that $f(x) = 1 + 5a_{n+1} - 6a_n$. Therefore, function f is also constant on $[2^{n+2}, 2^{n+3})$ taking the value $a_{n+2} = 1 + 5a_{n+1} - 6a_n$, which can be rewritten

$$a_{n+2} - \frac{1}{2} = 5\left(a_{n+1} - \frac{1}{2}\right) - 6\left(a_n - \frac{1}{2}\right).$$

Because the roots of the characteristic polynomial $X^2 - 5X + 6$ are 2 and 3, there exist two real numbers u, v such that

$$a_n = \frac{1}{2} + 2^n u + 3^n v, \quad \text{for all } n \geq 1.$$

Because $a_1 = 6$ and $a_2 = 25$, we have $u = -4$ and $v = \frac{9}{2}$. We deduce that

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0, 2), \\ -2^{n+2} + \dfrac{3^{n+2} + 1}{2} & \text{if } x \in \left[2^n, 2^{n+1}\right) \text{ and } n \geq 1. \end{cases}$$

3. There are 2015 coins on a table. For $i = 1, 2, \ldots, 2015$ in succession, one must turn over exactly i coins. Prove that it is always possible either to make all of the coins face up or to make all of the coins face down, but not both.

Solution 1. We start by proving that it is always possible either to make all of the coins face up or to make all of the coins face down.

Notice that if we proceed by choosing the numbers $i = 1, 2, \ldots, 2015$ in this order or in any other order, the final result will depend only on the i coins choosen at each step. For the sake of simplicity, we will follow the order

$$i = 1007, 1008, 1006, 1009, 1005, 1010, ..., 2, 2013, 1, 2014, 2015.$$

By symmetry of the face up/face down positions, we can assume without loss of generality that at the begining, the number of coins face up is odd equal to $2k + 1$. Choose k of these face up coins and $1007 - k$ of the face down coins and turn them over to get 1008 coins face up and 1007 coins face down. Turn these new 1008 coins face up to get all the coins face down. After this, at each step, whenever we choose i coins, for $i = 1006, 1005, \ldots, 1, 2015$, in the next step we choose the $2015 - i$ remaining coins to make at the end all the coins face up.

Now we prove that for each starting positions, it is not possible to choose to make all the coins face up or all the coins face down.

Assume that starting with an odd number of face up coins we can make at the end all the coins face down. This means that each face up coin will be

turned over an odd number of times while each face down coin will be turned over an even number of times. Therefore, the total number of turning over coins is

$$odd \times odd + even \times even = odd.$$

But the total number of turning over coins is

$$1 + 2 + \cdots + 2015 = 2015 \times 1008 = even,$$

which is a contradiction. So it is only possible to make all the coins face up and not to make all of them face down. In a similar way, if we start with an odd number of face down coins it is only possible to make them all face down.

Solution 2. The statement works for any odd number of coins. We prove it by induction on the number n of coins, n being odd. The case $n = 1$ is obvious. Suppose the statement true for $n = 2k-1$, for some positive integer k. If we are given $n = 2k+1$ coins, we consider the following cases.

- *Case 1.* There is a coin C_1 facing up and another coin C_2 facing down. We consider the other $2k-1$ coins first. By our induction, we can turn the coins $1, 2, \ldots, 2k-1$ times in succession so all the $2k-1$ coins are in one direction. Without loss of generality, we assume that all the $2k-1$ coins are facing up. Then we turn all these coins together with C_1 and then turn all the $2k+1$ coins so they will be all facing up.
- Case 2: All the coins are in the same direction. We arrange the coins around a circle and number them $1, 2, \ldots, 2k+1$ in clockwise order. We first turn coin 1, then coins 2 and 3, and then 4, 5, and 6, and so on along the circle. Then we make a total of $1+2+\cdots+(2k+1) = (k+1)(2k+1)$ turnings and each coin has been turned $k+1$ times. Since they start in the same direction, they end in the same direction.

From the above argument, we can find a way to make all of the $2k+1$ coins facing in one direction after $2k+1$ operations regardless of the initial configuration. Hence our induction is complete.

To prove that it is impossible to achieve both final configurations, we proceed as in the first solution.

4. Two circles ω_1 and ω_2, with center O_1 and O_2 respectively, meet at points A and B. Let X and Y be points on ω_1. Lines XA and YA meet ω_2 at Z and W, respectively, such that A lies between X and Z and between Y and

W. Let M be the midpoint of O_1O_2, S be the midpoint of XA and T be the midpoint of WA. Prove that $MS = MT$ if and only if X, Y, Z and W are concyclic.

Solution suggested by the Student Alzubair Habibullah. Because M is the midpoint of O_1O_2, we have from Apollonius' theorem

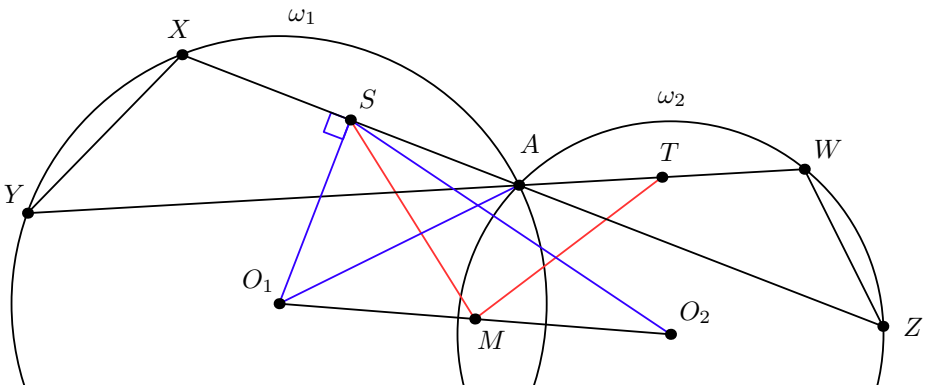
$$4MS^2 = 2SO_1^2 + 2SO_2^2 - O_1O_2^2.$$

Because S is the midpoint of segment AX, segment O_1S is perpendicular to segment AX and therefore $SO_1^2 = O_1A^2 - AS^2$.

On the other hand, we have from the power of point S with respect to circle ω_2, $SO_2^2 = SA \cdot SZ + O_2A^2 = SA^2 + SA \cdot AZ + O_2A^2$.

We deduce that

$$4MS^2 = 2O_1A^2 + 2O_2A^2 + 2SA \cdot AZ - O_1O_2^2 = 2O_1A^2 + 2O_2A^2 + XA \cdot AZ - O_1O_2^2.$$



We deduce in a similar way that

$$4MT^2 = 2O_2A^2 + 2O_1A^2 + WA \cdot AY - O_2O_1^2.$$

Therefore

$$4(MS^2 - MT^2) = XA \cdot AZ - WA \cdot AY.$$

This proves that $MS = MT$ if and only if $XA \cdot AZ = WA \cdot AY$, which is equivalent to saying that X, Y, Z and W are concyclic.

4.4 Solutions to the team selection test of day IV

1. Let $a_1, \ldots, a_n$ be a non increasing sequence of positive real numbers. Prove that
$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \leq a_1 + \frac{a_2}{\sqrt{2}+1} + \cdots + \frac{a_n}{\sqrt{n}+\sqrt{n-1}}.$$
When does equality hold?

Solution. We prove the inequality by induction on $n \geq 2$.

For $n = 2$, let $a_1 \geq a_2 > 0$. We have

$$\begin{aligned}\sqrt{a_1^2 + a_2^2} &= \sqrt{2a_2^2 + (a_1 - a_2)^2 + 2a_2(a_1 - a_2)} \\ &\leq \sqrt{2a_2^2 + (a_1 - a_2)^2 + 2\sqrt{2}a_2(a_1 - a_2)} = (a_1 - a_2) + \sqrt{2}a_2 \\ &\leq a_1 + \frac{a_2}{\sqrt{2}+1}.\end{aligned}$$

Assume the inequality true for any non increasing sequence of positive n real numbers and let $a_1 \geq a_2 \geq \cdots \geq a_{n+1} > 0$. Denote $q_n = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$. It is clear that $q_n \geq \sqrt{n}a_{n+1}$. We have

$$\begin{aligned}\sqrt{a_1^2 + a_2^2 + \cdots + a_{n+1}^2} &= \sqrt{q_n^2 + a_{n+1}^2} = \\ &= \sqrt{(n+1)a_{n+1}^2 + (q_n - \sqrt{n}a_{n+1})^2 + 2\sqrt{n}a_{n+1}(q_n - \sqrt{n}a_{n+1})} \\ &\leq \sqrt{(n+1)a_{n+1}^2 + (q_n - \sqrt{n}a_{n+1})^2 + 2\sqrt{n+1}a_{n+1}(q_n - \sqrt{n}a_{n+1})} \\ &\leq \sqrt{n+1}a_{n+1} + (q_n - \sqrt{n}a_{n+1}) = q_n + \frac{a_{n+1}}{\sqrt{n+1}+\sqrt{n}} \\ &\leq a_1 + \frac{a_2}{\sqrt{2}+1} + \cdots + \frac{a_{n+1}}{\sqrt{n+1}+\sqrt{n}}.\end{aligned}$$

The equality holds when $q_i = \sqrt{i}a_{k+1}$, for $i = 1, 2, \ldots, n-1$, that is $a_1 = a_2 = \cdots = a_n$.

2. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got

0 points, and each of the two players earned $\frac{1}{2}$ point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of his points against the other nine of the ten). What was the total number of players in the tournament?

Solution. Suppose that there were n players and assume $s_1 \geq s_2 \geq \cdots \geq s_n$ be the scores of each player. Because each player played exactly one game against each of the other players and each game worths 1 point, the total sum of scores is

$$s_1 + s_2 + \cdots + s_n = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Because each of the ten lowest scoring players earned half of his points against the other 9 of the 10, he earned the other half of his points against the $n-10$ highest scoring players. Therefore, the total sum of the scores earned in the games between the first $n-10$ players and the last 10 players is

$$10(n-10) = \frac{1}{2}s_1 + \frac{1}{2}s_2 + \cdots + \frac{1}{2}s_n = \frac{n(n-1)}{4}.$$

This implies that $n^2 - 41n + 400 = 0$, and thus $n = 16$ or 25.

Because each of the ten lowest scoring players earned half of his points against the other 9 of the 10, the total sum of the scores earned in the games between the last 10 players is

$$\frac{1}{2}s_{n-9} + \frac{1}{2}s_{n-8} + \cdots + \frac{1}{2}s_n = \binom{10}{2} = 45.$$

This gives an average of 9 points of score for each of the ten lowest scoring players. Therefore, each of the first $n-10$ players has earned at least 9 points.

Because each of the first $n-10$ has earned half of his score from the games against the $n-11$ of the $n-10$ players, we have

$$\frac{9}{2}(n-10) \leq \frac{1}{2}s_1 + \frac{1}{2}s_2 + \cdots + \frac{1}{2}s_{n-10} = \binom{n-10}{2} = \frac{(n-10)(n-11)}{2}.$$

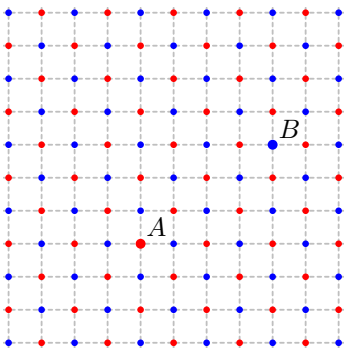
We deduce that $n \geq 20$ and hence $n = 25$.

Finally, we construct such a tournament with 25 players. In this tournament, there are 15 winners, each of them having 14 points of scores and 10 losers each of them having 9 points of score. Any game played between two winners or two loosers results in a tie. For games played between winners and losers, each winner wins in 6 games, loses in 2 games, and 2 games result in a tie and conversely each loser wins 3 games, loses 9 games and 3 games result in a tie. This makes such a tournament possible.

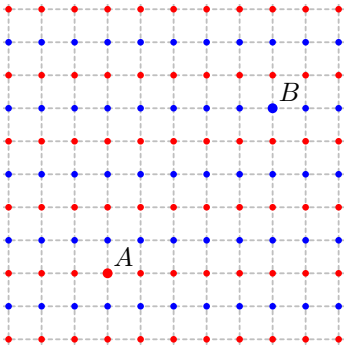
3. We are given a lattice and two pebbles A and B that are placed at two lattice points. At each step we are allowed to relocate one of the pebbles to another lattice point with the condition that the distance between pebbles is preserved. Is it possible after finite number of steps to switch positions of the pebbles?

Solution suggested by the student Mahdi Al-Shaikh Saleh. Consider the lattice $\mathbb{Z}^2$ and $(x_a, y_a), (x_b, y_b)$ the coordinates of positions of pebbles A, B respectively. If we relocate one of the pebbles, say B, at a new position, (x'_b, y'_b), because $(x_b - x_a)^2 + (y_b - y_a)^2 = (x'_b - x_a)^2 + (y'_b - y_a)^2$, by considering this equality in modulo 4, we deduce that the number of even terms among $x_b - x_a, y_b - y_a$ is preserved after each relocation. Whenever both terms are even, one can cancel the common 2 by considering a sublattice to end up with two terms with at most one even term. In this way, we can consider two cases:

- If only one term among $x_b - x_a, y_b - y_a$ is odd, we color the vertices of the lattice in blue and red by alternating the colors between each two consecutive vertices in a horizontal line and two consecutive vertices in a vertical line.



- If both terms $x_b - x_a, y_b - y_a$ are odd, we color the vertices of the lattice in blue and red by making the colors of all vertices in a same horizontal line the same and alternating the colors between each two consecutive horizontal lines.



In both cases, the colors of the vertex positions of pebbels A and B are different, and at each relocation, the colors of their vertex new positions remains unchanged. So, in both cases, it is impossible after finite number of steps to switch positions of the pebbles A and B.

4. Points A_1, B_1, C_1 lie on the sides BC, AC and AB of a triangle ABC, respectively, such that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let I_A, I_B, I_C be the incenters of triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively. Prove that the circumcenter of triangle $I_AI_BI_C$, is the incenter of triangle ABC.

Solution. If we orient positively the line AB from A to B, the line BC from B to C and the line CA from C to A, we get

$$\overline{B_1A} - \overline{AC_1} = \overline{A_1C} - \overline{CB_1} = \overline{C_1B} - \overline{BA_1}.$$

Let A_0, B_0, C_0 be the intouch points of the incircle of triangle ABC and the sides BC, CA, AB, respectively. We have

$$\overline{B_1A} - \overline{AC_1} = \overline{B_1B_0} + \overline{B_0A} - \overline{AC_0} - \overline{C_0C_1} = \overline{B_1B_0} + \overline{C_1C_0},$$

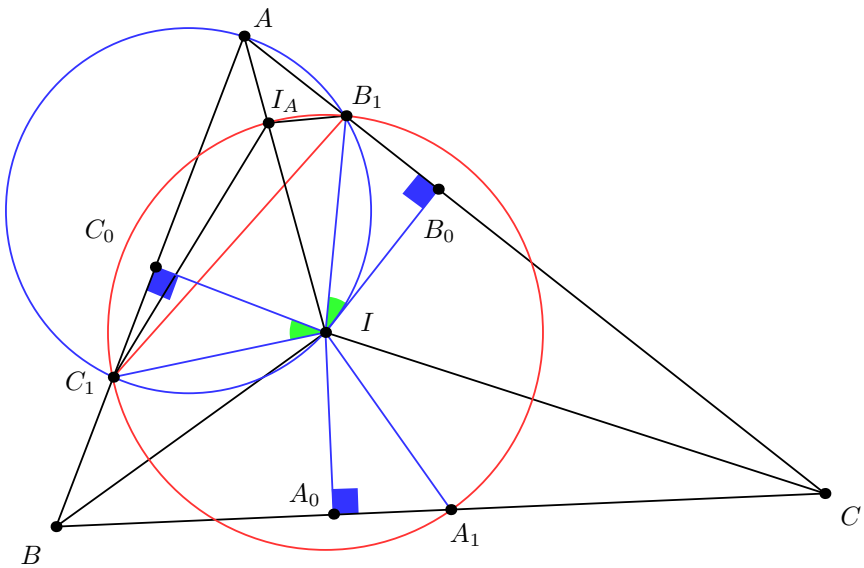
and similarly $\overline{A_1C} - \overline{CB_1} = \overline{A_1A_0} + \overline{B_1B_0}$, and $\overline{C_1B} - \overline{BA_1} = \overline{C_1C_0} + \overline{A_1A_0}$. We deduce that

$$\overline{B_1B_0} + \overline{C_1C_0} = \overline{A_1A_0} + \overline{B_1B_0} = \overline{C_1C_0} + \overline{A_1A_0},$$

which is equivalent to

$$\overline{A_0A_1} = \overline{B_0B_1} = \overline{C_0C_1}.$$

Because $IA_0 = IB_0 = IC_0$ and $\angle A_1A_0I = \angle B_1B_0I = \angle C_1C_0I$, the triangles IA_0A_1, IB_0B_1 and IC_0C_1 are congruent and I is the circumcircle of triangle $A_1B_1C_1$.



Because $\angle AB_0I = \angle BC_0I = 90°$, quadrilateral AC_0IB_0 is cyclic. Because $\angle B_0IB_1 = \angle C_0IC_1$, we have $\angle B_1IC_1 = \angle B_0IC_0 = 180° - \angle B_1AC_1$. We deduce that quadrilateral AC_1IB_1 is cyclic.

Because I is the midpoint of the minor arc $\widehat{C_1B_1}$ of the circumcircle of triangle AC_1B_1 not containing A, we deduce that $IB_1 = IC_1 = II_1$. Therefore I_A is on the circumcircle of triangle $A_1B_1C_1$ of center I.

Similarly, one can prove that I_B, I_C are on the circumcircle of triangle $A_1B_1C_1$. Hence the circumcenter of triangle $I_AI_BI_C$, is the incenter of triangle ABC.

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SAMC 2015

SAUDI ARABIA MATHEMATICAL COMPETITIONS

مسابقات الرّياضيات للمملكة العربيّة السّعوديّة

Malik Talbi

Riyadh, June 2015

مقدّمة

يحوي هذا الملف على مسائل استخدمت في التّدريب والاختيار خلال السّنة الأخيرة من إعداد الفريق الّذي يمثّل المملكة العربيّة السّعوديّة في الأولمبياد الدّولي. إنّ تدريب الفريق كان بدعم من وزارة التّعليم الّتي كلّفت مؤسّسة الملك عبدالعزيز ورجاله للموهبة والابداع - " موهبة " للقيام بالمهمّة. وتجدر الاشارة إلى التّعاون المنقطع النّظير الّذي لمسناه من جامعة الملك عبدالله للعلوم والتّقنيّة، حيث وفّرت لنا كلّ الإمكانيّات الّتي احتجنا لها في التّدريب في حرمها الجامعي. كما نشكر جامعة الملك سعود على تعاونها فيما يتعلّق بالمدرّبين من أساتذة قسم الرّياضيّات. وبودّنا أن نعبّر عن امتناننا البالغ لوزارة التّعليم على كلّ ما بذلته في المجالين الإداري والمالي.

تمّ عقد ثلاث ملتقيات تدريبيّة خلال العام الدّراسي ٢٠١٤ - ٢٠١٥، بالإضافة إلى فترة التّدريب المكثّف الّتي بدأت في شهر أبريل ٢٠١٥، كما شارك الفريق السّعودي في العديد من المسابقات، ومنها: أولمبياد الرّياضيات لدول آسيا والباسيفيك، أولمبياد دول الخليج في الكويت، أولمبياد الطّالبات للدّول الأوربيّة في روسيا البيضاء، وأولمبياد البلقان الّذي عقد في اليونان.

نأمل أن يكون هذا الملف، بما فيه من مسائل، هو إسهاما منّا لتقوية أواصر التّعاون وتبادل الخبرات بيننا والدّول المشاركة في الأولمبياد الدّولي.

د. فوزي بن أحمد الذكير

رئيس الفريق السّعودي للأولمبياد الدّولي

Introduction

This file contains problems used in the training and selection of the Saudi team for the International Mathematics Olympiad. The training was supported by The Ministry of Education, which commissioned **Mawhiba**, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. Our gratitude goes to King Abdullah University for Science and Technology KAUST for making its facilities, in its beautiful campus, available to us for our training. We also thank King Saud University for its cooperation regarding the trainers from the faculty of Mathematics. We are certainly indebted to the Ministry of education for its administrative and financial support.

The Saudi team had three main training camps during the academic year 20142015 beside the full-time training period that started in the first week of April 2015.

During this academic year, the team participated in the following contests: The Asian Pacific Mathematics Olympiad APMO, the Gulf Mathematical Olympiad, in Kuwait in the period April 1-5, 2015, the European Girls Mathematical Olympiad EGMO in Belarus, and the Balkan Mathematical Olympiad BMO, in Greece.

It is our pleasure to share these training and selection tests problems with other IMO teams, hoping it will contribute to future cooperation

Dr. Fawzi A. Al-Thukair
Leader of the Saudi Math Olympiad Team

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Part I

Problems in Arabic

اختبارات ترشيح الطّلاب
للتّدريب المكثّف 2015

مسائل اليوم الأوّل ـ 29 ذو القعدة 1435هـ

الوقت المتاح: 3 ساعات

1– ليكن ABC مثلّثا وD نقطة على ضلعه BC. النقطة E هي تناظر D بالنّسبة للمستقيم AB. النّقطة F هي تناظر E بالنّسبة للمستقيم AC. النقطة P هي تقاطع المستقيمين DF و AC. أثبت أنّ الرّباعي $AEDP$ دائري.

من تأليف مالك طالبي

2– كم توجد من كثيرات حدود P ، ذوات المعاملات الصّحيحة والدّرجة على الأكثر 4 ، الّتي تحقّق $0 \leq P(x) < 72$ لكلّ $x \in \{0, 1, 2, 3, 4\}$ ؟

هارفارد ـ ماساتشوستس 2011

3– جد كلّ الحلول الصّحيحة للمعادلة $x^2y^5 - 2^x5^y = 2015 + 4xy$.

من تأليف مالك طالبي

4– يتمّ تلوين خلايا جدول من القياس 8×8 بأحد اللّونين، أخضر أو أزرق، بحيث توجد a خليّة خضراء في كلّ مربّع قياسه 3×3 و b خليّة خضراء في كلّ مستطيل قياسه 2×4. جد كلّ القيّم الممكنة للزّوج (a, b).

من تأليف لي آن ڤن

مسائل اليوم الثّاني ـ 4 ذو الحجّة 1435هـ

الوقت المتاح: 3 ساعات

1– ليكن ABC مثلّثا و D نقطة على ضلعه BC. يقطع مماس الدّائرة المحيطة بالمثلّث ABD في D الضّلع AC عند E. يقطع مماس الدّائرة المحيطة بالمثلّث ACD في D الضّلع AB عند F. أثبت أنّ الرّأس A ومركزي الدّائرتين المحيطتين بالمثلّثين ABC و DEF على استقامة واحدة.

من تأليف مالك طالبي

2– جد كلّ الدّوال $f : \mathbb{R} \to \mathbb{R}$ الّتي تحقّق

$$f(x + y^2 - f(y)) = f(x),$$

لكلّ $x, y \in \mathbb{R}$.

من تأليف ڤو كوك با كان

3– جد كلّ الحلول الصّحيحة للمعادلة $14^x - 3^y = 2015$.

من تأليف مالك طالبي

4– كم توجد من متتالية أعداد صّحيحة

$$1 \leq a_1 \leq a_2 \leq \ldots \leq a_{11} \leq 2015$$

تحقّق $a_i \equiv i^2 \pmod{12}$ لكلّ $1 \leq i \leq 11$ ؟

من تأليف لي آن ڤن

مسائل اليوم الثّالث ـ 7 ذو الحجّة 1435هـ

الوقت المتاح: 3 ساعات

1– ليكن ABC مثلّثا، I مركزه الدّاخليّ، و D نقطة على القوس $\widehat{BC}$ للدّائرة المحيطة بالمثلّث ABC المقابل للرّأس A. يقطع منصّف الزّاوية $\angle ADB$ الضّلع AB في E. يقطع منصّف الزّاوية $\angle CDA$ الضّلع AC في F. أثبت أنّ النّقط E ، F ، I على استقامة واحدة.

من تأليف مالك طالبي

2– أثبت أنّه لا يمكن كتابة كثيرة الحدود $P(x) = (x^2 - 12x + 11)^4 + 23$ كحاصل ضرب ثلاث كثيرات حدود غير ثابتة، معاملاتها صحيحة.

من تأليف لي آن ڤن

3– لتكن $(a_n)_{n\geq 0}$ متتالية أعداد صحيحة موجبة تحقّق a_n^2 يقسم $a_{n-1}a_{n+1}$ ، لكلّ $n \geq 1$. أثبت أنّه إذا وجد عدد صحيح $k \geq 2$ يجعل a_k و a_1 أوّليّين نسبيّا، فإنّ a_1 يقسم a_0.

من تأليف مالك طالبي

4– هناك 22 كرسيّا على طاولة مستديرة. جد أقلّ قيمة ممكنة للعدد n بحيث كلّما جلست مجموعة مكوّنة من n شخص على الكراسي، يمكننا دائما أن نجد من بينهم شخصين يفصل بينهما كرسيّان أو ثمان كراسي.

من تأليف لي آن ڤن

اختبارات ترشيح الطّلاب لأولمبياد الرّياضيّات للدّول الأعضاء في مكتب التّربية العربي لدول الخليج 2015

مسائل اليوم الأوّل - 1 ربيع الثّاني 1436هـ

الوقت المتاح: 3 ساعات

1- لتكن a ، b ، c أعدادا حقيقيّة موجبة تحقّق $a+b+c=1$. أثبت أنّ

$$2\left(\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\right)+1\geq 6(ab+bc+ca).$$

من تأليف تران نام زونغ

2- لدى سلمان في حقيبته مجموعة حجر، وزن كلّ حجر لا يزيد على 0.5 كلغم، ووزن مجموع الحجر لا يزيد على 2.5 كلغم. أثبت أنّه يمكن سلمان أن يوزّع حجره إلى 4 مجموعات بحيث لا يزيد وزن كلّ مجموعة على 1 كلغم.

من تأليف تران نام زونغ

3- ليكن BD وCE ارتفاعين في مثلّث ABC ملتقى ارتفاعاته H ومركز دائرته المحيطة O. النقطتان M و N هما منتصفا الضّلعين AB و AC ، على التّرتيب، والنّقطة P هي نقطة تقاطع المستقيمين MN وDE. أثبت أنّ المستقيمين AP وOH عموديّان.

من تأليف ليانا توپان

4- ليكن p عددا أوّليّا فرديّا. أثبت أنّ هناك عددا صحيحا وحيدا k يحقّق $0\leq k\leq p^2$ ويجعل المقدار $k(k+1)(k+2)\cdots(k+p-3)-1$ يقبل القسمة على p^2.

من تأليف مالك طالبي

مسائل اليوم الثّاني ـ 5 ربيع الثّاني 1436هـ

الوقت المتاح: 3 ساعات

1– لنعتبر المتتالية $\{x_n\}$ المعرّفة بـ $x_1 = 1$ و $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$ لكلّ $n = 1, 2, 3, \ldots$ ، حيث يرمز $\lfloor x \rfloor$ لأكبر عدد صحيح أقل من أو يساوي x. أثبت أنّه لكلّ عدد صحيح موجب n لدينا

$$x_n x_{n+2} - x_{n+1}^2 = 4^{n-1}.$$

من تأليف تران نام زونغ

2– جد عدد متتاليات الأعداد الصّحيحة الموجبة المتزايدة تماما والّتي تبدأ بـ 0 وتنتهي بـ 15 ، وفي كلّ عنصرين متتاليين يوجد عنصر زوجي وحيد.

من تأليف لي آن ڤن

3– ليكن ABC مثلّثا وG مركزه المتوّسط، وG_a ، G_b ، G_c مساقط G على الأضلاع BC ، CA ، AB ، على التّرتيب. إذا كانت S_a ، S_b ، S_c هي مناظرات G_a ، G_b ، G_c ، على التّرتيب، بالنّسبة للمركز G ، فأثبت أنّ المستقيمات AS_a ، BS_b ، CS_c تلتقي في نقطة واحدة.

من تأليف ليانا توپان

4– نعرّف، لكلّ عدد صحيح موجب n ، المقدار $s(n) = \sum_{k=0}^{n} r_k$ ، حيث r_k هو باقي قسمة $\binom{n}{k}$ على 3. جد كلّ الأعداد الصّحيحة الموجبة n الّتي تحقّق $s(n) \geq n$.

من تأليف مالك طالبي

مسائل اليوم الثّالث ـ 8 ربيع الثّاني 1436هـ

الوقت المتاح: 3 ساعات

1– جد كلّ الدّوال $f:\mathbb{R}\to\mathbb{R}$ الّتي تحقّق الشّروط

$$(a)\ f(1)=1,\quad (b)\ \forall(x,y)\in\mathbb{R}^2,\ f(x+y)=f(x)+f(y),$$

$$(c)\ \forall x\in\mathbb{R}\setminus\{0\},\ f\left(\tfrac{1}{x}\right)=\tfrac{f(x)}{x^2}.$$

من تأليف تران نام زونغ

2– ماهو أكبر عدد من أحجار الفيلة الّتي يمكننا وضعها على طاولة شطرنج من القياس 8×8 بحيث لا يتواجد أكثر من ثلاث فيلة على نفس القطر؟

هارفارد ـ ماساشوستس 2015

3– ليكن ABC مثلّثا يحقّق $AB<AC$ ، و D قدم ارتفاعه عند A ، و M منتصف BC ، و B' تناظر B بالنّسبة للنّقطة D. يقطع المستقيم العمودي على BC عند B' الضّلع AC في النّقطة P. أثبت أنّه إذا كان BP و AM عموديّين فإنّ المثلّث ABC قائم.

من تأليف ليانا توپان

4– ليكن p ، q عددين أوّليّين فرديين مختلفين و n عددا صحيحا بحيث يقسم pq المقدار $n^{pq}+1$. أثبت أنّه إذا قسم p^3q^3 المقدار $n^{pq}+1$ فإنّ p^2 يقسم $n+1$ أو q^2 يقسم $n+1$.

من تأليف مالك طالبي

اختبارات ترشيح الطّلاب
لأولمبياد الرّياضيّات لدول البلقان 2015

مسائل اليوم الأوّل ـ 10 جمادى الآخرة 1436هـ

الوقت المتاح: 4 ساعات

1– أثبت أنّه لكلّ عدد صحيح $n \geq 2$ ، توجد متتالية متهية وحيدة $x_0, x_1, \ldots, x_n$ من الأعداد الحقيقيّة والّتي تحقّق $x_0 = x_n = 0$ و $x_{i+1} - 8x_i^3 - 4x_i + 3x_{i-1} + 1 = 0$ ، لكلّ $i = 1, 2, \ldots, n-1$. أثبت إضافة أنّ $|x_i| \leq \frac{1}{2}$ لكلّ $i = 1, 2, \ldots, n-1$.

من تأليف نوين دوي تاي سن

2– لدينا 2015 مجموعة جزئيّة $A_1, A_2, \ldots, A_{2015}$ من المجموعة $\{1, 2, \ldots, 1000\}$ والّتي تحقّق $|A_i| \geq 2$ لكلّ $i \geq 1$ و $|A_i \cap A_j| \geq 1$ لكلّ $1 \leq i < j \leq 2015$. أثبت أنّ $k = 3$ هو أصغر عدد من الألوان بحيث يمكننا دائما تلوين عناصر المجموعة $\{1, 2, \ldots, 1000\}$ بـ k لون بحيث تشتمل المجموعة الجزئيّة A_i على الأقل على عنصرين بلونين مختلفين لكلّ $i \geq 1$.

من تأليف لي آن ڤن

3– ليكن ABC مثلّثا، و Γ دائرته المحيطة، و I ملتقى منصّفاته، و ω دائرة مماسّة للمستقيم AI في I وللضّلع BC. أثبت أنّ الدّائرتين Γ و ω متماسّتان.

من تأليف مالك طالبي

4– ليكن $n \geq 2$ عددا صحيحا و $p_1 < p_2 < \cdots < p_n$ أعدادا أوّليّة. أثبت أنّه يوجد عدد صحيح k أوّليّ نسبيّا مع $p_1 p_2 \cdots p_n$ ويحقّق $\gcd(k + p_1 p_2 \cdots p_i, p_1 p_2 \cdots p_n) = 1$ لكلّ $i = 1, 2, \ldots, n-1$.

من تأليف مالك طالبي

مسائل اليوم الثّاني ـ 12 جمادى الآخرة 1436هـ

الوقت المتاح: 4 ساعات

1– جد كلّ الدّوال $f:\mathbb{Z}\to\mathbb{R}$ المتزايدة تماما والّتي لكلّ $m,n\in\mathbb{Z}$ يوجد $k\in\mathbb{Z}$ بحيث $f(k)=f(m)-f(n)$.

من تأليف نوين دوي تاي سن

2– جد عدد السّداسيّات $(a_1,a_2,a_3,a_4,a_5,a_6)$ من الأعداد الصحيحة الموجبة المختلفة والّتي تحقّق الشّرطين التّاليين:

ا. $a_1+a_2+a_3+a_4+a_5+a_6=30$؛

ب. يمكننا كتابة a_1,a_2,a_3,a_4,a_5,a_6 على أضلع سداسيّ بحيث بعد اختيار عدد منته من المرّات رأسا للسّداسي وإضافة 1 للعددين المكتوبين على الضّلعين المجاورين للرّأس، نحصل في الأخير على سداسي بأرقام كلّها متساوية.

من تأليف لي آن ڤن

3– ليكن ABC مثلّثا، و H_a ، H_b ، H_c أقدام ارتفاعاته عند الرّؤوس A ، B ، C ، على التّرتيب، و T_a ، T_b ، T_c نقط تماس دائرته المحاطة بالأضلاع BC ، CA ، AB ، على التّرتيب. تتقاطع الدّائرتان المحيطتان بالمثلّثين AH_bH_c و AT_bT_c مرّة ثانية في A' ، والدّائرتان المحيطتان بالمثلّثين BH_cH_a و BT_cT_a مرّة ثانية في B' ، والدّائرتان المحيطتان بالمثلّثين CH_aH_b و CT_aT_b مرّة ثانية في C'. أثبت أنّ النّقط A',B',C' على استقامة واحدة.

من تأليف مالك طالبي

4– أثبت وجود عدد غير منته من الأعداد الصّحيحة الموجبة المركّبة n الّتي تجعل المقدار $7^{n-1}-3^{n-1}$ يقبل القسمة على n.

من تأليف لي آن ڤن

اختبارات ترشيح الطّلاب لأولمبياد الرّياضيّات الدّولي 2015

مسائل اليوم الأوّل ـ 1 شعبان 1436هـ

الوقت المتاح: 4 ساعات ونصف

1– جد كلّ الدّوال $f : \mathbb{R}_{>0} \to \mathbb{R}$ التّي تحقّق

$$f\left(\frac{x}{y}\right) = f(x) + f(y) - f(x)f(y)$$

لكلّ $x, y \in \mathbb{R}_{>0}$، حيث يرمز $\mathbb{R}_{>0}$ لمجموعة الأعداد الحقيقيّة الموجبة.

من تأليف نوين دوي تاي سن

2– ليكن ABC مثلّثا وH ملتقى ارتفاعاته، وP نقطة على المستوى، وΩ الدائرة الّتي يكون فيها AP قطرا. تقطع الدّائرة Ω المستقيمين CA وAB مرّة ثانية في E و F، على التّرتيب. يقطع المستقيم PH الدّائرة Ω مرّة ثانية في G. النّقطة T هي تقاطع المستقيمين المماسّين للدّائرة Ω في النّقطتين E وF. النقطة M هي منتصف الضّلع BC وL هي النقطة على MG الّتي تحقّق $AL \parallel MT$. أثبت أنّ $LA \perp LH$.

من تأليف لي فك لو

3– ليكن n وk عددين صحيحين موجبين. أثبت أنّه إذا كان n أوّليّا نسبيّا مع 30، فإنّه يوجد عددان صحيحان a وb، كلّ منهما أوّليّ نسبيّا مع n، يجعلان المقدار $\dfrac{a^2 - b^2 + k}{n}$ عددا صحيحا.

من تأليف مالك طالبي

مسائل اليوم الثّاني ـ 2 شعبان 1436هـ

الوقت المتاح: 4 ساعات ونصف

4– ليكن ABC مثلّثا حادّا محاطا بالدّائرة (O)، و H قدم ارتفاعه عند الرّأس A، و P نقطة داخل المثلّث ABC تقع على منصّف الزّاوية $\angle BAC$. تقطع الدّائرة الّتي قطرها AP الدّائرة (O) مرّة ثانية في G. ليكن L إسقاط النّقطة P على AH. أثبت أنّه إذا كان المستقيم GL ينصّف القطعة HP فإنّ P هو المركز الدّاخلي للمثلّث ABC.

من تأليف لي فك لو

5– يلعب حمزة وماجد اللّعبة التّالية على طاولة شطرنج أفقيّة من القياس 3×2015 ذات مربّعات بيضاء. يلعب حمزة وماجد بالتّناوب ويبدأ حمزة أوّلا. يختار حمزة في دوره ثلاث مربّعات متراصّة أفقيّا تشكّل مستطيلا من القياس 1×3 فيلوّنها ثمّ يعطي الدّور لماجد. يختار ماجد في دوره ثلاث مربّعات متراصّة عموديّا تشكّل مستطيلا من القياس 3×1 فيلوّنها ثمّ يعطي الدّور لحمزة، وهكذا. لا يسمح لأحد من اللّاعبين أن يختار مربّعا سبق تلوينه فيلوّنه مرّة ثانية. آخر اللّاعبين الّذي يمكنه تلوين مستطيله هو الفائز. من من اللّاعبين، حمزة أو ماجد، يمكنه أن يضمن فوزا في لعبته وذلك مهما كانت لعبة صديقه، وما هي استراتيجيّته؟

من تأليف لي آن ڤن

6– لتكن a_1، a_2، ... ، a_n أعدادا حقيقيّة موجبة تحقّق

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2.$$

أثبت أنّ

$$\sum_{1 \le i < j \le n} a_i a_j (1 - a_i a_j) \ge 0.$$

من تأليف ڤو كوك با كان

مسائل اليوم الثّالث ـ 8 شعبان 1436هـ

الوقت المتاح: 4 ساعات ونصف

7– ليكن S عددا صحيحا موجبا يقبل القسمة على كلّ الأعداد 1، 2، ...، 2015 وa_1، a_2، ...، a_k أعدادا في $\{1, 2, \ldots, 2015\}$ تحقّق

$$2S \leq a_1 + a_2 + \ldots + a_k,$$

أثبت أنّه يمكننا اختيار من بين الأعداد a_1، a_2، ...، a_k أعددا بحيث يساوي مجموع هذه الأعداد المختارة S.

من تأليف لي آن ڤن

8– ليكن ABC مثلّثا و$\mathcal{C}$ دائرته المحيطة. تقع النّقطة D على القوس $\widehat{BC}$ للدّائرة $\mathcal{C}$ (الّذي لا يشمل A) وهي مختلفة عن النّقط B، C ومنتصف القوس $\widehat{BC}$. يقطع المستقيم المماس للدّائرة $\mathcal{C}$ في D المستقيمات BC، CA، AB في A'، B'، C'، على التّرتيب. يتقاطع المستقيمان BB' وCC' في E. يقطع المستقيم AA' الدّائرة $\mathcal{C}$ مرّة ثانية في F. أثبت أنّ النّقط D، E، F على استقامة واحدة.

من تأليف مالك طالبي

9– جد عدد المتتاليات S المكوّنة من 2015 رقم 0 أو 1 بحيث كلّما اخترنا من نفس المتتالية S رزمتين I_1 ، I_2 من الأرقام المتصافّة لهما نفس الطّول، وجدناهما تحقّقان

- الفارق بين مجموع أرقام الرّزمتين I_1 و I_2 لا يتجاوز 1 ؛
- إذا كانت الرّزمة I_1 مكوّنة من الأرقام على أقصى يسار S فإنّ مجموع أرقام I_1 لا يكبر مجموع أرقام I_2 ؛
- إذا كانت الرّزمة I_2 مكوّنة من الأرقام على أقصى يمين S فإنّ مجموع أرقام I_2 لا يصغر مجموع أرقام I_1.

من تأليف لي آن ڤن

مسائل اليوم الرّابع ـ 9 شعبان 1436هـ

الوقت المتاح: 4 ساعات ونصف

10– لتكن a ، b ، c ، d أعدادا صحيحة موجبة تجعل المقدار $ac+bd$ يقبل القسمة على المقدار a^2+b^2. أثبت أنّ $\gcd(c^2+d^2, a^2+b^2) > 1$.

من تأليف تران نام زونغ

11– عدد اللّغات المتكلّم بها في كاوست هو n. لكلّ عدد صحيح موجب $k \leq n$ يرمز A_k لمجموعة الأسخاص في كاوست الّتي تتكلّم على الأقل k لغة؛ ويرمز B_k لمجموعة الأشخاص P في كاوست الّتي لها الخاصيّة التّالية، لكلّ k لغة مختلفة متكلّم بها في كاوست، يستطيع P أن يتكلّم على الأقل واحدا من هذه اللّغات الّتي عددها k. أثبت أنّه

ا. إذا كان $2k \geq n+1$ فإنّ $A_k \subseteq B_k$؛

ب. إذا كان $2k \leq n+1$ فإنّ $A_k \supseteq B_k$.

من تأليف نوين دوي تاي سن

12– لتكن a، b، c أعدادا حقيقيّة موجبة تحقّق

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 10.$$

جد أكبر وأصغر قيمتين للمقدار

$$T = (x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right).$$

من تأليف تران نام زونغ

Part II

Problems in English

Chapter 1

Preselection Tests for the Full-time Training

The students who attended the Full-Time Training that started on April 7, 2015, were chosen according to their combined performance in this set of tests during the Autumn Camp at KAUST University in Thuwal from September 20 to October 1, 2014. The students have been trained during this Camp by: Lê Anh Vinh (co-chair), Liana Topan and Malik Talbi (co-chair).

1.1 Day I - September 24, 2014

Allowed time: 3 hours

1. Let ABC be a triangle and D a point on the side BC. Point E is the symmetric of D with respect to AB. Point F is the symmetric of E with respect to AC. Point P is the intersection of line DF with line AC. Prove that the quadrilateral $AEDP$ is cyclic.

   Suggested by Malik Talbi

2. How many polynomials P of integer coefficients and degree at most 4 satisfy $0 \leq P(x) < 72$ for all $x \in \{0, 1, 2, 3, 4\}$?

   Harvard-MIT Mathematics Tournament 2011

3. Find all integer solutions of the equation $x^2y^5 - 2^x5^y = 2015 + 4xy$.

   Suggested by Malik Talbi

4. We color each unit square of a 8×8 table into green or blue such that there are a green unit squares in each 3×3 square and b green unit squares in each 2×4 rectangle. Find all possible values of (a, b).

   Suggested by Lê Anh Vinh

1.2 Day II - September 28, 2014

Allowed time: 3 hours

1. Let ABC be a triangle and D a point on the side BC. The tangent line to the circumcircle of the triangle ABD at the point D intersects the side AC at E. The tangent line to the circumcircle of the triangle ACD at the the point D intersects the side AB at F. Prove that the point A and the circumcenters of the triangles ABC and DEF are collinear.

   Suggested by Malik Talbi

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

   $$f(x + y^2 - f(y)) = f(x),$$

   for all $x, y \in \mathbb{R}$.

   Suggested by Võ Quôc Bá Cãn

3. Find all integer solutions of the equation $14^x - 3^y = 2015$.

   Suggested by Malik Talbi

4. How many sequences of integers

   $$1 \leq a_1 \leq a_2 \leq \ldots \leq a_{11} \leq 2015$$

   that satisfy $a_i \equiv i^2 \pmod{12}$ for all $1 \leq i \leq 11$ are there?

   Suggested by Lê Anh Vinh

1.3 Day III - October 1, 2014

Allowed time: 3 hours

1. Let ABC be a triangle, I its incenter, and D a point on the arc $\widehat{BC}$ of the circumcircle of ABC not containing A. The bisector of the angle $\angle ADB$ intesects the segment AB at E. The bisector of the angle $\angle CDA$ intesects the segment AC at F. Prove that the points E, F, I are collinear.

   Suggested by Malik Talbi

2. Prove that the polynomial $P(X) = (X^2 - 12X + 11)^4 + 23$ can not be written as the product of three non-constant polynomials with integer coefficients.

   Suggested by Lê Anh Vinh

3. Let $(a_n)_{n\geq 0}$ be a sequence of positive integers such that a_n^2 divides $a_{n-1}a_{n+1}$, for all $n \geq 1$. Prove that if there exists an integer $k \geq 2$ such that a_k and a_1 are relatively prime, then a_1 divides a_0.

   Suggested by Malik Talbi

4. There are 22 chairs in a round table. Find the minimum n such that for any group of n people sitting in the table, we always can find two people with exactly 2 or 8 chairs between them.

   Suggested by Lê Anh Vinh

Chapter 2

Team Selection Tests for the Gulf Mathematical Olympiad 2015

The KSA 2015 Gulf MO team members were chosen according to their combined performance in this set of tests during the Winter Camp at KAUST University in Thuwal from January 17 to 28, 2015.

Members of the KSA 2015 Gulf MO team were Alzubair Habibullah, Shaden Al-Shammari, Omar Alrabiah, Majid Almarhumi, Renad Abualjmal, Hamza Shafi. The team was ranked 1st place among the 7 participating teams. The team's individual performances were as follows:

Alzubair Habibullah	GOLD Medallist
Shaden Al-Shammari	GOLD Medallist
Omar Alrabiah	SILVER Medallist
Majid Almarhumi	SILVER Medallist
Renad Abualjmal	SILVER Medallist
Hamza Shafi	SILVER Medallist

The students have been trained during the Winter Camp by: Liana Topan, Malik Talbi (co-chair) and Trân Nam Dũng.

2.1 Day I - January 21, 2015

Allowed time: 3 hours

1. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$2\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) + 1 \geq 6(ab + bc + ca).$$

Suggested by Trân Nam Dũng

2. In his bag, Salman has a number of stones. The weight of each stone is not greater than 0.5kg and the total weight of the stones is not greater than 2.5kg. Prove that Salman can divide his stones into 4 groups, each group has a total weight not greater than 1kg.

Suggested by Trân Nam Dũng

3. Let BD and CE be altitudes of an arbitrary scalene triangle ABC with orthocenter H and circumcenter O. Let M and N be the midpoints of sides AB, respectively AC, and P the intersection point of lines MN and DE. Prove that lines AP and OH are perpendicular.

Suggested by Liana Topan

4. Let p be an odd prime number. Prove that there exists a unique integer k such that $0 \leq k \leq p^2$ and p^2 divides $k(k+1)(k+2)\cdots(k+p-3) - 1$.

Suggested by Malik Talbi

2.2 Day II - January 25, 2015

Allowed time: 3 hours

1. Let be given the sequence (x_n) defined by $x_1 = 1$ and $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$ for all $n = 1, 2, 3, \ldots$, where $\lfloor x\rfloor$ denotes the greatest integer that does not exceed x. Prove that for any positive integer n we have

$$x_n x_{n+2} - x_{n+1}^2 = 4^{n-1}.$$

Suggested by Trân Nam Dũng

2. Find the number of strictly increasing sequences of nonnegative integers with the first term 0 and the last term 15, and among any two consecutive terms, exactly one of them is even.

Suggested by Lê Anh Vinh

3. Let ABC be a triangle and G its centroid. Let G_a, G_b and G_c be the orthogonal projections of G on sides BC, CA, respectively AB. If S_a, S_b and S_c are the symmetrical points of G_a, G_b, respectively G_c with respect to G, prove that AS_a, BS_b and CS_c are concurrent.

Suggested by Liana Topan

4. For each positive integer n, define $s(n) = \sum_{k=0}^{n} r_k$, where r_k is the remainder when $\binom{n}{k}$ is divided by 3. Find all positive integers n such that $s(n) \geq n$.

Suggested by Malik Talbi

2.3 Day III - January 28, 2015

Allowed time: 3 hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

$$(a)\ f(1) = 1, \quad (b)\ \forall (x, y) \in \mathbb{R}^2,\ f(x + y) = f(x) + f(y),$$

$$(c)\ \forall x \in \mathbb{R} \setminus \{0\},\ f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}.$$

Suggested by Trân Nam Dũng

2. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

Harvard-MIT Mathematics Tournament 2015

3. Let ABC be a triangle, with $AB < AC$, D the foot of the altitude from A, M the midpoint of BC, and B' the symmetric of B with respect to D. The perpendicular line to BC at B' intersects AC at point P. Prove that if BP and AM are perpendicular then triangle ABC is right-angled.

Suggested by Liana Topan

4. Let p, q be two different odd prime numbers and n an integer such that pq divides $n^{pq} + 1$. Prove that if p^3q^3 divides $n^{pq} + 1$ then either p^2 divides $n + 1$ or q^2 divides $n + 1$.

Suggested by Malik Talbi

Chapter 3

Team Selection Tests for the Balkan Mathematical Olympiad 2015

The KSA 2015 Balkan MO team members were chosen according to their combined performance in this set of tests during the Spring Camp at KAUST University in Thuwal from March 27 to April 2, 2015.

Members of the KSA 2015 Balkan MO team were Alzubair Habibullah, Alrabiah Omar, Alnufaili Alhamzah, Saleh Salman, Renad Abualjmal. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Tarek S. Shehata (Mawhiba, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh) as observer A of the KSA delegation. The total team scores was 79 out of 200, and was ranked 12th place among the 21 participating teams. The team's individual performances were as follows:

Alzubair Habibullah	BRONZE Medallist
Omar Alrabiah	BRONZE Medallist
Alnufaili Alhamzah	BRONZE Medallist
Saleh Salman	BRONZE Medallist
Renad Abualjmal	

The students have been trained during the Spring Camp by: Lê Anh Vinh (co-chair), Liana Topan, Malik Talbi (co-chair) and Nguyên Duy Thái Son.

3.1 Day I - March 30, 2015

Allowed time: 4 hours

1. Prove that for any integer $n \geq 2$, there exists a unique finite sequence $x_0, x_1, \ldots, x_n$ of real numbers which satisfies $x_0 = x_n = 0$ and $x_{i+1} - 8x_i^3 - 4x_i + 3x_{i-1} + 1 = 0$ for all $i = 1, 2, \ldots, n-1$. Prove moreover that $|x_i| \leq \frac{1}{2}$ for all $i = 1, 2, \ldots, n-1$.

   Suggested by Nguyên Duy Thái Son

2. Given 2015 subsets $A_1, A_2, \ldots, A_{2015}$ of the set $\{1, 2, \ldots, 1000\}$ such that $|A_i| \geq 2$ for every $i \geq 1$ and $|A_i \cap A_j| \geq 1$ for every $1 \leq i < j \leq 2015$. Prove that $k = 3$ is the smallest number of colors such that we can always color the elements of the set $\{1, 2, \ldots, 1000\}$ by k colors with the property that the subset A_i has at least two elements of different colors for every $i \geq 1$.

   Suggested by Lê Anh Vinh

3. Let ABC be a triangle, Γ its circumcircle, I its incenter, and ω a tangent circle to the line AI at I and to the side BC. Prove that the circles Γ and ω are tangent.

   Suggested by Malik Talbi

4. Let $n \geq 2$ be an integer and $p_1 < p_2 < \cdots < p_n$ prime numbers. Prove that there exists an integer k relatively prime with $p_1 p_2 \cdots p_n$ and such that $\gcd(k + p_1 p_2 \cdots p_i, p_1 p_2 \cdots p_n) = 1$ for all $i = 1, 2, \ldots, n-1$.

   Suggested by Malik Talbi

3.2 Day II - April 1, 2015

Allowed time: 4 hours

1. Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{R}$ such that for any $m, n \in \mathbb{Z}$ there exists a $k \in \mathbb{Z}$ such that $f(k) = f(m) - f(n)$.

Suggested by Nguyên Duy Thái Son

2. Find the number of 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of distinct positive integers satisfying the following two conditions:

   (a) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 30$;

   (b) We can write $a_1, a_2, a_3, a_4, a_5, a_6$ on sides of a hexagon such that after a finite number of time choosing a vertex of the hexagon and adding 1 to the two numbers written on two sides adjacent to the vertex, we obtain a hexagon with equal numbers on its sides.

Suggested by Lê Anh Vinh

3. Let ABC be a triangle, H_a, H_b and H_c the feet of its altitudes from A, B and C, respectively, T_a, T_b, T_c its intouch points on the sides BC, CA and AB, respectively. The circumcircles of triangles AH_bH_c and AT_bT_c intersect again at A'. The circumcircles of triangles BH_cH_a and BT_cT_a intersect again at B'. The circumcircles of triangles CH_aH_b and CT_aT_b intersect again at C'. Prove that the points A', B', C' are collinear.

Suggested by Malik Talbi

4. Prove that there exist infinitely many non prime positive integers n such that $7^{n-1} - 3^{n-1}$ is divisible by n.

Suggested by Lê Anh Vinh

Chapter 4

Team Selection Tests for the International Mathematical Olympiad 2015

The KSA 2015 IMO team members were chosen according to their combined performance in this set of tests during the Full-time Training Camp at KAUST University in Thuwal from April 7 to May 29, 2015. The students have been trained during the Full-Time Training Camp by: Lê Anh Vinh (co-chair), Lê Phúc Lũ, Malik Talbi (co-chair), Nguyên Duy Thái Son, Pham Kim Hung, Trân Nam Dũng, Võ Quôc Bá Cãn, Vũ Thé Khôi and Wei Li.

Members of the KSA 2015 IMO team were Alhamzah Alnufaili, Alzubair Habibullah, Ammar Alqatari, Omar Alrabiah, Salman Saleh and Shaden Al-Shammari. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Abdulrahman Alguwaizani (Mawhiba, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Sultan Albarakati (MAWHIBA, Riyadh), Dr. Abdulrahman Albarak (Ministry of Education) and Amjad Almajed (MAWHIBA, Riyadh) as observers of the KSA delegation.

Members of the KSA 2014 IMO team were Omar Alrabiah, Mahdi Alshaikh Saleh, Ibraheem Khan, Alzubair Habibullah, Shaden Alshammari and Salman Saleh. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulrahman Alguwaizani (MAWHIBA, Riyadh), Sultan

Albarakati (MAWHIBA, Riyadh), Mansour Almoaigel and Dr. Abdulrahman Albarak (Ministry of Education) as observers of the KSA delegation. The total team scores was 103 out of 252, and was ranked 43nd place among the 101 participating teams. The team's individual performances were as follows:

Omar Alrabiah	BRONZE Medallist
Mahdi Alshaikh Saleh	BRONZE Medallist
Ibraheem Khan	BRONZE Medallist
Alzubair Habibullah	BRONZE Medallist
Shaden Alshammari	Honourable mention
Salman Saleh	Honourable mention

4.1 Day I - May 19, 2015

Allowed time: 4 hours and half

1. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$f\left(\frac{x}{y}\right) = f(x) + f(y) - f(x)f(y)$$

for all $x, y \in \mathbb{R}_{>0}$. Here, $\mathbb{R}_{>0}$ denotes the set of all positive real numbers.

Suggested by Nguyên Duy Thái Son

2. Let ABC be a triangle with orthocenter H. Let P be any point of the plane of the triangle. Let Ω be the circle with the diameter AP. The circle Ω cuts CA and AB again at E and F, respectively. The line PH cuts Ω again at G. The tangent lines to Ω at E, F intersect at T. Let M be the midpoint of BC and L be the point on MG such that AL and MT are parallel. Prove that LA and LH are orthogonal.

Suggested by Lê Phúc Lũ

3. Let n and k be two positive integers. Prove that if n is relatively prime with 30, then there exist two integers a and b, each relatively prime with n, such that $\dfrac{a^2 - b^2 + k}{n}$ is an integer.

Suggested by Malik Talbi

4.2 Day II - May 20, 2015

Allowed time: 4 hours and half

4. Let ABC be an acute-angled triangle inscribed in the circle (O), H the foot of the altitude of ABC at A and P a point inside ABC lying on the bisector of $\angle BAC$. The circle of diameter AP cuts (O) again at G. Let L be the projection of P on AH. Prove that if GL bisects HP then P is the incenter of the triangle ABC.

Suggested by Lê Phúc Lũ

5. Hamza and Majid play a game on a horizontal 3×2015 white board. They alternate turns, with Hamza going first. A legal move for Hamza consists of painting three unit squares forming a horizontal 1×3 rectangle. A legal move for Majid consists of painting three unit squares forming a vertical 3×1 rectangle. No one of the two players is allowed to repaint already painted squares. The last player to make a legal move wins. Which of the two players, Hamza or Majid, can guarantee a win no matter what strategy his opponent chooses and what is his strategy to guarantee a win?

Suggested by Lê Anh Vinh

6. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Prove that

$$\sum_{1 \le i < j \le n} a_i a_j (1 - a_i a_j) \ge 0.$$

Suggested by Võ Quôc Bá Cãn

4.3 Day III - May 26, 2015

Allowed time: 4 hours and half

7. Let S be a positive integer divisible by all the integers $1, 2, \ldots, 2015$ and $a_1, a_2, \ldots, a_k$ numbers in $\{1, 2, \ldots, 2015\}$ such that

$$2S \leq a_1 + a_2 + \ldots + a_k,$$

Prove that we can select from $a_1, \ldots, a_k$ some numbers so that the sum of these selected numbers is equal to S.

Suggested by Lê Anh Vinh

8. Let ABC be a triangle and $\mathcal{C}$ its circumcircle. Point D lies on the arc $\widehat{BC}$ (not containing A) of $\mathcal{C}$ and is different from B, C and the midpoint of $\widehat{BC}$. The tangent line to $\mathcal{C}$ at D intersects lines BC, CA, AB at A', B', C' respectively. Lines BB' and CC' intersect at E. Line AA' intersects again circle $\mathcal{C}$ at F. Prove that the three points D, E, F are colinear.

Suggested by Malik Talbi

9. Find the number of binary sequences S of length 2015 such that for any two segments I_1, I_2 of S of the same length, we have

- The sum of digits of I_1 differs from the sum of digits of I_2 by *at most* 1;
- If I_1 begins on the left end of S then the sum of digits of I_1 is *not greater than* the sum of digits of I_2;
- If I_2 ends on the right end of S then the sum of digits of I_2 is *not less than* the sum of digits of I_1.

Suggested by Lê Anh Vinh

4.4 Day IV - May 27, 2015

Allowed time: 4 hours and half

10. Let a, b, c, d be positive integers such that $ac+bd$ is divisible by a^2+b^2. Prove that $\gcd(c^2+d^2, a^2+b^2) > 1$.

Suggested by Trân Nam Dũng

11. The total number of languages used in KAUST is n. For each positive integer $k \leq n$, let A_k be the set of all those people in KAUST who can speak at least k languages; and let B_k be the set of all people P in KAUST with the property that, for any k pairwise different languages (used in KAUST), P can speak at least one of these k languages. Prove that

    (a) If $2k \geq n+1$ then $A_k \subseteq B_k$;

    (b) If $2k \leq n+1$ then $A_k \supseteq B_k$.

Suggested by Nguyên Duy Thái Son

12. Let a, b, c be positive real numbers satisfying the condition

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 10.$$

Find the greatest value and the least value of

$$T = (x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right).$$

Suggested by Trân Nam Dũng

Part III

Solutions

Chapter 1

Solutions to The Preselection Tests for the Full-time Training

1.1 Solutions to The Preselection Test of Day I

1. Let ABC be a triangle and D a point on the side BC. Point E is the symmetric of D with respect to AB. Point F is the symmetric of E with respect to AC. Point P is the intersection of line DF with line AC. Prove that the quadrilateral $AEDP$ is cyclic.

   Solution. Let $\alpha = \angle BAC$ and $\theta = \angle BAD$. Because E is the symmetric of D with respect to AB, we have $AD = AE$ and DE is perpendicular to AB. We deduce that $\angle EAD = 2\theta$ and $\angle DEA = 90^\circ - \theta$.

   Because F is the symmetric of E with respect to AC, we have $AE = AF$ and EF is perpendicular to AC. We deduce that

   $$\angle CAF = \angle EAC = \angle EAB + BAC = \theta + \alpha$$

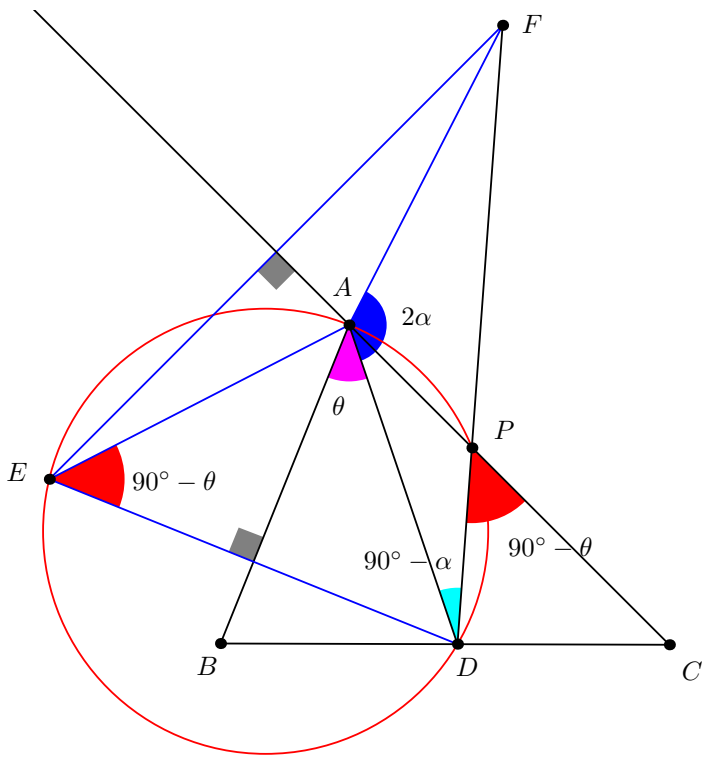
   and

   $$\begin{aligned} \angle PDA &= 90^\circ - \tfrac{1}{2}\angle DAF = 90^\circ - \tfrac{1}{2}\left(\angle DAC + CAF\right) \\ &= 90^\circ - \tfrac{1}{2}\left((\alpha - \theta) + (\alpha + \theta)\right) = 90^\circ - \alpha. \end{aligned}$$

   We deduce that

   $$\angle DPC = \angle PDA + \angle DAC = (90^\circ - \alpha) + (\alpha - \theta) = 90^\circ - \theta = \angle DEA.$$

   This proves that quadrilateral $AEDP$ is cyclic.



Remarks. 1) We could deduce that $\angle DAF = 2\alpha$ directly from the fact that the composition of the two symmetries is a rotation of center A and angle 2α. 2) All angles considered here are oriented. So if $90° - \alpha$ or $90° - \theta$ are negative one should rather consider the oriented angles $\angle ADP$, $\angle AED$ and $\angle CPD$.

2. How many polynomials P of integer coefficients and degree at most 4 satisfy $0 \leq P(x) < 72$ for all $x \in \{0, 1, 2, 3, 4\}$?

Solution. Remember that if $P_0(X), P_1(X), P_2(X), \ldots$ are monic polynomials with integer coefficients such that $\deg P_k(X) = k$ for all $k = 0, 1, 2, \ldots$ then any polynomial $P(X)$ with integer coefficients can be written in a unique way as

$$P(X) = a_0 P_0(X) + a_1 P_1(X) + \cdots + a_n P_n(X),$$

where $a_0, a_1, \ldots, a_n$ are integers and $a_n \neq 0$. This can be proved by induction on the degree of $P(X)$ using Euclidean algorithm.
Therefore, let us write our polynomial as:

$$P(x) = a_0 x(x-1)(x-2)(x-3) + a_1 x(x-1)(x-2) + a_2 x(x-1) + a_3 x + a_4.$$

Since $0 \le a_4 = P(0) < 72$, there are 72 possible choices for a_4.
For any choice of a_4, we have $-a_4 \le a_3 = P(1) - a_4 < 72 - a_4$. Hence, we also have 72 possible choices for a_3.
For each choice of a_3 and a_4 we have

$$-\frac{2a_3 + a_4}{2} \le a_3 = \frac{P(2) - 2a_3 - a_4}{2} < \frac{72}{2} - \frac{2a_3 + a_4}{2}.$$

Hence, there are 36 possible choices for a_2.
Keep doing that way, the total number of polynomials is

$$72 \times 72 \times (72/2) \times (72/6) \times (72/24) = 72 \times 72 \times 36 \times 12 \times 3 = 2592^2.$$

3. Find all integer solutions of the equation $x^2y^5 - 2^x5^y = 2015 + 4xy$.

Solution. Notice that if $x < 0$ or $y < 0$ then the term 2^x5^y is not an integer while $x^2y^5 - 2015 - 4xy$ is an integer which is impossible. If $x = 0$ or $y = 0$ then the equation becomes $-2^x5^y = 2015$ which has no solution. We deduce that both x, y are positive integers. Moreover, because $2^x5^y + 2015 + 4xy$ is an odd integer, both x, y are odd integers.

Working modulo 5, we get $x^2y \equiv 4xy$ and therefore either $x \equiv 0, 4$ or $y \equiv 0 \pmod 5$.

Working modulo 8, and because x, y are odd, we get $y - 2^x \cdot 5 \equiv -1 + 4$ which is equivalent to $y \equiv 5(2^x - 1) \pmod 8$. Therefore, either $x = 1$ and $y \ge 5$ or $x \ge 3$ and $y \equiv 3 \pmod 8$.

If $x = 1$ and $y \ge 5$ then the equation becomes $y^5 - 2 \cdot 5^y = 2015 + 4y$ and has no solution since the left hand side is negative (We prove it by induction on $y \ge 5$).

If $x = 3$ then $y \equiv 3 \pmod 8$. But $y \equiv 0 \pmod 5$. Therefore $y \ge 35$ and $9 \cdot y^5 - 8 \cdot 5^y$ is negative so the equation has no solution.

If $x \ge 5$ and $y \ne 3$ then $y \ge 5$. In this case $x^2 < 2^x$ and $y^5 \le 5^y$ (We prove it by induction) and therefore $x^2y^5 - 2^x5^y$ is negative and the equation has no solution.

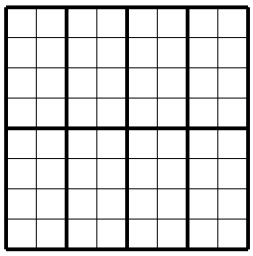
If $x \ge 7$ and $y = 3$ then because $x \equiv 0, 4 \pmod 5$, $x \ge 9$. Therefore $x^2 \cdot 3^5 - 2^x5^3 < 3^5(x^2 - 2^{x-2})$ is negative and therefore the equation has no solution.

If $x = 5$ and $y = 3$, the equation is satisfied.

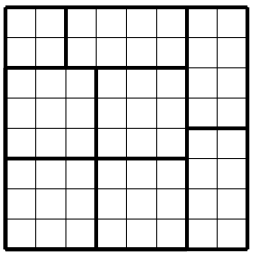
Hence the unique solution is $(5, 3)$.

4. We color each unit square of a 8×8 table into green or blue such that there are a green unit squares in each 3×3 square and b green unit squares in each 2×4 rectangle. Find all possible values of (a, b).

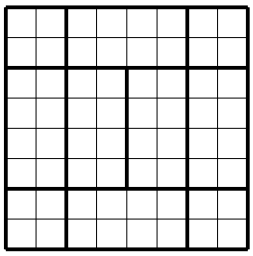
   Solution. By tiling our 8×8 table by eight 2×4 rectangles like in Tiling (1) we find that the total number of green unit squares in the table is $8b$.
   By tiling our 8×8 table by four 3×3 squares, three 2×4 rectangles and one 2×2 square, like in Tiling (2), we find that the total number of green unit squares in the table is $4a+3b+x$, where x is the number of green unit square in the 2×2 square in the left upper corner. Notice that by just rotating our tiling (2) so that the 2×2 square occupies at each time the four corners of the table, we deduce that the four 2×2 unit squares in the four corners, all have the same number of green unit squares x.
   Finally, by tiling our 8×8 table by six 2×4 rectangles and four 2×2 squares, like in Tiling (3), we find that the total number of green unit squares in the table is $6b + 4x$



Tiling (1)



Tiling (2)



Tiling (3)

   Putting together all these totals, we obtain the relations

$$8b = 4a + 3b + x = 6b + 4x,$$

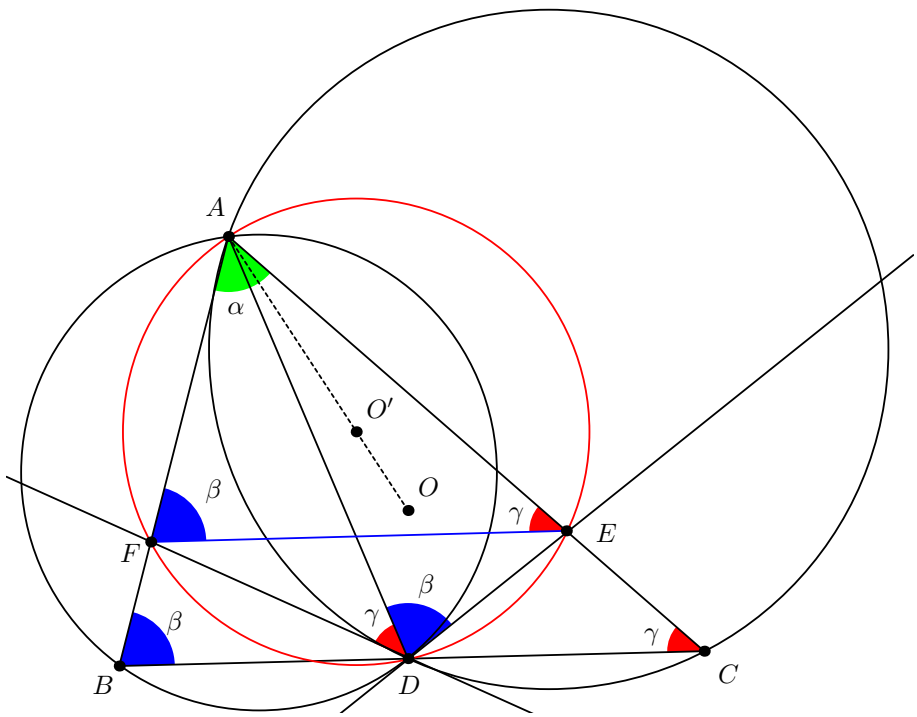
   from which we deduce that $b = 2x$ and $9x = 4a$. Since $0 \leq a \leq 9$, either $a = 0$ and $b = 0$ or $a = 9$ and $b = 8$. In other words, either we color the whole table or we dont color at all.

1.2 Solutions to The Preselection Test of Day II

1. Let ABC be a triangle and D a point on the side BC. The tangent line to the circumcircle of the triangle ABD at the point D intersects the side AC at E. The tangent line to the circumcircle of the triangle ACD at the the point

D intersects the side AB at F. Prove that the point A and the circumcenters of the triangles ABC and DEF are collinear.

Solution 2. Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$ and $\theta = BAD$. Because line DE is tangent to the circumcircle of triangle ABD, we have $\angle EDA = \angle DBA = \beta$. Because line DF is tangent to the circumcircle of triangle ADC, we have $\angle ADF = \angle ACD = \gamma$.



Therefore

$$\angle EDF + \angle FAE = \angle EDA + \angle ADF + \angle BAC = \gamma + \beta + \alpha = 180^\circ.$$

This proves that quadrilateral $AFDE$ is cyclic and hence $\angle EFA = \angle EDA = \beta$ and $\angle AEF = \angle ADF = \gamma$. This proves that sides EF and BC are parallel. Let O and O' be circumcenters of triangle ABC and AFE, respectively. We have

$$\angle FAO' = 90^\circ - \angle AEF = 90^\circ - \gamma = \angle BAO.$$

This proves that A, O and O' are collinear.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x + y^2 - f(y)) = f(x),$$

for all $x, y \in \mathbb{R}$.

Solution. Notice that the function f, defined by $f(x) = x^2$ for all $x \in \mathbb{R}$, is a solution of the equation.

Assume that there exists $a \in \mathbb{R}$ such that $f(a) \neq a^2$ and let $b = a^2 - f(a) \neq 0$. Then for all $x \in \mathbb{R}$ we have $f(x + b) = f(x)$. Therefore, for all $x, y \in \mathbb{R}$, let $z = \dfrac{y - x - b^2}{2b}$ we have

$$\begin{aligned} f(x) &= f(x + (z + b)^2 - f(z + b)) = f(x + 2bz + b^2 + z^2 - f(z)) \\ &= f(x + 2bz + b^2) = f(y). \end{aligned}$$

We deduce that f is constant. Conversely, any constant function satisfied the given functional equation.

3. Find all integer solutions of the equation $14^x - 3^y = 2015$.

Solution. Notice that $14^x \geq 2015$ which implies that $x \geq 3$. On the other hand $3^y = 14^x - 2015 \geq 14^3 - 2015 = 729 = 3^6$. We deduce that $y \geq 6$.

We have $3^y = 14^x - 2015 \equiv 1 \pmod 7$. But $\mathrm{Ord}_7 3 = 6$. We deduce that $y = 6k$ for some positive integer k.

We have $0 \equiv 2015 \equiv 14^x - 3^{6k} \equiv (-1)^x - (-1)^k \pmod 5$. We deduce that x and k have the same parity.

Because $y \geq 6$, we have $5^x \equiv 14^x - 3^y \equiv 2015 \equiv -1 \pmod 9$. This implies that $x \equiv 3 \pmod 6$ and therefore both x and k are odd numbers.

Now, assume that $x > 3$. This means that $14^x \equiv 0 \pmod{16}$ which implies that $9^k \equiv 3^{6k} \equiv 1 \pmod{16}$. But $\mathrm{Ord}_{16} 9 = 2$, which means that k is even, which is a contradiction.

We deduce that $x = 3$ and hence $y = 6$ which gives the unique solution of the equation.

4. How many sequences of integers

$$1 \leq a_1 \leq a_2 \leq \ldots \leq a_{11} \leq 2015$$

that satisfy $a_i \equiv i^2 \pmod{12}$ for all $1 \leq i \leq 11$ are there?

Solution. Let r_i be the remainder when i^2 is divided by 12, and $a_i = 12k_i + r_i$ for some nonnegative integer k_i, when $1 \leq i \leq 11$. From the following table showing the values of r_i

i	1	2	3	4	5	6	7	8	9	10	11
r_i	1	4	9	4	1	0	1	4	9	4	1

we deduce that the inequality

$$1 \leq a_1 \leq a_2 \leq \ldots \leq a_{11} \leq 2015$$

is equivalent to the inequality

$$0 \leq k_1 \leq k_2 \leq k_3 < k_4 < k_5 < k_6 \leq k_7 \leq k_8 \leq k_9 < k_{10} < k_{11} \leq 167$$

which is equivalent to

$$0 \leq k_1 < k_2 + 1 < k_3 + 2 < k_4 + 2 < k_5 + 2 < k_6 + 2 <$$

$$< k_7 + 3 < k_8 + 4 < k_9 + 5 < k_{10} + 5 < k_{11} + 5 \leq 172.$$

Therefore, there are $\binom{173}{11}$ such sequences.

1.3 Solutions to The Preselection Test of Day III

1. Let ABC be a triangle, I its incenter, and D a point on the arc $\widehat{BC}$ of the circumcircle of ABC not containing A. The bisector of the angle $\angle ADB$ intesects the segment AB at E. The bisector of the angle $\angle CDA$ intesects the segment AC at F. Prove that the points E, F, I are collinear.

Solution. We give 3 different solutions to this problem.

First solution. Let P be the intersection point of the bisector of the angle $\angle BAC$ with the segment EF. Because both I and P are on the bisector of $\angle BAC$, the problem is equivalent to showing that $AP = AI$.

Because DE is the biscetor of $\angle ADB$ we have from the bisector theorem

$$\frac{AE}{AD} = \frac{BE}{BD} = \frac{AB}{AD + BD}.$$

Because DF is the biscetor of $\angle CDA$ we have from the bisector theorem

$$\frac{AF}{AD} = \frac{CF}{CD} = \frac{AC}{AD + CD}.$$

We deduce that

$$\begin{aligned}\frac{1}{AE} + \frac{1}{AF} &= \frac{AD \cdot AC + BD \cdot AC + AD \cdot AB + CD \cdot AB}{AB \cdot AC \cdot AD} \\ &= \frac{AD \cdot AC + AD \cdot AB + AD \cdot BC}{AB \cdot AC \cdot AD} = \frac{AB + BC + CA}{AB \cdot AC}\end{aligned}$$

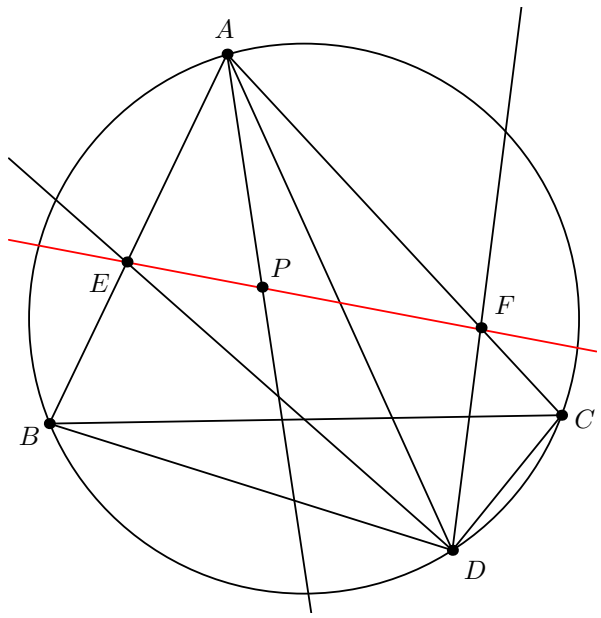
from Ptolemy's theorem applied to the cyclic quadrilateral $ABDC$.

Now from the areas of triangles AEP, APF and AEF, we have

$$\frac{1}{2}AE \cdot AP \sin\frac{\alpha}{2} + \frac{1}{2}AF \cdot AP \sin\frac{\alpha}{2} = \frac{1}{2}AE \cdot AF \sin\alpha,$$

from which we deduce

$$\begin{aligned}AP &= \frac{2AE \cdot AF \cos\frac{\alpha}{2}}{AE + AF} = \frac{2AB \cdot AC \cos\frac{\alpha}{2}}{AB + BC + CA} = \frac{AB \cdot AC \cos\frac{\alpha}{2}}{s} \\ &= \frac{4R\Delta \cos\frac{\alpha}{2}}{s \cdot BC} = \frac{2r \cos\frac{\alpha}{2}}{\sin\alpha} = \frac{r}{\sin\frac{\alpha}{2}} = AI,\end{aligned}$$



where $\alpha = \angle BAC$, $s = \frac{AB+BC+CA}{2}$ is the semi-perimeter of triangle ABC, R its circumradius, r its inradius and $\Delta = \frac{AB \cdot BC \cdot CA}{4R} = rs$ its area.

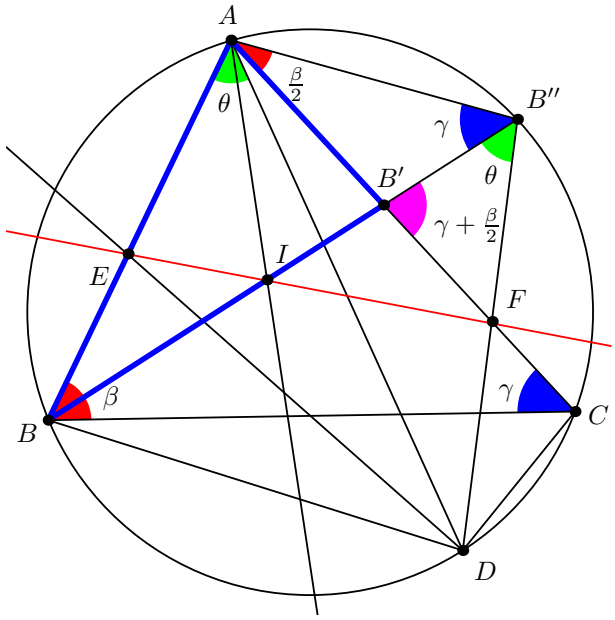
Second Solution. Let B' and B'' be the intersection points of the bisector of the angle $\angle CBA$ with the side BC and the arc $\widehat{BC}$. Denote $\beta = \angle CBA$, $\gamma = \angle ACB$ and $\theta = \angle BAD$.

We have $\angle B'B''F = \angle BB''D = \angle BAD = \theta$ and $\angle FB'B'' = \angle CAB'' + \angle AB''B = \angle CBB'' + \angle ACB = \frac{\beta}{2} + \gamma$. We deduce from sine law in triangle $B'FB''$ that

$$\frac{B'F}{FB''} = \frac{\sin\theta}{\sin\left(\frac{\beta}{2}+\gamma\right)}.$$

We have $\angle AB''F = \angle AB''B + \angle BB''D = \angle ACB + \angle BAD = \gamma + \theta$ and $\angle FAB'' = \angle CAB'' = \angle CBB'' = \frac{\beta}{2}$. We deduce from sine law in triangle AFB'' that

$$\frac{FA}{FB''} = \frac{\sin(\gamma+\theta)}{\sin\frac{\beta}{2}}.$$



We deduce that

$$\frac{\overline{B'F}}{\overline{FA}} = -\frac{\sin\theta\sin\frac{\beta}{2}}{\sin\left(\frac{\beta}{2}+\gamma\right)\sin(\gamma+\theta)}.$$

Because DE is the bisector of angle $\angle ADB$, we have from the theorem of bisectors

$$\frac{\overline{AE}}{\overline{EB}} = \frac{AD}{DB} = \frac{\sin\angle AB''D}{\sin\angle DAB} = \frac{\sin(\gamma+\theta)}{\sin\theta}.$$

Because AI is the bisector of angle $\angle BAB'$, we have from the theorem of bisectors and sine law in triangle ABB'

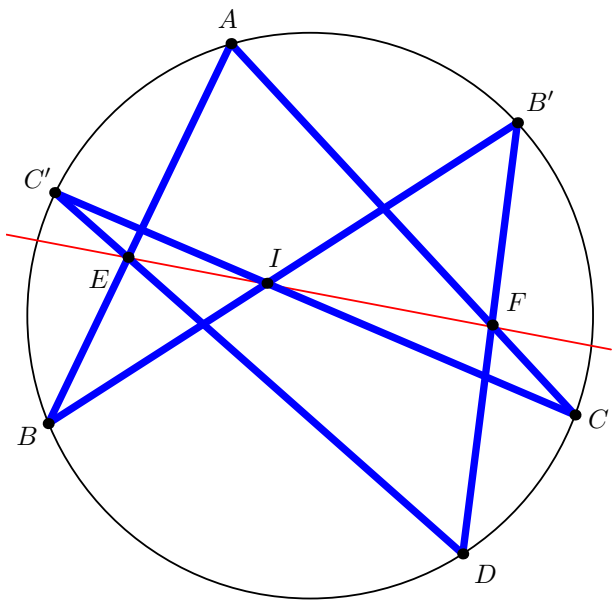
$$\frac{\overline{BI}}{\overline{IB'}} = \frac{AB}{AB'} = \frac{\sin\angle AB'B}{\sin\angle B'BA} = \frac{\sin\left(\frac{\beta}{2}+\gamma\right)}{\sin\frac{\beta}{2}}.$$

Therefore

$$\frac{\overline{AE}}{\overline{EB}} \cdot \frac{\overline{BI}}{\overline{IB'}} \cdot \frac{\overline{B'F}}{\overline{FA}} = -1$$

and from the theorem of Menelaus applied to the triangle ABB', the points E, I, F are collinear.

Third Solution. Let B' and C' be the midpoints of the arcs $\widehat{CA}$ and $\widehat{AB}$, respectively. Because DB' and DC' are the bisectors of the angles $\angle ADB$ and $\angle CDA$, respectively, point E is the intersection of AB and DC', point I is the intersection of BB' and $C'C$ and point F is the intersection of $B'D$ and CA. We deduce from Pascal's theorem applied to the cyclic hexagon $ABB'DC'C$ that the three points E, I and F are collinear.



2. Prove that the polynomial $P(X) = (X^2 - 12X + 11)^4 + 23$ can not be written as the product of three non-constant polynomials with integer coefficients.

   Solution. Suppose for the sake of contradiction that

$$P(X) = Q(X)H(X)R(X),$$

   where $Q(X), H(X), R(X)$ are non-constant polynomials with integer coefficients. Since $P(x) > 0$ for every $x \in \mathbb{R}$, the degrees of $Q(X), H(X), R(X)$ are all even. It implies that two of these three polynomials are quadratic. Suppose that $\deg Q(X) = \deg H(X) = 2$.

   Now, $P(1) = P(11) = 23$, implies that $Q(1), Q(11)$ are divisors of 23. This means that $Q(1), Q(11) \in \{\pm 1, \pm 23\}$. But because 10 divides $Q(11) - Q(1)$ we have $Q(1) = Q(11)$. Similarly, we have $H(1) = H(11)$.

   Besides, $Q(1)H(1)$ is a divisor of 23 so at least one of $Q(1)$ or $H(1)$ is ± 1. Suppose without loss of generality that $Q(1) = \pm 1$ then $Q(11) = Q(1) = \pm 1$. This implies that $Q(X) = (X - 1)(X - 11) \pm 1$. But this implies that $Q(X)$ has a real root while $P(x)$ is positive for all $x \in \mathbb{R}$, which is a contradiction.

3. Let $(a_n)_{n \geq 0}$ be a sequence of positive integers such that a_n^2 divides $a_{n-1}a_{n+1}$, for all $n \geq 1$. Prove that if there exists an integer $k \geq 2$ such that a_k and a_1 are relatively prime, then a_1 divides a_0.

   Solution. Assume, for the sake of contradiction, that there exists an integer $k \geq 2$ such that a_k and a_1 are relatively prime and that a_1 does not divide a_0. We deduce that there exists a prime number p such that $0 \leq v_p(a_0) < v_p(a_1)$, where $v_p(a)$ is the $p-$adic valuation of the integer a, that is the greatest integer m such that p^m divides a.

   Assume that there exists an integer $m \geq 0$, for which we have $v_p(a_m) < v_p(a_{m+1})$. Because a_{m+1}^2 divides $a_m a_{m+2}$, we have $2v_p(a_{m+1}) \leq v_p(a_m) + v_p(a_{m+2})$, and therefore $v_p(a_{m+1}) < v_p(a_{m+2})$. This proves by induction that the sequence $v_p(a_n)$ is increasing, and therefore $0 < v_p(a_1) < v_p(a_k)$. This implies that p divides $\gcd(a_1, a_k) = 1$, which is a contradiction.

4. There are 22 chairs in a round table. Find the minimum n such that for any group of n people sitting in the table, we always can find two people with exactly 2 or 8 chairs between them.

   Solution. We consider the graph with 22 vertices $0, 1, \ldots, 21$. Two vertices x and y are connected by an edge if $x - y \equiv \pm 3, \pm 9 \mod 22$. We consider only ± 3 module 22, then we have a cycle through all vertices. If we choose any $n \geq 12$ vertices, there are two vertices (corresponding to 2 people) with exactly

2 chairs between them. If $n = 11$, we can choose 11 vertices: $0, 6, 12, \ldots, 60 \mod 22$. It is clear that $10x \not\equiv \pm 3, \pm 9 \mod 22$ for every x. It means there is an arrangement for 11 people such that no two people with exactly 2 or 8 chairs between them.

Chapter 2

Solutions to The Team Selection Tests for the Gulf Mathematical Olympiad 2015

2.1 Solutions to The Team Selection Test of Day I

1. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$2\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) + 1 \geq 6(ab + bc + ca).$$

Solution. Using the condition $a + b + c = 1$, we have

$$\begin{aligned}\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) &= (a+b+c)\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right)\\ &= (ab+bc+ca) + abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)\\ &\geq (ab+bc+ca) + \frac{9abc}{2(a+b+c)}\\ &\geq (ab+bc+ca) + \tfrac{9}{2}abc,\end{aligned}$$

by Cauchy-Schwarz inequality. Therefore, it remains to prove that

$$9abc + 1 \geq 4(ab + bc + ca).$$

Using again the condition $a + b + c = 1$, the last inequality becomes

$$9abc + (a+b+c)^2 \geq 4(a+b+c)(ab+bc+ca),$$

which is equivalent, by simple algebraic manipulations, to

$$9abc + a^2 + b^2 + c^2 \geq 2(ab+bc+ca).$$

Using a third time the condition $a+b+c = 1$, the obtained inequality becomes

$$9abc + (a+b+c)(a^2+b^2+c^2) \geq 2(a+b+c)(ab+bc+ca),$$

which is equivalent, by simple algebraic manipulations, to

$$3abc + a^3 + b^3 + c^3 \geq a^2(b+c) + b^2(c+a) + c^2(a+b).$$

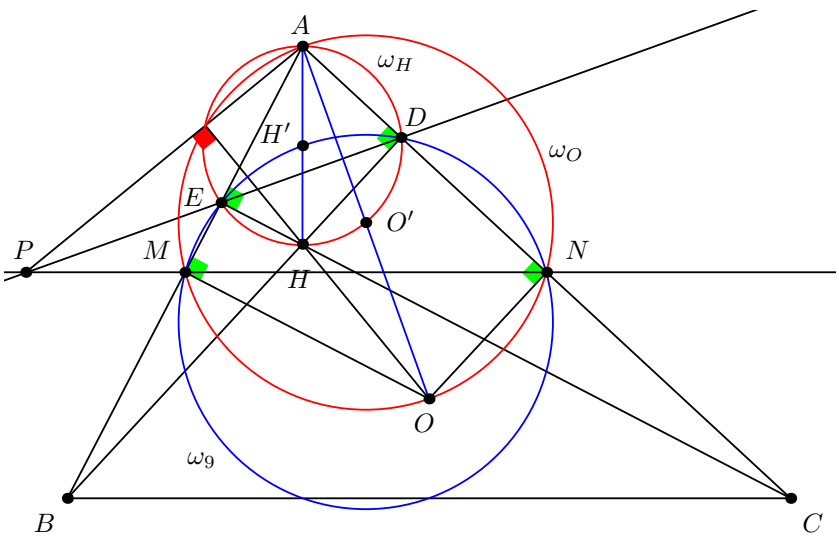
This last inequality is nothing but Schur's inequality. This solves the problem.

2. In his bag, Salman has a number of stones. The weight of each stone is not greater than 0.5kg and the total weight of the stones is not greater than 2.5kg. Prove that Salman can divide his stones into 4 groups, each group has a total weight not greater than 1kg.

Solution. Let k be the number of stones. Note that the sum of weights of any two stones is not greater than 1kg.
Thus, if $k \leq 8$ we can divide these stones into 4 (or less) groups, each group has 1 or 2 stones. This division obviously fulfills the condition.
If $k = 9$, we take 3 lightest stones. Their total weight must not be greater than $\frac{2.5}{3} < 1$kg. We put these stones together into one group and distribute the other 6 into 3 groups, 2 stones in each. This division again fulfills the condition.
Now, consider the general case. Whenever two or more stones have the total weight less or equal to 0.5 kg then we merge them into one new stone. Since the number of stones is finite, this process must terminate and our new stones have the property that the sum of the weights of any two stones is greater than 0.5kg. We deduce that the number of our new stones is less or equal to 9, otherwise, we will have at least 5 pairs of stones with total weight greater than 0.5kg each, and the total weight of all the stones will be greater than 2.5kg. Applying what we have done in the first two cases, we can divide these new stones as required. Obviously, the same division applies for the original unmerged stones.

3. Let BD and CE be altitudes of an arbitrary scalene triangle ABC with orthocenter H and circumcenter O. Let M and N be the midpoints of sides AB, respectively AC, and P the intersection point of lines MN and DE. Prove that lines AP and OH are perpendicular.

Solution. Because $\angle ADH = \angle HEA = 90°$, quadrilateral $AEHD$ is cyclic with AH a diameter of its circumcircle ω_H.
Because $\angle ANO = \angle OMA = 90°$, quadrilateral $AMON$ is cyclic with AO a diameter of its circumcircle ω_O.
Let ω_9 be the nine point circle of the triangle ABC. It is the circumcircle of quadrilateral $DEMN$.



Because the circles ω_H and ω_9 intersect at D and E, the line DE is radical axis of ω_H and ω_9. Because the circles ω_O and ω_9 intersect at M and N, the line MN is radical axis of ω_O and ω_9. We deduce that P is the radical center of the circles ω_H, ω_O and ω_9, and therefore, the line AP is the radical axis of the circles ω_H and ω_H, which is perpendicular to the line $H'O'$, where H' is the center of ω_H and O' the center of ω_O. But H' is the midpoint of AH and O' is the midpoint of AO. We deduce that AP is perpendicular to HO.

4. Let p be an odd prime number. Prove that there exists a unique integer k such that $0 \le k \le p^2$ and p^2 divides $k(k+1)(k+2)\cdots(k+p-3)-1$.

Solution. Assume that there exists an integer k such that $0 \leq k \leq p^2$ and p^2 divides $k(k+1)(k+2)\cdots(k+p-3)-1$. Because p does not divide any of the $p-2$ consecutive integers $k, (k+1), (k+2), \ldots, (k+p-3)$, either $k \equiv 1$ or $2 \bmod p$. If $k \equiv 2 \bmod p$ then, by Wilson's theorem,

$$k(k+1)(k+2)\cdots(k+p-3)-1 \equiv (p-1)!-1 \equiv -2 \not\equiv 0 \bmod p.$$

We deduce that $k \equiv 1 \bmod p$.
On the other hand, we know from Wilson's theorem that $(p-2)!-1 \equiv 0 \bmod p$. This means that there exists a unique integers $i_0 \in \{1, \ldots, p\}$ such that

$$(p-2)!-1 \equiv i_0 p \bmod p^2.$$

Let $k = ip+1$ for some integer $i \in \{0, 1, \ldots, p-1\}$. We have

$$\begin{aligned} k(k+1)(k+2)\cdots(k+p-3)-1 &\equiv (ip+1)(ip+2)\cdots(ip+p-2)-1 \\ &\equiv (p-2)!-1+ip\sum_{j=1}^{p-2}\frac{(p-2)!}{j} \bmod p^2, \end{aligned}$$

and

$$\sum_{j=1}^{p-2}\frac{(p-2)!}{j} \equiv (p-2)!\sum_{j=1}^{p-2} j^{-1} \equiv \sum_{j=1}^{p-2} j \equiv \frac{(p-2)(p-1)}{2} \equiv 1 \bmod p.$$

We deduce that

$$k(k+1)(k+2)\cdots(k+p-3)-1 \equiv (i_0+i)p \bmod p^2.$$

This proves that $k(k+1)(k+2)\cdots(k+p-3)-1 \equiv 0 \bmod p^2$ only for $k = (p-i_0)p+1$.

Remark. By using Hensel's lemma applied to the polynomial $P(X) = X(X+1)(X+2)\cdots(X+p-3)-1$, we can prove that for any positive integer n there exists a unique integer k such that $0 \leq k \leq p^n$ and p^n divides $k(k+1)(k+2)\cdots(k+p-3)-1$.

2.2 Solutions to The Team Selection Test of Day II

1. Let be given the sequence (x_n) defined by $x_1 = 1$ and $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$ for all $n = 1, 2, 3, \ldots$ where $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x. Prove that for any positive integer n we have

$$x_n x_{n+2} - x_{n+1}^2 = 4^{n-1}.$$

Solution. It is easy to prove by induction that x_n is an integer for all $n \geq 1$. Therefore, we have $x_{n+1} = \lfloor (3+\sqrt{5})x_n \rfloor$ for all $n \geq 1$, and we deduce that

$$\begin{aligned} x_{n+2} &= \lfloor (3+\sqrt{5})x_{n+1} \rfloor = 6x_{n+1} + \lfloor -(3-\sqrt{5})\lfloor (3+\sqrt{5})x_n \rfloor \rfloor \\ &= 6x_{n+1} + \left\lfloor -4\frac{\lfloor (3+\sqrt{5})x_n \rfloor}{(3+\sqrt{5})} \right\rfloor = 6x_{n+1} - 4x_n. \end{aligned}$$

Indeed, for the last equality we have

$$(3+\sqrt{5})x_n - 1 < \lfloor (3+\sqrt{5})x_n \rfloor \leq (3+\sqrt{5})x_n,$$

from which we deduce

$$-4x_n \leq -4\frac{\lfloor (3+\sqrt{5})x_n \rfloor}{(3+\sqrt{5})} < -4x_n + \frac{4}{3+\sqrt{5}} < -4x_n + 1.$$

Let $y_n = x_n x_{n+2} - x_{n+1}^2$, for all $n \geq 1$. We will prove by induction that $y_n = 4^{n-1}$, for all $n \geq 1$.

For $n = 1$, we have $x_1 = 1$, $x_2 = \lfloor (3+\sqrt{5}) \rfloor = 5$ and $x_3 = \lfloor 5(3+\sqrt{5}) \rfloor = 26$, and therefore $y_1 = 4^0$. Assume $y_n = 4^{n-1}$ for some positive integer n. We have

$$\begin{aligned} y_{n+1} &= x_{n+1}x_{n+3} - x_{n+2}^2 = x_{n+1}(6x_{n+2} - 4x_{n+1}) - x_{n+2}(6x_{n+1} - 4x_n) \\ &= 4y_n = 4^n. \end{aligned}$$

This ends the induction and solves the problem.

2. Find the number of strictly increasing sequences of nonnegative integers with the first term 0 and the last term 15, and among any two consecutive terms, exactly one of them is even.

Solution. Let A_n be the set of such sequences with the last term is n instead of 15, and $a_n = |A_n|$. We will show that (a_n) is in fact the Fibonacci sequence and deduce that $a_{15} = 610$.

We can check easily that $a_1 = a_2 = 1$. For $n \geq 1$, we consider the second-last term of each sequence in A_{n+2}. We have two cases.

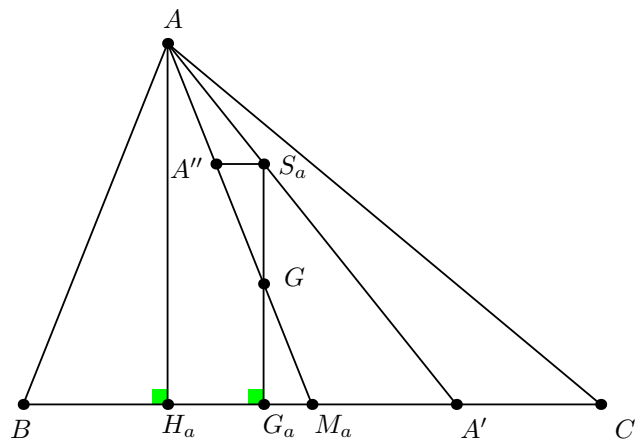
Case 1. The second-last term is $n+1$. Then we can leave out the last term to get an element of A_{n+1}. Conversely, for each element of A_{n+1}, we can add $n+2$ to the end to have an element of A_{n+2}. Hence, in this case, we have a_{n+1} sequences.

Case 2. The second-last term is at most n. But n and $n+2$ have the same parity, so the second-last term is at most $n-1$. When we substitute the last term $n+2$ with n, we have an element of A_n. Conversely, if we replace n in an element of A_n by $n+2$, we have an element of A_{n+2}. Hence, in this case, we have a_{n+2} sequences.

This proves that $a_{n+2} = a_{n+1} + a_n$, and therefore (a_n) is the Fibonacci sequence.

3. Let ABC be a triangle and G its centroid. Let G_a, G_b and G_c be the orthogonal projections of G on sides BC, CA, respectively AB. If S_a, S_b and S_c are the symmetrical points of G_a, G_b, respectively G_c with respect to G, prove that AS_a, BS_b and CS_c are concurrent.

Solution. Let H_a be the foot of altitude from A, M_a the midpoint of side BC, A' the intersection point of line AS_a and side BC and A'' the intersection point of the parallel line to BC passing through S_a with AM_a.



Because GG_a and AH_a are parallel, we have

$$\frac{H_aM_a}{G_aM_a} = \frac{AM_a}{GM_a} = 3.$$

Because S_aA'' and BC are parallel, we have

$$\frac{G_aM_a}{S_aA''} = \frac{G_aG}{S_aG} = 1.$$

We also have

$$\frac{S_aA''}{A'M_a} = \frac{AA''}{AM_a} = 1 - \frac{M_aA''}{AM_a} = 1 - 2\frac{M_aG}{AM_a} = \frac{1}{3}.$$

Multiplying these three relations we deduce that $A'M_a = M_aH_a$, which means that AS_a and AH_a are isotomic conjugate. Similarly, BS_b and BH_b are isotomic conjugate and CS_c and CH_c are isotomic conjugate. Since the altitudes of a triangle are concurrent, so are AS_a, BS_b and CS_c.

4. For each positive integer n, define $s(n) = \sum_{k=0}^{n} r_k$, where r_k is the remainder when $\binom{n}{k}$ is divided by 3. Find all positive integers n such that $s(n) \geq n$.

Solution. Let $0 \leq k \leq n$ be two integers written in basis 3 as

$$n =^{(3)} \overline{n_m n_{m-1} \cdots n_1 n_0}, \qquad k =^{(3)} \overline{k_m k_{m-1} \cdots k_1 k_0},$$

such that $n_m \neq 0$. We know from Lucas's theorem that

$$r_k \equiv \binom{n}{k} \equiv \prod_{i=0}^{m} \binom{n_i}{k_i} \pmod 3.$$

If there exists $0 \leq i \leq m$ such that $k_i > n_i$ then $\binom{n_i}{k_i} = 0$ and therefore $r_k = 0$. Therefore, from now on we will restrict to $0 \leq k \leq n$ with $k_i \leq n_i$ for all $0 \leq i \leq m$.

Let $0 \leq i \leq m$. If $n_i = 0$ then $k_i = 0$. If $n_i = 1$ then $k_i \in \{0, 1\}$. In all these cases $\binom{n_i}{k_i} = 1$. If $n_i = 2$ then $k_i \in \{0, 1, 2\}$ and $\binom{n_i}{k_i} = 1$ if $k_i \neq 1$ and $\binom{n_i}{k_i} = 2$ if $k_i = 1$. Therefore, if the number a of pairs $(n_i, k_i) = (2, 1)$, when $0 \leq i \leq m$, is odd then $\binom{n}{k} \equiv 2^a \equiv 2 \pmod 3$, othewise $\binom{n}{k} \equiv 1 \pmod 3$. We deduce that

$$s(n) = 2A(n) + B(n),$$

where $A(n)$ is the number of pairs of integers (n, k) such that $0 \leq k_i \leq n_i$ for all $0 \leq i \leq m$ and the number of pairs $(n_i, k_i) = (2, 1)$, when $0 \leq i \leq m$, is odd, while $B(n)$ is the number of pairs of integers (n, k) such that $0 \leq k_i \leq n_i$

for all $0 \le i \le m$ and the number of pairs $(n_i, k_i) = (2, 1)$, when $0 \le i \le m$, is even.

Denote by r the number of $0's$ in $n_0, n_1, n_2, \ldots, n_m$, by s the number of $1's$ and by t the number of $2's$. We have

$$A(n) = 2^s \left(\binom{t}{1} 2^{t-1} + \binom{t}{3} 2^{t-3} + \binom{t}{5} 2^{t-5} + \cdots \right)$$

and

$$B(n) = 2^s \left(\binom{t}{0} 2^{t} + \binom{t}{2} 2^{t-2} + \binom{t}{4} 2^{t-4} + \cdots \right).$$

We deduce that

$$\begin{aligned} s(n) &= 2^s \left(\left(\binom{t}{0} + \binom{t}{1} \right) 2^t + \left(\binom{t}{2} + \binom{t}{3} \right) 2^{t-2} + \cdots \right) \\ &= 2^s \left(\binom{t+1}{1} 2^t + \binom{t+1}{3} 2^{t-2} + \cdots \right) \\ &= 2^s \cdot \frac{(2+1)^{t+1} - (2-1)^{t+1}}{2} = 2^{s-1} \left(3^{t+1} - 1 \right). \end{aligned}$$

(a) If the first digit on the left of n is 2 then

$$\begin{aligned} n \ge \overline{2 \underbrace{00 \cdots 0}_{r} \underbrace{11 \cdots 1}_{s} \underbrace{22 \cdots 2}_{t-1}} &= 2 \cdot 3^{r+s+t-1} + \frac{3^s - 1}{2} \cdot 3^{t-1} + 2 \cdot \frac{3^{t-1} - 1}{2} \\ &= \frac{4 \cdot 3^{r+s} + 3^s + 1}{2} \cdot 3^{t-1} - 1. \end{aligned}$$

Therefore

$$n - s(n) \ge \frac{(4 \cdot 3^r + 1) \cdot 3^s - 9 \cdot 2^s + 1}{2} \cdot 3^{t-1} - 1 + 2^{s-1}.$$

- If $r \ge 1$ then $n - s(n) \ge \dfrac{13 - 9 + 1}{2} \cdot \dfrac{1}{3} - 1 + \dfrac{1}{2} > 0$
- If $r = 0$ and $s \ge 2$ then

$$n - s(n) \ge \frac{45 \cdot 3^{s-2} - 36 \cdot 2^{s-2} + 1}{2} \cdot 3^{t-1} - 1 + 2 > 0.$$

- If $r = 0$ and $s = 1$ then $n \le \overline{\underbrace{22 \cdots 2}_{t} 1} = 3^{t+1} - 2 < 3^{t+1} - 1 = s(n)$.

- If $r = s = 0$ then $n = \overline{\underbrace{22\cdots2}_{t}} = 3^t - 1 < \dfrac{3^{t+1}-1}{2} = s(n)$.

(b) If the first digit on the left of n is 1 then $s \geq 1$ and

$$n \geq \overline{1\underbrace{00\cdots0}_{r}\underbrace{11\cdots1}_{s-1}\underbrace{22\cdots2}_{t}} = 3^{r+s+t-1} + \frac{3^{s-1}-1}{2}\cdot 3^t + 2\cdot\frac{3^t-1}{2}$$

$$= \frac{2\cdot 3^{r+s-1} + 3^{s-1} + 1}{2}\cdot 3^t - 1.$$

Therefore

$$n - s(n) \geq \frac{(2\cdot 3^r + 1)\cdot 3^{s-1} - 6\cdot 2^{s-1} + 1}{2}\cdot 3^t + 2^{s-1} - 1.$$

- If $r \geq 1$ and because $s \geq 1$ we have

$$n - s(n) \geq \frac{7\cdot 3^{s-1} - 6\cdot 2^{s-1} + 1}{2}\cdot 3^t + 2^{s-1} - 1 \geq 1 + 1 - 1 > 0.$$

- If $r = 0$ and $s \geq 3$ then

$$n - s(n) \geq \frac{3\cdot\left(3^{s-1} - 2^s\right) + 1}{2}\cdot 3^t + 2^{s-1} - 1 \geq 2 + 4 - 1 > 0.$$

- If $r = 0$ and $s = 2$ then $n \leq \overline{1\underbrace{22\cdots2}_{t}1} = 2\left(3^{t+1} - 1\right) = s(n)$, and the equality is satisfied precisely when $n = \overline{1\underbrace{22\cdots2}_{t}1}$.

- If $r = 0$ and $s = 1$ then

$$n = \overline{1\underbrace{22\cdots2}_{t}} = 3^{t+1} - 3^t - 1 < 3^{t+1} - 1 = s(n).$$

Finally, we deduce that if in the digits of n written in basis 3,

(a) there is at least one 0, or three 1's, or no 0's and two 1's with the digit on the left is 2, then $s(n) < n$;

(b) $n = \overline{122\cdots21}$, then $s(n) = n$;

(c) there are no 0's and two 1's with the digit on the left is 1 and the digit on the right is 2, or there are no 0's and at most one 1, then $s(n) > n$.

2.3 Solutions to The Team Selection Test of Day III

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

$$(a)\ f(1) = 1, \quad (b)\ \forall (x, y) \in \mathbb{R}^2,\ f(x + y) = f(x) + f(y),$$

$$(c)\ \forall x \in \mathbb{R} \setminus \{0\},\ f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}.$$

Solution. Put $x = y = 0$ into equation (b) to get $f(0) = 0$. Put $y = -x$ into equation (b) to get $f(-x) = -f(x)$ and deduce that $f(x - y) = f(x) - f(y)$. Let $x \in \mathbb{R} \setminus \{0, 1\}$. We have

$$f\left(\frac{1}{x-1}\right) = \frac{f(x-1)}{(x-1)^2} = \frac{f(x)-1}{(x-1)^2}.$$

In the other hand, we

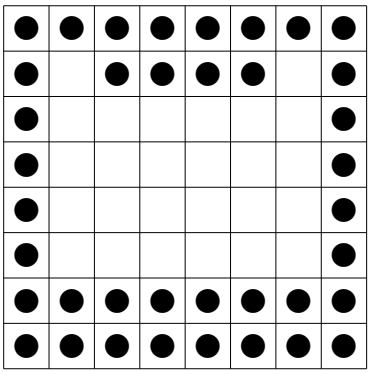
$$\begin{aligned} f\left(\frac{1}{x-1}\right) &= f\left(\frac{1}{x-1} + 1\right) - 1 = f\left(\frac{x}{x-1}\right) - 1 = \frac{f\left(\frac{x-1}{x}\right)}{\left(\frac{x-1}{x}\right)^2} - 1 \\ &= \frac{1 - f\left(\frac{1}{x}\right)}{\left(\frac{x-1}{x}\right)^2} - 1 = \frac{x^2 - f(x)}{(x-1)^2} - 1 = \frac{2x - 1 - f(x)}{(x-1)^2}. \end{aligned}$$

We deduce from these two relations that for all $x \in \mathbb{R} \setminus \{0, 1\}$, we have $f(x) - 1 = 2x - 1 - f(x)$ which is equivalent to $f(x) = x$. Therefore, $f(x) = x$ for all $x \in \mathbb{R}$. Conversely, this function satisfies all the conditions of the problem.

2. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

Solution. If the chessboard is colored black and white as usual, then any diagonal is a solid color. So we may consider bishops on black and white squares separately.

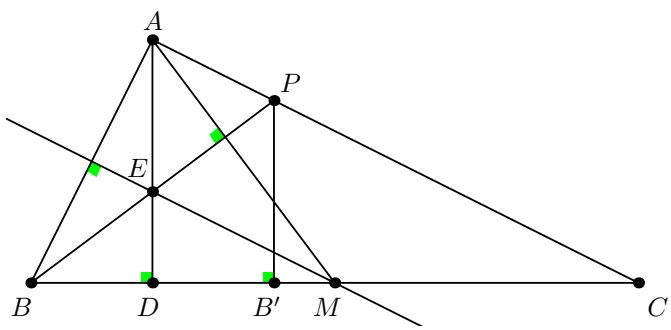
In one direction, the lengths of the black diagonals are 2, 4, 6, 8, 6, 4, and 2. Each of these can have at most three bishops, except the first and last diagonals which can have at most two, giving a total of at most $2 + 3 + 3 + 3 + 3 + 3 + 2 = 19$ bishops on black squares. Likewise there can be at most 19 bishops on white squares for a total of at most 38 bishops.



Conversely, if we place 38 bishops on the four boundaries of the table and on the second and seventh rows except the second and seventh square of the second row, as shown in the picture, one can check that this arrangement satisfies the condition of the problem.

3. Let ABC be a triangle, with $AB < AC$, D the foot of the altitude from A, M the midpoint of BC, and B' the symmetric of B with respect to D. The perpendicular line to BC at B' intersects AC at point P. Prove that if BP and AM are perpendicular then triangle ABC is right-angled.

   Solution. Let E be the intersection point of AD and BP. Because AD is perpendicular to BM and BP is perpendicular to AM, the point E is the orthocenter of triangle ABM and therefore ME is perpendicular to AB.



Because point D is the midpoint of the segment BB' and DE and $B'P$ are parallel, we deduce that point E is the midpoint of segment BP. But M is the midpoint of segment BC. We deduce that lines ME and BC are parrallel.

It follows that AB and AC are perpendicular and triangle ABC is right-angled.

4. Let p, q be two different odd prime numbers and n an integer such that pq divides $n^{pq}+1$. Prove that if p^3q^3 divides $n^{pq}+1$ then either p^2 divides $n+1$ or q^2 divides $n+1$.

   Solution. Because pq divides $n^{pq}+1$, neither p nor q divides n.

   Assume $p < q$. We have from Fermat's little theorem

$$0 \equiv n^{pq}+1 \equiv n^q+1 \pmod{p}.$$

   Therefore $n^{2q} \equiv 1 \pmod{p}$. But $n^{p-1} \equiv 1 \pmod{p}$ and $\gcd(p-1, q) = 1$, since $p-1 < q$ and q is prime. We deduce that $n^2 \equiv 1 \pmod{p}$ and hence p divides $n+1$.

   Let $n+1 = ap$ for an integer a. We have

$$\begin{aligned} n^{pq-1} - n^{pq-2} + \cdots - n + 1 &\equiv \sum_{i=0}^{pq-1} (1-ap)^i \equiv \sum_{i=0}^{pq-1} (1-iap) \\ &\equiv pq - \frac{pq-1}{2} ap^2 q \equiv pq \not\equiv 0 \pmod{p^2}. \end{aligned}$$

   Therefore p^2 does not divide $n^{pq-1} - n^{pq-2} + \cdots - n + 1$ and since p^3 divides

$$n^{pq}+1 = (n+1)\left(n^{pq-1} - n^{pq-2} + \cdots - n + 1\right),$$

   then p^2 divides $n+1$.

Chapter 3

Solutions to The Team Selection Tests for the Balkan Mathematical Olympiad 2015

3.1 Solutions to The Team Selection Test of Day I

1. Prove that for any integer $n \geq 2$, there exists a unique finite sequence $x_0, x_1, \ldots, x_n$ of real numbers which satisfies $x_0 = x_n = 0$ and $x_{i+1} - 8x_i^3 - 4x_i + 3x_{i-1} + 1 = 0$ for all $i = 1, 2, \ldots, n-1$. Prove moreover that $|x_i| \leq \frac{1}{2}$ for all $i = 1, 2, \ldots, n-1$.

   Solution. Let $P_1(X) = X$, $P_2(X) = 8X^3 + 4X - 1$ and define by induction $P_{k+1}(X)$ by $P_{k+1}(X) = 8P_k(X)^3 + 4P_k(X) - 3P_{k-1}(X) - 1$ for all integer $k \geq 2$. Clearly, $P_k(X)$ is a polynomial of odd degree for all $k \geq 1$.

   Let $n \geq 2$ be an integer and a a real zero of the polynomial $P_n(X)$. The real a exists since the degree of $P_n(X)$ is odd.

   Consider the finite sequence $x_0, x_1, \ldots, x_n$ of real numbers defined by $x_0 = 0$, $x_1 = a$ and $x_{i+1} = 8x_i^3 + 4x_i - 3x_{i-1} - 1$, for all $1 \leq i \leq n-1$. Clearly, $x_1 = P_1(a)$ and $x_2 = P_2(a)$. Assume that for $2 \leq k \leq n-1$, $x_{k-1} = P_{k-1}(a)$ and $x_k = P_k(a)$. We have

   $$x_{k+1} = 8x_k^3 + 4x_k - 3x_{k-1} - 1 = 8P_k(a)^3 + 4P_k(a) - 3P_{k-1}(a) - 1 = P_{k+1}(a).$$

   This proves that $x_k = P_k(a)$ for all $1 \leq k \leq n$ and in particular $x_n = P_n(a) = 0$. This proves the existence of the sequence $x_0, x_1, \ldots, x_n$ satisfying $x_0 = x_n = 0$ and $x_{i+1} - 8x_i^3 - 4x_i + 3x_{i-1} + 1 = 0$ for all $i = 1, 2, \ldots, n-1$.

Conversely, if such a sequence $x_0, x_1, \ldots, x_n$ exists then x_1 is a real zero of the polynomial $P_n(X)$. Therefore, proving the uniquiness of the sequence is equivalent to proving that $P_n(X)$ has a unique real zero.

Let x, y be two real numbers. We have $|P_2(x) - P_2(y)| = |4(x-y)(2(x^2 + xy + y^2) + 1)| \geq |x - y| = |P_1(x) - P_1(y)|$, since $x^2 + xy + y^2 \geq 0$. Assume that $|P_k(x) - P_k(y)| \geq |P_{k-1}(x) - P_{k-1}(y)|$, for some integer $k \geq 2$. We have

$$|P_{k+1}(x) - P_{k+1}(y)| \geq$$

$$\begin{aligned} \geq 4|P_k(x) - P_k(y)| \cdot |2(P_k^2(x) + P_k(x)P_k(y) + P_k^2(y)) + 1| \\ -3|P_{k-1}(x) - P_{k-1}(y)| \end{aligned}$$

$$\geq 4|P_k(x) - P_k(y)| - 3|P_{k-1}(x) - P_{k-1}(y)| \geq |P_k(x) - P_k(y)|.$$

Hence, the sequence $(|P_k(x) - P_k(y)|)_{k\geq 1}$ is non decreasing and if $x \neq y$ then $P_k(x) \neq P_k(y)$ for all $k \geq 1$. We deduce that $P_n(X)$ has a unique real zero, and therefore the sequence $x_0, x_1, \ldots, x_n$ is unique.

Let $c = \max\{|x_i| : 1 \leq i \leq n-1\}$. There exists $k \in \{1, 2, \ldots, n-1\}$ such that $c = |x_k|$. Because x_k and x_k^3 have the same sign, it follows that

$$\begin{aligned} c + 2c^3 &= |x_k| + 2|x_k|^3 = |x_k + 2x_k^3| = \tfrac{1}{4}|1 + x_{k+1} + 3x_{k-1}| \\ &\leq \frac{1}{4} + \frac{1}{4}(|x_{k+1}| + 3|x_{k-1}|) \leq \frac{1}{4} + c, \end{aligned}$$

which implies that $c \leq \dfrac{1}{2}$.

2. Given 2015 subsets $A_1, A_2, \ldots, A_{2015}$ of the set $\{1, 2, \ldots, 1000\}$ such that $|A_i| \geq 2$ for every $i \geq 1$ and $|A_i \cap A_j| \geq 1$ for every $1 \leq i < j \leq 2015$. Prove that $k = 3$ is the smallest number of colors such that we can always color the elements of the set $\{1, 2, \ldots, 1000\}$ by k colors with the property that the subset A_i has at least two elements of different colors for every $i \geq 1$.

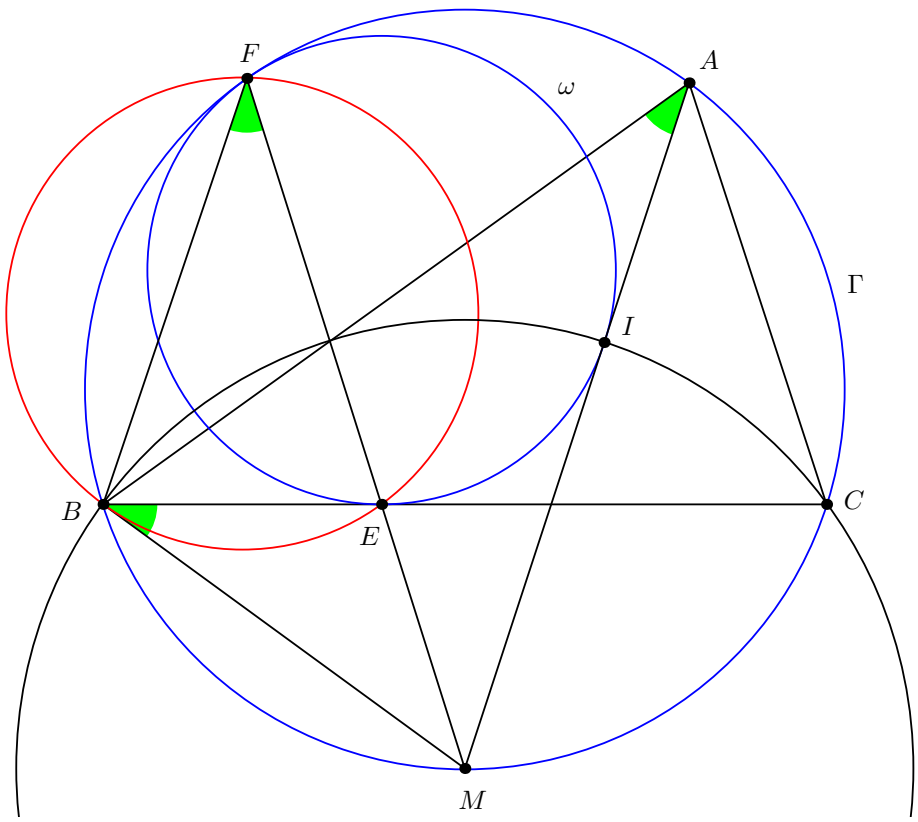
   Solution. Consider the collection: $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{1, 3\}$, $A_4, \ldots, A_{2015}$ be any 2011 subsets of $\{1, 2, \ldots, 1000\}$ that contain all $1, 2, 3$. One can check that this collection satisfies the given condition. For any 2-coloring of the set $\{1, 2, \ldots, 1000\}$, at least one of A_1, A_2, A_3 will contain two elements of the same color. Hence, we need more than 2 colors.

   Now we show that we always can color $\{1, 2, \ldots, 1000\}$ by 3 colors such that the set A_i contains at least two elements of different color for all i. We choose a set A_{i_0} with least number of elements. We color one element of A_{i_0} by red

and all the rest by blue. We color by green all the elements $\{1, 2, \ldots, 1000\}$ which do not belong to A_{i_0}. It is clear that A_{i_0} contains two elements of different color. For any subset A_i with $i \neq i_0$, $|A_i \cap A_1| \geq 1$ so A_i contains at least one red or blue color element. Moreover, because $|A_i| \geq |A_{i_0}|$, either A_i contains another element of A_{i_0} with the other color or contains a green element. Hence, the subset A_i contains two elements of different color.

3. Let ABC be a triangle, Γ its circumcircle, I its incenter, and ω a tangent circle to the line AI at I and to the side BC. Prove that the circles Γ and ω are tangent.

Solution. Let M be the midpoint of arc BC not containing A, E the tangent point of BC to ω, F the second intersection point of EM with ω. Remember that M is the circumcenter of triangle BCI and therefore $MI = MB$.



Because MI is tangent to ω, we have from the power of the point M with

respect to ω

$$ME \cdot MF = MI^2 = BM^2.$$

We deduce that the line BM is tangent to the circumcircle of triangle BEF. Therefore

$$\angle BFM = \angle MBE = \angle MAC = \angle BAM.$$

This means that point F is on the circle Γ.

Because the tangent line to Γ at M is parallel to the tangent line to ω at E and F, E, M are collinear and F is an intersection point of Γ and ω, F is the center of the homothety of the two circles ω and Γ and therefore, they are tangent.

4. Let $n \geq 2$ be an integer and $p_1 < p_2 < \cdots < p_n$ prime numbers. Prove that there exists an integer k relatively prime with $p_1p_2\cdots p_n$ and such that $\gcd(k + p_1p_2\cdots p_i, p_1p_2\cdots p_n) = 1$ for all $i = 1, 2, \ldots, n-1$.

   Solution. The problem is equivalent to proving that k relatively prime with $p_1p_2\cdots p_n$ and such that for all $i = 2, \ldots, n$, none of the integers $k, k + p_1, \ldots, k + p_1 \cdots p_{i-1}$ is divisible by p_i.

   Let $i \in \{2, \ldots, n\}$. Notice that $p_i \geq i + 1$. So the numbers $-p_1$, $-p_1p_2$, ..., $-p_1p_2\cdots p_{i-1}$ will not cover all the non-zero residues in modular p_i. Let k_i be one of these non covered non-zero residues. Clearly, none of the numbers $k_i, k_i + p_1, \ldots, k_i + p_1 \cdots p_{i-1}$ is congruent to 0 modular p_i.

   This defines $k_2, k_3, \ldots, k_n$ and let $k_1 = 1$. By the Chinese remainder theorem there exists an integer k such that $k \equiv k_i \not\equiv 0 \pmod{p_i}$ for all $i = 1, 2, \ldots, n$. The integer k is relatively prime with $p_1p_2\cdots p_n$. On the other hand, for all $1 \leq j < i \leq n$, we have $k + p_1p_2\cdots p_j \equiv k_i + p_1p_2\cdots p_j \not\equiv 0 \pmod{p_i}$.

3.2 Solutions to The Team Selection Test of Day II

1. Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{R}$ such that for any $m, n \in \mathbb{Z}$ there exists a $k \in \mathbb{Z}$ such that $f(k) = f(m) - f(n)$.

   Solution. Let $f : \mathbb{Z} \to \mathbb{R}$ be such a function. We prove that $f(n+1) - f(n)$ is a positive constant a independent of the integer n.

   Indeed, assume that there exist two integers n_0, n_1 such that

$$f(n_0 + 1) - f(n_0) < f(n_1 + 1) - f(n_1).$$

Let k_0, k_1 be integers such that $f(k_0) = f(n_0+1) - f(n_0)$ and $f(k_1) = f(n_1+1) - f(k_0)$. Because f is strictly increasing, $f(k_0) > 0$. We deduce that $f(n_1) < f(k_1) = f(n_1+1) - f(k_0) < f(n_1+1)$, which is impossible since f is strictly increasing and n_1, n_1+1 are consecutive integers.

Moreover, there exists an integer n_0 such that $f(n_0) = f(0) - f(0) = 0$.

Let $n > n_0$. We have

$$f(n) = f(n_0) + \sum_{m=0}^{n-n_0-1} (f(n_0+m+1) - f(n_0+m)) = a(n-n_0).$$

We obtain a similar formula for $n < n_0$ and therefore

$$f(n) = a(n-n_0), \quad \text{for all } n \in \mathbb{Z}.$$

Conversely, let $a > 0$ be a positive real number and n_0 an integer and define $f(n) = a(n-n_0)$ for all $n \in \mathbb{Z}$. The function f is strictly increasing and for any integers $m, n \in \mathbb{Z}$ we have $f(m) - f(n) = f(m-n+n_0)$.

2. Find the number of 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of distinct positive integers satisfying the following two conditions:

   (a) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 30$;

   (b) We can write $a_1, a_2, a_3, a_4, a_5, a_6$ on sides of a hexagon such that after a finite number of time choosing a vertex of the hexagon and adding 1 to the two numbers written on two sides adjacent to the vertex, we obtain a hexagon with equal numbers on its sides.

Solution. We label the vertex of the hexagon by $1, 2, 3, 4, 5, 6$ and suppose that six numbers are written in the order a, b, c, d, e, f on the edges $(1,2), (2,3), \ldots, (6,1)$, respectively.

We first notice that by choosing a vertex of the hexagon and adding 1 to the two numbers written on two adjacent sides to the vertex, the difference $(a+c+e) - (b+d+e)$ is an invariant. Thus, if we want to obtain a hexagon with equal numbers on its sides, we must have $a+c+e = b+d+f$ at the beginning.

We will show that it is also sufficient. Let N be a big integer, for example, we can take $N > 30$. We choose the vertices $1, 2, 4, 5$ and adding 1 to the two numbers written on two sides of those vertices by $N-f$, $N-b$, $N-c$ and $N-e$ times, respectively. Then, we obtain a hexagon with six numbers

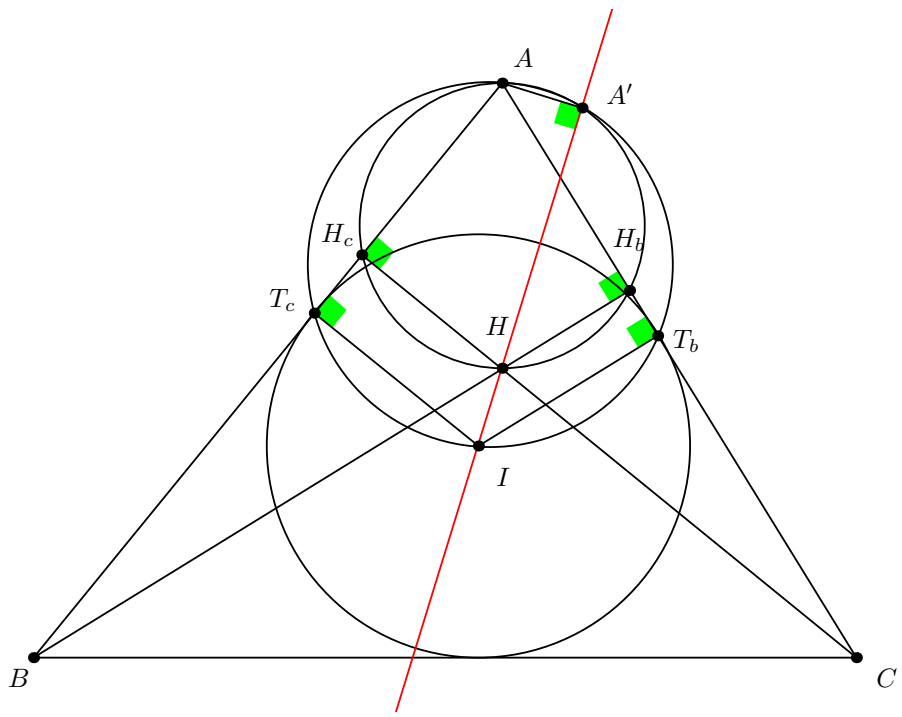
$a+2N-b-f, N, N, d+2N-e-c, N, N$. Since $a+c+e=b+d+f$, we have $a+2N-b-f=d+2N-e-c=M>N$ (since $N>30=a+b+c+d+e+f$). We now can choose the vertices $3,6$ for $M-N$ times each then we obtain a hexagon with numbers M on all of its sides.

Now, we count how many 6-tuples (a,b,c,d,e,f) of distinct positive integers with $a+c+e=b+d+f=15$.

We list all triples of distinct positive integers with sum 15: $(1,2,12)$, $(1,3,11)$, $(1,4,10)$, $(1,5,9)$, $(1,6,8)$, $(2,3,10)$, $(2,4,9)$, $(2,5,8)$, $(2,6,7)$, $(3,4,8)$, $(3,5,7)$, $(4,5,6)$. We then check that there are exactly 19 times we can pair these triples to obtain 6 distinct numbers.

Since we can permute the numbers on each tuple, the total number of 6-tuples satisfying the given conditions is $19\times 6!$.

3. Let ABC be a triangle, H_a, H_b and H_c the feet of its altitudes from A, B and C, respectively, T_a, T_b, T_c its intouch points on the sides BC, CA and AB, respectively. The circumcircles of triangles AH_bH_c and AT_bT_c intersect again at A'. The circumcircles of triangles BH_cH_a and BT_cT_a intersect again at B'. The circumcircles of triangles CH_aH_b and CT_aT_b intersect again at C'. Prove that the points A', B', C' are collinear.



Solution. Let H and I be the orthocenter and the incenter of triangle ABC, respectively. Because $\angle AH_bH = \angle HH_cA = 90°$, AH is a diameter of the circumcircle of AH_cHH_b, and therefore $\angle AA'H = 90°$. Because $\angle AT_bI = \angle IT_cA = 90°$, AI is a diameter of the circumcircle of AT_cIT_b, and therefore $\angle AA'I = 90°$. We deduce that point A' is on the line HI.

Similarly, we prove that the points B' and C' are on the line HI, which prove that the points A', B', C' are collinear.

4. Prove that there exist infinitely many non prime positive integers n such that $7^{n-1} - 3^{n-1}$ is divisible by n.

Solution. We will look for integers of the form $n = 7^a - 3^a$ with a dividing $n - 1$. Clearly, if a exists then $n = 7^a - 3^a$ divides $7^{n-1} - 3^{n-1}$.

Let $a = 3^r$ for $r \geq 1$. We have $n = 7^a - 3^a \equiv= (-1)^{3^r} - 3^{3^r} \equiv 4 \pmod 8$. We deduce that n is a composite number.

On the other hand, because 3 divides $7 - 1$, we have from the Lifting The Exponent theorem that $v_3(7^a - 1) = v_3(7 - 1) + v_3(a) = r + 1$. We deduce that $a = 3^r$ divides $7^{3^r} - 1$ and therefore it divides $n - 1 = 7^{3^r} - 1 + 3^{3^r}$. But there are infinitely many such a. This solves the problem.

Chapter 4

Solutions to The Team Selection Tests for the International Mathematical Olympiad 2015

4.1 Solutions to The Team Selection Test of Day I

1. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$f\left(\frac{x}{y}\right) = f(x) + f(y) - f(x)f(y)$$

for all $x, y \in \mathbb{R}_{>0}$. Here, $\mathbb{R}_{>0}$ denotes the set of all positive real numbers.

Solution. Define $g : \mathbb{R}_{>0} \to \mathbb{R}$ by $g(x) = 1 - f(x)$ for all $x > 0$. The functional equation can be rewritten $g(x/y) = g(x)g(y)$ for all $x, y > 0$.

Putting $y = x = 1$, we get $g(1) = g(1)^2$. This means that $g(1) = 0$ or $g(1) = 1$.

1. Assume $g(1) = 0$. For all $x > 0$, we have $g(x)^2 = g(x/x) = g(1) = 0$. Hence $g(x) = 0$ for all $x > 0$. Conversely, $g = 0$ is a solution of the given functional equation.
2. Assume $g(1) = 1$. In this case, for all $x > 0$, we have $g(x) = g(1)g(x) = g(1/x)$.

Let $x > 0$. We have

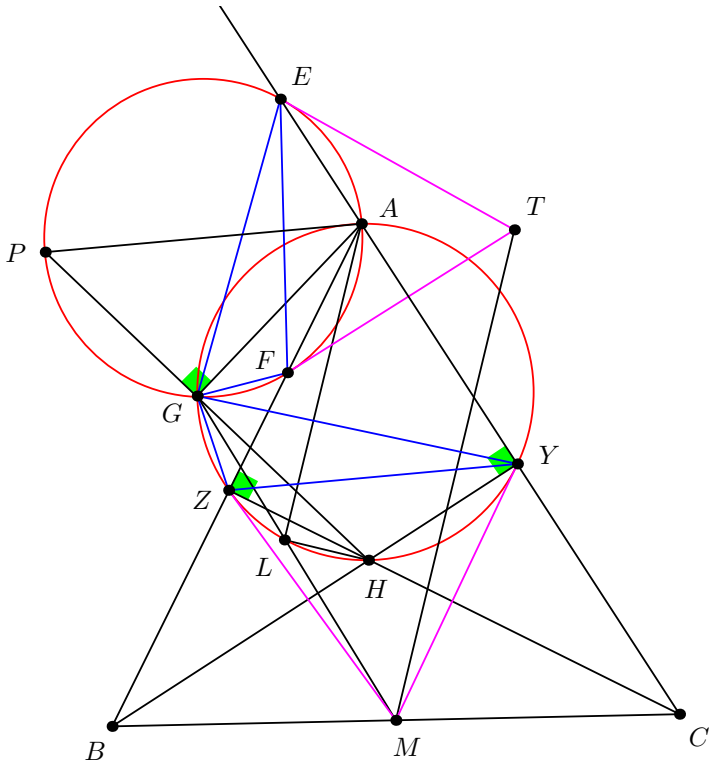
$$g(x) = g\left(\frac{\sqrt{x}}{1/\sqrt{x}}\right) = g(\sqrt{x})g(1/\sqrt{x}) = g(\sqrt{x})^2 = g(\sqrt{x}/\sqrt{x}) = g(1) = 1.$$

Conversely, $g = 1$ is a solution of the given functional equation.

In conclusion, $f(x)$ is identically 0 or identically 1.

2. Let ABC be a triangle with orthocenter H. Let P be any point of the plane of the triangle. Let Ω be the circle with the diameter AP. The circle Ω cuts CA and AB again at E and F, respectively. The line PH cuts Ω again at G. The tangent lines to Ω at E, F intersect at T. Let M be the midpoint of BC and L be the point on MG such that AL and MT are parallel. Prove that LA and LH are orthogonal.

Solution. Let BY, CZ be altitudes of ABC. Points Y, Z lie on the circle of diameter AH. The line HP cuts the circle Ω again at G. Since AP is a diameter in Ω, the lines AG and PH are perpendicular and therefore point G lies on the circle of diameter AH.



Because A, E, G, F are concyclic and A, G, Z, H, Y are concyclic, we have $\angle GEF = \angle GAF = \angle GAZ = \angle GYZ$ and $\angle FGE = \angle ZAY = \angle ZGY$. We deduce that triangles GEF and GYZ are similar.

We know that $\angle MZC = \angle ZCB = \angle ZAH$. We deduce that ZM is tangent to the circumcircle of triangle GZY. Similarly, MY is tangent to the circumcircle of triangle GZY. But FT and ET are tangent to the circumcircle of triangle GFE. We deduce that quadrilaterals $GFTE$ and $GZMY$ are similar and therefore $\angle GZA = \angle GZF = \angle GMT = \angle GLA$, since LA is parallel to MT. THis means that point L lies on the circle of diameter AH and therefore LA and LH are orthogonal.

3. Let n and k be two positive integers. Prove that if n is relatively prime with 30, then there exist two integers a and b, each relatively prime with n, such that $\dfrac{a^2 - b^2 + k}{n}$ is an integer.

Solution. We will proceed by steps.

- *First Step:* We prove it for $n = p$ a prime number, with $p \geq 7$:
  If p divides k then $a = b = 1$ satisfy the condition. If p does not divide k then let us consider two cases:

  (a) If k is a quadratic residue modulo p, and c and $c+1$ are two consecutive quadratic residues modulo p such that $1 \leq c < c+1 \leq p-1$, then both kc and $kc + k$ are quadratic residues relatively prime with p. This means that there exist a and b, each relatively prime with p, such that $a^2 \equiv ck$ and $b^2 \equiv ck + k$ modulo p and therefore $a^2 - b^2 + k \equiv 0 \pmod{p}$.

  (b) If k is a quadratic nonresidue modulo p, and c and $c+1$ are two consecutive quadratic nonresidues modulo p such that $1 \leq c < c+1 \leq p-1$, then both kc and $kc + k$ are quadratic residues relatively prime with p. This means that there exist a and b, each relatively prime with p, such that $a^2 \equiv ck$ and $b^2 \equiv ck + k$ modulo p and therefore $a^2 - b^2 + k \equiv 0 \pmod{p}$.

  Thus, this first step reduces to proving that among $1, 2, \ldots, p-1$ there are two consecutive quadratic residues and two consecutive quadratic nonresidues modulo p. We will prove this fact in two different ways:

  First way. Notice that both 1 and 4 are quadratic residues.

  If 2 or 3 is a quadratic residue then there exists at least two consecutive quadratic residues among $1, 2, 3, 4$. But exactly half of the numbers $1, 2, \ldots, p-1$ are quadratic nonresidues. So there exist at

least $\frac{p-3}{2}$ quadratic nonresidues among the $p-5$ consecutive numbers $5, 6, \ldots, p-1$. So, by a pigeonhole principle, at least two of them must be consecutive.

If 2 and 3 are consecutive quadratic nonresidues, we have two posibilities: Either -1 is a quadratic nonresidue and in this case $p-3$ and $p-2$ are consecutive quadratic residues, or -1 is a quadratic residue, which means that $p \equiv 1 \pmod 4$, $p \geq 13$, and for each integer m the integers m and $p-m$ are either both quadratic residues or both quadratic nonresidues. This means that there are $\frac{p-1}{4}$ quadratic residues between $1, 2, \ldots, \frac{p-1}{2}$.

If $\frac{p-1}{2}$ is a quadratic residue then $\frac{p-1}{2}, \frac{p+1}{2}$ are two consecutive quadratic residues.

If $\frac{p-1}{2}$ is a quadratic nonresidue then there are $\frac{p-5}{4}$ quadratic residues among the $\frac{p-9}{2}$ consecutive numbers $4, 5, \ldots, \frac{p-3}{2}$. But $\frac{p-9}{2}$ is an even number. So, by a pigeonhole principle, at least two of the quadratic residues must be consecutive.

Second way. In this second way we will compute the exact number of consecutive pairs of quadratic residues and the exact number of consecutive pairs of quadratic nonresidues.

Consider the partition

$$\{1, 2, \ldots, p-1\} = Q_1 \cup N_1 \cup Q_2 \cup N_2 \cup \cdots \cup Q_k \cup N_k$$

where Q_1 is the set of all the first consecutive quadratic residues, possibly reduced to the singleton $\{1\}$ when 2 is a quadratic nonresidue, N_1 is the set of all the first consecutive quadratic nonresidues, possibly reduced to a singleton, Q_2 is the set of all the first consecutive quadratic residues after the elements of N_1, and so on... Notice that N_k is the only possibly empty set when $p-1$ is a quadratic residue.

Let $a \in \{1, 2, \ldots, p-2\}$. Notice that $\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = 1$ if and only if $a, a+1$ are both quadratic residues or both quadratic nonresidues, where $\left(\frac{a}{p}\right)$ is the Legendre symbol. This means that $\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = -1$ if and only if $a, a+1$ do not belong to the same subset of the partition. This means that there are either $2k-2$ or $2k-1$ pairs $(a, a+1)$, with

$1 \leq a \leq p-2$ such that $\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = -1$, depending on whether N_k is empty or not. But $\left(\frac{a}{p}\right)\left(\frac{a+1}{p}\right) = \left(\frac{a}{p}\right)^2\left(\frac{1+a^{-1}}{p}\right) = \left(\frac{1+a^{-1}}{p}\right)$. But when a covers all the residues $1, 2, \ldots, p-2$, its inverse a^{-1} covers all the residues $1, 2, \ldots, p-2$. Because there are exactly $\frac{p-1}{2}$ quadratic nonresidues among the residues $2, 3, \ldots, p-1$, we deduce that $k = \frac{p+3}{4}$ if $p \equiv 1 \pmod 4$ and $k = \frac{p+1}{4}$ if $p \equiv 3 \pmod 4$.
Now, for $1 \leq i \leq k$, there are $\mathrm{Card} Q_i - 1$ consecutive pairs of quadratic residues in Q_i and $\mathrm{Card} N_i - 1$ consecutive pairs of quadratic nonresidues in N_i. We deduce that the total number of consecutive pairs of quadratic residues in $\{1, 2, \ldots, p-1\}$ is

$$\sum_{i=1}^{k} (\mathrm{Card} Q_i - 1) = \mathrm{Card} \cup_{i=1}^{k} Q_i - k = \frac{p-1}{2} - k = \left\lfloor \frac{p-3}{4} \right\rfloor \geq 1,$$

and the total number of consecutive pairs of quadratic nonresidues in $\{1, 2, \ldots, p-1\}$ is

$$\sum_{i=1}^{k-1} (\mathrm{Card} N_i - 1) = \mathrm{Card} \cup_{i=1}^{k-1} N_i - k + 1 = \frac{p-1}{2} - k + 1 = \left\lfloor \frac{p-1}{4} \right\rfloor \geq 1,$$

if $p \equiv 1 \pmod 4$ and

$$\sum_{i=1}^{k} (\mathrm{Card} N_i - 1) = \mathrm{Card} \cup_{i=1}^{k} N_i - k = \frac{p-1}{2} - k = \left\lfloor \frac{p-1}{4} \right\rfloor \geq 1,$$

if $p \equiv 3 \pmod 4$.

- *Second Step:* We prove it for $n = p^\alpha$, with $p \geq 7$ is a prime number and $\alpha \geq 1$. We procede by induction on α:
For $\alpha = 1$, we have already proved in the first step that there exist two integers a_1, b_1, each relatively prime with p, such that p divides $a_1^2 - b_1^2 + k$.
For $\alpha \geq 1$, assume that there exists an integer $a_\alpha \equiv a_1 \pmod p$ such that p^α divides $a_\alpha^2 - b_1^2 + k$. Consider the polynomial $P(X) = X^2 - b_1^2 + k$. Because $P(a_\alpha) \equiv 0 \pmod{p^\alpha}$ and $P'(a_\alpha) = 2a_\alpha \equiv 2a_1 \not\equiv 0 \pmod p$, there exists, from Hensel's lemma, an integer $a_{\alpha+1} \equiv a_\alpha \pmod{p^\alpha}$, and therefore $a_{\alpha+1} \equiv a_1 \pmod p$, such that $P(a_{\alpha+1}) = a_{\alpha+1}^2 - b_1^2 + k \equiv 0 \pmod{p^{\alpha+1}}$. This proves the induction.

- *Last Step:* We prove the general case when $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$, where $7 \le p_1 < p_2 < \cdots < p_l$ are prime numbers. We know from the second step, that for each $i = 1, 2, \ldots, l$, there exist two integers a_i, b_i, relatively prime with p_i, such that $p_i^{\alpha_i}$ divides $a_i^2 - b_i^2 + k$. From the Chinese remainder theorem, there exist two integers a, b such that $a \equiv a_i \pmod{p_i^{\alpha_i}}$ and $b \equiv b_i \pmod{p_i^{\alpha_i}}$, for $i = 1, 2, \ldots, l$. Therefore,

  $$a^2 - b^2 + k \equiv a_i^2 - b_i^2 + k \equiv 0 \pmod{p_i^{\alpha_i}}$$

  for all $i = 1, 2, \ldots, l$, and hence

  $$a^2 - b^2 + k \equiv 0 \pmod{n}.$$

  On the other hand,

  $$\gcd(n, a) = \prod_{i=0}^{l} \gcd(p_i^{\alpha_i}, a_i) = 1 \text{ and } \gcd(n, b) = \prod_{i=0}^{l} \gcd(p_i^{\alpha_i}, b_i) = 1.$$

  This completes the proof.

4.2 Solutions to The Team Selection Test of Day II

4. Let ABC be an acute-angled triangle inscribed in the circle (O), H the foot of the altitude of ABC at A and P a point inside ABC lying on the bisector of $\angle BAC$. The circle of diameter AP cuts (O) again at G. Let L be the projection of P on AH. Prove that if GL bisects HP then P is the incenter of the triangle ABC.

   Solution. Let AP intersect BC at F and (O) again at D. Because LP and BC are parallel, we have

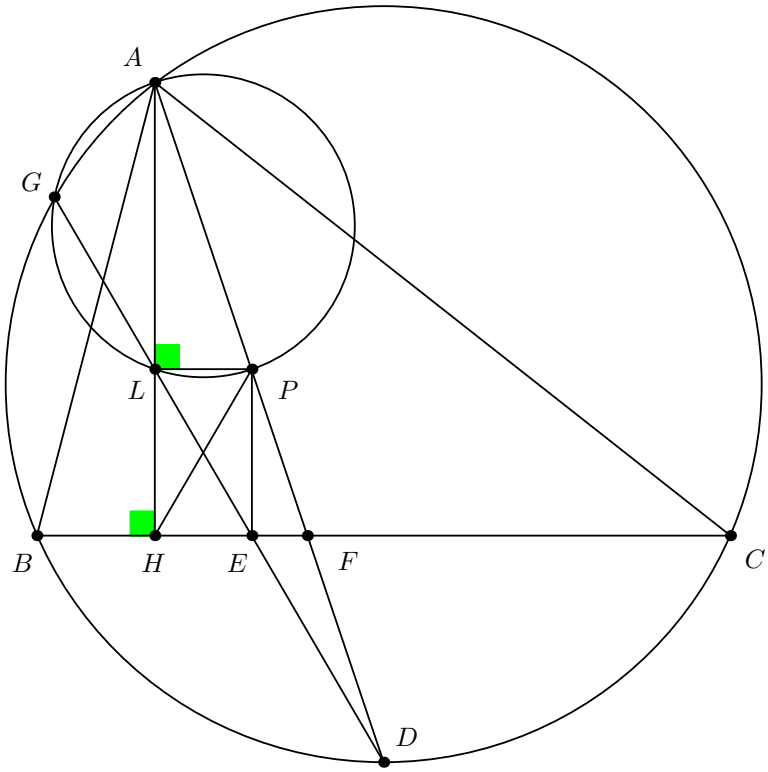
   $$\begin{aligned} \angle LGA &= \angle LPD = \angle BFD = \angle BAD + \angle CBA \\ &= \angle DAC + \angle CBA = \angle DBA = \angle DGA. \end{aligned}$$

   This implies that G, L, D are collinear.

   Let GL cut BC at E. Because LE bisects PH and $\angle EHL = \angle HLP = 90^\circ$, quadrilateral $PLHE$ is a rectangle. We deduce, using Thales's theorem, that

   $$\frac{DP}{DA} = \frac{PE}{AL} = \frac{LH}{AL} = \frac{PF}{AP}.$$

   Therefore, $\dfrac{DP}{DA} = 1 - \dfrac{AP}{DA} = 1 - \dfrac{PF}{DP} = \dfrac{DF}{DP}$. We deduce that $DP^2 = DF.DA$.



On the other hand, $\angle DBF = \angle DAC = \angle BAF$. This means that line BD is tangent to the circumcircle of triangle ABF and therefore $DB^2 = DF \cdot DA = DP^2$. But D is the circumcenter of triangle BCI, where I is the incenter of ABC. We deduce that P is the incenter of triangle ABC.

5. Hamza and Majid play a game on a horizontal 3×2015 white board. They alternate turns, with Hamza going first. A legal move for Hamza consists of painting three unit squares forming a horizontal 1×3 rectangle. A legal move for Majid consists of painting three unit squares forming a vertical 3×1 rectangle. No one of the two players is allowed to repaint already painted squares. The last player to make a legal move wins. Which of the two players, Hamza or Majid, can guarantee a win no matter what strategy his opponent chooses and what is his strategy to guarantee a win?

   Solution. Hamza has a winning stratery.

   We divide the rectangle into 671 squares of size 3×3 and a small rectangle of size 3×2.

Hamza will play as follow: He will paint at each time a horizontal 1×3 rectangle in a white 3×3 square (all unit squares in this 3×3 square are not colored yet) until he cannot continue to do so. In each step, Majid can only paint a 3×1 rectangle in a square of size 3×3 or in the small rectangle 3×2). So the number of 3×3 squares that will be used by a rectangle of Hamza is at least 336.

Note that, whenever Hamza paints a horizontal 1×3 in a 3×3 square then Majid cannot paint a vertical 3×1 in such a square. Hence, Hamza can play at least $336 \times 3 = 1008$ steps. Since the game ends after at most 2015 steps so Majid can play at most $2015 - 2008 = 2007 < 2008$ steps. In other words, Hamza has a winning strategy.

6. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Prove that

$$\sum_{1 \le i < j \le n} a_i a_j (1 - a_i a_j) \ge 0.$$

Solution. The original inequality is equivalent to

$$2 \sum_{1 \le i < j \le n} a_i a_j \ge 2 \sum_{1 \le i < j \le n} a_i^2 a_j^2,$$

or

$$\left(\sum_{i=1}^{n} a_i \right)^2 - \sum_{i=1}^{n} a_i^2 \ge \left(\sum_{i=1}^{n} a_i^2 \right)^2 - \sum_{i=1}^{n} a_i^4.$$

Using the given hypothesis, it can be rewritten as

$$a_1^4 + a_2^4 + \cdots + a_n^4 \ge a_1^2 + a_2^2 + \cdots + a_n^2.$$

To prove this last inequality, we have from Holder's inequality

$$(a_1^4 + a_2^4 + \cdots + a_n^4)(a_1 + a_2 + \cdots + a_n)^2 \ge (a_1^2 + a_2^2 + \cdots + a_n^2)^3.$$

Using the given hypothesis again, we get the result.

4.3 Solutions to The Team Selection Test of Day III

7. Let S be a positive integer divisible by all the integers $1, 2, \ldots, 2015$ and $a_1, a_2, \ldots, a_k$ numbers in $\{1, 2, \ldots, 2015\}$ such that

$$2S \leq a_1 + a_2 + \ldots + a_k,$$

Prove that we can select from $a_1, \ldots, a_k$ some numbers so that the sum of these selected numbers is equal to S.

Solution. Notice that because S is divisible by $2013, 2014$ and 2015, we have $S \geq 2013 \times 2014 \times 2015$.

Assume that each integer $1, 2, \ldots, 2015$ appears at most 2013 in the list of numbers $a_1, a_2, \ldots, a_k$. We have

$$2S \leq a_1 + \cdots + a_k \leq 2013(1 + 2 + \cdots + 2015) < 2013 \times 2014 \times 2015,$$

which is a contradiction.

Therefore, there exists an integer s which appears at least 2014 times in the list of numbers $a_1, a_2, \ldots, a_k$. Put 2014 of this number s aside from the list. The sum of the remaining numbers in the list is equal to $2S - 2014s \geq 2S - 2014 \times 2015 > S$.

Start from the empty set and choose randomly s integers $b_1, b_2, \ldots, b_s$ from the remaining numbers in the list. If there exists $1 \leq i \leq s$ such that $b_1 + b_2 + \cdots + b_i$ is divisible by s put $b_1, b_2, \ldots, b_i$ in the set and put back the other numbers in the remaining list of numbers. If not, by the pigeonhole principle, there exist $1 \leq i < j \leq s$ such that $b_1 + b_2 + \cdots + b_i \equiv b_1 + b_2 + \cdots + b_j \pmod{s}$ and therefore $b_{i+1} + b_{i+2} + \cdots + b_j$ is divisible by s. In this case put $b_{i+1}, b_{i+2}, \ldots, b_j$ in the set and put back the other numbers in the remaining list of numbers. Keep repeating this process whenever the sum of the numbers in the set is less or equal to $S - 2015s$.

Notice that each time the sum of the numbers in the set is divisible by s and each time it increases by a number which is less or equal to $2015s$. Moreover, whenever the sum of the numbers in the set is less or equal to $S - 2015s$ it is always possible to choose randomly s integers from the remaining numbers. Indeed, the sum of the remining numbers is less or equal to $2S - 2014s - (S - 2015s) = S + s > 2015 \times 2015 \geq 2015s$.

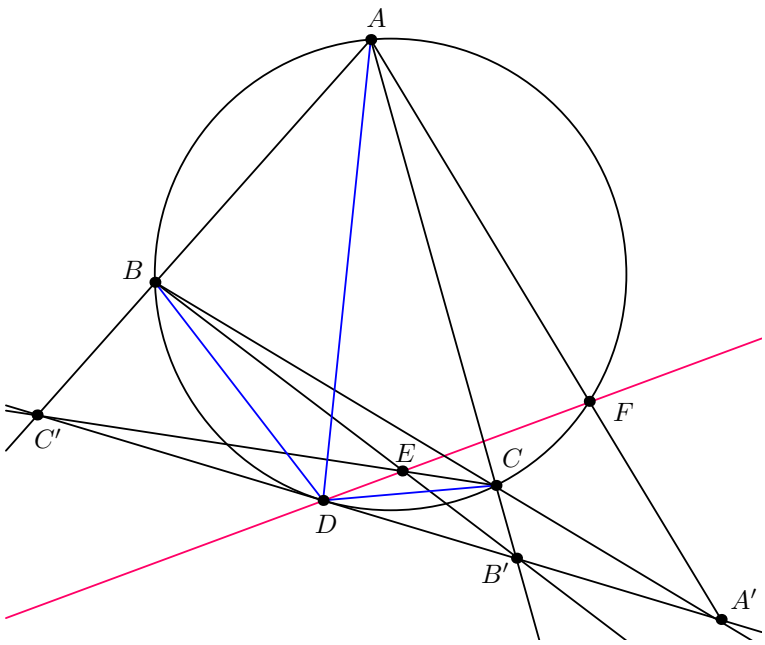
Once the sum T of the numbers in the set is strictly greater than $S - 2015s$, in this case we have $S \geq T > 2015s$. Because both S and T are divisible by s, we have $S - T = ms$ where $m \in \{0, 1, 2, \ldots, 2014\}$. Take m from the 2014

of the number s put aside and put them in the set to obtain a sum equal to S.

8. Let ABC be a triangle and $\mathcal{C}$ its circumcircle. Point D lies on the arc $\widehat{BC}$ (not containing A) of $\mathcal{C}$ and is different from B, C and the midpoint of $\widehat{BC}$. The tangent line to $\mathcal{C}$ at D intersects lines BC, CA, AB at A', B', C' respectively. Lines BB' and CC' intersect at E. Line AA' intersects again circle $\mathcal{C}$ at F. Prove that the three points D, E, F are colinear.

First Solution. The problem is equivalent to proving that

$$\angle CDE = \angle CDF.$$



For the sake of brevity, let us denote

$$\alpha_1 = \angle BAD,\ \alpha_2 = \angle DAC,\ \beta = \angle CBA,\ \gamma = \angle ACB.$$

Because we are considering the tangent line at D, we have

$$\angle BDC' = \angle BCD = \alpha_1 \quad \text{and} \quad \angle B'DC = \angle DBC = \alpha_2.$$

By applying the law of sines to triangles $BB'C$ and BDB', we obtain

$$\frac{\sin\angle DBB'}{\sin\angle B'BC} = \frac{DB' \cdot \frac{\sin\angle B'DB}{BB'}}{B'C \cdot \frac{\sin\angle BCB'}{BB'}} = \frac{DB'\sin\angle BDC'}{B'C\sin\angle ACB} = \frac{DB'\sin\alpha_1}{B'C\sin\gamma}.$$

We obtain, in a similar way

$$\frac{\sin\angle C'CD}{\sin\angle BCC'} = \frac{DC'\sin\alpha_2}{C'B\sin\beta}.$$

By applying trigonometric form of Ceva' theorem for the point E in triangle BCD, we obtain

$$\begin{aligned}\frac{\sin\angle CDE}{\sin\angle EDB} &= \frac{\sin\angle EBC}{\sin\angle DBE}\cdot\frac{\sin\angle ECD}{\sin\angle BCE} = \frac{\sin\angle B'BC}{\sin\angle DBB'}\cdot\frac{\sin\angle C'CD}{\sin\angle BCC'}\\ &= \frac{B'C\cdot DC'\sin\gamma\sin\alpha_2}{DB'\cdot C'B\sin\beta\sin\alpha_1}.\end{aligned}$$

On the other hand, by applying the law sines to the triangles $AB'A'$ and $AC'A'$, and then to the triangles CDB' and $BC'D$, and finaly to the triangle BDC, we obtain

$$\begin{aligned}\frac{\sin\angle B'AA'}{\sin\angle C'AA'} &= \frac{B'A'\cdot\frac{\sin\angle A'B'A}{AA'}}{C'A'\cdot\frac{\sin\angle A'C'A}{AA'}} = \frac{B'A'\sin\angle CB'D}{C'A'\sin\angle DC'B}\\ &= \frac{B'A'\cdot DC\cdot\frac{\sin\angle DCB'}{DB'}}{C'A'\cdot BD\cdot\frac{\sin\angle C'BD}{DC'}} = \frac{B'A'\cdot DC'\sin\alpha_2}{C'A'\cdot DB'\sin\alpha_2},\end{aligned}$$

since $\angle C'BD = 180^\circ - \angle DCB'$, by concyclicity.

By applying Menelaus' theorem for the line BC and triangle $AB'C'$, and then the law of sines in triangle ABC, we obtain

$$\frac{B'A'}{A'C'} = \frac{B'C}{CA}\cdot\frac{AB}{BC'} = \frac{B'C\sin\gamma}{BC'\sin\beta}.$$

Therefore

$$\frac{\sin\angle CDE}{\sin\angle EDB} = \frac{\sin\angle B'AA'}{\sin\angle C'AA'} = \frac{\sin\angle CAF}{\sin\angle BAF} = \frac{\sin\angle CDF}{\sin\angle FDB},$$

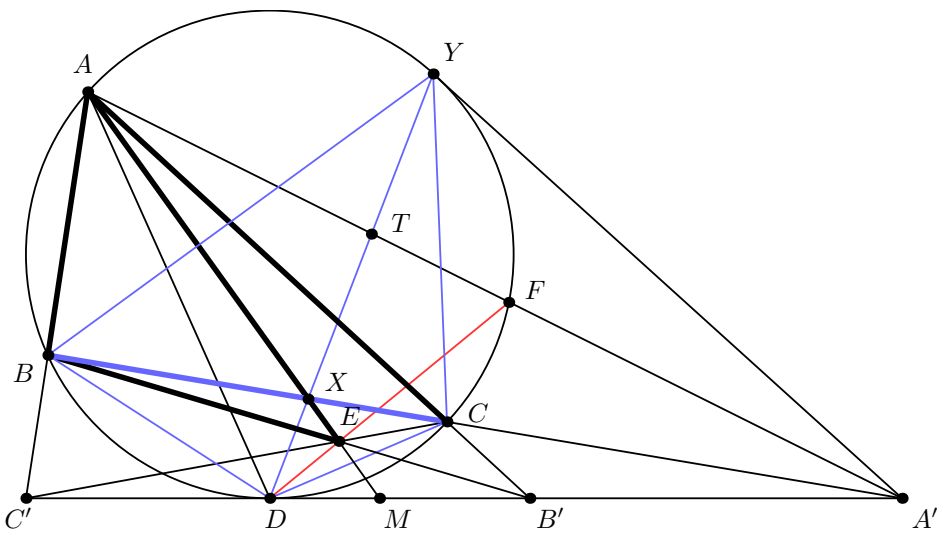
by concyclicity. But $\angle CDE + \angle EDB = \angle CDF + \angle FDB$. We deduce that $\angle CDE = \angle CDF$, as wanted.

Second Solution (By Shaden Al-Shammari). Let Y be the second tangent point on $\mathcal{C}$ of the line passing through A' and different from $A'D$. The line AE cuts the lines BC and $B'C'$ at X and M respectively.

The points D, X, Y are collinear. Indeed, because AX, BB', CC' are concurrent at E and $B'C'$ intersects BC at A', by using Ceva's and Menelaus's theorems, we have

$$\frac{\overline{BX}}{\overline{XC}} = \frac{\overline{BB'}}{\overline{B'A}} \cdot \frac{\overline{AC'}}{\overline{C'C}} = -\frac{\overline{BA'}}{\overline{A'C}},$$

which means that the cross ratio $(A', X, C, B) = -1$, that is (A', X, C, B) is a harmonic bundle.



Moreover, because $A'D, A'Y$ are two tangent lines to $\mathcal{C}$ and BC passes through A', the quadrilateral $CYBD$ is harmonic. Therefore X is the intersection point of DY and BC. This proves that D, X, Y are collinear.

Consider the triangle ABE. The cevians AB', BX and EC' intersect at C. On the other hand, the line $B'C'$ intersects the line AE at M. We deduce from Ceva's and Menelaus's theorems that (M, X, E, A) is a harmonic bundle.

On the other hand, let T be the intersection point of the lines DY and AA'. Because the lines AF, DA', YA' are concurrent and both DA', YA' are tangent to $\mathcal{C}$, the quadrilateral $ADFY$ is harmonic and therefore (A', T, F, A) is a harmonic bundle.

Therefore, (M, X, E, A) and (A', T, F, A) are harmonic bundles, points D, M, A' are collinear and points D, X, T are collinear. We deduce that points D, E, F are collinear.

9. Find the number of binary sequences S of length 2015 such that for any two segments I_1, I_2 of S of the same length, we have

- The sum of digits of I_1 differs from the sum of digits of I_2 by *at most* 1;
- If I_1 begins on the left end of S then the sum of digits of I_1 is *not greater than* the sum of digits of I_2;
- If I_2 ends on the right end of S then the sum of digits of I_2 is *not less than* the sum of digits of I_1.

Solution. Note that if the sequence S ends with zero then all digits of S will be zero.

Now suppose that the sequence S ends with one. Let k be the number of 1's in S then $1 \leq k \leq 2015$. Let $\alpha = \frac{2015}{k} \geq 1$. We consider the sequence $S_k = a_1 a_2 \ldots a_{2015}$ where $a_i = 1$ if and only if $i \in \{\lceil \alpha \rceil, \lceil 2\alpha \rceil, \ldots, \lceil k\alpha \rceil\}$.

One can check that this sequence has exactly k ones and satisfies all the given conditions.

Now we need to prove that those are all possible sequences.

Suppose that S is a sequence with exactly k ones and also satisfying all the given conditions. We will show that we have only one possible arrangement for k ones and $(2015 - k)$ zeros.

Let $\beta = \frac{2015-k}{2015}$. Starting from the right, we will show that for any $1 \leq s \leq 2015$, among the first s rightmost digits, there are exactly $\lfloor \beta s \rfloor$ zeros.

It is clear that the statement holds for $s = 1$ since the rightmost digit is a unit.

Suppose that the statement holds for s. We consider two cases.

Case 1. If $\lfloor \beta(s+1) \rfloor = \lfloor \beta s \rfloor + 1$. We need to show that the $(s+1)^{\text{th}}$ rightmost digit of S is zero. Otherwise, it follows that we have exactly $\lfloor \beta(s+1) \rfloor - 1$ in the $(s+1)$ rightmost digits and any $s+1$ consecutive digits, there are at most $\lfloor \beta(s+1) \rfloor$ zeros. We divide the sequence from the right into t segments of length $s+1$ and possibly one leftmost segment of length s' with $0 \leq s' \leq s$.

The number of zeros is at most

$$\begin{aligned} \lfloor \beta(s+1) \rfloor - 1 \quad &+ \quad (t-1) \lfloor \beta(s+1) \rfloor + \lfloor \beta s' \rfloor \\ &\leq \quad \beta(t(s+1) + s') - 1 = 2014 - k, \end{aligned}$$

which is a contradiction.

Case 2. If $\lfloor \beta(s+1) \rfloor = \lfloor \beta s \rfloor$. We need to show that the $(s+1)^{\text{th}}$ rightmost digit of S is a unit. Otherwise, it follows that for any $s+1$ consecutive digits, there are at least $\lfloor \beta(s+1) \rfloor + 1$ zeros. We divide the sequence from the right

into t segments of length $s+1$ and possibly one leftmost segment of length s' with $0 \le s' \le s$. Note that in the first s' rightmost digits, we have exactly $\lfloor \beta s' \rfloor$ zeros. If in the leftmost segment of length s' we have $\lfloor \beta s' \rfloor + 1$ then the number of zeros is at least

$$t(\lfloor \beta(s+1) \rfloor + 1) + \lfloor \beta s' \rfloor > \beta(t(s+1) + s') = 2015 - k,$$

which is a contradiction.

WE deduce that, in each segment of length s' we have exactly $\lfloor \beta s' \rfloor$ zeros. We divide the sequence into segments of length s' and possible one shorter segment from the left. Using the same argument as in the previous case, we do not have enough zeros, which is again a contradiction.

Therefore, the statement holds for all $1 \le s \le 2015$, which implies that for any $0 \le k \le 2015$, there exists a unique binary sequence with k units satisfying the given conditions.

4.4 Solutions to The Team Selection Test of Day IV

10. Let a, b, c, d be positive integers such that $ac+bd$ is divisible by a^2+b^2. Prove that $\gcd(c^2+d^2, a^2+b^2) > 1$.

Solution. Suppose for the sake of contradiction that $\gcd(c^2+d^2,\ a^2+b^2) = 1$. We first notice that

$$(ac+bd)^2 + (ad-bc)^2 = (a^2+b^2)(c^2+d^2).$$

Since $ac+bd$ is divisible by a^2+b^2, it follows that $(ad-bc)^2$ is also divisible by a^2+b^2.

Let p be a prime divisor of a^2+b^2 and $n = v_p(a^2+b^2)$. We have

$$v_p\big((ac+bd)^2 + (ad-bc)^2\big) = n,$$

since c^2+d^2 is not divisible by p. Note that,

$$v_p\big((ac+bd)^2\big) = 2v_p(ac+bd) \ge 2v_p(a^2+b^2) = 2n > n.$$

This implies that $v_p\big((ad-bc)^2\big) = n$, which means that n is even. This proves that $a^2+b^2 = u^2$ for some positive integer u, and therefore both $ac+bd$ and $ad-bc$ are divisible by u.

But

$$a(c^2 + d^2) = c(ac + bd) + d(ad - bc)$$

and

$$b(c^2 + d^2) = d(ac + bd) - c(ad - bc)$$

are also divisible by u. Because $\gcd(u,\, c^2 + d^2) = 1$, we deduce that u divides both a and b. But $u^2 = a^2 + b^2$. We have a contradiction.

11. The total number of languages used in KAUST is n. For each positive integer $k \leq n$, let A_k be the set of all those people in KAUST who can speak at least k languages; and let B_k be the set of all people P in KAUST with the property that, for any k pairwise different languages (used in KAUST), P can speak at least one of these k languages. Prove that

    (a) If $2k \geq n + 1$ then $A_k \subseteq B_k$;

    (b) If $2k \leq n + 1$ then $A_k \supseteq B_k$.

    Solution. Let $\mathcal{L}$ be the set of all the languages used in KAUST.

    a) Assume $2k \geq n + 1$. Consider $P \in A_k$, and let $\mathcal{L}_P \subseteq \mathcal{L}$ be the set of all languages spoken by P. The set $\mathcal{L}_P$ contains at least k languages.

    Let $\mathcal{L}' \subseteq \mathcal{L}$ be a set of k languages. We have

    $$|\mathcal{L}_P \cap \mathcal{L}'| = |\mathcal{L}_P| + |\mathcal{L}'| - |\mathcal{L}_P \cup \mathcal{L}'| \geq 2k - |\mathcal{L}| = 2k - n \geq 1.$$

    This means that $\mathcal{L}_P \cap \mathcal{L}' \neq \emptyset$ and therefore P can speak at least one language in $\mathcal{L}'$. Hence $P \in B_k$.

    b) Assume $2k \leq n + 1$. Consider $P \in B_k$, and let $\mathcal{L}_P \subseteq \mathcal{L}$ be the set of all languages spoken by P and $\mathcal{L}' = \mathcal{L} \setminus \mathcal{L}_P$. Because there is no language in $\mathcal{L}'$ spoken by P we have $|\mathcal{L}'| \leq k - 1$. We deduce that

    $$|\mathcal{L}_P| = |\mathcal{L}| - |\mathcal{L}'| \geq n - (k - 1) \geq k.$$

    This means that P speaks at least k languages and therefore $P \in A_k$.

12. Let a, b, c be positive real numbers satisfying the condition

    $$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 10.$$

    Find the greatest value and the least value of

    $$T = (x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

First Solution. Using the homogenuity, we can restate the original problem as follow: *Let x, y, z be positive real numbers satisfying the conditions $x+y+z=10$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Find the greatest and the least values of T.*

From the condition, we have $1 < z < 10$ and

$$x+y=10-z, \quad \frac{1}{x}+\frac{1}{y}=1-\frac{1}{z}.$$

By AM-GM, we have

$$(10-z)\left(1-\frac{1}{z}\right)=(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)\geq 4,$$

which implies that $z^2-10z+7\leq 0$, or $2\leq z\leq 5$. Similarly, we also have $2\leq x, y\leq 5$. Now we have $(x-2)(y-2)(z-2)\geq 0$. Expanding the expression and using the conditions $x+y+z=10$ and $xyz=xy+yz+zx$, we get $xyz\leq 32$.

Similarly, from $(x-5)(y-5)(z-5)\leq 0$, we get $xyz\geq \frac{125}{4}$.

Finally, denote $t=xyz=xy+yz+zx$, we have

$$\begin{aligned} T &= (x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \\ &= (100-2t)\cdot\frac{t^2-20t}{t^2} \\ &= 140-\left(2t+\frac{2000}{t}\right) \end{aligned}$$

where $\frac{125}{4}\leq t\leq 32$. From which, we get $T_{\min}=\frac{27}{2}$, achieved when $t=32$ or $\frac{125}{4}$, that is $(2,\,4,\,4)$, $\left(5,\,\frac{5}{2},\,\frac{5}{2}\right)$ and their permutations, and $T_{\max}=140-40\sqrt{10}$, achieved when $t=10\sqrt{10}$, that is when (x,y,z) are the three roots of the polynomial $X^3-10X^2+10\sqrt{10}X-10\sqrt{10}$.

Second Solution (By Vo Quoc Ba Can). First, we will find $\mathbf{max}\,\boldsymbol{T}$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\sqrt{\left(\sum x^2\right)\left(\sum\frac{1}{x^2}\right)}+2\sqrt{\left(\sum xy\right)\left(\sum\frac{1}{xy}\right)}\leq \\ &\qquad\leq\sqrt{\left(\sum x^2+2\sum xy\right)\left(\sum\frac{1}{x^2}+2\sum\frac{1}{xy}\right)} \\ &\qquad=\left(\sum x\right)\left(\sum\frac{1}{x}\right)=10. \end{aligned}$$

But, we also have

$$\left(\sum xy\right)\left(\sum \frac{1}{xy}\right) = \left(\sum \frac{1}{x}\right)\left(\sum x\right) = 10.$$

Therefore, the above estimation gives us $\sqrt{T} \le 10 - 2\sqrt{10}$, so

$$T \le \left(10 - 2\sqrt{10}\right)^2 = 140 - 40\sqrt{10}.$$

Equality holds if and only if

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 10 \quad \text{and} \quad \frac{x^2+y^2+z^2}{xy+yz+zx} = \frac{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}{\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}},$$

which is satisfied (for example) when $x = \sqrt{yz}$ and $\frac{y}{z} + \frac{z}{y} = 9 - 2\sqrt{10}$.

Now, we will find $\mathbf{min}\, \boldsymbol{T}$**.** From the given hypothesis, we have

$$\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) + \left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right) = 7.$$

And we have the following remarks:

$$\left(\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}\right) + 2\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2,$$
$$\left(\frac{y^2}{x^2}+\frac{z^2}{y^2}+\frac{x^2}{z^2}\right) + 2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) = \left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)^2.$$

Therefore,

$$\begin{aligned} T &= \sum \frac{x^2}{y^2} + \sum \frac{y^2}{x^2} + 3 \\ &= \left(\sum \frac{x}{y}\right)^2 + \left(\sum \frac{y}{x}\right)^2 - 2\left(\sum \frac{y}{x} + \sum \frac{x}{y}\right) + 3 \\ &\ge \frac{1}{2}\left(\sum \frac{y}{x} + \sum \frac{x}{y}\right)^2 - 2\left(\sum \frac{y}{x} + \sum \frac{x}{y}\right) + 3 \\ &= \frac{49}{2} - 14 + 3 = \frac{27}{2}. \end{aligned}$$

Equality holds if and only if

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \quad \text{and} \quad \frac{y}{x}+\frac{z}{y}+\frac{x}{z}+\frac{x}{y}+\frac{y}{z}+\frac{z}{x} = 7,$$

which is satisfied (for example) when $x = 2y = 2z$.

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مؤسسة الملك عبدالعزيز ورجاله للموهبة والإبداع
King Abdulaziz & his Companions Foundation For Giftedness & Creativity

موهبة

مسابقات الرياضيات للمملكة العربية السعودية

SAUDI ARABIAN

MATHEMATICAL COMPETITIONS 2016

SAMC 2016

Riyadh, June 2016

Saudi Arabian Mathematical Competitions 2016

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The Saudi Arabian team at **IMO 2016**

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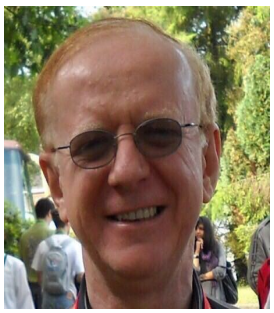
Team Training Administrators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مقدمة

يحوي هذا الملف على مسائل استخدمت في التدريب والاختيار خلال السنة الأخيرة من إعداد الفريق الذي يمثل المملكة العربية السعودية في الاولمبياد الدولي.

إن تدريب الفريق كان بدعم من وزارة التعليم بالتعاون مؤسسة الملك عبدالعزيز ورجالة للموهبة والإبداع "موهبة" .

وتجدر الاشارة الى التعاون المنقطع النظير الذي لمسناه من جامعة الملك عبدالله للعلوم والتقنية، حيث وفرت لنا كل الإمكانيات التي احتجنا لها في التدريب في حرمها الجامعي. كما نشكر جميع الرعاة الرسميين.

تم عقد ثلاث ملتقيات تدريبية خلال العام الدراسي 2015-2016، بالإضافة الى فترة التدريب المكثف التي بدأت في شهر ابريل 2016، كما شارك الفريق السعودي في العديد من المسابقات ومنها : أولمبياد الرياضيات لدول أسيا والباسيفيك، أولمبياد دول الخليج في المدينة المنورة ، أولمبياد الطالبات للدول الأوربية في رومانيا، وأولمبياد البلقان الذي عقد في ألبانيا.

نأمل أن يكون هذا الملف ، بما فيه من مسائل، هو إسهاما منا لتقوية أواصر التعاون وتبادل الخبرات بيننا والدول المشاركة في الأولمبياد الدولي

د. فوزي بن أحمد الذكير
رئيس الفريق السعودي للأولمبياد الدولي

Saudi Arabian Mathematical Competitions 2016

Introduction

This booklet contains problems used in the training and selection of the Saudi team for the International Mathematics Olympiad 2016. The training was supported by The Ministry of Education, which commissioned Mawhiba, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. Our gratitude goes to King Abdullah University for Science and Technology KAUST for making its facilities, in its beautiful campus, available to us for our training.

The Saudi team had three main training camps during the academic year 2015-2016 beside the full-time training period that started in the first week of April 2016.

During this academic year, the team participated in the following contests: The Asian Pacific Mathematics Olympiad APMO, the Gulf Mathematical Olympiad in Saudi Arabia, the European Girls Mathematical Olympiad EGMO in Romania, and the Balkan Mathematical Olympiad BMO, in Albania.

It is our pleasure to share these training and selection tests problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Fawzi A. Al-Thukair
Leader of the Saudi Math Olympiad Team

Saudi Arabian Mathematical Competitions 2016

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Part 1

PROBLEM STATEMENTS

Saudi Arabian Mathematical Competitions 2016

1.1. Preselection Tests for Full time training

1.1.1. Level 4, day 1

Problem 1. Let x, y, z be positive real numbers satisfy the condition $x^2 + y^2 + z^2 = 2(xy + yz + zx)$. Prove that

$$x + y + z + \frac{1}{2xyz} \geq 4.$$

Problem 2. Let ABC be a non isosceles triangle inscribed in a circle (O) and BE, CF are two angle bisectors intersect at I with E belongs to segment AC and F belongs to segment AB. Suppose that BE, CF intersect (O) at M, N respectively. The line d_1 passes through M and perpendicular to BM intersects (O) at the second point P, the line d_2 passes through N and perpendicular to CN intersect (O) at the second point Q. Denote H, K are two midpoints of MP and NQ respectively.

1. Prove that triangles IEF and OKH are similar.

2. Suppose that S is the intersection of two lines d_1 and d_2. Prove that SO perpendicular to EF.

Problem 3. A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too. Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)

Problem 4. Let p be a given prime. For each prime r, we defind the function as following

$$F(r) = \frac{(p^{rp} - 1)(p - 1)}{(p^r - 1)(p^p - 1)}.$$

1. Show that $F(r)$ is a positive integer for any prime $r \neq p$.

2. Show that $F(r)$ and $F(s)$ are coprime for any primes r and s such that $r \neq p, s \neq p$ and $r \neq s$.

3. Fix a prime $r \neq p$. Show that there is a prime divisor q of $F(r)$ such that $p \mid q - 1$ but $p^2 \nmid q - 1$.

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1.1.2. Level 4, day 2

Problem 1.

Given three numbers x, y, z, and set $x_1 = |x-y|$, $y_1 = |y-z|$, $z_1 = |z-x|$. From x_1, y_1, z_1, form in the same fashion the numbers x_2, y_2, z_2, and so on. It is known that $x_n = x, y_n = y, z_n = z$ for some n. Find all possible values of (x, y, z).

Problem 2. Ten vertices of a regular 20-gon $A_1A_2 \ldots A_{20}$ are painted blacks and the other ten vertices are painted blue. Consider the set consisting of diagonal A_1A_4 and all other diagonals of the same length.

1. Prove that in this set, the number of diagonals with two black endpoints is equal to the number of diagonals with two blue endpoints.

2. Find all possible numbers of the diagonals with two black endpoints.

Problem 3. Let u and v be positive rational numbers with $u \neq v$. Assume that there are infinitely many positive integers n with the property that $u^n - v^n$ are integers. Prove that u and v are integers.

Problem 4. Let ABC be a non isosceles triangle with circumcircle (O) and incircle (I). Denote (O_1) as the circle that external tangent to (O) at A' and also tangent to the lines AB, AC at A_b, A_c respectively. Define the circles $(O_2), (O_3)$ and the points $B', C', B_c, B_a, C_a, C_b$ similarly.

1. Denote J as the radical center of $(O_1), (O_2), (O_3)$ and suppose that JA' intersects (O_1) at the second point X, JB' intersects (O_2) at the second point Y, JC' intersects (O_3) at the second point Z. Prove that the circle (XYZ) is tangent to $(O_1), (O_2), (O_3)$.

2. Prove that AA', BB', CC' are concurrent at the point M and 3 points I, M, O are collinear.

1.1.3. Level 4+, day 1

Problem 1. Let ABC be an acute, non isosceles triangle, AX, BY, CZ are the altitudes with X, Y, Z belong to BC, CA, AB respectively. Respectively denote (O_1), (O_2), (O_3) as the circumcircles of triangles AYZ, BZX, CXY. Suppose that (K) is a circle that internal tangent to (O_1), (O_2), (O_3). Prove that (K) is tangent to circumcircle of triangle ABC.

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Problem 2. Let a, b, c be positive numbers such that $a^2+b^2+c^2+abc=4$. Prove that

$$\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b} \geq a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Problem 3. Let a, b be two positive integers such that

$$b+1|a^2+1,\ a+1|b^2+1$$

Prove that a, b are odd numbers.

Problem 4. The natural numbers $0, 1, 2, 3, \ldots$ are written on the square table 2015×2015 in a circular order (anti-clockwise) such that 0 is in the center of the table. The rows and columns are labelled from bottom to top and from left to right respectively. (see figure below)

1. The number 2015 is in which row and which column?
2. We are allowed to perform the following operations: First, we replace the number 0 in the center by 14; after that, each time, we can add 1 to each of 12 numbers on 12 consecutive unit squares in a row, or 12 consecutive unit squares in a column, or 12 unit squares in a rectangle 3×4. After a finite number of steps, can we make all numbers on the table are multiples of 2016?

	20	19	18	17	16	
	21	6	5	4	15	
	22	7	0	3	14	
	23	8	1	2	13	
	24	9	10	11	12	

1.1.4. Level 4+, day 2

Problem 1.

1) Prove that there are infinitely many positive integers n such that there exists a permutation of $1, 2, 3, \ldots, n$ with the property that the difference between any two adjacent numbers is equal to either 2015 or 2016.

2) Let k be a positive integer. Is the statement in 1) still true if we replace the numbers 2015 and 2016 by k and $k+2016$, respectively?

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Problem 2. Given four numbers x, y, z, t, let (a, b, c, d) be a permutation of (x, y, z, t) and set $x_1 = |a-b|$, $y_1 = |b-c|$, $z_1 = |c-d|$, and $t_1 = |d-a|$. From x_1, y_1, z_1, t_1, form in the same fashion the numbers x_2, y_2, z_2, t_2, and so on. It is known that $x_n = x, y_n = y, z_n = z, t_n = t$ for some n.

Find all possible values of (x, y, z, t).

Problem 3. Let ABC be a non isosceles triangle with circumcircle (O) and incircle (I). Denote (O_1) as the circle that internal tangent to (O) at A_1 and also tangent to segments AB, AC at A_b, A_c respectively. Define the circles $(O_2), (O_3)$ and the points $B_1, C_1, B_c, B_a, C_a, C_b$ similarly.

1. Prove that AA_1, BB_1, CC_1 are concurrent at the point M and 3 points I, M, O are collinear.

2. Prove that the circle (I) is inscribed in the hexagon with 6 vertices $A_b, A_c, B_c, B_a, C_a, C_b$.

Problem 4. Let n be a given positive integer. Prove that there are infinitely many pairs of positive integers (a, b) with $a, b > n$ such that

$$\prod_{i=1}^{2015}(a+i) \mid b(b+2016); \quad \prod_{i=1}^{2015}(a+i) \nmid b; \quad \prod_{i=1}^{2015}(a+i) \nmid (b+2016).$$

1.2. Team selection tests for Gulf Mathematical Olympiad 2016

1.2.1. Level 4, day 1

Problem 1. Let $f(x) = x^2 + ax + b$ be a quadratic function with real coefficients a, b. It is given that the equation $f(f(x)) = 0$ has 4 distinct real roots and the sum of 2 roots among these roots is equal to -1. Prove that $b \leq \frac{-1}{4}$.

Problem 2. Let $(O_1), (O_2)$ be given two circles intersecting at A and B. The tangent lines of (O_1) at A, B intersect at O. Let I be a point on the circle (O_1) but outside the circle (O_2). The lines IA, IB intersect circle (O_2) at C, D. Denote by M the midpoint of CD. Prove that I, M, O are collinear.

Problem 3. Find all positive integer n such that there exists a permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, 3, \ldots, n)$ satisfying the condition:

$$a_1 + a_2 + \cdots + a_k \text{ is divisible by } k \text{ for each } k = 1, 2, 3, \ldots, n.$$

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Problem 4. There are totally 16 teams participating in a football tournament; each team playing with every other exactly 1 time. In each match, the winner gains 3 points, the loser gains 0 point and each teams gain 1 point for the tie match. Suppose that at the end of the tournament, each team gains the same number of points. Prove that there are at least 4 teams that have the same number of winning matches, the same number of losing matches and the same number of tie matches.

1.2.2. Level 4, day 2

Problem 1. Let $S = x+y+z$ where x, y, z are three nonzero real numbers satisfying the following system of inequalities:

$$\begin{cases} xyz > 1 \\ x+y+z > \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}. \end{cases}$$

Prove that S can take on any real values when x, y, z vary.

Problem 2. Let c be a given real number. Find all polynomials P with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x) = c \quad \text{for all} \quad x \in \mathbb{R}.$$

Problem 3. Let ABC be a triangle whose incircle (I) is tangent to AB, AC at D, E respectively. Denote by Δ_b, Δ_c the lines symmetric to the lines AB, AC with respect to CD, BE correspondingly. Suppose that Δ_b, Δ_c meet at K.

1. Prove that $IK \perp BC$.
2. If $I \in (KDE)$, prove that $BD + CE = BC$.

1.2.3. Level 4, day 3

Problem 1. Let ABC be an acute, non-isosceles triangle which is inscribed in a circle (O). A point I belongs to the segment BC. Denote by H and K the projections of I on AB and AC, respectively. Suppose that the line HK intersects (O) at M, N (H is between M, K and K is between H, N). Prove the following assertions:

1. If A is the center of the circle (IMN), then BC is tangent to (IMN).
2. If I is the midpoint of BC, then BC is equal to 4 times of the distance between the centers of two circles (ABK) and (ACH).

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Problem 2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(2m + f(m) + f(m)f(n)) = nf(m) + m$$

for any integers m, n.

Problem 3. In a school there are totally $n > 2$ classes and not all of them have the same numbers of students. It is given that each class has one head student. The students in each class wear hats of the same color and different classes have different hat colors. One day all the students of the school stand in a circle facing toward the center, in an arbitrary order, to play a game. Every minute, each student put his hat on the person standing next to him on the right. Show that at some moment, there are 2 head students wearing hats of the same color.

1.2.4. Level 4+, day 1

Problem 1. Let $f(x) = x^2 + ax + b$ be a quadratic function with real coefficients a, b. It is given that the equation $f(f(x)) = 0$ has 4 distinct real roots and the sum of 2 roots among these roots is equal to -1. Prove that $b \leq \frac{-1}{4}$.

Problem 2. There are totally 16 teams participating in a football tournament; each team playing with every other exactly 1 time. In each match, the winner gains 3 points, the loser gains 0 point and each teams gain 1 point for the tie match. Suppose that at the end of the tournament, each team gains the same number of points. Prove that there are at least 4 teams that have the same number of winning matches, the same number of losing matches and the same number of tie matches.

Problem 3. Let ABC be an acute, non-isosceles triangle with the circumcircle (O). Denote D, E as the midpoints of AB, AC respectively. Two circles (ABE) and (ACD) intersect at K differs from A. Suppose that the ray AK intersects (O) at L. The line LB meets (ABE) at the second point M and the line LC meets (ACD) at the second point N.

1. Prove that M, K, N collinear and MN perpendicular to OL.

2. Prove that K is the midpoint of MN.

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1.2.5. Level 4+, day 2

Problem 1. Let $S = x+y+z$ where x, y, z are three nonzero real numbers satisfying the following system of inequalities:

$$\begin{cases} xyz > 1 \\ x+y+z > \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}. \end{cases}$$

Prove that S can take on any real values when x, y, z vary.

Problem 2. Let a, b be given two real number with $a \neq 0$. Find all polynomials P with real coefficients such that

$$xP(x-a) = (x-b)P(x) \text{ for all } x \in \mathbb{R}.$$

Problem 3. In a school, there are totally n students, with $n \geq 2$. The students take part in m clubs and in each club, there are at least 2 members (a student may take part in more than 1 club). Eventually, the Principal notices that: If 2 clubs share at least 2 common members then they have different numbers of members. Prove that

$$m \leq (n-1)^2.$$

1.2.6. Level 4+, day 3

Problem 1. Let ABC be an acute, non-isosceles triangle which is inscribed in a circle (O). A point I belongs to the segment BC. Denote by H and K the projections of I on AB and AC, respectively. Suppose that the line HK intersects (O) at M, N (H is between M, K and K is between H, N). Let X, Y be the centers of the circles $(ABK), (ACH)$ respectively. Prove the following assertions:

1. If I is the projection of A on BC, then A is the center of circle (IMN).
2. If $XY \parallel BC$, then the orthocenter of XOY is the midpoint of IO.

Problem 2. Let $n \geq 1$ be a fixed positive integer. We consider all the sets S which consist of sub-sequences of the sequence $0, 1, 2, \ldots, n$ satisfying the following conditions:
i) If $(a_i)_{i=0}^k$ belongs to S, then $a_0 = 0, a_k = n$ and $a_{i+1} - a_i \leq 2$ for all $0 \leq i \leq k-1$.
ii) If $(a_i)_{i=0}^k$ and $(b_j)_{j=0}^h$ both belong to S, then there exist $0 \leq i_0 \leq k-1$ and $0 \leq j_0 \leq h-1$ such that $a_{i_0} = b_{j_0}$ and $a_{i_0+1} = b_{j_0+1}$.

Find the maximum value of $|S|$ (among all the above-mentioned sets S).

Problem 3. Find all polynomials $P, Q \in \mathbb{Z}[x]$ such that every positive integer is a divisor of a certain nonzero term of the sequence $(x_n)_{n=0}^{\infty}$ given by the conditions:

$$x_0 = 2016,\ x_{2n+1} = P(x_{2n}),\ x_{2n+2} = Q(x_{2n+1}) \text{ for all } n \geq 0.$$

1.3. Team selection tests for Balkan Mathematical Olympiad 2016

1.3.1. Level 4, day 1

Problem 1. Given that the polynomial $P(x) = x^5 - x^2 + 1$ has 5 roots r_1, r_2, r_3, r_4, r_5. Find the value of the product

$$Q(r_1)Q(r_2)Q(r_3)Q(r_4)Q(r_5),$$

where $Q(x) = x^2 + 1$.

Problem 2. Let A be a point outside the circle ω. Two points B, C lie on ω such that AB, AC are tangent to ω. Let D be any point on ω (D is neither B nor C) and M the foot of perpendicular from B to CD. The line through D and the midpoint of BM meets ω again at P. Prove that $AP \perp CP$.

Problem 3. Show that there are infinitely many positive integers n such that n has at least two prime divisors and $20^n + 16^n$ is divisible by n^2.

Problem 4. On a checkered square 10×10 the cells of the upper left 5×5 square are black and all the other cells are white. What is the maximal n such that the original square can be dissected (along the borders of the cells) into n polygons such that in each of them the number of black cells is three times less than the number of white cells? (The polygons need not be congruent or even equal in area.)

1.3.2. Level 4, day 2

Problem 1. Let $P_i(x) = x^2 + b_i x + c_i$; $i = 1, 2, \cdots, n$ be pairwise distinct polynomials of degree 2 with real coefficients so that for any $0 \leq i < j \leq n$; $i, j \in \mathbb{N}$, the polynomial $Q_{i,j}(x) = P_i(x) + P_j(x)$ has only one real root. Find the greatest possible value of n.

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Problem 2. A circle with center O passes through points A and C and intersects the sides AB and BC of triangle ABC at points K and N, respectively. The circumcircles of triangles ABC and KBN meet at distinct points B and M. Prove that $\angle OMB = 90^\circ$.

Problem 3. Let m, n be odd integers such that $(n^2 - 1)$ is divisible by $m^2 + 1 - n^2$. Prove that $|m^2 + 1 - n^2|$ is a perfect square.

Problem 4. How many ways are there to color the vertices of a square with n colors $1, 2, \ldots, n$. (The colorings must be different so that we can't get one from the other by a rotation.)

1.3.3. Level 4, day 3

Problem 1. Let $a > b > c > d$ be positive integers such that $a^2 + ac - c^2 = b^2 + bd - d^2$. Prove that $ab + cd$ is a composite number.

Problem 2. Let ABC be a triangle and I its incenter. The point D is on segment BC and the circle ω is tangent to the circumcirle of triangle ABC but is also tangent to DC, DA at E, F, respectively. Prove that E, F and I are collinear.

Problem 3. Does there exist a polynomial $P(x)$ with integral coefficients such that

1. $P(\sqrt[3]{25} + \sqrt[3]{5}) = 220\sqrt[3]{25} + 284\sqrt[3]{5}$?
2. $P(\sqrt[3]{25} + \sqrt[3]{5}) = 1184\sqrt[3]{25} + 1210\sqrt[3]{5}$?

Problem 4. On a chessboard 5×9 squares, the following game is played. Initially, a number of frogs are randomly placed on some of the squares, no square containing more than one frog. A turn consists of moving all of the frogs subject to the following rules:

- Each frog may be moved one square up, down, left, or right;
- If a frog moves up or down on one turn, it must move left or right on the next turn, and vice versa;
- At the end of each turn, no square can contain two or more frogs.

The game stops if it becomes impossible to complete another turn. Prove that if initially 33 frogs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 frogs on the board so that the game can continue forever.

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1.3.4. Level 4+, day 1

Problem 1. Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of real coefficients. Suppose that $P(x)$ has n real roots (not necessarily distinct), and there exists a positive integer k such that $a_k = a_{k-1} = 0$. Prove that $P(x)$ has a real root of multiplicity $k+1$.
(**Note**: *we call a real number x_0 a root of multiplicity s of a polynomial $R(x)$ of real coefficients if there exists a polynomial $Q(x)$ such that $R(x) = (x-x_0)^s Q(x)$ and $Q(x_0) \neq 0$.*)

Problem 2. Let ABC be a triangle with $AB \neq AC$. The incirle of triangle ABC is tangent to BC, CA, AB at D, E, F, respectively. The perpendicular line from D to EF intersects AB at X. The second intersection point of circumcircles of triangles AEF and ABC is T. Prove that $TX \perp TF$.

Problem 3. For any positive integer n, show that there exists a positive integer m such that n divides $2016^m + m$.

Problem 4. There are 64 towns in a country, and some pairs of towns are connected by roads but we do not know these pairs. We may choose any pair of towns and find out whether they are connected by a road. Our aim is to determine whether it is possible to travel between any two towns using roads. Prove that there is no algorithm which would enable us to do this in less than 2016 questions.

1.3.5. Level 4+, day 2

Problem 1. Given two non-constant polynomials $P(x), Q(x)$ with real coefficients. For a real number a, we define

$$P_a = \{z \in \mathbb{C} : P(z) = a\}; \quad Q_a = \{z \in \mathbb{C} : Q(z) = a\}.$$

Denote by K the set of real numbers a such that $P_a = Q_a$. Suppose that the set K contains at least two elements, prove that $P(x) = Q(x)$.

Problem 2. Let I_a be the excenter of triangle ABC with respect to A. The line AI_a intersects the circumcircle of triangle ABC at T. Let X be a point on segment TI_a such that $XI_a^2 = XA.XT$. The perpendicular line from X to BC intersects BC at A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

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Problem 3. Let d be a positive integer. Show that for every integer S, there exist a positive integer n and a sequence $a_1, \ldots, a_n \in \{-1, 1\}$ such that

$$S = a_1(1+d)^2 + a_2(1+2d)^2 + \cdots + a_n(1+nd)^2.$$

Problem 4. Determine all positive integers $n \geq 3$ such that we can divide a convex n-polygon into triangles by using some diagonals of this polygon such that the number of the used diagonals of every vertex is even.

1.3.6. Level 4+, day 3

Problem 1. Let p, q be given primes and the sequence $(p_n)_{n\geq 1}$ defined recursively as follows: $p_1 = p; p_2 = q$ and p_{n+2} is the largest prime divisor of the number $(p_n + p_{n+1} + 2016)$ for all $n \geq 1$. Prove that this sequence is bounded, that is there exists a positive real number M such that $a_n < M$ for all positive integers n.

Problem 2. Let I be the incenter of an acute triangle ABC. Assume that K_1 is the point such that $AK_1 \perp BC$ and the circle with center K_1 of radius K_1A is internally tangent to the incircle of triangle ABC at A_1. The points B_1, C_1 are defined similarly.

1. Prove that AA_1, BB_1, CC_1 are concurrent at a point P.
2. Let $\omega_1, \omega_2, \omega_3$ be the excircles of triangle ABC with respect to A, B, C, respectively. The circles $\gamma_1, \gamma_2, \gamma_3$ are the reflections of $\omega_1, \omega_2, \omega_3$ with respect to the midpoints of BC, CA, AB, respectively. Prove that P is the radical center of $\gamma_1, \gamma_2, \gamma_3$.

Problem 3. Find all integers n such that there exists a polynomial $P(x)$ with integer coefficients satisfying

$$P(\sqrt[3]{n^2} + \sqrt[3]{n}) = 2016n + 20\sqrt[3]{n^2} + 16\sqrt[3]{n}.$$

Problem 4. Given six three-element subsets of the set X with at least 5 elements, show that it is possible to color the elements of X in two colors such that none of the given subsets is all in one color.

1.4. Team Selection Tests for the International Mathematical Olympiad 2016

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1.4.1. Level 4, day 1

Problem 1. Let $n \geq 3$ and $x_1, x_2, \ldots, x_n$ be n distinct integers. Prove that

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_n - x_1)^2 \geq 4n - 6.$$

Problem 2. Given a set of 2^{2016} cards with the numbers $1, 2, \ldots, 2^{2016}$ written on them. We divide the set of cards into pairs arbitrarily; from each pair, we keep the card with larger number and discard the other. We now again divide the 2^{2015} remaining cards into pairs arbitrarily; from each pair, we keep the card with smaller number and discard the other. We now have 2^{2014} cards, and again divide these cards into pairs and keep the larger one in each pair. We keep doing this way, alternating between keeping the larger number and keeping the smaller number in each pair, until we have just one card left. Find all possible values of this final card.

Problem 3. Given two circles (O_1) and (O_2) intersect at A and B. Let d_1 and d_2 be two lines through A and be symmetric with respect to AB. The line d_1 cuts (O_1) and (O_2) at $G, E\,(\neq A)$, respectively; the line d_2 cuts (O_1) and (O_2) at $F, H\,(\neq A)$, respectively; such that E is between A, G and F is between A, H.

Let J be the intersection of EH and FG. The line BJ cuts (O_1), (O_2) at $K, L\,(\neq B)$, respectively. Let N be the intersection of O_1K and O_2L. Prove that the circle (NLK) is tangent to AB.

1.4.2. Level 4, day 2

Problem 1. Let k be a positive integer. Prove that there exist integers x, y, neither of which divisible by 7 such that $x^2 + 6y^2 = 7^k$.

Problem 2. Let ABC be a triangle inscribed in the circle (O) and P is a point inside the triangle ABC. Let D be a point on (O) such that $AD \perp AP$. The line CD cuts the perpendicular bisector of BC at M. The line AD cuts the line passing through B and is perpendicular to BP at Q. Let N be the reflection of Q through M. Prove that $CN \perp CP$.

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x\big[f(x+y) - f(x-y)\big] = 4yf(x)$$

for any real numbers x, y.

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1.4.3. Level 4, day 3

Problem 1. Define the sequence $a_1, a_2, \ldots$ as follows: $a_1 = 1$, and for every $n \geq 2$, $a_n = n-2$ if $a_{n-1} = 0$ and $a_n = a_{n-1} - 1$, otherwise. Find the number of $1 \leq k \leq 2016$ such that there are non-negative integers r, s and a positive integer n satisfying $k = r + s$ and $a_{n+r} = a_n + s$.

Problem 2. Find all pairs of polynomials $P(x), Q(x)$ with integer coefficients such that

$$P(Q(x)) = (x-1)(x-2)\ldots(x-9)$$

for all real numbers x.

Problem 3. Let $n \geq 4$ be a positive integer and there exist n positive integers that are arranged on a circle such that:

- The product of each pair of two non-adjacent numbers is divisible by $2015 \cdot 2016$.
- The product of each pair of two adjacent numbers is not divisible by $2015 \cdot 2016$.

Find the maximum value of n.

1.4.4. Level 4, day 4

Problem 1. Let ABC be a triangle whose incircle (I) touches BC, CA, AB at D, E, F, respectively. The line passing through A and parallel to BC cuts DE, DF at M, N, respectively. The circumcircle of triangle DMN cuts (I) again at L.

1. Let K be the intersection of NE and MF. Prove that K is the orthocenter of the triangle DMN.
2. b) Prove that A, K, L are collinear.

Problem 2. Let a be a positive integer. Find all prime numbers p with the following property: there exist exactly p ordered pairs of integers (x, y), with $0 \leq x, y \leq p-1$, such that p divides $y^2 - x^3 - a^2 x$.

Problem 3. Find the number of permutations $(a_1, a_2, \ldots, a_{2016})$ of the first 2016 positive integers satisfing the following two conditions:

1. $a_{i+1} - a_i \leq 1$ for all $i = 1, 2, 3, \ldots, 2015$.
2. There are exactly two indices $i < j$ with $1 \leq i < j \leq 2016$ such that $a_i = i$ and $a_j = j$.

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1.4.5. Level 4+, day 1

Problem 1. Call a positive integer $N \geq 2$ "special" if for every k such that $2 \leq k \leq N$, N can be expressed as a sum of k positive integers that are relatively prime to N (although not necessarily relatively prime to each other). Find all special positive integers.

Problem 2. Given a set of 2^{2016} cards with the numbers $1, 2, \ldots, 2^{2016}$ written on them. We divide the set of cards into pairs arbitrarily; from each pair, we keep the card with larger number and discard the other. We now again divide the 2^{2015} remaining cards into pairs arbitrarily; from each pair, we keep the card with smaller number and discard the other. We now have 2^{2014} cards, and again divide these cards into pairs and keep the larger one in each pair. We keep doing this way, alternating between keeping the larger number and keeping the smaller number in each pair, until we have just one card left. Find all possible values of this final card.

Problem 3. Given triangle ABC inscribed in (O). Two tangents at B, C of (O) intersects at P. The bisector of angle A intersects (P, PB) (the circle with center P and radius PB) at point E lying inside triangle ABC. Let M, N be the midpoints of two arcs BC of (O) such that M and A are on different sides of BC. The circle with diameter BC intersects the line segment EN at F. Prove that the orthocenter of triangle EFM is belong to BC.

1.4.6. Level 4+, day 2

Problem 1. Let ABC be a triangle inscribed in the circle (O). The bisector of $\angle BAC$ cuts the circle (O) again at D. Let DE be the diameter of (O). Let G be a point on arc AB which does not contain C. The lines GD and BC intersect at F. Let H be a point on the line AG such that $FG \parallel AE$. Prove that the circumcircle of triangle HAB passes through the orthocenter of triangle HAC.

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions:

1. $f(x+1) \geq f(x) + 1$ for all $x \in \mathbb{R}$; (1)
2. $f(xy) \geq f(x)f(y)$ for all $x, y \in \mathbb{R}$. (2)

Problem 3. Given two positive integers $r > s$, and let $\mathscr{F}$ be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r - 1$ that shares at least s elements with each set in $\mathscr{F}$.

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1.4.7. Level 4+, day 3

Problem 1. Define the sequence $a_1, a_2, \ldots$ as follows: $a_1 = 1$, and for every $n \geq 2$, $a_n = n - 2$ if $a_{n-1} = 0$ and $a_n = a_{n-1} - 1$, otherwise. Find the number of $1 \leq k \leq 2016$ such that there are non-negative integers r, s and a positive integer n satisfying $k = r + s$ and $a_{n+r} = a_n + s$.

Problem 2. Let $n \geq 4$ be a positive integer and there exist n positive integers that are arranged on a circle such that:

- The product of each pair of two non-adjacent numbers is divisible by $2015 \cdot 2016$.
- The product of each pair of two adjacent numbers is not divisible by $2015 \cdot 2016$.

Find the maximum value of n.

Problem 3. Let $P \in \mathbb{Q}[x]$ be a polynomial of degree 2016 whose leading coefficient is 1. A positive integer m is "nice" if there exists some positive integer n such that $m = n^3 + 3n + 1$. Suppose that there exist infinitely many positive integers n such that $P(n)$ are nice. Prove that there exists an arithmetic sequence (n_k) of arbitrary length such that $P(n_k)$ are all nice for $k = 1, 2, 3, \ldots$

1.4.8. Level 4+, day 4

Problem 1. On the Cartesian coordinate system Oxy, consider a sequence of points $A_n(x_n, y_n)$ in which $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ are two sequences of positive numbers satisfing the following conditions:

$$x_{n+1} = \sqrt{\frac{x_n^2 + x_{n+2}^2}{2}}, \; y_{n+1} = \left(\frac{\sqrt{y_n} + \sqrt{y_{n+2}}}{2}\right)^2 \quad \forall n \geq 1.$$

Suppose that O, A_1, A_{2016} belong to a line d and A_1, A_{2016} are distinct. Prove that all the points $A_2, A_3, \ldots, A_{2015}$ lie on one side of d.

Problem 2. Let $ABCDEF$ be a convex hexagon with $AB = CD = EF$, $BC = DE = FA$ and $\angle A + \angle B = \angle C + \angle D = \angle E + \angle F$. Prove that all angles of this hexagon are equal.

Problem 3. Find the number of permutations $(a_1, a_2, \ldots, a_{2016})$ of the first 2016 positive integers satisfing the following two conditions:

1. $a_{i+1} - a_i \leq 1$ for all $i = 1, 2, 3, \ldots, 2015$.
2. There are exactly 2 indices $i < j$ such that $a_i = i$ and $a_j = j$.

Part 2

OFFICIAL SOLUTIONS

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2.1. Preselection Tests for Full time training

2.1.1. Level 4, day 1

Problem 1. Let x, y, z be positive real numbers satisfy the condition $x^2 + y^2 + z^2 = 2(xy + yz + zx)$. Prove that

$$x + y + z + \frac{1}{2xyz} \geq 4.$$

Solution.

Without loss of generality, assume that $z = \min\{x, y, z\}$. From the condition $x^2 + y^2 + z^2 = 2(xy + yz + zx)$, we get

$$(x + y)^2 - 2z(x + y) + z^2 = 4xy,$$

or

$$(x + y - z)^2 = 4xy.$$

Using the AM-GM inequality, we have

$$\frac{x + y - z}{2} + \frac{x + y - z}{2} + 2z + \frac{1}{2xyz} \geq 4\sqrt[4]{\frac{(x + y - z)^2 z}{4xyz}} = 4.$$

Hence

$$x + y + z + \frac{1}{2xyz} \geq 4.$$

The equality holds if $x = 2,\ y = z = \frac{1}{2}$, or any its cyclic permutation.

□

Problem 2. Let ABC be a non isosceles triangle inscribed in a circle (O) and BE, CF are two angle bisectors intersect at I with E belongs to segment AC and F belongs to segment AB. Suppose that BE, CF intersect (O) at M, N respectively. The line d_1 passes through M and perpendicular to BM intersects (O) at the second point P, the line d_2 passes through N and perpendicular to CN intersect (O) at the second point Q. Denote H, K are two midpoints of MP and NQ respectively.

1. Prove that triangles IEF and OKH are similar.
2. Suppose that S is the intersection of two lines d_1 and d_2. Prove that SO perpendicular to EF.

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Solution.

1) Denote the projections of I on AC, AB are X, Y respectively.

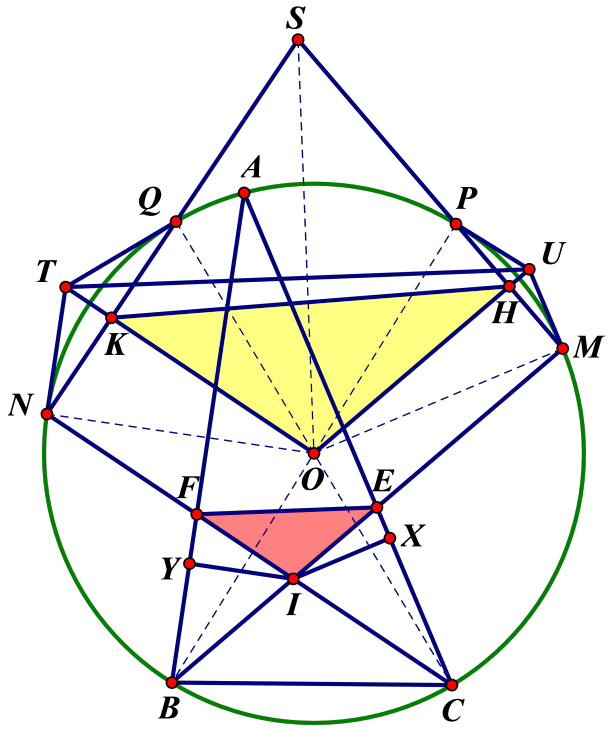
Note that B, O, P collinear because $\angle BMP = 90°$. Because O, H are two midpoints of PB, PM so $OH \parallel BM$ and $OH = \frac{1}{2}BM$.

Similarly we have $OK \parallel CN$ and $OK = \frac{1}{2}CN$. Hence, $\angle HOK = \angle EIF$. In the other hand,

$$\frac{OH}{OK} = \frac{BM}{CN} = \frac{\sin\left(A + \frac{B}{2}\right)}{\sin\left(A + \frac{C}{2}\right)},$$

$$\frac{IF}{IE} = \frac{IY}{\sin\angle IFY} : \frac{IX}{\sin\angle IEX} = \frac{\sin\angle IEX}{\sin\angle IFY} = \frac{\sin\left(A + \frac{B}{2}\right)}{\sin\left(A + \frac{C}{2}\right)}$$

So $\frac{OH}{OK} = \frac{IF}{IE}$ implies that $\Delta OHK \sim \Delta IFE$.



2) It is easy to see that

$$OK \cdot OT = R^2, OH \cdot OU = R^2$$

with R is the radius of (O). Then $OK \cdot OT = OH \cdot OU$ or K, T, H, O cyclic.

From this, we can see $\angle IEF = \angle IKH = \angle IUT$ so $EF \parallel UT$.

The antipole line of T, U pass through S so the antipole line of S is TU, leads to $TU \perp SO$. From these result, we get $SO \perp EF$

□

Problem 3. A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too. Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)

Solution.

The problem is solved if there is a way to change the orientation of any specified single key, without changing any of the others. This is equivalent to finding a way to switch the chosen key an odd number of times, while switching all other keys a even number of times.

This can be done by switching all keys on the same row and column of the chosen key (including the chosen key). To see why, choose a key K. If we switch it and all of its sisters that share the same row and column, K will be switched 7 times. Now, examine the other 15 keys in the lock. There are two cases: either the key is a sister of K, or not.

Suppose that L is a sister of K, say, sharing a row with K. Then L will be switched 4 times. For the other case, suppose M is not a sister of K.

Then M will be switched twice, because among the 6 sisters of K which are turned, exactly two of them share a row or column with M. Consequently, of all the keys in the lock, only K is switched an odd number of times.

All other keys are switched either 2 or 4 times, leaving their orientation unchanged. Thus we will be able to open the lock by selecting each horizontal key one-by-one, and turning it and all of its sister keys. □

Problem 4. Let p be a given prime. For each prime r, we define the function as following

$$F(r) = \frac{(p^{rp}-1)(p-1)}{(p^r-1)(p^p-1)}.$$

1. Show that $F(r)$ is a positive integer for any prime $r \neq p$.

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2. Show that $F(r)$ and $F(s)$ are coprime for any primes r and s such that $r \neq p, s \neq p$ and $r \neq s$.

3. Fix a prime $r \neq p$. Show that there is a prime divisor q of $F(r)$ such that $p \mid q-1$ but $p^2 \not| q-1$.

Solution.
Notice that with positive integers a, m, n and $a > 1$, we have

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1.$$

Let $f(r) = \frac{p^r-1}{p-1}$ with p is a prime and r is the positive integer.

1. Let $x = \gcd(p^r - 1, p^p - 1)$ and $y = \operatorname{lcm}(p^r - 1, p^p - 1)$, then

$$(p^r - 1)(p^p - 1) = xy.$$

From the above lemma, we get $x = p^{\gcd(r,p)} - 1 = p - 1$. Hence $(p^r - 1)(p^p - 1) = (p-1)y$, this implies that $F(r) = \frac{p^{pr}-1}{y}$. But $p^r - 1 \mid p^{pr} - 1$ and $p^p - 1 \mid p^{pr} - 1$, so $y \mid p^{pr} - 1$ leads to $F(r)$ is the positive integer.

2. We can see that

$$\gcd(p^{rp} - 1, p^{sp} - 1) = p^{\gcd(rp,sp)} - 1 = p^p - 1.$$

Let $p^{rp} - 1 = (p^p - 1)r_1, p^{sp} - 1 = (p^p - 1)s_2$ with $\gcd(r_1, s_1) = 1$, then

$$\begin{aligned}\gcd(F(r), F(s)) &= \gcd\left(\frac{(p^p-1)r_1(p-1)}{(p^r-1)(p^p-1)}, \frac{(p^p-1)s_1(p-1)}{(p^s-1)(p^p-1)}\right)\\ &= \gcd\left(\frac{r_1}{f(r)}, \frac{s_1}{f(s)}\right) = 1\end{aligned}$$

3. At first, we will show that with any prime divisor q of $F(r)$, we always have $p \mid q-1$. (*)

Indeed, from Fermat's theorem, $q | p^{q-1} - 1$ and because $q \left| \frac{p^{rp}-1}{(p^r-1)f(p)} \right.$ so $q | p^{rp} - 1$.

These imply that

$$q \left| \gcd\left(p^{q-1} - 1, p^{rp} - 1\right) = p^{\gcd(q-1,rp)} - 1 \right.$$

Since r, p are two primes, $d = \gcd(q-1, rp) \in \{1, r, p, rp\}$. We have to consider 4 following cases:

1. If $d = p$ or $d = rp$, we get $p | q-1$ and (*) follows.

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2. If $d = 1$, we have $q|p-1$ or $p \equiv 1(\text{mod}q)$. We also have

$$0 \equiv \frac{p^{rp}-1}{p^p-1} = p^{p(r-1)} + p^{p(r-2)} + \cdots + p^p + 1 \equiv p \equiv 1(\text{mod}q),$$

contradiction.

3. If $d = r$, we have $q|p^r-1$ or $p^r \equiv 1(\text{mod}q)$. We also have

$$0 \equiv \frac{p^{rp}-1}{p^r-1} = p^{r(p-1)} + p^{r(p-2)} + \cdots + p^r + 1 \equiv r(\text{mod}q),$$

contradiction.

Hence (*) is true. Finally, assume that with all the prime $q|F(r)$, we always have $p^2|q-1$.

Because all divisor of $F(r)$ congruent to 1 modulo p^2, hen $F(r) \equiv 1(\text{mod } p^2)$. Note that

$$F(r)(p^p-1)(p^r-1) = (p^{rp}-1)(p-1) = p^{rp+1}-p^{rp}-(p-1) \equiv -(p-1)(\text{ mod } p^2)$$

and

$$F(r)(p^p-1)(p^r-1) = F(r)\left(p^{p+r}-p^r-p^p+1\right) \equiv F(r) \equiv 1(\text{mod}p^2).$$

These imply that

$$1 \equiv -(p-1)(\text{mod}p^2) \Leftrightarrow p \equiv 0(\text{mod}p^2),$$

which is contradiction.

Therefore, there exist a prime satisfies all the given condition.

□

2.1.2. Level 4, day 2

Problem 1.

Given three numbers x, y, z, and set $x_1 = |x-y|$, $y_1 = |y-z|$, $z_1 = |z-x|$. From x_1, y_1, z_1, form in the same fashion the numbers x_2, y_2, z_2, and so on. It is known that $x_n = x, y_n = y, z_n = z$ for some n. Find all possible values of (x, y, z).

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Solution.
First, consider 3 sequences $(x_n), (y_n), (z_n), n \geq 1$ with

$$x_1 = |x-y|, y_1 = |y-z|, z_1 = |z-x| \text{ and } \begin{cases} x_{n+1} = |x_n - y_n| \\ y_{n+1} = |y_n - z_n| \\ z_{n+1} = |z_n - x_n| \end{cases}$$

From this, it is easy to see that $x_n, y_n, z_n \geq 0$ for all n.

Let $w_n = \max\{x_n, y_n, z_n\}, n \geq 1$. We have

$$\begin{aligned} w_n = \max\{x_{n+1}, y_{n+1}, z_{n+1}\} &= \max\{|x_n - y_n|, |y_n - z_n|, |z_n - x_n|\} \\ &\leq \max\{x_n, y_n, z_n\} = w_n. \end{aligned}$$

The equality occurs when there is at least one number among x_n, y_n, z_n is equal to 0. Hence, the sequence (w_n) non increasing. Suppose that k is a positive integer such that $(x_k, y_k, z_k) = (x, y, z)$, then

$$w_1 \geq w_2 \geq \ldots \geq w_k = w_1 \text{ which implies that } w_1 = w_2 = \ldots = w_k.$$

In the other hand, for all $i = \overline{1,k}$, there are at least one number among (x_i, y_i, z_i) is equal to 0. Without loss of generality, we may assume that $x \geq y \geq z = 0$ and we can see that $x_1 = |x-y|, y_1 = |y-z| = y, z_1 = |z-x| = x$ so we have some cases:

1. If $|x-y| = 0 \Leftrightarrow x = y$, we have the tuple $(x, y, z) = (a, a, 0)$ with $a \geq 0$. It is easy to check this tuple satisfy the given condition.

2. If $y = 0$, for separating to the previous case, we consider $x > 0$, we have the tuple $(x, y, z) = (a, 0, 0)$ with $a > 0$. But $x_1 = a, y_1 = 0, z_1 = a$ and from this, we cannot obtain $(a, 0, 0)$ anymore. Hence, this case is not satisfy the given condition.

3. If $x = 0$, we have $x = y = z = 0$ which was mentioned above.

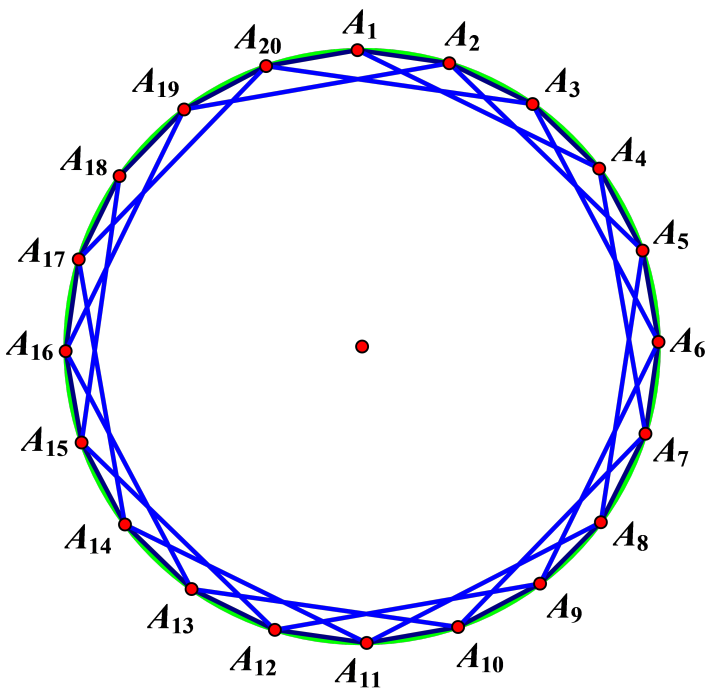
Therefore, the tuples $(x, y, z) = (a, a, 0)$ and its permutation with $a \geq 0$ satisfy the given condition. □

Problem 2. Ten vertices of a regular 20-gon $A_1A_2 \ldots A_{20}$ are painted blacks and the other ten vertices are painted blue. Consider the set consisting of diagonal A_1A_4 and all other diagonals of the same length.

1. Prove that in this set, the number of diagonals with two black endpoints is equal to the number of diagonals with two blue endpoints.

2. Find all possible numbers of the diagonals with two black endpoints.

Solution.



1) Define the diagonal with two black endpoints as the black diagonal and the diagonal with two blue endpoints as the blue diagonal. Firstly, consider the sequence of vertices:.

$$A_1 \to A_4 \to A_7 \to A_{10} \to A_{13} \to A_{16} \to A_{19} \to A_2 \to A_5 \to A_8 \to A_{11}$$
$$\to A_{14} \to A_{17} \to A_{20} \to A_3 \to A_6 \to A_9 \to A_{12} \to A_{15} \to A_{18}$$

The diagonal connect two adjacent vertices has the same length as A_1A_4.

Because the vertices are painted by black and blue, then they can be divided into some blocks of black vertices and some blocks of blue vertices. It is easy to see that one black block lies between two blue blocks and one blue block lies between two black blocks, which implies that the number of black blocks and blue blocks are equal, denote this number as k.

Notice that in each black block of size a, the number of black diagonal is $a-1$. By summing all the black diagonals among $k$black block(s), we can see that the number of black diagonal equal to the total black points minus the number of block, i.e. $10-k$.

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Similarly, the number of blue diagonal is $10-k$. Therefore, the number of black diagonal is equal the number of blue diagonal.

2) From part 1), the number of black diagonal is equal $10-k$ with $1 \leq k \leq 10$. It is easy to see that with each $k \in \{1,2,3,\ldots,10\}$, we can find the way to paint the vertices such that there are exactly k black block(s). □

Problem 3. Let u and v be positive rational numbers with $u \neq v$. Assume that there are infinitely many positive integers n with the property that $u^n - v^n$ are integers. Prove that u and v are integers.

Solution.
Let $u = x/z$ and $v = y/z$ with x, y, z positive integers, and $(x,y,z) = 1$. The statement is equivalent to saying that $x^n \equiv y^n \pmod{z^n}$. If $z > 1$, assume that there is an odd prime divisor $p \mid z$. Let r be the least positive integer such that $x^r \equiv y^r \pmod{p} \Rightarrow r \mid n \Rightarrow n = rk$. Let $a = v_p(n)$ and $b = v_p(x^r - y^r)$. By LTE, we have

$$v_p(x^n - y^n) = v_p((x^r)^k - (y^r)^k) = v_p(x^r - y^r) + v_p(k) \leqslant b + v_p(n) = a + b.$$

But $v_p(x^n - y^n) \geqslant v_p(z^n) \leqslant n \Rightarrow n \leqslant a + b \Rightarrow p^n \leqslant p^a \cdots p^b = np^b$. This cannot true for infinitely many positive integers n.
If z has no odd prime divisor, then z is a power of 2. From this we have x, y are both odd.
If n is odd then

$$2^n \mid x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

But the second factor is odd since n is odd we get $2^n \mid x - y$ for $x \neq y$. It is clear that there are only finitely many such n.
If n is even, let $s = v_2(x^2 - y^2)$ and $c = v_2(n)$. By LTE we have

$$v_2(x^n - y^n) = c + s - 1 \Rightarrow n \leqslant c + s - 1 \Rightarrow n \leqslant \log_2 n + s - 1.$$

Which does not hold for sufficiently large even values of n. □

Problem 4. Let ABC be a non isosceles triangle with circumcircle (O) and incircle (I). Denote (O_1) as the circle that external tangent to (O) at A' and also tangent to the lines AB, AC at A_b, A_c respectively. Define the circles $(O_2), (O_3)$ and the points $B', C', B_c, B_a, C_a, C_b$ similarly.

1. Denote J as the radical center of $(O_1), (O_2), (O_3)$ and suppose that JA' intersects (O_1) at the second point X, JB' intersects (O_2) at the second point Y, JC' intersects (O_3) at the second point Z. Prove that the circle (XYZ) is tangent to $(O_1), (O_2), (O_3)$.

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2. Prove that AA_1, BB_1, CC_1 are concurrent at the point M and 3 points I, M, O are collinear.

Solution.

1) Consider the inversion with center J and ratio is equal the power of J to three circles $(O_1), (O_2), (O_3)$ as a function f.

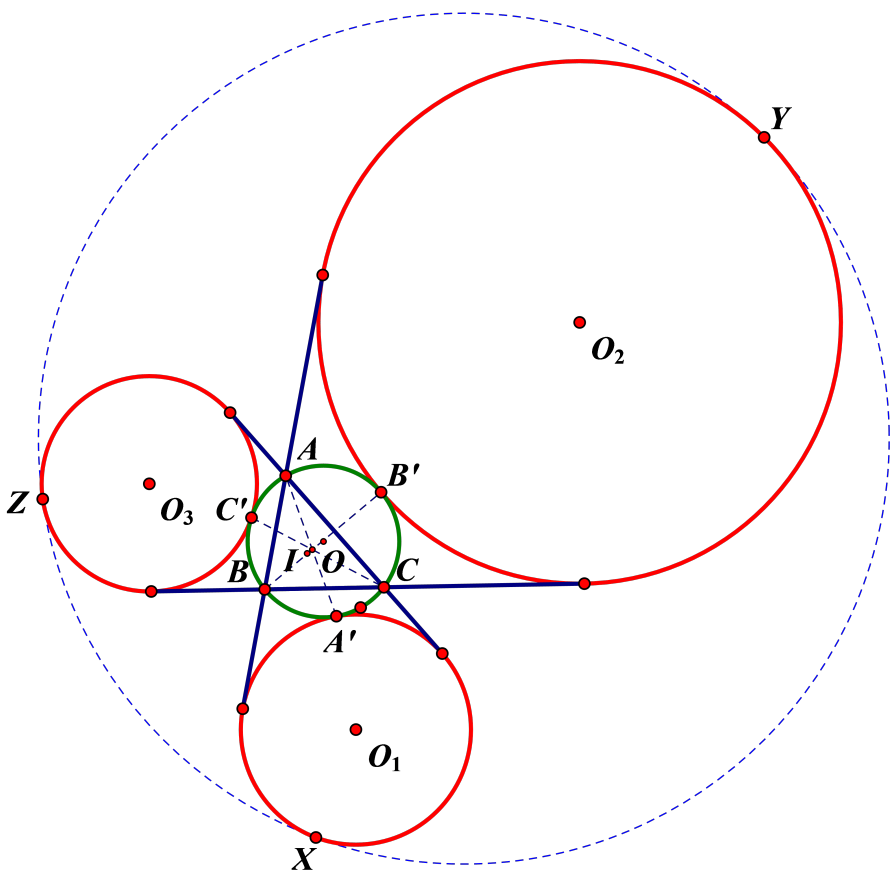
It is easy to see that

$$f((O_1)) = (O_1), f((O_2)) = (O_2), f((O_3)) = (O_3).$$

In the other hand $f(X) = A_1, f(Y) = B_1, f(Z) = C_1$ thus

$$f((XYZ)) = (A_1B_1C_1) = (O).$$

Because (O) is tangent to all of three circles $(O_1), (O_2), (O_3)$ then (XYZ) is also tangent to $(O_1), (O_2), (O_3)$, base on the property of inversion.



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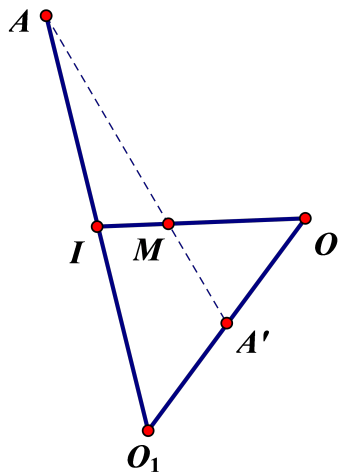
2) We will prove that AA', BB', CC' are concurrent at the homothety center of (O) and (I). Indeed,

Suppose that AA' intersects IO at M. It is easy to see that A, I, O_1 are collinear and O, O_1, A' are also collinear. Denote r_A as the radius of (O_1).



From the ratio of radius, we can see that:

$$\frac{\overline{AI}}{\overline{AO_1}} = \frac{r}{r_A} \text{ and } \frac{\overline{A'O}}{\overline{A'O_1}} = -\frac{R}{r_A}.$$

By applying Menelaus theorem, we have

$$\frac{\overline{AI}}{\overline{AO_1}}\frac{\overline{A'O_1}}{\overline{A'O}}\frac{\overline{MO}}{\overline{MI}} = 1 \Leftrightarrow \frac{\overline{MO}}{\overline{MI}} = \frac{\overline{AO_1}}{\overline{AI}}\frac{\overline{A'O}}{\overline{A'O_1}} = \frac{r_A}{r}\left(-\frac{R}{r_A}\right) = -\frac{R}{r}.$$

Hence the line AA' passes through the point M that internally divide the segment OI with ratio $\frac{R}{r}$. Similarly with BB', CC'. Therefore, AA', BB', CC' are concurrent at a point M belongs to the segment IO. □

2.1.3. Level 4+, day 1

Problem 1. Let ABC be an acute triangle, AX, BY, CZ are the altitudes with X, Y, Z belong to BC, CA, AB respectively. Denote (O_1), (O_2), (O_3) as circumcircles of triangles AYZ, BZX, CXY respectively. Suppose that (K) is a circle that internal tangent to (O_1), (O_2), (O_3). Prove that (K) is tangent to circumcircle of triangle ABC.

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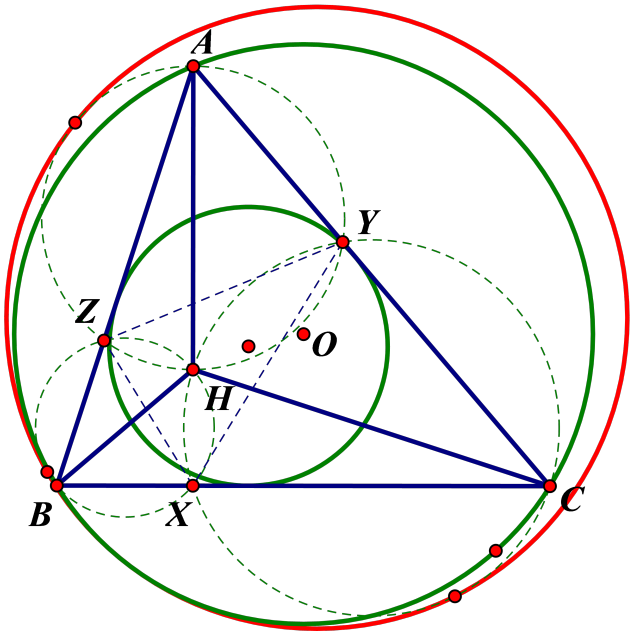
Solution.
Let H be the orthocenter of triangle ABC.



We can see that $HA \cdot HX = HB \cdot HY = HC \cdot HZ = k$. We consider the inversion with center H and ratio $-k$ as the function f.

It is easy to see that

$$f(A) = X, f(B) = Y, f(C) = Z$$

so $f((O)) = (O')$ with (O') is the 9-point circle (which also passes through X, Y, Z).

In the other hand, because $\angle HEA = \angle HFA = 90^\circ$ hence $H \in (O_1)$. After the inversion, the circles (O_1) become a line passes through images of X, Y; indeed, this line is BC or $f((O_1)) = BC$.

Similarly, we also have $f((O_2)) = CA$ and $f((O_3)) = AB$. So the circle K that tangent to $(O_1), (O_2), (O_3)$ will become the incircle (I) of triangle ABC. But base on Feuerbach's theorem, two circles $(O'), (I)$ are tangent to each other.
Therefore, the circles (K) and (O) are also tangent to each other. □

Problem 2. Let a, b, c be positive numbers such that $a^2+b^2+c^2+abc = 4$. Prove that

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

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Solution.
Without loss of generality, assume that $a \geq b \geq c$. From the condition, we get $a \geq 1 \geq c > 0$. Write the inequality as

$$\frac{a+b+c}{c} + \frac{a+b+c}{b} + \frac{a+b+c}{c} \geq a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+3,$$

which is equivalent to

$$(a+b+c-1)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right) \geq 4.$$

Since $(a-1)(1-c) \geq 0$, we obtain $a+c \geq 1+ac$ and thus $\frac{1}{a}+\frac{1}{c} \geq 1+\frac{1}{ac}$. Therefore,

$$\begin{aligned}&(a+b+c-1)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)\\ &\geq (b+ac)\left(\frac{1}{b}+\frac{1}{ac}\right) = 2+\frac{b}{ac}+\frac{ac}{b} \geq 2+2=4.\end{aligned}$$

This completes the proof. Equality holds for $a=b=c=1$. □

Problem 3. Let a, b be two positive integers such that

$$b+1 | a^2+1,\ a+1 | b^2+1.$$

Prove that a, b are odd numbers.

Solution.
Suppose by contrary that one of a, b is even, say a. So a^2+1 is odd. This means that $b+1$ must be odd, or equivalently, b is even. We will prove that this cannot happen.

Indeed, put $d = gcd(a+1, b+1)$ then $d | a+1 | b^2+1$, hence $d | b^2-b = b(b+1)+2b$. From this, we have $d|2b$ and $d|2(b+1) = 2b+2$ so $d|2$. But $a+1, b+1$ are odd numbers so d odd, which gives $d=1$.

On the other hand, by assumption $a+1 | a^2+b^2, b+1 | a^2+b^2$, we get $a^2+b^2 = (a^2+1)+(b^2-1) = (a^2-1)+(b^2+1)$ is divisible by both $a+1$ and $b+1$. Therefore $(a+1)(b+1) | a^2+b^2$ (since $a+1$ and $b+1$ are coprime).

In other words, there exists a positive integer k such that

$$a^2+b^2 = k(a+1)(b+1).$$

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We now prove that this equation has no integer solution. Fix a value of n, assume that this equation has a solution of positive integers. We can choose positive integers x_0, y_0 satisfying the equation with the property that the sum $x_0 + y_0$ is smallest and $x_0 \geq y_0$.

Consider the quadratic equation

$$X^2 - X \cdot n(y_0 + 1) + y_0^2 - ny_0 - n = 0. (*)$$

This equation has a solution x_0 so it also has another solution, say x_1. From Vieta's theorem, we have

$$x_0 + x_1 = n(y_0 + 1), x_0 x_1 = y_0^2 - ny_0 - n$$

We consider the following cases:

1. If $x_1 < 0$, i.e. $x_1 \leq -1$, then from $(*)$ we see that $x_1^2 - x_1 \cdot n(y_0 + 1) + y_0^2 - ny_0 - n = 0$ or

$$0 \geq x_1^2 + n(y_0 + 1) + y_0^2 - ny_0 - n = x_1^2 + y_0^2 > 0,$$

   which is clearly a contradiction.

2. If $x_1 > 0$ then (x_1, y_0) is also another solution of $(*)$, from the choice of x_0, we have $x_1 > x_0$. Moreover, since $x_0 x_1 = y_0^2 - ny_0 - n$ it follows that $x_0^2 < x_0 x_1 = y_0^2 - ny_0 - n < y_0^2$, this is also a contradiction because of the inequality $x_0 \geq y_0$.

Hence, $x_1 = 0$, and in this case $y_0^2 = n(y_0 + 1)$, which implies that $y_0 + 1 \mid y_0^2$. This cannot hold for $y_0 \geq 2$.

Therefore, we conclude that a, b are odd numbers, this ends the proof. □

Problem 4. The natural numbers $0, 1, 2, 3, \ldots$ are written on the square table 2015×2015 in a circular order (anti-clockwise) such that 0 is in the center of the table. The rows and columns are labelled from bottom to top and from left to right respectively. (see figure below)

1. The number 2015 is in which row and which column?

2. We are allowed to perform the following operations: First, we replace the number 0 in the center by 14; after that, each time, we can add 1 to each of 12 numbers on 12 consecutive unit squares in a row, or 12 consecutive unit squares in a column, or 12 unit squares in a rectangle 3×4. After a finite number of steps, can we make all numbers on the table are multiples of 2016?

	20	19	18	17	16	
	21	6	5	4	15	
	22	7	0	3	14	
	23	8	1	2	13	
	24	9	10	11	12	

Solution.
1. Consider the square table with size $(2k+1)\times(2k+1)$ and the center cell contains 0. From the way filling the number to the table, it's easy to see that the number on the left bottom cell of that table is $(2k+1)^2-1$, which is the maximum number among all number on that sub table.

Hence, 2015 belongs the square table with size 45×45 because $2015 < 45^2-1=2024$. Note that $2024-2015=9$ then 2015 belongs to the cell that same column and 10 cells above in comparison with 2024.

If we number the row and column of the given table from left to right and from bottom to top, then the leftmost and lowest cell belongs to column 1 and row 1. Because the size of table is 2015×2015 then the number 0 belongs to cell at 1008 and column 1008.

Since $1008-22=986$, the number 2024 belongs to 986th row and 986th column.

Therefore number 2015 belongs to $986-9=977^{\text{th}}$ row and 986th column.

2. We will prove that we never get the situation such that all numbers on the table are multiples of 2016. Indeed,

The sum of all number on the table when we change 0 to 14 is equal to

$$14+(1+2+3+\cdots+2015^2-1)=14+\frac{2015^2(2015^2-1)}{2}\equiv 2 \pmod 4$$

When we perform the operation, the sum of all number on the table increase by 12 which divisible by 4, that not change the remainder of the sum when divide by 4.

If we reach the state that all number on the table are multiples of 2016, then the sum will divisible by 4, which a contradiction. □

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2.1.4. Level 4+, day 2

Problem 1.

1) Prove that there are infinitely many positive integers n such that there exists a permutation of $1, 2, 3, \ldots, n$ with the property that the difference between any two adjacent numbers is equal to either 2015 or 2016.

2) Let k be a positive integer. Is the statement in 1) still true if we replace the numbers 2015 and 2016 by k and $k + 2016$, respectively?

Solution by Omer Alrabiea.
1) First, replace 2015 and 2016 by 3 and 4 respectively for convenience. We consider the permutation of $\{1, 2, 3, \ldots, 7\}$ as follows:

$$1, 5, 2, 6, 3, 7, 4.$$

The number increase by 4 and decrease by 3, then the sequence covers all number from 1 to 7. Add 1 to each term of the above sequence, we get

$$8, 12, 9, 13, 10, 14, 11.$$

Combine two sequences, we have

$$1, 5, 2, 6, 3, 7, 4, 8, 12, 9, 13, 10, 14, 11.$$

This is another permutation of $\{1, 2, 3, \ldots, 14\}$. By this method, we can construct the desired permutation for each number of form $7k, k \in \mathbb{Z}$ satisfy the condition.

In the same way with two given numbers 2015 and 2016:

$$1, 1 + 2016, 1 + 2016 - 2015, \ldots$$

Increase by 2016 then decrease by 2015 and so on, we can get the desired permutation for each number of form $(2015 + 2016)k, k \in \mathbb{Z}$.

2) Now it is easy to see that all number k such that $\gcd(k, 2016) > 1$ do not satisfy the given condition. Indeed,

Suppose that $\gcd(k, 2016) = d > 1$ and these exist a number n such that the permutation of $\{1, 2, 3, \ldots, n\}$ satisfies the given condition. And all numbers in this sequence have the same remainder when divide by d, which is a contradiction.

Now we will show that every number coprime with 2016 will satisfy the given condition by proving the general statement:

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If m, k are two positive numbers and $\gcd(m,k) = 1$, then there are infinitely many number n such that there exist a permutation with the property that the difference between any two adjacent numbers is equal to either m or k.

Suppose that $m > k$ and $m = ak + b, 0 \le b \le k-1$. We will prove that each number n which is a multiple of $t(m+k) = (a+1)k + b$ satisfies the given condition. Indeed,

Partition the set $\{1, 2, 3, \ldots, n\}$ into k sets and each set contains the numbers share the same remainder when divide by k. And each of them will be arranged decreasing from the largest to the smallest (except the set contains number 1).

$$1 \to 1 + (ak+b) \to 1 + (a-1)k + b \to \ldots \to 1 + b \to \ldots$$
$$1 + ak + 2b \to 1 + (a-1)k + 2b \to \ldots \to 1 + 2b \to \ldots$$
$$\ldots$$
$$1 + ak + (k-1)b \to 1 + (a-1)k + (k-1)b \to \ldots \to 1 + (k-1)b \to \ldots$$

It is easy to see that the difference between two adjacent numbers in each sequence is k and two sequences can be connected by add the smallest number in one set to the biggest number in another set. Notice that when we take a number $i \in \{2, 3, 4, \ldots, k\}$ and add m, we get $i + m$ which has the remainder differs from i when divide by k and since $\gcd(m,k) = 1$ then the set

$$\{i, i+m, i+2m, \ldots, i+(k-1)m\}$$

form a complete system of residue modulo k then when we connect all sets to get the permutation of $\{1, 2, 3, \ldots, n\}$. □

Problem 2. Given four numbers x, y, z, t, let (a, b, c, d) be a permutation of (x, y, z, t) and set $x_1 = |a-b|$, $y_1 = |b-c|$, $z_1 = |c-d|$, and $t_1 = |d-a|$. From x_1, y_1, z_1, t_1, form in the same fashion the numbers x_2, y_2, z_2, t_2, and so on. It is known that $x_n = x, y_n = y, z_n = z, t_n = t$ for some n. Find all possible values of (x, y, z, t).

Solution.
First, consider 4 sequences $(x_n), (y_n), (z_n), (t_n)$ with $x_1 = |a-b|, y_1 = |b-c|, z_1 = |c-d|, t_1 = |d-a|$, (a, b, c, d) is permutation of (x, y, z, t). And

$$\begin{cases} x_{n+1} = |a_n - b_n| \\ y_{n+1} = |b_n - c_n| \\ z_{n+1} = |c_n - d_n| \\ t_{n+1} = |d_n - a_n| \end{cases}$$

with (a_n, b_n, c_n, d_n) is a permutation of (x_n, y_n, z_n, t_n).

From this, it is easy to see that $x_n, y_n, z_n, t_n \geq 0$ for all n.

Let $w_n = \max\{x_n, y_n, z_n, t_n\}, n \geq 1$. We have

$$\begin{aligned} w_n &= \max\{x_{n+1}, y_{n+1}, z_{n+1}, t_{n+1}\} \\ &= \max\{|a_n - b_n|, |b_n - c_n|, |c_n - d_n|, |d_n - a_n|\} \\ &\leq \max\{a_n, b_n, c_n, d_n\} = \max\{x_n, y_n, z_n, t_n\} = w_n \end{aligned}$$

The equality occurs when there is at least one number among a_n, b_n, c_n, d_n is equal to 0.

Hence the sequence (w_n) non increasing. Suppose that k is a positive integer such that $(x_k, y_k, z_k, t_k) = (x, y, z, t)$, then $w_1 \geq w_2 \geq \cdots \geq w_k = w_1$ which implies that $w_1 = w_2 = \cdots = w_k$. So there at least one number among (x, y, z, t) is equal to 0. Base on the definition of the sequences, we can see that for each $i = 1, 2, 3, \ldots, k$, the tuple (x_i, y_i, z_i, t_i) have some common properties as follow:

1. There is at least one number is 0.
2. There are two numbers equal.
3. In the permutation to obtain the next term, two equal numbers must be adjacent.

It is easy to check that 3 tuples as follow satisfy the given conditions:

i. If $(x, y, z, t) = (0, 0, 0, 0)$.

ii. If $(x, y, z, t) = (a, a, 0, 0)$ with $a > 0$ and its permutation.

iii. If $(x, y, z, t) = (2a, a, a, 0)$ with $a > 0$ and its permutation.

Indeed, when (x, y, z, t) is one of 3 above tuples, then form of tuple (x_n, y_n, z_n, t_n) will not change. Continue, we will prove that if (x, y, z, t) is a tuple satisfies condition, then it must be one of 3 above forms. $(*)$

Without loss of generality, we may assume that $x \geq y \geq z \geq t = 0, x > 0$. Since there are two numbers equal among 4 numbers, we consider some cases:

1. If $x = y$ then we have $(x, y, z, t) = (a, a, b, 0)$ with $a \geq b \geq 0, a > 0$. Base on the 3rd property, (x_1, y_1, z_1, t_1) can only be $(0, a - b, b, a)$. Apply the 2nd property, we have:

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- If $a = a - b, b = 0$, we have $(x, y, z, t) = (a, a, 0, 0)$, which satisfies the given condition.
- If $a = b, a - b = 0$, then we have $(x, y, z, t) = (a, a, a, 0)$ with $a > 0$. So (x_1, y_1, z_1, t_1) will be $(0, 0, a, a)$. But from this, the form will not change, so we cannot obtain the original form anymore. This case doesn't satisfy.
- If $a - b = b$ then $a = 2b$, we have $(x, y, z, t) = (2b, 2b, b, 0)$ and $(x_1, y_1, z_1, t_1) = (0, b, b, 2b)$. But the form of this tuple will not change and similarly, this case doesn't satisfy.

2. If $y = z$, then we have $(x, y, z, t) = (a, b, b, 0)$ with $a \geq b > 0$. So (x_1, y_1, z_1, t_1) can be $(a-b, 0, b, a)$. Apply the 2nd property, we have:
   - If $a - b = 0, a = b$, we have $(x, y, z, t) = (a, a, a, 0)$, not satisfy.
   - If $a - b = a, b = 0$, we have the tuple $(x, y, z, t) = (a, 0, 0, 0)$ and $(x_1, y_1, z_1, t_1) = (a, 0, 0, a)$. It is easy to see that this case doesn't satisfy.
   - If $a - b = b$, we have $(x, y, z, t) = (2b, b, b, 0)$ which satisfy.
3. If $z = 0$ then we have $(x, y, z, t) = (a, b, 0, 0)$ with $a \geq b \geq 0)$. So (x_1, y_1, z_1, t_1) can be $(a-b, 0, b, a)$. Apply the 2nd property, we have:
   - If $a - b = 0, a = b$, we have $(x, y, z, t) = (a, a, 0, 0)$, satisfies.
   - If $a - b = a, b = 0$, we have $(x, y, z, t) = (a, 0, 0, 0)$, not satisfy.
   - If $a - b = b$, we have the tuple $(x, y, z, t) = (2b, b, 0, 0)$ and $(x_1, y_1, z_1, t_1) = (b, b, 0, 2b)$. It is easy to see that this case doesn't satisfy.

Therefore, there are 3 tuples satisfy the given condition as above. □

Problem 3. Let ABC be a non isosceles triangle with circumcircle (O) and incircle (I). Denote (O_1) as the circle that internal tangent to (O) at A_1 and also tangent to segments AB, AC at A_b, A_c respectively. Define the circles $(O_2), (O_3)$ and the points $B_1, C_1, B_c, B_a, C_a, C_b$ similarly.

1. Prove that AA_1, BB_1, CC_1 are concurrent at the point M and 3 points I, M, O are collinear.
2. Prove that the circle (I) is inscribed in the hexagon with 6 vertices $A_b, A_c, B_c, B_a, C_a, C_b$.

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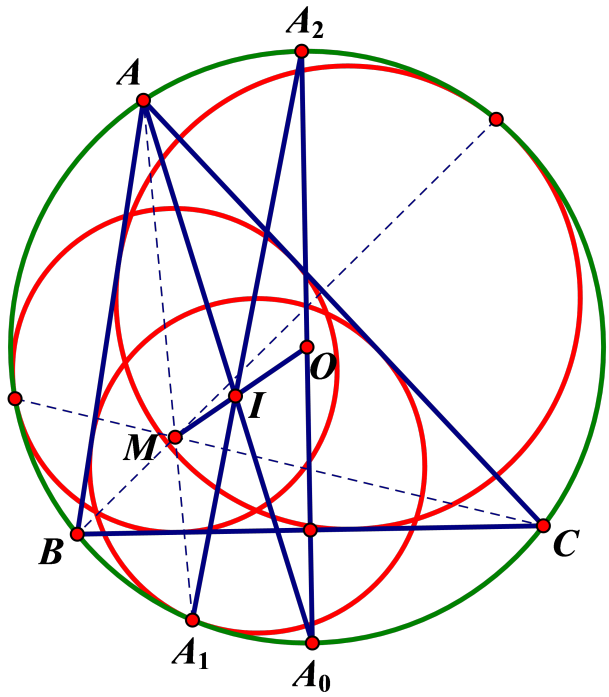
Solution.

1) We use inversion to solve this problem.



Suppose that AI, A_1I intersect (O) at A_0, A_1. Because AI is the bisector then A_0 is the midpoint of the minor arc BC. Base on the property of Mixtilinear circle, we also have A_1 the midpoint of the major arc BC.

Consider the inversion of center I and ratio equal the power of I to (O) as the function f.

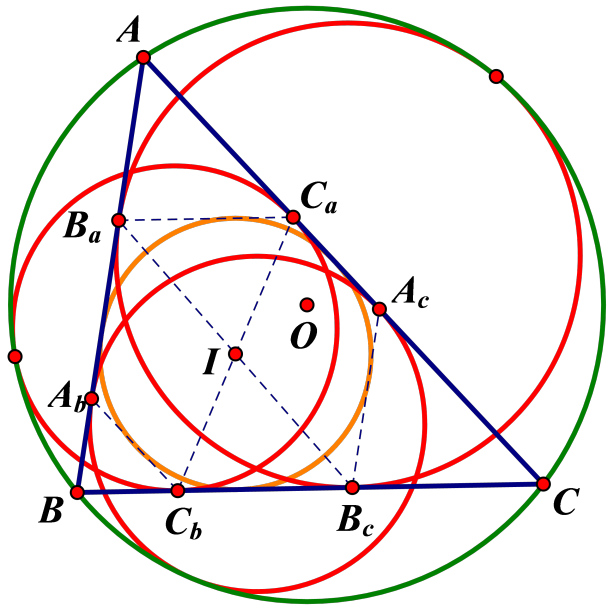
We have $f(A) = A_0, f(A_1) = A_2$ then $f(AA_1) = (IA_0A_2)$. Define B_0, B_2, C_0, C_2 similarly then $f(BB_1) = (IB_0B_2), f(CC_1) = (IC_0C_2)$.

It is easy to see that three circles $(IA_0A_2), (IB_0B_2)$ and (IC_0C_2) share the common point I. In the other hand, power of O to three circle are also equal to $-R^2$ which R is the radius of the circumcircle.

Hence, three circles have two common points and one of them is I which is the center of inversion. Then AA_1, BB_1, CC_1 are concurrent at a point M and M, I, O collinear.

2) From the property of Mixtilinear circle, we have B_aB_c and C_aC_b have the common midpoint I then $B_aC_aB_cC_b$ is a parallelogram, which implies

that B_aC_a parallel to BC and the distance from I to B_aC_a and BC are the same. Hence B_aC_a is tangent to (I). Similarly, we also have A_bC_b and B_cA_c are also tangent to (I).



It is easy to see that C_bB_c is concide with BC then it is tangent to (I). Similarly, we also have C_aA_c and A_bB_a are tangent to (I).

Hence, 6 sides of the hexagon $C_bB_cA_cC_aB_aA_b$ are tangent to the circle (I), which finish the solution. □

Problem 4. Let n be a given positive integer. Prove that there are infinitely many pairs of positive integers (a, b) with $a, b > n$ such that

$$\prod_{i=1}^{2015}(a+i) \mid b(b+2016); \quad \prod_{i=1}^{2015}(a+i) \nmid b; \quad \prod_{i=1}^{2015}(a+i) \nmid (b+2016).$$

Solution.
The given problem can be generalized as follows:

Given three positive integers k, m, n. Let $k_1, k_2, \ldots, k_m$ be any positive integer. Prove that there are infinitely many pair of positive integers (a, b) such that

$$\prod_{k=1}^{m}(a+k_i) \,\Big|\, b(b+k) \text{ but } \prod_{k=1}^{m}(a+k_i) \not| b \text{ and } \prod_{k=1}^{m}(a+k_i) \not| b+k.$$

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Proof. Let $k < p_1 < p_2 < \ldots < p_m$ be any m distinct primes and denote

$$M = \prod_{k=1}^{m} (a + k_i).$$

By Chinese remainder theorem, there are infinitely many positive integers $a > n$ such that

$$a \equiv -k_i (\mathrm{mod} p_i), \forall i = \overline{1, m}.$$

Hence $M \equiv 0 (\mathrm{mod} p_1 p_2 \ldots p_m)$.

Then we write

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m} q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$$

with $\alpha_i \geq 1, i = \overline{1, m}$; $\beta_j \geq 1, j = \overline{1, s}$ and $q_1, q_2, \ldots, q_s$ are s prime divisors of M which are difference from $p_1, p_2, \ldots, p_m$.

By Chinese remainder theorem, these exist infinitely many positive integers $b > m$ such that $\begin{cases} b \equiv 0 \left(\mathrm{mod} \dfrac{M}{p_m^{\alpha_m}} \right) \\ b \equiv -k \left(\mathrm{mod} p_m^{\alpha_m} \right) \end{cases}$. This implies that

$$M \not| b, M \not| b + k \text{ and } M | b(b + k).$$

□

2.2. Solution of TST for Gulf Mathematical Olympiad 2016

2.2.1. Level 4, day 1

Problem 1. Let $f(x) = x^2 + ax + b$ be a quadratic function with real coefficients a, b. It is given that the equation $f(f(x)) = 0$ has 4 distinct real roots and the sum of 2 roots among these roots is equal to -1. Prove that $b \leq -\frac{1}{4}$.

Solution.
First, we will prove that $f(x) = 0$ has some solutions (maybe not distinct).

Indeed, if $f(x) = 0$ has no root, then it can be written as $f(x) = (x-c)^2 + d$ with $d > 0$ and

$$f(f(x)) = ((x - c)^2 + d - c)^2 + d > 0.$$

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It means $f(f(x)) = 0$ has no solution, which is a contradiction.

Now, denote $c_1 \geq c_2$ as the solution of $f(x) = 0$ and x_1, x_2 as the solutions of $f(f(x)) = 0$ in such a way that $x_1 + x_2 = -1$. By Vieta's theorem, note that $c_1 + c_2 = -a$ and $c_1 c_2 = b$.

It is easy to see that $f(f(x)) = 0$ is equivalent to $f(x) = c_1, f(x) = c_2$. We need to consider 2 cases:

1. If x_1, x_2 are the solutions of one equation, by Vieta's theorem, then $a = 1$. Thus $c_2 \leq -\frac{1}{2}$.

   Consider equation $f(x) - c_2 = 0$, we have $\Delta = 1 - 4(b - c_2) > 0$, which implies that $b < -\frac{1}{4}$.

2. If x_1, x_2 are solutions of two equations, then $x_i^2 + ax_i + b_i = c_i$ with $i = 1, 2$. Sum these two identities, we get

$$x_1^2 + x_2^2 - a + 2b = -a \Leftrightarrow x_1^2 + x_2^2 + 2b = 0.$$

   Hence, $b = -\frac{x_1^2 + x_2^2}{2} \leq -\frac{(x_1 + x_2)^2}{4} = -\frac{1}{4}$.

Therefore, in all case, we always have $b \leq -\frac{1}{4}$. □

Problem 2. Let $(O_1), (O_2)$ be given two circles intersecting at A and B. The tangent lines of (O_1) at A, B intersect at O. Let I be a point on the circle (O_1) but outside the circle (O_2). The lines IA, IB intersect circle (O_2) at C, D. Denote by M the midpoint of CD. Prove that I, M, O are collinear.

Solution.
Denote N as the midpoint of AB. Because A, B, C, D belong to the same circle, then we have

$$\Delta IAB \sim \Delta IDC.$$

Since M is the midpoint of CD and N is the midpoint of AB then IM, IN are isogonal conjugate with respect to the angle $\angle CID$.

In the other hand, O is the intersection of two tangent line of (O_2) at A, B, then IO is the symmedian of triangle IAB. It means IO, IN are isogonal conjugate with respect to $\angle AIB$.

Hence, I, M, O are collinear.

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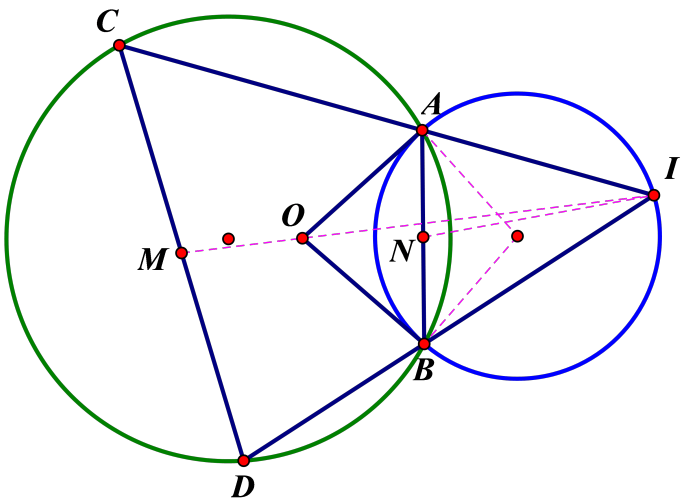

□

Problem 3. Find all positive integer n such that there exists a permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, 3, \ldots, n)$ satisfying the condition:

$$a_1 + a_2 + \cdots + a_k \text{ is divisible by } k \text{ for each } k = 1, 2, 3, \ldots, n.$$

Solution.
It is easy to see that $n = 1, n = 3$ satisfy the given condition, while $n = 2$ is not. We will show that all number $n > 3$ not satisfy.

First, we can see that

$$a_1 + a_2 + \cdots + a_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

then $\frac{n(n+1)}{2}$ is divisible by n, which means $\frac{n+1}{2} \in \mathbb{Z}$ or n is the odd number. We also have

$$n - 1 \mid a_1 + a_2 + \cdots + a_{n-1} = \frac{n(n+1)}{2} - a_n = \frac{(n+1)(n-1)}{2} + \frac{n+1}{2} - a_n$$

Thus, $\frac{n+1}{2} - a_n$ is divisible by $n - 1$.

In the other hand,

$$-(n-1) < \frac{n+1}{2} - a_n < n - 1$$

then we must have

$$\frac{n+1}{2} - a_n = 0 \Leftrightarrow a_n = \frac{n+1}{2}.$$

It implies that

$$a_1 + a_2 + \cdots + a_{n-1} = \frac{n^2-1}{2}.$$

Continue, we have

$$n-2 \mid a_1 + a_2 + \cdots + a_{n-2} = \frac{n^2-1}{2} - a_{n-1} = \frac{(n-2)(n+1)}{2} + \frac{n+1}{2} - a_{n-1}.$$

Similarly, we get $a_{n-1} = \frac{n+1}{2}$, then $a_{n-1} = a_n$, which is a contradiction.

Hence, there are only two satisfied value are $n = 1$ and $n = 3$. □

Problem 4. There are totally 16 teams participating in a football tournament; each team playing with every other exactly 1 time. In each match, the winner gains 3 points, the loser gains 0 point and each teams gain 1 point for the tie match. Suppose that at the end of the tournament, each team gains the same number of points. Prove that there are at least 4 teams that have the same number of winning matches, the same number of losing matches and the same number of tie matches.

Solution.
First, we can see that if a team got x win matches, y lose matches and z tie matches then they got $3x + z$ points and $x + y + z = 15$. We call two team have the same number of win matches, lose matches and tie matches are "relate".

Denote the number of win matches, lose matches and tie matches of two teams A, B by $(x_1, y_1, z_1), (x_2, y_2, z_2)$ respectively then

$$3x_1 + y_1 = 3x_2 + y_2,$$

then the difference between the tie matches is

$$|z_1 - z_2| = 3\,|x_1 - x_2| \text{ which is divisible by 3.}$$

And two teams are relate if and only if they have the same number of tie matches.

The number of their tie matches lies between 0 and 15 and all are congruent modulo 3. So we can divide these team into some group by the number of tie matches (the teams in a group have the same number of tie matches). Hence, by Pigeonhole principle, the number of group can not be exceed $\left\lceil \frac{15}{3} \right\rceil = 6$.

If there are some groups have greater than or equal 4 members, i.e 4 teams, then we are done.

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Otherwise, each group contains 3 members or fewer. Then the number of group must be 6, if not, the number of team is not greater than 15, which is a contradiction.

But the only way to obtain 6 groups like that is the number of tie matches of 16 teams are $0, 3, 6, 9, 12, 15$. It means there are some teams have all match are tie and there are also some teams have all match are not tie, which is a contradiction.

Hence, there are some groups have at least 4 members.

□

2.2.2. Level 4, day 2

Problem 1. Let $S = x + y + z$ where x, y, z are three nonzero real numbers satisfying the following system of inequalities:

$$\begin{cases} xyz > 1 \\ x + y + z > \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}. \end{cases}$$

Prove that S can take on any real values when x, y, z vary.

Solution.
First, if $z = \frac{1}{y}$, then the given system becomes

$$\begin{cases} x \cdot y \cdot \dfrac{1}{y} > 1 \\ x + y + \dfrac{1}{y} > \dfrac{1}{x} + \dfrac{1}{y} + y \end{cases} \Leftrightarrow x > 1.$$

So, all triples $\left(x, y, \dfrac{1}{y}\right)$, with $x > 1$ and $y \neq 0$, satisfy the given system of inequalities.

Next, note that the range of $f(y) := y + \dfrac{1}{y}$ is $(-\infty, -2] \cup [2, +\infty)$ and that, for such triples, $S = x + f(y)$. Then for each value $a \in \mathbb{R}$, we need only consider the following cases:

1. If $a > -1$, we can choose $x = a + 2$ (> 1), $y = -1$, and get $S = a + 2 + f(-1) = a$.

2. If $a < -1$, we let $x = -a$ (> 1) and choose $y \neq 0$ such that $f(y) = 2a$ $(\in (-\infty, -2])$. Then $S = -a + f(y) = a$.

3. If $a = -1$, we choose $x = 1.5$ $(> 1), y = -2$ and get $S = 1.5 + f(-2) = -1 = a$.

Therefore, S can take on any real values when x, y, z vary. □

Problem 2. Let c be a given real number. Find all polynomials P with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x) = c \quad \text{for all} \quad x \in \mathbb{R}.$$

Solution.
In terms of $G(x) := P(x) - \frac{c}{2}$, the given condition can be rewritten as

$$(x+1)G(x-1) = (x-1)G(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

It follows immediately (with $x = \pm 1$) that $G(0) = 0$ and $G(-1) = 0$. Therefore, $G(x) = x(x+1)Q(x)$ for some $Q \in \mathbb{R}[x]$. Then

$$Q(x-1) = Q(x) \quad \text{for all} \quad x \in \mathbb{R} \setminus \{-1, 0, 1\}.$$

This is true iff Q is a constant; i.e.,

$$P(x) = kx(x+1) + \frac{c}{2} \quad \forall x \in \mathbb{R}$$

where k is real constant. □

Remark. We have $(x+2)G(x) = xG(x+1)$ for all $x \in \mathbb{R}$. This is a special case of Problem 2 in the test for Level 4+ (where $a = -1$, $b = -2$).

Problem 3. Let ABC be a triangle whose incircle (I) is tangent to AB, AC at D, E respectively. Denote by Δ_b, Δ_c the lines symmetric to the lines AB, AC with respect to CD, BE correspondingly. Suppose that Δ_b, Δ_c meet at K.

1. Prove that $IK \perp BC$.

2. If $I \in (KDE)$, prove that $BD + CE = BC$.

Solution.
1. Denote by G, H the intersections of KE, KF and BC respectively. Consider triangle CEG: it is easy to see that

- CI is the internal bisector of $\angle C$

- EI is the external bisector of $\angle E$.

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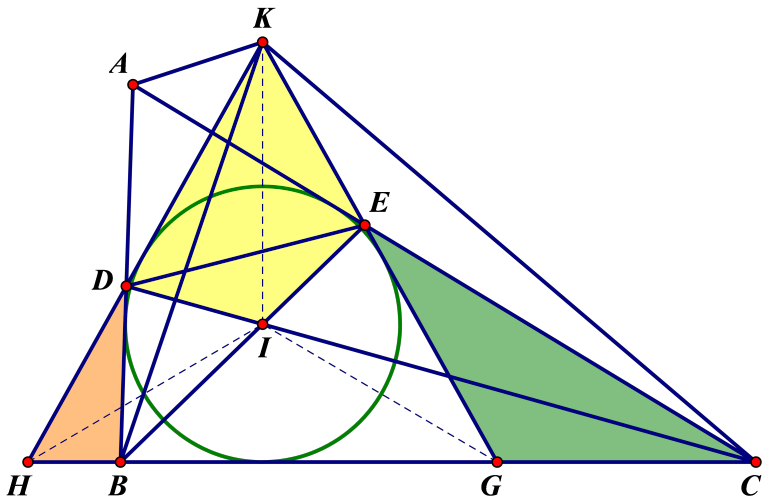
Hence, I is the excenter of this triangle, which implies that GI is the bisector of $\angle KGH$.



Similarly, we can also see that HI is the bisector of $\angle GHK$. So I is the incenter of triangle GHK and KI is the bisector of $\angle GKH$.

In the other hand, BA, BC are symmetric with respect to the line BE and CA, KG are also symmetric with respect to the line BE. From these facts, we can conclude that $\angle KGH = \angle BAC$.

Similarly, we also have $\angle KHG = \angle BAC$; therefore, KGH is an isosceles triangle.

Since I is the incenter of the isosceles triangle KGH, KI is also an altitude, i.e. $KI \perp BC$.

2. Denote $BC = a, CA = b, AB = c$ and $A = \alpha$. Then $\angle DIE = 90^\circ + \dfrac{\alpha}{2}$.

In the isosceles triangle KGH, we have $\angle KGH = \angle KHG = \alpha$, so $\angle DKE = 180^\circ - 2\alpha$. Hence, if $I \in (KDE)$, then

$$90^\circ + \frac{\alpha}{2} + 180^\circ - 2\alpha = 180^\circ$$

and $\alpha = 60^\circ$. From the cosine law in triangle ABC, we have

$$a^2 = b^2 + c^2 - bc.$$

By calculating, it is also easy to check that $BD = \dfrac{ac}{a+b}, CE = \dfrac{ab}{a+c}$; thus,

$$BD + CE = BC \Leftrightarrow \frac{ac}{a+b} + \frac{ab}{a+c} = a \Leftrightarrow b^2 + c^2 - bc = a^2,$$

which is true. $\square$

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2.2.3. Level 4, day 3

Problem 1. Let ABC be an acute, non-isosceles triangle which is inscribed in a circle (O). A point I belongs to the segment BC. Denote by H and K the projections of I on AB and AC, respectively. Suppose that the line HK intersects (O) at M, N (H is between M, K and K is between H, N). Prove the following assertions:

1. If A is the center of the circle (IMN), then BC is tangent to (IMN).
2. If I is the midpoint of BC, then BC is equal to 4 times of the distance between the centers of two circles (ABK) and (ACH).

Solution.
See the solution to Problem 1 in the test of level 4+. □

Problem 2. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(2m + f(m) + f(m)f(n)) = nf(m) + m$$

for any integers m, n.

By Hamzar Shafi.
Let $f : \mathbb{Z} \to \mathbb{Z}$ satisfy the given functional equation. Putting $n = 0$ in this equation, we get

$$f(2m + f(m) + f(m)f(0)) = m \quad \forall m \in \mathbb{Z};$$

therefore, f is surjective, so there exists $u \in \mathbb{Z}$ such that $f(u) = -1$.

With $m = u$, the given equation gives us

$$f(2u - 1 - f(n)) = -n + u.$$

Now, if a, b are some integers such that $f(a) = f(b)$, then

$$u - a = f(2u - 1 - f(a)) = f(2u - 1 - f(b)) = u - b,$$

which implies that $a = b$. Hence, f is also injective.

Next, putting $n = u$, we have

$$f(2m + f(m) - f(m)) = uf(m) + m \Leftrightarrow f(2m) = uf(m) + m \ (*)$$

for all $m \in \mathbb{Z}$. In $(*)$, letting $m = 0$, we see that $f(0) = uf(0)$, so $u = 1$ or $f(0) = 0$.

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If $f(0) = 0$, then with $m = u$, $(*)$ would imply $f(2u) = -u + u = 0 = f(0)$, i.e. $2u = 0 \Leftrightarrow u = 0$ (since f is injective), and thus $f(0) = -1$, a contradiction!

Hence, $u = 1 \Leftrightarrow f(1) = -1$ and $(*)$ becomes

$$f(2m) = f(m) + m$$

for all $m \in \mathbb{Z}$. Here, letting $m = 1$, we obtain $f(2) = 0$.

In the given functional equation, putting $m = n = 0$, we have

$$f(f(0) + f(0)^2) = 0 \Leftrightarrow f(0) + f(0)^2 = 2 \Leftrightarrow f(0) = 1 \text{ or } f(0) = -2.$$

If $f(0) = 1$, then it would follow from the given equation with $m = 0, n = 2$ that $f(1) = 2$, a contradiction (because $f(1) = -1$)!

Hence, $f(0) = -2$. In the functional equation, putting $n = 0$, we get

$$f(2m - f(m)) = m \quad \forall m \in \mathbb{Z}.$$

Using this, we see that if $f(m) = m - 2$ then $f(m + 2) = m$; but $f(0) = -2, f(1) = -1$, we can easily prove by induction that $f(n) = n - 2$ for all $n \geq 0$.

In the given equation, letting $m = 1$, we obtain

$$f(2 - 1 - f(n)) = -n + 1 \Leftrightarrow f(1 - f(n)) = 1 - n \quad \forall n \in \mathbb{Z}.$$

Now replacing n by $n + 3$ and letting $n \geq -3$, we have

$$f(1 - f(n + 3)) = -n - 2 \Leftrightarrow f(1 - (n + 1)) = -n - 2 \Leftrightarrow f(-n) = -n - 2.$$

So $f(n) = n - 2$ for all $n \in \mathbb{Z}$.

It is easy to check that this function satisfies the given condition. □

Problem 3. In a school there are totally $n > 2$ classes and not all of them have the same numbers of students. It is given that each class has one head student. The students in each class wear hats of the same color and different classes have different hat colors. One day all the students of the school stand in a circle facing toward the center, in an arbitrary order, to play a game. Every minute, each student put his hat on the person standing next to him on the right. Show that at some moment, there are 2 head students wearing hats of the same color.

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Solution.
Suppose that there are m students in total. We number the students standing in the circle by $1, 2, 3, \ldots, m$, clockwise (starting with an arbitrary student). Denote by $a_1 < a_2 < a_3 < \cdots < a_n$ the "positions" of the n head students obtained in this numbering.

We consider an $n \times m$—table and fill numbers in its cells by the following way: For each $1 \le i \le n$, in the ith row, we start with a_i (filling in the first cell of this row with a_i) and continue with

$$a_i + 1, a_i + 2, \ldots, m, 1, 2, \ldots, a_i - 1$$

to the right. We will focus on the hat colors of the students in the first column.

a_1	a_1+1	a_1+2	$\ldots$	a_1-1
a_2	a_2+1	a_2+2	$\ldots$	a_2-1
a_3	a_3+1	a_3+2	$\ldots$	a_3-1
a_n	a_n+1	a_n+2	$\ldots$	a_n-1

At the first minute, each student in the first column wears his own hat.

At the second minute, based on the game rule, each student in the first column will wear the hat of the student in the same row but in the second column.

Similarly, at the kth minute, each student in the first column will wear the hat of the student in the same row but in the kth column, for each $1 \le k \le m$.
Suppose, on the contrary, that we could never find 2 head students (in the first column) wearing hats of the same color. This would imply that at the beginning all the students in each column also wear hats of (pairwise) different colors, so they belong to distinct classes. Hence, the numbers of students of n classes are the same, which is a contradiction to the hypothesis.

Therefore, at some moment, there are 2 head students wearing hats of the same color. □

2.2.4. Level 4+, day 1

Problem 1. Let $f(x) = x^2 + ax + b$ be a quadratic function with real coefficients a, b. It is given that the equation $f(f(x)) = 0$ has 4 distinct

real roots and the sum of 2 roots among these roots is equal to -1. Prove that $b \leq \frac{-1}{4}$.

Solution.
See the solution to Problem 1 in the test of level 4. □

Problem 2. There are totally 16 teams participating in a football tournament; each team playing with every other exactly 1 time. In each match, the winner gains 3 points, the loser gains 0 point and each teams gain 1 point for the tie match. Suppose that at the end of the tournament, each team gains the same number of points. Prove that there are at least 4 teams that have the same number of winning matches, the same number of losing matches and the same number of tie matches.

Solution.
See the solution to Problem 1 in the test of level 4. □

Problem 3. Let ABC be an acute, non-isosceles triangle with the circumcircle (O). Denote D, E as the midpoints of AB, AC respectively. Two circles (ABE) and (ACD) intersect at K differs from A. Suppose that the ray AK intersects (O) at L. The line LB meets (ABE) at the second point M and the line LC meets (ACD) at the second point N.

1. Prove that M, K, N collinear and MN perpendicular to OL.
2. Prove that K is the midpoint of MN.

Solution of Omer Alrabiea.

1) Denote G as the intersection of BE, CD, then G is the centroid of triangle ABC. We assume that $AB < AC$ and the solution is similar to all other cases.

Since $ABLC$ is cyclic, we have $180^\circ - \angle AKN = \angle ACN = \angle ABM = \angle AKM$, hence

$$\angle AKM + \angle AKN = 180^\circ,$$

which mean M, K, N are collinear.

Since $AKNC$ and $AKBM$ are both cyclic, then by the power of a point to circle, we have

$$LN \cdot LC = LK \cdot LA = LB \cdot LM,$$

thus $MBNC$ is concyclic and we can see $\angle CBL = \angle LNM$.

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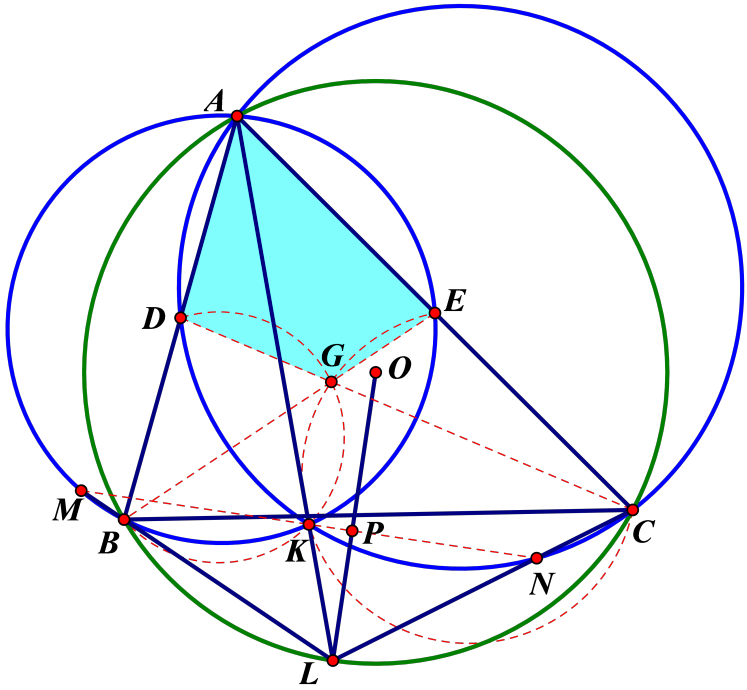
Denote P as the intersection of OL and MN. Since O is the circumcenter of $(ABLC)$ then

$$\angle PLN = 90^\circ - \frac{\angle LOC}{2} = 90^\circ - \angle CBL = 90^\circ - \angle MNL$$

Therefore, OL is perpendicular to MN.



2) Notice that $ADGEBC$ is a complete quadrilateral and K is the Miquel point so K belongs to two circles $(BDG), (CEG)$. Thus we have

$$\angle DKB = \angle DGB = \angle EGC = \angle EKC, \angle KDB = \angle KGB = \angle KCE.$$

So $\Delta BKD \sim \Delta EKC$, then by sin law, we have

$$\frac{AB}{AC} = \frac{BD}{CE} = \frac{BK}{KE} = \frac{\sin BAK}{\sin EAK} = \frac{\sin BAL}{\sin CAL}$$

which means $ABLC$ is the harmonic quadrilateral and LA is the symmedian of triangle LBC.

Finally, because MN is the antiparallel to BC with respect to $\angle BAC$ then the symmedian of triangle LBC is the median of triangle LMN, which means LA passes through the midpoint of MN or K is the midpoint of the segment MN. □

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2.2.5. Level 4+, day 2

Problem 1. Let $S = x+y+z$ where x, y, z are three nonzero real numbers satisfying the following system of inequalities:

$$\begin{cases} xyz > 1 \\ x+y+z > \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}. \end{cases}$$

Prove that S can take on any real values when x, y, z vary.

Solution.
See the solution to Problem 1 in the test of level 4. □

Problem 2. Let a, b be given two real number with $a \neq 0$. Find all polynomials P with real coefficients such that

$$xP(x-a) = (x-b)P(x) \text{ for all } x \in \mathbb{R}.$$

Solution.
Let $P \in \mathbb{R}[x]$ satisfy the given condition.

First, we notice that if $b = 0$ then $xP(x-a) = xP(x) \Rightarrow P(x-a) = P(x)$ $(\forall x \neq 0)$, hence $P(x) \equiv$ a constant (since $a \neq 0$), and we can recheck that any constant polynomial satisfies the given condition. Now, let $b \neq 0$, and put $\frac{b}{a} = k$. We need only consider 2 cases.

1. $k \in \mathbb{N}$: Substituting $x = 0$ into the given condition, we have

$$-bP(0) = 0 \Rightarrow P(0) = 0.$$

Substituting $x = a, 2a, 3a, \ldots, ka$ (in succession) into the given condition, we also have

$$\begin{aligned} &aP(0) = (a-b)P(a) \Rightarrow P(a) = 0 \\ &2aP(a) = (2a-b)P(2a) \Rightarrow P(2a) = 0 \\ &3aP(2a) = (3a-b)P(3a) \Rightarrow P(3a) = 0 \\ &\ldots \\ &(k-1)aP((k-2)a) = ((k-1)a-b)P((k-1)a) \Rightarrow P((k-1)a) = 0. \end{aligned}$$

Thus, $0, a, 2a, \ldots, (k-1)a$ are roots of $P(x) = 0$; therefore, by Bezout's theorem, $P(x)$ can be written in the form

$$P(x) = x(x-a)(x-2a)\cdots(x-(k-1)a)Q(x)$$

for some $Q \in \mathbb{R}[x]$.

The given condition is then equivalent to:

$$\begin{aligned}
&x(x-a)\cdots(x-ka)Q(x-a) = x(x-a)\cdots(x-ka)Q(x)\\
&\Leftrightarrow Q(x-a) = Q(x)\ \ \forall x \notin \{0, a, 2a, \ldots, ka\}\\
&\Leftrightarrow Q(x) \equiv c\\
&\Leftrightarrow P(x) = cx(x-a)(x-2a)\cdots(x-(k-1)a)\ \ \forall x \in \mathbb{R}
\end{aligned}$$

for any real constant c.

2. $k \notin \mathbb{N}$: In the same way, it is easy to check that all numbers ma, with $m \in \mathbb{N}$, are roots of $P(x) = 0$; which implies that $P(x) \equiv 0$; and we can re-check that the zero polynomial satisfies the given condition.

In summary:

- If $b = 0$, then $P(x) \equiv c$.
- If $\frac{b}{a} = k \in \mathbb{N}$, then $P(x) \equiv cx(x-a)(x-2a)\cdots(x-(k-1)a)$.
- If $\frac{b}{a} = k \notin \mathbb{N}$, then $P(x) \equiv 0$.

□

Problem 3. In a school, there are totally n students, with $n \geq 2$. The students take part in m clubs and in each club, there are at least 2 members (a student may take part in more than 1 club). Eventually, the Principal notices that: If 2 clubs share at least 2 common members then they have different numbers of members. Prove that

$$m \leq (n-1)^2.$$

Solution.
Let a_i be the number of clubs that have i members. Here, $2 \leq i \leq n$ and

$$m = a_2 + a_3 + \cdots + a_n.$$

We will count the tuples (A, B, C) in which, the students A, B take part in the same clubs C that have i members.

1. The first way of counting:
   - Choosing 1 club among the clubs that have i members, we have a_i ways.

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- Choosing 2 students that take part in that club, we have $\binom{i}{2}$ ways.

2. The second way of counting:

- Choosing 2 students among all students of the school, we have $\binom{n}{2}$ ways.
- Choose 1 club in which these two students take part. By assumption, there is at most 1 such club.

So we get the following relation:

$$a_i\binom{i}{2} \le \binom{n}{2} \Leftrightarrow a_i \le \frac{\binom{n}{2}}{\binom{i}{2}}.$$

Hence, we have

$$m = a_2 + a_3 + \cdots + a_n \le \binom{n}{2}\left(\frac{1}{\binom{2}{2}} + \frac{1}{\binom{3}{2}} + \cdots + \frac{1}{\binom{n}{2}}\right).$$

Note that

$$\sum_{i=2}^{n} \frac{1}{\binom{i}{2}} = \sum_{i=2}^{n} \frac{2}{i(i-1)} = 2\sum_{i=2}^{n}\left(\frac{1}{i-1} - \frac{1}{i}\right) = 2\left(1 - \frac{1}{n}\right) = \frac{2(n-1)}{n}.$$

Therefore, $m \le \frac{n(n-1)}{2} \cdot \frac{2(n-1)}{n} = (n-1)^2$. □

2.2.6. Level 4+, day 3

Problem 1. Let ABC be an acute, non-isosceles triangle which is inscribed in a circle (O). A point I belongs to the segment BC. Denote by H and K the projections of I on AB and AC, respectively. Suppose that the line HK intersects (O) at M, N (H is between M, K and K is between H, N). Let X, Y be the centers of the circles $(ABK), (ACH)$ respectively. Prove the following assertions:

1. If I is the projection of A on BC, then A is the center of circle (IMN).

2. If $XY \parallel BC$, then the orthocenter of XOY is the midpoint of IO.

Solution.
1) We will use the inversion to solve this problem.

Note that $\Delta AHI \sim \Delta AIB$ and that $\Delta AKI \sim \Delta AIC$, hence

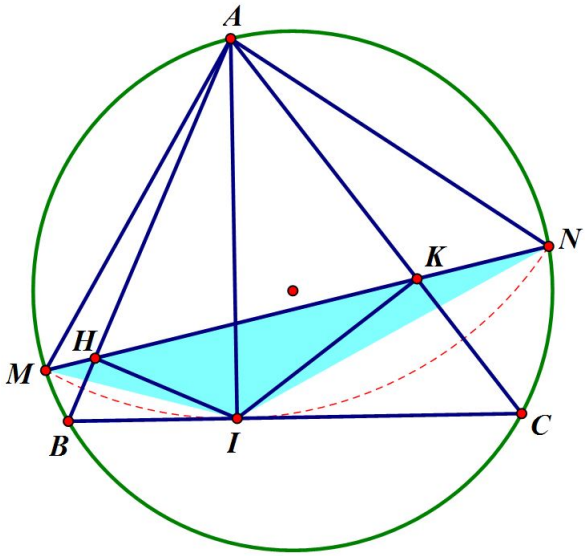
$$AI^2 = AH \cdot AB = AK \cdot AC =: k.$$

Let f be the inversion with center A and power k. Then

$$f(I) = I, f(H) = B, f(B) = H, f(K) = C, f(C) = K.$$

So this inversion sends the line HK to (O), which implies that the intersection of HK and (O) are fixed points under f.

Therefore, $AM^2 = AN^2 = k = AI^2$; therefore, A is the center of the circle (IMN).



2) First, we will prove the following lemma: Let D be a point on the segment BC and H be the projection of D on AC. Suppose that I is the circumcenter of triangle ABH. Denote by M, N the midpoints of AB, AC respectively. Let MN meet AD at K. Then $IK \perp AC$.

Denote by L the projection of B on AC. It is easy to see that I, M, O are collinear. Further,

$$\angle AIM = \angle LHB, \angle AOM = \angle LCB,$$

hence $\Delta AIM \sim \Delta BHL$ and $\Delta AOM \sim \Delta CBL$.

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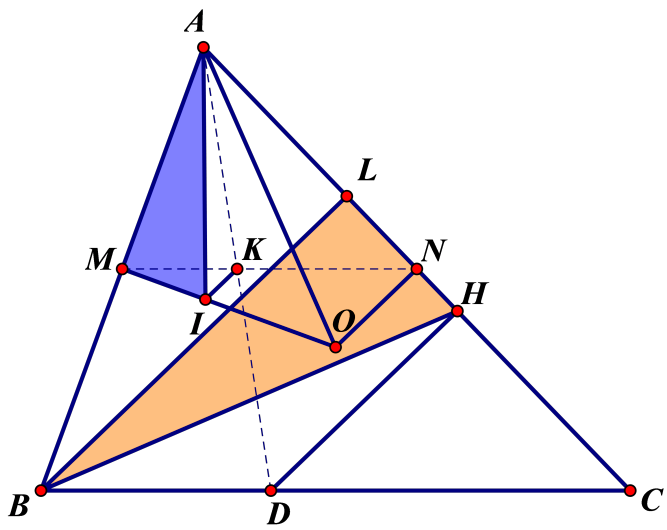
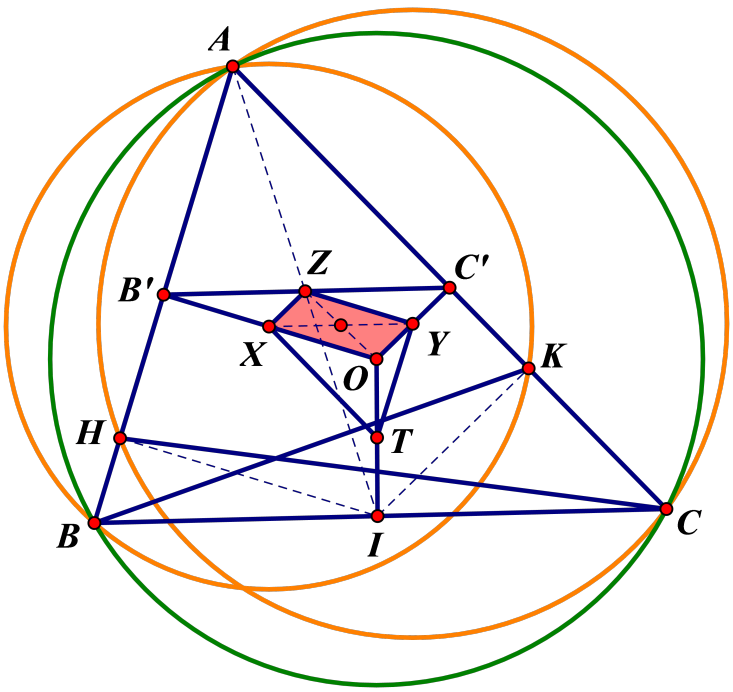


It follows that

$$\frac{IM}{IO} = \frac{LH}{LC} = \frac{BD}{BC} = \frac{MK}{MN};$$

which implies that $IK \parallel ON$, i.e. $IK \perp AC$.

Going back to the original problem, we denote by B', C' the midpoints of AB, AC and by Z the intersection of $B'C'$ and AI.



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Applying the lemma, we have $XZ \parallel OC'$ and $YZ \parallel OB'$; therefore, $OXZY$ is a parallelogram.

Hence, OZ passes through the midpoint of XY. But $XY \parallel B'C'$, so Z is the midpoint of $B'C'$ and XY is a medial line of triangle $OB'C'$.

In the other hand, $B'C' \parallel BC$, hence I is the midpoint of BC.

Let T be the midpoint of the segment OI. Then $XT \parallel B'I \parallel AC$. But $OC' \perp AC$, thus $OY \perp XT$.

Similarly, $OX \perp YT$. Therefore, T is the orthocenter of triangle XOY. $\square$

Problem 2. Let $n \geq 1$ be a fixed positive integer. We consider all the sets S which consist of sub-sequences of the sequence $0, 1, 2, \ldots, n$ satisfying the following conditions:
i) If $(a_i)_{i=0}^k$ belongs to S, then $a_0 = 0, a_k = n$ and $a_{i+1} - a_i \leq 2$ for all $0 \leq i \leq k-1$.
ii) If $(a_i)_{i=0}^k$ and $(b_j)_{j=0}^h$ both belong to S, then there exist $0 \leq i_0 \leq k-1$ and $0 \leq j_0 \leq h-1$ such that $a_{i_0} = b_{j_0}$ and $a_{i_0+1} = b_{j_0+1}$.
Find the maximum value of $|S|$ (among all the above-mentioned sets S).

Solution.
Firstly, denote by A_n the set of all sub-sequences $(a_i)_{i=0}^k$ of $(0, 1, 2, \ldots, n)$ such that:

- $a_0 = 0, a_1 = 1, a_k = n$,
- $a_{i+1} - a_i \leq 2$ for all $0 \leq i \leq k-1$.

Obviously, A_n is one of the sets S satisfying Conditions i)-ii).

It is also clear that $|A_1| = |A_2| = 1$.

If $n > 2$, then for each $(a_i)_{i=0}^k \in A_n$, let us define $F\left((a_i)_{i=0}^k\right) = (a_i)_{i=0}^{k-1}$. In so doing, we have clearly obtained a bijection $F : A_n \to A_{n-1} \cup A_{n-2}$, hence,

$$|A_n| = |A_{n-1}| + |A_{n-2}|$$

when $n > 2$. So, $|A_n|$ is the nth Fibonacci number, which is denoted by F_n. We will show that F_n is the maximum value of $|S|$, among all the sets S satisfying Conditions i)-ii). Indeed, for such a set S, we need only prove that

$$|S| \leq F_n.$$

Let $S_1 = S \cap A_n$. Then $S = S_1 \cup (S \setminus S_1)$. We define a mapping $f : S \setminus S_1 \to A_n$ by partitioning every $w \in S \setminus S_1$ into parts of the form $(m, m+2, m+$

$3,\ldots,m+2k)$ or $(m,m+2,m+3,\ldots,m+2k+1)$ in which $k\geq 1$ and then f changes each part by the following rule:

$$(m,m+2,m+3,\ldots,m+2k)\mapsto(m,m+1,m+2,m+4,m+6,\ldots,m+2k)$$

and

$$(m,m+2,m+3,\ldots,m+2k+1)\mapsto(m,m+1,m+3,m+5,\ldots,m+2k+1).$$

We can see that:

- f is one-to-one.
- The two sequences w and $f(w)$ do not satisfy Condition ii); so,

$$f(w)\notin S \quad \forall w\in S\setminus S_1.$$

Hence, $S_1\cap f(S\setminus S_1)=\emptyset$. Since $S_1\cup f(S\setminus S_1)\subset A_n$, it follows that

$$|S|=|S_1\cup f(S\setminus S_1)|\leq|A_n|=F_n.$$

□

Problem 3. Find all polynomials $P,Q\in\mathbb{Z}[x]$ such that every positive integer is a divisor of a certain nonzero term of the sequence $(x_n)_{n=0}^{\infty}$ given by the conditions:

$$x_0=2016,\ x_{2n+1}=P(x_{2n}),\ x_{2n+2}=Q(x_{2n+1})\ \text{ for all }\ n\geq 0.$$

Solution.
Suppose that $P,Q\in\mathbb{Z}[x]$ satisfy the given requirement. Step by step we will draw some conclusions.

Step 1. $\deg P\geq 1$, $\deg Q\geq 1$.

Suppose, on the contrary, that one of P,Q were a constant polynomial c.

1. If $P(x)=c$ $(\forall x\in\mathbb{Z})$, then

$$x_n\in\{2016,c,Q(c)\}\quad\forall n\geq 0.$$

2. If $Q(x)=c$ $(\forall x\in\mathbb{Z})$, then

$$x_n\in\{2016,P(2016),c,P(c)\}\quad\forall n\geq 0.$$

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In both cases, the requirement "every positive integer is a divisor of a certain nonzero term of the sequence $(x_n)_{n=0}^{\infty}$" couldn't be satisfied. Hence, $\deg P \geq 1$, $\deg Q \geq 1$.

Step 2. $\deg Q = 1$.

Suppose, on the contrary, that $\deg Q > 1$.

By Step 1, $\deg P \geq 1$, so there exist positive numbers M and $k \geq 1$ such that $k\,|P(x)| \geq |x|$ for all $x \in \mathbb{Z}$ with $|x| \geq M$. In particular, $\lim\limits_{|x|\to\infty} |P(x)| = \infty$.

But $\deg Q > 1$, which implies that

$$\lim_{|x|\to\infty} \frac{|Q(x)|}{|x|} = \infty;$$

therefore, if $N \in (M, \infty)$ is chosen large enough, then we have:

$$|Q(x)| > |x|$$

and

$$|Q(P(x))| > 2k\,|P(x)| = k\,|P(x)| + k\,|P(x)| \geq k\,|P(x)| + |x|$$

when $|x| \geq N$.

Now, in accordance with the given requirement, $\lim\limits_{n\to\infty} \max\limits_{0\leq\ell\leq 2n} |x_\ell| = \infty$. Further, for large n, let $0 \leq i = i(n) \leq 2n$ be an index such that

$$|x_i| = \max_{0\leq\ell\leq 2n} |x_\ell| > N.$$

If i were odd, say $i = 2j-1$, then we would get

$$\left|x_{2j}\right| = \left|Q(x_{2j-1})\right| > \left|x_{2j-1}\right| = \max_{0\leq\ell\leq 2n} |x_\ell|,$$

a contradiction (since $2j \leq 2n$). So, i should be even (when n is large enough), say $i = 2j$. For such a j, take $m = \left|x_{2j+2} - x_{2j}\right|$. Then

$$m = \left|Q(P(x_{2j})) - x_{2j}\right| \geq \left|Q(P(x_{2j}))\right| - \left|x_{2j}\right| > k\left|P(x_{2j})\right| \geq \left|x_{2j}\right|,$$

and $m > k\left|P(x_{2j})\right| \geq \left|P(x_{2j})\right| = \left|x_{2j+1}\right|$. Therefore, among the first $2j+2$ terms $x_0, x_1, \ldots, x_{2j}, x_{2j+1}$, there do not exist any term which is nonzero and which is divisible by m. It should also be noted here that $x_{2j}x_{2j+1} \neq 0$ (since $\left|x_{2j+1}\right| = \left|P(x_{2j})\right| \geq \dfrac{\left|x_{2j}\right|}{k} > \dfrac{N}{k} > 0$). Hence, neither x_{2j} nor x_{2j+1} is divisible by m.

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On the other hand,

$$(x_{2\ell+2}-x_{2\ell}) \mid (Q(P(x_{2\ell+2}))-Q(P(x_{2\ell}))) = x_{2\ell+4}-x_{2\ell+2}$$

and

$$(x_{2\ell+2}-x_{2\ell}) \mid (P(x_{2\ell+2})-P(x_{2\ell})) = x_{2\ell+3}-x_{2\ell+1}.$$

Thus, if $\ell \geq j$, then $x_{2\ell+2}-x_{2\ell}$ and $x_{2\ell+3}-x_{2\ell+1}$ are both divisible by $m = \left|x_{2j+2}-x_{2j}\right|$. But, as we have shown above, neither x_{2j} nor x_{2j+1} is divisible by m. It follows that neither $x_{2\ell+2}$ nor $x_{2\ell+3}$ is divisible by m when $\ell \geq j$.

Therefore, m could not be a divisor of any nonzero term of the sequence $(x_n)_{n=0}^{\infty}$, a contradiction again! So, $\deg Q = 1$.

Step 3. $\deg P = 1$.

The proof is similar to that in Step 2.

Step 4. Now $P(x) = ax+b, Q(x) = cx+d$ with $a,b,c,d \in \mathbb{Z}$ and $ab \neq 0$. We will prove that $ac = 1$. By definition,

$$\begin{cases} x_{2n+1} = ax_{2n}+b \\ x_{2n+2} = cx_{2n+1}+d \end{cases} \quad \forall n \geq 0.$$

Hence, $x_{2n+2} = acx_{2n}+bc+d$ and $x_{2n+3} = acx_{2n+1}+ad+b$ for all $n \geq 0$. These 2 sub-sequences share a common recurrence relation of the form $y_{n+1} = ry_n+s$ with $s \in \mathbb{Z}, r = ac$.

Suppose, on the contrary, that $r \neq 1$. Then

$$y_n = r^n y_0 + s\frac{r^n-1}{r-1} \quad \forall n \geq 0.$$

If $r = -1$, then $y_n \in \{y_0, -y_0+s\}$ $(\forall n)$; therefore, the given requirement could not be satisfied. So, $|r| > 1$.

The requirement implies that one of the above-mentioned 2 sub-sequences has the following property: for each $q \in \mathbb{N}_0$ there is an $n = n(q) \in \mathbb{N}_0$ such that $r^q \mid y_n \neq 0$. Of course, $n \to \infty$ as $q \to \infty$. Moreover,

$$r^{\min\{q,n\}} \mid (y_n - r^n y_0) = s\frac{r^n-1}{r-1}.$$

Since $\gcd\left(r, \dfrac{r^n-1}{r-1}\right) = 1$, it follows that $r^{\min\{q,n\}} \mid s$ (for all q). But $|r| > 1$ and $\min\{q,n\} \to \infty$ as $q \to \infty$, we see that $s = 0$. Hence, $y_n = r^n y_0$ $(\forall n)$,

and therefore, the requirement could not be satisfied. This contradiction shows that $r = ac = 1$.

Step 5. Finally, $P(x) = \pm x + b, Q(x) = \pm x + d$ where $b, d \in \mathbb{Z}$ are to be found. We have to consider two cases:

- $P(x) = x + b$ and $Q(x) = x + d$ with $b, d \in \mathbb{Z}$. In this case, by induction, we can show that

  $$x_{2n} = 2016 + n(b+d) \text{ and } x_{2n+1} = 2016 + b + n(b+d) \quad \forall n \geq 0.$$

  So, a necessity condition for the requirement to be satisfied is $b+d \neq 0$. Under this condition, the requirement says that for each $m \in \mathbb{N}$ one of the linear congruences

  $$(b+d)x \equiv -2016 (\text{mod } m), (b+d)x \equiv -(2016+b) (\text{mod } m)$$

  has (infinitely many) solutions $x = n$ in $\mathbb{N}_0$. Equivalently,

  $$\gcd(b+d, m) \mid 2016 \text{ or } \gcd(b+d, m) \mid (2016+b) \text{ for each } m \in \mathbb{N}.$$

  It suffices to consider $m = |b+d|$ and obtain the condition:

  $$(b+d) \mid 2016 \text{ or } (b+d) \mid (2016+b), \text{ where } b+d \neq 0.$$

- $P(x) = -x + b$ and $Q(x) = -x + d$ with $b, d \in \mathbb{Z}$. In almost the same manner, we see that the requirement is satisfied if and only if

  $$(b-d) \mid 2016 \text{ or } (b-d) \mid (-2016+b), \text{ where } b-d \neq 0.$$

□

Remark. We are going to give here another proof of the conclusions in Steps 2-3 (the proof of that in Step 1 and Steps 4-5 will be the same). We need the following lemma.

Lemma. Let $a \in \mathbb{Z}$ and $T \in \mathbb{Z}[x]$ be given. A sequence (y_n) is defined as

$$y_0 = a, y_{n+1} = T(y_n) \quad \forall n \geq 0.$$

Suppose that each positive integer m is a divisor of some nonzero term of (y_n). Then $\deg T = 1$.

Proof. It is easy to check that any constant polynomial T does not satisfy the given condition. We suppose on the contrary that $\deg T > 1$. Then there exists $c > 0$ such that $|T(x)| > 2|x|$ whenever $|x| > c$.

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By assumption, $\lim_{m\to\infty} \max_{0\le \ell\le m} |y_\ell| = \infty$. Further, let $0 \le n = n(m) \le m$ be the smallest index such that $|y_n| = \max_{0\le\ell\le m} |y_\ell|$. Then $n \to \infty$ as $m \to \infty$, and $|y_n| > |y_i|$ for all $0 \le i < n$. Hence, we can choose an $N \in \mathbb{N}$ such that $|y_N| > \max\{c, |y_0|, |y_1|, \ldots, |y_{N-1}|\}$. In particular, this implies that

$$|y_{N+1}| = |T(y_N)| > 2|y_N|.$$

Set $m = |y_{N+1} - y_N|$. Then

$$m \ge |y_{N+1}| - |y_N| > |y_N| > \max\{|y_0|, |y_1|, \ldots, |y_{N-1}|\}.$$

Therefore, y_N is not divisible by m. Moreover, m is not a divisor of any nonzero term among $y_0, y_1, \ldots, y_{N-1}$.
On the other hand, $y_{n+1} - y_n$ is divisible by $y_n - y_{n-1}$ for all $n \ge 1$. So, $y_{n+1} - y_n$ is divisible by $m = |y_{N+1} - y_N|$ for all $n \ge N$. It follows that

$$y_n - y_N = (y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \ldots + (y_{N+1} - y_N)$$

is divisible by m for all $n > N$. But y_N is not divisible by m, so y_n is not divisible by m for all $n > N$, which is a contradiction. This completes the proof of the lemma.

We are now in a position to prove the conclusions of Steps 2-3.

Let $H(x) = P(Q(x))$ and $K(x) = Q(P(x))$. Suppose, on the contrary, that either $\deg P \ge 2$ or $\deg Q \ge 2$. Then $\deg H \ge 2$ and $\deg K \ge 2$ (since, by Step 1, $\deg P \ge 1$, $\deg Q \ge 1$).

Consider the sub-sequence $(x_0, x_2, x_4, \ldots)$, $x_{2(n+1)} = K(x_{2n})$ for all $n \ge 0$. Because $\deg K \ge 2$, the sequence $(y_n) = (x_{2n})$ cannot satisfy the condition given in the lemma. Thus, there exists a positive integer m such that none of $m, 2m, 3m, \ldots$ can be a divisor of a certain nonzero term x_{2n}.

This implies that for each $k \in \mathbb{N}$ there exists a nonzero term x_{2n_k+1} divisible by km. Therefore, the sub-sequence

$$(x_1, x_3, x_5, \ldots), x_{2n+3} = H(x_{2n+1})$$

(for all $n \ge 0$), satisfies the condition given in the lemma. According to this lemma, $\deg H = 1$, a contradiction.

This contradiction shows that $\deg P = 1$, $\deg Q = 1$.

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2.3. Solution of TST for Balkan Mathematical Olympiad 2016

2.3.1. Level 4, day 1

Problem 1. Given that the polynomial $P(x) = x^5 - x^2 + 1$ has 5 roots r_1, r_2, r_3, r_4, r_5. Find the value of the product

$$Q(r_1)Q(r_2)Q(r_3)Q(r_4)Q(r_5),$$

where $Q(x) = x^2 + 1$.

Solution.
Since $r_1, \ldots, r_5$ are the roots of $P(x) = x^5 - x^2 + 1$, by factorization theorem we have

$$P(x) = \prod_{i=1}^{5}(x - r_i).$$

It follows that

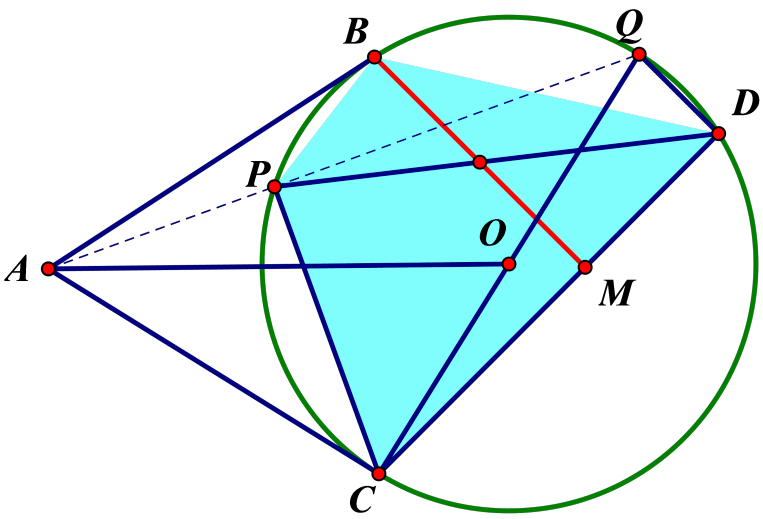
$$\prod_{j=1}^{5} Q(r_j) = \prod_{j=1}^{5}(r_j^2) = \prod_{j=1}^{5}(r_j + i)\prod_{j=1}^{5}(r_j - i) = P(i)P(-i),$$

where $i^2 = -1$. This gives $P(i) = i^5 - i^2 + 1 = -i + 1 + 1 = 2 - i$ and $P(-i) = (-i)^5 - (-i)^2 + 2 = 2 + i$. Hence, $P(i)P(-i) = (2+i)(2-i) = 4 - i^2 = 5$. $\square$

Problem 2. Let A be a point outside the circle ω. Two points B, C lie on ω such that AB, AC are tangent to ω. Let D be any point on ω (D is neither B nor C) and M the foot of perpendicular from B to CD. The line through D and the midpoint of BM meets ω again at P. Prove that $AP \perp CP$.

Solution.
Let O be the center of ω and Q the intersection point of CO and ω.

Since QD and BM are perpendicular to CD then $BM \parallel QD$. Because N is the midpoint of BM, which implies that $D(M, B, N, Q) = -1$.
Consider this harmonic quartet with circle ω, one has $CPBQ$ is a harmonic quadrilateral. This implies that A, P, Q are collinear and $AP \perp CP$.



□

Problem 3. Show that there are infinitely many positive integers n such that n has at least two prime divisors and $20^n + 16^n$ is divisible by n^2.

Solution.

We will construct (by induction on k) the infinite increasing sequence $(n_k)_{k\geq 1}$ of odd positive integers such that any n_k satisfies $4^{n_k} + 5^{n_k}$ is divisible by n_k^2.

We take $n_1 = 1$ and $n_2 = 3$.

Assume we already have n_k, that is $4^{n_k}+5^{n_k} = an_k^2$ for some positive integer a which must be odd, and greater than 1 (since $n_k \geqslant n_2 = 3$, it follows that $4^{n_k} + 5^{n_k} > n_k^2$).

Take p an odd prime divisor of a. Substitute 4^{n_K} by x and 5^{n_k} by y. Then, obviously

$$x^p + y^p = (x + y)(x^{p-1} - yx^{p-2} + \cdots - y^{p-2}x + y^{p-1})$$

is divisible by pn_k^2. Furthermore, it is clear that

$$x^{p-1}, -yx^{p-2}, \ldots, -y^{p-2}x, y^{p-1}$$

are congruent to each other modulo p, and the number of them is p.

So, $x^{p-1} - yx^{p-2} + \cdots - y^{p-2}x + y^{p-1}$ must be divisible by p. It follows that $x^p + y^p$ is divisible by $(pn_k)^2$.

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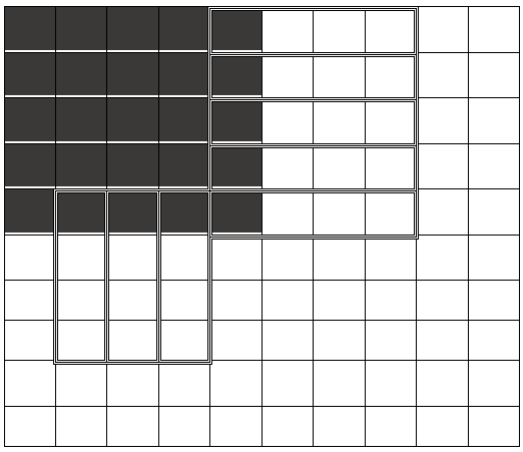
Now we take $n_{k+1} = pn_k$. As $p > 1$, $n_{k+1} > n_k$, and $4^{n_{k+1}} + 5^{n_{k+1}}$ is divisible by n_{k+1}^2.

Finally, it is clear that the number $n = 2n_k$ satisfies $n^2 \mid 20^n + 16^n$ for any positive integer k, which completes the solution. □

Problem 4. On a checkered square 10×10 the cells of the upper left 5×5 square are black and all the other cells are white. What is the maximal n such that the original square can be dissected (along the borders of the cells) into n polygons such that in each of them the number of black cells is three times less than the number of white cells? (The polygons need not be congruent or even equal in area.)

Solution.
The answer is 9. We can see that there are only 9 cells on the border of black square that connect to the white area. Each of them belongs to at most 1 polygon, so there are at most 9 polygons.

An example as follows (each of cells belongs to one part that has the ratio of black:white is 1 : 3)



□

2.3.2. Level 4, day 2

Problem 1. Let $P_i(x) = x^2 + b_i x + c_i$; $i = 1, 2, \cdots, n$ be pairwise distinct polynomials of degree 2 with real coefficients so that for any $0 \le i < j \le n$; $i, j \in \mathbb{N}$, the polynomial $Q_{i,j}(x) = P_i(x) + P_j(x)$ has only one real root. Find the greatest possible value of n.

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Solution.
The answer $n = 3$. In fact, the polynomials $P_1(x) = x^2 - 4, P_2(x) = x^2 - 4x + 6$ and $P_3(x) = x^2 - 8x + 12$ satisfy the conditions $P_1 + P_2 = 2(x - 1)^2, P_1 + P_3 = 2(x - 2)^2, P_2 + P_3 = 2(x - 3)^2$. Suppose that there are four polynomials P_1, P_2, P_3, P_4 satisfying the conditions. Then $P_1 + P_2 = 2(x - t_{12})^2, P_3 + P_4 = 2(x - t_{34})^2, P_1 + P_3 = 2(x - t_{13})^2, P_2 + P_4 = 2(x - t_{24})^2$, where t_{ij} is multiple root of the polynomials $P_i + P_j$. Let $Q = P_1 + P_2 + P_3 + P_4$. Then Q has two representations

$$Q = 2(x - t_{12})^2 + 2(x - t_{34})^2 \text{ and } Q = 2(x - t_{13})^2 + 2(x - t_{24})^2.$$

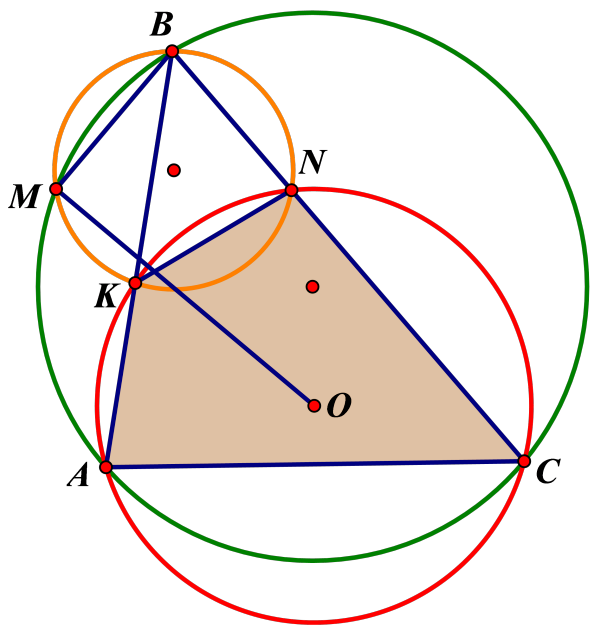
By considering linear and constant terms of both expressions of Q, we get

$$t_{12} + t_{34} = t_{13} + t_{24} \text{ and } t_{12}^2 + t_{34}^2 = t_{13}^2 + t_{24}^2.$$

This shows that $(t_{12}, t_{34}) = (t_{24}, t_{13})$. But, if $t_{12} = t_{13} \Rightarrow P_2 = P_3$, and if $t_{12} = t_{24} \Rightarrow P_1 = P_4$, in both cases, we get a contradiction. □

Problem 2. A circle with center O passes through points A and C and intersects the sides AB and BC of triangle ABC at points K and N, respectively. The circumcircles of triangles ABC and KBN meet at distinct points B and M. Prove that $\angle OMB = 90°$.

Solution.



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Consider an inversion with center B and radius $r > 0$. The points A', B', C' are collinear and the points K', N', M' are, too.

The circumcircle of quadrilateral $ACNK$ becomes the circumcircle ω of quadrilateral $A'C'N'K'$. In case B is outside ω, let X, Y be the tangent points of 2 tangents from B to ω, then O' is the midpoint of XY.

Since M', X, Y lies on the polar of B relative to ω then $\angle M'O'B = 90°$, this implies that $\angle OMB = 90°$.

In the other cases, we also have O' lies on the polar of B relative to ω, and the argument is similar.

□

Remark. This problem can be solve by using the spiral similarity of center M or by using the Brocard's theorem.

Problem 3. Let m, n be odd integers such that $(n^2 - 1)$ is divisible by $m^2 + 1 - n^2$. Prove that $|m^2 + 1 - n^2|$ is a perfect square.

Solution.

By assumption, there is an integer t such that $n^2 - 1 = t(m^2 + 1 - n^2)$. Put $k = t + 1$. It is clear that $k \neq 0$. We have $k(n^2 - 1) = tm^2$. Note that $(k, t) = 1$ we get there is an integer l such that $n^2 - 1 = lt$ (which gives $l = m^2 + 1 - n^2$) and $m^2 = lk = k(m^2 + 1 - n^2) = (t + 1)(m^2 + 1 - n^2)$. Therefore, it suffices to prove that k is a perfect square.

Let S be the set of ordered pairs (x, y) of integers such that

$$(x + y)^2 = k(1 + 4xy).$$

Then, the pair $\left(\frac{m+n}{2}, \frac{m-n}{2}\right) \in S$, and this shows that S is non-empty. Put

$$a = \min\{|x|; \exists (x, y) \in S\}.$$

We show that $a = 0$ whence $k = y^2$.

If $(x, y) \in S \Rightarrow (-x, -y) \in S$, then $(a, y) \in S$. We have $(a + y)^2 = k(1 + 4ay)$. The two roots b_1, b_2 of the equation are integers with $|b_1|, |b_2| \geqslant a$. (Since $(a, b_i) \in S \Rightarrow (b_i, a) \in S$). On the other hand,

$$b_1 + b_2 = 4ak - 2a, \quad b_1 b_2 = a^2 - k.$$

We get $(a + b_1)(a + b_2) = (4a^2 - 1)k$.

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If $k < 0$ then $b_1 b_2 > 0, b_1 + b_2 \leqslant 0 \Rightarrow b_1, b_2 < 0$. Since $|b_i| > a$, it follows that $a + b_i < 0 \Rightarrow (a + b_1)(a + b_2) > 0$. This means that $(4a^2 - 1)k > 0$. Since $k < 0$, we get $a = 0$. Now, assume that $k > 0$ and $a \neq 0$ $(a > 0)$.

In this case, $b_1 + b_2 > 0, (a + b_1)(a + b_2) > 0 \Rightarrow b_1, b_2 > 0$. By the choice of a, we get $a^2 \leqslant b_1 b_2$, which contradicts to the equality $b_1 b_2 = a^2 - k$. So $a = 0$. □

Problem 4. How many ways are there to color the vertices of a square with n colors $1, 2, \ldots, n$. (The colorings must be different so that we can't get one from the other by a rotation.)

Solution.
Consider the cases :

1. Use 1 color: $\binom{n}{1} = n$,
2. Use 2 colors: $4 \times \binom{n}{2} = 2n(n-1)$,
3. Use 3 colors: $9 \times \binom{n}{3} = 3\dfrac{n(n-1)(n-2)}{2}$,
4. Use 4 colors: $6 \times \binom{n}{4} = \dfrac{n(n-1)(n-2)(n-3)}{4}$

The sum up will give : $\dfrac{n^4 + n^2 + 2n}{4}$. □

Remark. We can rotate the coloring respect the angles $0, \dfrac{2\pi}{4}, 2\dfrac{2\pi}{4}, 3\dfrac{2\pi}{4}$. We get $4\times$(the number S of the rotationally distinct colorings). The arbitrary rotation of the colorings with only 1 color gives the same coloring, thus we have to add $3 \times \binom{n}{1}$ and similarly $2 \times \binom{n}{2}$. Thus, we get

$$4 \times S = n^4 + 3 \times \binom{n}{1} + 2 \times \binom{n}{2} = n^4 + n^2 + 2n \Rightarrow S = \frac{n^4 + n^2 + 2n}{4}.$$

2.3.3. Level 4, day 3

Problem 1. Let $a > b > c > d$ be positive integers such that $a^2 + ac - c^2 = b^2 + bd - d^2$. Prove that $ab + cd$ is a composite number.

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Solution.
By contrary, assume that $p = ab + cd$ is a prime. In particular $(b,c) = (b,d) = 1$, and $ab \equiv -cd \pmod p$. We get

$$\begin{aligned} 0 &= b^2(b^2+bd-d^2) - b^2(a^2+ac-c^2) \\ &= b^2(b^2+bd-d^2) - (ab)^2 - (ab)(bc) + b^2c^2. \\ &\equiv (b^2+c^2)(b^2+bd-d^2) \pmod p \end{aligned}$$

Therefore, one of the numbers b^2+c^2 and b^2+bd-d^2 is divisible by p. If $p \mid b^2+c^2$, then $b^2+c^2 = p$ since $0 < b^2+c^2 < 2(ab+cd) = 2p$. This shows $ab+cd = b^2+c^2$ and then $b \mid c(c-d)$. But b,c are coprime, we get $b \mid c-d$ which is impossible.

If $p \mid b^2+bd-d^2$, then $b^2+bd-d^2 = p$ since $0 < b^2+bd-d^2 < 2b^2 < 2ab < 2p$. This shows that $ab+cd = b^2+bd-d^2$ and then

$$b(a-b) = bd - cd - d^2 = d(b-c-d).$$

In particular, $b-c-d > 0$ and $b \mid b-c-d$ (since b,d are coprime) which is impossible, again.

Hence, $ab+cd$ must be a composite number. □

Problem 2. Let ABC be a triangle and I its incenter. The point D is on segment BC and the circle ω is tangent to the circumcirle of triangle ABC but is also tangent to DC, DA at E, F, respectively. Prove that E, F and I are collinear.

Solution.

Denote ω the circumcircle of $\triangle ABC$ and γ the circle tangent to ω, DA, DC. Let ω touches γ at K and M be the midpoint of arc BC on ω not containing K. One has the dilation with center K sent γ to ω and BC to the tangent of ω at M. Hence K, E, M are collinear. We also have A, I, M are collinear and $MI = MC$.

Let EI meet γ again at F'. Since ω, γ have the same tangent at K then $\angle KF'I = \angle EF'K = \angle MAK = \angle KAI \Rightarrow AKIF'$ is cyclic. One has $\angle MCB = \angle MBC = \angle MKC \Rightarrow \triangle MEC \sim \triangle MCK$. Hence

$$MI^2 = MC^2 = ME.MK \Rightarrow \triangle MEI \sim \triangle MIK.$$

Therefore, $\angle KEI = \angle AIK = \angle AF'K$. This implies that AF' is tangent to γ, and $F = F'$.

Hence E, I, F are collinear.

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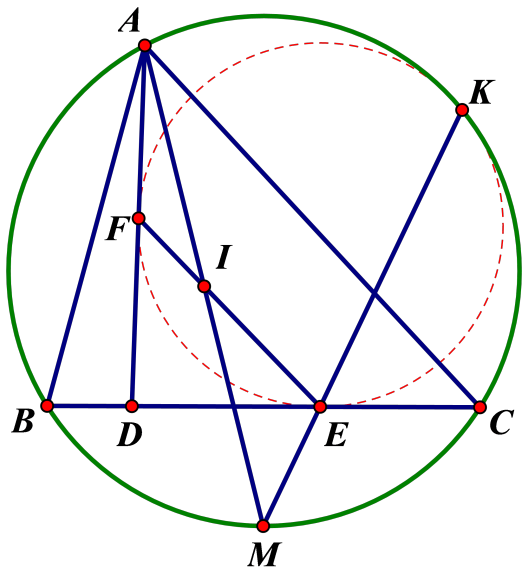

□

Problem 3. Does there exist a polynomial $P(x)$ with integral coefficients such that

1. $P(\sqrt[3]{25}+\sqrt[3]{5}) = 220\sqrt[3]{25}+284\sqrt[3]{5}$?
2. $P(\sqrt[3]{25}+\sqrt[3]{5}) = 1184\sqrt[3]{25}+1210\sqrt[3]{5}$?

Solution.
First, we shall prove two following lemmas:

Lemma 1. If a is an integer numbers that is not a perfect cube, and m, n, p are integer numbers such that $m + n\sqrt[3]{a} + p\sqrt[3]{a^2} = 0$, then

$$m = n = p = 0.$$

Proof of Lemma 1. Since $m + n\sqrt[3]{a} + p\sqrt[3]{a^2} = 0$, then $\sqrt[3]{a}$ is the root of $P(x) = px^2 + nx + m$ with integer coefficients.

Consider the polynomial $Q(x) = x^3 - a$, then $\sqrt[3]{a}$ is also a root of $Q(x)$. Assume that $Q(x)$ is reducible, then one of the factor must have degree 1, that is $Q(x)$ has a rational root. But we know that $\sqrt[3]{a}$ is irrational since a is not a perfect cube (by assumption). Hence, $Q(x)$ is irreducible.

In other words, $Q(x)$ is the minimal polynomial of $\sqrt[3]{a}$, and this leads to $P(x) = 0$, or $m = n = p = 0$.

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Lemma 2. Given that $P(x)$ is a polynomial with integer coefficients, and a is an integer such that $\sqrt[3]{a}$ is not an integer. If

$$P(\sqrt[3]{a^2}+\sqrt[3]{a}) = m + n\sqrt[3]{a} + p\sqrt[3]{a^2},$$

then $a-1 \mid n-p$.

Proof of Lemma 2. We will prove by induction on the degree of $P(x)$ which is denoted by k.

For $k=1$, it is trivial.

Assume that the statement is true for $k=t$, consider $P(x)$ of degree $t+1$, we write $P(x) = xQ(x)+s$, with $Q(x)$ of degree t and $s = P(0)$ is an integer.

Assume that $Q(\sqrt[3]{a^2}+\sqrt[3]{a}) = d+e\sqrt[3]{a}+f\sqrt[3]{a^2}$, then by induction hypothesis $a-1 \mid e-f$. This gives

$$\begin{aligned} P(\sqrt[3]{a^2}+\sqrt[3]{a}) &= (\sqrt[3]{a^2}+\sqrt[3]{a})(d+e\sqrt[3]{a}+f\sqrt[3]{a^2})+s \\ &= s+ea+fa+(d+fa)\sqrt[3]{a}+(d+e)\sqrt[3]{a^2}. \end{aligned}$$

Hence, $(d+fa)-(d+e) = fa-e = (fa-f)+(f-e)$, which is a multiple of $a-1$.

Now we turn to the problem.

For part a), we show that $P(x)$ does exist, for example we chose $P(x) = ux^2+vx+w$, where u,v,w must satisfy

$$u+v = 220, 5u+v = 284, 10u+w = 0.$$

That is, $P(x) = 16x^2+204x-160$.

For the part b), since $1210-1184 = 26$, which is not a multiple of 4, in this case there is no $P(x)$ satisfying the condition of the problem. □

Problem 4. On a chessboard 5×9 squares, the following game is played. Initially, a number of frogs are randomly placed on some of the squares, no square containing more than one frog. A turn consists of moving all of the frogs subject to the following rules:

- Each frog may be moved one square up, down, left, or right;
- If a frog moves up or down on one turn, it must move left or right on the next turn, and vice versa;
- At the end of each turn, no square can contain two or more frogs.

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The game stops if it becomes impossible to complete another turn. Prove that if initially 33 frogs are placed on the board, the game must eventually stop. Prove also that it is possible to place 32 frogs on the board so that the game can continue forever.

Solution.

If 32 frogs are placed in an 4×8 rectangle, they can all move down, right, up, left, down, etc.

∘	∘	∘	∘	∘	∘	∘	∘	
∘	∘	∘	∘	∘	∘	∘	∘	
∘	∘	∘	∘	∘	∘	∘	∘	
∘	∘	∘	∘	∘	∘	∘	∘	

To show that a game with 33 frogs must stop, label the board as shown:

1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1
2	3	2	3	2	3	2	3	2
1	2	1	2	1	2	1	2	1

Note that a frog on 1 goes to a 3 after two moves, a frog on 2 goes to 1 or 3 immediately, and a frog on 3 goes to a 2 immediately. Thus if k frogs start on 1 and $k > 8$, the game stops because there are not enough 3s to accommodate these frogs. Thus we assume $k \leq 8$, in which case there are at most 16 squares on 1 or 3 at the start, and so at least 17 on 2.

Of these 17, at most 8 can move onto 3 after one move, so at least 9 end up on 1; these frogs will not all be able to move onto 3 two moves later, so the game will stop.

□

2.3.4. Level 4+, day 1

Problem 1. Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of real coefficients. Suppose that $P(x)$ has n real roots (not necessarily distinct), and there exists a positive integer k such that $a_k = a_{k-1} = 0$.

Prove that $P(x)$ has a real root of multiplicity $k+1$.
(**Note**: *we call a real number x_0 a root of multiplicity s of a polynomial $R(x)$ of real coefficients if there exists a polynomial $Q(x)$ such that $R(x) = (x-x_0)^s Q(x)$ and $Q(x_0) \neq 0$.*)

Solution.
We will show that $a_k = a_{k-1} = a_{k-2} = \cdots = a_0 = 0$ by induction on n the degree of $P(x)$.

In fact, we may assume that the leading coefficient of $P(x)$ is 1.For $n = 1, 2$, the result follows immediately.

Assume that the induction hypothesis is true for every $n < m$, we shall prove it is also true for $n = m$. Denote by $P_m(x) = x^m + \cdots + a_1 x + a_0$, and for some $k < m$, $a_k = a_{k-1} = 0$.

By taking derivative of $P_m(x)$, we obtain $P'_m(x) = b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, for some real numbers $b_{m-1}, \ldots, b_0$.

Since $a_k = a_{k-1} = 0$, we conclude that $b_{k-1} = b_{k-2} = 0$.

This together with $P_m(x)$ has only real roots, implies that $P'_m(x)$ also has only real roots.

Hence, by induction hypothesis, we get $b_{k-3} = \cdots = b_0 = 0$. In other words, $a_{k-2} = \cdots = a_1 = 0$. It remains to show that $a_0 = 0$. Assume that $a_0 \neq 0$, then if $r_1, \ldots, r_m$ are the roots of $P_m(x)$, then by Vieta's theorem,

$$r_1 \cdots r_m = (-1)^m a_0, r_1 + r_2 + \cdots + r_m = 0$$

and

$$\sum_{i,j} \frac{1}{r_i r_j} = 0 \Rightarrow \sum_i \frac{1}{r_i^2} = 0,$$

a contradiction. Therefore, $a_0 = 0$, the induction process is completed.

Obviously from that, we get 0 is the root of $P(x)$ with multiplicity at least $k+1$. □

Problem 2. Let ABC be a triangle with $AB \neq AC$. The incirle of triangle ABC is tangent to BC, CA, AB at D, E, F, respectively. The perpendicular line from D to EF intersects AB at X. The second intersection point of circumcircles of triangles AEF and ABC is T. Prove that $TX \perp TF$.

Solution.

Let G be the intersection point of DX and EF.

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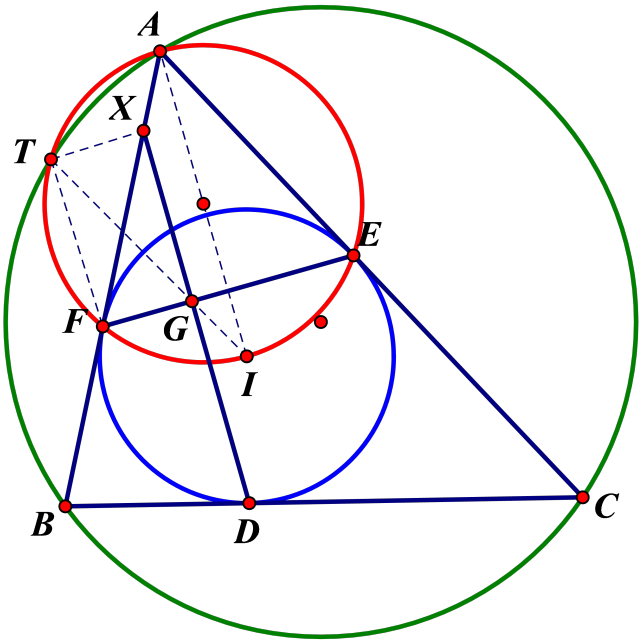

Consider the inversion $I_I^{ID^2}$, then the circumcircle of $\triangle ABC$ is sent to the nine-point circle of $\triangle DEF$ and the line EF is sent to the circumcircle of $\triangle AEF$. Hence I, T, and G are collinear.

Then $AI \parallel GX$ since they are both perpendicular to EF. We have

$$\angle GTF = \angle ITF = \angle IFG = \angle IAE = \angle FAI = \angle FXG$$

so, quadrilateral $TXGF$ is cyclic, this implies that

$$\angle XTF = \angle XGF = 90^\circ.$$

□

Problem 3. For any positive integer n, show that there exists a positive integer m such that n divides $2016^m + m$.

Solution.
We generalize the problem to the following problem:

Let a be a given positive integer. For every natural n, there is a positive integer m such that n divide $a^m + m$.

In fact, we proceed by induction on n. Obviously this statement holds for $n = 1$. Now assume $n > 1$ and this statement holds for every natural number less than n. Consider two cases:

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Case 1: $n = p$ is a prime. If $p \mid a$ we are done. If not, take $m = (a+1)(p-1)+1$. Then

$$a^m + m = a^{(a+1)(p-1)+1} + (a+1)(p-1) + 1 \equiv (a+1)p \equiv 0 \pmod{p}.$$

Case 2: n is a composite. Let p be the largest prime divisor of n. Write $n = p^b q_1^{a_1} \cdots q_k^{a_k}$ as the prime factorization of n and put $n = pn_1$.

By induction hypothesis (to $n_1 < n$), there is a positive integer k such that $a^k + k \equiv 0 \pmod{n_1}$. This leads to $a^k + k = n_1 q$, and we represent $q = pq_1 + r$ with $0 \leqslant r < p$. Thus, $a^k + k = n_1(pq_1 + r) \equiv rn_1 \pmod{n}$. Now, put $A = (q_1 - 1)\cdots(q_k - 1)$, or 1 if n has only one prime divisor.

Since p is the largest prime divisor of n, it follows that A and p are coprime. Hence, there is a positive integer c such that

$$Ac \equiv 1 \pmod{p} \Longleftrightarrow (p-1)rAc \equiv (p-1)r \equiv -r \pmod{p}.$$

This leads to

$$p^{b-1} q_1^{a_1} \cdots q_k^{a_k} r(p-1)Ac \equiv -p^{b-1} q_1^{a_1} \cdots q_k^{a_k} r \pmod{p^b q_1^{a_1} \cdots q_k^{a_k}}.$$

Hence, we have

$$q_1 \cdots q_k \varphi(n) rc \equiv -n_1 r \pmod{n}.$$

Finally, define $d = q_1 \cdots q_k c$ or c, if n has one prime divisor. We get $r\varphi(n)d \equiv -n_1 r \pmod{n}$. Put $m = r\varphi(n)d + k$. Then,

$$\begin{aligned} a^m + m &\equiv a^{r\varphi(n)d+k} + r\varphi(n)d + k \equiv a^k + k - n_1 r \\ &\equiv n_1 r + r\varphi(n)d \equiv n_1 r - n_1 r \equiv 0 \pmod{n}. \end{aligned}$$

□

Problem 4. There are 64 towns in a country, and some pairs of towns are connected by roads but we do not know these pairs. We may choose any pair of towns and find out whether they are connected by a road. Our aim is to determine whether it is possible to travel between any two towns using roads. Prove that there is no algorithm which would enable us to do this in less than 2016 questions.

Solution.

In general, we can replace 64 and 2016 by n and $\dfrac{n(n-1)}{2}$.

The given problem can be converted into the graph theory by considering each town as a vertex in the graph G and each road connects two towns as the edge. So there are exactly $\frac{n(n-1)}{2}$ undirected edges in this graph.

We need to prove that in all cases, we need to do the operation of checking connection between two vertices exactly $\frac{n(n-1)}{2}$ times to make sure the connectivity of each pair of vertices.

This can be done by using induction on number n or using the candy distribution idea. □

Remark. This problem is inspired by the Problem 3 of the International Olympiad in Informatics (IOI) 2014.

2.3.5. Level 4+, day 2

Problem 1. Given two non-constant polynomials $P(x), Q(x)$ with real coefficients. For a real number a, we define

$$P_a = \{z \in \mathbb{C} : P(z) = a\}; \quad Q_a = \{z \in \mathbb{C} : Q(z) = a\}.$$

Denote by K the set of real numbers a such that $P_a = Q_a$. Suppose that the set K contains at least two elements, prove that $P(x) = Q(x)$.

Solution.
First we see that if the polynomial $P(x) - a$ has a root α with multiplicity k, then $P'(x)$ also has the root α with multiplicity $k-1$.

Assume that a, b are two distinct element of K and $r_1, r_2, \ldots, r_i$ are the roots of $P(x) - a$ with multiplicity $k_1, k_2, \ldots, k_i$, respectively.

Then $r_1, r_2, \ldots, r_i$ are also the roots of $Q(x) - a$.

Let $t_1, t_2, \ldots, t_j$ be the roots of $P(x) - b$ with multiplicity $s_1, s_2, \ldots, s_j$, respectively.

Then $t_1, t_2, \ldots, t_j$ are also the roots of $Q(x) - b$.

Assume that $\deg P(x) \geq \deg Q(x)$, and $R(x) = P(x) - Q(x)$ is not identical zero. Thus, $\deg P(x) = k_1 + \cdots + k_i = s_1 + \cdots + s_j \geq \deg R(x) \geq i + j$, then we can get

$$\begin{aligned}\deg P'(x) &\geq (k_1 - 1) + \cdots + (k_i - 1) + (s_1 - 1) + \cdots (s_j - 1) \\ &= (k_1 + \cdots + k_i) + (s_1 + \cdots + s_j) - (i + j) \geq \deg P(x),\end{aligned}$$

a contradiction. □

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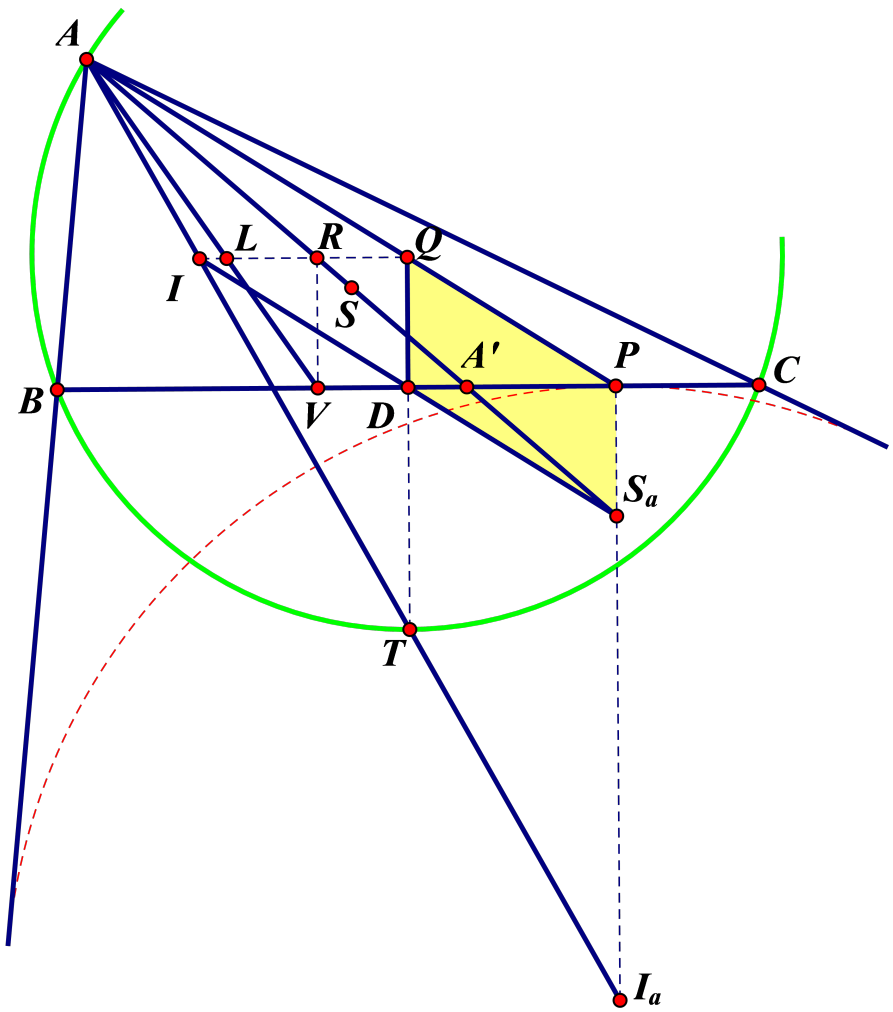
Problem 2. Let I_a be the excenter of triangle ABC with respect to A. The line AI_a intersects the circumcircle of triangle ABC at T. Let X be a point on segment TI_a such that $XI_a^2 = XA.XT$. The perpendicular line from X to BC intersects BC at A'. Define B' and C' in the same way. Prove that AA', BB' and CC' are concurrent.

Solution.

Let P be the foot of perpendicular from I_a to BC, D the midpoint of BC and S_a the reflection of A with respect to S. The line through I parallel to BC cuts AP, AA', AV at Q, R, L, respectively.

Redefine A' as follow: Let I be the incenter and S the Spieker point of triangle ABC. Assume that $A' \equiv AS \cap BC$ and V is the reflection of P with respect to A'. Then we need to prove that A' is also satisfy the given construction.



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Since D is clearly the midpoint of IS_a, then A' is the midpoint of RS_a, so $PS_a \parallel RV$ and $PS_a \perp BC$ and since $DI \parallel AP$, $PQDS_a$ is a parallelogram. This implies that $DQ \parallel PS_a$ and $DQ \perp BC$.

As a result $Q \in DT$ and VR is perpendicular bisector of LQ then

$$V(L,Q,R,P) = -1.$$

If RV cuts AI_a, AP at X_a, P_a, we also get

$$V(L,Q,R,P) = (A,Q,P_a,P) = (A,T,X_a,I_a) = -1$$

Then X is midpoint of I_aX_a which implies that

$$XA' \parallel I_aP \parallel X_aV \text{ and } XA' \perp BC.$$

It's similar to B' and C'. So AA', BC', CC' are concurrent at the Spieker point S of $\triangle ABC$. □

Problem 3. Let d be a positive integer. Show that for every integer S, there exist a positive integer n and a sequence $a_1, \ldots, a_n \in \{-1, 1\}$ such that

$$S = a_1(1+d)^2 + a_2(1+2d)^2 + \cdots + a_n(1+nd)^2.$$

Solution.
Let $U_k = (1+kd)^2$. We calculate $U_{k+3} - U_{k+2} - U_{k+1} + U_k$. This turns out to be $4d^2$, a constant. Changing signs, we obtain the sum $-4d^2$.

Thus if we have found an expression for a certain number S_0 as a sum of the desired type, we can obtain an expression of the desired type for $S_0 + (4d^2)q$, for any integer q. It remains to show that for any S, there exists an integer S' such that $S' \equiv S \pmod{4d^2}$ and S' can be expressed in the desired form.

Look at the sum $(1+d)^2 + (1+2d)^2 + \cdots + (1+Nd)^2$, where N is large. We can choose N so that the sum is odd, or that the sum is even.

By changing the sign in front of $(1+kd)^2$ to a minus sign, we decrease the sum by $2(1+kd)^2$. In particular, if $k \equiv 0 \pmod{2d}$, we decrease the sum by $2 \pmod{4d^2}$. So if N is large enough, there are many $k < N$ such that k is a multiple of $2d$.

By switching the sign in front of r of these, we change the congruence class modulo $4d^2$ by $2r$.

By choosing N so that the original sum is odd, and choosing suitable $r < 2d^2$, we can obtain numbers congruent to all odd numbers modulo $4d^2$.

By choosing N so that the original sum is even, we can obtain the numbers congruent to all even numbers modulo $4d^2$. This completes the proof. □

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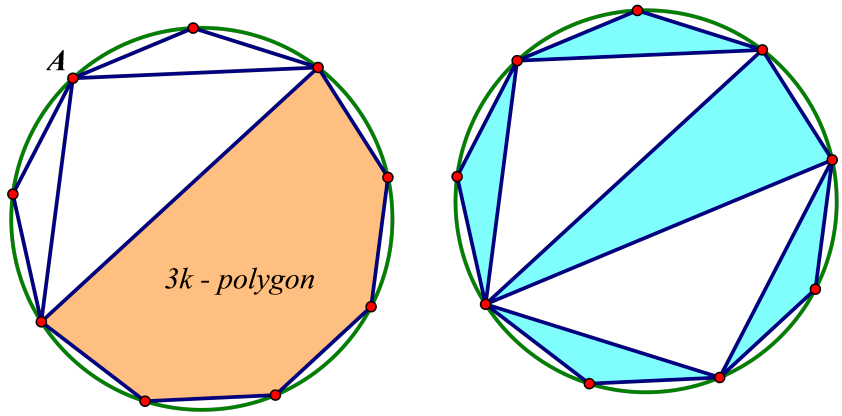
Problem 4. Determine all positive integers $n \geq 3$ such that we can divide a convex n-polygon into triangles by using some diagonals of this polygon such that the number of the used diagonals of every vertex is even.

Solution.
The answer is $3|n$.

1) We show that if $n = 3k$ then we can divide every convex n-polygon (without overlapping) into triangles by using some diagonals of this polygon such that the number of the used diagonals of every vertex is an even integer. The proof is by induction in k. The figure shows how to reduce from $3(k+1)$-polygon to $3k$-polygon.



2) We show that if a convex n-polygon can divide (non-overlap) a convex n-polygon into triangles by using some diagonals of this polygon such that the number of the used diagonals of every vertex is an even integer, then $3|n$.
First note that the number of triangles is exact $n-3$ using the sum of the angles (this sum is $\pi \times (n-2)$ in any convex n-polygon). Thus, the number of used diagonals is $n-3$ (by adding all the numbers of sides of the $n-2$ triangles we get $3(n-2) = n+2\times$ number of diagonals). Now, we can color the triangles with white and black in such a way that: One side of the n-polygon is black and no adjacent (having one side in common) triangles have the same color.

Since the diagonals of each vertex is even, we easily see that all sides of the n-polygon are black. Let x denote the number of the black triangles, the sum of the sides of all black triangles is

$$3x = n + (n-3) \Rightarrow 3|n.$$

□

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2.3.6. Level 4+, day 3

Problem 1. Let p, q be given primes and the sequence $(p_n)_{n\geq 1}$ defined recursively as follows: $p_1 = p; p_2 = q$ and p_{n+2} is the largest prime divisor of the number $(p_n + p_{n+1} + 2016)$ for all $n \geq 1$. Prove that this sequence is bounded, that is there exists a positive real number M such that $a_n < M$ for all positive integers n.

Solution.

Put $k = 2016$ and $a_n = \max\{p_n, p_{n+1}\}$ for each $n \geqslant 1$.

We shall show that $a_{n+1} \leqslant a_n + k + 2$. In fact, since $p_{n+1} \leqslant a_n$, it remains to show that $p_{n+2} \leqslant a_n + k + 2$.

If $p_{n+2} = 2$ then the inequality is trivially true. It suffices to consider the case $p_{n+2} > 2$.

If either $p_n = 2$ or $p_{n+1} = 2$ then $p_{n+2} \leqslant p_n + p_{n+1} + k = a_n + k + 2$. Otherwise, both primes p_n and p_{n+1} are odd, in particular $p_n + p_{n+1} + k$ is even. But $p_{n+2} > 2$, we get

$$p_{n+2} \leqslant \frac{p_n + p_{n+1} + k}{2} < a_n + k + 2.$$

It is clear that there is a positive integer s such that $a_1 \leqslant s(k+3)! + 1$. We show, by induction that $a_n \leqslant s(k+3)! + 1$. Indeed, suppose this is true for n, then

$$a_{n+1} \leqslant a_n + k + 2 \leqslant s(k+3)! + k + 3.$$

If $a_{n+1} > s(k+3)! + 1$, then $1 < m \leqslant k + 3$ with $m = a_{n+1} - q(k+3)!$. Hence, m must be a divisor of $(k+3)!$, that is m is a proper divisor of $s(k+3)! + m = a_{n+1}$. However, this cannot happen since a_{n+1} is a prime.

Thus, $a_n \leqslant s(k+3)! + 1$, and this completes the solution. □

Problem 2. Let I be the incenter of an acute triangle ABC. Assume that K_1 is the point such that $AK_1 \perp BC$ and the circle with center K_1 of radius K_1A is internally tangent to the incircle of triangle ABC at A_1. The points B_1, C_1 are defined similarly.

1. Prove that AA_1, BB_1, CC_1 are concurrent at a point P.
2. Let $\omega_1, \omega_2, \omega_3$ be the excircles of triangle ABC with respect to A, B, C, respectively. The circles $\gamma_1, \gamma_2, \gamma_3$ are the reflections of $\omega_1, \omega_2, \omega_3$ with respect to the midpoints of BC, CA, AB, respectively. Prove that P is the radical center of $\gamma_1, \gamma_2, \gamma_3$.

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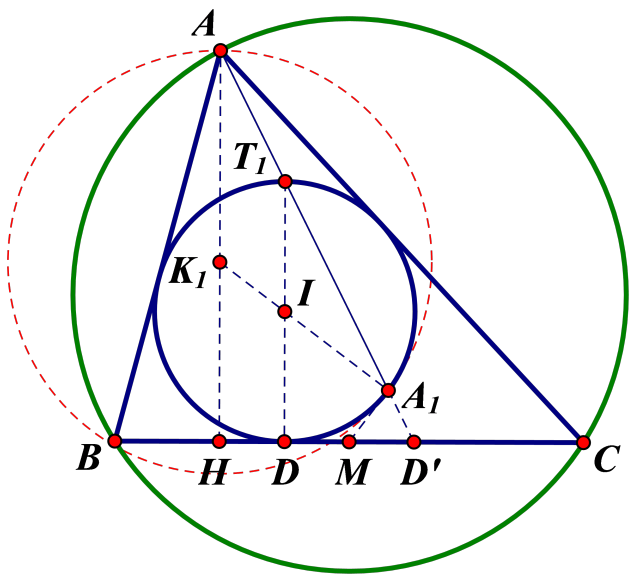
Solution.

1) Let ω be the incirle and D, E, F the tangent points of ω with BC, CA, AB, respectively.

The inversion with center A and radius $AE = AF$ sent ω to itself. The circle with center $K_1$is sent to a line parallel to BC and tangent to ω at T_1, so A_1 is sent to T_1.



Hence $AA_1 \equiv AT_1$ is the A-Nagel ray of $\triangle ABC \Rightarrow AA_1, BB_1, CC_1$ are concurrent at the Nagel point P of $\triangle ABC$.

2) Let I_1, I_2, I_3 be the excenter of $\triangle ABC$ with respect to A, B, C, respectively. The points J_1, J_2, J_3 are the reflections of I_1, I_2, I_3 with respect to the midpoints of BC, CA, AB,respectively. The points D', E', F' are the reflections of D, E, F with respect to the midpoints of BC, CA, AB, respectively.

The circles γ_3, γ_2 are tangent to AB, AC at F, E, respectively, so I_2AJ_2C and I_3AJ_3B are parallelograms $\Rightarrow BJ_3 \perp IA$ and $AJ_3 \perp BI \Rightarrow J_3$ is the orthocenter of $\triangle AIB$.

Similarly, J_2 is the orthocenter of $\triangle AIC \Rightarrow J_2, J_3$ are the poles of the $B-$midline and $C-$midline of $\triangle ABC$ relative to ω. Hence J_2J_3 is the polar of the midpoint M_1 of BC. Then the circle with diameter DD', which passes through A_1, is orthogonal to $\omega \Rightarrow$ the tangent of ω at A_1 passes through M_1, then $J_2J_3 \equiv DA_1 \Rightarrow AD' \perp J_2J_3$.

Since A has the same power $AF^2 = AE^2$ with respect to the circles γ_2, γ_3 then AD' is the radical axis of γ_2, γ_3. Because D' is the tangent point of the excirle with respect to A of $\triangle ABC$ with BC then $P \in AD'$.

Therefore P, which lies on the radical axes, is the radical center of $\gamma_1, \gamma_2, \gamma_3$. □

Problem 3. Find all integers n such that there exists a polynomial $P(x)$ with integer coefficients satisfying

$$P(\sqrt[3]{n^2} + \sqrt[3]{n}) = 2016n + 20\sqrt[3]{n^2} + 16\sqrt[3]{n}.$$

Solution.

First, we prove 2 lemmas as the problem in Level 4' s Test. Then turn to the problem, since $P(\sqrt[3]{n^2} + \sqrt[3]{n}) = 2016n + 20\sqrt[3]{n^2} + 16\sqrt[3]{n}$, then by the second lemma, we obtain $n - 1 \mid 20 - 16 = 4$.

That is, $n \in \{5, 3, 2, 0, -1, -3\}$, and we can easily check that these numbers satisfy the condition of the problem. □

Problem 4. Given six three-element subsets of the set X with at least 5 elements, show that it is possible to color the elements of X in two colors such that none of the given subsets is all in one color.

Solution.

Let $A_1, \ldots, A_6$ be the subsets. We induct on the number n of elements of X.

If $n = 5$, since $\binom{5}{3} = 10 > 6$, we can find a three-element subset Y of X not equal to any of $A_1, \ldots, A_6$; coloring the elements of Y in one color and the other elements.

If $n = 6$, since $\binom{6}{3} = 20 > 6 \times 2 = 12$, we can find a three-element subset Y of X not equal to any of $A_1, \ldots, A_6$ or its complements; coloring the elements of Y in one color and the other elements in the other color meets the desired condition.

Now suppose $n \geq 7$. There must be two elements $u; v$ of X such that $\{u; v\}$ is not a subset of any A_i, since there are at least $\binom{7}{2} = 21$ pairs, and at most

$$6 \times |\{(x, y) | x, y \in A_i, i = 1, \ldots, 6\}| = 6 \times 3 = 18$$

lie in an A_i. Replace all occurrences of u and v by a new element w, and color the resulting elements using the induction hypothesis.

Now color the original set by giving u and v the same color given to w. □

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2.4. Solution of TST for IMO 2016

2.4.1. Level 4, day 1

Problem 1. Let $n \geq 3$ and $x_1, x_2, \ldots, x_n$ be n distinct integers. Prove that

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + \ldots + (x_n - x_1)^2 \geq 4n - 6.$$

Solution.
We prove by induction on n.

When $n = 3$, we can assume that $x_1 > x_2 > x_3$ (only when $n = 3$) and the $LHS \geq 1^2 + 1^2 + 2^2 = 6$.

Moving from n to $n+1$, the change in LHS is

$$C_{n+1} = (x_n - x_{n+1})^2 + (x_{n+1} - x_1)^2 - (x_n - x_1)^2.$$

We need to show that $C_{n+1} \geq 4$. We can see that it is not true in general. However, due to the cyclicity, we can assume that x_{n+1} is the least number.

Moreover, since the LHS is unchanged under translation, we can assume also that $x_{n+1} = 0$. Thus, we have

$$C_{n+1} = x_n^2 + x_1^2 - (x_n - x_1)^2 = 2x_1x_n \geq 4.$$

□

Problem 2. Given a set of 2^{2016} cards with the numbers $1, 2, \ldots, 2^{2016}$ written on them. We divide the set of cards into pairs arbitrarily; from each pair, we keep the card with larger number and discard the other. We now again divide the 2^{2015} remaining cards into pairs arbitrarily; from each pair, we keep the card with smaller number and discard the other. We now have 2^{2014} cards, and again divide these cards into pairs and keep the larger one in each pair. We keep doing this way, alternating between keeping the larger number and keeping the smaller number in each pair, until we have just one card left. Find all possible values of this final card.

Solution.
Note that the remaining number is kept 1008 times as the larger one of the pair. So it is bigger than at least $2^{1008} - 1$ numbers.

Similarly, the remaining number is kept 1008 times as the smaller one of the pair so it is smaller than at least $2^{1008} - 1$ numbers.

Therefore, the remaining number x satisfies

$$2^{1008} \leq x \leq 2^{2016} - 2^{2018} + 1.$$

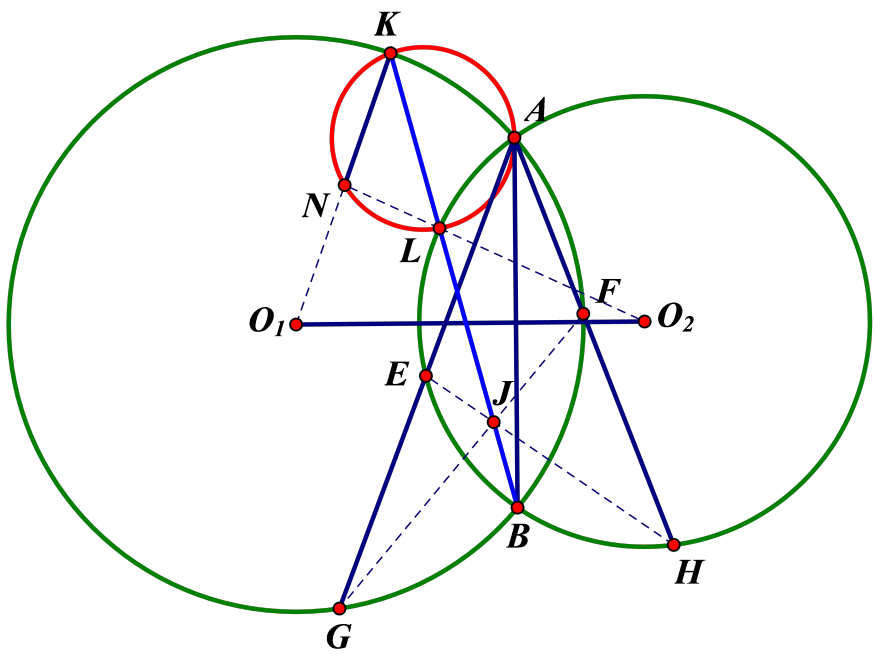
To prove that any x satisfies the above inequalities is true, we can carry out the pairing inductively so that after $2i$ steps, the following condition is satisfied: if the numbers remaining are

$$a_1 < a_2 < \ldots < a_{2^{2(n-i)}}$$

then x is one of these, and there are at least $2^{n-i} - 1$ numbers smaller than x and at least $2^{n-i} - 1$ numbers larger than x. □

Problem 3. Given two circles (O_1) and (O_2) intersect at A and B. Let d_1 and d_2 be two lines through A and be symmetric with respect to AB. d_1 cuts (O_1), (O_2) at G, E, respectively, d_2 cuts (O_1), (O_2) at F, H, respectively, such that E is between A and G, F is between A and H. EH cuts FG at J. BJ cuts (O_1), (O_2) at K, L, respectively, O_1K cuts O_2L at N. Prove that (NLK) is tangent to AB.

Solution.



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We have

$$\angle AKO_1 = 90^\circ - \frac{1}{2}\angle AO_1K = 90^\circ - \angle ABK = 90^\circ - \angle ABL$$
$$= 90^\circ - \frac{1}{2}\angle AO_2L = \angle ALO_2.$$

Therefore $ALNK$ is a cyclic quadrilateral.

Let O be the circumcenter of quadrilateral $ALNK$. Since B is the intersection of (AGF) and (AEH) then B is Miquel point of complete quadrilateral $AEJF.GH$ or $B \in (JFH)$.

We obtain

$$\angle BAO = \angle BAK + \angle KAO = \angle BAF + \angle FAK + 90^\circ - \angle ALK$$
$$= \frac{1}{2}(\overset{\frown}{BF} + \overset{\frown}{FK}) + 90^\circ - \angle AHB = \frac{1}{2}(\overset{\frown}{BG} + \overset{\frown}{FK}) - \angle GJB + 90^\circ = 90^\circ.$$

From this, AB is a tangent of (O). We are done. □

2.4.2. Level 4, day 2

Problem 1. Let k be a positive integer. Prove that there exist integers x, y, neither of which divisible by 7 such that $x^2 + 6y^2 = 7^k$.'

Solution.
Take $x_1 = y_1 = -1$, and the following relations

$$x_{k+1} = x_k - 6y_k, y_{k+1} = x_k + y_k.$$

for $k = 1, 2, 3, \ldots$

One can check that

$$x_{k+1}^2 + 6y_{k+1}^2 = 7(x_k^2 + 6y_k^2)$$

and

$$x_k \equiv y_k \equiv (-1)^k \pmod 7.$$

Hence, there exist the integer $x = x_k, y = y_k$ satisfy the given condition. □

Problem 2. Let ABC be a triangle inscribed in the circle (O) and P is a point inside the triangle ABC. Let D be a point on (O) such that $AD \perp AP$. The line CD cuts the perpendicular bisector of BC at M. The line AD cuts the line passing through B and is perpendicular to BP at Q. Let N be the reflection of Q through M. Prove that $CN \perp CP$.

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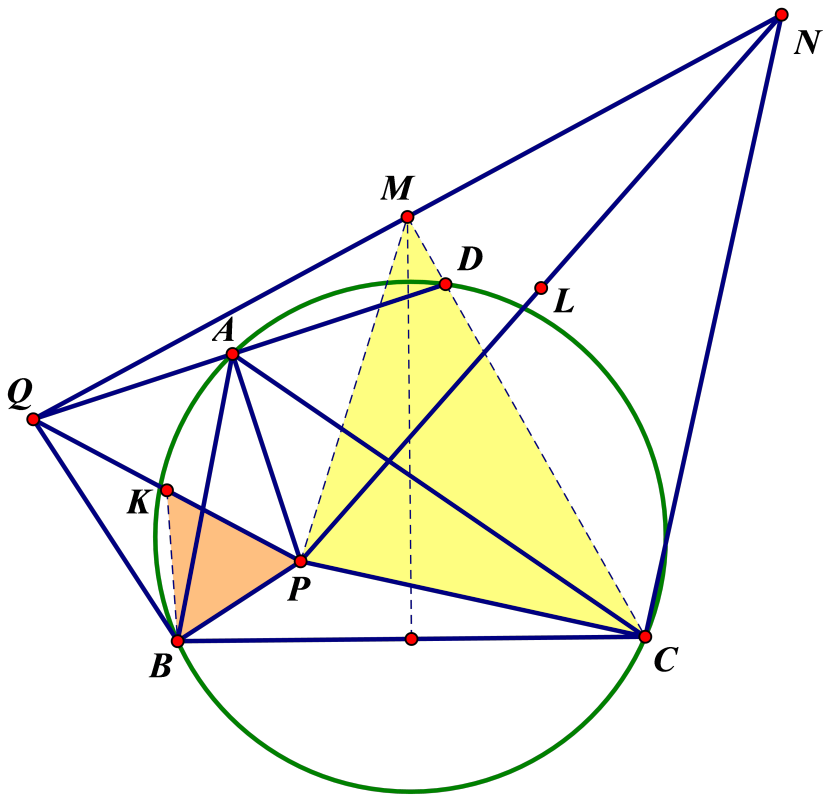
Solution.



Let K, L be the midpoints of PQ, PN.

Since $PBQA$ is cyclic

$$\angle KPB = \angle BAQ = \angle DCB.$$

Hence the isosceles triangles KPB and MCB are similar. The line BM cuts PQ at R. We have

$$\angle LMC = \angle LMB - \angle BMC = \angle MRQ - \angle BKP = \angle KBM.$$

Easily seen $KB = KP = ML$ and $BM = MC$.

Hence, $\triangle KBM = \triangle LMC$. We deduce $LC = KM = LP = LN$.

This, triangle PCN is right at C. We are done. □

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x\big[f(x+y) - f(x-y)\big] = 4yf(x)$$

for any real numbers x, y.

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Solution.

Denote $(*)$ as the given condition.

Let $a = f(0)$. Taking $x = y$ in $(*)$, we get

$$f(2x) = a + 4f(x)$$

for any $x \neq 0$. Now, by replacing x, y by $2x$, $2y$ in $(*)$ and using the above property (here we choose x, y such that $x+y$, $x-y$, $x \neq 0$), we get

$$2x\big[4f(x+y) - 4f(x-y)\big] = 8y\big[a + 4f(x)\big],$$

or

$$f(x+y) - f(x-y) = \frac{ay}{x} + \frac{4y}{x}f(x).$$

Combining this with $(*)$, we get $a = 0$. It follows that $f(0) = 0$ and $f(2x) = 4f(x)$ for any x.

Replacing $y = -x$ in $(*)$, we have

$$f(-2x) = 4f(x) = f(2x)$$

for any $x \neq 0$. Since $f(0) = 0$, we conclude that f is an even function.

Now, replacing x by x in (1), we get

$$x\big[f(-x-y) - f(y-x)\big] = 4yf(-x),$$

or

$$x\big[f(y+x) - f(y-x)\big] = 4yf(x).$$

It follows that

$$xy\big[f(y+x) - f(y-x)\big] = 4y^2 f(x).$$

From $(*)$, we get $y\big[f(y+x) - f(y-x)\big] = 4xf(y)$. Combine with the above equation, we get

$$x^2 f(y) = y^2 f(x)$$

for any real numbers x, y. This shows that

$$f(x) = ax^2,$$

where a is a constant, which is truly a solution. $\square$

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2.4.3. Level 4, day 3

Problem 1. Define the sequence $a_1, a_2, \ldots$ as follows: $a_1 = 1$, and for every $n \geq 2$, $a_n = n-2$ if $a_{n-1} = 0$ and $a_n = a_{n-1} - 1$, otherwise. Find the number of $1 \leq k \leq 2016$ such that there are non-negative integers r, s and a positive integer n satisfying $k = r+s$ and $a_{n+r} = a_n + s$.

Solution.
Let $N = n+r$ and $M = n$, then

$$r = N - M, s = a_N - a_M \text{ and } k = r+s = (a_N + N) - (a_M + M).$$

We need to find the number of possible values of $(a_N + N) - (a_M + M)$, where $N \geq M$ and $a_N \geq a_M$.

It is easy to see by induction that $a_{2^k} = 0$ and thus $a_{2^k+1} = 2^k - 1$ for all $k \geq 1$. The sequence is as the following

$$1, 0, 1, 0, 3, 2, 1, 0, 7, 6, 5, 4, 3, 2, 1, 0, \ldots$$

We divide the sequence into blocks with k^{th} block contains a_i for $2^{k-1} < i \leq 2^k$. Within each block, the value $a_n + n$ is constant, and for the kth block ($k \geq 1$) it equals 2^k.

Therefore, $d = (a_N + N) - (a_M + M)$ is the different of two powers of 2. Now it's not difficult to show that there are 51 such possible values. □

Problem 2. Find all pairs of polynomials $P(x), Q(x)$ with integer coefficients such that

$$P(Q(x)) = (x-1)(x-2)\ldots(x-9)$$

for all real numbers x.

Solution.
We consider 3 cases.

1) Suppose that $\deg Q = 1$. By looking at the leading cooeficient, we have $Q(x) = \pm x + a$. If $Q(x) = x+1$, we have

$$P(x) = (x-(a+1))(x-(a+2))\ldots(x-(a+9)).$$

And if $Q(x) = -x + a$ we have

$$P(x) = (a-1-x)(a-2-x)\ldots(a-9-x).$$

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2) Suppose that $\deg P = 1$. By looking at the leading cooeficient, we have $P(x) = \pm x + a$. In this case

$$Q(x) = \pm(x-1)(x-2)\dots(x-9) - a.$$

3) Suppose $\deg P, \deg Q > 1$. Then, the only option is $\deg P = \deg Q = 3$.

Since $1, 2, \dots 9$ are 9 roots of $P(Q(x))$, we have $P(x)$ have 3 roots a, b, c. For each root, for example a, the equation $Q(x) = a$ has 3 roots a_1, a_2, a_3 then $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ form a permutation of $1, 2, \dots, 9$.

By Viete's formula, we have

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 \text{ and } a_1a_2 + a_2a_3 + a_3a_1 = b_1b_2 + b_2b_3 + b_3b_1$$

so $a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2$. Since $x^2 \equiv 0$ or $1 \pmod 4$ so the number of odd numbers in a_1, a_2, a_3 is the same as the number of odd numbers in b_1, b_2, b_3.

This implies that the number of odd numbers in $1, 2, \dots, 9$ is divisible by 3, which is a contradiction.

□

Problem 3. Let $n \geq 4$ be a positive integer and there exist n positive integers that are arranged on a circle such that:

- The product of each pair of two non-adjacent numbers is divisible by $2015 \cdot 2016$.
- The product of each pair of two adjacent numbers is not divisible by $2015 \cdot 2016$.

Find the maximum value of n.

Solution.
Denote $v_p(k)$ as the exponent of prime p in prime factorization of positive integer n. It is easy to see that $v_p(ab) = v_p(a) + v_p(b)$ for all positive integers a, b.

We have $2015 = 5 \cdot 13 \cdot 31, 2016 = 2^5 \cdot 3^2 \cdot 7$.

Denote $M = 2015 \cdot 2016$ and

$$N = \{2, 3, 5, 7, 13, 31\}.$$

Suppose there exist n positive integers $x_1, x_2, x_3, \dots, x_n$ are arranged on the circle satisfying the given condition and denote $x_{n+1} \equiv x_1$ and $x_{n+2} \equiv$

x_2. Since $x_i x_{i+1}$ is not divisible by $M$with $i = \overline{1,n}$, then there exist a prime $p_i \in N$ such that

$$v_{p_i}(x_i x_{i+1}) < v_{p_i}(M) \Leftrightarrow v_{p_i}(x_i) + v_{p_i}(x_{i+1}) < v_{p_i}(M) \ (*)$$

These primes form a sequence $p_1, p_2, \ldots, p_n$ and the element can be repeated. We will prove that $p_i \neq p_j$ with i, j are two non-adjacent indices.

Indeed, if there are some non-adjacent indices i, j such that $p_i = p_j = p$. Denote $a = v_p(M)$ and from (*), we have $v_p(x_i) + v_p(x_{i+1}) < a$ and $v_p(x_j) + v_p(x_{j+1}) < a$.

Hence, we have

$$v_p(x_i) + v_p(x_{i+1}) + v_p(x_j) + v_p(x_{j+1}) < 2a.$$

In the other hand, since $(i, j), (i+1, j+1)$ are pair of non-adjacent indices then

$$v_p(x_i) + v_p(x_j) \geq a, v_p(x_{i+1}) + v_p(x_{j+1}) \geq a.$$

This implies that

$$v_p(x_i) + v_p(x_{i+1}) + v_p(x_j) + v_p(x_{j+1}) \geq 2a,$$

which is a contradiction.

Continue, suppose that there exist index i such that $p_i = p_{i+1} = p$. If $v_p(M) = 1$, because $x_i x_{i+1}, x_{i+1} x_{i+2}$ are not divisible by M, we can conclude that x_i, x_{i+1}, x_{i+2} are not divisible by p.Hence $x_i x_{i+2} \not| M$, contradiction because x_i, x_{i+2} are not adjacent.

So if $p_i = p_{i+1} = p$, then we must have $v_p(M) \geq 2$.

Now we can conclude that the sequence $p_1, p_2, \ldots, p_n$ has some properties as follow:

- Each prime $p_i \in N$ appears at most 2 times.
- If some prime $p_i \in N$ appears 2 times, then $p_i^2 | M$.

Hence, we have

$$p_1 p_2 \ldots p_n | M.$$

Since $M = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 31$, we can see that $n \leq 8$ (in the sequence (p_i), prime 2 appears 2 times, prime 3 appears 2 times and the rest appear 1 time). It is easy to check that if we choose 8 numbers as follows:

$$\frac{M}{2^5}, \frac{M}{2 \cdot 3}, \frac{M}{3^2}, \frac{M}{3 \cdot 5}, \frac{M}{5 \cdot 7}, \frac{M}{7 \cdot 13}, \frac{M}{13 \cdot 31}, \frac{M}{31 \cdot 2}$$

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and the correspondence sequence is

$$p_1 = 2, p_2 = 3, p_3 = 3, p_4 = 5, p_5 = 7, p_6 = 13, p_7 = 31, p_8 = 2.$$

Therefore, the maximum value of n is 8.

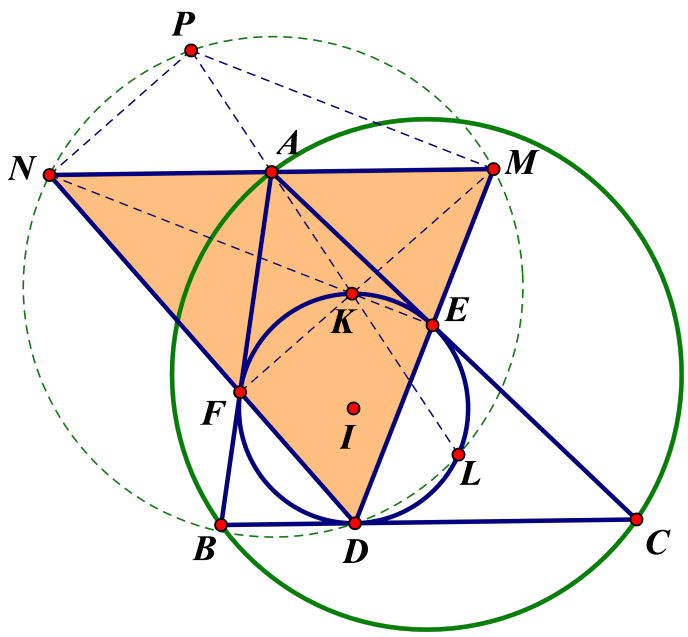
□

2.4.4. Level 4, day 4

Problem 1. Let ABC be a triangle whose incircle (I) touches BC, CA, AB at D, E, F, respectively. The line passing through A and parallel to BC cuts DE, DF at M, N, respectively. The circumcircle of triangle DMN cuts (I) again at L.

1. Let K be the intersection of NE and MF. Prove that K is the orthocenter of the triangle DMN.
2. Prove that A, K, L are collinear.

Solution.



1) Because $MN \parallel BC$ so $\angle ANF = \angle FDB = \angle DFB = \angle AFN$, this deduces $AN = AF$. Similarly, $AM = AE = AF = AN$.

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Thus, M, N, E, F lie on the circle center A. Let MF cuts NE at K, because E, F lie on circle diameter MN so

$$\angle KFD = \angle KED = 90^\circ,$$

this means DK is diameter of (AEF). So K lies on (I). Easily seen, K is orthocenter of triangle DMN.

2) Let P be the reflection of K through A then $KMPN$ is parallelogram, so $PM \parallel NK \perp MD$ and $PN \parallel KM \perp DN$. Thus, DP is diameter of (DMN). But DK is diameter of (I) so $\angle DLK = 90^\circ$, we deduce L lies on AK.

□

Problem 2. Let a be a positive integer. Find all prime numbers p with the following property: there exist exactly p ordered pairs of integers (x, y), with $0 \le x, y \le p-1$, such that p divides $y^2 - x^3 - a^2x$.

Solution.

The case $p = 2$ works with solutions $(0,0)$ and $(1,1)$ if a is even; $(0,0)$ and $(0,1)$ if a is odd.

We claim that any odd prime divisor p of a also works. Of course for $0 \le x, y \le p-1$, the congruence $y^2 \equiv x^3 \pmod p$ implies that $x = 0$ if and only if $y = 0$. For any $0 < x \le p-1$, it's clear that $x^3 = x^2 \cdot x$ is a quadratic residue modulo p if and only if x is.

Therefore, if x is not a quadratic residue then we can not find any y such that $y^2 \equiv x^3 \pmod p$ and if x is a quadratic residue the congruence $y^2 \equiv x^3 \pmod p$ has exactly 2 solutions in the set $\{1, 2, \ldots, p-1\}$ (the condition p is odd ensures that 2 solutions are distinct).

Since there are $(p-1)/2$ quadratic residues, the number of (x, y) with the required property is

$$1 + 2 \cdot (p-1)/2 = p.$$

If $p \equiv 3 \pmod 4$ then -1 is not a quadratic residue, so if $x^3 + a^2x \neq 0$, then exactly one of $x^3 + a^2x$ and $-x^3 - a^2x$ is a square and it gives two solutions. Together with $(0,0)$, it gives us exactly p solutions.

If $p \equiv 1 \pmod 4$, let i be a square root of $-1 \pmod p$, i.e. $i^2 = -1$. Then we have

$$y^2 \equiv x(x+ai)(x-ai) \pmod p.$$

For $x = 0, ai, -ai$, we have one choice of $y = 0$. For others choice of x, we have either 0 or 2 choices of y. Replacing x by $-x$, since -1 is a quadratic

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residue, we also have two choices for y. Hence, the total number of pairs is 3 (mod 4), which cannot be exactly p.

Thus, the answer is $p = 2$, all the odd prime divisors of a and $p \equiv 3 \pmod 4$.

□

Problem 3. Find the number of permutations $(a_1, a_2, \ldots, a_{2016})$ of the first 2016 positive integers satisfing the following two conditions:

1. $a_{i+1} - a_i \leq 1$ for all $i = 1, 2, 3, \ldots, 2015$.
2. There are exactly two indices $i < j$ with $1 \leq i < j \leq 2016$ such that $a_i = i$ and $a_j = j$.

Solution.
For each positive integer $n \geq 1$, we denote s_n as the number of permutations of the first n positive integer that satisfying the condition

$$a_{i+1} - a_i \leq 1, i = \overline{1, n-1}.$$

We call these permutation are nice.

First, we shall prove that $s_n = 2^{n-1}$ for all $n \geq 1$.

Let k be the index such that $a_k = n$. We have $a_{k-1} \geq a_k - 1 = n - 1$ which implies that $a_{k-1} = n - 1$, and so on, we have $a_1 = n - k + 1$.

All numbers after a_k are $1, 2, 3, \ldots, n - k$ form a nice permutation so in case $a_k = n$, the number of nice permutation is s_{n-k}.

Note that if $k = n$, we have only 1 nice permutation is $1, 2, 3, \ldots, n$.

Therefore, we get the following formula

$$s_n = 1 + \sum_{k=1}^{n-1} s_{n-k} = s_1 + s_2 + \ldots + s_{n-1} + 1.$$

It is easy to compute $s_1 = 1, s_2 = 2$, then by induction, we have $s_n = 2^{n-1}$ for all $n \geq 1$.

Back to the original problem, we suppose that $a_i = i$ and $a_j = j$. Thus,

$$a_{i+1} \leq a_i + 1 = i + 1, a_{i+2} \leq a_{i+1} + 1 \leq i + 2,$$

so on, then we have $a_j \leq j$. But in fact, $a_j = j$ which implies that the equality must occur.

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So $a_k = k$ for all $k = i, i+1, i+2, \ldots, j$. But there are only 2 indices i, j like that so we have $j = i+1$.

Continue, $a_{i-1} \geq a_i - 1 = i - 1$, but $a_i = i, a_{i+1} = i+1$ so $a_{i-1} \geq i+2$. Then,

$$a_{i-2} \geq a_{i-1} - 1 \geq i+1 \text{ or } a_{i-2} \geq i+2.$$

By the same way, it is clearly to check that $a_1, a_2, \ldots, a_{i-1} \geq i+2$. Similarly, we also have $a_{i+2}, a_{i+3}, \ldots, a_{2016} \leq i-1$.

Therefore, the length of two sequences $a_1, a_2, \ldots, a_{i-1}$ and $a_{i+2}, a_{i+3}, \ldots, a_n$ are the same. Hence,

$$i - 1 = 2015 - i \Leftrightarrow i = 1008.$$

So the given nice permutation looks like

$$\left(a_1, a_2, \ldots, a_{1007}, 1008, 1009, a_{1010}, a_{1011}, \ldots, a_{2016}\right).$$

Clearly, the subsequence $(a_{1010}, a_{1011}, \ldots, a_{2016})$ form a nice permutation of length 1007 and the same with the subsequences

$$(a_1 - 1009, a_2 - 1009, \ldots, a_{1007} - 1009).$$

Therefore, the number of permutation satisfy the given condition is

$$s_{1007} \cdot s_{1007} = \left(2^{1006}\right)^2 = 2^{2012}.$$

□

2.4.5. Level 4+, day 1

Problem 1. Call a positive integer $N \geq 2$ "special" if for every k such that $2 \leq k \leq N$, N can be expressed as a sum of k positive integers that are relatively prime to N (although not necessarily relatively prime to each other). Find all special positive integers.

Solution.
We claim that all odd numbers are special, and the only special even number is 2. For any even $N > 2$, the number relatively to N must be odd, and N cannot be expressed as a sum of 3 positive odd numbers.

Now, suppose that N is odd. We consider the binary decomposition of N

$$N = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_i}.$$

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Note that $(2^t, N) = 1$ and $i < \log_2 N + 1$.

For any $i \leq k < N - 1$, suppose that N can be expressed as a sum of k powers of 2, i.e. $N = 2^{x_1} + \ldots + 2^{x_k}$. Then at least one of $x_j > 0$ (otherwise, $k = N$). Suppose that $x_1 > 0$ then

$$N = 2^{x_1 - 1} + 2^{x_1 - 1} + 2^{x_2} + \ldots + 2^{x_k}.$$

So N can be expressed a sum of $k+1$ powers of 2. This implies that, for any $k < \log_2 N + 1$, N can be expressed as a sum of k positive integers that are relatively prime to N.

Now, start from $k = 2$. Let 2^a be the largest power of 2 such that $2^a < N$. Then $N = 2^a + (N - 2^a)$. Using the above argument, we can write 2^a as the sum of k powers of 2 for ay $1 < k < 2^a$. Since $2^a > N/2$, we have N can be expressed as a sum of k positive integers that are relatively prime to N for any $2 \leq k \leq N/2$.

Since $N/2 + 1 > \log_2 N$ for all $N \geq 3$, we conclude that all $N > 3$ odd are special. □

Problem 2. Given a set of 2^{2016} cards with the numbers $1, 2, \ldots, 2^{2016}$ written on them. We divide the set of cards into pairs arbitrarily; from each pair, we keep the card with larger number and discard the other. We now again divide the 2^{2015} remaining cards into pairs arbitrarily; from each pair, we keep the card with smaller number and discard the other. We now have 2^{2014} cards, and again divide these cards into pairs and keep the larger one in each pair. We keep doing this way, alternating between keeping the larger number and keeping the smaller number in each pair, until we have just one card left. Find all possible values of this final card.

Solution.
See the solution in the test of level 4. □

Problem 3. Given triangle ABC inscribed in circle (O). Two tangents at B, C of (O) intersects at P. The bisector of angle A intersects (P, PB) (the circle with center P and radius PB) at point E lying inside triangle ABC. Let M, N be the midpoints of two arcs BC of (O) such that M and A are on different sides of BC. The circle of diameter BC intersects the line EN at F. Prove that the orthocenter of triangle EFM is belong to BC.

Solution.

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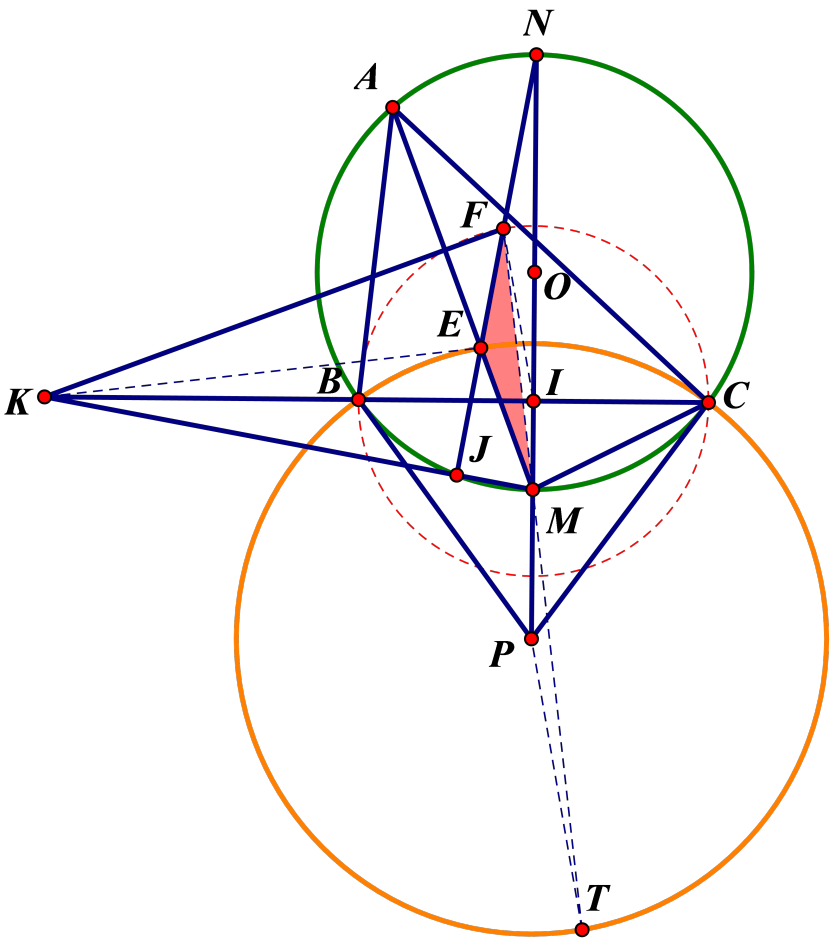

Let I be the midpoint of BC. We have $\angle ICM = \angle MAC = \angle MCP$ then CM is the bisector of $\angle ICP$. This follows that M is the insimilicenter of (I) and (P). However, $\angle MCN = 90^\circ$ hence N is the exsimilicenter of (I) and (P).

Let L, T be the intersections of FM and (P) (L is between F and M). Consider the homothety center at M, ratio $\dfrac{-PC}{IC}$, $\mathscr{H}_M^{\frac{-PC}{IC}} : F \mapsto T$, we get $IF \parallel PT$.

By the same way, consider the homothety $\mathscr{H}_N^{\frac{PC}{IC}} : F \mapsto E$, we get $IF \parallel PE$.

Therefore E, P, T are collinear or ET is the diameter of (P). This means $\angle ELP = 90^\circ$.

Denote J the second intersection of FN and (O).

Since 3 circles $(O), (P), (ELMJ)$ have EL, BC, JM as they radical axes then EL cuts JM at point K lying on BC.

In conclusion, orthocenter K of triangle EFM lies on BC. $\square$

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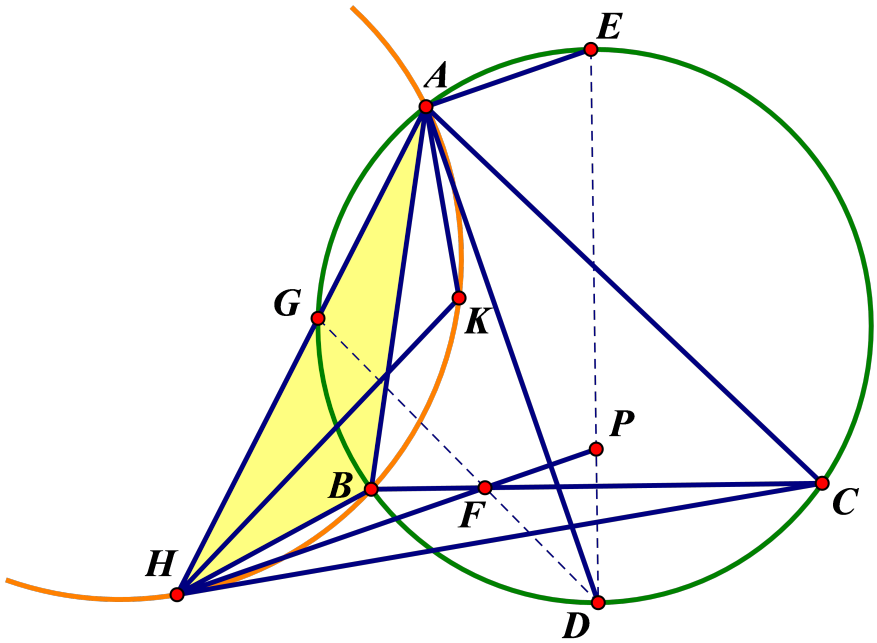
2.4.6. Level 4+, day 2

Problem 1. Let ABC be a triangle inscribed in (O). The bisector of $\angle BAC$ cuts (O) again at D. Let DE be the diameter of (O). Let G be a point on arc AB which does not contain C. The lines GD and BC intersect at F. Let H be a point on the line AG such that $FH \parallel AE$. Prove that the circumcircle of triangle HAB passes through the orthocenter of triangle HAC.

Solution.
Let HF cut DE at P. We have $\angle HGD = \angle E = \angle HPD$ so $HGPD$ is cyclic, we deduce $FH.FP = FG.FD = FB.FC$ so $HBPC$ is cyclic.



Easily seen DE is perpendicular bisector of BC so $PB = PC$. Hence HP is bisector of $\angle BHC$. From this,

$$\begin{aligned}\angle HBA + \angle HCA &= 180^\circ - \angle BHA - \angle BAH + 180^\circ - \angle CHA - \angle ACH \\ &= 360^\circ - (\angle AHB + \angle AHC) - (\angle HAB + \angle HAC) \\ &= 360^\circ - 2\angle AHF - 2\angle HAD \\ &= 360^\circ - 2\angle GDE - 2(\angle GAE - 90^\circ) = 180^\circ.\end{aligned}$$

Now let K be orthocenter of triangle HAC then

$$\angle AKH = 180^\circ - \angle ACH = \angle ABH$$

so K lies on (HAB). □

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Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

1. $f(x+1) \geq f(x)+1$ for all $x \in \mathbb{R}$; (1);
2. $f(xy) \geq f(x)f(y)$ for all $x, y \in \mathbb{R}$. (2)

Solution.
Let $x = y$ in (2), we have $f(x^2) \geq f^2(x) \geq 0$. This implies that $f(x) \geq 0$ for all $x \geq 0$. Let $x = 0$ in (1), we have $f(1) \geq f(0)+1 \geq 1$. Let $x = y = 1$ in (1), we have $f(1) \geq f^2(1)$, tức $0 \leq f(1) \leq 1$. These inequalities imply that $f(1) = 1$, and hence $f(0) = 0$. From (1) and (2), by induction, we have

$$f(x+n) \geq f(x)+n \ (*)$$

for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, and

$$f(x^n) \geq f^n(x) \ (**)$$

for all $x \geq 0$ and $n \in \mathbb{N}^*$. Using $(*)$, for each $x > 0$, we have

$$f(x) = f\big(\{x\}+\lfloor x \rfloor\big) \geq f\big(\{x\}\big)+\lfloor x \rfloor \geq \lfloor x \rfloor > x-1.$$

From $(**)$ and (2), for each $x > 1$ and $n \in \mathbb{N}^*$, we have

$$(x^n-1)f^n\left(\frac{1}{x}\right) \leq f(x^n)f\left(\frac{1}{x^n}\right) \leq f(1) = 1.$$

Hence,

$$f\left(\frac{1}{x}\right) \leq \frac{1}{\sqrt[n]{x^n-1}}$$

for all $x > 1$, $n \in \mathbb{N}^*$. Fix $x > 1$ and let $n \to +\infty$, we obtain $f\left(\frac{1}{x}\right) \leq \frac{1}{x}$, or (note that $f(0) = 0$ and $f(1) = 1$),

$$f(x) \leq x$$

for $x \in [0, 1]$. On the other hand, we can prove by induction from (1) that

$$f(x-n) \leq f(x)-n$$

for $n \in \mathbb{N}$. For each $x < 0$, we have $\lfloor x \rfloor < 0$ so

$$f(x) = f\big(\{x\}+\lfloor x \rfloor\big) \leq f\big(\{x\}\big)+\lfloor x \rfloor \leq \{x\}+\lfloor x \rfloor = x.$$

In (2), set $x = y = t$ with $t \in [-1, 0]$ then

$$f^2(t) \leq f(t^2) \leq t^2.$$

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This implies that $t \le f(t) \le -t$. Hence, $f(t) = t$ for all $t \in [-1, 0]$. Now, for each $x < -1$, we have

$$\frac{1}{x^2} f^2(x) = f^2(x) f^2\left(\frac{1}{x}\right) \le f(x^2) f\left(\frac{1}{x^2}\right) \le f(1) = 1$$

so $f^2(x) \le x^2$, implying that $x \le f(x) \le -x$. So $f(x) = x$ for all $x \le 0$. Let $y = -1$ in (2) and consider $x > 0$, we have

$$-x = f(-x) \ge f(x) f(-1) = -f(x),$$

implying $f(x) \ge x$ for all $x > 0$. Hence, we have $f(x) = x$ for all $x \in (0, 1)$. Finally, suppose that $x > 1$, we have

$$\frac{1}{x} f(x) = f(x) f\left(\frac{1}{x}\right) \le f(1) = 1$$

so $f(x) \le x$. Since $f(x) \ge x$, we have $f(x) = x$ for all $x > 1$. Inclusion, $f(x) = x$ for every x is the only function satisfying given conditions. □

Problem 3. Given two positive integers $r > s$, and let $\mathscr{F}$ be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r - 1$ that shares at least s elements with each set in $\mathscr{F}$.

Solution.

We call a set S s-meets $\mathscr{F}$ if it shares at least s elements with each set in $\mathscr{F}$. Suppose no such set of size (at most) $r-1$ exists. (Each $S \in \mathscr{F}$ s-meets $\mathscr{F}$ by the problem hypothesis.)

Let T be a maximal set such that $T \subseteq S$ for infinitely many $S \in \mathscr{F}$, which form $\mathscr{F}' \subseteq S$ (such T exists, since the empty set works).

Clearly $|T| < r$, so by assumption, T does not s-meet $\mathscr{F}$, and there exists $U \in \mathscr{F}$ with $|U \cap T| \le s - 1$.

But U s-meets $\mathscr{F}'$, so by pigeonhole, there must exist $u \in U \backslash T$ belonging to infinitely many $S \in \mathscr{F}'$, contradicting the maximality of T. □

Remark. Let X be an infinite set, and $a_1, a_2, \ldots, a_{2r-2-s}$ elements not in X. Then the set

$$\mathscr{F} = \{B \cup \{x\} : B \subseteq \{a_1, a_2, \ldots, a_{2r-2-s}\}, |B| = r - 1, x \in X\}$$

shows we cannot replace $r - 1$ with any smaller number.

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2.4.7. Level 4+, day 3

Problem 1. Define the sequence $a_1, a_2, \ldots$ as follows: $a_1 = 1$, and for every $n \geq 2$, $a_n = n - 2$ if $a_{n-1} = 0$ and $a_n = a_{n-1} - 1$, otherwise. Find the number of $1 \leq k \leq 2016$ such that there are non-negative integers r, s and a positive integer n satisfying $k = r + s$ and $a_{n+r} = a_n + s$.

Solution.
Let $N = n+r$ and $M = n$, then $r = N - M$ and $s = a_N - a_M$, and $k = r + s = (a_N + N) - (a_M + M)$. We need to find the number of possible values of $(a_N + N) - (a_M + M)$, where $N \geq M$ and $a_N \geq a_M$.
It is easy to see by induction that $a_{2^k} = 0$ and thus $a_{2^k+1} = 2^k - 1$ for all $k \geq 1$. The sequence is as the following

$$1, 0, 1, 0, 3, 2, 1, 0, 7, 6, 5, 4, 3, 2, 1, 0, \ldots$$

We divide the sequence into blocks with k^{th} block contains a_i for $2^{k-1} < i \leq 2^k$. Within each block, the value $a_n + n$ is constant, and for the kth block $(k \geq 1)$ it equals 2^k. Therefore, $d = (a_N + N) - (a_M + M)$ is the different of two powers of 2. Now it's not difficult to show that there are 51 such possible values. □

Problem 2. Let $n \geq 4$ be a positive integer and there exist n positive integers that are arranged on a circle such that:

- The product of each pair of two non-adjacent numbers is divisible by $2015 \cdot 2016$.
- The product of each pair of two adjacent numbers is not divisible by $2015 \cdot 2016$.

Find the maximum value of n.

Solution.
See the solution in the test of level 4. □

Problem 3. Let $P \in \mathbb{Q}[x]$ be a polynomial of degree 2016 whose leading coefficient is 1. A positive integer m is "nice" if there exists some positive integer n such that

$$m = n^3 + 3n + 1.$$

Suppose that there exist infinitely many positive integers n such that $P(n)$ are nice. Prove that there exists an arithmetic sequence (n_k) of arbitrary length such that $P(n_k)$ are all nice for $k = 1, 2, 3, \ldots$

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Solution.
For convenience, denote $3d = 2016$.

Lemma. There exist a polynomial $Q(x) \in \mathbb{Q}[x]$ such that

$$\lim_{x\to+\infty}\left[\sqrt[3]{P(x)}-Q(x)\right]=0.$$

Proof. Suppose that $S(x) \in \mathbb{Q}[x]$ is a polynomial such that $\deg\left(P(x)-S^3(x)\right)$ is minimize (if there are many $S(x)$ satisfy, we choose one of them).

It is easy to check that:

- $\deg\left(P(x)-S^3(x)\right) < \deg P(x) = 3d$.
- $S(x)$ has the leading coefficient is 1.
- $\deg\left(P(x)-S^3(x)\right) < 2d$.

For iii), denote $P(x)-S^3(x) = a_l x^l + a_{l-1}x^{l-1} + \cdots + a_0$ and suppose on contradiction that $l = \deg\left(P(x)-S^3(x)\right) \geq 2d$. Consider polynomial $S*(x) = S(x) + \frac{a_l}{3}x^{l-2d}$ then

$$P(x)-(S*(x))^3 =$$
$$\left(P(x)-S^3(x)-a_l x^{l-2d}S^2(x)\right)-\left(\frac{1}{3}a_l^2x^{2l-4d}S(x)+\frac{1}{27}a_l^3x^{3l-6d}\right).$$

The result is a polynomial has degree less than l. This means

$$\deg\left(P(x)-(S*(x))^3\right) < l = \deg\left(P(x)-S^3(x)\right),$$

which is a contradiction. Then, take $Q(x) = S(x)$ to get

$$\sqrt[3]{P(x)}-Q(x) = \frac{P(x)-Q^3(x)}{\sqrt[3]{P^2(x)}+\sqrt[3]{P(x)}Q(x)+Q^2(x)}.$$

From iii), we can see that

$$\deg\left(\sqrt[3]{P(x)}-Q(x)\right) < 2d = \deg\left(\sqrt[3]{P^2(x)}+\sqrt[3]{P(x)}Q(x)+Q^2(x)\right)$$

then $\lim\limits_{x\to+\infty}\left[\sqrt[3]{P(x)}-Q(x)\right]=0$. The lemma is proved.

Since there exist infinitely many number a such that $P(a)$ is nice, we can choose some strictly increasing sequence of positive integers (a_n) such that $P(a_i)$ are nice for all $i \in \mathbb{N}*$.

Clearly, for each a_i, there exist $N_i \in \mathbb{Z}^+$ such that $P(a_i) = N_i^3 + 3N_i + 1$.

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Since $\lim\limits_{x\to+\infty} a_i = +\infty$, which implies that

$$\lim_{x\to+\infty} P(a_i) = +\infty \text{ and } \lim_{x\to+\infty} N_i = +\infty,$$

then

$$\lim_{x\to+\infty} \left[\sqrt[3]{N_i^3 + 3N_i + 1} - N_i\right] = 0.$$

Now, let $Q(x)$ be the polynomial described in lemma.

Because $Q(x) \in \mathbb{Q}[x]$, there exist a number $M \in \mathbb{Z}^+$ such that $M \cdot Q(x) \in \mathbb{Z}[x]$. We have

$$\lim_{i\to+\infty} M\left(Q(a_i) - N_i\right) = \lim_{i\to+\infty} M\left(Q(a_i) - \sqrt[3]{P(a_i)} + \sqrt[3]{P(a_i)} - N_i\right) = 0.$$

Hence, there is a number $i_0 \in \mathbb{Z}^+$ such that

$$-1 < M(Q(a_i) - N_i) < 1, \forall i > i_0.$$

In the other hand, $MQ(a_i) \in \mathbb{Z}, N_i \in \mathbb{Z}$ so $M(Q(a_i) - N_i) = 0$ for all $i > i_0$, then $Q(a_i) - N_i = 0 \Leftrightarrow P(a_i) = Q^3(a_i) + 3Q(a_i) + 1$ for all $i > i_0$. This implies that $P(x) = Q^3(x) + 3Q(x) + 1$ for all $x \in \mathbb{R}$.

Choose number N such that $Q(x) > 0, \forall x > a_N$ then we will prove that the arithmetic sequence which is given by the formula $n_k = a_N + Mk$ satisfying the given requirement.

It is needed to clarify $Q(a_1 + Mk) \in \mathbb{Z}^+$ for all $k \in \mathbb{Z}^+$.

Indeed, let $R(x) = MQ(x)$ then $R(x) \in \mathbb{Z}[x]$, then $R(n_1) = Ma_1$ is divisible by M. So

$$(n_k - n_1) \,|R(n_k) - R(n_1)\,,$$

leads to $M\,|R(n_k)$ for all $k \in \mathbb{Z}^+$. This finishes the proof. □

2.4.8. Level 4+, day 4

Problem 1. On the Cartesian coordinate system Oxy, consider a sequence of points $A_n(x_n, y_n)$ in which $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ are two sequences of positive numbers satisfing the following conditions:

$$x_{n+1} = \sqrt{\frac{x_n^2 + x_{n+2}^2}{2}},\ y_{n+1} = \left(\frac{\sqrt{y_n} + \sqrt{y_{n+2}}}{2}\right)^2 \quad \forall n \geq 1.$$

Suppose that O, A_1, A_{2016} belong to a line d and A_1, A_{2016} are distinct. Prove that all the points $A_2, A_3, \ldots, A_{2015}$ lie on one side of d.

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Solution.
From O, A_1, A_{2016} are collinear, there exist some positive number k such that

$$\frac{x_1}{y_1} = \frac{x_{2016}}{y_{2016}} = k > 0.$$

We shall prove that for all $i = \overline{2, 2015}$, $\frac{x_i}{y_i} > k$. Indeed,

Notice that

$$x_{n+1} = \sqrt{\frac{x_n^2 + x_{n+2}^2}{2}} \Leftrightarrow x_{n+1}^2 = \frac{x_n^2 + x_{n+2}^2}{2},$$

which implies that (x_n^2) form an arithmetic sequence.

Hence, for all $i = \overline{2, 2015}$, let

$$\alpha_i = \frac{2016 - i}{2015}, \beta_i = \frac{i-1}{2015}$$

then $\alpha_i + \beta_i = 1, 2 \le i \le 2015$ and

$$x_i^2 = \alpha_i x_1^2 + \beta_i x_{2016}^2 \Leftrightarrow x_i = \sqrt{\alpha_i x_1^2 + \beta_i x_{2016}^2}.$$

Similarly, we also have

$$y_i = \left(\alpha_i \sqrt{y_1} + \beta_i \sqrt{y_{2016}}\right)^2 \text{ and } k y_i = \left(\alpha_i \sqrt{x_1} + \beta_i \sqrt{x_{2016}}\right)^2.$$

We need to prove

$$\sqrt{\alpha_i x_1^2 + \beta_i x_{2016}^2} > \left(\alpha_i \sqrt{x_1} + \beta_i \sqrt{x_{2016}}\right)^2$$

for all $2 \le i \le 2015$ $(*)$.

By applying the Cauchy-Schwarz's inequality, we have

$$\begin{aligned}(\alpha_i + \beta_i)\left(\alpha_i x_1^2 + \beta_i x_{2016}^2\right) &\ge (\alpha_i x_1 + \beta_i x_{2016})^2 \\ &= ((\alpha_i + \beta_i)(\alpha_i x_1 + \beta_i x_{2016}))^2 \\ &\ge \left(\alpha_i \sqrt{x_1} + \beta_i \sqrt{x_{2016}}\right)^4.\end{aligned}$$

So the inequality $(*)$ are true for each $2 \le i \le 2015$. Since all points are distinct, then $A_1 \ne A_{2016} \Leftrightarrow x_1 \ne x_{2016}$ then equality does not occur in $(*)$.

So all the point $A_2, A_3, \ldots, A_{2015}$ belong to the same side respect to d. □

Problem 2. Let $ABCDEF$ be a convex hexagon with $AB = CD = EF$, $BC = DE = FA$ and $\angle A + \angle B = \angle C + \angle D = \angle E + \angle F$. Prove that all angles of this hexagon are equal.

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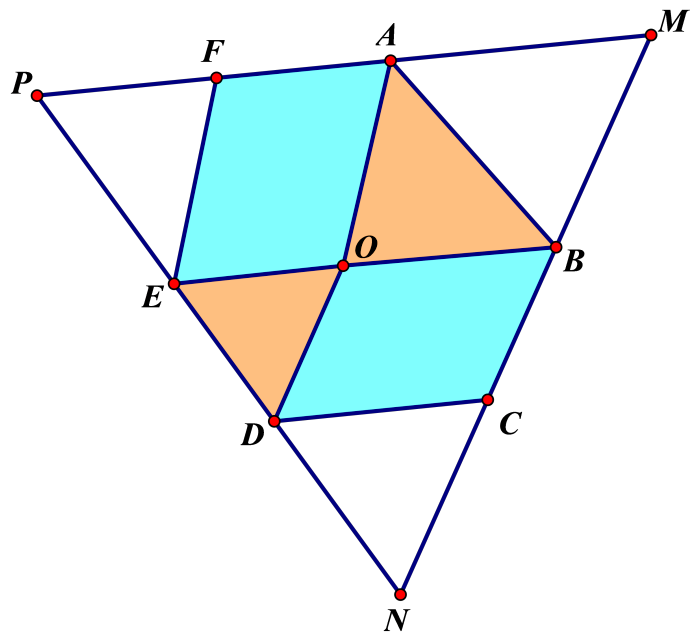
Solution.
Let BC, DE, AF intersect bound triangle MNP. Because sum of angle in hexagon is $720°$ so $\angle A + \angle B = \angle C + \angle D = \angle E + \angle F = 240°$.



Therefore, we easily seen triangle MNP is equilateral. To build the equilateral triangle DEO with O lie inside hexagon. We see $EOAF$ and $DOBC$ are the parallelogram so $AB = EF = OA = CD = OB$, thus OAB is an equilateral triangle. Hence,

$$\angle AFE + \angle BCD = \angle AOE + \angle BOD = 360° - 60° - 60° = 240°.$$

But $\angle EDC + \angle BCD = 240°$. These imply $\angle AFE = \angle EDC$.

Similarly, $\angle EDC = \angle CBA$. Similarly, $\angle BAF = \angle FED = \angle EDC$.

Now from the condition $\angle A + \angle B = \angle C + \angle D = \angle E + \angle F$ we deduce that, all angles of this hexagon are equal $120°$. □

Problem 3. Find the number of permutations $(a_1, a_2, \ldots, a_{2016})$ of the first 2016 positive integers satisfing the following two conditions:

1. $a_{i+1} - a_i \leq 1$ for all $i = 1, 2, 3, \ldots, 2015$.
2. There are exactly two indices $i < j$ with $1 \leq i < j \leq 2016$ such that $a_i = i$ and $a_j = j$.

Solution.
See the solution in the test of level 4. □

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موهبة

مسابقات الرياضيات للمملكة العربية السعودية

SAUDI ARABIAN
Mathematical Competitions
2017

SAMC 2017

Riyadh, June 2017

Saudi Arabian Mathematical Competitions 2017

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SAUDI ARABIAN

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SAMC 2017

Riyadh, June 2017

Saudi Arabian Mathematical Competitions 2017

جامعة الملك عبدالله
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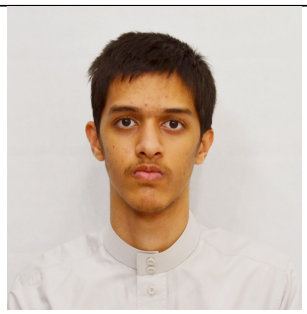
The Saudi Arabian team at **IMO 2017**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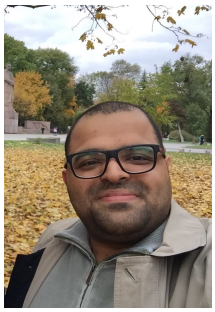			
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مقدمة

يحوي هذا الكتيب على مسائل التصفيات لمسابقة البلقان و مسابقة البلقان للناشئين و تصفيات الاولمبياد الدولي للرياضيات ٢٠١٧.

ان تدريب الفريق كان بدعم من وزارة التعليم بالتعاون مع مؤسسة " الملك عبد العزيز و رجاله للموهبة و الابداع " موهبة

وتجدر الاشارة الى التعاون الفعّال من جامعة الملك عبدالله للعلوم و التقنية، حيث وفرت لنا كل الامكانات التي احتجنا لها في التدريب في حرمها الجامعي. كما نشكر الرعاة الرسميين لنا في دعم التدريب.

تم عقد ثلاثة ملتقيات تدريبية خلال العام الدراس ٢٠١٦-٢٠١٧ ، بالاضافة الى فترة التدريب المكثف التي بدأت في شهر ابريل ٢٠١٧ الى نهاية شهر يونيو. كما شارك الطلبة المتميزون في العديد من المسابقات الإقليمية و منها: اولمبياد الرياضيات لدول آسيا و الباسيفيك ، اولمبياد دول الخليج في الرياض، اولمبياد الطالبات للدول الأوربية في سويسرا ، اولمبياد البلقان في ماقدونيا و اولمبياد الناشئين لدول البلقان في بلغاريا.

نأمل ان يكون محتوى هذا الكتيب إسهاماً منا لتقوية اواصر التعاون و تبادل الخبرات بيننا و الدول المشاركة في الاولمبياد الدولي.

د. فوزي بن أحمد الذكير
رئيس الفريق السعودي للأولمبياد الدولي

Saudi Arabian Mathematical Competitions 2017

Introduction

This booklet contains the Team Selection Tests of the Saudi teams to the Balkan Mathematics Olympiad, Balkan Junior Mathematics Olympiad, and the International mathematics Olympiad. The training was supported by the Ministry of Education, which commissioned Mawhiba, the main establishment in Saudi Arabia that cares for the gifted students , to do the task.

We would like to express our gratitude to King Abdullah University of Science and Technology KAUST for making its facilities in its beautiful campus available to us for our training. Our thanks go to the sponsors whose logos appear in this booklet.

The Saudi team had three main training camps during the academic year 2016-2017. In addition, the team had an intensive training period in the period , April to the end of June 2017.

During this academic year, the selected students participated in the following contests: The Asia Pacific Mathematics Olympiad, the Gulf mathematics Olympiad in Riyad, the European Girls Mathematics Olympiad in Switzerland , Balkan Mathematics Olympiad in Macedonia and the Junior Balkan mathematics Olympiad in Bulgaria. It is our pleasure to share the selection tests problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Fawzi A Al-Thukair
Leader of the Saudi team in IMO 2017

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Part 1

PROBLEM STATEMENTS

Saudi Arabian Mathematical Competitions 2017

1.1. Team selection tests for BMO 2017

1.1.1. Day 1

Problem 1. Let $n = p_1p_2\dots p_{17}$ be the positive integer where $p_1, p_2, \dots, p_{17}$ are 2017 distinct odd primes. A triangle is called nice if it is a right triangle with integer side lengths and the inradius is n. Find the number of nice triangles (two triangles are consider different if their tuples of length of sides are different).

Problem 2. Let ABC be an acute triangle with AT, AS respectively are the internal, external angle bisector of ABC and $T, S \in BC$. On the circle with diameter TS, take an arbitrary point P that lies inside the triangle ABC. Denote D, E, F, I as the incenter of triangle PBC, PCA, PAB, ABC. Prove that four lines AD, BE, CF and IP are concurrent.

Problem 3. How many ways are there to insert plus signs + between the digits of number $111111\dots111$ which includes thirty of digits 1 so that the result will be a multiple of 30?

Problem 4. Fibonacci sequences is defined as

$$\begin{cases} f_1 = 1, f_2 = 2, \\ f_{n+1} = f_n + f_{n-1}, n \geq 2. \end{cases}$$

1. Prove that every positive integer can be represented as sum of several distinct Fibonacci number.

2. A positive integer is called **Fib-unique** if the way to represent it as sum of several distinct Fibonacci number is unique.
   Example: 13 is not Fib-unique because $13 = 13 = 8 + 5 = 8 + 3 + 2$.
   Find all Fib-unique.

1.1.2. Day 2

Problem 1. Find the smallest prime q such that

$$q = a_1^2 + b_1^2 = a_2^2 + 2b_2^2 = a_3^2 + 3b_3^2 = \dots = a_{10}^2 + 10b_{10}^2$$

where $a_i, b_i (i = 1, 2, \dots, 10)$ are positive integers.

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Problem 2. Polynomial $P(x)$ with integer coefficient is called **cube-presented** if it can be represented as sum of several cube of polynomials with integer coefficients.

Examples: $3x + 3x^2$ is cube-represented because $3x + 3x^2 = (x+1)^3 + (-x)^3 + (-1)^3$.

1. Is $3x^2$ a cube-represented polynomial?
2. How many quadratic polynomial $P(x)$ with integer coefficients belong to the set $\{1, 2, 3, \ldots, 2017\}$ which is cube-represented?

Problem 3. We put four numbers $1, 2, 3, 4$ around a circle in order. One starts at the number 1 and every step, he moves to an adjacent number on either side. How many ways he can move such that sum of the numbers he visits in his path (including the starting number) is equal to 21?

Problem 4. Let ABC be a triangle with A is an obtuse angle. Denote BE as the internal angle bisector of triangle ABC with $E \in AC$ and suppose that $\angle AEB = 45^\circ$. The altitude AD of triangle ABC intersects BE at F. Let O_1, O_2 be the circumcenter of triangles FED, EDC. Suppose that EO_1, EO_2 meet BC at G, H respectively. Prove that $\dfrac{GH}{GB} = \tan\dfrac{\alpha}{2}$.

1.2. Team selection tests for JBMO 2017

1.2.1. Day 1

Problem 1. Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality

$$\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \geq 1 + a + b + c.$$

Problem 2. Find all prime numbers p such that $\dfrac{3^{p-1}-1}{p}$ is a perfect square.

Problem 3. On the table, there are 1024 marbles and two students A, B alternatively take a positive number of marble(s). The student A goes first, B goes after that and so on. On the first move, A takes k marbles with $1 < k < 1024$. On the moves after that, A and B are not allowed to take more than k marbles or 0 marble. The student takes the last marble(s)

on the table will be the winner. Find all values of k the student A should choose to make sure that there is a strategy for him to win the game.

Problem 4. Let ABC be an acute, non isosceles triangle and (O) be its circumcircle (with center O). Denote by G the centroid of the triangle ABC, by H the foot of the altitude from A onto the side BC and by I the midpoint of AH. The line IG intersects BC at K.

1. Prove that $CK = BH$.
2. The ray GH intersects (O) at L. Denote by T the circumcenter of the circle (BHL). Prove that AO and BT intersect on the circle (O).

1.2.2. Day 2

Problem 1. Given a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx$. It is known that each of the equations $f(x) = 1$ and $f(x) = 2$ has four real roots (not necessary distinct). Prove that if the roots of the first equation satisfy the equality $x_1 + x_2 = x_3 + x_4$, then the same equation holds for the roots of the second equation.

Problem 2. A positive integer $k > 1$ is called nice if for any pair (m, n) of positive integers satisfying the condition $kn + m | km + n$ we have $n | m$.

1. Prove that 5 is a nice number;
2. Find all nice numbers.

Problem 3. Let (O) be a circle, and BC be a chord of (O) such that BC is not a diameter. Let A be a point on the larger arc BC of (O), E and F be two feet of the perpendiculars from B and C to AC, AB respectively.

1. Prove that two tangents of (AEF) at E and F intersect to each other at a fixed point M when A moves on the larger arc BC of (O).
2. Let T be the intersection of EF and BC, H be the orthocenter of ABC. Prove that TH is perpendicular to AM.

Problem 4. Find the number of ways one can put numbers 1 or 2 in each cell of a 8×8 chessboard in such a way that the sum of numbers in each column and in each row is an odd number (two ways are considered different if the number on some cell in the first way is different from the number on the cell at correspondent position in the second way).

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1.2.3. Day 3

Problem 1. Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that we have inequality

$$\frac{a(a-b^2)}{a+b^2}+\frac{b(b-c^2)}{b+c^2}+\frac{c(c-a^2)}{c+a^2}\geq 0.$$

Problem 2. Find all pairs of positive integers (p, q) such that both equations $x^2-px+q=0$ and $x^2-qx+p=0$ have integral solutions.

Problem 3. Let BC be a chord of a circle (O) such that BC is not a diameter. Let AE be a diameter perpendicular to BC such that A belongs to the larger arc BC of (O). Let D be a point on the larger arc BC of (O) which is different from A. Suppose that AD intersects BC at S, ED intersects BC at T. Let F be the midpoint of ST and I be the second intersection of the circle (ODF) with BC.

1. Let the line passing I and parallel to OD intersect AD and ED at M and N respectively. Find the maximum value of the area of triangle MDN when D moves on the larger arc BC of (O) (such that $D\neq A$).

2. Prove that the perpendicular from D to ST passes through the midpoint of MN.

Problem 4. Given a set S of 200 points on the plane: 100 points are the vertices of a convex polygon A, and 100 other points are in the interior of the polygon. Moreover, there does not exist 3 collinear points. A triangulation is a way to partition the polygon A into triangles by drawing the edges between some two points of S such that any two edges do not intersect in the interior, and each point in S is the vertices of at least one triangle.

1. Prove that number of edges does not depend on the triangulation.

2. Show that for any triangulation, one can draw each triangle by 1 of 3 given colors such that 2 adjacent triangles have different colors.

1.3. Mock tests for BMO 2017

1.3.1. Day 1

Problem 1. Prove that there are infinitely many positive integer n such that $n!$ is divisible by n^3-1.

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Problem 2. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all function $f : \mathbb{R}^+ \to \mathbb{R}$ such that, for all positive real number x and y, the following conditions are satisfied:

i) $2f(x) + 2f(y) \leq f(x+y)$.

ii) $(x+y)[yf(x) + xf(y)] \geq xyf(x+y)$.

Problem 3. Let ABC be an acute triangle and (O) be its circumcircle. Denote by H its orthocenter and I the midpoint of BC. The lines BH, CH intersect AC, AB at E, F respectively. The circles (IBF) and (ICE) meet again at D.

1. Prove that D, I, A are collinear and HD, EF, BC are concurrent.

2. Let L be the foot of the angle bisector of $\angle BAC$ on the side BC. The circle (ADL) intersects (O) again at K and intersects the line BC at S out of the side BC. Suppose that AK, AS intersects the circles (AEF) again at G, T respectively. Prove that $TG = TD$.

Problem 4. Consider the set $X = \{1, 2, 3, \ldots, 2018\}$. How many positive integers k with $2 \leq k \leq 2017$ that satisfy the following conditions:

i) There exists some partition of the set X into 1009 disjoint pairs which are $(a_1, b_1), (a_2, b_2), \ldots, (a_{1009}, b_{1009})$ with $|a_i - b_i| \in \{1, k\}$.

ii) For all partitions satisfy the condition (i), the sum $T = \sum_{i=1}^{1009} |a_i - b_i|$ has the right most digit is 9.

1.3.2. Day 2

Problem 1. Let a, b, c be positive real numbers. Prove that

$$\frac{a(b^2+c^2)}{(b+c)(a^2+bc)} + \frac{b(c^2+a^2)}{(c+a)(b^2+ca)} + \frac{c(a^2+b^2)}{(a+b)(c^2+ab)} \geq \frac{3}{2}.$$

Problem 2. Solve the following equation in positive integers x, y:

$$x^{2017} - 1 = (x-1)(y^{2015} - 1).$$

Problem 3. Let $ABCD$ be a cyclic quadrilateral and triangles ACD, BCD are acute. Suppose that the lines AB and CD meet at S. Denote by E the intersection of AC, BD. The circles (ADE) and (BCE) meet again at F.

1. Prove that $SF \perp EF$.

2. The point G is taken out side of the quadrilateral $ABCD$ such that triangle GAB and FDC are similar. Prove that $GA + FB = GB + FA$.

Problem 4. Let p be a prime number and a table of size $(p^2+p+1)\times(p^2+p+1)$ which is divided into unit cells. The way to color some cells of this table is called nice if there are no four colored cells that form a rectangle (the sides of rectangle are parallel to the sides of given table).

1. Let k be the number of colored cells in some nice coloring way. Prove that $k \leq (p+1)(p^2+p+1)$. Denote this number as k_{max}.

2. Prove that all ordered tuples (a,b,c) with $0 \leq a,b,c < p$ and $a+b+c > 0$ can be partitioned into p^2+p+1 sets $S_1, S_2, \ldots S_{p^2+p+1}$ such that two tuples (a_1,b_1,c_1) and (a_2,b_2,c_2) belong to the same set if and only if $a_1 \equiv ka_2, b_1 \equiv kb_2, c_1 \equiv kc_2 \pmod p$ for some $k \in \{1,2,3,\ldots,p-1\}$.

3. For $1 \leq i,j \leq p^2+p+1$, if there exist $(a_1,b_1,c_1) \in S_i$ and $(a_2,b_2,c_2) \in S_j$ such that $a_1a_2+b_1b_2+c_1c_2 \equiv 0 \pmod p$, we color the cell (i,j) of the given table. Prove that this coloring way is nice with k_{max} colored cells.

1.4. Team Selection Tests for the International Mathematical Olympiad 2017

1.4.1. Day 1

Problem 1. Let ABC be a triangle inscribed in circle (O), with its altitudes BE, CF intersect at orthocenter H $(E \in AC, F \in AB)$. Let M be the midpoint of BC, K be the orthogonal projection of H on AM. EF intersects BC at P. Let Q be the intersection of tangent of (O) which passes through A with BC, T be the reflection of Q through P. Prove that $\angle OKT = 90°$.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(xf(y)-y)+f(xy-x)+f(x+y)=2xy$$

for all $x, y \in \mathbb{R}$.

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Problem 3. Prove that there are infinitely many positive integers n such that n divides $2017^{2017^n-1} - 1$ but n does not divide $2017^n - 1$.

1.4.2. Day 2

Problem 1. In the garden of Wonderland, there are 2016 apples, 2017 bananas and 2018 oranges.Two monkeys Adu and Bakar play the following game: alternatively each of them takes and eats one fruit of any kind except for the one that he took in previous turn (in the first turn, each of them can take a fruit of any kind). Who can not take a fruit is the loser. Which monkey has the winning strategy if Adu plays first?

Problem 2. Let $ABCD$ be the circumscribed quadrilateral with the incircle (I). The circle (I) touches AB, BC, CD, DA at M, N, P, Q respectively. Let K and L be the circumcenters of the triangles AMN and APQ respectively. The line KL cuts the line BD at R. The line AI cuts the line MQ at J. Prove that $RA = RJ$.

Problem 3. Find the greatest positive real number M such that for all positive real sequence (a_n) and for all real number $m < M$, it is possible to find some index $n \geq 1$ that satisfies the inequality $a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} > ma_n$.

1.4.3. Day 3

Problem 1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2017$, the integer $P(n)$ is positive and $S(P(n)) = P(S(n))$.

Problem 2. Let $ABCD$ be a quadrilateral inscribed a circle (O). Assume that AB and CD intersect at E,AC and BD intersect at K, and O does not belong to the line KE. Let G and H be the midpoints of AB and CD respectively. Let (I) be the circumcircle of the triangle GKH. Let (I) and (O) intersect at M, N such that $MGHN$ is convex quadrilateral. Let P be the intersection of MG and HN, Q be the intersection of MN and GH.

1. Prove that IK and OE are parallel.

2. Prove that PK is perpendicular to IQ.

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Problem 3. The 64 cells of an 8×8 chessboard have 64 different colours. A Knight stays in one cell. In each move, the Knight jumps from one cell to another cell (the 2 cells on the diagonal of an 2×3 board) also the colours of the 2 cells interchange. In the end, the Knight goes to a cell having common side with the cell it stays at first. Can it happen that: there are exactly 3 cells having the colours different from the original colours?

1.4.4. Day 4

Problem 1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Problem 2. Denote by $\{x\}$ the fractional part of a real number x, that is $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the maximum integer not greater than$x$. Prove that

1. For every integer n, we have $\left\{n\sqrt{17}\right\} > \frac{1}{2\sqrt{17}\cdot n}$.

2. The value $\frac{1}{2\sqrt{17}}$ is the largest constant c such that the inequality $\left\{n\sqrt{17}\right\} > cn$ holds for all positive integers n.

Problem 3. For integer $n > 1$, consider n cube polynomials $P_1(x), \ldots, P_n(x)$ such that each polynomial has 3 distinct real roots. Denote S as the set of roots of following equation

$$P_1(x)P_2(x)P_3(x)\ldots P_n(x) = 0.$$

It is also known that for each $1 \leq i < j \leq n$, $P_i(x)P_j(x) = 0$ has 5 distinct real roots.

1. Prove that if for each $a, b \in S$, there is exactly one $i \in \{1, 2, 3, \ldots, n\}$ such that $P_i(a) = P_i(b) = 0$ then $n = 7$.

2. Prove that if $n > 7$ then $|S| = 2n + 1$.

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1.5. Training problem set for JBMO and IMO 2017

1.5.1. Training problem for IMO Team

Problem 1. Let m, n, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that either $m = 1$ and $l = 2k$; or $l \mid k$ and $m = \dfrac{n^{k-l} - 1}{n^l - 1}$.

Problem 2. There are 4950 ants. Assume that, for any three ants A, B and C, if the ant A is the boss of the ant B, and the ant B is the boss of the ant C then the ant A is also the boss of the ant C. We want to divide the ants into n groups so that in any group, either any two ants have the boss relationship or any two ants do not have the boss relationship. Find the smallest of n we can always do in any case.

Problem 3. Let $ABCD$ be a convex quadrilateral. Ray AD meets ray BC at P. Let O, O' be the circumcenters of triangles PCD, PAB, respectively, H, H' be the orthocenters of triangles PCD, PAB, respectively. Prove that circumcircle of triangle DOC is tangent to circumcircle of triangle $AO'B$ if and only if circumcircle of triangle DHC is tangent to circumcircle of triangle $AH'B$.

Problem 4. Does there exist an integer $n \geq 3$ and an arithmetic sequence $a_0, a_1, \ldots, a_n$ such that the polynomial $a_n x^n + \ldots + a_1 x + a_0$ has n roots which also form an arithmetic sequence?

Problem 5. Let ABC be an acute triangle inscribed in circle (O), with orthocenter H. Median AM of triangle ABC intersects circle (O) at A and N. AH intersects (O) at A and K. Three lines KN, BC and line through H and perpendicular to AN intersect each other and form triangle XYZ. Prove that the circumcircle of triangle XYZ is tangent to (O).

Problem 6. A convex polygon is divided into some triangles. Let V and E be respectively the set of vertices and the set of egdes of all triangles (each vertex in V may be some vertex of the polygon or some point inside the polygon). The polygon is said to be *good* if the following conditions hold:

i. There are no 3 vertices in V which are collinear;

ii. Each vertex in V belongs to an even number of edges in E.

Find all good polygon.

Problem 7. Find all pairs of integers (x, y) such that

$$y^3 = 8x^6 + 2x^3y - y^2.$$

Problem 8. here are 2017 points on the plane, no three of them are collinear. Some pairs of the points are connected by n segments. Find the smallest value of n so that there always exists two disjoint segments in any case.

Problem 9. Let ABC be a triangle inscribed in circle (O), with its altitudes BH_b, CH_c intersect at orthocenter H $(H_b \in AC, H_c \in AB)$. H_bH_c meets BC at P. Let N be the midpoint of AH, L be the orthogonal projection of O on the symmedian with respect to angle A of triangle ABC. Prove that $\angle NLP = 90°$.

1.5.2. Training problem for JBMO Team

f **Problem 1.** For each pair of positive integers (x, y) a nonnegative integer $x\Delta y$ is defined. It's known that for all positive integers a and b the following equalities hold:

i. $(a+b)\Delta b = a\Delta b + 1$.

ii. $(a\Delta b)\cdot(b\Delta a) = 0$.

Find values of the expressions $2016\Delta 121$ and $2016\Delta 144$.

Problem 2. Let ABC be a triangle inscribed in circle (O) such that two points B, C are fixed, A moves on major arc BC of (O). Two tangents through B, C of (O) intersect at P. Circle with diameter OP intersects AC, AB at D, E, respectively. Prove that DE is tangent to a fixed circle whose radius is half the radius of (O).

Problem 3. Find all pairs of primes (p, q) such that

$$p^3 - q^5 = (p+q)^2.$$

Problem 4. Let $S = \{-17, -16, \ldots, 16, 17\}$. We call a subset T of S a *good set* if $-x \in T$ for any $x \in T$ and if $x, y, z \in T$ (x, y, z may be equal) then $x + y + z \neq 0$. Find the largest number of elements in a good set.

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Problem 5. Let a, b, c be positive real numbers such that $a+b+c=6$. Prove that

$$\frac{1}{a^2b+16}+\frac{1}{b^2c+16}+\frac{1}{c^2a+16}\geq\frac{1}{8}.$$

Problem 6. Find all pairs of prime numbers (p,q) such that $p^2+5pq+4q^2$ is a square.

Problem 7. Let ABC be a triangle inscribed in circle (O), with orthocenter H. Let d be an arbitrary line which passes through H and intersects (O) at two points P and Q. Draw diameter AA' of circle (O). $A'P$,$A'Q$ meet BC at K,L, respectively. Prove that O,K,L,A' are concyclic.

Problem 8. A chessboard has 64 cells painted black and white in the usual way. A *bishop path* is a sequence of distinct cells such that two consecutive cells have exactly one common point. At least how many bishop paths can the set of all white cells be divided into?

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Part 2

SOLUTION FOR THE TESTS

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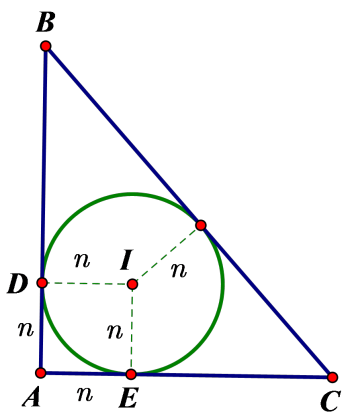
2.1. Solution of TST for BMO 2017

2.1.1. Day 1

Problem 1. Let $n = p_1 p_2 \dots p_{17}$ be the positive integer where $p_1, p_2, \dots, p_{2017}$ are 2017 distinct odd primes. A triangle is called nice if it is a right triangle with integer side lengths and the inradius is n. Find the number of nice triangles (two triangles are consider different if their tuples of length of sides are different).

Solution.
Consider the right triangle ABC with incircle (I) and D, E are tangent points of (I) on sides AB, AC. It is easy to see that $ADIE$ is a square of side n.



Denote $BD = x, CE = y$ then we can see that from a pair (x, y) with $x > n, y > n$, we can construct a right triangle satisfies the given condition. Thus, to count the number of triangles, we will count the number of pairs (x, y). Notice that

$$[ABC] = \frac{1}{2} AB \cdot AC = \frac{(x+n)(y+n)}{2}$$

and

$$[ABC] = \frac{1}{2} n(AB + BC + CA) = \frac{1}{2} n(x + n + y + n + x + y) = n(x + y + n).$$

Hence,

$$(x+n)(y+n) = 2n(x+y+n) \Leftrightarrow n^2 = xy - (nx + ny).$$

This implies that $(x-n)(y-n) = 2n^2$. Since $2n^2 = 2p_1^2p_2^2\cdots p_{2017}^2$ then $2n^2$ has $2\cdot 3^{2017}$ positive divisors.

These divisors can be partitioned into 3^{2017} pairs of form $\left(d, \frac{2n^2}{d}\right)$, and from each pair, we can find one pair (x, y) which implies that there are exactly 3^{2017} triangles satisfy the given conditions.

□

Problem 2. Let ABC be an acute triangle with AT, AS respectively are the internal, external angle bisector of ABC and $T, S \in BC$. On the circle with diameter TS, take an arbitrary point P that lies inside the triangle ABC. Denote D, E, F, I as the incenter of triangle PBC, PCA, PAB, ABC. Prove that four lines AD, BE, CF and IP are concurrent.

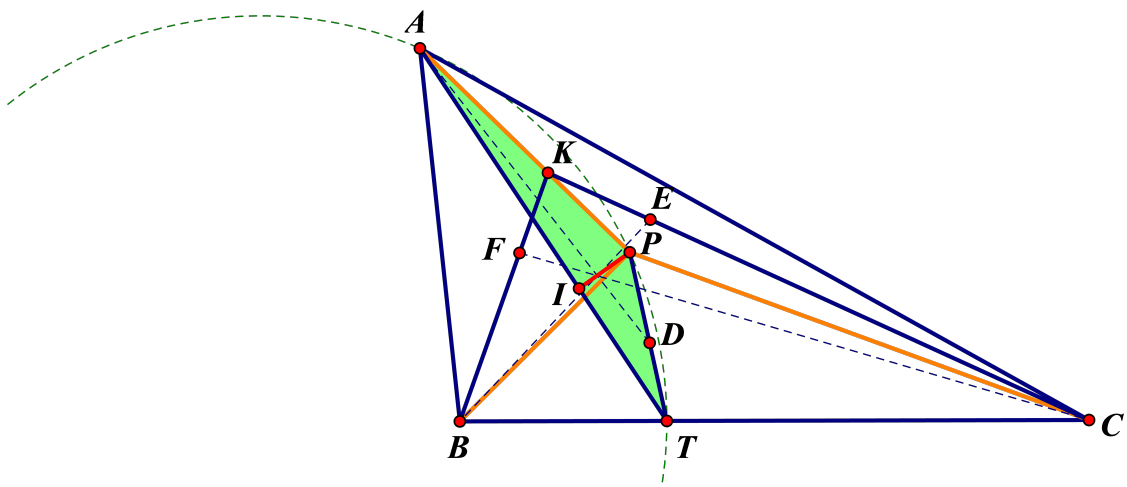
Solution.
First, we note that the circle of diameter TS is the Apollonius circle of triangle ABC then

$$\frac{BP}{CP} = \frac{BA}{CA} = \frac{BT}{CT}$$

or

$$\frac{BP}{BA} = \frac{CP}{CA},$$

which implies that the bisector of angle B in triangle ABP and the bisector of angle C in triangle ACP pass through the same point on AP. Denote that point is K. So CE, BF, AP are concurrence at K.



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It is easy to see that $I \in AT$. Consider triangle APT and we have

$$\frac{IA}{IT} \cdot \frac{KP}{KA} \cdot \frac{DT}{DP} = \frac{BA}{BT} \cdot \frac{BP}{BA} \cdot \frac{BT}{BP} = 1$$

then by applying the Ceva's theorem, we can see that three lines AD, IP, KT are concurrent. (1)

Continue, consider triangle KBC and we have

$$\frac{FK}{FB} \cdot \frac{EC}{EK} = \frac{PK}{PB} \cdot \frac{PC}{PK} = \frac{PC}{PB} = \frac{TC}{TB}$$

or

$$\frac{FK}{FB} \cdot \frac{EC}{EK} \cdot \frac{TB}{TC} = 1$$

then KT, BE, CF are also concurrent. (2)

Finally, suppose that BE, CF, AD are concurrent at X then denote $BE \cap CF = X$, we will have $X \in KT$ base on (2), but $X \in AD$ then $X \in KT \cap AD$.

Therefore, base on (1), we have $X \in IP$, then the result will follow. □

Problem 3. How many ways are there to insert plus signs + between the digits of number $111111\ldots111$ which includes thirty of digits 1 so that the result will be a multiple of 30?

Solution.

Let S be the final sum, then the necessary conditions are $3 \mid S, 10 \mid S$.

One can check that for every positive integer n then $S(n) \equiv n \pmod 9$ which implies that for all way to insert the plus signs, we have

$$S \equiv 1 + 1 + 1 + \cdots + 1 \equiv 0 \pmod 3.$$

Then the condition that divisible by 3 is always satisfied.

Suppose that we inserted m signs in total, then we have $m+1$ part. Each part is a positive integer that ends by 1 so the right most digit of S is equal to or congruent to $m+1$ modulo 10. Then we must have $m \in \{9, 19, 29\}$.

It is easy to see that the given condition is also the sufficiency condition. Denote $a_1, a_2, \ldots, a_{m+1}$ as the number of digits of each part that partitioned from m signs then

$$a_1 + a_2 + a_3 + \cdots + a_{m+1} = 30.$$

This equation has $\binom{29}{m}$ solutions. Therefore the number of ways to insert the plus signs is $\binom{29}{9} + \binom{29}{19} + \binom{29}{29}$.

□

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Problem 4. Fibonacci sequences is defined as

$$\begin{cases} f_1 = 1, f_2 = 2, \\ f_{n+1} = f_n + f_{n-1}, n \geq 2 \end{cases}$$

1. Prove that every positive integer can be represented as sum of several distinct Fibonacci number.

2. A positive integer is called **Fib-unique** if the way to represent it as sum of several distinct Fibonacci number is unique.
   Example: 13 is not Fib-unique because $13 = 13 = 8 + 5 = 8 + 3 + 2$.
   Find all Fib-unique.

Solution.
1. We will prove by induction that every positive integer k can be written as sum of several distinct Fibonacci numbers.

The case $k = 1$ is obvious, we have $1 = f_1$.

Assume that every positive integer less than n can be written as sum of several distinct Fibonacci numbers, consider $k = n$.

We can find f_i is the greatest Fibonacci number which is less than n, we have

$$n = f_i + (n - f_i).$$

According to the induction hypothesis, $n - f_i$ is a positive integer less than n, therefore it can be written as

$$n - f_i = f_{j_1} + f_{j_2} + \cdots + f_{j_t},\ j_1 < j_2 < \ldots < j_t.$$

We must have $i > j_t$ because otherwise, $n = f_i + f_{j_t} + \cdots > f_i + f_{i-1} = f_{i+1}$, contradiction to the choice of i.

So, $n = f_{j_1} + f_{j_2} + \cdots + f_{j_t} + f_i$, which is a representation of n as sum of distinct Fibonacci numbers, we are done.

2. According to the part 1., an integer n can be written as

$$n = f_{i_1} + f_{i_2} + \cdots + f_{i_k},\ i_1 < i_2 < \cdots < i_k.$$

We need to find all n of which representation is unique.
First, if $i_1 \geq 3$, then we can replace f_{i_1} by $f_{i_1-1} + f_{i_1-2}$ and get another representation of n. So if n is a "Fib-unique", $i_1 \in \{1, 2\}$.

Second, if there is some t such that $i_{t+1} - i_t \geq 3$, then we can replace $f_{i_{t+1}}$ by $f_{i_{t+1}-1} + f_{i_{t+1}-2}$ and get another representation of n.

Hence, for a "Fib-unique" integer n, we have $i_{t+1} - i_t \leq 2$ for every $0 \leq t \leq k-1$.

Third, if we have some t such that $i_{t+1} - i_t = 1$, chose that t which is the largest. Then we can replace $f_{i_{t+1}} + f_{i_t}$ by $f_{i_{t+1}+1}$, and get another representation of n. So for a "fib-unique" n we must have $i_{t+1} - i_t = 2$ for every $0 \leq t \leq k-1$.

So, every possible "Fib-unique" positive n are f_1, f_2 and those in these forms:

i . $n = f_1 + f_3 + \cdots + f_{2k-1} = f_{2k} - 1$

ii . $n = f_2 + f_4 + \cdots + f_{2k} = f_{2k+1} - 1$

In conclusion, every $Fib-unique$ is f_1, f_2 and $f_k - 1$ with $k \geq 3$. □

2.1.2. Day 2

Problem 1. Find the smallest prime q such that

$$q = a_1^2 + b_1^2 = a_2^2 + 2b_2^2 = a_3^2 + 3b_3^2 = \ldots = a_{10}^2 + 10b_{10}^2$$

where $a_i, b_i (i = 1, 2, \ldots, 10)$ are positive integers.

Solution.
Since $q = a_1^2 + b_1^2$ then q must have the form $4k+1$; otherwise, $a_1^2 + b_1^2$ is divisible by $q = 4k+3$ which implies that $q|a_1, q|b_1$, a contradiction.

Note that $q = a_2^2 + 2b_2^2$ so in modulo 8, number q is congruent to either 1 or 3 modulo 8, but $q \equiv 1 \pmod 4$ then $q \equiv 1 \pmod 8$.

Continue, $q = a_3^2 + 3b_3^2$ so by taking modulo 3, it gives q is congruent to 1 in modulo 3. Similarly, by consider two equations $q = a_5^2 + 5b_5^2$ and $q = a_7^2 + 7b_7^2$, we have $q \equiv 1, 4 \pmod 5$ and $q \equiv 1, 2, 4 \pmod 7$.

These congruence can be put together as q is congruent to

$$1, 121, 169, 289, 361, 529 \text{ in modulo } 840.$$

The first smallest values that satisfy the previous congruence are

$$1, 121, 361, 169, 289, 529, 841, 961, 1009, \ldots$$

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and it is easy to check that the smallest prime solution is 1009.

And it does satisfy the problem requirement by following equalites

$$\begin{aligned}
&1009\\
&= 28^2 + 15^2 = 19^2 + 2(18^2)\\
&= 31^2 + 3(4^2) = 15^2 + 4(19^2)\\
&= 17^2 + 5(12^2) = 25^2 + 6(8^2)\\
&= 1^2 + 7(12^2) = 19^2 + 8(9^2)\\
&= 28^2 + 9(5^2) = 3^2 + 10(10^2)
\end{aligned}$$

Therefore, the answer is 1009. □

Problem 2. Polynomial $P(x)$ with integer coefficient is called **cube-presented** if it can be represented as sum of several cube of polynomials with integer coefficients.

Examples: $3x + 3x^2$ is cube-represented because $3x + 3x^2 = (x+1)^3 + (-x)^3 + (-1)^3$.

1. Is $3x^2$ a cube-represented polynomial?
2. How many quadratic polynomial $P(x)$ with integer coefficients belong to the set $\{1, 2, 3, \ldots, 2017\}$ which is cube-represented?

Solution.

A polynomial with integer coefficients is called "nice" if the coefficients in the terms whose degree is not a multiple of 3, are multiples of 3, and their sum is multiple of 6. Clearly, the sum of nice polynomials is a nice polynomial, and the sum of "cube-represented" polynomials is a "cube-represented" polynomial. We shall prove the following lemma:

Lemma. A polynomial is "cube-represented" if and only if it is "nice".

Consider the cube of a polynomial $P(x)$ with integer coefficients, where:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

After expanding, all term of $P^3(x)$ are either of the form $a_i^3 x^{3i}$, or $3a_i^2 a_j x^{2i+j}$ with i different from j, or of the form $6a_i a_j a_k x^{i+j+k}$ with i, j, k are distinct.

We have $3|2i + j$ if and only if $3|2j + i$, and $a_i^2 a_j + a_j^2 a_i = a_i a_j (a_i + a_j)$ which is always an even number. From here clearly a "cube-represented" polynomial has to be a "nice"polynomial, which is defined as above.

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Second, consider a "nice" polynomial $P(x)$ we will prove by induction on n that $P(x)$ is "cube-represented", with n is the integer that the degree of $P(x)$ does not exceed $3n$, but is larger than $3n-3$.

For the base case where $n=1$, $P(x)=a_3x^3+3a_2x^2+3a_1x+a_0$, where $2|a_1-a_2$.

We can write

$$P(x)=\frac{a_1+a_2}{2}(x+1)^3+\frac{a_1-a_2}{2}(x-1)^3+(a_3-a_1)x^3+(a_0-a_2)1^3$$

which obviously a "cube-represented" polynomial.

Assume the result is true for $n-1$, if $P(x)=a_{3n}x^{3n}+3a_{3n-1}x^{3n-1}+3a_{3n-2}x^{3n-2}+Q(x)$ is nice, where degree of $Q(x)$ is at most $3n-3$.

Then either $a_{3n-1}+a_{3n-2}$ is even and $Q(x)$ is good, or $a_{3n-1}+a_{3n-2}$ is odd, and $Q(x)-3x^{3n-4}$ is good.

In the first case

$$P(x)=\frac{a_{3n-1}+a_{3n-2}}{2}(x^n+x^{n-1})^3+\frac{a_{3n-2}-a_{3n-1}}{2}(x^n-x^{n-1})^3+ (a_{3n}-a_{3n-2})x^{3n}+Q(x)-a_2(x^{n-1})^3$$

In the second case

$$P(x)=\frac{a_{3n-1}+a_{3n-2}-1}{2}(x^n+x^{n-1})^3+\frac{a_{3n-2}-a_{3n-1}-1}{2}(x^n-x^{n-1})^3 +(x^n+x^{n-2})^3+(a_{3n}-a_{3n-2})x^{3n}+(Q(x)-3x^{3n-4})-a_2x^{3n-3}$$

so in both cases, we have $P(x)$ is a "nice" polynomial. Our lemma has been proved.

Back to our problem,

1. With the lemma, we can easily verify that $P(x)=3x^2$ is not a "nice" polynomial, so it can not be a "cube-represented" polynomial.

2. We need to count the number of triples (a,b,c) which are belong to the set $\{1,2,...,2017\}$ such that $P(x)=ax^2+bx+c$ is "cube-represented", that mean it's "nice".

$P(x)$ is nice if and only if a,b are multiple of 3 and $a+b$ is multiple of 2.

We have $a,b\in\{3,6,\dots,2016\}$ and $a+b$ is even, there are 336^2 such a pair. Value c can be arbitrary, so the number of pair (a,b,c) is $2017\cdot 336^2$, which is the number of quadratic "cube-represented" polynomials satisfy the problem.

□

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Problem 3. We put four numbers $1, 2, 3, 4$ around a circle in order. One starts at the number 1 and every step, he moves to an adjacent number on either side. How many ways he can move such that sum of the numbers he visits in his path (including the starting number) is equal to 21?

Solution.
Let a_i, b_i, c_i, d_i be the number of paths that end by $1, 2, 3, 4$ respectively and has the sum equal to i. These paths all start from 1. So it is easy to check that

$$a_1 = 1, b_1 = c_1 = d_1 = 0, a_2 = b_2 = c_2 = d_2 = 0,$$
$$a_3 = 0, b_3 = 1, c_3 = d_3 = 0, a_4 = 1, b_4 = c_4 = d_4 = 0.$$

We can visit 1 from 2 or 4, visit 2 from 1 or 3, visit 3 from 2 or 4, visit 4 from 3 or 1 then we have the following relations

$$\begin{cases} a_n = b_{n-1} + d_{n-1} \\ b_n = a_{n-2} + c_{n-2} \\ c_n = b_{n-3} + d_{n-3} \\ d_n = c_{n-4} + a_{n-4} \end{cases} \text{with } n \geq 5.$$

We have to calculate $a_{21}+b_{21}+c_{21}+d_{21}$. Denote $u_n = a_n+c_n$ and $v_n = b_n+d_n$ for $n \geq 1$ then we come to another relation

$$\begin{cases} u_n = v_{n-1} + v_{n-3} \\ v_n = u_{n-2} + u_{n-4} \end{cases}$$

We also have $u_1 = 1, u_2 = 0, u_3 = 0, u_4 = 1$ and $v_1 = 0, v_2 = 0, v_3 = 1, v_4 = 0$. Then $v_n = v_{n-3} + 2v_{n-5} + v_{n-7}$ and $u_n = u_{n-3} + 2u_{n-5} + u_{n-7}$.

By putting $s_n = u_n + v_n$, we get

$$s_n = s_{n-3} + 2s_{n-5} + s_{n-7}$$

with $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 1, s_5 = 1, s_6 = 3, s_7 = 1$. By directly calculation, we have

$$s_8 = 4, s_9 = 5, s_{10} = 4, s_{11} = 11,$$
$$s_{12} = 8, s_{13} = 15, s_{14} = 22, s_{15} = 20,$$
$$s_{16} = 42, s_{17} = 42, s_{18} = 61, s_{19} = 94$$
$$s_{20} = 97, s_{21} = 167.$$

Therefore, we have 167 paths that have the sum of numbers on these path is equal to 21.

□

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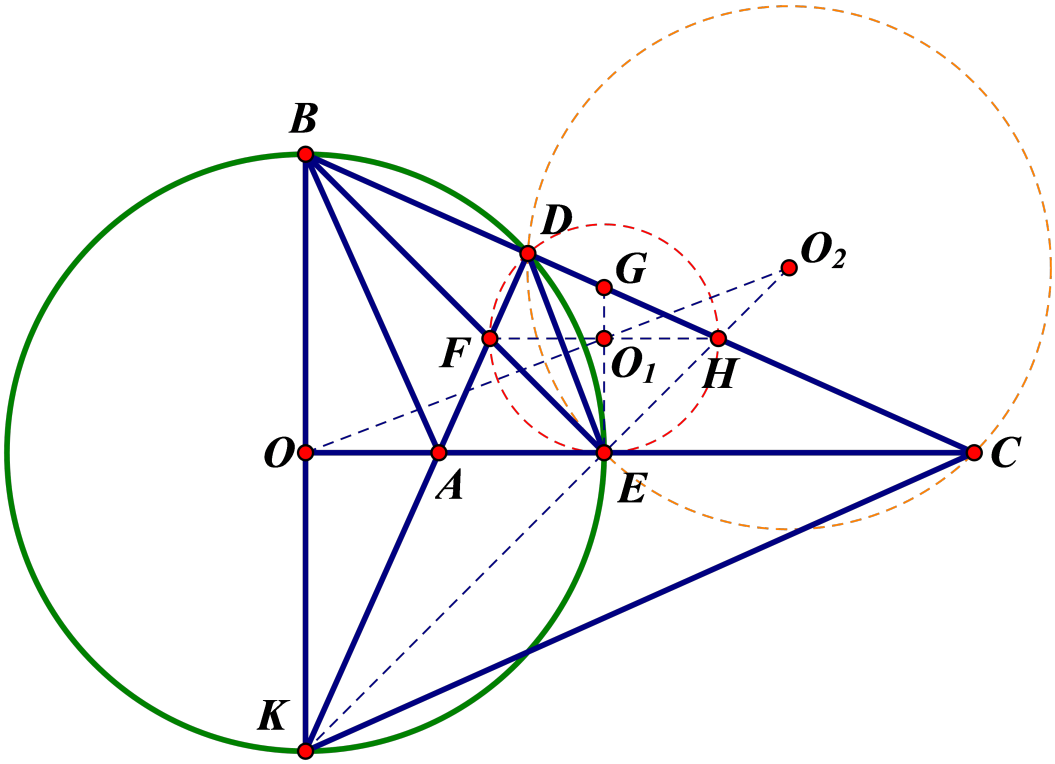
Problem 4. Let ABC be a triangle with A is an obtuse angle. Denote BE as the internal angle bisector of triangle ABC with $E \in AC$ and suppose that $\angle AEB = 45^\circ$. The altitude AD of triangle ABC intersects BE at F. Let O_1, O_2 be the circumcenter of triangles FED, EDC. Suppose that EO_1, EO_2 meet BC at G, H respectively. Prove that $\dfrac{GH}{GB} = \tan\dfrac{\alpha}{2}$.

Solution.
Denote O as the projection of B on AC and BK is the diameter of (O, OB). It is easy to see that $E \in (O)$. We have

$$\angle BCA = \angle AEB - \angle CBE = \angle OBE - \angle ABE = \angle OBA = \angle OKA.$$

This implies that $ODCK$ is the inscribed quadrilateral and $\angle KDC = 90^\circ$. Hence, K, A, D are collinear and $D \in (O)$.

Triangle CBK is isosceles then CE is the angle bisector of $\angle BCK$. Note that DE is the angle bisector of $\angle ADC$. Then E is the incenter of triangle KDC.



So O_2 is the midpoint of minor arc CD of the circumcircle of KDC, then K, E, O_2 are collinear.

We have

$$\Delta BAD \sim \Delta KCD \Rightarrow \frac{BD}{BA} = \frac{KD}{KC}.$$

Base on the property of bisector in triangle, we have

$$\frac{FD}{FA} = \frac{BD}{BA} = \frac{KD}{KC} = \frac{HD}{HC}.$$

Hence, $FH \parallel AC$. We have O_1 is the midpoint of FH also is the circumcenter of quadrilateral $DFEH$.

From this, we can conclude that EG is the angle bisector in triangle BEH and

$$\frac{GH}{GB} = \frac{EH}{EB} = \tan\frac{B}{2} = \tan\frac{\alpha}{2}.$$

□

2.2. Solution of TST for JBMO 2017

2.2.1. Day 1

Problem 1. Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality

$$\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \geq 1 + a + b + c.$$

Solution.
First, we can see that among three numbers $a-1, b-1, c-1$, there are two numbers have the same sign. Without loss of generality, we may assume that $a-1, b-1$ have the same sign, which means $(a-1)(b-1) \geq 0$.

By applying Cauchy-Schwarz inequality, we have

$$\sqrt{2(1+c^2)} = \sqrt{(1^2+1^2)(1+c^2)} \geq 1 + c$$

and

$$\sqrt{(1+a^2)(1+b^2)} \geq 1 + ab.$$

Hence, we need to prove $(1+c)(1+ab) \geq 1+a+b+c$ which is equivalent to

$$1 + ab + c + abc \geq 1 + a + b + c \text{ or } (a-1)(b-1) \geq 0.$$

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The last inequality is true which implies that the given inequality is also true. The equality occurs when $a = b = c = 1$.

□

Problem 2. Find all prime numbers p such that $\dfrac{3^{p-1}-1}{p}$ is a perfect square.

Solution.
Let p be a prime satisfying the condition of the problem. By assumption, there is a positive integer A such that $3^{p-1}-1 = pA^2$.
It is clear that $p = 2$ is a solution. Now, we consider $p > 2$. Put $p-1 = 2k$, one has $(3^k-1)(3^k+1) = pA^2$. Since $(3^k-1, 3^k+1) = 2$, it follows that there are positive integers B and C such that either

$$3^k - 1 = 2pB^2,\ 3^k + 1 = 2C^2$$

or

$$3^k - 1 = B^2,\ 3^k + 1 = 2pC^2.$$

But, the first case cannot hold since $2C^2 = 3^k + 1 \equiv 1 \pmod 3$, we get $C^2 \equiv 2 \pmod 3$ which is impossible.
For the second case, if k is odd then $4 \mid 3^k + 1 = 2pC^2$, hence $2 \mid C$ (since p is odd). This turns out to be that $3^k + 1 = 2pC^2$ is divisible by 8 which contradicts to $3^k + 1 \equiv 4 \pmod 8$ (since k is odd). Thus, k must be even. Put $k = 2m$, then

$$2B^2 = 3^k - 1 = (3^m - 1)(3^m + 1).$$

Again, since $(3^m-1, 3^m+1) = 2$, there are positive integers D, E such that either

$$3^m - 1 = E^2,\ 3^m + 1 = 2D^2$$

or

$$3^m - 1 = 2E^2,\ 3^m + 1 = D^2.$$

As above, the equality $3^m + 1 = 2D^2$ leads to a contradiction. Hence, $3^m + 1 = D^2$, that is

$$3^m = D^2 - 1 = (D-1)(D+1).$$

Therefore, there are non-negative integers $t > s$ such that $D - 1 = 3^s$ and $D + 1 = 3^t$. This gives,

$$2 = (D+1) - (D-1) = 3^t - 3^s = 3^s(3^{t-s} - 1).$$

This happens if and only if $3^s = 1$ and $3^{t-s} - 1 = 2$, i.e. $s = 0$ and $t = 1$, we find that $p = 5$ (satisfied).
In conclusion, $p = 2; 5$. □

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Problem 3. On the table, there are 1024 marbles and two students A, B alternatively take a positive number of marble(s). The student A goes first, B goes after that and so on. On the first move, A takes k marbles with $1 < k < 1024$. On the moves after that, A and B are not allowed to take more than k marbles or 0 marble. The student takes the last marble(s) on the table will be the winner. Find all values of k the student A should choose to make sure that there is a strategy for him to win the game.

Solution.
After the first move, there are $1024-k$ marbles remain and both students A, B cannot take more than k marbles during their turn.

We shall prove that if $k+1|1024-k$ then the student A will have the strategy to win the game.

Indeed, let $1024-k = m(k+1)$ with $m \in \mathbb{Z}^+$. On each turn, if B choose x marble(s) then A will choose $k+1-x$ marble(s) on his next turn. Since $x \in \{1,2,3,\ldots,k\}$ then $k+1-x \in \{1,2,3,\ldots,k\}$ which implies that A always can perform his turn. After m turns of A and m turns of B, student A takes the last marble then wins the game.

If $1024-k$ is not divisible by $k+1$ then we put $1024-k = m(k+1)+n$ with $0 < n < k+1$. In this case, the student B can take n marbles on the next turn. The number of remaining marbles are $m(k+1)$ then by applying the same argument above, the student B have the strategy to win the game.

Hence, the condition we have to find is $k+1|1024-k$ or $k+1|1025$. Note that the divisors of 1025 are $1, 5, 25, 41, 205, 1025$ then

$$k+1 \in \{1,5,25,41,205,1025\} \text{ and } 0 < k < 1024.$$

This means that all values of k we need to find are $4, 24, 40, 204$. □

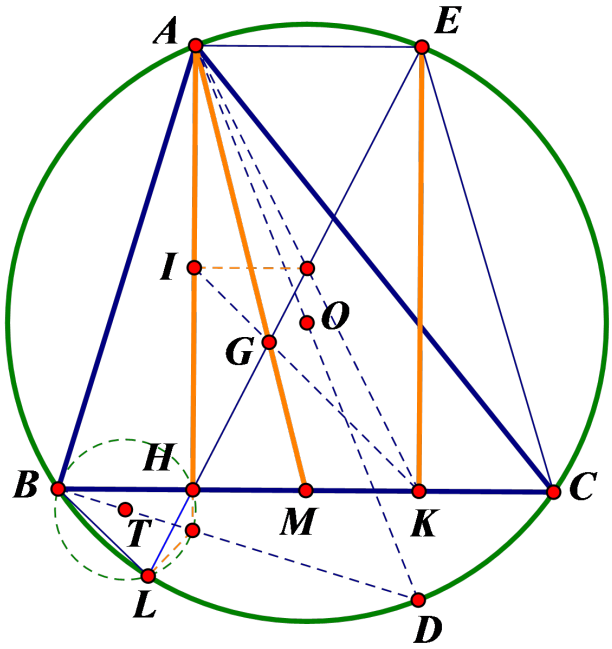
Problem 4. Let ABC be an acute, non isosceles triangle and (O) be its circumcircle (with center O). Denote by G the centroid of the triangle ABC, by H the foot of the altitude from A onto the side BC and by I the midpoint of AH. The line IG intersects BC at K.

1. Prove that $CK = BH$.
2. The ray GH intersects (O) at L. Denote by T the circumcenter of the circle (BHL). Prove that AO and BT intersect on the circle (O).

Solution.
1) Let M be the midpoint of BC then $G \in AM$ and $\dfrac{AG}{AM} = \dfrac{2}{3}$. Take the

point $K' \in BC$ such that M is the midpoint of HK' then AM is the median of triangle AHK' and G is its centroid.

Then $K'G$ is the median of triangle AHK' or $K'G$ passes through the midpoint of AH. This implies that $K \equiv K'$ and we have $BH = CK$.



2) The line passes through A and parallel to BC intersects (O) at E differs from A. Then by the symmetric through the perpendicular bisector of BC, it is easy to check that $AHKE$ is a rectangle. Since $\dfrac{GA}{GM} = \dfrac{AE}{HM} = 2$, we have H, G, E, L are collinear.

Then $\angle BLH = \angle BLE = \angle BCE = \angle ABC$ which implies that

$$\angle ABT = \angle ABC + \angle CBT = \angle BLH + \angle CBT = 90^\circ.$$

Thus if we denote $D = BT \cap (O)$ then AD is the diameter of (O), then $O \in AD$. Therefore, BT and AO intersect at a point that belongs to (O).

□

2.2.2. Day 2

Problem 1. Given a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx$. It is known that each of the equations $f(x) = 1$ and $f(x) = 2$ has four real roots (not

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necessary distinct). Prove that if the roots of the first equation satisfy the equality $x_1 + x_2 = x_3 + x_4$, then the same equation holds for the roots of the second equation.

Solution.
Consider the equation $f(x) = 1 \Leftrightarrow x^4 + ax^3 + bx^2 + cx = 1$, since it has four roots x_1, x_2, x_3, x_4 then we can write it as

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0.$$

Note that $x_1 + x_2 = x_3 + x_4 = -\frac{a}{2}$ then

$$\left(x^2 - (x_1 + x_2)x + x_1x_2\right)\left(x^2 - (x_3 + x_4)x + x_3x_4\right) = 0$$

or

$$\left(x^2 + \frac{ax}{2} + x_1x_2\right)\left(x^2 + \frac{ax}{2} + x_3x_4\right) = 0.$$

Continue with the equation $f(x) = 2 \Leftrightarrow \left(x^2 + \frac{ax}{2} + x_1x_2\right)\left(x^2 + \frac{ax}{2} + x_3x_4\right) = 1$. By putting $X = x^2 + \frac{ax}{2}$, we get

$$(X + x_1x_2)(X + x_3x_4) = 1.$$

Since $f(x) = 2$ has four real roots then this equation has two roots like $X = \alpha, X = \beta$. This implies that

$$\begin{cases} x^2 + \dfrac{ax}{2} = \alpha \\ x^2 + \dfrac{ax}{2} = \beta \end{cases}$$

Then four roots x'_1, x'_2, x'_3, x'_4 of the $f(x) = 2$ can be divided into two pairs that have the sum equal to $-\frac{a}{2}$ which mean $x'_1 + x'_2 = x'_3 + x'_4$. This finishes the proof.

□

Problem 2. A positive integer $k > 1$ is called nice if for any pair (m, n) of positive integers satisfying the condition $kn + m | km + n$ we have $n | m$.

1. Prove that 5 is a nice number;
2. Find all nice numbers.

Solution.
1) For $k = 5$, we need to prove that for all m, n satisfy $5n + m | 5m + n$ then $n | m$. Note that $5n + m \le 5m + n$ or $n \le m$, then $1 \le \dfrac{5m + n}{5n + m} < 5$.

Then $A = \frac{5m+n}{5n+m} \in \{1, 2, 3, 4\}$. We consider some cases

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- If $A = 1$ then $m = n$.
- If $A = 2$ then $5m + n = 10n + 2m \Leftrightarrow m = 3n$.
- If $A = 3$ then $5m + n = 15n + 3m \Leftrightarrow m = 7n$.
- If $A = 4$ then $5m + n = 20n + 4m \Leftrightarrow m = 19n$.

So in all case, we always have $n|m$, which implies that $k = 5$ is a nice number.

2) We can directly check that $k = 2$ is a nice number. Consider some nice number $k > 2$. By similar way, we can check that $n \leq m$ and

$$1 \leq \frac{km+n}{kn+m} < k.$$

Thus $A = \frac{km+n}{kn+m} \in \{1, 2, 3, \ldots, k-1\}$. In case $A = 2$, we have

$$\frac{km+n}{kn+m} = 2 \Leftrightarrow km + n = 2m + 2kn \Leftrightarrow \frac{m}{n} = \frac{2k-1}{k-2}.$$

We must have $\dfrac{m}{n} \in \mathbb{Z}^+$ then $\dfrac{2k-1}{k-2} = 2 + \dfrac{3}{k-2} \in \mathbb{Z}^+$. Since $k > 1$, this means that $k - 2 \in \{1, 3\}$ or $k \in \{3, 5\}$.

It is easy to check that $k = 3$ is also a nice number. Therefore, all nice numbers are $2, 3, 5$.

□

Problem 3. Let (O) be a circle, and BC be a chord of (O) such that BC is not a diameter. Let A be a point on the larger arc BC of (O), E and F be two feet of the perpendiculars from B and C to AC, AB respectively.

1. Prove that two tangents of (AEF) at E and F intersect to each other at a fixed point M when A moves on the larger arc BC of (O).
2. Let T be the intersection of EF and BC, H be the orthocenter of ABC. Prove that TH is perpendicular to AM.

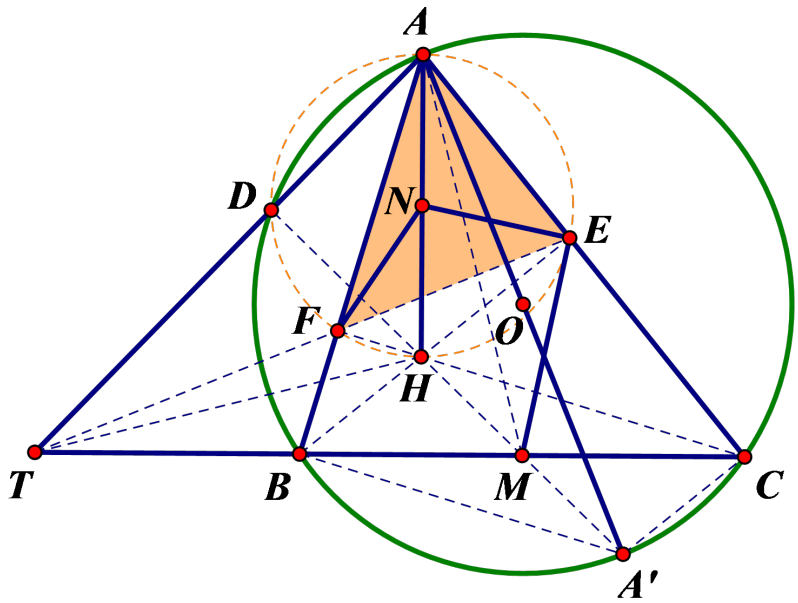
Solution.

Denote M as the midpoint of BC. Since $\angle AEH = \angle AFH = 90^\circ$ then A, H, E, F belong to the same circle and the center of this circle is midpoint N of AH. It is easy to check that $NE = NH$ and $ME = MB$, so

$$\angle MEN = \angle MEB + \angle NEB = \angle MBE + \angle NHE = \angle EAH + \angle AHE = 90^\circ.$$

Then ME is the tangent line of (AEF) at E.

By same way, we have FM is the tangent line of (AEF) at F. These imply that the tangent lines of (AEF) at E, F meet at the fixed point M.



2) Consider the diameter AA' of (O) then it is easy to see that $BHCA'$ is a parallelogram and H, M, A' are collinear.

Suppose that $MH \cap (O) = D \neq A'$ then $\angle ADH = \angle ADA' = 90^\circ$ which means $D \in (AEF)$.

By consider three radical axis of three circles $(O), (BFEC), (AEHF)$, we have AD, EF, BC are concurrent at T, which is the radical center. In triangle ATM, we have $AH \perp TM$ and $MH \perp AT$, then H is the orthocenter. From this, we can conclude that $TH \perp AM$.

□

Problem 4. Find the number of ways one can put numbers 1 or 2 in each cell of a 8×8 chessboard in such a way that the sum of numbers in each column and in each row is an odd number (two ways are considered different if in two ways, the numbers put in correspondent position are different).

Solution.
Consider the leftmost column and lowest row of table, we color all these cells. We can see that for all ways to put the number of the sub square 7×7 that not colored, we can choose the number for the correspondent

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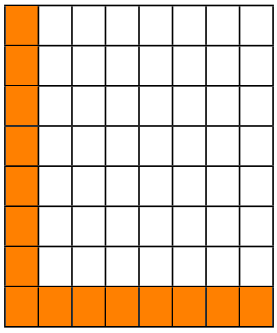
colored position at same column or row. Indeed, if sum of the 7 numbers is odd, then we put 1 on the remain cell; otherwise, we put 2.



Finally, the number for the cell at corner can choose base on the parity of the sum of all number in square 7×7. These mean that the way to fill in the square 7×7 uniquely define the numbers on the rest cells.

Since we can fill each cell among 49 cells of the square 7×7 by 1 or 2 in any way then the number of way is 2^{49}. □

2.2.3. Day 3

Problem 1. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that we have inequality

$$\frac{a(a-b^2)}{a+b^2} + \frac{b(b-c^2)}{b+c^2} + \frac{c(c-a^2)}{c+a^2} \geq 0.$$

Solution.
Note that

$$\frac{a(a-b^2)}{a+b^2} = \frac{a(a+b^2)-2ab^2}{a+b^2} = a - \frac{2ab^2}{a+b^2} \leq a - \frac{2ab^2}{2b\sqrt{a}} = a - b\sqrt{a}.$$

Similarly we have $\frac{b(b-c^2)}{b+c^2} \leq b - c\sqrt{b}$ and $\frac{c(c-a^2)}{c+a^2} \leq c - a\sqrt{c}$. Then we need to prove

$$a + b + c \geq b\sqrt{a} + c\sqrt{b} + a\sqrt{c}.$$

By applying Cauchy-Schwarz inequality, we get

$$\left(b\sqrt{a} + c\sqrt{b} + a\sqrt{c}\right)^2 \leq (ab + bc + ca)(a + b + c).$$

Then we have to prove that

$$a + b + c \geq ab + bc + ca.$$

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But this is true because

$$(a+b+c)^2 \geq 3(ab+bc+ca) \geq (a+b+c)(ab+bc+ca).$$

The given inequality is proved. The equality occurs when $a=b=c=1$.

□

Problem 2. Find all pairs of positive integers (p,q) such that both equations $x^2-px+q=0$ and $x^2-qx+p=0$ have integral solutions.

Solution.
Since $x^2-px+q=0$ has integral solution, then we may assume that

$$\begin{cases} p=a+b \\ q=ab \end{cases} \text{ with } a,b \in \mathbb{Z}^+.$$

Note that $q-p+1=ab-(a+b)+1=(a-1)(b-1)\geq 0$ then $q+1\geq p$.

By the same way on the second equation $x^2-qx+p=0$, then we have $p+1\geq q$. Thus $p=q$ or $p+1=q$ or $q+1=p$. We consider three cases:

1. If $p=q$ then $x^2-px+p=0$ then we have

   $$\Delta = p^2-4p = k^2 \Leftrightarrow (p-2)^2-k^2=4.$$

   Hence, $(p-2-k)(p-2+k)=4$. Note that $(p-2-k)+(p-2+k)=2(p-2)$ then these numbers have the same parity, so we just need to consider $p-2-k=p-2+k=2$ or $p=4, k=0$. In this case, we have one pair satisfies the given condition $(p,q)=(4,4)$.

2. If $p+1=q$ then $x^2-px+p+1=0$, we have

   $$\Delta = p^2-4p-4 = k^2 \Leftrightarrow (p-2)^2-k^2=8.$$

   Hence $(p-2-k)(p-2+k)=8$. Similarly, these numbers have the same parity then $p-2-k=2, p-2+k=4$ or $p=5, k=5$. In this case, we have one pair satisfies the given condition $(p,q)=(5,6)$.

3. If $q+1=p$, similarly, we have $(p,q)=(6,5)$.

Therefore, $(p,q)=(4,4),(5,6),(6,5)$.

□

Problem 3. Let BC be a chord of a circle (O) such that BC is not a diameter. Let AE be a diameter perpendicular to BC such that A belongs to the larger arc BC of (O). Let D be a point on the larger arc BC of (O) which is different from A. Suppose that AD intersects BC at S, ED intersects BC at T. Let F be the midpoint of ST and I be the second intersection of the circle (ODF) with the line BC.

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1. Let the line passing I and parallel to OD intersect AD and ED at M and N respectively. Find the maximum value of the area of the triangle MDN when D moves on the larger arc BC of (O) (such that $D \neq A$).

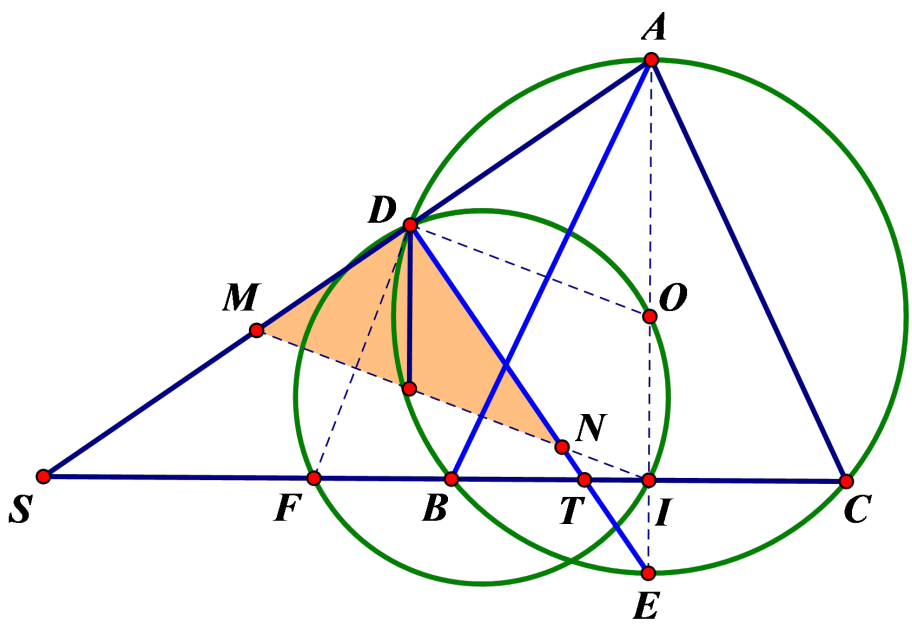
2. Prove that the perpendicular from D to ST passes through the midpoint of MN.

Solution.

1) First, note that $\angle ADE = 90^\circ$ then DO, DF are two medians of the triangles ADE, SDT.

Then $\angle ODF = \angle ODT + \angle FDT = \angle OET + \angle FTD = 90^\circ$. Hence, $\angle OIF = 180^\circ - \angle ODF = 90^\circ$ which implies that I is the midpoint of BC.

Since ODE is isosceles triangle then INE, IMA are also isosceles which implies that $IE = IN, IM = IA$. Hence $MN = IM - IN = IA - IE = const$.



Two triangles DMN and DAE are similar with the constant ratio, then to maximize the area of DMN, we have to maximize the area of triangle ADE. We have

$$[ADE] = \frac{1}{2} DA \cdot DE \leq \frac{DA^2 + DE^2}{2} = \frac{AE^2}{2}.$$

The equality occurs when $DA = DE$ or D lies on circle such that ADE is isosceles right triangle.

2) Denote P as the midpoint of MN then

$$\angle PDN = \angle PND = \angle INE = \angle IEN,$$

thus $DP \parallel AE$ or the perpendicular line from D to ST passes through the midpoint of MN.

□

Problem 4. Given a set S of 200 points on the plane: 100 points are the vertices of a convex polygon A, and 100 other points are in the interior of the polygon. Moreover, there does not exist 3 collinear points. A triangulation is a way to partition the polygon A into triangles by drawing the edges between some two points of S such that any two edges do not intersect in the interior, and each point in S is the vertices of at least one triangle.

1. Prove that number of edges does not depend on the triangulation.
2. Show that for any triangulation, one can draw each triangle by 1 of 3 given colors such that 2 adjacent triangles have different colors.

Solution.
1) Suppose that we have k triangles in some triangulation. By calculating the sum of all angles of these triangles, we have $180^\circ \cdot k$.

The sum of interior angles of A is $180^\circ \cdot 98$.

The sum of angle around each point among 100 points is $360^\circ \cdot 100$.

Hence, we have

$$180^\circ \cdot k = 180^\circ \cdot 98 + 360^\circ \cdot 100 \Leftrightarrow k = 298.$$

Each triangle gives 3 edges and among them, there are 100 edges of A. Note that the interior edges are doubled counted, then the number of edges in each triangulation is

$$\frac{3 \cdot 298 - 100}{2} + 100 = 497.$$

2) Fix the polygon A, we will prove this problem by induction on $m+n$ of total points, in which n is the number of vertices of A and m is the number of interior points.

For $m+n=3$, which implies that $n=3, m=0$, we just have one triangle with no interior point. We color this triangle by some color and finish this case.

For bigger $m+n$, whenever we add one more point and perform the triangulation, we have two cases:

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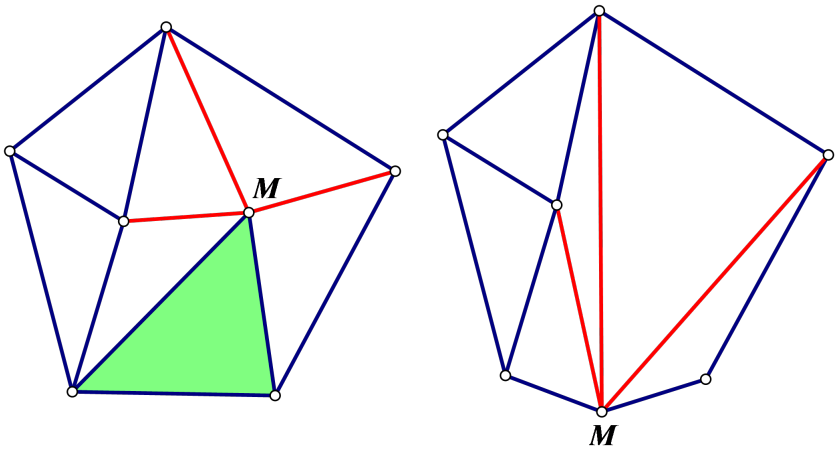
1. If there are some triangle that share exactly one edge with the polygon A, call triangle X. Then X is adjacent to at most two other triangles then if these two triangles was colored by the different color, we just need color X by the third color (in case two triangles was colored by the same color, we color X by random color among the rest). Since X has one vertex is the interior point, then we can move this point outside as the vertex of A. It is easy to see that polygon A still has n vertices and m decreases by 1. Then we can apply the induction hypothesis.



2. If for all triangles X as described above, the vertices of X are not the interior points then X is adjacent to exactly one triangle. We just color X by the color differs from the adjacent one and remove the vertex of A to make number of sides of A decreases by 2 and n decreases by 1. Then we can also apply the induction hypothesis.

Therefore, in all case, we can change the triangulation of $m+n$ points to triangulation of $m+n-1$ points, which means the problem can be solved by induction completely.
Then we always can draw the triangle by one of three colors that satisfy the given condition. □

2.3. Solution of Mock tests for BMO 2017

2.3.1. Day 1

Problem 1. Prove that there are infinitely many positive integer n such that $n!$ is divisible by n^3-1.

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Solution.
We have $n^3-1=(n-1)(n^2+n+1)$. We shall prove that there are infinitely many positive integers n such that n^2+n+1 can be represented as the product of some factors less than $n-1$ and then deduce that n^3-1 divides $n!$. In fact, we first choose $n=m^2$ for some positive integer m, one has

$$n^2+n+1=m^4+m^2+1=(m^2+1)^2-m^2=(m^2-m+1)(m^2+m+1).$$

Next, we choose $m=3k+1$, then

$$m^2+m+1=(3k+1)^2+(3k+1)+1=9k^2+6k+1+3k+1+1=3(3k^2+3k+1)$$

and

$$m^2-m+1=(3k+1)^2-(3k+1)+1=9k^2+6k+1-3k-1+1=9k^2+3k+1.$$

It is clear that

$$3<3k^2+3k+1<9k^2+3k+1<9k^2+6k=(3k+1)^2-1=n-1.$$

Therefore, for $n=(3k+1)^2$ with any positive integer k, it follows that $n!$ is divisible by

$$3(3k^2+3k+1)(9k^2+3k+1)(9k^2+6k)=n^3-1.$$

The problem is proved. □

Problem 2. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all function $f:\mathbb{R}^+\to\mathbb{R}$ such that, for all positive real number x and y, the following conditions are satisfied:

i) $2f(x)+2f(y)\le f(x+y)$.

ii) $(x+y)[yf(x)+xf(y)]\ge xyf(x+y)$.

Solution.
Replacing $x=y$ in i) and ii), we get $f(2x)=4f(x)$, $\forall x\in\mathbb{R}^+$. Therefore, by induction, we can prove that

$$f(2^nx)=4^nf(x),\quad \forall n\in\mathbb{Z}^+,\ x\in\mathbb{R}^+.$$

Now, by substituting $y=2x$ in i) and ii), we get $10f(x)\le f(3x)\le 9f(x)$, $\forall x\in\mathbb{R}^+$. It follows that $f(x)\le 0$, $\forall x\in\mathbb{R}^+$. Therefore, from ii), we have

$$xyf(x+y)\le(x+y)yf(x)\le xyf(x),$$

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or

$$f(x+y) \le f(x), \quad \forall x,\ y \in \mathbb{R}^+.$$

Next, we rewrite the condition ii) as

$$\frac{f(x+y)}{x+y} \le \frac{f(x)}{x} + \frac{f(y)}{y}.$$

By substituting $y = 2x,\ 3x,\ \ldots,\ nx$ $(n \in \mathbb{Z}^+)$, we can prove by induction that

$$f(nx) \le n^2 f(x), \quad \forall n \in \mathbb{Z}^+,\ x \in \mathbb{R}^+.\ (*)$$

From these inequalities, we get

$$\begin{aligned}\frac{2^n f(x)}{x} = \frac{f(2^n x)}{2^n x} &= \frac{f\left((2^n - n)x + nx\right)}{(2^n - n)x + nx} \\ &\le \frac{f\left((2^n - n)x\right)}{(2^n - n)x} + \frac{f(nx)}{nx} \\ &\le \frac{(2^n - n)f(x)}{x} + \frac{nf(x)}{x} = \frac{2^n f(x)}{x}\end{aligned}$$

for any positive integer n and any positive real number x. Therefore, the equality cases in $(*)$ must hold, i.e.

$$f(nx) = n^2 f(x), \quad \forall n \in \mathbb{Z}^+,\ x \in \mathbb{R}^+.$$

From this, we can easily prove that $f(x) = kx^2$, $\forall x \in \mathbb{Q}^+$ (where $k = f(1)$, and $k \le 0$). And then, by combining this result with $f(x+y) \le f(x), \forall x, y > 0$, we conclude that $f(x) = kx^2$, $\forall x \in \mathbb{R}^+$.

It is easy to check that this function satisfy the given conditions. □

Problem 3. Let ABC be an acute triangle and (O) be its circumcircle. Denote by H its orthocenter and I the midpoint of BC. The lines BH, CH intersect AC, AB at E, F respectively. The circles (IBF) and (ICE) meet again at D.

1. Prove that D, I, A are collinear and HD, EF, BC are concurrent.

2. Let L be the foot of the angle bisector of $\angle BAC$ on the side BC. The circle (ADL) intersects (O) again at K and intersects the line BC at S out of the side BC. Suppose that AK, AS intersects the circles (AEF) again at G, T respectively. Prove that $TG = TD$.

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Solution.

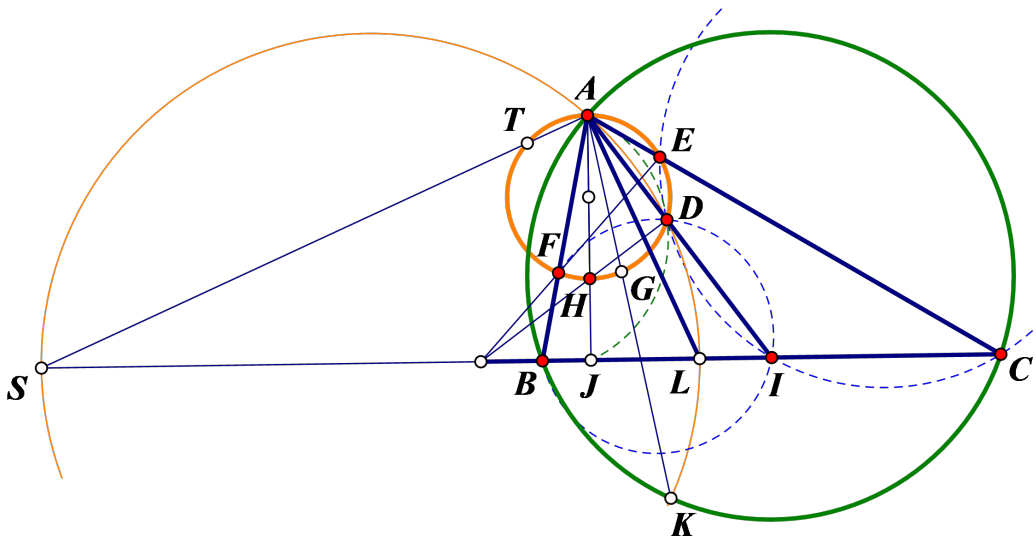
1) From the power of point to circle, we have

$$\mathscr{P}_{A/(BIF)} = AF \cdot AB \text{ and } \mathscr{P}_{A/(CIE)} = AE \cdot AC.$$

But it is easy to see that $AF \cdot AB = AE \cdot AC$ then A belongs to the radical axis of two circles $(BIF), (CIE)$.

Hence, A, D, I are collinear.

Denote $J = AH \cap BC$ then $AD \cdot AI = AH \cdot AJ$ which implies that H, D, I, J are concyclic, then $HD \perp AI$. Then by considering the radical axis of three circles $(AEF), (JFEI), (JHDI)$, we can conclude that BC, HD, EF are concurrent.



2) Using sine law in some triangle, we get

$$\frac{DB}{\sin\angle BID} = \frac{BI}{\sin\angle BDI} = \frac{BC}{2\sin\angle ABC}$$

and

$$\frac{DC}{\sin\angle CID} = \frac{CI}{\sin\angle CDI} = \frac{BC}{2\sin\angle ACB}.$$

Then

$$\frac{DB}{DC} = \frac{\sin\angle ACB}{\sin\angle ABC} = \frac{AB}{AC} = \frac{LB}{LC}.$$

This implies that (ADL) is the Apollonius circle of triangle ABC.

Thus $\frac{KB}{KC} = \frac{AB}{AC}$ or $ABKC$ is the harmonic quadrilateral, then AK is the symmedian of triangle ABC. Hence, AI, AK are isogonal conjugate in angle $\angle BAC$, also in angle $\angle EAF$. Since $G, D \in (AEF)$, we have $GD \parallel EF$.

Finally, AS is the external bisector of the angle $\angle BAC$, also of the angle $\angle EAF$ then T is the midpoint of major arc EF. From these, we can conclude that $TD = TG$.

□

Problem 4. Consider the set $X = \{1, 2, 3, \ldots, 2018\}$. How many positive integers k with $2 \le k \le 2017$ that satisfy the following conditions:

i) There exists some partition of the set X into 1009 disjoint pairs which are $(a_1, b_1), (a_2, b_2), \ldots, (a_{1009}, b_{1009})$ with $|a_i - b_i| \in \{1, k\}$.

ii) For all partitions satisfy the condition (i), the sum $T = \sum_{i=1}^{1009} |a_i - b_i|$ has the right most digit is 9.

Solution.
First, we notice that for all $2 \le k \le 2017$, we always can find a partition satisfy i). For example, we can choose $a_i = 2i - 1, b_i = 2i$ with $i = 1, 2, \ldots, 1009$.

Denote m as the number of pairs (a_i, b_i) with $|a_i - b_i| = k$ then

$$T = (1009 - m) \cdot 1 + m \cdot k = 1009 + (k - 1)m.$$

Hence, the necessary condition of k is $(k-1)m$ is divisible by 10. First, we can see that all number $k \equiv 1 \pmod 5$ are satisfy the given condition.

Indeed, we have some cases

1. If $k \equiv 1 \pmod{10}$ then $10 | k - 1$ and ii) is satisfied.

2. If $k \equiv 6 \pmod{10}$ then k is even. Thus, we need to prove m is even. This is true because we can write $m = m_1 + m_2$ with m_1, m_2 are the number of pairs with the difference is k that same parity and different parity. In X, the number of even and odd numbers are equal and all the pairs (a_i, b_i) with $|a_i - b_i| = 1$ also contain exactly 1 number in each type of even/odd. These imply that $m_1 = m_2$ or m is the even number.

   Hence, all number $k \equiv 1 \pmod 5$ satisfy the given condition.

   Next, we will prove that all other numbers k do not satisfy the given condition.

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3. If k is odd then $k-1$ is not divisible by 5. We choose the partition has $a_1 = 1, b_1 = k+1$ and the rest are divided into pairs with a_i, b_i are two consecutive numbers. It is easy to check that this partition is satisfy i) but not satisfy ii) because $T = 1008 + k \neq 9 \pmod{10}$.

4. If k is even, $k-1$ is not divisible by 5 then $k-1$ is an odd number that not divisible by 5. We choose the partition has exactly two pairs are $(1, k+1), (2, k+2)$ and the rest are divided into pairs with a_i, b_i are two consecutive numbers. Similarly with the previous case, we have $T = 1009 + 2(k-1) \neq 9 \pmod{10}$.

Therefore, the necessary and sufficient condition of k is $k \equiv 1 \pmod 5$. From 2 to 2017, we have 403 numbers in total satisfy it. □

2.3.2. Day 2

Problem 1. Let a, b, c be positive real numbers. Prove that

$$\frac{a(b^2+c^2)}{(b+c)(a^2+bc)} + \frac{b(c^2+a^2)}{(c+a)(b^2+ca)} + \frac{c(a^2+b^2)}{(a+b)(c^2+ab)} \geq \frac{3}{2}.$$

Solution.
Note that

$$(b+c)(a^2+bc) = a^2b + a^2c + b^2c + bc^2 = b(a^2+c^2) + c(a^2+b^2).$$

Then by putting $x = a(b^2+c^2), y = b(c^2+a^2), z = c(a^2+b^2)$ then

$$\frac{a(b^2+c^2)}{(b+c)(a^2+bc)} = \frac{x}{y+z} \text{ and } x, y, z > 0.$$

Similarly, we also have

$$\frac{b(c^2+a^2)}{(c+a)(b^2+ca)} = \frac{y}{z+x}, \frac{c(a^2+b^2)}{(a+b)(c^2+ab)} = \frac{z}{x+y}.$$

So we just need to prove $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$, which is true by Nesbit's inequality. The equality occurs when $x = y = z$ or $a = b = c$.

□

Problem 2. Solve the following equation in positive integers x, y:

$$x^{2017} - 1 = (x-1)(y^{2015} - 1).$$

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Solution.
It is clear that $x = 1$ is a solution of the problem (with y any positive integer). We now show that for $x > 1$, the given equation has no solution. In fact, suppose that there are $x > 1$ and y are positive integers satisfying the equation. Then, one has $y > 1$ and

$$x^{2016} + x^{2015} + \cdots + x + 1 = y^{2015} - 1 = (y-1)(y^{2014} + y^{2013} + \cdots + y + 1).$$

Put $p = 2017$, and note that p is a prime. Let q be a prime factor of $x^{p-1} + \cdots + x + 1$. Then $(x, q) = 1$ and by Fermat's little theorem we get $q \mid x^{q-1} - 1$. But $q \mid x^p - 1$, it follows that the order of x modulo q is a divisor of $(p, q-1)$ which is either 1 or p. This shows that $q \mid x - 1$ or $p \mid q - 1$.

If $q \mid x - 1$ then $x^{p-1} + \cdots + x + 1 \equiv p \pmod{q}$, i.e. $q \mid p$, this gives $p = q$. This means that any prime divisor of $x^{p-1} + \cdots + x + 1$ is congruent either to 0 or to $1 \pmod{p}$, hence any positive divisor of $x^{p-1} + \cdots + x + 1$ is too. In particular, $y - 1$ and $y^{2014} + y^{2013} + \cdots + y + 1$ (which are positive divisors of $x^{p-1} + \cdots + x + 1$) are congruent either to 0 or to $1 \pmod{p}$. But, if $p \mid y - 1$ then

$$y^{2014} + y^{2013} + \cdots + y + 1 \equiv 2015 \not\equiv 0, 1 \pmod{p},$$

we get a contradiction.
Hence, $y - 1 \equiv 1 \pmod{p}$, i.e. $y \equiv 2 \pmod{p}$, then

$$y^{2014} + y^{2013} + \cdots + y + 1 \equiv 2^{2014} + 2^{2013} + \cdots + 2 + 1 \equiv 2^{2015} - 1 \pmod{p}.$$

This shows that $2^{2015} - 1 \equiv 0 \pmod{p}$ or $2^{2015} - 1 \equiv 1 \pmod{p}$. That is, either $p \mid 2^{2015} - 1$ or $p \mid 2^{2014} - 1$. On the other hands, by Fermat's little theorem again, $p \mid 2^{p-1} - 1 = 2^{2016} - 1$. From this, we easily get $p \mid 1$ or $p \mid 3$ which is imposible. The problem is therefore solved. □

Problem 3. Let $ABCD$ be a cyclic quadrilateral and triangles ACD, BCD are acute. Suppose that the lines AB and CD meet at S. Denote by E the intersection of AC, BD. The circles (ADE) and (BCE) meet again at F.

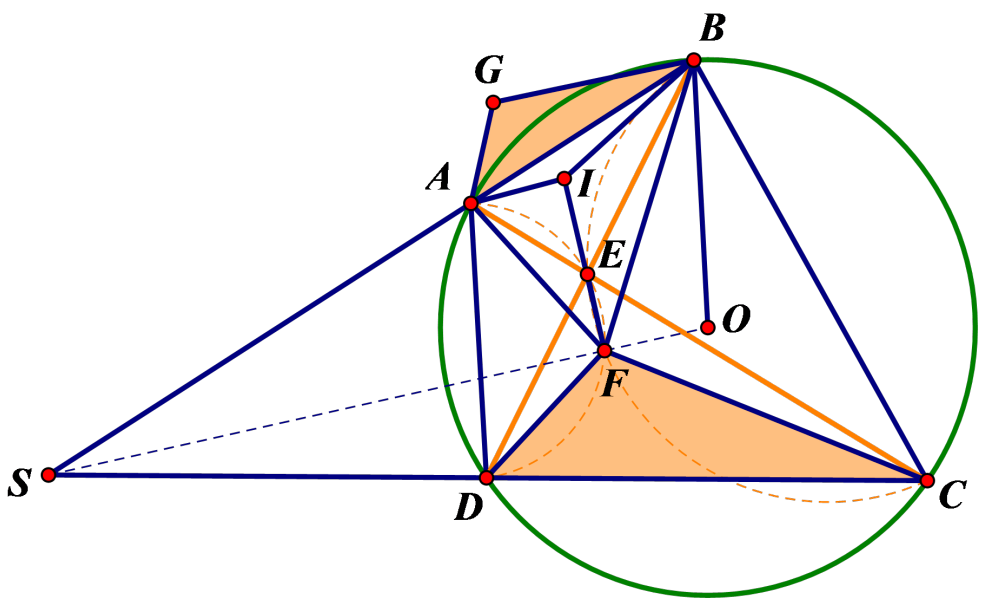
1. Prove that $SF \perp EF$.
2. The point G is taken out side of the quadrilateral $ABCD$ such that triangle GAB and FDC are similar. Prove that $GA + FB = GB + FA$.

Solution.
1) First, notice that $\angle AFB = \angle AFE + \angle BFE = \angle ADB + \angle ACB = \angle AOB$, then A, F, O, B are concyclic. Similarly, D, F, O, C are also concyclic.

By consider three radical axis of three circles $(O),(ABOF),(CDFO)$, we can see that three lines AB, CD, FO are concurrent at S which implies that S, F, O are collinear. We have

$$\begin{aligned}\angle EFO &= \angle AFO - \angle AFE = 180^\circ - \angle ABO - \angle ADB \\ &= 180^\circ - \angle ABO - \frac{1}{2}\angle AOB = 90^\circ.\end{aligned}$$

Then $SF \perp EF$.



2) Suppose that the angle bisectors of $\angle GAF, \angle GBF$ intersect at I. We shall prove that I, E, F are collinear. Indeed, we have

$$\begin{aligned}&\angle IAG = \angle IAF \Leftrightarrow \angle GAB + \angle IAB = \angle IAE + \angle EAF \\ &\Leftrightarrow \angle FDC + \angle EDF + \angle IAB = \angle IAE + 2\angle EAF \\ &\Leftrightarrow \angle EDC - \angle IAE + \angle IAB = 2\angle EAF \\ &\Leftrightarrow 2\angle IAB = 2\angle EAF \Leftrightarrow \angle IAB = \angle EAF\end{aligned}$$

Hence, IA, EA are isogonal conjugate in angle $\angle BAF$. Similarly, IB, EB isogonal conjugate in angle $\angle ABF$. Then IF, EF also are isogonal conjugate in angle $\angle AFB$. But EF is the bisector of angle $\angle AFB$ then I, E, F are collinear and IF is the angle bisector of $\angle AFB$.

In quadrilateral $GAFB$, we have three internal angle bisectors are concurrent at I then this quadrilateral is circumscribed, which means $GA + FB = GB + FA$.

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□

Problem 4. Let p be a prime number and a table of size $(p^2+p+1)\times(p^2+p+1)$ which is divided into unit cells. The way to color some cells of this table is called nice if there are no four colored cells that form a rectangle (the sides of rectangle are parallel to the sides of given table).

1. Let k be the number of colored cells in some nice coloring way. Prove that $k \le (p+1)(p^2+p+1)$. Denote this number as k_{max}.

2. Prove that all ordered tuples (a,b,c) with $0 \le a,b,c < p$ and $a+b+c > 0$ can be partitioned into p^2+p+1 sets $S_1, S_2, \ldots S_{p^2+p+1}$ such that two tuples (a_1,b_1,c_1) and (a_2,b_2,c_2) belong to the same set if and only if $a_1 \equiv ka_2, b_1 \equiv kb_2, c_1 \equiv kc_2 \pmod p$ for some $k \in \{1,2,3,\ldots,p-1\}$.

3. For $1 \le i,j \le p^2+p+1$, if there exist $(a_1,b_1,c_1) \in S_i$ and $(a_2,b_2,c_2) \in S_j$ such that $a_1a_2+b_1b_2+c_1c_2 \equiv 0 \pmod p$, we color the cell (i,j) of the given table. Prove that this coloring way is nice with k_{max} colored cells.

Solution.
1) We will count value S, number of the tuples (A,B,C) with column A,B and row C in relation $A-C, B-C$ intersect at the colored cell.

First way, denote x_i as the number of colored cells on i^{th} row, then

$$S = \sum_{i=1}^{p^2-p+1} \binom{x_i}{2}.$$

Second way, choose two columns among p^2+p+1 columns, we have $\binom{p^2+p+1}{2}$ way. And then, we have at most 1 row C that cuts A,B at two colored cells (since there are no rectangle) then

$$S \le \binom{p^2+p+1}{2}.$$

Notice that $\sum_{i=1}^{p^2+p+1} x_i = k$. Hence, we have

$$\binom{p^2+p+1}{2} \ge \sum_{i=1}^{p^2+p+1} \binom{x_i}{2} \ge \frac{1}{2}\left(\frac{1}{p^2+p+1}k^2 - k\right)$$

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or $k \leq (p+1)(p^2+p+1)$. 2) Consider the tuples (a,b,c) in which $0 \leq a,b,c \leq p-1$ and $a+b+c>0$, then we have p^3-1 tuples in total.

Since two tuples (a,b,c) andd (d,e,f) are considered belong to the same set if and only if $\exists k \in \{1,2,3,\ldots,p-1\}$ such that $a \equiv kd, b \equiv ke, c \equiv kf \pmod p$. We have $p-1$ ways to choose k then $\frac{p^3-1}{p-1} = p^2+p+1$ sets satisfy the given condition.

3) From the way to color the table, one can check that on each row, there are exactly $p+1$ colored cells. Indeed, we need to prove that $ax+by+cz \equiv 0 \pmod p$ has exactly $p+1$ solutions that not belong to the same set $S_1, S_2, \ldots, S_{p^2+p+1}$.

There are p^2-1 pairs (x,y) and for each pair, there is unique number z such that $ax+by+cz$ is divisible by p. Each set has $p-1$ solutions then there are $p+1$ solution that not belong to the same set. Multiply this number with the number of columns, we have verified that there are $(p+1)(p^2+p+1)$ colored cells on the table.

Then, we shall prove that there are no rectangle. For two tuples (a,b,c) and (d,e,f) do not belong to the same set, we need to prove that the following system of equations has no more than one solution

$$\begin{cases} ax+by+cz \equiv 0 \pmod p \\ dx+ey+fz \equiv 0 \pmod p \end{cases}$$

Without loss of generality, we can suppose that $\frac{b}{e} - \frac{c}{f}$ is not divisible by p or $bf \neq ce \pmod p$ then for all $x_0 = \overline{1, p-1}$, there is exactly one value y_0, z_0 such that

$$by_0 + cz_0 = ax_0, ey_0 + fz_0 = dx_0 \pmod p.$$

On the other hand, there are p value x_0 and we do not consider the tuple $(0,0,0)$ then there are at most $p-1$ solution.

Note that if (x,y,z) is a solution, then the same with (kx,ky,kz), then the above system has no more than one solution.

This finishes our proof and $k_{\max} = (p+1)(p^2+p+1)$.

□

2.4. Team Selection Tests for IMO 2017

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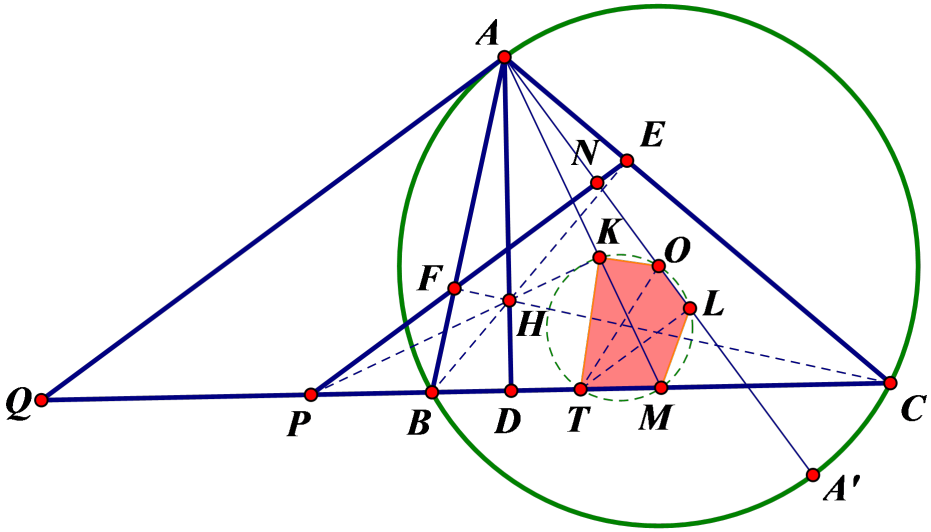
2.4.1. Day 1

Problem 1. Let ABC be an acute non isosceles triangle inscribed in circle (O), with its altitudes BE, CF intersect at orthocenter H $(E \in AC, F \in AB)$. Let M be the midpoint of BC, K be the orthogonal projection of H on AM. EF intersects BC at P. Let Q be the intersection of tangent of (O) which passes through A with BC, T be the reflection of Q through P. Prove that $\angle OKT = 90°$.

Solution.
Let AD be the altitude of triangle ABC, AA' be the diameter of (O). AO meets EF at N. Let L be the reflection of A through N.



We have $AO \perp EF$ then quadrilateral $NECA'$ is cyclic. Then

$$AO \cdot AL = \frac{1}{2}AA' \cdot 2AN = AA' \cdot AN = AE \cdot AC = AH \cdot AD = AK \cdot AM.$$

From this we obtain $KOLM$ is cyclic.

On the other side, $AQ \parallel EF$ and P is the midpoint of QT then $ALTQ$ is a trapezoid with its midline PN. We get

$$\angle LTC = \angle EPC = \angle HAO = \angle MOL.$$

Hence $OMTL$ is cyclic, which follows that O, K, M, T, L are concyclic or K lies on circle with diameter OT.

In other words, $\angle OKT = 90°$. □

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Problem 2. Let $\mathbb{R}$ be the set of real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the condition

$$f(xf(y)-y)+f(xy-x)+f(x+y)=2xy$$

for all $x,y\in\mathbb{R}$.

Solution.
Let denote by $P(x,y)$ the equation

$$f(xf(y)-y)+f(xy-x)+f(x+y)=2xy.$$

$P(0,y)$ gives us $f(-y)+f(y)=0,\forall y$. Thus f is an odd function.

$P(-1,y)$ follows

$$f(-f(y)-y)+f(-y+1)+f(-1+y)=-2y.$$

From this, since f is odd, we have $f(-f(y)-y)=-2y$ and thus $f(f(y)+y)=2y$. So f is surjective.

Since f is surjective, there is real number a such that $f(a)=-1$.

$P(x,a)$ gives us

$$f(-x-a)+f(ax-x)+f(x+a)=2ax.$$

From this, again since f is odd we have $f(ax-x)=2ax$ for all $x\in\mathbb{R}$. $(*)$
If $a=1$ then from above equation, we have $f(0)=2x,\forall x\in\mathbb{R}$, which is a contradiction. Thus, $a\neq 1$. Replace $x=\frac{t}{a-1}$ in $(*)$, we have

$$f(t)=\frac{2a}{a-1}t=ct \text{ with } c=\frac{2a}{a-1}.$$

Replace back into the original equation, we have

$$c(cxy-y)+c(xy-x)+c(x+y)=2xy.$$

From which $(c^2+c-2)xy=0,\forall x,y\in\mathbb{R}$ which implies that $c\in\{1,-2\}$.

So $f(x)=x$ or $f(x)=-2x$ for all $x\in\mathbb{R}$.

It is easy to check that these two functions satisfy the condition. □

Problem 3. Prove that there are infinitely many positive integers n such that n divides $2017^{2017^n-1}-1$ but n does not divide 2017^n-1.

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Solution.
We will prove that if $a > 1$ is a positive integer such that $a-1$ is not a power of 2, then there are infinitely many positive integers n such that n divides $a^{a^n-1}-1$ but n does not divide a^n-1. Therefore, the given problem follows directly as an application with $a = 2017$.

In fact, there is a odd prime divisor p of $a-1$. Put $s = v_p(a-1) \geq 1$, for any positive integer k, by LTE we have $v_p(a^{p^k}-1) = v_p(a-1)+k$, this means that $n = \dfrac{a^{p^k}-1}{p^{s+1}}$ is a positive integer. By LTE again,

$$v_p(a^n-1) = v_p(a-1)+v_p(n) = s+v_p(a-1)+k-(s+1) = s+k-1 \geq k.$$

Thus, $p^k \mid a^n-1$, we get $n \mid a^{p^k}-1 \mid a^{a^n-1}-1$. It remains to show that $n \mid a^n-1$. By above, we can write $a^n-1 = p^{s+k-1}A$ and $n = p^{k-1}B$ for some positive integers A, B coprime to p. This gives

$$(n, a^n-1) = (p^{k-1}A, p^{k-1}B) = \frac{1}{p^s}(p^{s+k-1}A, p^{s+k-1}B) = \frac{1}{p^s}(\frac{a^{p^k}-1}{p}, a^n-1)$$

which is clearly a divisor of

$$\frac{1}{p^s}(a^{p^k}-1, a^n-1) = \frac{1}{p^s}(a^{(p^k,n)}-1) = \frac{1}{p^s}(a^{p^{k-1}}-1) < \frac{a^{p^k}-1}{p^{s+1}} = n.$$

This shows that n cannot divide a^n-1. The proof is done. □

2.4.2. Day 2

Problem 1. In the garden of Wonderland, there are 2016 apples, 2017 bananas and 2018 oranges. Two monkeys Adu and Bakar play the following game: alternatively each of them takes and eats one fruit of any kind except for the one that he took in previous turn (in the first turn, each of them can take a fruit of any kind). Who can not take a fruit is the loser. Which monkey has the winning strategy if Adu plays first?

Solution.
Adu has the strategy to win the game.

First, he takes one banana then we have 2016 apples, 2016 bananas and 2018 oranges. In the next turns, we have some cases:

1. If Bakar takes apple or orange, then Adu takes the same. We can see that the number of fruits in these type are always even, then whenever Bakar can takes, so does Adu.

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2. If on the next turn, Bakar takes banana, then Adu cannot takes banana anymore base on the constraint. Then Adu takes apple. So the number of apples, bananas and oranges are $2015, 2015, 2018$. Then continue, Adu will play with following strategy:

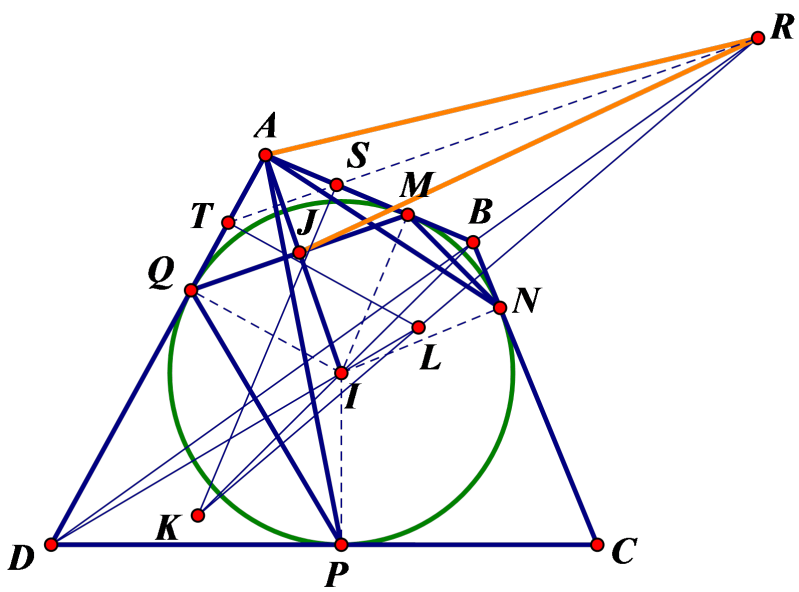
   - Bakar takes banana then Adu takes apple.
   - Bakar takes apple then Adu takes banana.
   - Bakar takes orange then Adu takes orange.

   Similar with the first case, Adu always can perform his turn.

Then in all cases, Adu has the strategy to win this game. □

Problem 2. Let $ABCD$ be the circumscribed quadrilateral with the incircle (I). The circle (I) touches AB, BC, CD, DA at M, N, P, Q respectively. Let K and L be the circumcenters of the triangles AMN and APQ respectively. The line KL cuts the line BD at R. The line AI cuts the line MQ at J. Prove that $RA = RJ$.

Solution.
Denote S, T as the midpoints of AM, AQ respectively. Hence, KS is the perpendicular bisector of AM and TL is the perpendicular bisector of AQ.



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These imply that AI, SK, TL are concurrent at the circumcenter G of triangle AMQ.

Note that BI is the perpendicular bisector of MN then $K \in BI$, DI is the perpendicular bisector of PQ then $L \in DI$.

Consider two triangles IBD and GST with BS, DT, IG are concurrent at A then by applying the Desargues theorem, we have S, T, R are collinear.

In other word, R belongs to ST which is the perpendicular bisector of AJ then $RA = RJ$. □

Problem 3. Find the greatest positive real number M such that for all positive real sequence (a_n) and for all real number $m < M$, it is possible to find some index $n \geq 1$ that satisfies the inequality $a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} > ma_n$.

Solution.
First, consider the sequence $a_n = 2^n, n = 1, 2, 3, \ldots$ and

$$a_1 + a_2 + a_3 + \cdots + a_{n+1} = 2^1 + 2^2 + 2^3 + \cdots + 2^{n+1} = 4 \cdot 2^n - 1.$$

So for all $m < M$, we need to have $4 \cdot 2^n - 1 > m \cdot 2^n$ for some n. It is equivalent to $m < \frac{4 \cdot 2^n - 1}{2^n} = 4 - \frac{1}{2^n}$.

If $M > 4$ then we choose $m = 4 < M$, but the above inequality cannot be satisfied for all n. This implies that $M \leq 4$. We shall prove that $M = 4$ satisfy the given condition.

Consider some positive real sequences (a_n) and some value $m < 4$. If $m \leq 0$, it's obviously to see that the given inequality is true. So we need to check with $m > 0$.

Suppose on contrary that $a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} \leq ma_n$ for all n. Consider $b_n = \sum_{i=1}^{n} a_i$ then b_n is strictly increasing (since $a_n > 0$ for all n). The above inequality can be written as

$$b_{n+1} \leq m(b_n - b_{n-1}) \Leftrightarrow b_{n+1} + mb_{n-1} < mb_n \Leftrightarrow \frac{b_{n+1}}{b_n\sqrt{m}} + \frac{b_{n-1}\sqrt{m}}{b_n} < \sqrt{m}.$$

Put $c_n = \frac{b_{n+1}}{b_n\sqrt{m}}$ for all $n \geq 1$ then we have $\frac{1}{c_n} + c_{n-1} < \sqrt{m}$ for all $n > 1$. From this, we can see that

$$n\sqrt{m} > \sum_{i=1}^{n}\left(\frac{1}{c_i} + c_{i-1}\right) > \sum_{i=1}^{n-1}\left(\frac{1}{c_i} + c_i\right) + \frac{1}{c_n} > \sum_{i=1}^{n-1} 2 = 2(n-1).$$

This implies that $\sqrt{m} > 2 - \frac{2}{n}$ for all $n \geq 1$. Since $m < 4$, this inequality will be wrong for n is big enough, contradiction. Therefore, the greatest value of M is 4. □

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2.4.3. Day 3

Problem 1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2017$, the integer $P(n)$ is positive and $S(P(n)) = P(S(n))$.

Solution.
We consider the degree of polynomial P:

Case 1: If $\deg P = 0$ then $P(x) \equiv c$ for some $c \in \mathbb{Z}$, the given condition becomes $S(c) = c$ which holds if and only if $1 \leq c \leq 9$.

Case 2: If $\deg P = 1$. We notice that $S(m+n) \leq S(m)+S(n)$ for all positive integers m, n and the equality occurs when there is no carry in the addition $m+n$. Let $P(x) = ax+b$ for $a, b \in \mathbb{Z}$ and $a \neq 0$. Since $S(P(n)) = P(S(n))$ for $n \geq 2017$ then a must be positive. The given condition becomes $S(an+b) = aS(n)+b$ for $n \geq 2017$. Putting $n = 2025$ and $n = 2020$, we get

$$S(2025a+b) - S(2020a+b) = (aS(2025)+b) - (aS(2020)+b) = 5a.$$

We also have

$$S(2025a+b) = S((2020a+b)+5a) \leq S(2020a+b) + S(5a).$$

These imply that $S(5a) \geq 5a$, as $a \geq 1$, this holds only when $a = 1$. Then we have $S(n+1+b) - S(n+b) = S(n+1) - S(n)$.

1. If $b > 0$ then choose n such that $n+1+b = 10^k$ for some k big enough then all digits of $n+b$ are 9 then the left hand side is $1-9k$. Also note that n is a positive integer less than 10^k-1, then $S(n) < 9k$, which means $S(n+1) - S(n) \geq 1-(9k-1) = 2-9k > 1-9k$, a contradiction.

2. If $b < 0$, similarly, we also come to a contradiction.

Thus $b = 0$ and $P(x) = x$, which is trivially satisfied.

Case 3: If $\deg P = d \geq 2$ then the leading term of P is $a_d n^d$ with $a_d \neq 0$, then similarly we have $a_d > 0$. Choose $n = 10^k - 1$ for some $k \in \mathbb{Z}^+$ big enough, then $S(P(n)) = P(9k)$.

Since $P(9k)$ grows approximately as $(9k)^d$ while $S(P(n))$ grows approximately as a constant multiple of k, the given equality cannot hold for sufficiently large k since $d \geq 2$.

In conclusion, $P(x) = c$ for some $c \in \{1, 2, 3, \ldots, 9\}$ or $P(x) = x, \forall x$.

□

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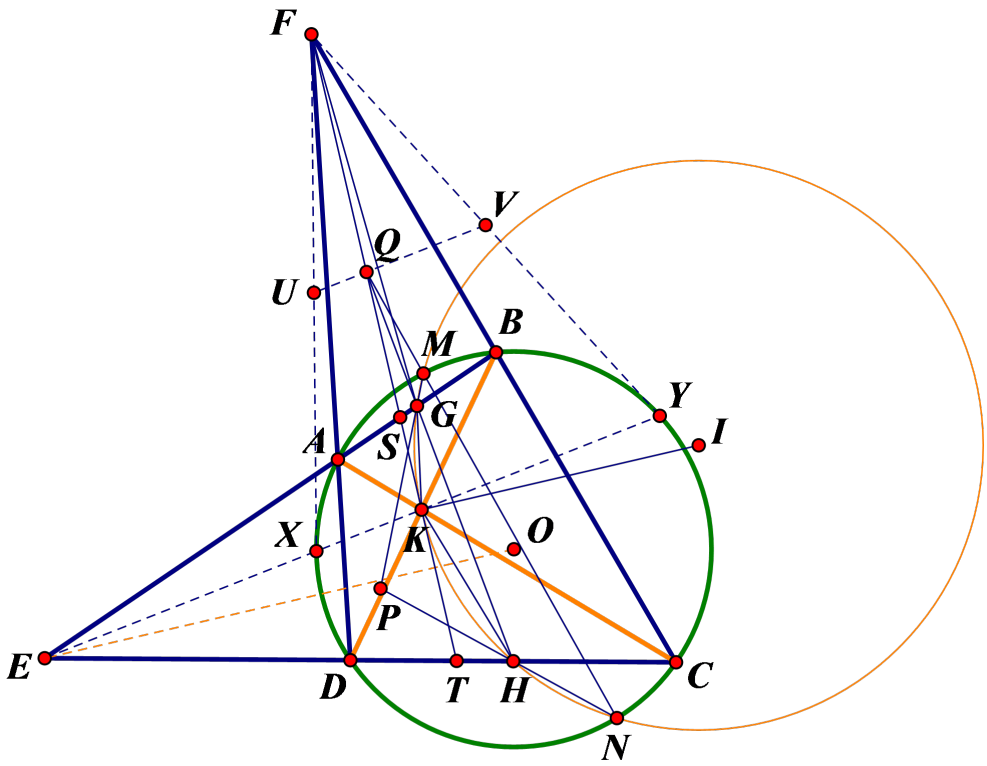
Problem 2. Let $ABCD$ be a quadrilateral inscribed a circle (O). Assume that AB and CD intersect at E,AC and BD intersect at K, and O does not belong to the line KE. Let G and H be the midpoints of AB and CD respectively. Let (I) be the circumcircle of the triangle GKH. Let (I) and (O) intersect at M,N such that $MGHN$ is convex quadrilateral. Let P be the intersection of MG and HN,Q be the intersection of MN and GH.

1. Prove that IK and OE are parallel.
2. Prove that PK is perpendicular to IQ.

Solution.
Denote F as the intersection of AD,BC. Suppose that FK intersects AB,CD at S,T respectively. Since the harmonic points of complete quadrilateral, we have $(E,T,D,C)=-1$. Then $ET\cdot EH=ED\cdot EC$. Similarly, $ES\cdot EG=EA\cdot EB$. But $EA\cdot EB=ED\cdot EC$ then $ES\cdot EG=ET\cdot EH$ which implies that G,H,T,S are concyclic.



Denote Q' as the midpoint of FK then Q',G,H belong to the Gauss line of the complete quadrilateral $AFBKCD$. We shall prove that Q' belongs to the radical axis of $(O),(I)$.

It is easy to see that EK is the antipole of F then denote $EK \cap (O) = \{X, Y\}$ then FX, FY are the tangent lines of (O). Let U, V be midpoints of FX, FY.

We consider the power of point to circle (O) and degenerate circle F.

$$\mathscr{P}_{U/(O)} = UX^2, \mathscr{P}_{U/(F)} = UF^2$$

but $UX = UF$ then $\mathscr{P}_{U/(O)} = \mathscr{P}_{U/(F)}$. This implies that U belongs to the radical axis of $(O), (F)$. Similarly with the point V then UV is the radical axis of $(O), (F)$. But $Q' \in UV$ which is the midline of triangle FXY, then

$$\mathscr{P}_{Q'/(F)} = \mathscr{P}_{Q'/(O)} \Leftrightarrow \mathscr{P}_{Q'/(O)} = Q'F^2 = Q'K^2 = Q'S \cdot Q'T = Q'G \cdot Q'H = \mathscr{P}_{Q'/(I)}.$$

Then Q' belongs to the radical axis of $(I), (O)$ which means $Q' \in MN$ or $Q' \equiv Q$.

Thus $QK^2 = \mathscr{P}_{Q/(I)}$ implies that QK is the tangent line of (I). Hence, $QK \perp KI$.

Using the Brocard's theorem, we have K is the orthocenter of triangle OEF then $EO \perp FK$. Combining all these results, we get $EO \parallel IK$.

2) Two lines pass through Q and intersect (I) at M, N and G, H then P is the intersection of MG, NH which means P belongs to the antipole of Q.

Note that QK is tangent to (I) which means K also belong to the antipole of the pole Q respect to circle (I).

Then PK is the antipole of Q respect to the circle (I). This implies that $PK \perp IQ$. □

Problem 3. The 64 cells of an 8×8 chessboard have 64 different colours. A Knight stays in one cell. In each move, the Knight jumps from one cell to another cell (the 2 cells on the diagonal of an 2×3 board) also the colours of the 2 cells interchange. In the end, the Knight goes to a cell having common side with the cell it stays at first. Can it happen that: there are exactly 3 cells having the colours different from the original colours?

Solution.
The answer is no. Suppose that there are exactly 3 cells having the colours different from the original colours.

Assign each colour with an integer from 1 to 64 and colour again the chessboard by black and white as usual. Arrange the numbers from left to right and up to down, so each move gives us a new permutation of the set $S = \{1, 2, ..., 64\}$.

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In each permutation, we call a pair (x, y) *bad* if x is in the left of y but $x > y$.
Claim 1. Each move interchanges 2 numbers in the old permutation, and the number of bad pairs changes the parity.

Claim 2. Each move the Knight goes from a black cell to a white cell or vice versa.

The first cell and the last cell the Knight stays have colours black and white different. So by claim 2, there were totally an odd number of moves. By claim 1, the number of bad pairs in the first permutation and the number of bad pairs in the last permutation have different parities.

On the other hand, there are exactly 3 positions change from the first permutation to the last permutation. We can check to see that, the difference of the numbers of bad pairs are always 2. That means the number of bad pairs in the first permutation and the number of bad pairs in the last permutation have the same parity. A contradiction. □

2.4.4. Day 4

Problem 1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Solution.
Let $f(x) = 1 + x^2$. The inequality can be rewritten as

$$f(a)f(b)f(c) \leq f^3\left(\frac{a+b+c}{3}\right).$$

with the condition $ab, bc, ca \geq 1$. First, we shall prove the following lemma

Lemma. If $ab \geq 1$ then

$$f(a)f(b) \leq f^2\left(\frac{a+b}{2}\right).$$

Proof. We have

$$\left(1+\left(\frac{a+b}{2}\right)^2\right)^2 - (1+a^2)(1+b^2) \geq 0 \Leftrightarrow (a-b)^2((a+b)^2+4ab-8) \geq 0.$$

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Since $(a+b)^2+4ab \geq 8ab \geq 8$ then last inequality holds.

Now, let $c \geq b \geq a$ then $c \geq 1$ and $c \cdot \frac{a+b+c}{3} = \frac{ca+cb+c^2}{3} \geq 1$ and we can apply lemma for $c, \frac{a+b+c}{3}$ to get

$$f(c)f(\frac{a+b+c}{3}) \leq f^2\left(\frac{4c+a+b}{6}\right).$$

From here we have

$$f(a)f(b)f(c)\left(\frac{a+b+c}{3}\right) \leq f^2\left(\frac{a+b}{2}\right)f^2\left(\frac{4c+a+b}{6}\right).$$

since

$$\frac{a+b}{2}.\frac{4c+a+b}{6} = \frac{4ac+4bc+(a+b)^2}{12} \geq 1.$$

We can apply lemma for $\frac{a+b}{2}, \frac{4c+a+b}{6}$ to get

$$f\left(\frac{a+b}{2}\right)f\left(\frac{4c+a+b}{6}\right) \leq f^2\left(\frac{\frac{a+b}{2}+\frac{4c+a+b}{6}}{2}\right) = f^2\left(\frac{a+b+c}{3}\right).$$

From these results, we get done.

□

Problem 2. Denote by $\{x\}$ the fractional part of a real number x, that is $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the maximum integer not greater than$x$. Prove that

1. For every integer n, we have $\left\{n\sqrt{17}\right\} > \frac{1}{2\sqrt{17}\cdot n}$.
2. The value $\frac{1}{2\sqrt{17}}$ is the largest constant c such that the inequality $\left\{n\sqrt{17}\right\} > cn$ holds for all positive integers n.

Solution.

1) For all $n \in \mathbb{Z}^+$, we have $n\sqrt{17} \notin \mathbb{Z}$ then $\left[n\sqrt{17}\right] < n\sqrt{17}$ or

$$\left[n\sqrt{17}\right]^2 < (n\sqrt{17})^2 \forall n.$$

This implies that

$$\begin{aligned}
&17n^2 - \left[n\sqrt{17}\right]^2 \geq 1 \\
&\Leftrightarrow 17n^2 - \left(n\sqrt{17} - \left\{n\sqrt{17}\right\}\right)^2 \geq 1 \\
&\Leftrightarrow 17n^2 - \left(17n^2 - 2n\sqrt{17}\left\{n\sqrt{17}\right\} + \left\{n\sqrt{17}\right\}^2\right) \geq 1 \\
&\Leftrightarrow \left\{n\sqrt{17}\right\} \geq \frac{1+\left\{n\sqrt{17}\right\}^2}{2n\sqrt{17}} > \frac{1}{2n\sqrt{17}}
\end{aligned}$$

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2) Consider the Pell equation $m^2-17n^2=-1$, since 17 is the prime of form $4k+1$ then this equation has infinitely many positive integer solutions.

Thus $n\sqrt{17}=\sqrt{m^2+1}$ then $\left[n\sqrt{17}\right]=m$ for all $m,n\in\mathbb{Z}^+$.

Then we have $\left\{n\sqrt{17}\right\}=n\sqrt{17}-m$ which mean

$$n\sqrt{17}-m>\frac{c}{n}\Leftrightarrow\frac{17n^2-m^2}{n\sqrt{17}+m}>\frac{c}{n}\Leftrightarrow c<\frac{1}{\sqrt{17}+\frac{m}{n}}.$$

Note that $\frac{m}{n}=\sqrt{17-\frac{1}{n^2}}$ then when $m,n\to+\infty$, we have $\frac{m}{n}\to\lim\sqrt{17-\frac{1}{n^2}}=\sqrt{17}$. This implies that

$$c\le\frac{1}{\sqrt{17}+\lim\limits_{m,n\to+\infty}\frac{m}{n}}=\frac{1}{2\sqrt{17}}.$$

Hence, $c=\frac{1}{2\sqrt{17}}$ is the maximum constant that satisfies the given condition. □

Problem 3. For integer $n>1$, consider n cube polynomials $P_1(x),\dots,P_n(x)$ such that each polynomial has 3 distinct real roots. Denote S as the set of roots of following equation

$$P_1(x)P_2(x)P_3(x)\dots P_n(x)=0.$$

It is also known that for each $1\le i<j\le n$, $P_i(x)P_j(x)=0$ has 5 distinct real roots.

1. Prove that if for each $a,b\in S$, there is exactly one $i\in\{1,2,3,\dots,n\}$ such that $P_i(a)=P_i(b)=0$ then $n=7$.
2. Prove that if $n>7$ then $|S|=2n+1$.

Solution.

1) For $i\in\{1,2,3,\dots,n\}$, denote A_i as the set of roots of $P_i(x)$ then since the given condition, we can see that $|A_i|=3$ and for all $i,j\in\{1,2,3,\dots,n\}$ then $\left|A_i\cup A_j\right|=5\Leftrightarrow\left|A_i\cap A_j\right|=1$.

Suppose that $|S|=m$ and $S=\{a_1,a_2,\dots,a_m\}$. Denote k as the number of set A_i that contains a_1 then we will count $\#(u,v)$ in which $u\in S,v\in\{1,2,3,\dots,n\},u\ne a_1$ and $u\in A_v,a_1\in A_v$.

- Choose u first, we have $\#(u,v)=(m-1)\cdot 1$.
- Choose v first, we have $\#(u,v)=k\cdot 2$.

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Then $2k = m-1 \Leftrightarrow k = \frac{m-1}{2}$. This argument can be applied the same for $a_2, a_3, \ldots, a_m$ so for all $a_i \in S$, there are exactly $\frac{m-1}{2}$ sets A_i that contain a_i.

Then by counting $\#\left(a_i, a_j, A_k\right)$ such that $a_i \in A_k, a_j \in A_k$ we have

$$\binom{m}{2} = n\binom{3}{2} \Leftrightarrow m(m-1) = 3n.$$

By counting $\#\left(A_i, A_j, a_k\right)$ such that $a_k \in A_i, a_k \in A_j$ we have

$$\binom{n}{2} = m\binom{(m-1)/2}{2} \Leftrightarrow n(n-1) = m \cdot \frac{m-1}{2} \cdot \frac{m-3}{2}.$$

By solving this system of equation, we have $m = n = 7$.

2) For $n > 7$, denote $A_1 = \{a, b, c\}$ and by pigeonhole principle, there some element of A_1 that appears in at least 3 sets among $A_2, A_3, \ldots, A_n$. Without loss of generality, we can suppose that $a \in A_2, A_3, A_4$. We shall prove that $a \in A_i$ for all $i \in \{1, 2, 3, \ldots, n\}$. If there some index $k \in \{5, 6, \ldots, n\}$ with $a \notin A_k$ then we can write $A_k \cap A_i = \{x_i\}$ for $i = 1, 2, 3, 4$.

It is clearly to check that $x_i \neq a$ for all $i = 1, 2, 3, 4$ and $x_i \neq x_j$ for $i \neq j$ since all set A_1, A_2, A_3, A_4 contain a. This implies that $|A_k| \geq 4$, which is a contradiction.

Hence, all sets $A_1, A_2, \ldots, A_n$ share one common element a. This means

$$|S| = |A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = 1 + n(3-1) = 2n + 1.$$

In general, we can state this problem as follows:

For integers $m, n, k > 1$, we consider $S = \{1, 2, 3, \ldots, m\}$ and n subsets $A_1, A_2, \ldots, A_n$ of S such that $|A_i| = k$ for $i = \overline{1, n}$. Suppose that $\left|A_i \cup A_j\right| = 1$ for all $i \neq j$.

1. If for each $a, b \in S$, there is exactly 1 subset A_i such that $a, b \in A_i$ then $m = n = k^2 - k + 1$.

2. If $n > k^2 - k + 1$ then there is exactly 1 element $a \in S$ that appears in all sets $A_1, A_2, \ldots, A_n$ and $m = (k-1)n + 1$.

The way to solve this problem is the same as solution for original problem. □

We do not include the solutions for training the IMO and JBMO team in this booklet.

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Ministry of Education

مؤسسة الملك عبدالعزيز ورجاله للموهبة والإبداع
King Abdulaziz & his Companions Foundation For Giftedness & Creativity

موهبة

مسابقات الرياضيات للمملكة العربية السعودية

SAUDI ARABIAN
Mathematical Competitions
2018

SAMC 2018

Riyadh, June 2018

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جامعة الملك عبدالله
للعلوم والتقنية
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The Saudi Arabian team at **IMO 2018**

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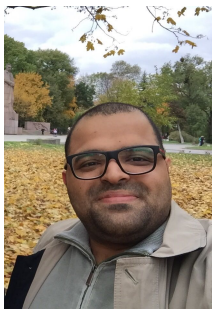
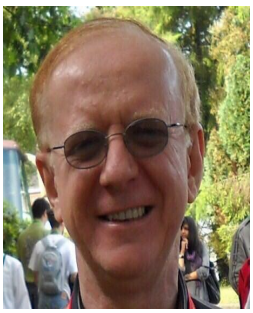
Team Training Administrators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مقدمة

المقدمة

يحوي هذا الكتيب على مسائل التصفيات لمسابقة البلقان و مسابقة البلقان للناشئين و تصفيات الاولمبياد الدولي للرياضيات ٢٠١٨.

ان تدريب الفريق كان بدعم من وزارة التعليم بالتعاون مع مؤسسة الملك عبد العزيز و رجاله للموهبة و الابداع " موهبة "

وتجدر الاشارة الى التعاون الفعّال من جامعة الملك عبدالله للعلوم و التقنية (كاوست)، حيث وفرت لنا كل الامكانات التي احتجنا لها في التدريب في حرمها الجامعي و كذلك الهيئة الملكية بالجبيل و ينبع و جامعة نورة.

كما تم عقد ثلاثة ملتقيات تدريبية خلال العام الدراس ٢٠١٧-٢٠١٨ ، بالاضافة الى فترة التدريب المكثف التي بدأت في شهر ابريل ٢٠١٨ الى نهاية شهر يونيو. كما شارك الطلبة المتميزون في العديد من المسابقات الإقليمية و منها: اولمبياد الرياضيات لدول آسيا و الباسيفيك ، اولمبياد الطالبات للدول الأوربية في إيطاليا ، اولمبياد البلقان في صربيا و اولمبياد الناشئين لدول البلقان في اليونان.

نأمل ان يكون محتوى هذا الكتيب إسهاماً منا لتقوية أواصر التعاون و تبادل الخبرات بيننا و الدول المشاركة في الاولمبياد الدولي.

د. فوزي بن أحمد الذكير

رئيس الفريق السعودي للاولمبياد الدولي ٢٠١٨

Saudi Arabian Mathematical Competitions 2018

Introduction

This booklet contains the Team Selection Tests of the Saudi teams to the Balkan Mathematics Olympiad, Balkan Junior Mathematics Olympiad, and the International mathematics Olympiad.

The training was supported by the Ministry of Education, which commissioned Mawhiba, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. We would like to express our gratitude to King Abdullah University of Science and Technology KAUST for making its facilities in its beautiful campus available to us for our training.

The Saudi team had three main training camps during the academic year 2017-2018. In addition, the team had an intensive training period in the period, April to the end of June 2018. During this academic year, the selected students participated in the following contests: The Asia Pacific Mathematics Olympiad, the European Girls Mathematics Olympiad in Italy, Balkan Mathematics Olympiad in Serbia and the Junior Balkan mathematics Olympiad in Greece.

It is our pleasure to share the selection tests problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Fawzi A Al-Thukair
Leader of the Saudi team in IMO 2018

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Part 1

PROBLEM STATEMENTS

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1.1. Team selection tests for GMO 2018

The first day

Problem 1. Let $\{x_n\}$ be a sequence defined by $x_1 = 2$ and

$$x_{n+1} = x_n^2 - x_n + 1$$

for $n \geq 1$. Prove that

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}$$

for all n.

Problem 3. Let p be a prime number of the form $9k+1$. Show that there exists an integer n such that $p \mid n^3 - 3n + 1$.

Problem 4. Let I, O be the incenter, circumcenter of triangle ABC and A_1, B_1, C_1 be arbitrary points on the segments AI, BI, CI respectively. The perpendicular bisectors of AA_1, BB_1, CC_1 intersect each other at X, Y and Z. Prove that the circumcenter of triangle XYZ coincides with O if and only if I is the orthocenter of triangle $A_1B_1C_1$.

Problem 4. In each of the cells of a 13×13 board is written an integer such that the integers in adjacent cells differ by 1. If there are two 2s and two 24s on this board, how many 13s can there be?

The second day

Problem 1. Let n be an odd positive integer with $n > 1$ and let $a_1, a_2, \ldots, a_n$ be positive integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Let $d = \gcd(a_1^n + a_1 \cdot a_2 \cdots a_n, a_2^n + a_1 \cdot a_2 \cdots a_n, \ldots, a_n^n + a_1 \cdot a_2 \cdots a_n)$. Show that the possible values of d are $d = 1, d = 2$.

Problem 2. Two positive integers m and n are called *similar* if one of them can be obtained from the other one by swapping two digits (note that a 0-digit cannot be swapped with the leading digit). Find the greatest integer N such that N is divisible by 13 and any number similar to N is not divisible by 13.

Problem 3. Let C be a point lies outside the circle (O) and CS, CT are tangent lines of (O). Take two points A, B on (O) with M is the midpoint of

the minor arc AB such that A, B, M differ from S, T. Suppose that MS, MT cut line AB at E, F. Take $X \in OS$ and $Y \in OT$ such that EX, FY are perpendicular to AB. Prove that XY and CM are perpendicular.

Problem 4. In a graph with 8 vertices that contains no cycle of length 4, at most how many edges can there be?

1.2. Team selection tests for BMO 2018

The first day

Problem 1. Find the smallest positive integer n which can not be expressed as $n = \dfrac{2^a - 2^b}{2^c - 2^d}$ for some positive integers a, b, c, d.

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(2x^3 + f(y)\right) = y + 2x^2 f(x),$$

for all real numbers x, y.

Problem 3. The partition of $2n$ positive integers into n pairs is called square-free if the product of numbers in each pair is not a perfect square. Prove that if for $2n$ distinct positive integers, there exists one square-free partition, then there exists at least $n!$ square-free partitions.

Problem 4. Let ABC be an acute, non isosceles with I is its incenter. Denote D, E as tangent points of (I) on AB, AC, respectively. The median segments respect to vertex A of triangles ABE and ACD meet (I) at P, Q, respectively. Take points M, N on the line DE such that $AM \perp BE$ and $AN \perp CD$ respectively.

1. Prove that A lies on the radical axis of (MIP) and (NIQ).

2. Suppose that the orthocenter H of triangle ABC lies on (I). Prove that there exists a line which is tangent to three circles of center A, B, C and all pass through H.

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The second day

Problem 1. Let ABC be a triangle with M, N, P as midpoints of the segments BC, CA, AB respectively. Suppose that I is the intersection of angle bisectors of $\angle BPM, \angle MNP$ and J is the intersection of angle bisectors of $\angle CNM, \angle MPN$. Denote (ω_1)as the circle of center I and tangent to MP at D, (ω_2)as the circle of center J and tangent to MN at E.

1. Prove that DE is parallel to BC.
2. Prove that the radical axis of two circles $(\omega_1), (\omega_2)$ bisects the segment DE.

Problem 2. Suppose that 2018 numbers 1 and -1 are written around a circle. For every two adjacent numbers, their product is taken. Suppose that the sum of all 2018 products is negative. Find all possible values of sum of 2018 given numbers.

Problem 3. Find all positive integers n such that $\varphi(n)$ is a divisor of n^2+3.

Problem 4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y)-x)+y^2f(2x-f(y)) = \frac{(f(x))^2}{x} + f(yf(y)),$$

for all $x,\ y \in \mathbb{Z},\ x \neq 0$.

1.3. Team selection tests for JBMO 2018

The first day

Problem 1. Determine if there exists a triangle whose side lengths x, y, z satisfy

$$x^3 + y^3 + z^3 = (x+y)(y+z)(z+x).$$

Problem 2. Let M and N are two nine-digit palindromes without leading zeros (a palindrome is a number that reads the same backward as forward) such that $M < N$ and there are no other nine-digit palindromes strictly between M and N. What are all possible values of $N - M$?

Problem 3. Let ABC be a triangle inscribed in circle (O) with incenter I. The lines IB and IC intersect (O) again at J and L. Circumcircle (ω) of

triangle IBC meets CA, AB again at E, F. Prove that EL and FJ intersect on (ω).

Problem 4. Let $n \geq 2$ be a positive integer. A subset of positive integers S is called *comprehensive* if for every integer $0 \leq x < n$, there is a subset of S who sum of elements has remainder x when divided by n. Note that the empty set has sum 0. Show that if a set S is comprehensive then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

The second day

Problem 1. Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. Find A.

Problem 2. Let a, b, c be real numbers such that $a+b+c+ab+bc+ca+abc \geq 7$. Prove that

$$\sqrt{a^2+b^2+2} + \sqrt{b^2+c^2+2} + \sqrt{c^2+a^2+2} \geq 6.$$

Problem 3. The $n \times n \times n$ cubic is constructed from n^3 unit cubic $1 \times 1 \times 1$ such that at least one of unit cubic is black. Show that we can always cut the $n \times n \times n$ cubic into rectangular boxes such that each box contains exactly one black unit cubic.

Problem 4. Let ABC be an acute triangle with O, H are its circumcenter, orthocenter. Take a point M belongs to the minor arc BC of (O) (not coincide to B, C) and denote D, E, F as reflection of M through OA, OB, OC. Suppose that BF meets CE at K and let I be the incenter of triangle DEF.

1. Prove that perpendicular bisectors of EF and IK meet on circle (O).

2. Prove that three points H, K, I are collinear.

The third day

Problem 1. Assume that n is a composite positive integer. For each proper divisor d of n, we write down $d+1$ on the board. Find all values of n such that the written numbers appear to be exactly all the proper divisors of

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some positive integer m. Note that a proper divisor of a positive integer $a > 1$ is any positive divisor of a, distinct from 1 and a.

Problem 2. Let $ABCD$ be a square inscribed in circle (O). Let P be a point lies on minor arc CD of (O). Line PB intersects AC at E. Line PA intersects DB at F. Circumcircle of triangle PEF cuts (O) again at Q. Prove that PQ is parallel to CD.

Problem 3. Prove that in every triangle, there are two sides x, y such that

$$\frac{\sqrt{5}-1}{2} \leq \frac{x}{y} \leq \frac{\sqrt{5}+1}{2}.$$

Problem 4. Let $n > 2$ be a positive integer. Consider n bags of candy, each of them has exactly 1 candy. Ali and Omar take turn play the following game (Ali moves first): At each turn, the player takes two bags contain the numbers of candy as x, y for some coprime integers x, y and then merges them to one bag. Who cannot perform this action will be the loser. Who has the strategy to win this game?

1.4. Team selection tests for IMO 2018

The first day

Problem 1. Consider the infinite, strictly increasing sequence of positive integer (a_n) such that

i. All terms of sequences are pairwise coprime.

ii. The sum $\frac{1}{\sqrt{a_1a_2}} + \frac{1}{\sqrt{a_2a_3}} + \frac{1}{\sqrt{a_3a_4}} + \cdots$ is unbounded.

Prove that this sequence contains infinitely many primes.

Problem 2. Let n be an even positive integer. We fill in a number on each cell of a rectangle table of n columns and multiple rows as following:

i. Each row is assigned to some positive integer a and its cells are filled by 0 or a (in any order);

ii. The sum of all numbers in each row is n.

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Note that we cannot add any more row to the table such that the conditions (i) and (ii) still hold. Prove that if the number of 0's on the table is odd then the maximum odd number on the table is a perfect square.

Problem 3. Let $ABCD$ be a convex quadrilateral inscibed in circle (O) such that $DB = DA + DC$. The point P lies on the ray AC such that $AP = BC$. The point E is on (O) such that $BE \perp AD$. Prove that DP is parallel to the angle bisector of $\angle BEC$.

The second day

Problem 1. Let ABC be an acute, non isosceles triangle with M, N, P are midpoints of BC, CA, AB, respectively. Denote d_1 as the line passes through M and perpendicular to the angle bisector of $\angle BAC$, similarly define for d_2, d_3. Suppose that $d_2 \cap d_3 = D, d_3 \cap d_1 = E, d_1 \cap d_2 = F$. Let I, H be the incenter and orthocenter of triangle ABC. Prove that the circumcenter of triangle DEF is the midpoint of segment IH.

Problem 2. *Problem about arithmetic and geometric sequence.*

1. For integer $n \geq 3$, suppose that $0 < a_1 < a_2 < \ldots < a_n$ is a arithmetic sequence and $0 < b_1 < b_2 < \ldots < b_n$ is a geometric sequence with $a_1 = b_1, a_n = b_n$. Prove that $a_k > b_k$ for all $k = 2, 3, \ldots, n-1$.

2. Prove that for every positive integer $n \geq 3$, there exist an integer arithmetic sequence (a_n) and an integer geometric sequence (b_n) such that
$$0 < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n.$$

Problem 3. Two sets of positive integers A, B are called *connected* if they are not empty and for all $a \in A$, $b \in B$, number $ab + 1$ is a perfect square.

1. Given $A = \{1, 2, 3, 4\}$. Prove that there does not exist any set B such that A, B are connected.

2. Suppose that A, B are connected with $|A|, |B| \geq 2$. For any $a_1 > a_2 \in A$ and $b_1 > b_2 \in B$, prove that $a_1 b_1 > 13 a_2 b_2$.

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The third day

Problem 1. Denote S as the set of prime divisors of all integers of form $2^{n^2+1}-3^n, n \in \mathbb{Z}^+$. Prove that S and $\mathscr{P} \backslash S$ are both contains infinitely many elements ($\mathscr{P}$ is set of prime numbers).

Problem 2. A non-empty subset of $\{1, 2, \ldots, n\}$ is called *arabic* if arithmetic mean of its elements is an integer. Show that the number of arabic subsets of $\{1, 2, \ldots, n\}$ has the same parity as n.

Problem 3. Consider the function

$$f(x) = (x - F_1)(x - F_2)\ldots(x - F_{3030})$$

with (F_n) is the Fibonacci sequence, which defined as $F_1 = 1, F_2 = 2, F_{n+2} = F_{n+1} + F_n, n \geq 1$. Suppose that on the range (F_1, F_{3030}), the function $|f(x)|$ takes on the maximum value at $x = x_0$. Prove that $x_0 > 2^{2018}$.

The fourth day

Problem 1. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying $f(1) = 2,\ f(2) \neq 4$, and

$$\max\{f(m) + f(n), m + n\} \,|\min\{2m + 2n, f(m + n) + 1\}$$

for all $m, n \in \mathbb{Z}^+$.

Problem 2. Let ABC be an acute-angled triangle inscribed in circle (O). Let G be a point on the small arc AC of (O) and (K) be a circle passing through A and G. Bisector of $\angle BAC$ cuts (K) again at P. The point E is chosen on (K) such that AE is parallel to BC. The line PK meets the perpendicular bisector of BC at F. Prove that $\angle EGF = 90°$.

Problem 3. Find all positive integers k such that there exists some permutation of $(1, 2, \ldots, 1000)$ namely $(a_1, a_2, \ldots, a_{1000})$ and satisfy $|a_i - i| = k$ for all $i = \overline{1, 1000}$.

Part 2

SOLUTION FOR THE TESTS

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2.1. Solution of TST for GMO 2018

The first day

Problem 1. Let $\{x_n\}$ be a sequence defined by $x_1 = 2$ and

$$x_{n+1} = x_n^2 - x_n + 1$$

for $n \geq 1$. Prove that

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}$$

for all n.

Solution.
The sequence is increasing since $x_{n+1} - x_n = (x_n - 1)^2 \geq 0$. We also have $x_{n+1} - 1 = x_n(x_n - 1)$ which gives us

$$\frac{1}{x_{n+1} - 1} = \frac{1}{x_n(x_n - 1)} = \frac{1}{x_n - 1} - \frac{1}{x_n}$$

This identity is valid because given $x_1 = 2$ we have $x_n \geq 2$ and therefore $x_n - 1 \neq 0$ Now given that $\frac{1}{x_n} = \frac{1}{x_n - 1} - \frac{1}{x_{n+1} - 1}$ we obtain the identity

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 - \frac{1}{x_{n+1} - 1}$$

So we have to prove

$$1 - \frac{1}{2^{2^{n-1}}} < 1 - \frac{1}{x_{n+1} - 1} < 1 - \frac{1}{2^{2^n}}$$

or equivalently

$$2^{2^{n-1}} < x_{n+1} - 1 < 2^{2^n}$$

We will prove this by induction.

For $n = 1$ we simply get $2 < x_2 - 1 < 4$ which is true since $x_2 = 3$. Assume the statement is true for $n = k$, i.e.

$$2^{2^{k-1}} < x_{k+1} - 1 < 2^{2^k}$$

For $n = k + 1$ we have

$$x_{k+2} - 1 = x_{k+1}(x_{k+1} - 1) > 2^{2^{k-1}}(2^{2^{k-1}} + 1) = 2^{2^k} + 2^{2^{k-1}} > 2^{2^k}$$

by the induction hypothesis. On the other hand we know that x_n's are all integers and therefore $x_{k+1} \leq 2^{2^k}$ by the induction hypothesis. We obtain

$$x_{k+2} - 1 = x_{k+1}(x_{k+1} - 1) > 2^{2^k} \cdot 2^{2^k} = 2^{2^{k+1}}$$

which is what we wanted to prove. □

Problem 2. Let p be a prime number of the form $9k+1$. Show that there exists an integer n such that $p \mid n^3 - 3n + 1$.

Solution.
Note that the existence of an integer as described in the problem can be equivalently stated as follows: the polynomial $x^3 - 3x + 1$ has a root in $\mathbb{F}_p$. Following the classical method for solving cubic equations, we set $x = w + \frac{1}{w}$. Then

$$x^3 - 3x + 1 = (w + \frac{1}{w})^3 - 3(w + \frac{1}{w}) + 1 = w^3 + \frac{1}{w^3} + 1$$

We then observe that the solutions of the equation $\lambda + \frac{1}{\lambda} + 1 = 0$ are cubic roots of unity. If $p = 9k+1$ and g is a primitive root $\pmod p$, then g^{3k} is a cubic root of unity in $\mathbb{F}_p$. Thus we look for w such that $w^3 = g^{3k}$, so we let $w = g^k$.

It is now easy to verify that $x = w + \frac{1}{w} = g^k + g^{8k}$ is a root of the original cubic polynomial.

Remark. The other two roots are $g^{2k} + g^{7k}$ and $g^{4k} + g^{5k}$. Therefore, the polynomial not only has a root in $\mathbb{F}_p$, but it indeed splits. □

Problem 3. Let I, O be the incenter, circumcenter of triangle ABC and A_1, B_1, C_1 be arbitrary points on the segments AI, BI, CI respectively. The perpendicular bisectors of AA_1, BB_1, CC_1 intersect each other at X, Y and Z. Prove that the circumcenter of triangle XYZ coincides with O if and only if I is the orthocenter of triangle $A_1B_1C_1$.

Solution.
Denote $\angle B = \beta$ then we have

$$\angle XYZ = 180^\circ - \angle AIC = 180^\circ - \left(90^\circ + \frac{\beta}{2}\right) = 90^\circ - \frac{\beta}{2}.$$

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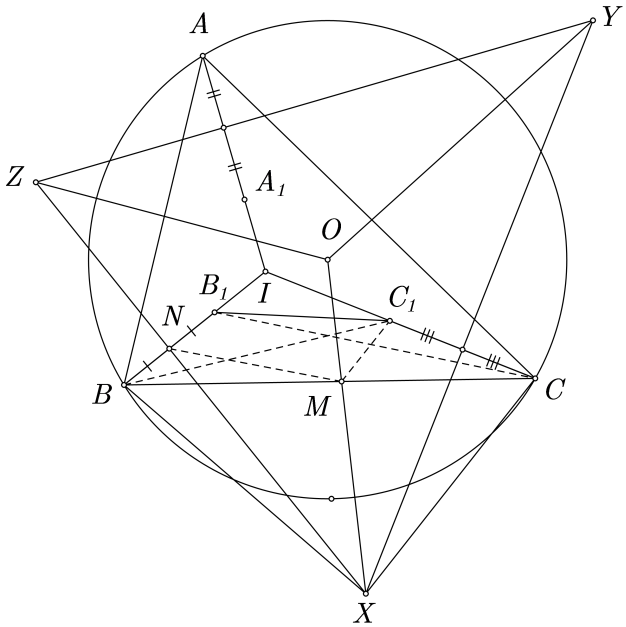
So if O is the circumcenter of XYZ then

$$\angle OXZ = 90^\circ - \left(90^\circ - \frac{\beta}{2}\right) = \frac{\beta}{2} = \angle IBC,$$

which implies that $OX \perp BC$. Similarly, we also have $OY \perp CA, OZ \perp AB$.



Denote M, N as the midpoints of BC, BB_1 respectively. From the cyclic quadrilateral and parallel line, we have $\angle BB_1C = \angle BNM = 180^\circ - \angle BXM$. Similarly, $\angle BC_1C = \angle 180^\circ - \angle CXM$. Since $MX \perp BC$, triangle XBC is isosceles, then $\angle BXM = \angle CXM$, thus $\angle BB_1C = \angle BC_1C$, which implies that BCC_1B_1 is cyclic.

Angle chasing again, we have $\angle IC_1B_1 = \angle CBB_1 = \frac{\beta}{2}$, but $\angle AIC = 90^\circ + \frac{\beta}{2}$, implies that $AI \perp B_1C_1$. By similar way, we get $BI \perp C_1A_1$ and $CI \perp A_1B_1$, hence I is the orthocenter of triangle $A_1B_1C_1$.

It is easy to check that these conditions are equivalent, which finishes the proof. □

Problem 4. In each of the cells of a 13×13 board is written an integer such that the integers in adjacent cells differ by 1. If there are two $2s$ and two $24s$ on this board, how many $13s$ can there be?

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Solution.
Let us define the *distance* between any two cells of the board to be the minimum number of steps required to get from one of the cells to the other one provided that one moves between adjacent cells in each step. Consequently, the distance of any cell to itself is 0, the distance between adjacent cells is 1 and the largest distance on the 13×13 board (that between two opposite corners) is 24. It is easy to observe that the numbers written on any two cells cannot differ by more than the distance between the cells.

Let us now assign coordinates to the cells of the board in the usual way: each cell is denoted (i, j) where $i, j \in \{0, 1, , \ldots, 12\}$. Let us also denote the number on the cell (i, j) by $x_{(i,j)}$. Thus the inequality mentioned above is expressed as

$$|x_{(i_1,j_1)} - x_{(i_2,j_2)}| \leq |i_1 - i_2| + |j_1 - j_2|.$$

Now there must be a distance of at least 22 between a 2 and a 24 and this places a strong restriction on the possible locations of $2s$ and $24s$. Namely, $x_{(i,j)} \in \{2, 24\}$ then we have

$$\min(i, 12 - i) + \min(j, 12 - j) \leq 2.$$

This roughly means that $2s$ and $24s$ can be found only very near the corners. Without loss of generality (by symmetry of the board), let us assume that there exists a cell (i, j) with $x_{(i,j)} = 2$ and $i + j \leq 2$. Now, $x_{(i,j)} = 24$ implies that $i + j \geq 22$, that is, both of the $24s$ must be near the opposite corner. Hence $x_{(i,j)} = 2$ thus $i + j \leq 2$, that is, both of the $2s$ must be near the same corner.

If there exists a cell (i, j) with $x_{(i,j)} = 2$ and $i + j = 2$, then $x_{(i,j)} = 24$ then $i + j \geq 24$, so $i = j = 12$. Thus there cannot be two distinct $24s$ on the board. We therefore conclude that

$$x_{(i,j)} = 2 \Rightarrow i + j \leq 1 \Rightarrow (i, j) \in \{(0, 0), (1, 0), (0, 1)\}.$$

Similarly,

$$x_{(i,j)} = 24 \Rightarrow i + j \geq 23 \Rightarrow (i, j) \in \{(12, 12), (11, 12), (12, 11)\}.$$

If $x_{(0,0)} = 2$, then $x_{(1,0)} \neq 2$ and $x_{(0,1)} \neq 2$, thus there cannot be two $2s$ on the board. We therefore conclude that

$$x_{(0,0)} \neq 2, x_{(1,0)} = x_{(0,1)} = 2$$

and similarly

$$x_{(12,12)} \neq 24, x_{(11,12)} = x_{(12,11)} = 24.$$

Now, if we consider a path (a sequence of adjacent cells) of length 22 between the cells $(1,0)$ and $(11,12)$ or between the cells $(0,1)$ and $(12,11)$, we see that the numbers on the cells on this path are uniquely determined. Since any cell of the board except $(0,0)$ and $(12,12)$ can be placed on such a path, almost the whole board is uniquely determined. We obtain the following formula:

$$1 \leq i+j \leq 23 \Rightarrow x_{(i,j)} = i+j+1$$

Furthermore, $x_{(0,0)} \in \{1,3\}$ and $x_{(12,12)} \in \{23,25\}$, therefore $x_{(i,j)} = 13 \Leftrightarrow i+j = 12$. Hence there are exactly thirteen $13s$ on the board.

Remark. There are exactly $16 = 4 \cdot 2 \cdot 2$ possible configurations of the numbers on the board: we have to multiply 4 (for the choice of the corner near which we place the $2s$), 2 (for the choice of $x_{(0,0)}$) and 2 (for the choice of $x_{(12,12)}$), otherwise the numbers are uniquely determined.

□

The second day

Problem 1. Let n be an odd positive integer with $n > 1$ and let $a_1, a_2, \ldots, a_n$ be positive integers such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. Let $d = \gcd(a_1^n + a_1 \cdot a_2 \cdots a_n, a_2^n + a_1 \cdot a_2 \cdots a_n, \ldots, a_n^n + a_1 \cdot a_2 \cdots a_n)$. Show that the possible values of d are $d = 1, d = 2$.

Solution.
Let p be a prime with $p \mid d$. We know that $p \mid a_i^n + a_1 \cdot a_2 \cdots a_n$ for all i. If $\exists i$ such that $p \mid a_i$, then $p \mid a_1 \cdot a_2 \cdots a_n$. Since $p \mid a_j^n + a_1 \cdot a_2 \cdots a_n$ for all j, we find $p \mid a_j$ for all j, which contradicts to the hypothesis.

Thus we consider the case $p \nmid a_i$ for all i, i.e. $p \nmid a_1 \cdot a_2 \cdots a_n$. We have the n congruences

$$a_1^n \equiv -a_1 \cdot a_2 \cdots a_n \pmod{p}, a_2^n \equiv -a_1 \cdot a_2 \cdots a_n \pmod{p}, \ldots,$$
$$a_n^n \equiv -a_1 \cdot a_2 \cdots a_n \pmod{p}$$

and by multiplying them, we obtain

$$(a_1 \cdot a_2 \cdots a_n)^n \equiv (-1)^n \cdot (a_1 \cdot a_2 \cdots a_n)^n \equiv -(a_1 \cdot a_2 \cdots a_n)^n \pmod{p}$$

which yields $p = 2$.

This shows that d must be a power of 2. Suppose, now, that $4 \mid d$. Clearly, $a_1, a_2, \ldots, a_n$ must be all odd. We consider two cases

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1. If $a_1 \cdot a_2 \cdots a_n \equiv 3 \pmod 4$, then $\exists i$ such that $a_i \equiv 3 \pmod 4$ then $a_i^n + a_1 \cdot a_2 \cdots a_n \equiv 3 + 3 \equiv 2 \pmod 4$ which contradicts with $4 \mid d$.

2. If $a_1 \cdot a_2 \cdots a_n \equiv 1 \pmod 4$, then $\exists i$ such that $a_i \equiv 1 \pmod 4$ then $a_i^n + a_1 \cdot a_2 \cdots a_n \equiv 1 + 1 \equiv 2 \pmod 4$ which also contradicts with $4 \mid d$. (In both cases, we made use of $3^n \equiv 3 \pmod 4$).

Therefore, $d = 1$ or $d = 2$. If $a_1, a_2, \ldots, a_n$ are all odd, then d must be even, hence $d = 2$. If some of $a_1, a_2, \ldots, a_n$ are even and some of them odd, then d must be odd, hence $d = 1$. As such, both of these values are realized. □

Problem 2. Two positive integers m and n are called *similar* if one of them can be obtained from the other one by swapping two digits (note that a 0-digit cannot be swapped with the leading digit). Find the greatest integer N such that N is divisible by 13 and any number similar to N is not divisible by 13.

Solution.

Let k be the number of digits of N and let $N = \sum_{i=0}^{k-1} 10^i \cdot d_i$ where d_i are digits. If M is obtained by swapping the i-th and the j-th digits, then

$$M - N = (10^j - 10^i) \cdot (d_i - d_j).$$

Therefore $13 \mid M - N$ implies that $d_i = d_j$ or $6 \mid i - j$. Hence, a number N satisfies the property given in the problem if and only if $13 \mid N$, $d_i \neq d_j$ for all $i \neq j$ and no swap between i-th and j-th digits are possible if $6 \mid i - j$.

- If $k \geq 8$, one can swap the 0th and the 6th digits, hence N does not satisfy the property.

- If $k = 7$, one can swap the 0th and the 6th digits unless the 0th digit is 0, but if the 0th digit is 0, then no swap is possible.

Hence, N satisfies the property if and only if $13 \mid N$, all the digits are distinct and the 0th digit is 0. Therefore, $N \leq 9876540$.

Considering all multiples of 13 that end with 0 below this level, one finds $N \in \{9876490, 9876360, 9876230, \ldots\}$. Since the two greatest elements of this set do not have all distinct digits, the greatest integer $N = 9876230$. □

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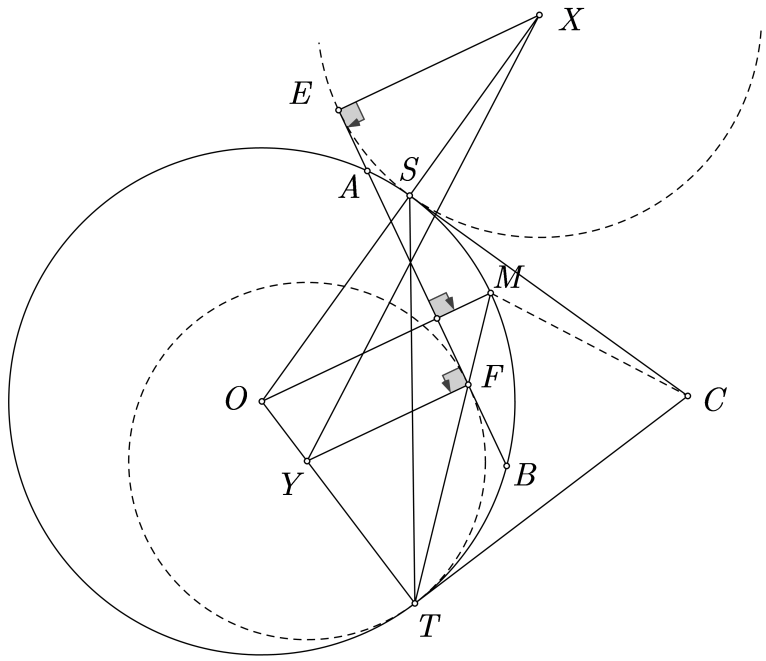
Problem 3. Let C be a point lies outside the circle (O) and CS, CT are tangent lines of (O). Take two points A, B on (O) with M is the midpoint of the minor arc AB such that A, B, M differ from S, T. Suppose that MS, MT cut line AB at E, F. Take $X \in OS$ and $Y \in OT$ such that EX, FY are perpendicular to AB. Prove that XY and CM are perpendicular.

Solution.
First, note that $OM \perp AB$ then $OM \parallel XE$. But OMS is isosceles triangle implies that triangle XES is also isosceles, or $XE = XS$. Similarly, $YE = YT$. Denote $(\omega_1), (\omega_2)$ as the circle of center X, radius XS and center Y, radius YT. Since $CS \perp XS$, we have CS is tangent to (ω_1) so $\mathscr{P}_{C/(\omega_1)} = CS^2$. On the other hand, $\mathscr{P}_{C/(\omega_2)} = CT^2$ and $CS = CT$ imply that C belongs to radical axis of two circles $(\omega_1), (\omega_2)$.



By similar triangles, we get $MA^2 = MS \cdot ME$ and $MB^2 = MF \cdot MT$, but $MA = MB$ then $MS \cdot ME = MF \cdot MT$, thus $\mathscr{P}_{M/(\omega_1)} = \mathscr{P}_{M/(\omega_2)}$. By combining these results, we obtain CM is the radical axis of two circles $(\omega_1), (\omega_2)$; hence, $CM \perp XY$. □

Problem 4. In a graph with 8 vertices that contains no cycle of length 4, at most how many edges can there be?

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Solution.
Let V denote the set of all vertices. Let d be the greatest degree of a vertex in the graph and $v \in V$ be a vertex with degree d and let $D \subset V$ be the set of d vertices connected to v.

In the full subgraph consisting of vertices in D, no vertex can have degree 2 or more, because this would make a cycle of length 4. Therefore, there are at most $\lfloor \frac{d}{2} \rfloor$ edges in this full subgraph.

Similarly, none of the $7-d$ vertices in $V \setminus (D \cup \{v\})$ can have 2 or more connections to D because this would also make a cycle of length 4. Therefore, there are at most $7-d$ edges between $V \setminus (D \cup \{v\})$ and $D \cup \{v\}$. Finally, the number of edges in $V \setminus (D \cup \{v\})$ is at most $\binom{7-d}{2}$, hence

$$e \leq d + \left\lfloor \frac{d}{2} \right\rfloor + 7 - d + \binom{7-d}{2} = 7 + \left\lfloor \frac{d}{2} \right\rfloor + \binom{7-d}{2}.$$

- If $d \geq 5$, $e \leq 7 + \left\lfloor \frac{d}{2} \right\rfloor + \binom{7-d}{2} \leq 10$.
- If $d = 4$, $e \leq 7 + \left\lfloor \frac{4}{2} \right\rfloor + \binom{7-4}{2} = 12$.
- If $d \leq 3$, $e = \dfrac{\sum_{u \in V} d(u)}{2} \leq \dfrac{8 \cdot 3}{2} = 12$.

Let us now prove that $e = 12$ is not possible by ruling out the two cases with $d = 4$ and $d = 3$.

1. If $e = 12$ and $d = 4$, then there are exactly 2 edges in the full subgraph consisting of vertices in D (each vertex in D is connected to exactly one other vertex in D), there are exactly 3 edges between $V \setminus (D \cup \{v\})$ and D (each vertex in $V \setminus (D \cup \{v\})$ is connected to exactly 1 vertex in D) and there are exactly 3 edges in $V \setminus (D \cup \{v\})$ (all 3 of the vertices in $V \setminus (D \cup \{v\})$ are connected).

   Let us denote the vertices in $V \setminus (D \cup \{v\})$ by a_1, a_2, a_3. a_i and a_j cannot be connected to the same vertex in D, otherwise this would make a cycle of length 4 (for example (a, c, b, x) cyclically connected).

   Therefore, exactly 3 of the 4 vertices in D are connected to $V \setminus (D \cup \{v\})$. Since each vertex in D is connected to exactly one other vertex in D, two of these 3 vertices are connected. Let us call them x, y and let us assume without loss of generality that they are connected to a_1 and a_2. Now, we have a cycle of length 4 as (x, y, a_2, a_1) are cyclically connected. Therefore, $e = 12$ and $d = 4$ is not possible.

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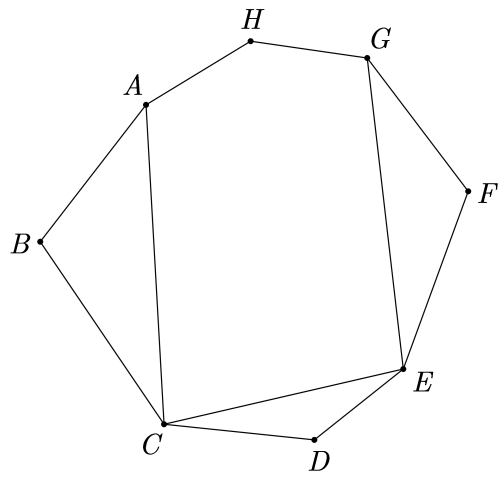
2. If $e = 12$ and $d = 3$, then all vertices in the graph must have degree 3. Let v be any vertex and D be defined as above. Similarly as above, no vertex in $V \setminus (D \cup \{v\})$ can have 2 or more connections to D as this would make a cycle of length 4.

   Hence, each vertex in $V \setminus (D \cup \{v\})$ is connected to at most 1 vertex in $D \cup \{v\}$ and thus connected to at least 2 vertices in $V \setminus (D \cup \{v\})$. If we consider the full subgraph consisting of the 4 vertices in $V \setminus (D \cup \{v\})$, each vertex has degree at least 2. In a graph with 4 vertices such that each vertex has degree at least 2, there must be a cycle of length 4. Therefore, $e = 12$ and $d = 3$ is also not possible.



Let us now give an example of a graph with $e = 11$. Consider an octagon $ABCDEFGH$. Now, draw the additional three edges AC, CE, EG. This is a graph with 8 vertices, 11 edges and no cycles of length 4. □

2.2. Solution of TST for BMO 2018

The first day

Problem 1. Find the smallest positive integer n which can not be expressed as $n = \dfrac{2^a - 2^b}{2^c - 2^d}$ for some positive integers a, b, c, d.

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Solution.

Let S be the set of positive integers which can be written as $s = \dfrac{2^a - 2^b}{2^c - 2^d}$ for some positive integers a, b, c, d.

Since $s > 0$, we can assume that $a > b, c > d$ and write $s = 2^{b-d}\dfrac{2^{a-b}-1}{2^{c-d}-1}$. It's now clear that $b - d = v_2(x)$.

So if we set $x = 2^{v_2(x)}y$ then $y = \dfrac{2^u - 1}{2^v - 1}$ (u, v are positive integers). In particular, the number we are looking for is odd. Clearly the numbers of the form $2^u - 1$ ($v = 0$) belong to S. More generally, we know that $2^v - 1 \mid 2^u - 1$ if and only if $v \mid u$. Set $u = dv$ then

$$y = \frac{2^u - 1}{2^v - 1} = \frac{2^{dv} - 1}{2^v - 1} = 2^{(d-1)v} + 2^{(d-2)v} + \cdots + 2^v + 1.$$

This means that the base 2 representation of y is of the form

$$y = 1\underbrace{0\ldots0}_{v-1}1\underbrace{0\ldots0}_{v-1}1\ldots\underbrace{0\ldots0}_{v-1}1$$

(d digits 1, every two consecutive 1's are separated by $m - 1$ 0's). So the number we are looking for is the smallest odd number not of this form (for some $d \geq 1, v \geq 1$). Note that

$$1 = \overline{1}, 3 = \overline{11}, 5 = \overline{101}, 7 = \overline{111}, 9 = \overline{1001}, 11 = \overline{1011}.$$

We can conclude that 11 is the smallest odd number whose the base 2 representation is not of the desired form. So the answer is $n = 11$.

□

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + 2f(x)f(y)) = f(x) + 2xf(y),$$

for all real numbers x, y.

Solution.

Denote $(*)$ as the given condition. Replacing $x = y = -\frac{1}{2}$ in $(*)$, we get

$$f\left(2\left(f\left(-\frac{1}{2}\right)\right)^2 - \frac{1}{2}\right) = 0.$$

If there exists some $a \neq 0$ such that $f(a) = 0$. Substituting $x = a$ in $(*)$, we get $f(y) = 0$ for any y, which is obviously a solution.

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Let us consider now the case $f(x) \neq 0$ for any nonzero real number x. Hence, $f(0) = 0$ and $f\left(-\frac{1}{2}\right) = \pm\frac{1}{2}$. On the other hand, it is easy to see that if f is solution, then $-f$ is also a solution. Therefore, we only need to consider the case $f(0) = 0$ and $f\left(-\frac{1}{2}\right) = -\frac{1}{2}$. From this, we have

$$\begin{aligned} f\left(-\frac{1}{2} - f(y)\right) &= -f(y) - \frac{1}{2}, \quad \forall y \in \mathbb{R} \\ f\left(\frac{1}{2}\right) &= \frac{1}{2} \end{aligned}$$

and

$$f(x - f(x)) = f(x) - x, \quad \forall x \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} f\left(x - \frac{1}{2} - f(x)\right) &= f\left(-\frac{1}{2} - f(x - f(x))\right) \\ &= -\frac{1}{2} - f(x - f(x)) = x - \frac{1}{2} - f(x), \end{aligned}$$

which implies $f\left(-f\left(\frac{1}{2}\right)\right) = -f\left(\frac{1}{2}\right)$. Now, substituting $x = \frac{1}{2}$ and $y = -f\left(\frac{1}{2}\right)$ in $(*)$, we get

$$f\left(\frac{1}{2} - 2\left(f\left(\frac{1}{2}\right)\right)^2\right) = 0.$$

Thus $f\left(\frac{1}{2}\right) = \pm\frac{1}{2}$. If $f\left(\frac{1}{2}\right) = -\frac{1}{2}$, then $\frac{1}{2} = -f\left(\frac{1}{2}\right) = f\left(-f\left(\frac{1}{2}\right)\right) = f\left(\frac{1}{2}\right) = -\frac{1}{2}$, contradiction. Therefore, $f\left(\frac{1}{2}\right) = \frac{1}{2}$. From here, it follows that

$$f\left(f(y) + \frac{1}{2}\right) = f(y) + \frac{1}{2}, \quad \forall y \in \mathbb{R}.$$

In this equation, substituting $y = x - \frac{1}{2} - f(x)$, we get

$$f(x - f(x)) = x - f(x), \quad \forall x \in \mathbb{R}.$$

Comparing this result with $f(x - f(x)) = f(x) - x$, we get $f(x) = x$ for any x, which is clearly a solution.

In conclusion, there are three solutions: $f(x) = 0$, $f(x) = x$ and $f(x) = -x$ for all real number x. □

Problem 3. The partition of $2n$ positive integers into n pairs is called square-free if the product of numbers in each pair is not a perfect square. Prove that if for $2n$ distinct positive integers, there exists one square-free partition, then there exists at least $n!$ square-free partitions.

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Solution.
For convenience, let call integers are not perfect square by "good" and integers are perfect square by "bad". We shall prove the given estimation by induction. Indeed,

For $n = 1$ we only have 1 square-free partition.

For $n = 2$ we have four numbers x_1, x_2, y_1, y_2 with x_1y_1 and x_2y_2 are good. Since $x_1^2 = \frac{x_1x_2 \cdot x_1y_2}{x_2y_2}$ and the number x_2y_2 is good, then two numbers x_1x_2, x_1y_2 cannot be both bad. We have some cases:

- If x_1x_2, x_1y_2 are both good, then we have two sub cases: if x_2y_1 is good, then we are done; otherwise, consider ratio $y_1y_2 = \frac{x_2y_1 \cdot x_2y_2}{x_2^2}$, then this number must be good, other partition $(x_1, x_2), (y_1, y_2)$ is square-free.
- Otherwise, assume that x_1x_2 is bad and x_1y_2 is good. By similar argument, we have the partition $(x_1, y_2), (x_2, y_1)$.

So in all cases, we always have one more partition, the statement is true for $n = 2$.

Suppose that for some $n \geq 2$, the statement is true. Consider $2n + 2$ numbers with one square-free partition

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{n+1}, y_{n+1}),$$

First, fix the pair (x_1, y_1) then apply the induction hypothesis for other n pairs, we have $n!$ square-free partitions. After that, take some index $2 \leq k \leq n+1$, then change four numbers x_1, y_1, x_k, y_k to another partition; so for $2(n-1)$ others pairs, we have at least $(n-1)!$ square-free partitions. In total, we get at least

$$n! + n \cdot (n-1)! = (n+1)! \text{ square-free partitions.}$$

Hence, the statement is also true for $n + 1$. This finishes our proof.

Remark. We can show an example satisfies the equality as: $p, p^2, p^3, \ldots, p^n$. We just can make pairs of different parity exponents. □

Problem 4. Let ABC be an acute, non isosceles triangle with I is its incenter. Denote D, E as tangent points of (I) on AB, AC, respectively. The median segments respect to vertex A of triangles ABE and ACD meet (I) at P, Q, respectively. Take points M, N on the line DE such that $AM \parallel BE$ and $AN \parallel CD$ respectively.

1. Prove that A lies on the radical axis of (MIP) and (NIQ).

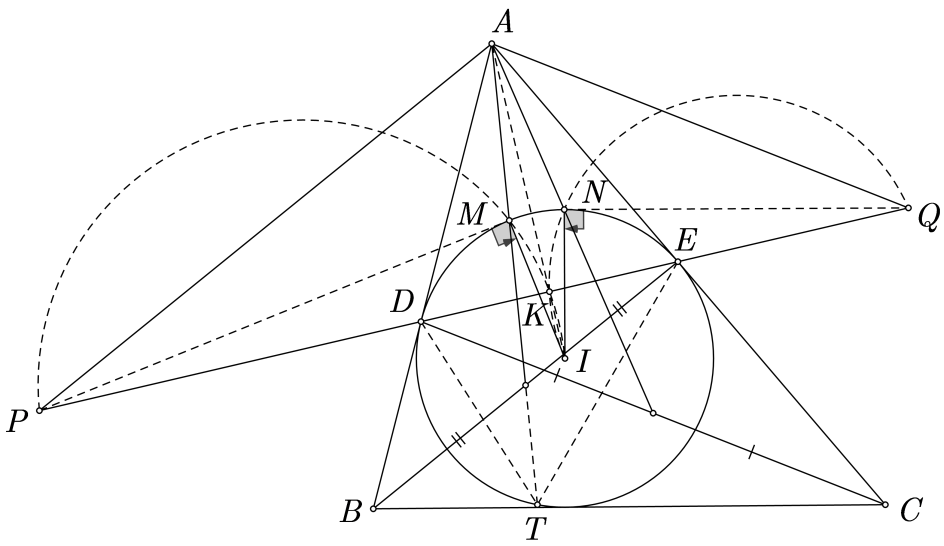
2. Suppose that the orthocenter H of triangle ABC lies on (I). Prove that there exists a line which is tangent to three circles of center A, B, C and all pass through H.

Solution.
1) Since AM passes through the midpoint of the segment BE and $AP \parallel BE$, we can see that

$$(AP, AM, AB, AC) = -1 \text{ or } (MP, MA, MD, ME) = -1.$$

Suppose that AM cuts (I) at the second point T then $MDTE$ is harmonic quadrilateral, then $(Mx, MT, MD, ME) = -1$ with Mx is the tangent line of (I).

From these results, we get $Mx \equiv MP$ which implies that PM is tangent to (I) or $\angle IMP = 90°$. Similarly, we also have $\angle INP = 90°$. Hence, if we denote K is the midpoint of DE then $IK \perp DE$ and $K \in (MIP), (NIQ)$. Since $A \in IK$ as the radical axis of these two circles, the result follows.



2) Denote A_1, B_1, C_1 as the projections from A, B, C to the opposite side, respectively then

$$\overline{HA} \cdot \overline{HA_1} = \overline{HB} \cdot \overline{HB_1} = \overline{HC} \cdot \overline{HC_1} = \frac{1}{2}\mathscr{P}_{H/(ABC)}.$$

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Consider the inversion of center H and the power equals to the value above, then we have

$$A \longleftrightarrow A_1, B \longleftrightarrow B_1, C \longleftrightarrow C_1,$$
$$(HB_1C_1) \longleftrightarrow BC, (HC_1A_1) \longleftrightarrow CA, (HA_1B_1) \longleftrightarrow AB$$

The image of (I) should be some line d since $H \in (I)$. Because BC, CA, AB are tangent to (I) then d is also tangent to $(HB_1C_1), (HC_1A_1), (HA_1B_1)$.

Consider the homothety of center H, ratio 2 which maps $d \to d'$ and the cirle (HB_1C_1) to the circle of center H and radius HA (since HA is the diameter of (HB_1C_1)). By the property of homothety, we have d' is also tangent to (A, AH). Similarly to these circles $(B, BH), (C, CH)$ so the line d' satisfies the given condition.

□

The second day

Problem 1. Let ABC be a triangle with M, N, P as midpoints of the segments BC, CA, AB respectively. Suppose that I is the intersection of angle bisectors of $\angle BPM, \angle MNP$ and J is the intersection of angle bisectors of $\angle CNM, \angle MPN$. Denote (ω_1)as the circle of center I and tangent to MP at D, (ω_2)as the circle of center J and tangent to MN at E.

1. Prove that DE is parallel to BC.
2. Prove that the radical axis of two circles $(\omega_1), (\omega_2)$ bisects the segment DE.

Solution.
1) Note that $\angle MNC = \angle MPB = \angle A$ then by angle chasing, we have $IP \parallel JN$. Denote $K = PJ \cap IN$ then K is the incenter of triangle MNP. Hence, MK is the angle bisector of $\angle NMP$, thus $MK \parallel IP$. Denote $X = IN \cap MP$ then

$$\frac{IP}{MK} = \frac{XP}{XM} = \frac{NP}{NM}.$$

Similarly, $\frac{JN}{MK} = \frac{NP}{MP}$.

Thus $\frac{IP}{JN} = \frac{MP}{MN}$. Since $\Delta IPD \sim \Delta JNE$ then $\frac{IP}{JN} = \frac{PD}{NE}$. Therefore, $\frac{MP}{MN} = \frac{PD}{NE}$ which implies that $DE \parallel NP \parallel BC$.

2) Suppose that DE cuts $(\omega_1), (\omega_2)$ at R, S respectively. We have

$$\frac{DS}{ER} = \frac{ID \cdot \cos\angle IDR}{JE \cdot \cos\angle JES} = \frac{PD \cdot \sin\angle PDR}{NE \cdot \sin\angle NES} = \frac{AC \cdot \sin C}{AB \cdot \sin B} = 1.$$

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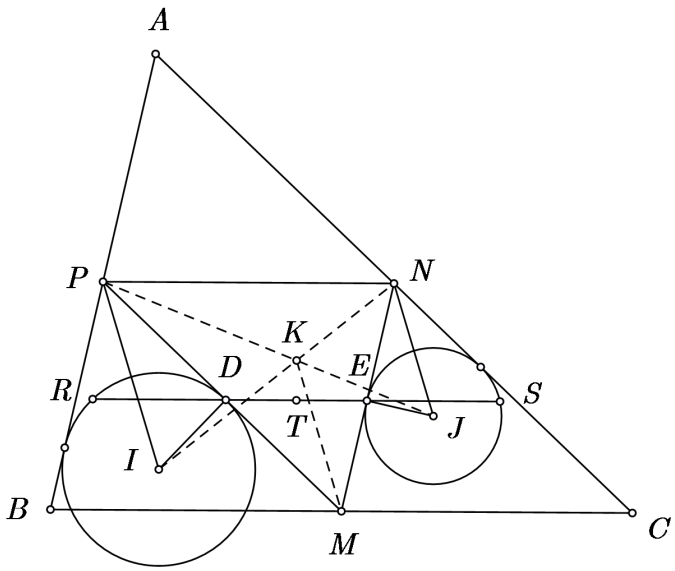

Hence $DS = ER$. Denote T as midpoint of DE then

$$\mathscr{P}_{T/(\omega_1)} = TD \cdot TR = TE \cdot TS = \mathscr{P}_{T/(\omega_2)},$$

which implies that T lies on the radical axis of $(\omega_1), (\omega_2)$.

□

Problem 2. Suppose that 2018 numbers 1 and -1 are written around a circle. For every two adjacent numbers, their product is taken. Suppose that the sum of all 2018 products is negative. Find all possible values of sum of 2018 given numbers.

Solution.

First, we show that the sum is at most 1008 and at least -1008.

Indeed, suppose that there are a numbers $+1$ in 2018 given numbers and b products equal -1. Then the sum of 2018 given numbers is $a + (-1)(2018 - a) = 2a - 2018$, and sum of 2018 products is

$$(2018 - b) + (-1)b = 2018 - 2b.$$

Since $2018 - 2b < 0$ we have $b \geq 1010$. And because each product of -1 comes from one 1 and one -1, and one -1 can contribute to at most two -1 products. Hence the number of -1 numbers is at least $1010/2 = 505$, or $2018 - a \geq 505$. Hence $a \leq 1513$, or $2a - 2018 \leq 1008$.

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Similarly, each product of -1 come from one 1 and one -1, and one 1 can contribute to at most two -1 products. Hence the number of 1 numbers is also at least 505. Hence, $a \geq 505$, or $2a - 2018 \geq -1008$. Combine these results, we showed that the sum is in range $[-1008; 1008]$.

It is clear that the sum is even, and we will show that the sum can be any even number in that range. We start from a configuration of 1009 numbers of $+1$ and 1009 numbers of -1 alternating around the circle. In this case, the sum of 2018 numbers is 0 and the sum of the product is -2018.

On one way, we eventually change one -1 to one 1 exactly 504 times. Each time, the sum of 2018 numbers increase by 2, so the sum will increase from 0 to 1008. Besides, the sum of 2018 products will increase by 4 (two -1 products change to two 1 products). Hence the sum of 2018 products is at most $-2018 + 504 \times 4 < 0$.

On another way, we eventually change one $+1$ to one -1 exactly 504 times. By doing these steps, the sum of 2018 numbers will decrease from 0 to -1008 and the sum of 2018 products is always negative.

Therefore, all even numbers from -1008 to 1008 are obtainable. □

Problem 3. Find all positive integers n such that $\varphi(n)$ is a divisor of n^2+3.

Solution.

First assume n is a prime number. Thus $\varphi(n) = n-1 \mid n^2+3 = (n-1)(n+1)+4$, implying $n-1 \mid 4$. We deduce that $n = 2, 3$ or 5.

From now on, assume n is composite and set $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where k is the number of distinct prime divisors of n. Since $n \geq 3$, $\phi(n)$ is even. In particular, the condition $\phi(n) \mid n^2+3$ shows that n is odd and so the p_i's are all odd. It follows that

$$2^k \mid \varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1).$$

So n^2+3 is divisible by 2^k implying that $k \leq 2$. Moreover, if $\alpha_i \geq 2$ then $p_i \mid \varphi(n)$, so $p_i \mid n^2+3$, forcing $p_i = 3$. In this case, $9 \nmid n^2+3$ and $9 \nmid \varphi(n)$. This shows that $\alpha_i = 2$. Our discussion shows that if the composite $n \geq 3$ satisfies the problem then $n = 3^2 = 9$ or $n = 9p$, or $n = pq$ for some odd primes p, q. Of course, $n = 9$ is a solution of our problem.

- Suppose $n = 9p$. Then $\varphi(n) = 6(p-1)$ and $n^2+3 = 81(p-1)(p+1)+84$. Therefore, $3(p-1) \mid 84$, or equivalently $p-1 \mid 28$. It follows that $p = 5$ or $p = 29$. But in both cases, we have $8 \mid \phi(n)$, contradiction.

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- Suppose $n = pq$, where p, q are odd primes. First, consider the case $q = 3$ (the case $p = 3$ is of course similar). Then $\varphi(n) = 2(p-1)$ so $n^2 + 3 = 9(p-1)(p+1) + 12$. It follows that $2(p-1) \mid 12$, and $p - 1 \mid 6$, implying$p = 3$ or $p = 7$. In this case we obtain a new solution $n = pq = 3 \times 7 = 21$ of our problem.

  Now, suppose $p, q > 3$. Thus, $3 \nmid n^2 + 3$ so $3 \nmid (p-1)(q-1)$. This shows that $p, q \equiv 2 \pmod 3$. Since $(p-1)(q-1) \mid p^2q^2 + 3 = (p^2 - 1)(q^2 - 1) + p^2 + q^2 + 2$ so $(p-1)(q-1) \mid p^2 + q^2 + 2$. Set

  $$p^2 + q^2 + 2 = k(p-1)(q-1),$$

  where k is a positive integer. Set $p = 2x + 1, y = 2y + 1$, then the above equation reads $x^2 + y^2 + x + y + 1 = kxy$, or equivalently,

  $$x^2 - (ky - 1)x + y^2 + y + 1 = 0.$$

  By a Vieta jumping technique, we deduce that $k = 5$. Thus, we have

  $$5(p-1)(q-1) = p^2 + q^2 + 2.$$

  But then $p \equiv q \equiv 2 \pmod 3$ and the left hand side is $\equiv 2 \pmod 3$, while the right hand side is $\equiv 1 \pmod 3$, contradiction.

Therefore, all number satisfy the given condition are $n \in \{1, 2, 3, 5, 9, 21\}$. □

Problem 4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{(f(x))^2}{x} + f(yf(y)),$$

for all $x,\ y \in \mathbb{Z},\ x \neq 0$.

Solution.
Let $a = f(0)$ and denote $(*)$ as the given condition. Assume that $a \neq 0$, substituting $x = 2a$ and $y = 0$ in $(*)$, we get

$$2a^2 = \frac{(f(2a))^2}{2a} + a,$$

or

$$4a - 2 = \left(\frac{f(2a)}{2a}\right)^2.$$

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This follows that $4a-2$ is a perfect square which is a contradiction since there is no perfect square congruent to 2 modulo 4. Thus $a=0$, or $f(0)=0$. Now, substituting $y=0$ in $(*)$, we get

$$x^2 f(-x) = (f(x))^2, \quad \forall x \in \mathbb{Z},\ x \neq 0.$$

It follows that

$$x^2 f(x) = (f(-x))^2, \quad \forall x \in \mathbb{Z},\ x \neq 0.$$

Therefore

$$(f(x))^4 = x^4 (f(-x))^2 = x^6 f(x), \quad \forall x \in \mathbb{Z},\ x \neq 0.$$

From this and $f(0)=0$, we get $f(x)=0$ or $f(x)=x^2$ for any integer x. Clearly, the function $f(x)=x^2$ satisfies the original equation. Let us consider now the case there exists some integer $c \neq 0$ such that $f(c)=0$. Replacing $y=c$ in $(*)$, we get

$$c^2 f(2x) = 0, \quad \forall x \in \mathbb{Z},\ x \neq 0.$$

Thus, for any even integer x, we have $f(x)=0$. Now, if there exists some odd number d such that $f(d)=d^2$, taking x even and $y=d$ in $(*)$, we get

$$d^2 f(2x-d^2) = f(d^3), \quad \forall x \in \mathbb{Z},\ x \neq 0,\ 2 \mid x.$$

If $f(d^3) \neq 0$, then $f(2x-d^2) \neq 0$ and hence $f(d^3)=d^6$, $f(2x-d^2)=(2x-d^2)^2$. Plug back into the above equation, we get a contradiction. Therefore, $f(d^3)=0$ and hence $f(2x-d^2)=0$ for any nonzero even integer x. It follows that $f(x)=0$ for any integer x such that $x \equiv -1 \pmod 4$ and $x \neq -d^2$.

In $x^2 f(-x) = (f(x))^2$, taking $x \equiv -1 \pmod 4$, $x \neq -d^2$, we get $f(-x)=0$. It follows that $f(x)=0$ for any integer x such that $x \equiv 1 \pmod 4$ and $x \neq d^2$. Thus $f(x)=0$ for any integer $x \neq \pm d^2$. However, we have $f(d)=d^2 \neq 0$, so $d=\pm d^2$ or $d=\pm 1$. It follows that $d=d^3$ and so $f(d)=f(d^3)=0$, which is a contradiction. Therefore, in this case, we have $f(x)=0$ for any integer x.

In conclusion, the equation has two solution: $f(x)=0$ and $f(x)=x^2$. $\square$

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2.3. Solution of TST for JBMO 2018

The first day

Problem 1. Determine if there exists a triangle whose side lengths x, y, z satisfy

$$x^3 + y^3 + z^3 = (x+y)(y+z)(z+x).$$

Solution.
Since x, y, z is lengths of some triangle, then

$$x + y > z, y + z > x, z + x > y.$$

Suppose that x, y, z are side lengths of a triangle, then

$$\begin{aligned}(x+y)(x+z)(y+z) &= x^2(y+z) + y^2(x+z) + z^2(x+y) + 2xyz \\ &> x^3 + y^3 + z^3.\end{aligned}$$

So the answer is no. □

Problem 2. Let M and N are two nine-digit palindromes without leading zeros (a palindrome is a number that reads the same backward as forward) such that $M < N$ and there are no other nine-digit palindromes strictly between M and N. What are all possible values of $N - M$?

Solution.
Denote the number M is $abcdedcba$. We consider cases:

1. If $e \geq 8$ then the next number N is $abcd(e+1)dcba$.
2. If $e = 9, d \geq 8$ then the next number N is $abc(d+1)9(d+1)cba$.
3. If $e = d = 9, c \leq 8$ then the next number N is $ab(c+1)999(c+1)ba$.
4. If $e = d = c = 9, b \leq 8$ then next number is $a(b+1)99999(b+1)a$.
5. If $e = d = c = b = 9$, then $a \leq 8$ (otherwise, there is no N). Then the next number is $(a+1)9999999(a+1)$.

So the difference are $10000, 10100, 1000100, 10000010, 100000001$. □

Problem 3. Let ABC be a triangle inscribed in circle (O) with incenter I. The lines IB and IC intersect (O) again at J and L. Circumcircle (ω) of triangle IBC meets CA, AB again at E, F. Prove that EL and FJ intersect on (ω).

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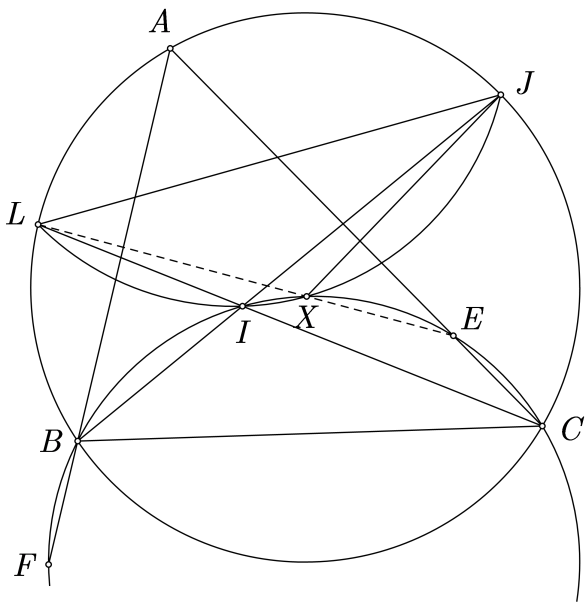
Solution.
Denote X as the other intersection of (IJL) and (ω). We shall prove that X is the intersection of EL and FJ.



By the cyclic quadrileteral, we have

$$\angle LXI = \angle LJI = \angle LJB = \angle LCB = \angle LCA = 180^\circ - \angle EXI.$$

Hence L, X, E are collinear, which means X belongs to EL.

Similarly, JF passes through X. These finish our proof. □

Problem 4. Let $n \geq 2$ be a positive integer. A subset of positive integers S is called *comprehensive* if for every integer $0 \leq x < n$, there is a subset of S who sum of elements has remainder x when divided by n. Note that the empty set has sum 0. Show that if a set S is comprehensive then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

Solution.
We will show that if $|S| \geq n$, we can remove one element from S and still have a comprehensive set. Doing this repeatedly will always allow us to find a comprehensive subset of size at most $n-1$.

Denote $S = \{s_1, s_2, \ldots, s_k\}$ for some $k \geq n$. Now start with the empty set and add in the elements s_i in order. During this process, we will keep track of all possible remainders of sums of any subset.

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If T is the set of current remainders at any time, and we add an element s_i, the set of remainders will be T and $\{t + s_i | t \in T\}$. In particular, the set of remainders only depends on the previous set of remainders and the element we add in.

At the beginning of our process, the set of possible remainders is $\{0\}$ for the empty set. Since we assumed that S is comprehensive, the final set is $\{0, 1, \ldots, n\}$. The number of elements changes from 1 to $n-1$.

However, since we added $k \geq n$ elements, at least one element did not change the size of our remainder set. This implies that adding this element did not contribute to making any new remainders and S is still comprehensive without this element, proving our claim. □

The second day

Problem 1. Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. Find A.

Solution.

Review the given condition as

$$pq(2r + 50) = pr(7q + 55) = qr(8p + 12) = A.$$

This implies that A is a multiple of p, q and r so the value $K = \frac{A}{pqr}$ is an integer. Dividing through, we have

$$K = 8 + \frac{12}{p} = 7 + \frac{55}{q} = 2 + \frac{50}{r}.$$

Hence, $p \mid 12, q \mid 55, r \mid 50$. So we have 3 cases as follow

- $p = 2, q = 11, r = 5$.
- $p = 3, q = 11, r = 5$.
- $p = 3, q = 5, r = 2$.

We can check that only $(p, q, r) = (3, 11, 5)$ works so $K = 12$ and the value of A is 1980. □

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Problem 2. Let a, b, c be real numbers such that $a+b+c+ab+bc+ca+abc \geq 7$. Prove that

$$\sqrt{a^2+b^2+2}+\sqrt{b^2+c^2+2}+\sqrt{c^2+a^2+2} \geq 6.$$

Solution.
First, by AM-GM we can show that

$$x^2+y^2+1 \geq xy+x+y \text{ for all } x, y, z \in \mathbb{R}.$$

Hence, $\sqrt{a^2+b^2+2} \geq \sqrt{|ab|+|a|+|b|+1} = \sqrt{(|a|+1)(|b|+1)}$. Construct similar inequalities and take the sum, we get

$$\begin{aligned}&\sqrt{a^2+b^2+2}+\sqrt{b^2+c^2+2}+\sqrt{c^2+a^2+2}\\ &\geq \sqrt{((|a|+1)(|b|+1)}+\sqrt{((|b|+1)(|c|+1)}+\sqrt{((|c|+1)(|a|+1)}.\end{aligned}$$

By AM-GM for three numbers, we get

$$\begin{aligned}&\sqrt{((|a|+1)(|b|+1)}+\sqrt{((|b|+1)(|c|+1)}+\sqrt{((|c|+1)(|a|+1)}\\ &\geq 3\sqrt[3]{(|a|+1)(|b|+1)(|c|+1)}\\ &= 3\sqrt[3]{|abc|+|ab|+|bc|+|ca|+|a|+|b|+|c|+1}\\ &\geq 3\sqrt[3]{abc+ab+bc+ca+a+b+c+1} \geq 3\sqrt[3]{7+1} = 6.\end{aligned}$$

By combining these two inequalities, we finish the proof. The equality occurs when $a=b=c=1$. □

Problem 3. The $n \times n \times n$ cubic is constructed from n^3 unit cubic $1 \times 1 \times 1$ such that at least one of unit cubic is black. Show that we can always cut the $n \times n \times n$ cubic into rectangular boxes such that each box contains exactly one black unit cubic.

Solution.
We shall prove the problem also true for all brick $a \times b \times c$ for any positive integers a, b, c by induction on the number of black boxes.

First, if the brick contains only one black cubic then no any cut is needed. Suppose that it contains at least two black cubics.

We choose a plane that divide the brick into two smaller bricks such that each smaller brick contains exactly one black cubic.

By induction hypothesis, we can divide each smaller bricks to make all black cubics are separated, then we are done!

Remark. This problem is easy as long as we notice that the problem is not about "big cubic", it's true for any arbitrary brick. □

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Problem 4. Let ABC be an acute triangle with O, H are its circumcenter, orthocenter. Take a point M belongs to the minor arc BC of (O) (not coincide to B, C) and denote D, E, F as reflection of M through OA, OB, OC. Suppose that BF meets CE at K and let I be the incenter of triangle DEF.

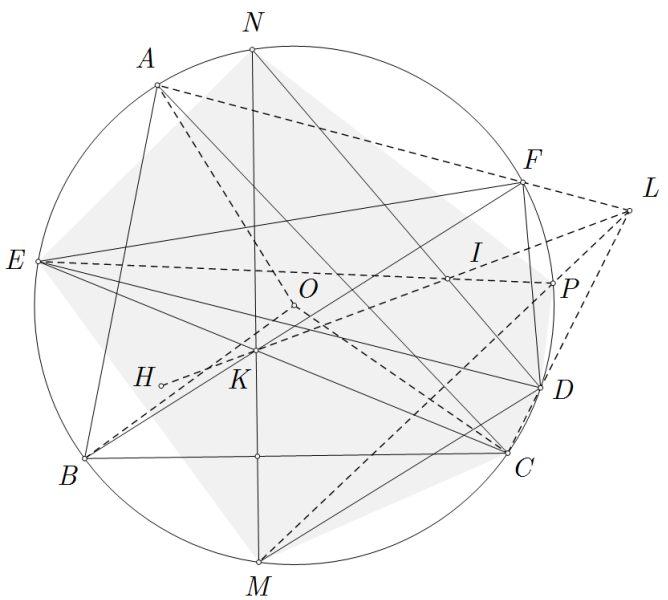
1. Prove that perpendicular bisectors of EF and IK meet on circle (O).

2. Prove that three points H, K, I are collinear.

Solution.
1) By definition of E, F we can see that EC is the angle bisector of $\angle MEF$ and FB is the angle bisector of $\angle MFE$. This implies that K is the incenter of triangle MEF.

Hence, MK is the angle bisector of $\angle EMF$ which passes through the midpoint N of the arc $EF$of (O).

So by applying the well known property, we have $NE = NK = NI = NF$.



This implies that E, F, K, I are concyclic and two perpendicular bisectors of EF, IK pass through the point $N \in (O)$.

2) We have $BK = BM = BE$ and $CK = CM = CE$. Hence, K is the reflection of M through BC. By similarly way, take $CD \cap AF = L$ then L is the reflection of M through AC and ML, EI intersect at midpoint P of the arc DF of (O).

Consider the hexagon $CDPNEM$ and apply Pascal's theorem, we can see that three intersections

$$L = CD \cap MP,\ I = DN \cap PE,\ K = CE \cap MN \text{ are collinear.}$$

We also know that KL is the Steiner's line of M respect to (O) then KL passes through orthocenter H of triangle ABC. This implies that H, I, K are collinear. □

The third day

Problem 1. Assume that n is a composite positive integer. For each proper divisor d of n, we write down $d+1$ on the board. Find all values of n such that the written numbers appear to be exactly all the proper divisors of some positive integer m.

Note that a proper divisor of a positive integer $a > 1$ is any positive divisor of a, distinct from 1 and a.

Solution.
The number 2 does not appear on the board since we do not consider divisor 1 so m is odd. Therefore, all numbers on the board are odd, which implies that all the $n = 2^k$ for some positive integer $k > 1$.

If $n = 4$ then we write number 3, then take $m = 9$ implies that this value satisfies.

If $n = 8$ then we write number $3, 5$, then take $m = 15$, also satisfies.

Suppose that some composite $n \geq 16$ is satisfy. All the proper divisors of n are $2, 4, 8, \ldots, 2^{k-1}$ and the numbers on the board are $3, 5, 9, \ldots, 2^{k-1}+1$. Since $3, 5, 9$ are divisors of some positive integer m then $45|m$ so 15 is the proper divisor of m. But the next smallest number should be $16+1 = 17$, a contradiction.

Therefore, $n = 4$ and $n = 8$ are the solutions. □

Problem 2. Let $ABCD$ be a square inscribed in circle (O). Let P be a point lies on minor arc CD of (O). Line PB intersects AC at E. Line PA intersects DB at F. Circumcircle of triangle PEF cuts (O) again at Q. Prove that PQ is parallel to CD.

Solution.
Let K be the circumcenter of triangle PEF. Note that

$$\angle APB = \angle ADB = 45^\circ$$

so triangle KEF is right isoscesles at K. Thus, quadrilateral $OEKF$ is cyclic.

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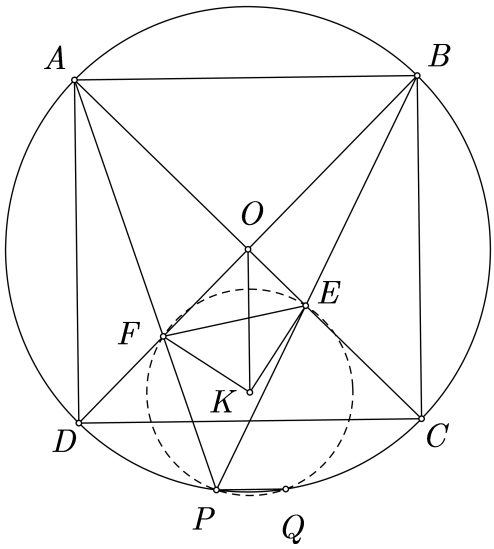

Hence, $\angle FOK = \angle FEK = 45°$, this means OK is bisector of angle DOC, it is also perpendidular bisector of CD. Since $KP = KQ, OP = OQ$, we have $OK \perp PQ$.

Therefore, PQ and CD are parallel. □

Problem 3. Prove that in every triangle, there are two sides x, y such that

$$\frac{\sqrt{5}-1}{2} \leq \frac{x}{y} \leq \frac{\sqrt{5}+1}{2}.$$

Solution.
Let a, b, c be the side lengths of the given triangle and we may assume that $a \geq b \geq c$. Denote $m = \frac{a}{b}$ and $n = \frac{b}{c}$. Note that

$$m, n \geq 1 > \frac{\sqrt{5}-1}{2}.$$

We will show that at least one of m, n is less than or equal to $\frac{\sqrt{5}+1}{2}$. Assume that both $m, n > \frac{\sqrt{5}+1}{2}$. Since $b + c > a$, it follows that $n + 1 > mn$. Hence,

$$n + 1 > mn > \frac{\sqrt{5}+1}{2}n, \text{ or } n < \frac{\sqrt{5}+1}{2},$$

which is a contradiction. Therefore, one of two numbers m, n is less than or equal $\frac{\sqrt{5}+1}{2}$, this finishes our proof. □

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Problem 4. Let $n > 2$ be a positive integer. Consider n bags of candy, each of them has exactly 1 candy. Ali and Omar take turn play the following game (Ali moves first): At each turn, the player takes two bags contain the numbers of candy as x, y for some coprime integers x, y and then merges them to one bag. Who cannot perform this action will be the loser. Who has the strategy to win this game?

Solution.
We shall prove that for all $n > 2$, Omar always has the strategy to win.

First, Ali has to merge some two bags of 1 candy into one bag of 2 candies. We prove that Omar can turn the state of all bags into: one bag contains odd number of candies and the others just have one candy each.

Indeed, in the second turn, Omar merges the bag of 2 candies with some bag of 1 candy to get one bag of 3 candies. Suppose that after a turn of Omar, there is a bag of $2k+1$ candies (with $k \in \mathbb{Z}^+$) and the others bag just have one candy each. At the next turn of Ali, there are two cases:

- If Ali merges two bags of 1 candy to one bag of 2 candies, then on the next turn, Omar merges that bag with the bag of $2k+1$ candies to get a bag of $2k+3$ candies.

- If Ali merges the bag of $2k+1$ candies with some bag of 1 candy to get another bag of $2k+2$, then on the next turn, Omar merges that bag with some bag of 1 candy to get a bag of $2k+3$ candies.

Hence, Omar always can control the state of bags like that. Note that after two turns of players, the number of bags reduces by 2. The finally, if after a turn of Omar, there is only one bag of $2k+1$ candies then Ali will be loser. Otherwise, there is a bag of $2k+1$ candies and 3 bags of 1 candy. There are two case:

- If Ali merges two bags of 1 candy then Omar merges the bag of $2k+1$ with the bag of 1 candy; then after that turn, there are a bag of 2 candies and a bag of $2k+2$ which implies that Ali loses.

- If Ali merges the bag of $2k+1$ candies with a bag of 1 candy then Omar merges two bags of 1 candy to make two bags of even number of candies as above.

Therefore, in all case of $n > 2$, Omar always can win the game.

□

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2.4. Solution of TST for IMO 2018

The first day

Problem 1. Consider the infinite, strictly increasing sequence of positive integer (a_n) such that

i. All terms of sequences are pairwise coprime.

ii. The sum $\dfrac{1}{\sqrt{a_1a_2}}+\dfrac{1}{\sqrt{a_2a_3}}+\dfrac{1}{\sqrt{a_3a_4}}+\cdots$ is unbounded.

Prove that this sequence contains infinitely many primes.

Solution.
Suppose on the contrary that there are infinite primes in this sequence, thus there is some index k such that a_n is composite for all $n>k$. Denote S as all prime divisors of the first k terms of given sequence.

By comparing a_{k+1} with the first prime p_1 that does not appear in S then we have $a_{k+1}\geq p_1^2$ (since a_{k+1} is composite and p_1 is not greater than the least prime divisor of this number).

Similarly define p_2 is the second prime that does not appear in S then $a_{k+2}\geq p_2^2$ (since all terms of sequence are pairwise coprime). In general, we get

$$a_{k+i}\geq p_i^2 \text{ for all } i\in\mathbb{Z}^+.$$

Note that $\dfrac{1}{\sqrt{a_{k+1}a_{k+2}}}<\dfrac{1}{a_{k+1}}\leq\dfrac{1}{p_1^2}, \dfrac{1}{\sqrt{a_{k+2}a_{k+3}}}<\dfrac{1}{a_{k+2}}\leq\dfrac{1}{p_2^2},\cdots$ and so on. Then the given sum in second condition can be divided into two parts: the first is the sum take on value from $a_1,a_2,\ldots,a_k$ which is finite, and the second is less than

$$\begin{aligned}\frac{1}{p_1^2}+\frac{1}{p_2^2}+\cdots&<\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots<1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\\&=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)<2.\end{aligned}$$

Hence, the given sum is bounded, which is a contradiction and this finishes our proof.

Remark. There is some sequence satisfies the given conditions in this problem. We can take (a_n) are all prime numbers. By applying a well-known result of Euler about the sum of reciprocal of primes tends to infinity, the second condition holds. □

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Problem 2. Let n be an even positive integer. We fill in a number on each cell of a rectangle table of n columns and multiple rows as following:

i. Each row is assigned to some positive integer a and its cells are filled by 0 or a (in any order);

ii. The sum of all numbers in each row is n.

Note that we cannot add any more row to the table such that the conditions (i) and (ii) still hold. Prove that if the number of 0's on the table is odd then the maximum odd number on the table is a perfect square.

Solution.
Denote $\sigma(n), \tau(n)$ as the sum of divisors and number of divisors of n, respectively.
Consider a row with assigned number is a, and suppose that there are b cells on that row are filled by a then $n = ab$ or a is a divisor of n. By condition ii), the number of rows of the table is exactly the number of positive divisors of n.
Number of nonzero on each row is $\frac{n}{a}$, thus the number of zero on each row is $n - \frac{n}{a}$. Number of zero on table is

$$\sum_{a|n}\left(n - \frac{n}{a}\right) = \sum_{a|n}(n-a) = n \cdot \tau(n) - \sum_{a|n} a = n \cdot \tau(n) - \sigma(n).$$

This number is odd and n is even which imply that $\sigma(n)$ is odd. Put $n = 2^k \cdot q$ with q is odd number then q is also the maximum odd number on the table. The sum of divisor $\sigma(n)$ of n is of form $\prod_{p^\alpha \| n} \sigma(p^\alpha)$.

- If $p = 2$ then $\sigma(2^k)$ is an odd number.
- If $p > 2$ then $\sigma(p^\alpha) = 1 + p + \cdots + p^\alpha$ must be odd, which implies that α is even.

From this, we can conclude that all odd prime divisors of n have even exponent, thus q is a perfect square. $\square$

Problem 3. Let $ABCD$ be a convex quadrilateral inscibed in circle (O) such that $DB = DA + DC$. The point P lies on the ray AC such that $AP = BC$. The point E is on (O) such that $BE \perp AD$. Prove that DP is parallel to the angle bisector of $\angle BEC$.

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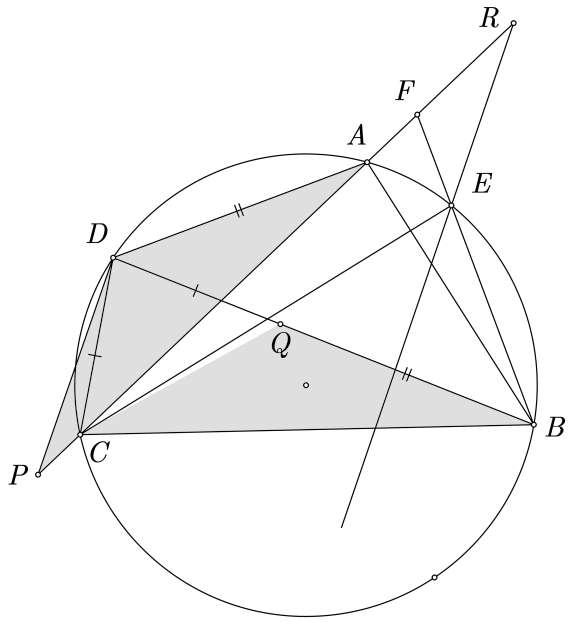
Solution.
Denote R, F as the intersections of AC with the angle bisector of $\angle BEC$ and BE, respectively. Let Q be a point on BD such that $DQ = DC$, hence $QB = DB - DQ = DB - DC = AD$.



Thus two triangles CQB and PDA are congruent (s.a.s). Note that $AD \perp BE$ then $\angle CAD + \angle AFE = 90^\circ$. From these, we have

$$\begin{aligned}\angle APD &= \angle BCQ = \angle CQD - \angle CBD \\ &= 90^\circ - \frac{1}{2}\angle BDC - \angle CAD = (90^\circ - \angle CAD) - \frac{1}{2}\angle BEC \\ &= \angle AFE - \angle BER = \angle ARE.\end{aligned}$$

Hence, we conclude that $DP \parallel ER$. □

The second day

Problem 4. Let ABC be an acute, non isosceles triangle with M, N, P are midpoints of BC, CA, AB, respectively. Denote d_1 as the line passes through M and perpendicular to the angle bisector of $\angle BAC$, similarly define for d_2, d_3. Suppose that $d_2 \cap d_3 = D, d_3 \cap d_1 = E, d_1 \cap d_2 = F$. Let I, H be the incenter and orthocenter of triangle ABC. Prove that the circumcenter of triangle DEF is the midpoint of segment IH.

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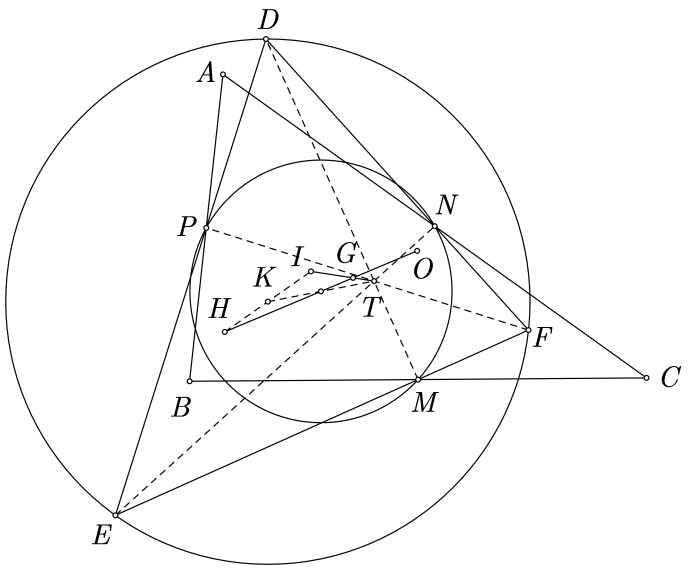
Solution.
Denote $(I_a),(I_b),(I_c)$ as ex-circle respect to angle A,B,C of triangle ABC. It is easy to check that $\mathscr{P}_{M/(I)} = \mathscr{P}_{M/(I_a)}$ and $\mathscr{P}_{M/(I_b)} = \mathscr{P}_{M/(I_c)}$. Let O,K be the circumcenter of triangle ABC, DEF.

Hence, d_1 is the radical axis of $(I),(I_a)$ since $d_1 \perp II_a$. Similar to d_2, d_3 which implies that $D = d_2 \cap d_3$ is the radical center of three circles $(I),(I_b),(I_c)$. Thus, MD is the radical axis of $(I_b),(I_c)$.

So we have $MD \perp I_bI_c$, but $I_bI_c \perp AI, EF \perp AI$ then $MD \perp EF$ leads to MD is the altitude in triangle DEF.



Similar to NE, PF so three lines MD, NE, PF concur at point T which is the orthocenter of DEF. Two triangles ABC and DEF share the same nine-point circle (MNP) then OH and KT share the midpoint, which is the circumcenter of triangle (MNP). This implies that $\overrightarrow{HK} = \overrightarrow{TO}$.

On the other hand, it is easy to check that T also is the incenter of triangle MNP. Then consider the homothety with center G as the centroid of triangle ABC, ratio -2 then $\mathscr{H} : M \to A,\ N \to B,\ P \to C$ implies that $\mathscr{H} : T \to I,\ O \to H$. Hence, $\overrightarrow{HI} = 2\overrightarrow{TO}$.

Combine with the previous equality, we have $\overrightarrow{HI} = 2\overrightarrow{HK}$, so K is the midpoint of the segment HI.

□

Problem 5. *Problem about arithmetic and geometric sequence.*

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1. For integer $n \geq 3$, suppose that $0 < a_1 < a_2 < \ldots < a_n$ is a arithmetic sequence and $0 < b_1 < b_2 < \ldots < b_n$ is a geometric sequence with $a_1 = b_1, a_n = b_n$. Prove that $a_k > b_k$ for all $k = 2, 3, \ldots, n-1$.

2. Prove that for every positive integer $n \geq 3$, there exist an integer arithmetic sequence (a_n) and an integer geometric sequence (b_n) such that
$$0 < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n.$$

Solution.
1) Put $a_1 = b_1 = x$ and $a_n = b_n = y$, then $0 < x < y$. Denote d as the difference and the ratio of two consecutive terms of $(a_n), (b_n)$ respectively.

We have $a_n = a_1 + (n-1)d$ and $a_k = a_1 + (k-1)d$ for any $k \in \{2, 3, \ldots, n-1\}$. Hence,
$$a_k = \frac{(n-k)a_1 + (k-1)a_n}{n-1} = \frac{(n-k)x + (k-1)y}{n-1}.$$
Similarly, $b_k = \sqrt[n-1]{x^{n-k}y^{k-1}}$ so we need to prove that
$$\frac{(n-k)x + (k-1)y}{n-1} > \sqrt[n-1]{x^{n-k}y^{k-1}},$$
which is true by AM-GM inequality.

2) With $2^k \leq m$, consider some estimations as follow
$$\left(1 + \frac{1}{m}\right)^k = 1 + \frac{\binom{k}{1}}{m} + \frac{\binom{k}{2}}{m^2} + \cdots \geq 1 + \frac{k}{m}$$
and
$$\left(1 + \frac{1}{m}\right)^k = 1 + \frac{\binom{k}{1}}{m} + \frac{\binom{k}{2}}{m^2} + \cdots \leq 1 + \frac{\binom{k}{1}}{m} + \frac{\binom{k}{2} + \cdots + \binom{k}{k}}{m}$$
$$< 1 + \frac{k}{m} + \frac{2^k}{m^2} \leq 1 + \frac{k+1}{m}$$
then $1 + \frac{k}{m} < \left(1 + \frac{1}{m}\right)^k < 1 + \frac{k+1}{m}$ or $m + k < m\left(1 + \frac{1}{m}\right)^k < m + k + 1$. This is true for all $k \leq \log_2 m$ so
$$m\left(1 + \frac{1}{m}\right) < m + 2 < m\left(1 + \frac{1}{m}\right)^2 < m + 3 < \cdots < m + k + 1.$$
To make all these numbers are integers, we multiple them by $m^{[\log_2 m]}$. Finally, take $\log_2 m > n$, the result will follow. $\square$

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Problem 6. Two sets of positive integers A, B are called *connected* if they are not empty and for all $a \in A$, $b \in B$, number $ab+1$ is a perfect square.

1. Given $A = \{1, 2, 3, 4\}$. Prove that there does not exist any set B such that A, B are connected.

2. Suppose that A, B are connected with $|A|, |B| \geq 2$. For any $a_1 > a_2 \in A$ and $b_1 > b_2 \in B$, prove that $a_1 b_1 > 13 a_2 b_2$.

Solution.
1) Assume on the contrary that there exist set B connects to A. Take $n \in B$, then we must have $n+1 = x^2, 4n+1 = y^2$ for some $x, y \in \mathbb{Z}$. Then

$$x^2 y^2 = (n+1)(4n+1) = 4n^2 + 5n + 1.$$

Note that $(2n+1)^2 < 4n^2+5n+1 < (2n+2)^2$ thus this number cannot be a perfect square, a contradiction.

2) Since $a_1 > a_2, b_1 > b_2$ then $(a_1 b_1 + 1)(a_2 b_2 + 1) > (a_1 b_2 + 1)(a_2 b_1 + 1)$ or

$$\sqrt{(a_1 b_1 + 1)}\sqrt{(a_2 b_2 + 1)} > \sqrt{(a_1 b_2 + 1)}\sqrt{(a_2 b_1 + 1)}.$$

Note that the expressions under square root are all perfect squares, then we can rewrite

$$\sqrt{(a_1 b_1 + 1)}\sqrt{(a_2 b_2 + 1)} \geq \sqrt{(a_1 b_2 + 1)}\sqrt{(a_2 b_1 + 1)} + 1.$$

Squaring both sides and expanding them, we get

$$\begin{aligned} a_1 b_1 + a_2 b_2 &\geq a_1 b_2 + a_2 b_1 + 2\sqrt{(a_1 b_2 + 1)(a_2 b_1 + 1)} + 1 \\ &> a_1 b_2 + a_2 b_1 + 2\sqrt{a_1 a_2 b_1 b_2}. \end{aligned}$$

By applying AM-GM inequality, we have $a_1 b_2 + a_2 b_1 \geq 2\sqrt{a_1 a_2 b_1 b_2}$, then

$$a_1 b_1 + a_2 b_2 > 4\sqrt{a_1 a_2 b_1 b_2}.$$

Rewrite this inequality as $\frac{a_1 b_1}{a_2 b_2} + 1 > 4\sqrt{\frac{a_1 b_1}{a_2 b_2}}$. Put $t = \sqrt{\frac{a_1 b_1}{a_2 b_2}}$ then $t^2 - 4t + 1 > 0$, so $t > 2 + \sqrt{3}$. From this we can conclude that

$$\frac{a_1 b_1}{a_2 b_2} = t^2 > (2+\sqrt{3})^2 = 7 + 4\sqrt{3} > 7 + 4 \cdot \frac{3}{2} = 13.$$

□

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The third day

Problem 7. Denote S as the set of prime divisors of all integers of form $2^{n^2+1}-3^n, n \in \mathbb{Z}^+$. Prove that S and $\mathscr{P} \backslash S$ are both contains infinitely many elements ($\mathscr{P}$ is set of prime numbers).

Solution.
First, suppose on the contrary that S is finite, then $S = \{p_1, p_2, \ldots, p_k\}$ for $k = |S|$. It is easy to check that $2, 3 \notin S$.

Consider number $N = (p_1-1)(p_2-1)\cdots(p_k-1)$ and $M = 2^{N^2+1}-3^N$. For some $p \in S$, by Euler's theorem, we have

$$2^{p-1} \equiv 1 \pmod{p} \text{ and } 3^{p-1} \equiv 1 \pmod{p}.$$

Then $2^{N^2+1} = 2 \cdot 2^{N^2} = 2 \cdot \left(2^{p-1}\right)^{\frac{N^2}{p-1}} \equiv 2 \pmod{p}$ and $3^N \equiv \left(3^{p-1}\right)^{\frac{N}{p-1}} \equiv 1 \pmod{p}$. Hence, $M \equiv 2-1 = 1 \pmod{p}$. So by Chinese remainder theorem, we also have $M \equiv 1 \pmod{N}$ and $M > 2$ then M has some prime divisor that differs from $p_1, p_2, \ldots, p_k$, which is a contradiction.

Thus, S has infinitely many number of elements.

To prove the second part, just consider some prime p such that there does not exist $x \in \mathbb{Z}$ such that either $x^2 \equiv 2 \pmod{p}$ or $x^2 \equiv 3 \pmod{p}$. Indeed, suppose that for such prime p, $p \in S$ the we have two cases:

- If n is even then $2 \cdot 2^{n^2} \equiv 3^n \pmod{p}$, which can be written as $2x^2 \equiv y^2 \pmod{p}$ for some x, y are coprime to p. Then take x' such that $xx' \equiv 1 \pmod{p}$, so $2 \equiv (x'y)^2 \pmod{p}$, contradict to 2 is not a quadratic residue modulo p.
- If n is odd then we have the similar argument.

Thus, we just need to show that there are infinitely many such prime. For Euler's criterion about quadratic residue, we need

$$\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = -1 \text{ and } \left(\frac{2}{p}\right) \equiv 2^{\frac{p-1}{2}} (\text{mod} p), \left(\frac{3}{p}\right) \equiv 3^{\frac{p-1}{2}} \pmod{p}.$$

So we can see that $p \equiv 3, 5 \pmod{8}$ and $p \equiv 2 \pmod{3}$ which implies that we can take any prime of form $p = 24k+5$ or $p = 24k+11$. Dirichlet's theorem states that the prime of form $an+b$ for any coprime positive integers a, b are infinite, so we finish our proof.

□

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Problem 8. A non-empty subset of $\{1, 2, \ldots, n\}$ is called *arabic* if arithmetic mean of its elements is an integer. Show that the number of arabic subsets of $\{1, 2, \ldots, n\}$ has the same parity as n.

Solution.
The solution is based on a simple fact, that if add the arithmetic mean of the sequence to the sequence, then the arithmetic mean of the sequence will not change, since

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_2 + \cdots + a_n + \frac{a_1+a_2+\cdots+a_n}{n}}{n+1}$$

Denote by $\mu(A)$ the arithmetic mean of the elements of A. Denote

$$P = \{A \subset \{1, 2, \ldots, n\} \quad : \quad \mu(A) \in Z_+, \mu(A) \in A\}$$

and

$$Q = \{A \subset \{1, 2, \ldots, n\} \quad : \quad \mu(A) \in Z_+, \mu(A) \notin A\}.$$

We need to prove that $|P| + |Q|$ is even. For that we will prove $|P| = |Q|$. Really, take any set from P and remove it's arithmetic mean. We will get an element from Q. Take any set from Q and add it's arithmetic mean. We will get an element from P. Since arithmetic mean of the set is defined uniquely, so we have bijection between P and Q. □

Problem 9. Consider the function

$$f(x) = (x - F_1)(x - F_2)\ldots(x - F_{3030})$$

with (F_n) is the Fibonacci sequence, which defined as $F_1 = 1, F_2 = 2, F_{n+2} = F_{n+1} + F_n, n \geq 1$. Suppose that on the range (F_1, F_{3030}), the function $|f(x)|$ takes on the maximum value at $x = x_0$. Prove that $x_0 > 2^{2018}$.

Solution.
We will prove that $x_0 \in (F_{3029}, F_{3030})$ by showing that for all $x^* \in (F_1, F_{3029}]$, there is some $x^{**} \in (F_{3029}, F_{3030})$ for $|f(x^{**})| > |f(x^*)|$.

Indeed, if

$$x^* \in \left\{F_1, F_2, \ldots, F_{3029}\right\}$$

then $|f(x^*)| = 0$ which is obvious.

Suppose that $x^* \in (F_k, F_{k+1})$ for $1 \leq k \leq 3028$ then put $m = x^* - x_k$, we choose $x^{**} = F_{3030} - m$. We need

$$|f(x^{**})| > |f(x^*)| \Leftrightarrow \left|\prod_{i=1}^{4030}(x^{**} - F_i)\right| > \left|\prod_{i=1}^{4030}(x^* - F_i)\right|.$$

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Each side has 4030 positive factors then we will make pair one of the left and one of the right such that the value of the right is bigger (except the case $i = k$ then $|x^* - F_k| = m = |x^{**} - F_{3030}|$). For details:

1. If $i = 1, 2, \ldots, k$, it is clearly that $x^{**} - F_i > x^* - F_i > 0$.
2. If $1 \le i \le 3030 - k$, we have $|x^* - F_{k+i}| < |x^{**} - F_{3030-i}|$. Note that we just consider the separated ranges, i.e $k+i \le 3030-i$. Otherwise, some ranges are overlap then we remove that part, then

$$\begin{aligned} F_{k+i} - x^* &= (F_{k+1} - x^*) + (F_{k+2} - F_{k+1}) + \cdots + (F_{k+i} - F_{k+i-1}) \\ &< (x^{**} - F_{3029}) + (F_{3029} - F_{3028}) + \cdots + (F_{3030-(i-1)} - F_{3030-i}) \\ &= x^{**} - F_{3030-i} \end{aligned}$$

Hence, the statement is proved. From here, we have

$$\begin{aligned} x_0 > F_{3029} &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{3030} - \left(\frac{1-\sqrt{5}}{2}\right)^{3030}\right] \\ &> \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{3029} > \left(\frac{1+\sqrt{5}}{2}\right)^{3027}. \end{aligned}$$

It is easy to check that $\frac{1+\sqrt{5}}{2} > 2^{\frac{2}{3}}$ then substitute into the above inequality, we get $x_0 > 2^{2018}$.

□

The fourth day

Problem 10. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying $f(1) = 2,\ f(2) \ne 4$, and

$$\max\{f(m) + f(n), m + n\} \,|\min\{2m + 2n, f(m+n) + 1\}$$

for all $m, n \in \mathbb{Z}^+$.

Solution.
In given condition, substitute $m = n$, we have

$$\max\{2f(n), 2n\} | \min\{4n, f(2n) + 1\},$$

thus $4n \ge 2f(n)$ or $f(n) \le 2n$.
Continue substitute $m = 1$, we have

$$\max\{2 + f(n), 1 + n\} \,|\min\{2 + 2n, f(1+n) + 1\} \ (*)$$

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Thus $f(n+1)+1 \geq f(n)+2$ or $f(n+1) \geq f(n)+1$. Since $f(1)=2$, we have $f(n) \geq n+1, \forall n \in \mathbb{Z}^+$.

From these, we can conclude that $f(2) \in \{3;4\}$. But we have $f(2) \neq 4$ then $f(2)=3$. We shall prove by induction that $f(n)=n+1, \forall n \geq 2$.

The conclusion is true for $n=1,2$. Suppose that $f(n)=n+1$ for some $n \geq 2$.

Back to $(*)$, note that $\max\{2+f(n), 1+n\} = f(n)+2 = n+3$ and $\dfrac{2n+2}{n+3} \in (1;2)$ which implies that $n+3 \not| 2n+2$. Hence

$$\min\{2+2n, f(1+n)+1\} = f(n+1)+1.$$

Since $n+2 \leq f(n+1) \leq 2n+2$, we must have $\dfrac{f(n+1)+1}{n+3} = 1 \Leftrightarrow f(n+1)=n+2$. By induction hypothesis, we get $f(n)=n+1$.

Check condition, for all $m,n \in \mathbb{Z}^+$ then

$$\min\{2m+2n, f(m+n)+1\} = \min\{2m+2n, m+n+2\} = m+n+2 \text{ and}$$

$$\max\{f(m)+f(n), m+n\} = \max\{m+1+n+1, m+n\} = m+n+2$$

Hence, the function $f(n)=n+1$ satisfies the given condition.

Remark. For the case $f(2)=4$, we can find another solution of this problem is $f(n)=2n$. The solution is a bit of tricky as follow.

Denote A as the set of positive integer a such that $f(a)=2a$, thus $A \neq \varnothing$ since $1 \in A$. Substitute $m=1$ into the given condition, we have

$$f(n)+2 \,|\, \min(2n+2, f(n+1)+1)$$

Note that if $f(n)<2n$ then $\min(2n+2, f(n+1)+1) = f(n+1)+1$ and $n+3 \leq f(n)+2 \,|\, f(n+1)+1 \leq 2n+3 < 2(n+3)$. So we must have $f(n)+2 = f(n+1)+1$ or $f(n+1) = f(n)+1 < 2(n+1)$, which implies that for all $N>n$, then $f(N)<2N$. Hence, we have two cases:

- *If $A=\mathbb{Z}^+$ then $f(n)=2n$ will be a solution.*
- *If $\exists n \in \mathbb{Z}^+$ but $n \notin A$, as above, we conclude that A has finite number of elements. Take $a>1$ as the smallest integer that not belong to A. So $f(n)=2n, \forall n<a$ and $f(a)=b \in [a+1, 2a-1] \cap \mathbb{Z}^+$.*

  Note that, for $n \geq a$ then $f(n+1)=f(n)+1$ so $f(n)=n+b-a$ for all $n \geq a$. Take $m,n \geq a$ and substitute to the given condition, we have

$$\max(m+n+2(b-a), m+n) \,|\, \min(2m+2n, m+n+b-a+1)$$

So $m+n+2(b-a)\,|\,m+n+b-a+1$. *But* $b \geq a+1$ *so* $b-a+1 \leq 2(b-a)$, *which means* $m+n+2(b-a) \geq m+n+b-a+1$. *Thus, we get* $b = a+1$. *Finally, choose* $m = n = a-1$, *then*

$$4(a-1)\,|\min(4(a-1), 2a) \Rightarrow 4(a-1) = 2a \Rightarrow a = 2.$$

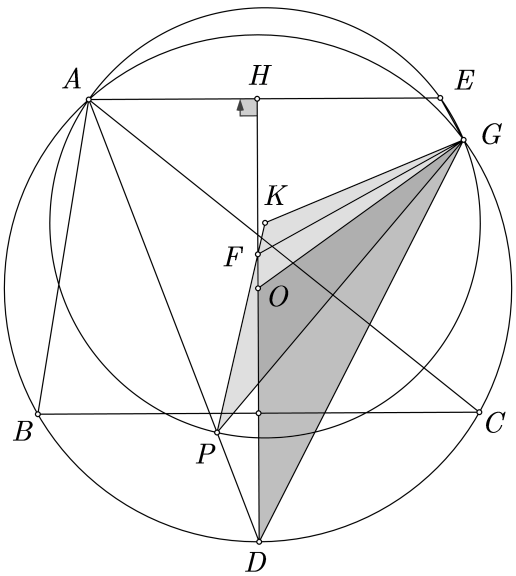
From this, we conclude that $f(n) = n+1$ *for all* $n \geq 1$.

Therefore, $f(n) = 2n$ *and* $f(n) = n+1$ *are two solution of this problem.*

□

Problem 11. Let ABC be an acute-angled triangle inscribed in circle (O). Let G be a point on the small arc AC of (O) and (K) be a circle passing through A and G. The angle bisector of $\angle BAC$ cuts (K) again at P. The point E is chosen on (K) such that AE is parallel to BC. The line PK meets the perpendicular bisector of BC at F. Prove that $\angle EGF = 90°$.

Solution.
Let D be the second intersection of AP and (O) and Hh be the intersection of OD and AE. Note that D is the midpoint of the minor arc BC of (O) then OD is the perpendicular bisector of BC. Since $AE \parallel BC$, we have $\angle DHE = 90°$.



On the other hand,

$$\angle DOG = 2\angle DAG = \angle PKG,$$

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which implies that two isosceles triangles KPG and ODG are similar. We get $\angle ODG = \angle KPG$ therefore $PFGD$ is cyclic.

From this, $\angle AEG = \angle DPG = \angle DFG$. This means $HEGF$ is cyclic, we deduce to $\angle EGF = 180^\circ - \angle EHF = 90^\circ$. □

Problem 12. Find all positive integers k such that there exists some permutation of $(1, 2, \ldots, 1000)$ namely $(a_1, a_2, \ldots, a_{1000})$ and satisfy $|a_i - i| = k$ for all $i = \overline{1, 1000}$.

Solution.
Note that $k = 1$ is an answer since we can choose a permutation with $a_{2i-1} = 2i,\ a_{2i} = 2i - 1$. Now we take $k > 1$, denote $S_0, S_1, S_2, \ldots, S_{k-1}$ as the subsets of $\{1, 2, 3, \ldots, 1000\}$ and all elements of S_i are congruent to i modulo k. We have:

- The number of elements of S_t is $m_t = \left\lceil \frac{1000-t}{k} \right\rceil$.
- Since $a_i = i \pm k$ and $a_i \equiv i \pmod{k}$, two numbers a_i and i belong to same subset.

Denote $t = x_1 < x_2 < \ldots < x_{m_t}$ as elements of S_t then they form an arithmetic progression with formular $x_i = t + (i-1)k$ for $1 \le i \le m_t$.

Put $y_i = a_{x_i}$ for $1 \le i \le m_t$ then $|y_i - x_i| = \left|a_{x_i} - x_i\right| = k$ which implies that $y_1 < y_2 < \ldots < y_{m_t}$ is a permutation of $x_1 < x_2 < \ldots < x_{m_t}$.

Set $r_i = \frac{x_i - t}{k} + 1, s_i = \frac{y_i - t}{k} + 1$ for $1 \le i \le m_t$ then $r_i = i$ and $|s_i - i| = 1$.

We have

$$\sum_{i=1}^{m_t} |r_i - s_i| + \sum_{i=1}^{m_t} |r_i + s_i| = 2\sum_{i=1}^{m_t} \max\{r_i, s_i\}$$

so $m_t + 2\sum_{i=1}^{m_t} r_i = 2\sum_{i=1}^{m_t} \max\{r_i, s_i\}$ and m_t is even. By the similar definition for $i = 0, 1, \ldots, k-1$, we have $m_0 \le m_1 \le \ldots \le m_{k-1}$ are all even and

$$m_0 - m_{k-1} = \left[\frac{1000}{k}\right] - \left[\frac{1000-(k-1)}{k}\right] \le 1.$$

Hence, $m_0, m_1, \ldots, m_{k-1}$ are all equal and even. This can be happened when $k|1000$ and $\frac{1000}{k}$ is an even number. It is easy to check that we can construct such a permutation satisfy with these k.

Therefore, k is a divisor of 500. □

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مؤسسة الملك عبدالعزيز ورجاله للموهبة والإبداع

King Abdulaziz & his Companions Foundation For Giftedness & Creativity

موهبة

IMO Booklet

SAUDI ARABIAN Mathematical Competitions 2019

SAMC 2019

Riyadh, June 2019

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مسابقات الرياضيات للمملكة العربية السعودية

SAUDI ARABIAN

MATHEMATICAL COMPETITIONS 2019

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وزارة التعليم
Ministry of Education

جامعة الملك عبدالله للعلوم والتقنية
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The Saudi Arabian team at **IMO 2019**

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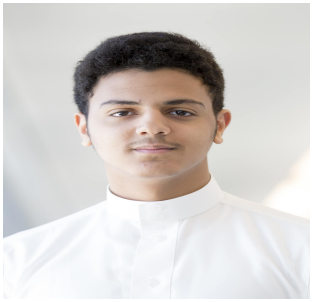

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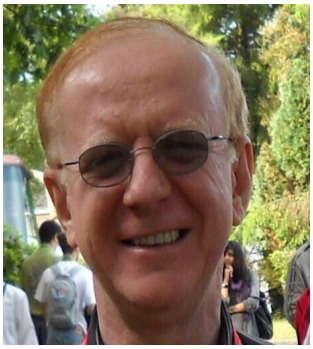

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Introduction

This booklet contains the Team Selection Tests of the Saudi teams to the Balkan Mathematics Olympiad, Balkan Junior Mathematics Olympiad, and the International Mathematics Olympiad.

The training was supported by the Ministry of Education, which commissioned Mawhiba, the main establishment in Saudi Arabia that cares for the gifted students, to do the task.

We would like to express our gratitude to King Abdullah University of Science and Technology KAUST for making its facilities on its beautiful campus available to us for our training.

The Saudi team had three main training camps during the academic year 2018-2019. In addition, the team had an intensive training period from March to the end of June 2019.

During this academic year, the selected students participated in the following contests: The Asia Pacific Mathematics Olympiad, the European Girls Mathematics Olympiad in Ukraine, Balkan Mathematics Olympiad in Moldova and the Junior Balkan mathematics Olympiad in Cyprus.

It is our pleasure to share the selection tests problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Fawzi A Al-Thukair
Leader of the Saudi team in IMO 2019

مقدمة

يحوي هذا الكتيب على مسائل التصفيات لمسابقة البلقان و مسابقة البلقان للناشئين و تصفيات الاولمبياد الدولي للرياضيات ٢٠١٩.

ان تدريب الفريق كان بدعم من وزارة التعليم بالتعاون مع مؤسسة الملك عبد العزيز و رجاله للموهبة و الابداع " موهبة "

وتجدر الاشارة الى التعاون و الاسهام الفعّال من جامعة الملك عبدالله للعلوم و التقنية، حيث وفرت لنا كل الامكانات التي احتجنا لها في التدريب في حرمها الجامعي الجميل.

تم عقد ثلاثة ملتقيات تدريبية خلال العام الدراسي ٢٠١٨-٢٠١٩ بالاضافة الى فترة التدريب المكثف التي بدأت في شهر مارس ٢٠١٩ الى نهاية شهر يونيو. كما شارك الطلبة المتميزون في العديد من المسابقات الإقليمية و منها: اولمبياد الرياضيات لدول آسيا و الباسيفيك ، اولمبياد الطالبات للدول الأوربية في اوكرانيا ، اولمبياد البلقان في مالدوفا و اولمبياد الناشئين لدول البلقان في قبرص.

نأمل ان يكون محتوى هذا الكتيب إسهاماً منا لتقوية اواصر التعاون و تبادل الخبرات بيننا و الدول المشاركة في الاولمبياد الدولي.

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رئيس الفريق السعودي للاولمبياد الدولي. ٢٠١٩

Selected problems from camps

1. January camp

1.1. Test 1

Problem 1. Suppose that x, y, z are non-zero real numbers such that

$$x = 2 - \frac{y}{z},\ y = 2 - \frac{z}{x},\ z = 2 - \frac{x}{y}.$$

Find all possible values of $T = x + y + z$.

Problem 2. Let $P(x)$ be a polynomial of degree $n \geq 2$ with rational coefficients such that $P(x)$ has n pairwise different real roots forming an arithmetic progression. Prove that among the roots of $P(x)$ there are two that are also the roots of some polynomial of degree 2 with rational coefficients.

Problem 3. Let $ABCDEF$ be a convex hexagon satisfying $AC = DF$, $CE = FB$ and $EA = BD$. Prove that the lines connecting the midpoints of opposite sides of the hexagon $ABCDEF$ intersect in one point.

1.2. Test 2

Problem 4. Suppose that a, b, c, d are pairwise distinct positive integers such that $a + b = c + d = p$ for some odd prime $p > 3$. Prove that $abcd$ is not a perfect square.

Problem 5. There are 3 clubs A, B, C with non-empty members. For any triplet of members (a, b, c) with $a \in A, b \in B, c \in C$, two of them are friend and two of them are not friend (here the friend relationship is bidirectional). Prove that one of these statements must be true

1. There exist one student from A that knows all students from B.
2. There exist one student from B that knows all students from C.
3. There exist one student from C that knows all students from A.

Problem 6. Let ABC be a triangle with A', B', C' are midpoints of BC, CA, AB respectively. The circle (ω_A) of center A has a big enough radius cuts $B'C'$ at X_1, X_2. Define circles $(\omega_B), (\omega_C)$ with Y_1, Y_2, Z_1, Z_2 similarly. Suppose that these circles have the same radius, prove that $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are concyclic.

1.3. Test 1

Problem 7. Let ABC be a triangle inscribed in a circle (ω) and I is the incenter. Denote D, E as the intersection of AI, BI with (ω). And DE cuts AC, BC at F, G respectively. Let P be a point such that $PF \parallel AD$ and $PG \parallel BE$. Suppose that the tangent lines of (ω) at A, B meet at K. Prove that three lines AE, BD, KP are concurrent or parallel.

Problem 8. It is given a graph whose vertices are positive integers and an edge between numbers a and b exists if and only if

$$a + b + 1 \mid a^2 + b^2 + 1.$$

Is this graph connected?

Problem 9. Define sequence of positive integers (a_n) as $a_1 = a$ and $a_{n+1} = a_n^2 + 1$ for $n \geq 1$. Prove that there is no index n for which

$$\prod_{k=1}^{n} \left(a_k^2 + a_k + 1\right)$$

is a perfect square.

2. March camp - BMO TST

2.1. Test 1

Problem 1. Let p be an odd prime number.

1. Show that p divides $n2^n + 1$ for infinitely many positive integers n.
2. Find all n satisfy condition above when $p = 3$.

Problem 2. Let I be the incenter of triangle ABC and J the excenter of the side BC. Let M be the midpoint of CB and N the midpoint of arc BAC of circle (ABC). If T is the symmetric of the point N by the point A, prove that the quadrilateral $JMIT$ is cyclic.

Problem 3. For $n \geq 3$, it is given an $2n \times 2n$ board with black and white squares. It is known that all border squares are black and no 2×2 subboard has all four squares of the same color. Prove that there exists a 2×2 subboard painted like a chessboard, i.e. with two opposite black corners and two opposite white corners.

2.2. Test 2

Problem 4. There are n people with hats present at a party. Each two of them greeted each other exactly once and each greeting consisted of exchanging the hats that the two persons had at the moment. Find all $n \geq 2$ for which the order of greetings can be arranged in such a way that after all of them, each person has their own hat back.

Problem 5. Let sequences of real numbers (x_n) and (y_n) satisfy $x_1 = y_1 = 1$ and

$$x_{n+1} = \frac{x_n + 2}{x_n + 1} \quad \text{and} \quad y_{n+1} = \frac{y_n^2 + 2}{2y_n} \text{ for } n = 1, 2, \ldots.$$

Prove that $y_{n+1} = x_{2^n}$ holds for $n = 0, 1, 2, \ldots$

Problem 6. The triangle ABC $(AB > BC)$ is inscribed in the circle Ω. On the sides AB and BC, the points M and N are chosen, respectively, so that $AM = CN$. The lines MN and AC intersect at point K. Let P be the center of the inscribed circle of triangle AMK, and Q the center of the excircle of the triangle CNK tangent to side CN. Prove that the midpoint of the arc ABC of the circle Ω is equidistant from the P and Q.

2.3. Test 3

Problem 7. Let 19 integer numbers are given. Let Hamza writes on the paper the greatest common divisor for each pair of numbers. It occurs that the difference between the biggest and smallest numbers written on the paper is less than 180. Prove that not all numbers on the paper are different.

Problem 8. Let $ABCD$ is a trapezoid with $\angle A = \angle B = 90^\circ$ and let E is a point lying on side CD. Let the circle ω is inscribed to triangle ABE and tangents sides AB, AE and BE at points P, F and K respectively. Let KF intersects segments BC and AD at points M and N respectively, as well as PM and PN intersect ω at points H and T respectively. Prove that $PH = PT$.

Problem 9. Let 300 students participate to the Olympiad. Between each 3 participants there is a pair that are not friends. Hamza enumerates participants in some order and denotes by x_i the number of friends of i-th participant. It occurs that

$$\{x_1, x_2, \ldots, x_{299}, x_{300}\} = \{1, 2, \ldots, N-1, N\}.$$

Find the biggest possible value for N.

3. April camp

3.1. Test 1

Problem 1. In a school there are 40 different clubs, each of them contains exactly 30 children. For every i from 1 to 30 define n_i as a number of children who attend exactly i clubs. Prove that it is possible to organize 40 new clubs with 30 children in each of them such, that the analogical numbers $n_1, n_2, \ldots, n_{30}$ will be the same for them.

Problem 2. Let Pascal triangle be an equilateral triangular array of number, consists of 2019 rows and except for the numbers in the bottom row, each number is equal to the sum of two numbers immediately below it. How many ways to assign each of numbers a_0, a_1, $\ldots$, a_{2018} (from left to right) in the bottom row by 0 or 1 such that the number S on the top is divisible by 1019.

Problem 3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\left(3\left(f(xy)\right)^2 + (xy)^2\right) = \left(xf(y) + yf(x)\right)^2 \text{ for any } x, y > 0.$$

3.2. Test 2

Problem 4. Let pairwise different positive integers a, b, c with $\gcd(a, b, c) = 1$ are such that

$$a \mid (b-c)^2, \quad b \mid (c-a)^2, \quad c \mid (a-b)^2.$$

Prove, that there is no non-degenerate triangle with side lengths a, b and c.

Problem 5. Let be given a positive integer $n > 1$. Find all polynomials $P(x)$ non constant, with real coefficients such that

$$P(x)P(x^2)\dots P(x^n) = P\left(x^{\frac{n(n+1)}{2}}\right)$$

for all $x \in \mathbb{R}$.

Problem 6. Let ABC be an acute, non isosceles triangle with O, H are circumcenter and orthocenter, respectively. Prove that the nine-point circles of AHO, BHO, CHO has two common points.

3.3. Test 3

Problem 7. Let $P(x)$ be a monic polynomial of degree 100 with 100 distinct non-integer real roots. Suppose that each of polynomials $P(2x^2 - 4x)$ and $P(4x - 2x^2)$ has exactly 130 distinct real roots. Prove that there exist non constant polynomials $A(x), B(x)$ such that $A(x)B(x) = P(x)$ and $A(x) = B(x)$ has no root in $(-1; 1)$.

Problem 8. Let ABC be a triangle, the circle having BC as diameter cuts AB, AC at F, E respectively. Let P a point on this circle. Let C_0, B_0 be the projections of P upon the sides AB, AC respectively. Let H be the orthocenter of the triangle AB_0C_0. Show that $\angle EHF = 90°$.

Problem 9. All of the numbers $1, 2, 3, ..., 1000000$ are initially colored black. On each move it is possible to choose the number x (among the colored numbers) and change the color of x and of all of the numbers that are not co-prime with x (black into white, white into black). Is it possible to color all of the numbers white?

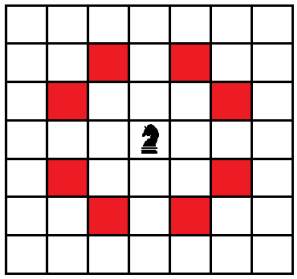
4. JBMO TST

4.1. Test 1

Problem 1. Find the smallest integer m for which there are positive integers $n > k > 1$ satisfying the equation

$$\underbrace{11\dots1}_{n} = \underbrace{11\dots1}_{k} \cdot m.$$

Problem 2. Chess horse attacks fields in distance $\sqrt{5}$. Let several horses are put on the board 12×12 such, that every square of size 2×2 contains at least one horse. Find the maximal possible number of cells that are not under attack (horse doesn't attack it's own cell).



Problem 3. How many integers n satisfy to the following conditions?

i) $219 \leq n \leq 2019$,

ii) there exist $x, y \in \mathbb{Z}$ such that $1 \leq x < n < y$ and y is divisible by all integers from 1 to n, except two numbers x and $x+1$.

Problem 4. Let AD be the altitude of the right angled triangle ABC with $\angle A = 90°$. Let DE be the altitude of the triangle ADB and DZ be the altitude of the triangle ADC respectively. Let N is chosen on the line AB such that CN is parallel to EZ. Let A' be the symmetric of A with respect to the line EZ and I, K the projections of A' into AB and AC respectively. Prove that $\angle NA'T = \angle ADT$, where T is the intersection point of IK and DE.

4.2. Test 2

Problem 1. In square $ABCD$ with side 1 point E lies on BC and F lies on CD such that $\angle EAB = 20°, \angle EAF = 45°$. Find the length of altitude AH of $\triangle AEF$.

Problem 2. Prove the inequality for non-negative a, b, c

$$a\sqrt{3a^2+6b^2} + b\sqrt{3b^2+6c^2} + c\sqrt{3c^2+6a^2} \geq (a+b+c)^2.$$

Problem 3. Find all primes p such that there exist integers m and n satisfying $p = m^2 + n^2$ and $p \mid m^3 + n^3 + 8mn$.

Problem 4. An 11×11 square is partitioned into 121 smaller 1×1 squares, 4 of which are painted black, the rest being white. We cut a fully white rectangle (possibly a square) out of the big 11×11 square. What is the maximal area of the rectangle we can obtain regardless of the positions of the black squares? It is allowed to cut the rectangle along the grid lines.

4.3. Test 3

Problem 1. Determine the maximal number of disjoint crosses (5 squares) which can be put inside 8×8 chessboard such that sides of a cross are parallel to sides of the chessboard.

Problem 2. Find all pairs of positive integers (m, n) such that

$$125 \cdot 2^n - 3^m = 271.$$

Problem 3. Let ABC be an acute, non isosceles triangle. Take two points D, E inside this triangle such that

$$\angle DAB = \angle DCB,\ \angle DAC = \angle DBC;$$
$$\angle EAB = \angle EBC,\ \angle EAC = \angle ECB.$$

Prove that triangle ADE is right.

Problem 4. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be any real numbers. Prove that there exists $m, k \in \{1, 2, \ldots, n\}$ such that

$$\left| \sum_{i=1}^{m} a_i - \sum_{i=m+1}^{n} a_i \right| \leq |a_k| .$$

4.4. Test 4

Problem 1. A set S is called neighboring if it has the following two properties:

i) S has exactly 4 elements,

ii) for every element $x \in S$ at least one of the $x - 1$ or $x + 1$ belongs to S.

Find the number of all neighboring subsets of the set $\{1, 2, \ldots, n\}$.

Problem 2. Prove that there are no positive integers x, y, z such that

$$(3x + 4y)(4x + 5y) = 7^z.$$

Problem 3. Let S be a given set of real numbers such that:

i) $1 \in S$,

ii) for any $a, b \in S$ (not necessary different), then $a - b \in S$,

iii) for $a \in S, a \neq 0$ then $\frac{1}{a} \in S$.

Prove that for any $a, b \in S$ then $ab \in S$.

Problem 4. In triangle ABC, such that $\angle ACB = 45°$ let O and H be circumcenter and orthocenter, respectively. Line passing through O and perpendicular to CO intersects AC and BC at K and L, respectively. Prove that perimeter of KLH is equal to diameter of circumcircle of triangle ABC.

5. May camp - IMO TST

5.1. Day 1

Problem 1. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

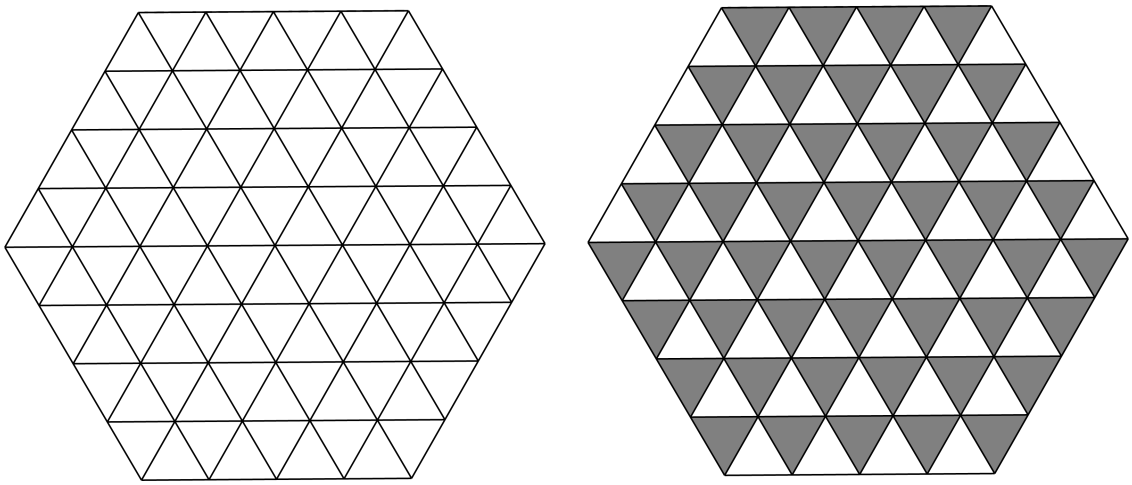
$$n^3 - n^2 \leq f(n) \cdot (f(f(n)))^2 \leq n^3 + n^2$$

for every n is positive integers.

Problem 2. Find all pair of integers (m, n) and $m \geq n$ such that there exist a positive integer s and

1. Product of all divisor of sm, sn are equal.
2. Number of divisors of sm, sn are equal.

Problem 3. Let regular hexagon is divided into $6n^2$ regular triangles. Let $2n$ coins are put in different triangles such, that no any two coins lie on the same layer (layer is area between two consecutive parallel lines). Let also triangles are painted like on the chess board.



Prove that exactly n coins lie on black triangles.

5.2. Day 2

Problem 4. Let a_0 be an arbitrary positive integer. Let (a_n) be infinite sequence of positive integers such that for every positive integer n, the term a_n is the smallest positive integer such that $a_0 + a_1 + \cdots + a_n$ is divisible by n. Prove that there exist N such that $a_{n+1} = a_n$ for all $n \geq N$.

Problem 5. Let non-constant polynomial $f(x)$ with real coefficients is given with the following property: for any positive integer n and k, the value of expression

$$\frac{f(n+1)f(n+2)\dots f(n+k)}{f(1)f(2)\dots f(k)} \in \mathbb{Z}.$$

Prove that $f(x)$ is divisible by x.

Problem 6. Let ABC be an acute nonisosceles triangle with incenter I and (d) is an arbitrary line tangent to (I) at K. The lines passes through I, perpendicular to IA, IB, IC cut (d) at A_1, B_1, C_1 respectively. Suppose that (d) cuts BC, CA, AB at M, N, P respectively. The lines through M, N, P and respectively parallel to the internal bisectors of A, B, C in triangle ABC meet each other to define a triange XYZ. Prove that three lines AA_1, BB_1, CC_1 are concurrent and IK is tangent to the circle (XYZ).

Solution to tests of January camp

1. Test 1

Problem 1. *Suppose that x, y, z are non-zero real numbers such that*

$$x = 2 - \frac{y}{z},\ y = 2 - \frac{z}{x},\ z = 2 - \frac{x}{y}.$$

Find all possible values of $T = x + y + z$.

Solution. From the given conditions, we have

$$xz = 2z - y,\ xy = 2x - z,\ yz = 2y - x.$$

Taking the sum of these equations, side by side, we have

$$xy + yz + zx = x + y + z = T.$$

From $xz = 2z - y$, we also can get $2z - x - y = xz - x \to 3z - T = x(z - 1)$. Make the similar equations and multiply them, we get

$$xyz(x-1)(y-1)(z-1) = (3x - T)(3y - T)(3z - T).$$

Note that $(x-1)(y-1)(z-1) = xyz - (xy + yz + zx) + (x + y + z) - 1 = xyz - 1$ and

$$\begin{aligned}(3x - T)(3y - T)(3z - T) &= 27xyz - 9T(xy + yz + zx) + T^2(3x + 3y + 3z) - T^3 \\ &= 27xyz - 9T^2 + 2T^3\end{aligned}$$

Thus

$$xyz(xyz - 1) = 27xyz - 9T^2 + 2T^3 \text{ or } (xyz)^2 - 28xyz = 2T^3 - 9T^2. \tag{1}$$

By the similar transformation, $xz = 2z + y \to xz + 2x = 2z + 2x - y \to x(z + 2) = 2T - 3y$. Make the similar equations and multiply them, side by side, we get

$$(xyz)^2 + 6(xyz)T + 35xyz = 18T^2 - 4T^3.$$

On the other hand, $xz = 2z - y \to xyz = 2yz - y^2$. Make the similar equations and multiply them, side by side, we get

$$3xyz = 2(xy + yz + zx) - (x^2 + y^2 + z^2) = 4(xy + yz + zx) - (x + y + z)^2 = 4T - T^2.$$

So we have $xyz = \frac{4T - T^2}{3}$, then by substituting to equation (1), we get

$$\left(\frac{4T - T^2}{3}\right)^2 - 28\left(\frac{4T - T^2}{3}\right) = 2T^3 - 9T^2 \tag{2}$$

Solving this equation, we have $T \in \{0, 3, 7, 16\}$.

1. If $T = 0$, then $xyz = 0$, a contradiction.
2. If $T = 16$, then $xyz = -64$ and this pair $(T, xyz) = (16, -64)$ does not satisfy equation (2).
3. If $T = 3$, then $xyz = 1$ and by Vieta's theorem, x, y, z are real roots of $t^3 - 3t^2 + 3t - 1 = 0$, which implies that $x = y = z = 1$.
4. If $T = 7$, we get $xyz = -7$ and x, y, z are roots of $t^3 - 7t^2 + 7t + 7 = 0$, this also have three distinct real roots.

Hence, all possible values of T are 3 or 7. □

Remark. In this problem, if we have condition x, y, z are positive, then it would be much more easier. We can use AM-GM inequality to prove $x = y = z = 1$.

Problem 2. *Let $P(x)$ be a polynomial of degree $n \geq 2$ with rational coefficients such that $P(x)$ has n pairwise different real roots forming an arithmetic progression. Prove that among the roots of $P(x)$ there are two that are also the roots of some polynomial of degree 2 with rational coefficients.*

Solution. Denote $x_1 < x_2 < \ldots < x_n$ as the roots of $P(x)$. Let $d = x_n - x_{n-1} = \cdots = x_2 - x_1 > 0$. Since $P(x)$ has rational coefficients then by applying Vieta's theorem, we have

$$\sum_{i=1}^{n} x_i \in \mathbb{Q} \text{ and } \sum_{1 \leq i < j \leq n} x_i x_j \in \mathbb{Q}.$$

Note that $\sum\limits_{i=1}^{n} x_i = \frac{n(x_1+x_n)}{2}$ so $x_1 + x_n \in \mathbb{Q}$. On the other hand,

$$\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - 2 \sum_{1 \leq i < j \leq n} x_i x_j \in \mathbb{Q}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} (x_1 + (i-1)d)^2 = nx_1^2 + n(n-1)x_1 d + \frac{n(n-1)(2n-1)}{6} d^2 \\
&= n\left(x_1 + \frac{(n-1)d}{2}\right)^2 + \frac{n^2(n^2-1)}{12} d^2 \\
&= \frac{n(x_1+x_n)^2}{4} + \frac{n(n^2-1)}{12} d^2.
\end{aligned}$$

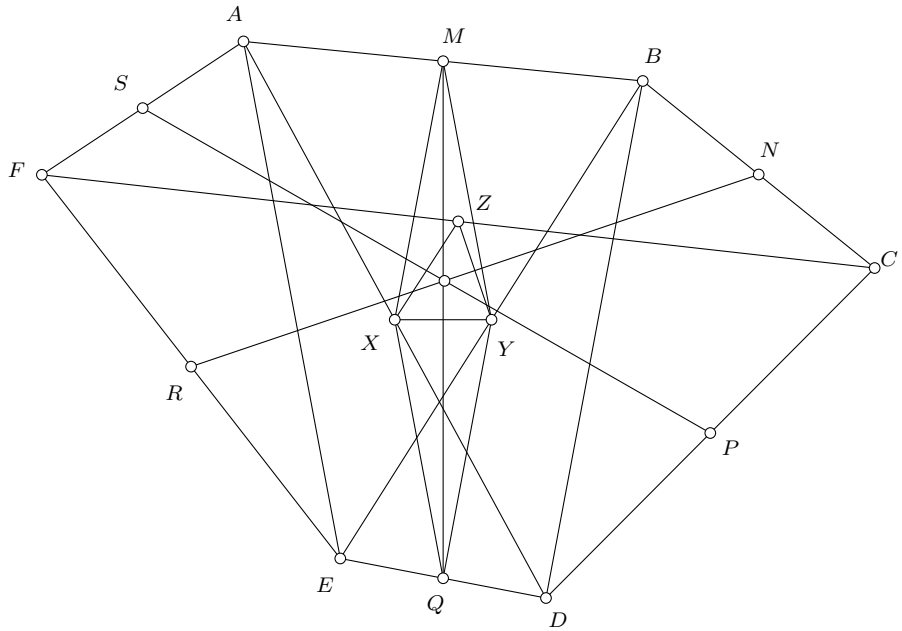
From these, we can conclude that $\frac{n(n^2-1)}{12} d^2 \in \mathbb{Q}$ so $d^2 \in \mathbb{Q}$. Thus

$$x_1 x_n = \frac{(x_1+x_n)^2 - (x_n - x_1)^2}{4} = \frac{(x_1+x_n)^2 - (n-1)^2 d^2}{4} \in \mathbb{Q}.$$

Therefore, $x_1 + x_n, x_1 x_n \in \mathbb{Q}$ which implies that two numbers x_1, x_n satisfy. □

Problem 3. *Let $ABCDEF$ be a convex hexagon satisfying $AC = DF$, $CE = FB$ and $EA = BD$. Prove that the lines connecting the midpoints of opposite sides of the hexagon $ABCDEF$ intersect in one point.*

Solution. Let M, N, P, Q, R, S be the midpoints of sides AB, BC, CD, DE, EF, FA, respectively, and X, Y, Z be the midpoints of AD, BE, CF.



Since $AE = BD$ and the midsegments in some triangles, we get

$$XQ = YM = \frac{1}{2} \cdot AE = \frac{1}{2} \cdot BD = XM = YQ,$$

so $XMYQ$ is a rhombus, then MQ is the perpendicular bisector of the segment XY. Similarly, NR, PS are the perpendicular bisectors of XZ, YZ, so MQ, NR, PS are concurrent at the circumcenter of the $\triangle XYZ$. $\square$

2. Test 2

Problem 4. *Suppose that a, b, c, d are pairwise distinct positive integers such that $a + b = c + d = p$ for odd prime $p > 3$. Prove that $abcd$ is not a perfect square.*

Solution. Suppose that $abcd = n^2$ for some $n \in \mathbb{Z}^+$. We can suppose that $a < c < d < b$. From this, we have $bd + ac - ad - bc = (b - a)(d - c) > 0$, thus

$$ad + bc < \frac{1}{2}(ad + bc + bd + ac) = \frac{(a + b)(c + d)}{2} = \frac{p^2}{2}.$$

Denote $\gcd(ad, bc) = k \in \mathbb{Z}^+$ then $ad = ku^2, bc = kv^2$ with $u, v \in \mathbb{Z}^+$, and $u < v, \gcd(u, v) = 1$. Since a, b, c, d are coprime to p then $\gcd(k, p) = 1$. We have

$$k(v^2 - u^2) = bc - ad = (p - a)c - a(p - c) = p(c - a).$$

So $p|v^2 - u^2$ since $\gcd(k, p) = 1$. Hence, $p|v - u$ or $p|v + u$. In both of cases, we always have $u + v \geq p$. From this, we can conclude that

$$ad + bc = k(u^2 + v^2) > k\frac{(u+v)^2}{2} \geq \frac{p^2}{2}$$

by AM-GM inequality. But this contradicts to the inequality we stated above, then $abcd$ cannot be a perfect square. □

Remark. This problem can be written as: Let p be a odd prime and $x, y \in \{1, 2, \ldots, \frac{p-1}{2}\}$. Prove that if $xy(p-x)(p-y)$ is a perfect square then $x = y$.

Problem 5. *There are 3 clubs A, B, C with non-empty members. For any triplet of members (a, b, c) with $a \in A, b \in B, c \in C$, two of them are friend and two of them are not friend (here the friend relationship is bidirectional). Prove that one of these conditions must be true*

1) There exists one student from A know all students from B.
2) There exists one student from B know all students from C.
3) There exists one student from C know all students from A.

Solution. We will prove the statement by induction on the maximum number of members in clubs A, B, C.

For $n = 1$, each club has exactly one member and the statement is obviously true.

Suppose that when the maximum numbers in three clubs is $n \geq 1$, then one of three above conditions holds. Assume that $a \in A$ is friend of all members in B. Consider a new member x to make the maximum number of members increased by 1. It is easy to see that if $x \in A$ or $x \in C$, then the conditions are still true (since the members in B remains unchanged).

If $x \in B$, denote $B = \{b_1, b_2, \ldots, b_n, b_{n+1}\}$ with $b_{n+1} \equiv x$. In case (a, x) are friend, the condition 1) is true and we are done. So we may assume that (a, x) are not friend. Partition C into two subsets, C_1 has members that are friend of a, and C_2 has members that are not friend of a. Take $c_1 \in C_1$ and $c_2 \in C_2$ (if any).

For $1 \leq k \leq n$, consider (a, b_k, c_1), we have (a, b_k) are friend (since before adding x, member a is friend of all members of B) and (a, c_1) are friend so (b_k, c_1) are not friend (by the given condition).

Consider (a, x, c_2), we have (x, c_2) are friend (since $(a, x), (a, c_2)$ are not friend).

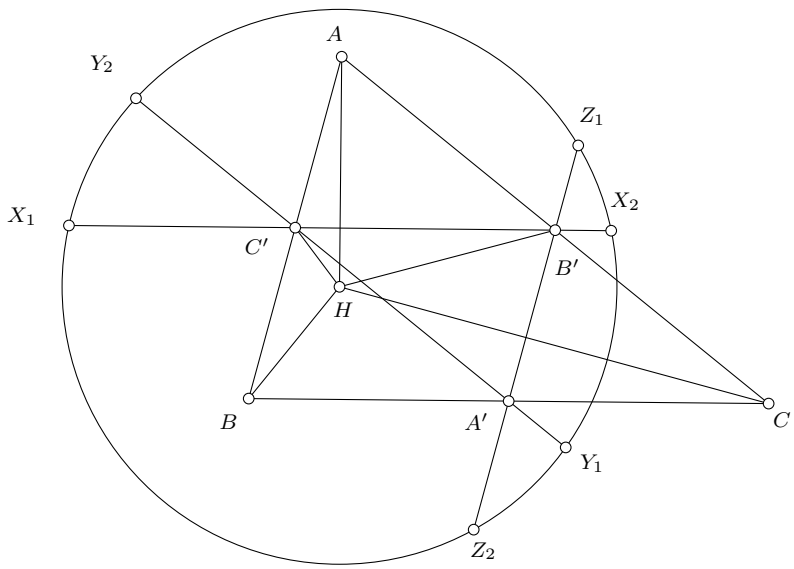
- If x is friend of all members in C_1 then x is friend of all members in $C = C_1 \cup C_2$ and condition 2) is true.
- Otherwise, $\exists c_1 \in C$ such that c_1 is not friend of x, also is not friend of all members in B. If $\exists a' \in A$ such that (a', c_1) are not friend, then consider (a', b_k, c_1) with $1 \leq k \leq n + 1$ then a' is friend of all b_k, condition 1) is true. Otherwise, c_1 are friend of all members in A and condition 3) is true.

Hence, in all cases, the condition is always true.

□

Problem 6. *Let ABC be a triangle with A', B', C' are midpoints of BC, CA, AB respectively. The circle (ω_A) of center A has a big enough radius cuts $B'C'$ at X_1, X_2. Define circles $(\omega_B), (\omega_C)$ with Y_1, Y_2, Z_1, Z_2 similarly. Suppose that these circles have the same radius, prove that $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are concyclic.*

Solution. Let H be the orthocenter of $\triangle ABC$. Since $AH \perp X_1X_2$ and $AX_1 = AX_2$, then $HX_1 = HX_2$. Similarly, $HY_1 = HY_2$ and $HZ_1 = HZ_2$. (1)



Denote R is the radius of 3 circles (ω_A), (ω_B), (ω_C). As $HB \perp Y_1C'$, $HC \perp Z_1B'$, we have

$$HY_1{}^2 - R^2 = HY_1{}^2 - BY_1{}^2 = HC'^2 - BC'^2,$$

$$HZ_1{}^2 - R^2 = HZ_1{}^2 - CZ_1{}^2 = HB'^2 - CB'^2.$$

In addition, $AH \perp B'C'$, so

$$HC'^2 - BC'^2 = HC'^2 - AC'^2 = HB'^2 - AB'^2 = HB'^2 - CB'^2.$$

From that $HY_1 = HZ_1$. Similarly, $HX_1 = HY_1 = HZ_1$ and $HX_2 = HY_2 = HZ_2$. (2)
From (1) and (2), the orthocenter H is indeed, the center of the circle which goes through 6 points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$. □

3. Test 3

Problem 7. *Let ABC be a triangle inscribed in a circle (ω) and I is the incenter. Denote D, E as the intersection of AI, BI with (ω) and DE cuts AC, BC at F, G respectively. Let P be a point such that $PF \parallel AD$ and $PG \parallel BE$. Suppose that the tangent lines of (ω) at A, B meet at K. Prove that three lines AE, BD, KP are concurrent or parallel.*

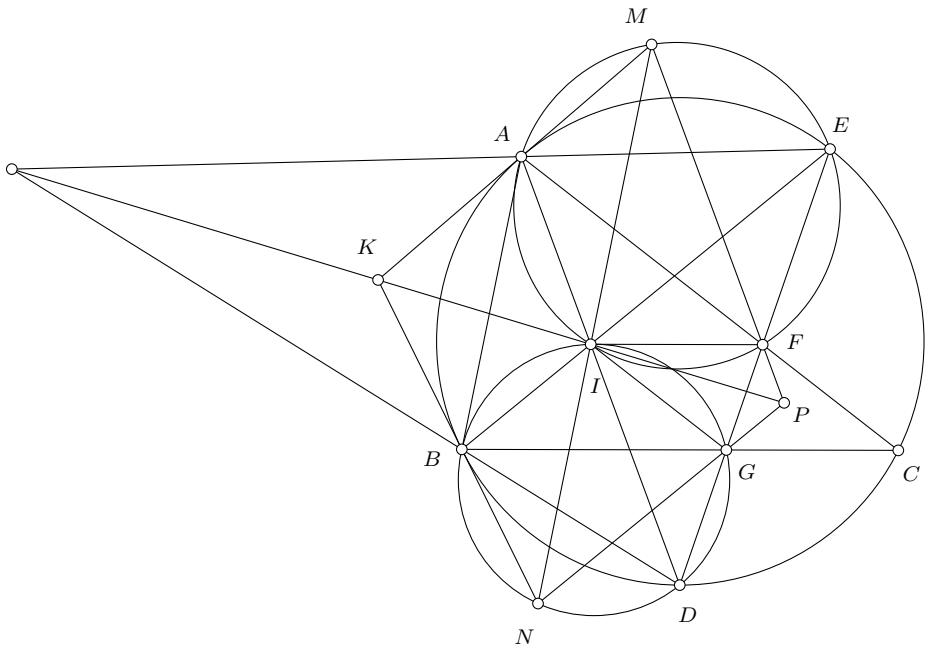
Solution. Suppose that KA cuts PF at M, KB cuts PG at N. By angle chasing, we have

$$\angle IEF = \angle BAI = \angle FAI,$$

then $AIFE$ is cyclic. In addition, we get

$$\begin{aligned}\angle AMF = \angle KAI &= \angle KAB + \angle BAI \\ &= \angle AEI + \angle FEI = \angle AEF,\end{aligned}$$

then $AIFM$ is cyclic.



Summarily, points A, I, F, E, M lie on a circle. And similarly, points B, I, G, D, N also lie on a circle. We have known that DE is perpendicular bisector of CI, so it is easy to see that $IFCG$ is a rhombus. From that above, we get

$$\angle AMI = \angle AEI = \angle KAB,$$

then $IM \parallel AB$. Similarly $IN \parallel AB$, implies that M, N, I are collinear. But from $PF \parallel AD$, $PG \parallel BE$, we have $AIFM$ and $BIGN$ are isosceles trapezoids, so

$$AM = IF = IG = BN,$$

then $ABNM$ is also a isosceles trapezoid, which means that it is cyclic. (1)

In the other hand, we get

$$\angle PFG = \angle MFE = \angle MIE = \angle BIN = \angle ING,$$

so $FGNM$ is cyclic. (2)

From (1) and (2), we have PK is the radical axis of two circles (AIE) and (BID). Now on, we can easily take the conclusion of this problem. □

Problem 8. *We are given a graph whose vertices are positive integers and an edge between numbers a and b exists if and only if*

$$a+b+1 \mid a^2+b^2+1.$$

Is this graph connected?

Solution. If $x, y \in \mathbb{Z}^+$, define $x \leftrightarrow y$ if and only if x, y are connected by some edge.

We have for all $a \in \mathbb{Z}^+$,

$$a^2+a+1 \mid (a^2-a+1)(a^2+a+1) = a^4+a^2+1.$$

Thus $a \leftrightarrow a^2$ for all a, then also true for $a+1 \leftrightarrow (a+1)^2$. Moreover,

$$\begin{aligned}(a^2)^2 + ((a+1)^2)^2 + 1 &= 2a^4+4a^3+6a^2+4a+2 \\ &= (2a^2+2a+2)(a^2+a+1).\end{aligned}$$

Thus $a^2+(a+1)^2+1 = 2a^2+2a+2$ divided $(a^2)^2+((a+1)^2)^2+1$, which implies that $a^2 \leftrightarrow (a+1)^2$. Hence,

$$a \leftrightarrow a^2 \leftrightarrow (a+1)^2 \leftrightarrow a+1,$$

this means that every two consecutive integers $a, a+1$ are connected. Thus the given infinity graph is connected. □

Problem 9. *Define sequence of positive integers $\{a_n\}$ as $a_1 = a$ and $a_{n+1} = a_n^2+1$ for $n \geq 1$. Prove that there is no index n for which*

$$\prod_{k=1}^{n}\left(a_k^2+a_k+1\right)$$

is a perfect square.

Solution. Denote p as an prime of $a_1^2+a_1+1$, note that a_1 is odd (since $a_1^2+a_1+1 = a_1(a_1+1)+1$ is an odd number) and pa_1. By induction, we can show that

$$a_n \equiv a_2 \equiv -a_1 \pmod{p} \text{ for any } n > 1.$$

Thus $a_n^2+a_n+1 \equiv a_1^2-a_1+1 \equiv -2a_1 \neq 0 \pmod{p}$ so

$$v_p\left(\prod_{k=1}^{n}(a_k^2+a_k+1)\right) = v_p(a_1^2+a_1+1).$$

Since $a_1^2 < a_1^2+a_1+1 < (a_1+1)^2$, then $a_1^2+a_1+1$ is not a perfect square.

This implies that there exist some prime p such that $v_p(a_1^2+a_1+1)$ is odd. This finishes the proof. □

Remark. One can show that

$$\prod_{k=1}^{n}(a_k^2 + a_k + 1) = a_n^4 + a_n^2 + 1$$

for all $n \geq 2$ by induction. And note that for any $a \in \mathbb{Z}^+$, the number $a^4 + a^2 + 1$ cannot be a perfect square since

$$\left(a^2\right)^2 < a^4 + a^2 + 1 < \left(a^2 + 1\right)^2.$$

Solution to tests of March camp

1. Test 1

Problem 1. *Let p be an odd prime number.*

1) Show that p divides $n2^n + 1$ for infinitely many positive integers n.
2) Find all such n when $p = 3$.

Solution. 1) We will show that for all $k > 0$, the number $n = (pk+1)(p-1)$ satisfies $p|n \cdot 2^n + 1$. Indeed, by Fermat Little's theorem, we have

$$2^{p-1} \equiv 1 \pmod{p} \text{ so } 2^n = 2^{(pk+1)(p-1)} \equiv 1 \pmod{p}.$$

And then

$$n \cdot 2^n + 1 \equiv (pk+1)(p-1) \cdot 1 + 1 = p^2k - pk + p \equiv 0 \pmod{p}.$$

Since there are infinitely many number of form $(pk+1)(p-1)$ so we get the conclusion.

2) Notice that n is periodic modulo 3, with a period of 3, and 2^n is periodic modulo 3, with period 2. Hence, $n \cdot 2^n + 1$ is periodic, with period (at most) 6, and only first 6 positive integers need to be analyzed.

The answer is $n = 6k + 1$ or $n = 6k + 2$. □

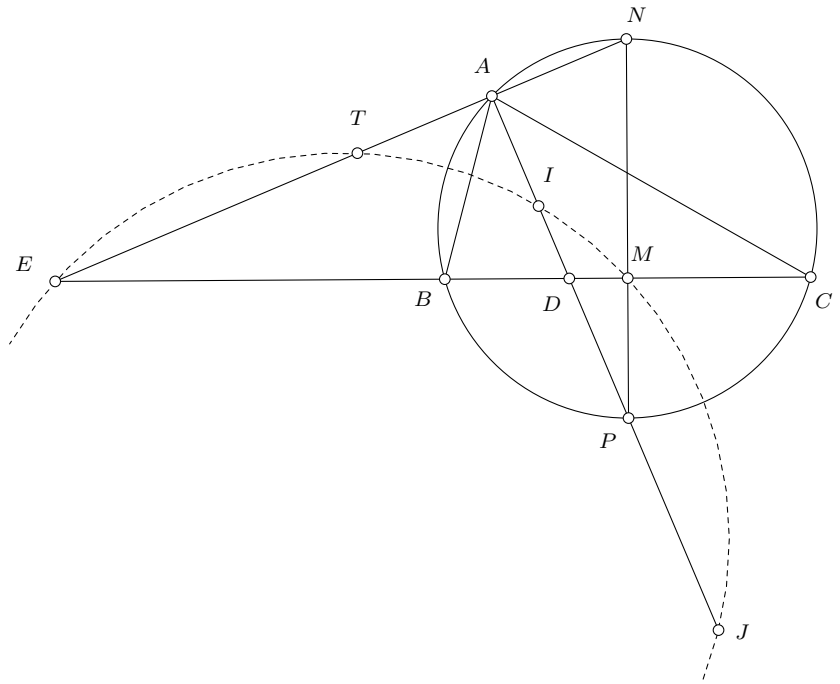
Problem 2. *Let I be the incenter of ABC and J the excenter of the side BC, let M be the midpoint of CB and N the midpoint of arc BC (with the point A). If T is the symmetric of the point N by the point A, prove that the quadrilateral $JMIT$ is cyclic.*

Solution. Suppose that AI cuts BC at D and cuts (ABC) again at P. We have known that B, C, I, J lie on circle $(P, \frac{IJ}{2})$.

Suppose that AN meets BC at E, then AE is the external angle bisector of $\angle BAC$, so $(DE, BC) = -1$. We have

$$\overline{DE} \cdot \overline{DM} = \overline{DB} \cdot \overline{DC} = \overline{DI} \cdot \overline{DJ},$$

means that $IMJE$ is cyclic. (1)



In the other hand, we have known that $(AD, IJ) = -1$. Combining with D is orthocenter of $\triangle NPE$, we get

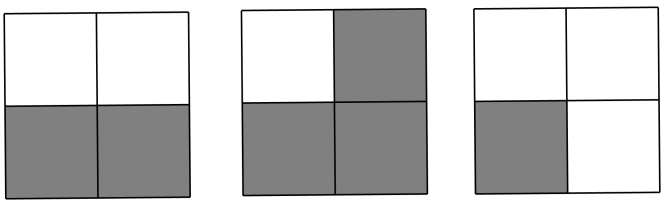
$$\overline{AT} \cdot \overline{AS} = -\overline{AN} \cdot \overline{AS} = \overline{AD} \cdot \overline{AP} = \overline{AI} \cdot \overline{AJ},$$

implies that $IJET$ is cyclic. (2)

From (1) and (2), then the points J, I, M, T, E lie on a circle. □

Problem 3. *For $n \geq 3$, given is an $2n \times 2n$ board with black and white squares. It is known that all border squares are black and no 2×2 subboard has all four squares of the same color. Prove that there exists a 2×2 subboard painted like a chessboard, i.e. with two opposite black corners and two opposite white corners.*

Solution. Assume for the sake of contradiction that there are no 2×2 square painted like a chessboard. Then all 2×2 subboards are of these paintings as follow (and their rotations). (*)



We will count the length of black-white border in two ways.

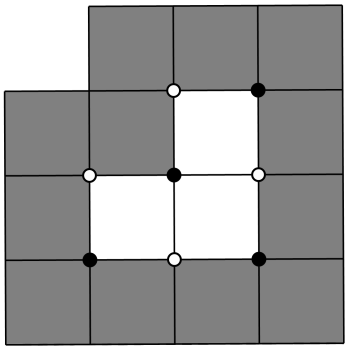
1. **First way.** Observe that from $(*)$, each subboard 2×2 has exactly 2 units of the black-white border. Since the border of whole board is black so each

black-white border appears in exactly two 2×2 subboards. The number of 2×2 subboards is $(2n-1)^2$, so the total length of black-white border is

$$\frac{2(2n-1)^2}{2} = (2n-1)^2,$$

which is an odd number.

2. **Second way.** Let color the lattice point in the given board by black-white like chessboard. When we move along the border of any white squares, we will get the black and white points alternatively, which mean any connected region of white squares has even perimeter.



The entire white region is the union of disjoint white regions, so the total length of black-white border is even.

This is a contradiction so we can get the conclusion that there exists some subboard 2×2 which colored like chessboard. □

2. Test 2

Problem 4. *There are n people with hats present at a party. Each two of them greeted each other exactly once and each greeting consisted of exchanging the hats that the two persons had at the moment. Find all $n \geq 2$ for which the order of greetings can be arranged in such a way that after all of them, each person has their own hat back.*

Solution. We may consider the changing of the hats of two people as the changing of their positions in some permutation on a straight line, and it is actually the swapping two terms in the permutation of numbers $1, 2, \ldots, n$. Firstly, we have $(1, 2, \ldots, n)$ and then, by applying exactly $\binom{n}{2}$ numbers of swapping, we get the original position.

Define a "disorder" of a integer sequence as a pair (a, b) such that $a > b$ and a lies on the left of b. And denote $\#d$ as the total number of disorders.

For example: $\#d$ of the sequence $(1, 4, 2, 3)$ is 2, which are $(4, 2), (4, 3)$.

We will prove the following lemma: If we swap two numbers in the sequence, then $\#d$ will increase or decrease by an odd value.

Proof. Indeed, suppose that we have the following sequence with distinct terms

$$a_1, a_2, \ldots, a_m, x, b_1, b_2, \ldots, b_n, y, c_1, c_2, \ldots, c_p.$$

We may assume that $x < y$ and by swapping $x \leftrightarrow y$, the number of disorders containing a_i, c_j remain unchanged and we have one new disorder is (y, x).

If among $b_1, b_2, \ldots, b_n$, there are k values less than x then by moving x, the number of disorders will change $(n-k)-k = n-2k$. Similarly, if among $b_1, b_2, \ldots, b_n$, there are t values greater than y then by moving y,the number of disorder will change $(n-t)-t = n-2t$. Thus, the number of disorders will change $(n-2k)+(n-2t) = 2(n-k-t)$ which is a even number. Hence, the lemma is proved. □

By applying this lemma, we can see that the original and final positions have the 0 number of disorder, hence $\binom{n}{2} = \frac{n(n-1)}{2} \equiv 0 \pmod 2$, which implies that $n \equiv 0, 1 \pmod 4$. Now we will construct the example by induction.

With $n = 4$, we may proceed as follow

$$(1 \leftrightarrow 2), (3 \leftrightarrow 4), (1 \leftrightarrow 4), (2 \leftrightarrow 3), (1 \leftrightarrow 3), (3 \leftrightarrow 4).$$

With $n = 5$, we may proceed as follow

$$\begin{aligned}&(5 \leftrightarrow 1), (3 \leftrightarrow 2), (5 \leftrightarrow 3), (4 \leftrightarrow 1), (5 \leftrightarrow 4),\\ &(1 \leftrightarrow 3), (2 \leftrightarrow 1), (2 \leftrightarrow 3), (3 \leftrightarrow 4), (4 \leftrightarrow 2).\end{aligned}$$

Suppose that we already have the construction for some $k \equiv 0, 1 \pmod 4$, consider $k+4$ numbers. Firstly, we swap $k+x$ with $x \in \{1, 2, 3\}$ with all of first k numbers $(k+x \leftrightarrow 1), (k+x \leftrightarrow 2), \ldots, (k+x \leftrightarrow k)$. Then

$$(k+4 \leftrightarrow k+1), (k+4 \leftrightarrow k), \ldots, (k+4 \leftrightarrow 1).$$

Finally

$$(k+1 \leftrightarrow k+3), (k+2 \leftrightarrow k+4), (k+4 \leftrightarrow k+3), (k+2 \leftrightarrow k+3), (k+2 \leftrightarrow k+1).$$

It is easily to check that this permutation works. So all numbers $n \equiv 0, 1 \pmod 4$ satisfy the given condition. □

Problem 5. *Let sequences of real numbers (x_n) and (y_n) satisfy $x_1 = y_1 = 1$,*

$$x_{n+1} = \frac{x_n + 2}{x_n + 1} \quad \text{and} \quad y_{n+1} = \frac{y_n^2 + 2}{2y_n} \text{ for } n = 1, 2, \ldots.$$

Prove that $y_{n+1} = x_{2^n}$ holds for $n = 0, 1, 2, \ldots$.

Solution. Note that, since $x_1 = 1, x_{n+1} = \frac{x_n+2}{x_n+1}$ we can write

$$\begin{aligned}x_{n+1} &= \frac{x_n \cdot x_1 + 2}{x_n + x_1} = \frac{\frac{x_{n-1}+2}{x_{n-1}+1} \cdot x_1 + 2}{\frac{x_{n-1}+2}{x_{n-1}+1} + x_1} = \frac{x_{n-1}(x_1+2) + 2(x_1+1)}{x_{n-1}(x_1+1) + (x_1+2)}\\ &= \frac{x_{n-1} \cdot \frac{x_1+2}{x_1+1} + 2}{x_{n-1} + \frac{x_1+2}{x_1+1}} = \frac{x_{n-1} \cdot x_2 + 2}{x_{n-1} + x_2}.\end{aligned}$$

So by the same way, we can prove that

$$x_n = \frac{x_k x_{n-k} + 2}{x_k + x_{n-k}} \text{ for any } 1 \le k \le n-1.$$

Let $a_{n+1} = x_{2^n}$ for $n \ge 0$ then $a_1 = x_1 = 1, a_2 = x_2, a_3 = x_4, \ldots$ Take $k = 2^{n-1}$ in above formula, we get

$$x_{2^n} = \frac{x_{2^{n-1}}^2 + 2}{2x_{2^{n-1}}} \text{ so } a_{n+1} = \frac{a_n^2 + 2}{2a_n}.$$

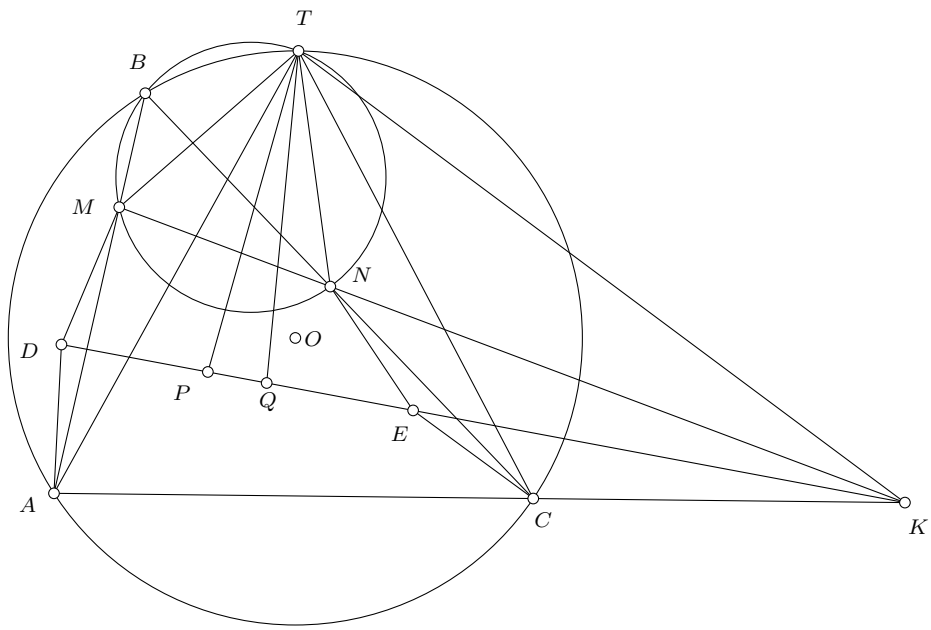
Thus by induction, one can get $a_n = y_n$ for all $n \ge 1$. Therefore, $y_{n+1} = a_{n+1} = x_{2^n}$ for any $n \ge 0$. $\square$

Problem 6. *The triangle ABC ($AB > BC$) is inscribed in the circle Ω. On the sides AB and BC, the points M and N are chosen, respectively, so that $AM = CN$. The lines MN and AC intersect at point K. Let P be the center of the inscribed circle of triangle AMK, and Q the center of the excircle of the triangle CNK tangent to side CN. Prove that the midpoint of the arc ABC of the circle Ω is equidistant from the P and Q.*

Solution. Let T be the second intersection of two circles (BMN) and (O). We have

$$\angle TAB = \angle TCB, \quad \angle TMB = \angle TNB,$$

and $AM = CN$, so $\triangle TAM \cong \angle TCN$. Then $TA = TB$, means that T is the midpoint of the arc BAC of circle (O).



In the other hand, we also have

$$\angle TCK = \angle TBA = \angle TNK,$$

then $TNCK$ is cyclic. Similarly, $TMAK$ is also cyclic.

Since $AM = CN$, easy to see that (KAM) and (KCN) are equal. So, if we call that D, E are the midpoints of arcs AM and CN of circles (KAM) and (KCN), respectively, then two isosceles triangles DAM and ECN are congruent. But we know that D, E are also the circumcenter of $\triangle PAM$ and $\triangle QCN$, so $DP = EQ$.

At last, from (KAM) and (KCN) are equal, we get $TD = TE$, then $TP = TQ$. □

Remark. Note that point T is the Miquel point of the completed quadrilateral $AMNC.BK$. In this problem, we have the following lemma (trillium theorem):

Let L be the midpoint of arc YZ (not containing point X) of the circumscribed circle of the triangle XYZ. Let I be the center of the incircle of XYZ, and I_x be the center of the X excircle of this triangle. Then

$$LY = LZ = LI = LI_x.$$

3. Test 3

Problem 7. *Let* 19 *integer numbers are given. Let Hamza writes on the paper the greatest common divisor for each pair of numbers. It occurs that the difference between the biggest and smallest numbers written on the paper is less than* 180*. Prove that not all numbers on the paper are different.*

Solution. Let $a_1, a_2, \ldots, a_{19}$ be the given numbers and suppose on the contrary that the set S of all $\binom{19}{2} = 171$ numbers, which are written on the paper are all different,

$$d_1 < d_2 < \ldots < d_{171}.$$

Denote k as the number of even value among 19 given numbers, and t as the number of even value in S. It is easy to see that for any $d \in S$, there exist a_x, a_y such that $d = \gcd(a_x, a_y)$; and this number d is even if and only if a_x, a_y are both even. Hence, we have $t = \binom{k}{2}$.

Then the number of odd value in S is $171 - t$. Since $d_{171} - d_1 < 180$ then the number of even value and odd value in S is not exceed 90, which implies that

$$\begin{cases} t \leq 90 \\ 171 - t \leq 90 \end{cases} \text{ so } 81 \leq \binom{k}{2} \leq 90.$$

Note that

$$\binom{13}{2} = 78 < 91 = \binom{14}{2},$$

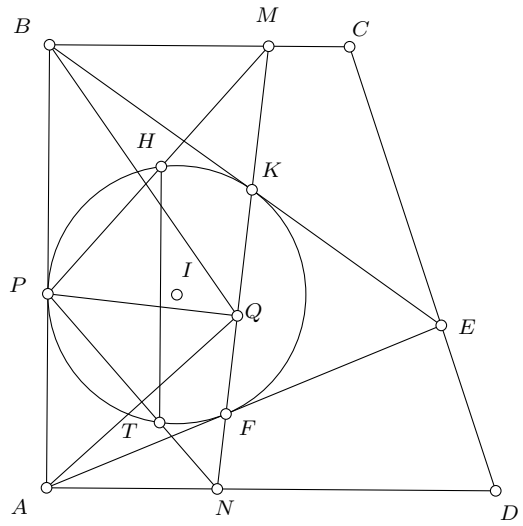
so there does not exist positive integer k such that $81 \leq \binom{k}{2} \leq 90$, contradiction.

Hence, the numbers written on the paper cannot be all equal.

□

Problem 8. *Let $ABCD$ is a trapezoid with $\angle A = \angle B = 90^o$ and let E is a point lying on side CD. Let the circle ω is inscribed to triangle ABE and tangents sides AB, AE and BE at points P, F and K respectively. Let KF intersects segments BC and AD at points M and N respectively, as well as PM and PN intersect ω at points H and T respectively. Prove that $PH = PT$.*

Solution. Let KF meets AB at S. We have known that EP, AK, BF are concurrent at Gergonne's point, then $(SP, AB) = -1$. Let Q be the projection of P on KF then $Q(SP, AB) = -1$, but $QS \perp QP$ so QP is the angle bisector of $\angle AQB$. From that, we get $\angle BQM = \angle AQN$.



In the other hand, easy to realize that $PQMB$ and $PQNA$ are cyclic, so

$$\angle BPM = \angle BQM = \angle AQN = \angle APN.$$

Combining with the truth that AB is the tagent of (I), we get

$$\angle PTH = \angle BPM = \angle APN = \angle PHT,$$

or $PH = PT$. □

Remark. If we denote X as the intersection of AT, MP and Y as the intersection of BH, TP, and Z as the intersection of XY, HT then AZ is parallel to XY.

Problem 9. *Let 300 students participate to the Olympiad. Between each 3 participants there is a pair that are not friends. Hamza enumerates participants in some order and denotes by x_i the number of friends of i-th participant. It occurs that*

$$\{x_1, x_2, \ldots, x_{299}, x_{300}\} = \{1, 2, \ldots, N-1, N\}.$$

Find the biggest possible value for N.

Solution. Firstly, we shall prove that if A, B are friend then the sum of friends of each one is not exceed 300. Indeed,

Suppose that A has $a \leq N$ friends and B has $b \leq N$ friends. Note that A and B cannot have any common friend; otherwise, take C is a friend of A, B and then the triple (A, B, C) does not satisfy the given condition. Thus

$$(a-1)+(b-1) \leq 298 \text{ and } a+b \leq 300.$$

Suppose that $N \geq 201$ and take some student X has 201 friends. Take Y has at least 100 friends, then as the remark above, X and Y are not friend. Note that we always can find 100 distinct students and each of them has exactly $100, 101, \ldots, 200$ friends, and they are not friend of X. So the number of friend of X is less than 200, contradiction. Therefore, $N \leq 200$.

We can give an examples as follows. Divide students into groups $A = \{A_1, A_2, \ldots, A_{200}\}$ and $B = \{B_1, B_2, \ldots, B_{100}\}$.

- Student B_1 is friends of all student in A.
- Student B_2 is friends of all student from A_2 to A_{200}.
- ...
- Student B_{100} is friends of students from A_{100} to A_{200}.
- Students in same group are not friend.

So it is easy to check that there are no any triplet of students that are friend of each other. And number of friends of students in A is $1, 2, \ldots, 100$ and number of friends of students in B is $200, 199, \ldots, 101$. $\square$

Solution to tests of April camp

1. Test 1

Problem 1. *In a school there are* 40 *different clubs, each of them contains exactly* 30 *children. For every* i *from* 1 *to* 30 *define* n_i *as a number of children who attend exactly* i *clubs. Prove that it is possible to organize* 40 *new clubs with* 30 *children in each of them such, that the analogical numbers* $n_1, n_2, \ldots, n_{30}$ *will be the same for them.*

Solution. We will do the following algorithm to rearrange the children.

- Put the children attend to exactly one club at the star of the line (in any order).
- Put the children attend to exactly 2 club at the star of the line (in any order), and so on.
- Finally, put the children attend to exactly 30 club at the star of the line (in any order).

Since each club contains 40 children, then the number of pairs (club, children) is equal to

$$1200 = 30 \times 40 = n_1 + 2n_2 + \cdots + 30n_{30}.$$

Now back to the line, we will count from $1 \to 1200$ and for each number, we will point at the children in the following way:

- We point at the children from the top to the bottom of the line.
- If a children attends to k clubs, we points at him k times (each time, we also count the numbers).

Finally, we create the new clubs $C_1, C_2, \ldots, C_{40}$ and add children to them using the rule: if we count a number i mod 40 while pointing at some children, then put that one into C_i for any i.

We will count $\frac{1200}{40} = 30$ numbers which are i mod 40 for each i, so indeed, each club has exactly 30 children, and no one appears in the same club since each children is counted at most 30 times in the row. And it is easy to check that the numbers $n_1, n_2, \ldots, n_{30}$ remains unchanged. $\square$

Problem 2. *Let Pascal triangle be an equilateral triangular array of number, consists of* 2019 *rows and except for the numbers in the bottom row, each number is equal to the sum of two numbers immediately below it. How many ways to assign each of numbers* a_0, a_1, ..., a_{2018} *(from left to right) in the bottom row by* 0 *or* 1 *such that the number* S *on the top is divisible by* 1019.

Solution. First, by induction, one can show that

$$S = \binom{n}{0}a_0 + \binom{n}{1}a_1 + \cdots + \binom{n}{n}a_n$$

if the Pascal triangle consist of n rows.

Note that for any odd prime p, we also have:

Claim 1. $\binom{2p}{p} \equiv 2 \pmod{p}$. Indeed,

$$\begin{aligned}\binom{2p}{p} - 2 &= \frac{(2p)!}{p!p!} - 2 = \frac{(p+1)(p+2)\dots(2p-1)(2p)}{p!} - 2\\ &= 2\frac{(p+1)(p+2)\dots(2p-1) - (p-1)!}{(p-1)!}\end{aligned}$$

The numerator is congruent to $1 \cdot 2 \cdot 3 \cdot (p-1) - (p-1)! = 0 \pmod{p}$ so $\binom{2p}{p} \equiv 2 \pmod{p}$.

Claim 2. $\binom{2p}{k} \equiv 0 \pmod{p}$ for $1 \le k \le 2p-1$ and $k \ne p$. Indeed,

Since $\binom{2p}{k} = \binom{2p}{2p-k}$ so we can suppose $1 \le k < p$. Similar calculation, we have

$$\binom{2p}{k} = \frac{(2p-k+1)(2p-k+2)\dots(2p-1)(2p)}{k!}.$$

The numerator is divisible by p while $(p, k!) = 1$ since $1 \le k < p$ so we are done. From this, we can conclude that, if $1009|S$ then $a_0 + a_{2018} + 2a_{1009}$ is divisible by 1009. This only happened when all of them are equal to 0.

The others number can be assigned any of 0 or 1 so the number of ways is 2^{2016}. □

Problem 3. *Find all functions* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$f\left(3\left(f\left(xy\right)\right)^2 + \left(xy\right)^2\right) = \left(xf\left(y\right) + yf\left(x\right)\right)^2.$$

Solution. By hypothesis of the problem, for all $x, y, z > 0$, we have

$$\sqrt{f\left(3f^2(xyz) + x^2y^2z^2\right)} = (yf(xz) + xzf(y)) = (yzf(x) + xf(yz)),$$

which implies that

$$yzf(x) + xf(yz) = yf(xz) + xzf(y).$$

Let $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ given by $g(x) = \dfrac{f(x)}{x}$, for all $x \in \mathbb{R}^+$. Then

$$g(x) + g(yz) = g(xz) + g(y), \text{ for all } x, y, z > 0.$$

Let $x = yz$, we have

$$2g(yz) = g(yz^2) + g(y), \text{ for all } y, z > 0$$

Hence $g(z^2) = 2g(z) - g(1)$ for all $z > 0$. Substitute y by y^2 we have

$$2g(y^2z) = g(y^2z^2) + g(y^2) = 2g(yz) + 2g(y) - 2g(1),$$

implies

$$g(yz) + g(y) - g(1) = g(y^2z) = g(y^2z) + g(z) - g(z) = 2g(yz) - g(z),$$

or equivalently

$$g(y) - g(1) + g(z) - g(1) = g(yz) - g(1), \text{ for all } y, z > 0.$$

Let $y = e^a, z = e^b$ and put $h(x) = g(e^x) - g(1)$, we deduce that $h : \mathbb{R} \longrightarrow (-g(1), +\infty)$ and

$$h(a + b) = h(a) + h(b), \text{ for all } a, b \in \mathbb{R}.$$

By induction, we prove that $h(na) = nh(a)$ for all $a \in \mathbb{R}$ and $n \in \mathbb{N}^*$. Since $h(na) > -g(1)$, we have

$$h(a) \geq -\frac{g(1)}{n}, \text{ for all } a \in \mathbb{R}, n \in \mathbb{N}^*.$$

Let $n \to +\infty$, we have $h(a) \geq 0$ for all $a \in \mathbb{R}$. Then for all $a > b$, we have

$$h(a) = h(a - b) + h(b) \geq h(b).$$

Thus, h is a monotone function. Combines with the additivity of f, we have $h(x) = kx$ on $\mathbb{R}$ with a is a constant. However, $h(x) \geq 0$ for all x then $kx \geq 0$ for all x, implying $k = 0$. So that $g(x) = g(1) = a$ and $f(x) = ax$ for all $x > 0$. Finally, substitute $f(x) = ax$ into the given condition, we have $a = 1$ or $a = \frac{1}{3}$. Thus $f(x) = x$ or $f(x) = \frac{x}{3}$ are solutions of this equation. □

2. Test 2

Problem 4. *Let pairwise different positive integers a, b, c with $gcd(a, b, c) = 1$ are such that $a \mid (b - c)^2$, $b \mid (c - a)^2$, $c \mid (a - b)^2$. Prove that there is no non-degenerate triangle with side lengths a, b and c.*

Solution. First, we will show that a, b, c are pairwise coprime. Denote $d = \gcd(b, c)$ and suppose that $d > 1$. Take p as a prime divisor of d. We have

$$\begin{cases} p|b \\ p|c \end{cases} \Rightarrow \begin{cases} p|(c-a)^2 \\ p|(a-b)^2 \end{cases} \Rightarrow \begin{cases} p|c-a \\ p|a-b \end{cases} \Rightarrow p|a.$$

This implies that $p|\gcd(a, b, c)$, a contradiction. So $\gcd(b, c) = 1$. Similar with $\gcd(c, a), \gcd(a, b)$. Suppose that a, b, c are side-lengths of some triangle then

$$b + c > a,\ c + a > b,\ a + b > c.$$

So by putting $x = (a + b - c)(b + c - a)(c + a - b) > 0$, we can see that $x > 0$. We have $a|(b - c)^2 - a^2 = (b - c - a)(b - c + a)$, thus $a|x$. Similarly, we have $b|x, c|x$ and since $\gcd(a, b) = \gcd(b, c) = \gcd(c, a) = 1$ so $abc|x$. Hence,

$$abc \leq (a + b - c)(b + c - a)(c + a - b).$$

From AM-GM, we have $(a + b - c)(b + c - a) \leq \left(\dfrac{a + b - c + b + c - a}{2}\right)^2 = b^2$ and similar inequalities as $(b + c - a)(c + a - b) \leq c^2, (c + a - b)(a + b - c) \leq a^2$ then

$$(a + b - c)(b + c - a)(c + a - b) \leq abc.$$

These imply the equality must be hold, so $a = b = c$, contradiction since a, b, c pairwise distinct.

□

Problem 5. *Let be given a positive integer $n > 1$. Find all polynomials $P(x)$ non constant, with real coefficients such that*

$$P(x)P(x^2)\dots P(x^n) = P\left(x^{\frac{n(n+1)}{2}}\right)$$

for all $x \in \mathbb{R}$.

Solution. Denote m as degree of $P(x)$ then by comparing the leading coefficients a of two sides, we get $a^n = a$. We consider two cases base on the parity of n.

1. If n is even then $a = 1$. Put $P(x) = x^m + Q(x)$ with $Q \in \mathbb{R}[x]$ and if $Q(x) \equiv 0$, we obtain $P(x) = x^m$ as a solution. Otherwise, suppose that $Q(x) \neq 0$ and $\deg Q(x) = k < m$. So

$$(x^m + Q(x))\left(x^{2m} + Q(x^2)\right)\cdots(x^{mn} + Q(x^n)) - x^{\frac{mn(n+1)}{2}} = Q\left(x^{\frac{n(n+1)}{2}}\right).$$

The degree of LHS is $\frac{mn(n+1)}{2} - (m - k)$, while the degree of RHS is $\frac{kn(n+1)}{2}$. Note that

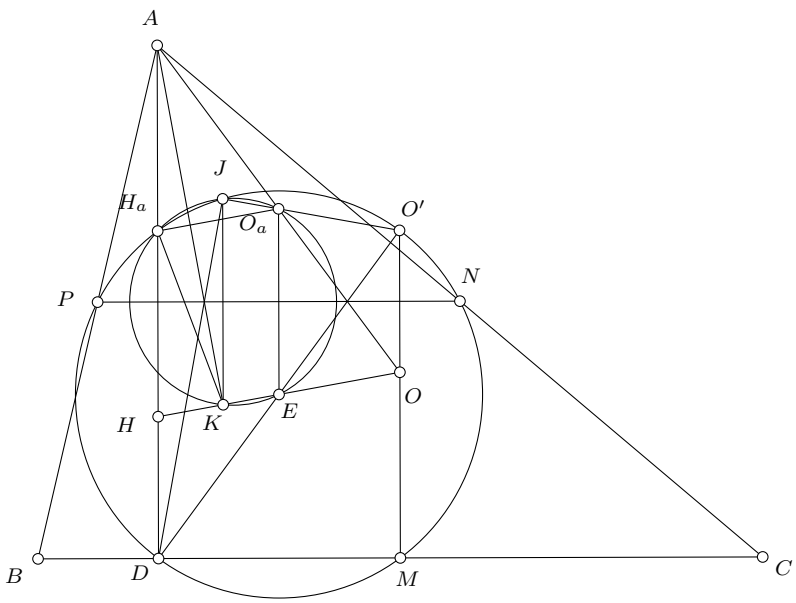
$$\frac{mn(n+1)}{2} - (m-k) - \frac{kn(n+1)}{2} = \frac{(m-k)(n+2)(n-1)}{2} > 0$$

which implies that the degree of both side cannot be equal. Thus $Q(x) \equiv 0$.

2. If n is odd then $a = \pm 1$ and the case $a = 1$ can be solved similarly as above. For the case $a = -1$, put $P(x) = -(x^m + Q(x))$ then process similarly, we get $P(x) = -x^m$ as another solution.

Hence, $P(x) = x^m$ for some $m \in \mathbb{Z}^+$ when n is even and $P(x) = \pm x^m$ when n is odd. $\square$

Problem 6. *Let ABC be an acute, non isosceles triangle with O, H are circumcenter and orthocenter, respectively. Prove that the nine-point circles of AHO, BHO, CHO has two common points.*



Solution. Let M, N, P be the midpoints of BC, CA, AB and D be the projection of A on BC. Denote (E) is the Euler circle of $\triangle ABC$ then M, N, P, D lie on (E) and E is midpoint of OH. Similarly, let H_a, O_a be the midpoints of AH, AO and K be the projection of A on OH then H_a, O_a, K, E lie on the Euler circle of $\triangle AOH$.

Denote J as symmetric with K through NP. Foremost, we will prove that J is the intersection of (KH_aO_a) and (E).

Indeed, easy to see that O, E is circumcenter and orthocenter of $\triangle MNP$. From midsegment, we have

$$\overrightarrow{EO_a} = \frac{1}{2} \cdot \overrightarrow{HA} = \overrightarrow{MO},$$

so O_a is symmetric with E through NP. Then EO_aJK is isosceles trapezoid, which means that $J \in (EO_aK)$. (1)

In the other hand, since O is orthocenter of $\triangle MNP$, if we denote O' as symmetric with O through NP, so $O' \in (E)$. In addition $MO' \perp MD$, then DO' as the diameter of (E). And by the symmetry, we get

$$\angle DJO' = \angle AKO = 90^\circ,$$

which implies that $J \in (E)$. (2)

From (1) and (2), we get J is the intersection of (KH_aO_a) and (E). Note that $J \in (E)$ and $O'J$ is symmetric to OH through NP, which implies that J is the anti-Steiner point of OH which respect to $\triangle MNP$. Similarly, J lies on the Euler circles of $\triangle BOH$, $\triangle COH$. $\square$

Remark. This problem is a corollary of three Fontené's theorem and the properties of the orthopole.

3. Test 3

Problem 7. *Let $P(x)$ be a monic polynomial of degree 100 with 100 distinct non-integer real roots. Suppose that each of polynomials $P(2x^2-4x)$ and $P(4x-2x^2)$ has exactly 130 distinct real roots. Prove that there exist non constant polynomials $A(x), B(x)$ such that $A(x)B(x) = P(x)$ and $A(x) = B(x)$ has no root in $(-1;1)$.*

Solution. Denote $P(x) = (x-a_1)(x-a_2)\cdots(x-a_{100})$ with $a_1, a_2, \ldots, a_{100}$ are real roots of polynomial $P(x)$. So

$$P(2x^2-4x) = 0 \Leftrightarrow 2x^2-4x-a_i = 0 \text{ for } 1 \le i \le 100.$$

This equation cannot have 1 root since $\Delta = 16+8a_i \neq 0$. So each equation can have 0 or 2 roots. Note that $P(2x^2-4x)$ has 130 roots so there are $\frac{130}{2} = 65$ equations have 2 roots, which mean there are 65 numbers $a_i > -2$ and 35 numbers $a_i < -2$.

By the same way, we have

$$P(4x-2x^2) = 0 \Leftrightarrow 2x^2-4x+a_i = 0$$

and $\Delta = 16-8a_i$ for any$1 \le i \le 100$. And there are 65 numbers $a_i < 2$ and 35 numbers $a_i > 2$.

By applying the principle of inclusion and exclusion, there are

$$65+65-100 = 30 \text{ numbers } a_i \in (-2;2).$$

Suppose that $a_1 < a_2 < \ldots < a_{35} < -2 < a_{36} < \ldots < a_{64} < 2 < a_{65} < \ldots < a_{100}$; and denote M, N, K as the subsets of these numbers with the indices $1 \to 35$, $36 \to 64$, $65 \to 100$.

Take $A_1(x) = \prod\limits_{m\in M}(x-m), B(x) = \prod\limits_{n\in N}(x-n), A_2(x) = \prod\limits_{k\in K}(x-k)$ then we will prove that

$$|A_1(x_0)\cdot A_2(x_0)| > |B(x_0)|$$

for any number $x_0 \in (-1;1)$.

Note that $\forall x_0 \in (-1;1)$ then $|x_0-m| > 1, \forall m < -2$ and $|x_0-k| > 1, \forall k > 2$. We can suppose that $x_0 > 0$ and for any $n \in N$, we have two cases:

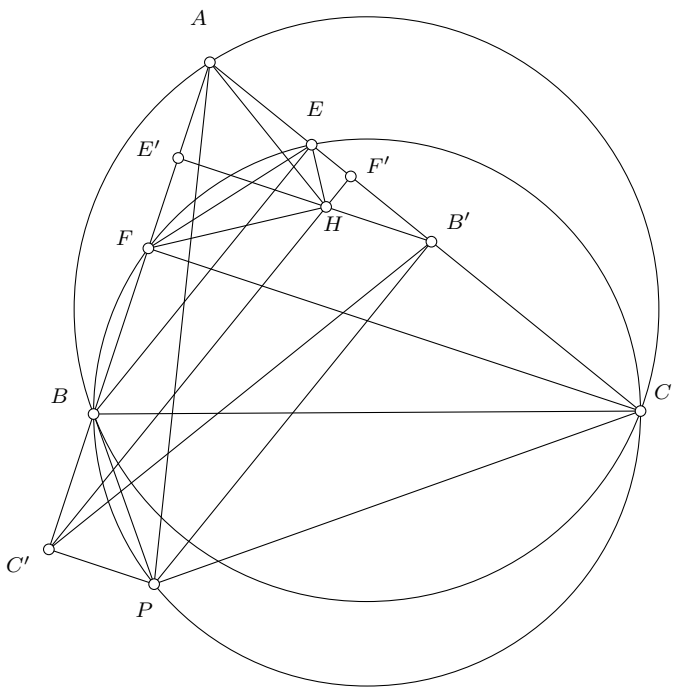
1. If $n > 0$ then $n \in (0;2)$ and $|x_0-n| < |x_0-2| < |x_0-k|$ with any $k > 2$.

2. If $n < 0$ then $n \in (-2; 0)$ and $|x_0 - n| < |x_0 - (-2)| < |x_0 - m|$ with any $m < -2$. So in all cases of n respect to the factor $x_0 - n$ in $B(x_0)$, we can choose factor from $A_1(x_0)$ or $A_2(x_0)$ with absolute value greater than it, and since $|M|, |P| > |N|$, we always can do that. Hence $|A_1(x_0) \cdot A_2(x_0)| > |B(x_0)|, \forall x_0 \in (-1; 1)$ which mean this equation has no solution.

Therefore, we can choose $A(x) = A_1(x)A_2(x)$ and $B(x)$ to satisfy the given condition. □

Problem 8. *Let ABC be a triangle, the circle having BC as diameter cuts AB, AC at F, E respectively. Let P a point on this circle. Let C_0, B_0 be the projections of P upon the sides AB, AC respectively. Let H be the orthocenter of the triangle AB_0C_0. Show that $\angle EHF = 90°$.*

Solution. Let $B'E', C'F'$ be the altitudes of $\triangle AB'C'$ and concur at H.
We have that $AB'PC'$ is inscribed in the circle with diameter AP, so AP passes through the circumcenter of $\triangle AB'C'$. Then AP, AH reflect each other by the angle bisector of $\angle B'AC'$. From that, easy to prove that $\triangle AHF' \sim \triangle APC'$.



But $BE \parallel C'F'$, as Thales's Theorem, we have $\dfrac{AE}{AF'} = \dfrac{AB}{AC'}$, implies that

$$\frac{HE}{PB} = \frac{AE}{AB} = \frac{EF}{BC}.$$

Similarly, we get $\dfrac{HF}{PC} = \dfrac{EF}{BC}$, thus $\triangle HEF \sim \triangle PBC$. Then we get

$$\angle EHF = \angle BPC = 90°.$$

□

Remark. We can solve this problem by applying the property of transformation "homothety - symmetric" like following:

Denote E' and F' as the projection of B' and C' onto AC, AB respectively. Consider the "homothety - symmetric" transformation with ratio $\frac{AB}{AF}$ maps $E' \to B'$ and $F' \to C'$, then it maps $H \to P$. We can show that it also maps

$$F \to C,\ E \to B$$

then $\angle EHF = \angle BPC = 90°$.

Problem 9. *All of the numbers* $1, 2, 3, ..., 1000000$ *are initially colored black. On each move it is possible to choose the number* x *(among the colored numbers) and change the color of* x *and of all of the numbers that are not co-prime with* x *(black into white, white into black). Is it possible to color all of the numbers white?*

Solution. The answer is YES. We will prove by induction that the procedure can be applied for any positive integer n.

The statement is true for $n = 1$. Suppose that it is also true for $n = k - 1 \geq 1$, which mean exist a way to change every number not exceed $k - 1$ from black to white, which called process A.

We shall prove the statement for k with some cases as follow

1. If k is not a square-free number. Put $k = p_1^{a_1} p_2^{a_2} \dots p_v^{a_v}$ and $k' = p_1 p_2 \dots p_v$ with $p_1, p_2, \dots, p_v$ are primes. And note that the number of changing the color of k and k' are equal (since the set of integers not coprime with them are the same), then k' can change the color if and only if k can. So after making process A with the sequence $1, 2, \dots, k - 1$, the number k will changes color from black to white.

2. If k is a square-free number. In case after making process A, number k changes color, then we are done. Otherwise, put $k = p_1 p_2 \dots p_t$ and we consider process B to select all divisors other than 1 of k. After that, number k will be affected $2^t - 1$ times and will change the color.

   Denote k' is some divisor greater than 1 of k, and k' has $s < t$ prime divisors. Take some number c is a multiple of k' but coprime to k/k'. After process A, number c has color white. And after process B, the number of changing the color of c is $(2^t - 1) - (2^{t-s} - 1) = 2^t - 2^{t-s}$ which is an even number. This implies that c will change the color.

So in all cases, we always can find a process to change color of all numbers from 1 to k from black to white.

Finally, by replacing $n = 10^6$, the problem is solved. □

Solution to JBMO tests

1. Test 1

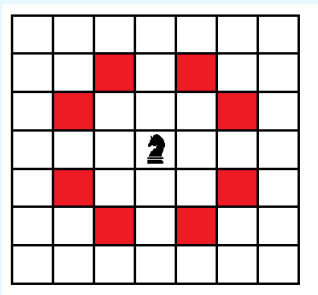
Problem 1. *Find the smallest integer m for which there are positive integers $n > k > 1$ satisfying the equation*

$$\underbrace{11\ldots1}_{n} = \underbrace{11\ldots1}_{k} \cdot m.$$

Solution. Obviously $m > 9$. If $m = \overline{ab}$, where $a \geq 1$ then we must have $b = 1$ to ensure last digit of $1 \cdot m$ is equal 1. In this case regardless of value of a the second last digit of $\underbrace{11\ldots1}_{k} \cdot m$ is equal to the last digit of $a + 1$ and can't be equal 1. So $m \geq 100$. Obviously $m = 100$ doesn't satisfy to the condition, but $m = 101$ satisfies, since $11 \cdot 101 = 1111$.

Hence, the answer to this problem is $m = 101$. □

Problem 2. *Chess horse attacks fields in distance $\sqrt{5}$. Let several horses are put on the board 12×12 such, that every square of size 2×2 contains at least one horse. Find the maximal possible number of cells that are not under attack (horse doesn't attack it's own cell).*

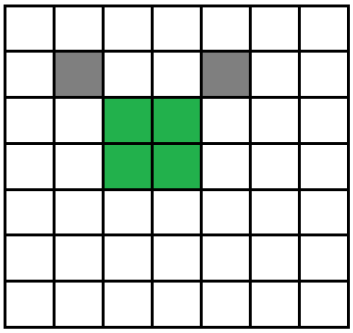


Solution. Let's note, that if we put a horse in any green cell, then it will attack a grey cell. Since green cells form a square 2×2, so one of them contains a horse, so at least one grey is under attack.

Now let's split the board into 72 pairs like in the figure. According what we said above, at least 72 cells are under attack. To get example with exactly 72 cell under

attack lets' paint board as chessboard and put horses in all black cells. Then all black cells will be free of attack.

Hence, the answer to this problem is 72.



1	2	3	1	2	3	4	5	6	4	5	6
7	8	9	7	8	9	10	11	12	10	11	12
67	68	69	67	68	69	70	71	72	70	71	72

□

Problem 3. *How many integers n satisfy to the following conditions?*

i) $219 \leq n \leq 2019$,
ii) There exist $x, y \in \mathbb{N}$ such that $1 \leq x < n < y$ and y is divisible by all integers from 1 to n, except two numbers x and $x+1$.

Solution. The answer is 292. We can see that if $x = pq$ for some integers $p, q > 1$ and $gcd(p, q) = 1$ then $1 < p, q < x$ which implies that $p \mid y$, $q \mid y$, then $pq \mid y$, contradiction.

Hence x and $x+1$ must be the powers of primes. But one of these numbers is even so one of them must be the power of 2. On the other hand, n must be less than $2x$, otherwise $2x \mid y$ leads to $x \mid y$, contradiction. With the existence of x we can easily choose y.

Thus, number n satisfies the given condition if and only if there exists an exponent of 2 less than n and bigger than $n/2$, namely x such that $x+1$ or $x-1$ is a power of some prime. We can check directly each range of numbers

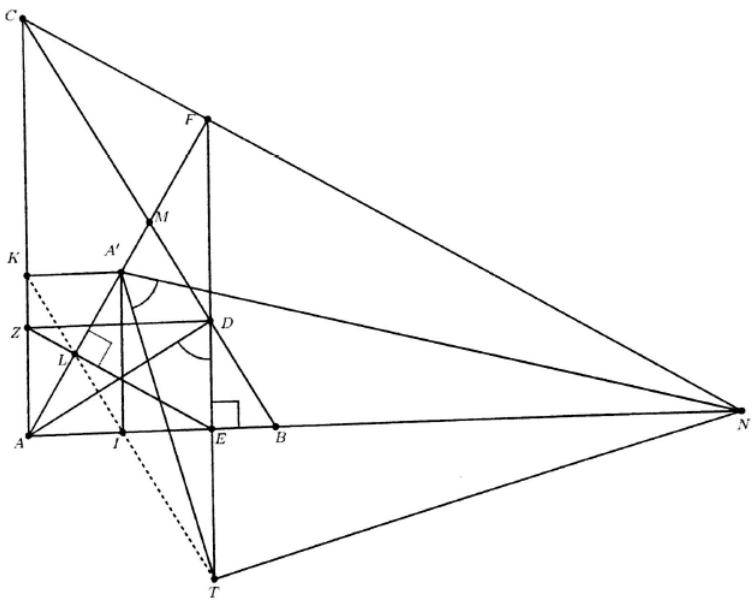
1. For each number $219 \leq n \leq 255$ we can choose $x = 127^1, x+1 = 2^7$.
2. For each number $257 \leq n \leq 511$ we can choose $x = 2^8$ and $x+1 = 257^1$
3. For each numbers from $513 \leq n \leq 1023$ we cannot choose any x since 511 and 513 are not the powers of prime.
4. For each numbers from $1025 \leq n \leq 2019$ we cannot choose any x since 1023 and 1025 are not powers of prime.

Therefore, the total number of integers we need to find is $37 + 255 = 292$. □

Problem 4. *Let AD be the altitude of the right angled triangle ABC with $\angle A = 90°$. Let DE be the altitude of the triangle ADB and DZ be the altitude of the triangle ADC respectively. Let N is chosen on the line AB such that CN is parallel to EZ. Let A' be the symmetric of A with respect to the line EZ and I, K the projections of A' into AB and AC respectively. Prove that $\angle NA'T = \angle ADT$, where T is the intersection point of IK and DE.*

Solution. Suppose that the line AA' intersects the lines EZ, BC and CN at the points L, M, F respectively. The line IK being diagonal of the rectangle $KA'IA$ passes through L, which by construction of A' is the middle of the other diagonal AA'. The triangles ZAL and ALE are similar, so $\angle ZAL = \angle AEZ$. By the similarity of the triangles ABC and DAB we get $\angle ACB = \angle BAD$. We have also that $\angle AEZ = \angle BAD$, therefore

$$\angle ZAL = \angle CAM = \angle ACB = \angle ACM.$$



Since $AF \perp CN$, we have that the right angled triangles AFC and CDA are equal. Thus the altitudes from the vertices F and D of triangles AFC and CDA are respectively equal. It follows that $FD \parallel AC$ and since $DE \parallel AC$ we get that the points E, D, F are collinear. In triangle LFT we have

$$A'I \parallel FT \text{ and } \angle LA'I = \angle LIA',$$

so $\angle LFT = \angle LTF$. Therefore the points F, A', I, T belong to the same circle. Also $\angle A'IN = \angle A'FN = 90°$ so the quadrilateral $IA'FN$ is cyclic. Thus the points F, A', I, N all lie on a circle. From the above, we infer that

$$\angle NA'T = \angle TFN = \angle ACF = \angle FEZ = \angle ADT.$$

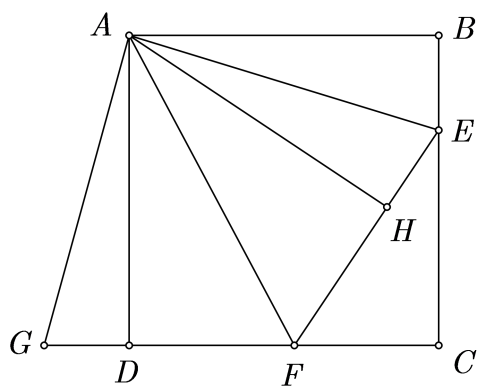
□

2. Test 2

Problem 1. *In square $ABCD$ with side 1 point E lies on BC and F lies on CD such that $\angle EAB = 20^\circ$ and $\angle EAF = 45^\circ$. Determine the height of triangle AEF from point A.*

Solution. Take point G on the opposite ray of ray DC such that $GD = BE$. Then two triangles ABE, ADG are congruent, implies that $AE = AG$ and $\angle EAB = \angle DAG = 20^\circ$. Thus

$$\angle FAG = \angle FAD + \angle DAG = \angle EAB + \angle DAG = 45^\circ.$$



Thus two triangle AEF, AGF has common side AF and $AE = AG$ and $\angle FAG = \angle FAE = 45^\circ$, thus $\triangle AEF \cong \triangle AGF$, then $AH = AD = 1$, which AH is the altitude respect to vertex A in triangle AEF. □

Problem 2. *Prove the inequality for non-negative a, b, c*

$$a\sqrt{3a^2+6b^2} + b\sqrt{3b^2+6c^2} + c\sqrt{3c^2+6a^2} \geq (a+b+c)^2.$$

Solution. Note that $3a^2 + 6b^2 \geq (a+2b)^2$ true for all real numbers a, b. Indeed after expanding and grouping we get $2a^2 - 4ab + 2b^2 \geq 0$ which is equivalent to $2(a-b)^2 \geq 0$. So with non-negative numbers a, b, c we have

$$\begin{aligned} &a\sqrt{3a^2+6b^2} + b\sqrt{3b^2+6c^2} + c\sqrt{3c^2+6a^2} \\ &\geq a(a+2b) + b(b+2c) + c(c+2a) = (a+b+c)^2. \end{aligned}$$

□

Problem 3. *Find all primes p such that there exist integers m and n satisfying relations $p = m^2 + n^2$ and $p \mid m^3 + n^3 + 8mn$.*

Solution. If $m = 0$ then $p = n^2$ is a prime, which impossible. Similar cas happens when $n = 0$. So we may assume $mn \neq 0$. Then $p > |m|$ and $p > |n|$. Note, that

$$m^3 + n^3 = (m+n)(m^2 - mn + n^2) \equiv -mn(m+n) \pmod{p}.$$

So the problem statement is equivalent to

$$mn(m+n-8) \equiv 0 \pmod{p}.$$

Since $p > m, n$ and p is prime, then $p \mid m+n-8$, so $m^2 + n^2 \leq\mid m+n-8$. We have two cases:

1. If $m+n-8 \geq 0$ then $m^2+n^2 \leq m+n-8$ which means $m(m-1)+n(n-1 \leq -8$. this is impossible, since $m(m-1)$ and $n(n-1)$ are integers and non-negative.
2. If $m+n-8 < 0$ then we get $m(m+1)+n(n+1) \leq 7$. So we conclude $m, n = \pm 3, \pm 2, \pm 1$.

By case works we get 3 solutions as follow

- $p = 2$ when $(m, n) = (1, 1)$,
- $p = 5$ when $(m, n) = (2, 1)$,
- $p = 13$ when $(m, n) = (-3, -2)$.

□

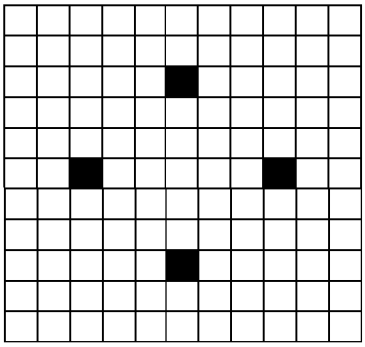
Problem 4. *An 11×11 square is partitioned into 121 smaller 1×1 squares, 4 of which are painted black, the rest being white. We cut a fully white rectangle (possibly a square) out of the big 11×11 square. What is the maximal area of the rectangle we can obtain regardless of the positions of the black squares? It is allowed to cut the rectangle along the grid lines.*

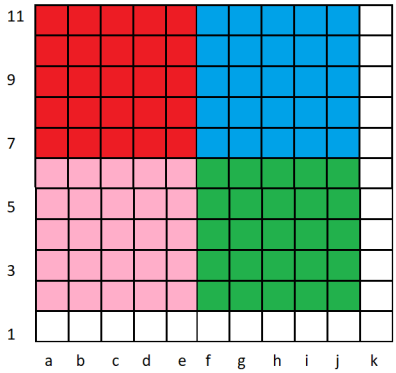
Solution. In the first image we have position for 4 black cells, such that the biggest rectangle without black is 25. Now let's prove that for any configuration there exists a rectangle of area 25. Assume for some configuration the biggest rectangle is at most 24. Let's divide the board into 4 squares of size 5 like in second image. Each of them must contain at least one black cell, so the last column and first row have no black cells.

Analogously first column and last row have no black cells. By considering squares $a1 - e5$, $a7 - e11$, $g1 - k5$ and $g7 - k11$ we conclude that 6-th row and f column doesn't contain black cell.

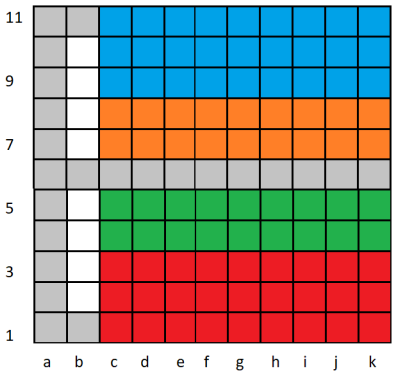
Now assumes that there is a black cell in column b. Then each of rectangles $c1 - k3$, $c9 - k11$ contain at least one black cell, so either orange or green rectangle doesn't contain a black cell. That rectangle together with layer $c6 - k6$ will form a rectangle of ares 27 that has no black cell. Contradiction. So column b has no black cell. The same for column l and rows 2 and 10. So we get the last image configuration, where in gray cells can not be painted black. Then each white square 3×3 must contains exactly one black cell.

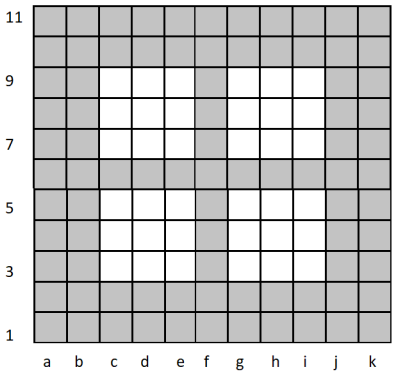
Assume $c9$ is black. Then consider $d7 - h11$ and $a6 - k8$. We conclude that black cell is in square $g7 - h8$. Same goes for $d4 - e5$. Then by looking at the rectangles $i1 - k11$ and $a1 - k3$ we conclude that $i3$ must be black.





From rectangles $a1-d8$ and $a1-h4$ follows that $d4$ and $h8$ are black. Be then $e1-g11$ is white and has ares 33. So we conclude that $c3, c9, i3, i9$ are white. Finally, each of rectangles $c4-c8$, $d9-h9$, $d3-h3$, $i4-i8$ must have exactly one black square. But then central square $d4-h8$ contains 25 white cells, a contradiction.

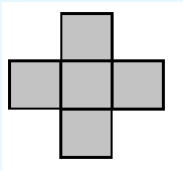




□

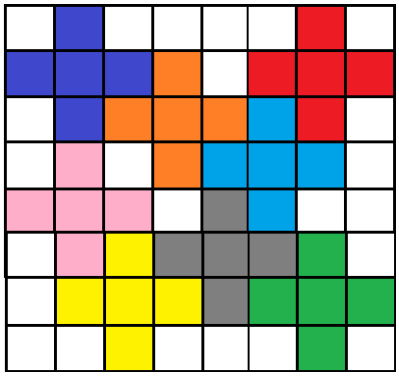
3. Test 3

Problem 1. *Determine the maximal number of disjoint crosses can be put on* 8×8 *chessboard such that sides of a crosses are parallel to sides of the chessboard.*



Solution. Let's note, that 4 corner cells can't be covered by crosses. So we assume that they are removed from the board and we have only 60 cells.

From the first row at most two cells can be covered by crosses. So at least 4 cells will be not covered. The same argument works for last row, first column and last column. So at most $60 - 4 \cdot 4 = 44$ cell can be covered, which means at most 8 crosses can be used. It remains to show the example with 8 crosses.



□

Problem 2. *Find all pairs of positive integers (m, n) such that the following equation holds*

$$125 \cdot 2^n - 3^m = 271.$$

Solution. Considering the equation mod 5 we get

$$3^m \equiv -1 \pmod 5,$$

so $m = 4k + 2$ for some positive integer k. Then, considering the equation mod 7 we get

$$-2^n - 9^{2k+1} \equiv 5 \pmod 7,$$

which means

$$2^n + 2^{2k+1} \equiv 2 \pmod 7.$$

Since $2^s \equiv 1, 2, 4 \pmod 7$ so the only possibility is $2^n \equiv 2^{2k+1} \equiv 1 \pmod 7$ so $3 \mid n$ and $3 \mid 2k+1$. From the last one we get $3 \mid m$ so we can write $n = 3x$ and $m = 3y$.

Therefore, the given equation takes the form $5^3 \cdot 2^{3x} - 3^{3y} = 271$, or

$$(5 \cdot 2^x - 3^y)(25 \cdot 2^{2x} + 5 \cdot 2^x \cdot 3^y + 3^{2y}) = 271.$$

It follows that $25 \cdot 2^{2x} + 5 \cdot 2^x \cdot 3^y + 3^{2y} \leq 271$ ans so $x < 2$. We conclude $x = 1$ and then $y = 2$. So $m = 6$ and $n = 3$. □

Problem 3. *Let ABC be an acute, non isosceles triangle. Take two points D, E inside this triangle such that*

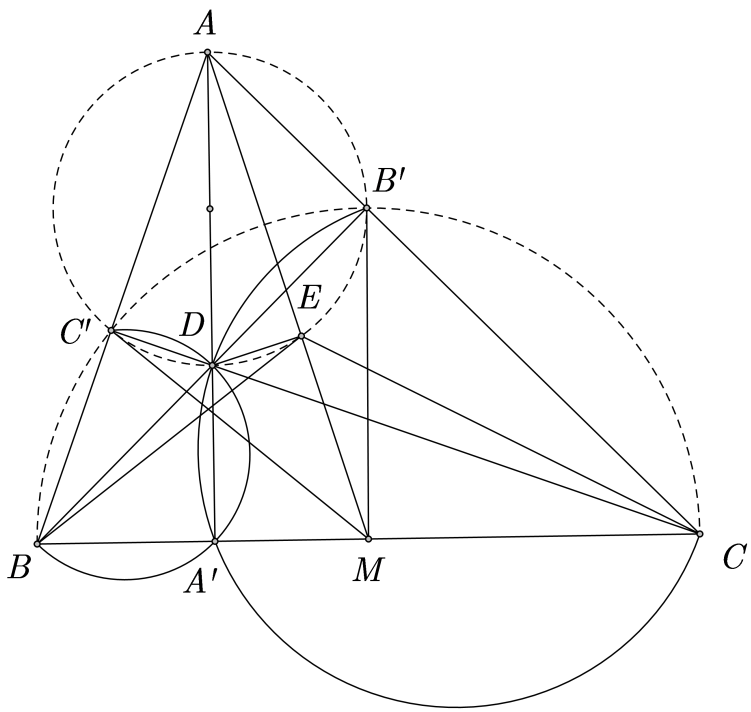
$$\angle DAB = \angle DCB, \ \angle DAC = \angle DBC;$$
$$\angle EAB = \angle EBC, \ \angle EAC = \angle ECB.$$

Prove that triangle ADE is right.

Solution. First, we will show that D is the orthocenter of triangle ABC. Denote $A' = AD \cap BC, B' = BD \cap CA, C' = CD \cap AB$. Since $\angle DAB = \angle DCB$, we have $ACA'C'$ is cyclic. Similarly, $ABA'B'$ is also cyclic. So we have

$$\angle DA'B = \angle DB'A, \angle DA'C = \angle DC'A \text{ and } DA \cdot DA' = DB \cdot DB' = DC \cdot DC',$$

so $BCB'C'$ is also cyclic. Thus $\angle DB'A = \angle DC'A$ which implies that $\angle DA'B = \angle DA'C$, but $\angle DA'B + \angle DA'C = 180^\circ \to AA' \perp BC$. It leads to $BB' \perp CA, CC' \perp AB$ then D is the orthocenter of triangle ABC.



Continue, denote M as the intersection of AE, BC then $\angle EBM = \angle EAB$ implies that MB is tangent to (ABE). Similarly, MB is tangent to (ACE) then

$$ME \cdot MA = MB^2 = MC^2 = MB'^2 = MC'^2$$

(since $MB = MC = MB' = MC'$). So MB' is tangent to (AEB'), thus $\angle AEB' = \angle MB'C = \angle C$. Similarly, $\angle AEC' = \angle B$ then

$$\angle B'EC' = \angle AEB' + \angle AEC' = \angle B + \angle C = 180^\circ - \angle A.$$

Thus $AB'EC'$ is cyclic, but $AB'DC'$ is cyclic, so five points A, B', C', D, E are concyclic, which implies that

$$\angle AED = \angle AB'D = \angle AC'D = 90^\circ.$$

Therefore, ADE is a right triangle. □

Remark. This problem is about property of Humpty's point. A similar way to state this is as follow: Take F is a point inside $\triangle ABC$ such that $\angle FAB = \angle FCB$ and $\angle FAC = \angle FBC$. Prove that $\angle AFO = 90^\circ$ with O is circumcenter.

Problem 4. *Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be any real numbers. Prove that there exists $m, k \in \{1, 2, \ldots, n\}$ such that*

$$\left| \sum_{i=1}^{m} a_i - \sum_{i=m+1}^{n} a_i \right| \leq | \, a_k \, | \, .$$

Solution. Denote

$$S_x = \left| \sum_{i=1}^{x} a_i - \sum_{i=x+1}^{n} a_i \right|.$$

Obviously $S_0 = -S_n$ and $S_{x+1} - S_x = 2a_{x+1}$. We may assume that $S_n \geq 0$ and consider m for which $S_m \leq 0$ and $S_{m+1} \geq 0$. Then

$$|S_m| + |S_{m+1}| = |S_m - S_{m+1}| = 2 \mid a_{m+1} \mid,$$

so either $|S_m|$ or $|S_{m+1}|$ is at most a_{m+1} which is at most $\max_i |a_i| = a_k$. This completes the proof. □

4. Test 4

Problem 1. *A set S is called neighbouring if it has the following two properties:*

i) S has exactly 4 elements,

ii) for every element $x \in S$ at least one of the numbers $x - 1$ or $x + 1$ belongs to S.

Find the number of all neighboring subsets of the set $\{1, 2, \ldots, n\}$.

Solution. Let x is the smallest index and y is the biggest index of the neighboring set. Then our set consists of elements

$$a_x, a_{x+1}, a_{y-1}, a_y.$$

So the number of neighboring sets is equal to the number of pairs (x, y) where $y - x \geq 3$. The number of such a sets is equal

$$1 + 2 + \cdots + (n - 3) = \frac{(n-3)(n-2)}{2}.$$

□

Problem 2. *Prove that there are no positive integers x, y, z such that*

$$(3x + 4y)(4x + 5y) = 7^z.$$

Solution. From the condition, we can see that both numbers $3x+4y$ and $4x+5y$ are powers of 7, so such must be also their division. However

$$1 < \frac{4x+5y}{3x+4y} < 2$$

and can't be power of 7. □

Remark. Another way to prove this problem is: $7|3x+4y, 7|4x+5y$ then

$$7|4(3x+4y) - 3(4x+5y) = y,$$

thus $7|y$ and $7|x$. Put $x = 7x', y = 7y'$ with $x', y' \in \mathbb{Z}^+$ then we also have $3x' + 4y', 4x' + 5y'$ are powers of 7. But this process can be repeat infinitely many times, a contradiction.

Problem 3. *Let S be a given set of real numbers such that*

i) $1 \in S$,

ii) For any $a, b \in S$ (not necessary different), one has $a - b \in S$,

iii) For $a \in S, a \neq 0$ one has $\frac{1}{a} \in S$.

Prove that for any $a, b \in S$, then $ab \in S$.

Solution. If $a = 0 \in S$ or $a = 1 \in S$ then for any $b \in S$ we have $ab \in S$ which is obvious. So we can suppose that $a, b \notin \{0; 1\}$.

From $1 \in S, a \in S$ we have $1 - a \in S \to (1-a) - 1 = -a \in S$, then

$$b - (-a) = a + b \in S.$$

We also have $a - 1 \in S$ so $\frac{1}{a}, \frac{1}{a-1} \in S$ then

$$\frac{1}{a-1} - \frac{1}{a} = \frac{1}{a(a-1)} \in S \to a(a-1) \in S.$$

Since $a \in S, a(a-1) \in S$ then $a^2 = a(a-1) + a \in S$. Similarly, we also have $b^2 \in S$. Thus $a, b \in S \to a + b \in S \to (a+b)^2 \in S$.

Finally, $(a+b)^2 - a^2 = 2ab + b^2 \in S$ and $(2ab + b^2) - b^2 = 2ab \in S$ which implies that

$$\frac{1}{2ab} + \frac{1}{2ab} = \frac{1}{ab} \in S \to ab \in S.$$

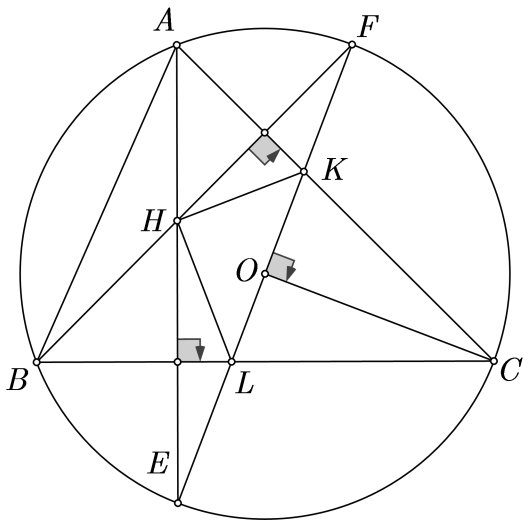
□

Problem 4. *In triangle ABC, such that $\angle ACB = 45^\circ$ let O and H be circumcenter and orthocenter, respectively. Line passing through O and perpendicular to CO intersects AC and BC at K and L, respectively. Prove that perimeter of KLH is equal to diameter of circumcircle of triangle ABC.*

Solution. Suppose that AH, BH cut (O) at the second points E, F. By angle chasing, we can see that H, E are symmetric respect to BC. Note that

$$\angle HBC = 90^\circ - \angle C = 45^\circ$$

so $\angle CBE = \angle CBH = 45^\circ$.



Thus $\angle CFE = 45^\circ$, which implies that $\angle COE = 90^\circ$. Similarly, $\angle COF = 90^\circ$, then four points E, L, K, F are collinear. From this, we conclude that $LH = LE$ and $KH = KF$ which mean

$$\begin{aligned} perimeter(HKL) &= HL + LK + KH \\ &= EL + LK + KF = EF = diameter(O). \end{aligned}$$

$\square$

Solution to IMO Team selection tests

1. Day 1

Problem 1. *Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that*

$$n^3 - n^2 \leq f(n) \cdot (f(f(n)))^2 \leq n^3 + n^2$$

for every n is positive integers.

Solution. Firstly, we prove that f is injective. Let two natural numbers a, b such that $a + 1 \leq b$ and $f(a) = f(b)$. By the given condition, we have

$$f(a)[f(f(a))]^2 \leq a^3 + a^2 < a^3 + 2a^2 + a = (a+1)^3 - (a+1)^2.$$

Since $a, b \in \mathbb{Z}^+$ and $a + 1 \leq b$, we have

$$(a+1)^3 - (a+1)^2 \leq b^3 - b^2 \leq f(b)[f(f(b))]^2.$$

Thus $f(a)[f(f(a))]^2 < f(b)[f(f(b))]^2$, a contradiction. This implies f is injective. In given condition, let $n = 1$, we have

$$0 \leq f(1)[f(f(1))]^2 \leq 2.$$

Since $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$, we have two cases

- $f(1) = 1$.
- $f(1) = 2$ and $f(2) = 1$.

We shall prove that if $\{f(1), \ldots, f(k)\} = \{1, \ldots, k\}$ then either $f(k+1) = k+1$ or $f(k+1) = k+2, f(k+2) = k+2$. Indeed,

Since f is injective, we have $f(k+1) \geq k+1$. If $f(k+1) = k+1$, we have

$$\{f(1), \ldots, f(k+1)\} = \{1, \ldots, k+1\}.$$

Otherwise, if $f(k+1) \geq k+2$, we have $[f(f(k+1))]^2 \geq (k+1)^2$. In given condition, let $n = k+1$, we have

$$(k+1)^3 + (k+1)^2 \geq f(k+1)[f(f(k+1))]^2 \geq (k+2)(k+1)^2.$$

Since $(k+1)^3 + (k+1)^2 = (k+2)(k+1)^2$, we deduce that $f(k+1) = k+2$ and $f(k+2) = k+1$. Thus $\{f(1), \ldots, f(k+2)\} = \{1, \ldots, k+2\}$. Now we can check both cases satisfy the conditions

$$\begin{aligned}&(k+2)^3 - (k+2)^2 \leq f(k+2)f(f(k+2))^2 \leq (k+2)^3 + (k+2)^2\\ &\Leftrightarrow (k+2)^2(k+1) \leq (k+1)(k+2)^2 \leq (k+2)^2(k+3).\end{aligned}$$

Therefore, the solutions of this problem are the functions have following properties

- There are some disjoint pairs $(k, k+1)$ with $f(k) = k+1$ and $f(k+1) = f(k)$.
- For other positive integer a, then $f(a) = a$.

$\square$

Problem 2. *Find all pair of integers (m, n) and $m \geq n$ such that there exist a positive integer s and satisfy*

1) The product of all divisors of sm, sn are equal.
2) The number of divisors of sm, sn are equal.

Solution. 1) Denote $d(x), \pi(x)$ as the number of divisors, the product of divisors of positive integer x.

Firstly, we can see that for any divisor y_k of x, $1 \leq k \leq d(x)$ then $\frac{x}{y_i}$ is also divisor of x, thus

$$\prod_{k=1}^{d(x)} y_k = \prod_{k=1}^{d(x)} \frac{x}{y_k} \text{ so } \left(\prod_{k=1}^{d(x)} y_k\right)^2 = x^{d(x)}, \text{ hence } \pi(x) = x^{\frac{d(x)}{2}}.$$

Thus for any $m, n \in \mathbb{Z}^+$ then $\pi(m) = \pi(n)$ implies that m, n share the common prime divisors set S.

Suppose that $d(m) \geq d(n)$ and take $p \in S$. Since $m^{d(m)} = n^{d(n)}$, we have $d(m) \cdot v_p(m) = d(n) \cdot v_p(n)$. Since $d(m) \geq d(n)$, we get $v_p(m) \leq v_p(n)$. And this is true for all $p \in S$, thus

$$d(m) = \prod_{p \in S} (v_p(m) + 1) \leq \prod_{p \in S} (v_p(n) + 1) = d(n).$$

So $d(m) = d(n)$, which implies that $m = n$. Therefore, we can find a positive integer such that product of all divisor of sm, sn are equal if and only if $m = n$.

2) Firstly, we can see that if $n|m$ then any divisor of sn is also divisor of sm, so $d(sn) < d(sm)$. We consider $n \nmid m$, and denote $p_1, p_2, \ldots, p_t$ be all prime dividing mn. Suppose that

$$m = \prod_{i=1}^{t} p_i^{\alpha_i} \text{ and } n = \prod_{i=1}^{t} p_i^{\beta_i}.$$

Now we looking for $s = \prod_{i=1}^{t} p_i^{\gamma_i}$ such that

$$\frac{d(sm)}{d(sn)} = \prod_{i=1}^{t} \frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = 1.$$

Note that if $\alpha_i = \beta_i$, then regardless of the value of γ_i, the corresponding factor equals to 1 and not affect the product. So we may assume that $\alpha_i \neq \beta_i$ for all $1 \leq i \leq t$.

Claim. Let $\alpha > \beta$ be nonnegative integers. Then for every $M \geq \beta + 1$, there exist a nonnegative integer γ such that

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = \frac{M + 1}{M}.$$

It is equivalent to $\gamma = M(\alpha - \beta) - (\beta + 1) \geq 0$, which is true.

Back to the original problem, we can assume that $\alpha_i > \beta_i$ for $i = 1, 2, \ldots, u$ and $\alpha_i < \beta_i$ for $i = u+1, u+2, \ldots, t$. Take some big enough X and choose γ_i such that

- $\dfrac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = \dfrac{uX + i}{uX + i - 1}$ for $1 \leq i \leq u$.
- $\dfrac{\beta_{u+i} + \gamma_{u+i} + 1}{\alpha_{u+i} + \gamma_{u+i} + 1} = \dfrac{(t-u)X + i}{(t-u)X + i - 1}$ for $1 \leq i \leq t - u$.

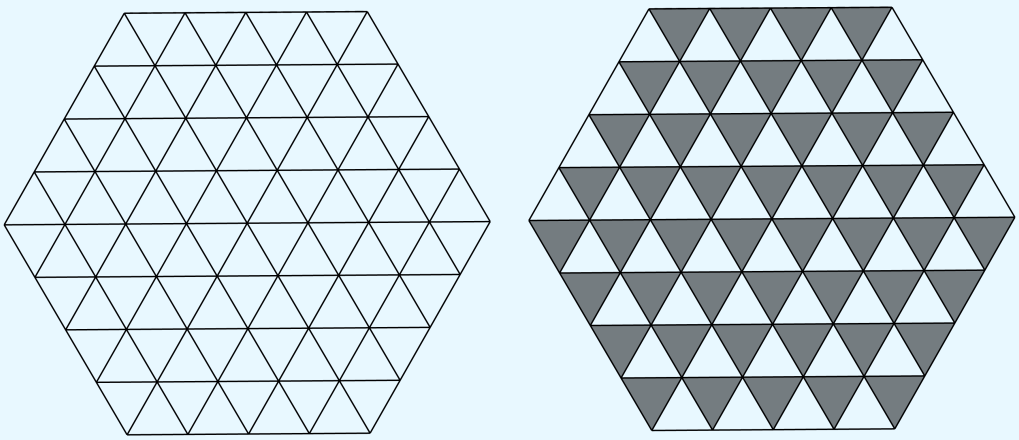
Then we have

$$\frac{d(sm)}{d(sn)} = \prod_{i=1}^{u} \frac{uX + i}{uX + i - 1} \cdot \prod_{i=1}^{t-u} \frac{(t-u)X + i - 1}{(t-u)X + i} = \frac{u(X+1)}{uX} \cdot \frac{(t-u)X}{(t-u)(X+1)} = 1.$$

The proof is complete. □

Remark. This problem takes the idea from problem N1, IMO Shortlisted 2018.

Problem 3. *Let regular hexagon is divided into $6n^2$ regular triangles. Let $2n$ coins are put in different triangles such, that no any two coins lie on the same layer (layer is area between two consecutive parallel lines). Let also triangles are painted like on the chess board.*



Prove that exactly n coins lie on black triangles.

Solution. Let's divide the perimeter of the hexagon to 3 parts, each of them containing pair of neighbor sides. Let's enumerate $2n$ segments in each pair from 0 to $2n - 1$ in counter-clockwise direction. Let's correspond to each triangle the sum of three numbers written on projection segments (taken in direction of rays).

Note, that every black triangle corresponds $3n - 1$ and to every white triangle corresponds $3n - 2$. Assume we have a black triangles having coins.

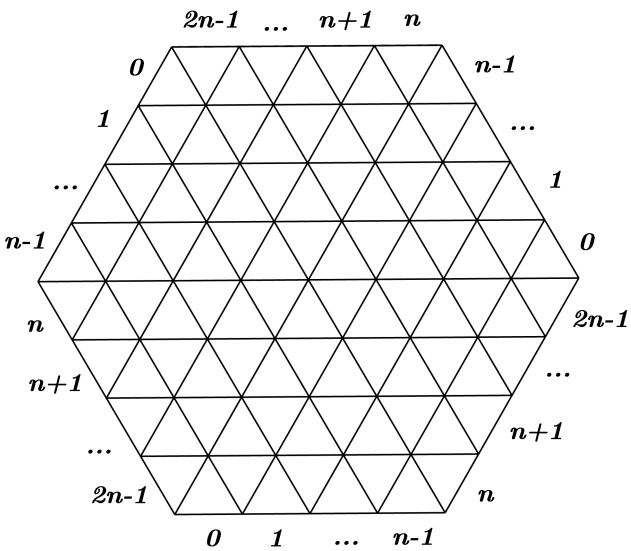
Now, let's do double-counting and count the sum of numbers that correspond to the triangles containing coin. From one side the sum is $a(3n-1)+(2n-a)(3n-2)$ and on other side the same sum is equal to

$$3(0+1+2+\ldots+(2n-1)),$$

so we have

$$a(3n-1)+(2n-a)(3n-2)=3n(2n-1),$$

which means $a=n$.



□

2. Day 2

Problem 4. *Let a_0 be an arbitrary positive integer. Let (a_n) be infinite sequence of positive integers such that for every positive integer n, the term a_n is the smallest positive integer such that $a_0+a_1+\cdots+a_n$ is divisible by n. Prove that there exist N such that $a_{n+1}=a_n$ for all $n\geq N$.*

Solution. Define $b_n=\dfrac{a_0+a_1+\cdots+a_n}{n}$ for every positive integer n. According to the condition, $b_n\in\mathbb{Z}^+$ for every positive integer n. Since a_{n+1} is the smallest positive integer such that $\dfrac{a_0+a_1+\cdots+a_n}{n}\in\mathbb{Z}^+$ and

$$\frac{a_0+a_1+\cdots+a_n+b_n}{n+1}=\frac{a_0+a_1+\cdots+a_n+\frac{a_0+a_1+\cdots+a_n}{n}}{n}=b_n$$

which is a positive integer, we get $a_{n+1}\leq b_n$ for every $n\in\mathbb{Z}^+$. Hence, the infinite sequence of positive integer $b_1,b_2,\ldots$ is non-increasing. So there exist a positive integer T such that for all $n\geq T$ then

$$b_{n+1}=b_n\Rightarrow\frac{a_0+a_1+\cdots+a_n+a_{n+1}}{n+1}=\frac{a_0+a_1+\cdots+a_n}{n}$$

thus

$$n(a_0 + a_1 + \cdots + a_n + a_{n+1}) = (n+1)(a_0 + a_1 + \cdots + a_n)$$

and

$$a_{n+1} = \frac{a_0 + a_1 + \cdots + a_n}{n} = b_n.$$

Similarly, we get $a_{n+2} = b_{n+1}$, which follows that

$$a_{n+2} = b_{n+1} = b_n = a_{n+1}.$$

Hence, taking $M = T + 1$, we can state that $a_{n+1} = a_n$ for every $n \geq M$.

□

Problem 5. *Let non-constant polynomial $f(x)$ with real coefficients is given with the following property: for any positive integer n and k, the value of expression*

$$\frac{f(n+1)f(n+2)\dots f(n+k)}{f(1)f(2)\dots f(k)} \in \mathbb{Z}.$$

Prove that $f(x)$ is divisible by x.

Solution. Without lose of generality one may assume that $f(1) \in \mathbb{Z}$. Since for all positive k, we have $\dfrac{f(k+1)}{f(1)}$ is integer, then we conclude that on all positive integer points our polynomial gets integer values. Assume that $deg(f) = d$ then, according to Lagrange interpolation formula we get

$$f(x) = \sum_{i=1}^{d+1} f(i) \frac{(x-1)\dots(x-(i-1))(x-(i+1))\dots(x-(d+1))}{(i-1)\dots 1(-1)\dots(i-(d+1))},$$

and all numbers are rational, so $p(x)$ is a polynomial with rational coefficients.

By multiplying to the constant we can get $f(x)$ with integer coefficients. If $f(0) = 0$ then we are done. Assume that $f(0) \neq 0$. Let's fix positive integer k and denote $f(n) = f(n+1)f(n+2)\dots f(n+k)$ and $M = f(0)$.

According to the problem condition, we have $\dfrac{f(n)}{M} \in \mathbb{Z}$ for all positive integer n. Since $f(x)$ is polynomial with integer coefficients, then

$$f(-1) \equiv f(2\,|M| - 1) \equiv 0 \pmod{M},$$

which means

$$\frac{f(-1)}{M} = \frac{f(0)}{f(k)} \in \mathbb{Z},$$

for all positive integers k. It means $f(0) = 0$ or $f(x)$ is divisible by x. □

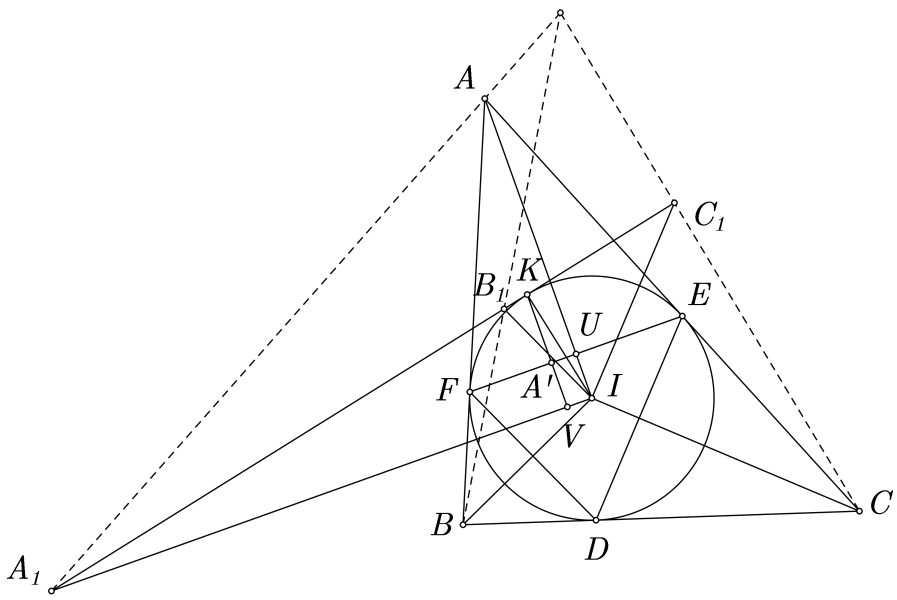
Problem 6. *Let ABC be an acute nonisosceles triangle with incenter I and (d) is an arbitrary line tangent to (I) at K. The lines passes through I, perpendicular to IA, IB, IC cut (d) at A_1, B_1, C_1 respectively. Suppose that (d) cuts BC, CA, AB at M, N, P respectively. The lines through M, N, P and parallel to the internal bisectors of A, B, C in triangle ABC meet each other to define a triange XYZ.*

Prove that three lines AA_1, BB_1, CC_1 are concurrent and IK is tangent to the circle (XYZ).

Solution. Suppose that (d) is tangent to (I) at K. Denote D, E, F as the tangent points of (I) with the sides BC, CA, AB. Let V be the projection of K onto A_1I and $U = AI \cap EF$. It is easy to see that

$$IU \cdot IA = IV \cdot IA_1 = r^2$$

with r is the radius of (I).



Consider the inversion Ω of center I, and power r^2 then $A \leftrightarrow U, A_1 \leftrightarrow V$ so $AA_1 \leftrightarrow (IUV)$.

Define B', C' similarly, then to prove the original problem, we just need to show that the circles of diameter IA', IB', IC' have a common point differs from I.

Note that A', B', C' belongs to the Simson's line of the point K respect to triangle DEF so by taking the homothety of center I, ratio $\frac{1}{2}$ then the center of above circles are collinear. Thus they share some other common point differs from I; the image of this point through the inversion is the concurrent point of three line AA_1, BB_1, CC_1.

Continue, consider the figure as below, the other case of position of points can be processed similarly. Denote S as the reflection of I throught K. Then we have $YZ \parallel AI, YX \parallel IC$ so

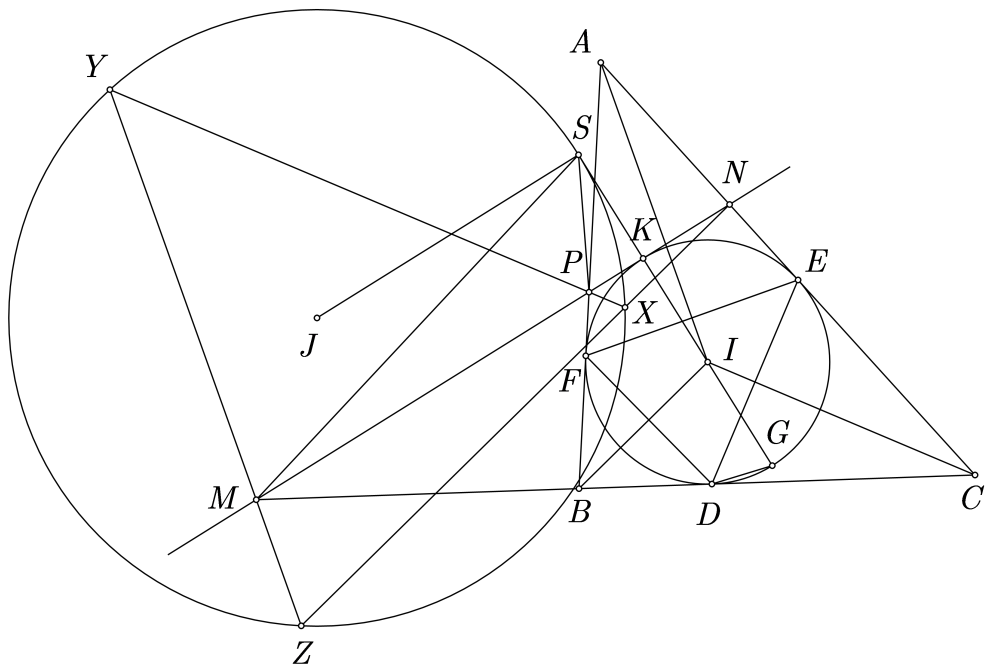
$$\angle MYP = \angle XYZ = (YX, YZ) = (IA, IC) = 90^\circ - \frac{\angle B}{2}.$$

On the other hand, we have

$$\angle MSP = \angle MIP = \angle KIM - \angle KIP = \frac{\angle KID - \angle KIE}{2} = \frac{1}{2}\angle DIF = 90^\circ - \frac{\angle B}{2},$$

so these points M, P, Y, S are concyclic which implies that $S \in (MPY)$.

Similarly, $S \in (NPX), S \in (MNZ)$. From this, we can conclude that S is the Miquel's point of the completed quadrilateral of 6 vertices M, N, P, X, Y, Z. Thus $S \in (XYZ)$.



Continue, note that by doing the similar angle chasing, we can see that two triangles XYZ, DEF are similar (with the same direction). Hence, there exist a spiral similarity Ω transforms $XYZ \to DEF$ with the angle 90° (since their sides are perpendicular pairwise). Draw the diameter KG of circle (I), then we have

$$\angle GDE = \angle GKE = \angle IKE = \angle KNI = \angle KNS = \angle PXS = \angle SXY.$$

Similarly, $\angle GED = \angle SYX$. Thus $\Omega : S \to G$ (two corresponding points). Denote J as the circumcenter of triangle XYZ then $\Omega : J \to I$ so we have $JS \perp IG$. But these points I, G, K, S are collinear so $\angle JSK = 90^\circ$ which implies that IK is tangent to the circle (XYZ). $\square$

Remark. This problem takes the idea from problem G7, IMO Shortlisted 2018.

Problems without solution

1. Senior level

Test 1

Problem 1. *Some $n > 2$ lamps are cyclically connected: lamp 1 with lamp 2, ..., lamp k with $k+1$, ..., lamp n with lamp 1. At the beginning, all lamps are off. When one pushes the switch of a lamp, that lamp and the two ones connected to it change status (from off to on, or vice-versa). Determine the number of configurations of lamps reachable from the initial one, through some set of switches being pushed.*

Problem 2. *Determine all arithmetic sequences $a_1, a_2, \ldots$ for which there exists integer $N > 1$ such that for any positive integer k the following divisibility holds*

$$a_1 a_2 \ldots a_k \mid a_{N+1} a_{N+2} \ldots a_{N+k}.$$

Problem 3. *Let $ABCD$ be a trapezoid with $\angle A = \angle B = 90^\circ$ and a point E lies on the segment CD. Denote (ω) as incircle of triangle ABE and it is tangent to AB, AE, BE respectively at P, F, K. Suppose that KF cuts BC, AD at M, N; and PM, PN cut (ω) at H, T. Prove that $PH = PT$.*

Test 2

Problem 4. *Let be given a positive integer $n \geq 3$. Consider integers $a_1, a_2, \ldots, a_n > 1$ with the product equals to A such that: for each $k \in \{1, 2, \ldots, n\}$ then the remainder when $\frac{A}{a_k}$ divided by a_k are all equal to r. Prove that $r \leq n - 2$.*

Problem 5. *A sequence $(a_1, a_2, \ldots, a_k)$ consisting of pairwise different cells of an $n \times n$ board is called a* cycle *if $k \geq 4$ and cell a_i shares a side with cell a_{i+1} for every $i = 1, 2, \ldots, k$, where $a_{k+1} = a_1$. We will say that a subset X of the set of cells of a board is* malicious *if every cycle on the board contains at least one cell belonging to X. Determine all real numbers C with the following property: for every integer $n \geq 2$ on an $n \times n$ board there exists a malicious set containing at most Cn^2 cells.*

Problem 6. *Consider equilateral triangle ABC and suppose that there exist three distinct points X, Y, Z lie inside triangle ABC such that*

i) $AX = BY = CZ$.

ii) The triplets of points $(A, X, Z), (B, Y, X), (C, Z, Y)$ are collinear in that order.

Prove that XYZ is an equilateral triangle.

Test 3

Problem 7. *In triangle ABC $\angle B = 60^\circ$, O is the circumcenter, and L is the foot of an angle bisector of angle B. The circumcirle of triangle BOL meets the circumcircle of ABC at point $D \neq B$. Prove that $BD \perp AC$.*

Problem 8. *Find all triples of real numbers (x, y, z) such that*

$$\begin{cases} x^4 + y^2 + 4 = 5yz \\ y^4 + z^2 + 4 = 5zx \\ z^4 + x^2 + 4 = 5xy. \end{cases}$$

Problem 9. *Let d be a positive divisor of a positive integer m and (a_l), (b_l) two arithmetic sequences of positive integers. It is given that $\gcd(a_i, b_j) = 1$ and $\gcd(a_k, b_n) = m$ for some positive integers i, j, k, and n. Prove that there exist positive integers t and s such that $\gcd(a_t, b_s) = d$.*

Test 4

Problem 10. *Find the smallest positive integer n with the following property: After painting black exactly n cells of a 7×7 board there always exists a 2×2 square with at least three black cells.*

Problem 11. *Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ that for all real numbers x, y, z satisfies to the equation*

$$f\big(f(x, z), f(z, y)\big) = f(x, y) + z.$$

Problem 12. *Let ABC be a triangle, let D be the touch point of the side BC and the incircle of the triangle ABC, and let J_b and J_c be the incentres of the triangles ABD and ACD, respectively. Prove that the circumcentre of the triangle AJ_bJ_c lies on the bisector of the angle BAC.*

Test 5

Problem 13. *Let n be a positive integer and $p > n+1$ a prime. Prove that p divides the following sum*

$$S = 1^n + 2^n + \ldots + (p-1)^n.$$

Problem 14. *Let the bisector of the outside angle of A of triangle ABC and the circum-circle of triangle ABC meet at point P. The circle passing through points A and P intersects segments BP and CP at points E and F respectively. Let AD is the bisector of triangle AD. Prove that $\angle PED = \angle PFD$.*

Problem 15. *Let x, y, z, a, b, c are pairwise different integers from the set $\{1, 2, 3, 4, 5, 6\}$. Find the smallest possible value for expression $xyz + abc - ax - by - cz$.*

2. Junior level

Test 1

Problem 1. *Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that*

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\geq 2\sqrt{2}\Big(\sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}}\Big).$$

Problem 2. *We call a tiling of an $m\times n$ rectangle with arabos (see figure below) "regular" if there is no sub-rectangle which is tiled with arabos. Prove that if for some m and n there exists a "regular" tiling of the $m\times n$ rectangle then there exists a "regular" tiling also for the $2m\times 2n$ rectangle.*

Problem 3. *Consider a triangle ABC and let M be the midpoint of the side BC. Suppose $\angle MAC=\angle ABC$ and $\angle BAM=105^\circ$. Find the measure of $\angle ABC$.*

Problem 4. *Let p be a prime number. Show that $7p+3^p-4$ is not a perfect square.*

Test 2

Problem 5. *There are 2016 digits written around a circle. If you write down these digits going around the circle clockwise and starting with a certain digit, you get a number divisible by 81. Prove that if you write down the digits going around the circle clockwise, but starting with any other digit, you will again get a number divisible by 81.*

Problem 6. *The altitudes AA_1 and BB_1 of an acute-angled triangle ABC meet at point O. Let A_1A_2 and B_1B_2 be the altitudes of triangles OBA_1 and OAB_1 respectively. Prove that A_2B_2 is parallel to AB.*

Problem 7. *Let n be a positive integer. Each number from 1, 2, 3, ..., 1000 was painted one of n colors. It turned out that every two distinct numbers, one of which is a divisor of the other one, have different colors. Find the smallest n for which such situation is possible.*

Problem 8. *Among numbers $a+b$, $a-b$, ab, $\frac{a}{b}$ exactly two are positive and exactly two are negative. Find the sign of b.*

Test 3

Problem 9. *Each cell of 8×8 board is either black or white. It is known that among any nine cells forming a 3×3 square there are evenly many white ones. What is the minimum possible number of black cells?*

Problem 10. *Let a, b, $c\geq 0$ satisfy $a+b+c=1$. Prove that*

$$\frac{\sqrt{a}}{b+1}+\frac{\sqrt{b}}{c+1}+\frac{\sqrt{c}}{a+1}>\frac{1}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Problem 11. *Determine whether there exists a positive integer n such that $n+2$ divides the following sum*

$$S = 1^{2019} + 2^{2019} + \ldots + n^{2019}.$$

Problem 12. *Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.*

Test 4

Problem 13. *Each cell of 8×8 board is either black or white. It is known that among any nine cells forming a 3×3 square there are evenly many white ones. What is the minimum possible number of black cells?*

Problem 14. *Let 6 pairwise different digits are given and all of them are different from 0. Prove that there exist 2 six-digit integers, such that their difference is equal to 9 and each of them contains all given 6 digits.*

Problem 15. *Let non-integer real numbers a, b, c, d are given, such that the sum of each 3 of them is integer. May it happen that $ab + cd$ is an integer.*

Problem 16. *Let 14 integer numbers are given. Let Hamza writes on the paper the greatest common divisor for each pair of numbers. It occurs that the difference between the biggest and smallest numbers written on the paper is less than 91. Prove that not all numbers on the paper are different.*

Test 5

Problem 17. *Let the bisector of the outside angle of A of triangle ABC and the circum-circle of triangle ABC meet at point P. The circle passing through points A and P intersects segments BP and CP at points E and F respectively. Let AD is the bisector of triangle AD. Prove that $\angle PED = \angle PFD$.*

Problem 18. *Find the largest prime number that divides both $2^{63} - 1$ and $2^{91} - 1$.*

Problem 19. *Real nonzero numbers x, y, z are such that $x + y + z = 0$. Moreover, it is known that*

$$A = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{x}{z} + \frac{z}{y} + \frac{y}{x} + 1.$$

Determine A.

Problem 20. *In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.*

Test 6

Problem 21. *Each point of the plane has some color. It is known that on every straight line there are points in* at most two *different colors. What is the maximum possible number of colors present on this plane?*

Problem 22. *An acute-angled triangle ABC is inscribed in a circle ω. The tangents to the circle ω at A and C intersect at F . The perpendicular bisector of segment AB intersects the line BC at E. Prove that the lines FE and AB are parallel.*

Problem 23. *Does there exist any integer a, b, c such that $a^2bc+2,\ ab^2c+2,\ abc^2+2$ are perfect squares?*

Problem 24. *For any numbers x, y, z prove the following inequality*

$$x^2(3y^2 + 3z^2 - 2yz) \geq yz(2xy + 2xz - yz).$$

Test 7

Problem 25. *Each integer is painted one of three colors. Prove that there exist two distinct numbers of the same color, whose difference is a perfect square.*

Problem 26. *Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Find the minimum for*

$$S = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{2\left(a^3 + b^3 + c^3\right)}{abc}.$$

Problem 27. *Let d be a positive divisor of the number $A = 1024^{1024}+5$ and suppose that d can be expressed as $d = 2x^2+2xy+3y^2$ for some integers x, y. Which reminder we can have when divide d by 20.*

Problem 28. *Positive integer n is called 'nice' if the sum of squares of its divisors is equal $(n + 3)^2$ (for example $287 = 7 \cdot 41$ is 'nice', since $1^2 + 7^2 + 41^2 + 287^2 = 84100 = (287 + 3)^2$. Prove that if for two primes p and q (not necessary distinct) the number $n = pq$ is 'nice', then both $n + 2$ and $2(n + 1)$ are perfect squares.*

Test 8

Problem 29. *There are 11 empty boxes. Each move consists of choosing 10 boxes and adding one stone to each of them. Two players alternate moves. The one to obtain a box with 21 stones wins. Which player has a winning strategy?*

Problem 30. *Quadrilateral $ABCD$ is circumscribed about a circle Λ and K, L, M, N are points of tangency of sides AB, BC, CD, DA with Λ respectively. Let S be the intersection point of lines KM and LN. If quadrilateral $SKBL$ is cyclic then show that $SNDM$ is also cyclic.*

Problem 31. *What is the biggest number of queens of two colors (black and white) one can place on a chessboard, such that no two of them beat one another (the queens do not beat through each other) ?*
<u>Note:</u> The queen is able to move any number of squares vertically, horizontally or diagonally.

Problem 32. *Let $a, b, c \in \mathbb{R}_+$. Prove that*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

Test 9

Problem 33. *Given positive integer n. Several cells of a $n \times n$ board were coloured green in a way that no two green cells share a common segment. Is it always true (i.e. no matter which cells were coloured green) that it is possible to place exactly n rooks in some not green cells of board such that no rook captures any other rook if:*

1. $n = 20$
2. $n = 19$ *?*

Problem 34. *Two circles with centers A and B intersect at points M and N. Radii AP and BQ are parallel (on opposide sides of AB). If the common external tangents meet AB at D, and PQ meets AB at C, prove that CND is a right angle.*

Problem 35. *Given* 10 *quadratic equations*

$$x^2 + a_1x + b_1 = 0,\ x^2 + a_2x + b_2 = 0,\ \ \ldots,\ x^2 + a_{10}x + b_{10} = 0,$$

each of them has two distinct real roots and the set of all roots is $S = \{\pm 1, \pm 2, \ldots, \pm 10\}$. Find the minimum value of of the sum $T = b_1 + b_2 + \cdots + b_{10}$.

Problem 36. *Find all integers $k > 1$ satisfy condition: there exists some non-negative integer n such that the number $A = 17^{18n} + 4 \cdot 17^{2n} + 7 \cdot 19^{5n}$ can be written as the product of k consecutive positive integers.*

Test 10

Problem 37. *Let E be a point lies inside the parallelogram $ABCD$ such that $\angle BCE = \angle BAE$. Prove that the circumcenters of triangles ABE, BCE, CDE, DAE are concyclic.*

Problem 38. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^3}{a^2 + bc} + \frac{b^3}{b^2 + ca} + \frac{c^3}{c^2 + ab} \geq \frac{(a^2 + b^2 + c^2)(ab + bc + ca)}{a^3 + b^3 + c^3 + 3abc}.$$

When will the equality hold?

Problem 39. *Find all positive integers of form $\overline{abcd}$ such that*

$$\overline{abcd} = a^{a+b+c+d} - a^{-a+b-c+d} + a.$$

Problem 40. *All the cells in a 8*8 board are colored white. Omar and Asaad play the following game: in the beginning Omar colors n cells red, then Asaad chooses 4 rows and 4 columns and colors them black. Omar wins if there is at least one red cell. Find the least possible value for n such that Omar can always win regardless of Asaad's move.*

Saudi Arabian Mathematical Competitions 2021

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King Abdullah University of Science and Technology

IMO Booklet

SAUDI ARABIAN Mathematical Competitions

2021

SAMC

Riyadh, June 2021

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مسابقات الرياضيات للمملكة العربية السعودية

SAUDI ARABIAN

MATHEMATICAL COMPETITIONS 2021

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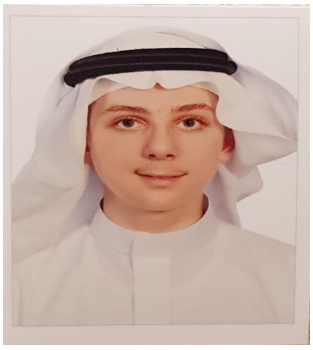
The Saudi Arabian team at **IMO 2021**



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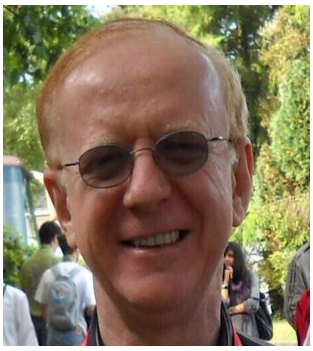

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Introduction

This booklet contains the Team Selection Tests of the Saudi teams to the, Balkan Junior Mathematics Olympiad, and the International Mathematics Olympiad.

The training was supported by the Ministry of Education, which commissioned Mawhiba, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. We would like to express our gratitude to King Abdullah University of Science and Technology KAUST for making its facilities on its beautiful campus available to us for our training, even during the current pandemic.

The Saudi team had three main training camps during the academic year 2020-2021. In addition, the team had an intensive training period from March to the end of June 2021.

During this academic year, the selected students participated in the following contests: The Asia Pacific Mathematics Olympiad, the European Girls Mathematics Olympiad , and the Junior Balkan mathematics Olympiad. The contests were all held online.

It is our pleasure to share the selection tests problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Fawzi A Al-Thukair
Leader of the Saudi team in IMO 2021

مقدمة

يحوي هذا الكتيب على مسائل التصفيات لمسابقة البلقان و مسابقة البلقان للناشئين و تصفيات الاولمبياد الدولي للرياضيات ٢٠١٩.

ان تدريب الفريق كان بدعم من وزارة التعليم بالتعاون مع مؤسسة الملك عبد العزيز و رجاله للموهبة و الابداع " موهبة "

وتجدر الاشارة الى التعاون و الاسهام الفعّال من جامعة الملك عبدالله للعلوم و التقنية، حيث وفرت لنا كل الامكانات التي احتجنا لها في التدريب في حرمها الجامعي الجميل.

تم عقد ثلاثة ملتقيات تدريبية خلال العام الدراسي ٢٠١٨-٢٠١٩ بالاضافة الى فترة التدريب المكثف التي بدأت في شهر مارس ٢٠١٩ الى نهاية شهر يونيو. كما شارك الطلبة المتميزون في العديد من المسابقات الإقليمية و منها: اولمبياد الرياضيات لدول آسيا و الباسيفيك ، اولمبياد الطالبات للدول الأوربية في اوكرانيا ، اولمبياد البلقان في مالدوفا و اولمبياد الناشئين لدول البلقان في قبرص.

نأمل ان يكون محتوى هذا الكتيب إسهاماً منا لتقوية اواصر التعاون و تبادل الخبرات بيننا و الدول المشاركة في الاولمبياد الدولي.

د.فوزي بن أحمد الذكير

رئيس الفريق السعودي للاولمبياد الدولي. ٢٠١٩

Selected problems from camps

1. BMO TST - March camp

1.1. Test 1

Problem 1. Do there exist two polynomials P and Q with integer coefficient such that

i) both P and Q have a coefficient with absolute value bigger than 2021,

ii) all coefficients of $P \cdot Q$ by absolute value are at most 1.

Problem 2. Let ABC be an acute, non-isosceles triangle with H is the orthocenter and M is the midpoint of AH. Denote O_1, O_2 as the centers of circles pass through H and respectively tangent to BC at B, C. Let X, Y be the ex-centers which respect to angle H in triangles HMO_1, HMO_2. Prove that XY is parallel to O_1O_2.

Problem 3. Let x, y and z be odd positive integers such that $\gcd(x, y, z) = 1$ and the sum $x^2 + y^2 + z^2$ is divisible by $x + y + z$. Prove that $x + y + z - 2$ is not divisible by 3.

Problem 4. In the popular game of Minesweeper, some fields of an $a \times b$ board are marked with a mine and on all the remaining fields the number of adjacent fields that contain a mine is recorded. Two fields are considered adjacent if they share a common vertex. For which $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is it possible for some a and b , $ab > 2021$, to create a board whose fields are covered in mines, except for 2021 fields who are all marked with k?

1.2. Test 2

Problem 5. There are $n \geq 2$ positive integers written on the whiteboard. A move consists of three steps: calculate the least common multiple N of all numbers then choose any number a and replace a by $\frac{N}{a}$.

Prove that, using a finite number of moves, you can always make all the numbers on the whiteboard equal to 1.

Problem 6. Let ABC be an acute triangle with $AB < AC$ and inscribed in the circle (O). Denote I as the incenter of ABC and D, E as the intersections of AI with $BC, (O)$ respectively. Take a point K on BC such that $\angle AIK = 90^\circ$ and KA, KE meet (O) again at M, N respectively. The rays ND, NI meet the circle (O) at Q, P.

1. Prove that the quadrilateral $MPQE$ is a kite.

2. Take J on IO such that $AK \perp AJ$. The line through I and perpendicular to OI cuts BC at R, cuts EK at S. Prove that $OR \parallel JS$.

Problem 7. Let a, b, and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

Problem 8. A set of n points in space is given, no three of which are collinear and no four of which are co-planar (on a single plane), and each pair of points is connected by a line segment. Initially, all the line segments are colorless. A positive integer b is given and Alice and Bob play the following game. In each turn Alice colors one segment red and then Bob colors up to b segments blue. This is repeated until there are no more colorless segments left. If Alice colors a red triangle, Alice wins. If there are no more colorless segments and Alice hasn't succeeded in coloring a red triangle, Bob wins. Neither player is allowed to color over an already colored line segment.

1. Prove that if $b < \sqrt{2n-2} - \frac{3}{2}$, then Alice has a winning strategy.
2. Prove that if $b \geq 2\sqrt{n}$, then Bob has a winning strategy.

2. JBMO TST - April camp

2.1. Test 1

Problem 1. Find all positive integers a, b, c and prime number p such that

$$73p^2 + 6 = 9a^2 + 17b^2 + 17c^2.$$

Problem 2. In a circle O, there are six points, A, B, C, D, E, F in a counterclockwise order. $BD \perp CF$, and CF, BE, AD are concurrent. Let the perpendicular from B to AC be M, and the perpendicular from D to CE be N. Prove that $AE \| MN$.

Problem 3. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 9$ and

$$a_{n+1} = \frac{(n+5)a_n + 22}{n+3}$$

for $n \geq 1$. Find all positive integers n for which a_n is a perfect square.

Problem 4. Let F is the set of all sequences $\{(a_1, a_2, \ldots, a_{2020})$ with $a_i \in \{-1, 1\}\}$ for all $i = 1, 2, \ldots, 2020$. Prove that there exists a set S, such that $S \subset F, |S| = 2020$ and for any $(a_1, a_2, \ldots, a_{2020}) \in F$ there exists $(b_1, b_2, \ldots, b_{2020}) \in S$, such that $\sum_{i=1}^{2020} a_i b_i = 0$.

2.2. Test 2

Problem 5. Let $(a_n)_{n \geq 1}$ be a sequence given by $a_1 = 45$ and

$$a_n = a_{n-1}^2 + 15a_{n-1}$$

for $n > 1$. Prove that the sequence contains no perfect squares.

Problem 6. In a triangle ABC, let K be a point on the median BM such that $CM = CK$. It turned out that $\angle CBM = 2\angle ABM$. Show that $BC = KM$.

Problem 7. We have $n > 2$ nonzero integers such that everyone of them is divisible by the sum of the other $n-1$ numbers, Show that the sum of the n numbers is precisely 0.

Problem 8. Let us call a set of positive integers nice if the number of its elements equals to the average of its numbers. Call a positive integer n an amazing number if the set $\{1, 2, , \ldots, n\}$ can be partitioned into nice subsets.

a) Prove that every perfect square is amazing

b) Show that there are infinitely many positive integers which are not amazing.

3. May camp - IMO TST

3.1. Test 1

Problem 1. For a non-empty set $\mathcal{T}$ denote by $p(\mathcal{T})$ the product of all elements of $\mathcal{T}$. Does there exist a set $\mathcal{T}$ of 2021 elements such that for any $a \in \mathcal{T}$ one has that $p(\mathcal{T}) - a$ is an odd integer. Consider two cases:

1. All elements of $\mathcal{T}$ are irrational numbers?

2. At least one element of $\mathcal{T}$ is a rational number?

Problem 2. Find all positive integers n, such that n is a perfect number and $\varphi(n)$ is power of 2. Note: a positive integer n is called perfect if the sum of all its positive divisors is equal to $2n$.

Problem 3. Let I and I_A be the incenter and the A-excenter of an acute-angled triangle ABC with $AB < AC$. Let the incircle meets BC at D, and the line AD meets BI_A and CI_A at E and F, respectively. Prove that the circumcircles of triangles AID and I_AEF are tangent to each other.

3.2. Test 2

Problem 4. In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, Q_2, \ldots, Q_{24}$ whose corners (vertices of the quadrilateral) are vertices of the 100 -gon, so that

- the quadrilaterals $Q_1, Q_2, \ldots, Q_{24}$ are pairwise disjoint, and
- every quadrilateral Q_i has three corners of one color and one corner of the other color.

Problem 5. Let ABC be a non isosceles triangle with incenter I and let the circumcircle of the triangle ABC has radius R. Let AL is the external angle bisector of $\angle BAC$ with $L \in BC$. Let K is the point on the perpendicular bisector of BC such that $IL \perp IK$. Prove that $OK = 3R$.

Problem 6. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f^{\left(a^2+b^2\right)}(a+b) = af(a) + bf(b)$$

for every $a, b \in \mathbb{Z}$. Here, f^n denotes the nth iteration of f, i. e. $f^{(0)}(x) = x$ and $f^{(n+1)}(x) = f\left(f^{(n)}(x)\right)$ for all $n \geq 0$.

3.3. Test 3

Problem 7. Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside the side AB such that $AD < DB$. Let P and Q be two points inside the sides BC and CA respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meets the line segment CQ at E and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E and F are collinear. Prove that $\angle ACB = 90°$.

Problem 8. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist $x, y \in S$, such that $x - y = F_k$.

Problem 9. For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime with n. Prove that that for any number C there exists an integer n for which

$$\frac{\varphi(d(n))}{d(\varphi(n))} > C.$$

3.4. Test 4

Problem 10. Given a positive integer k, show that there exists a prime p such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, 3, \ldots, k$.

Problem 11. Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$

Problem 12. Let p be an odd prime and let $N = \frac{1}{4}\left(p^3 - p\right) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leq N$, denote by $r(n)$ the fraction of integers in $\{1, 2, \ldots, n\}$ that are red (number of red numbers divided by n). Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq \frac{a}{p}$ for all $n = 1, 2, \ldots, N$

Solution to BMO selection tests

1. Day 1

Problem 1. *Do there exist two polynomials P and Q with integer coefficient such that:*

i) both P and Q have a coefficient with absolute value bigger than 2021,

ii) all coefficients of $P \cdot Q$ by absolute value are at most 1.

Solution. Note that the polynomial

$$(1-x^2)(1-x^4)(1-x^8)\dots(1-x^{2^n})$$

has all coefficients equal 0, +1 or −1. Also note, that

$$\begin{aligned}&(1-x^2)(1-x^4)(1-x^8)\dots(1-x^{2^n})\\ =&\prod_{i=0}^{n-1}\left(1-x^{2^i}\right)\cdot\prod_{i=0}^{n-1}\left(1+x^{2^i}\right)\\ =&(1-x)^n\cdot\prod_{i=0}^{n-1}(1+x^i)^{n-i}\end{aligned}$$

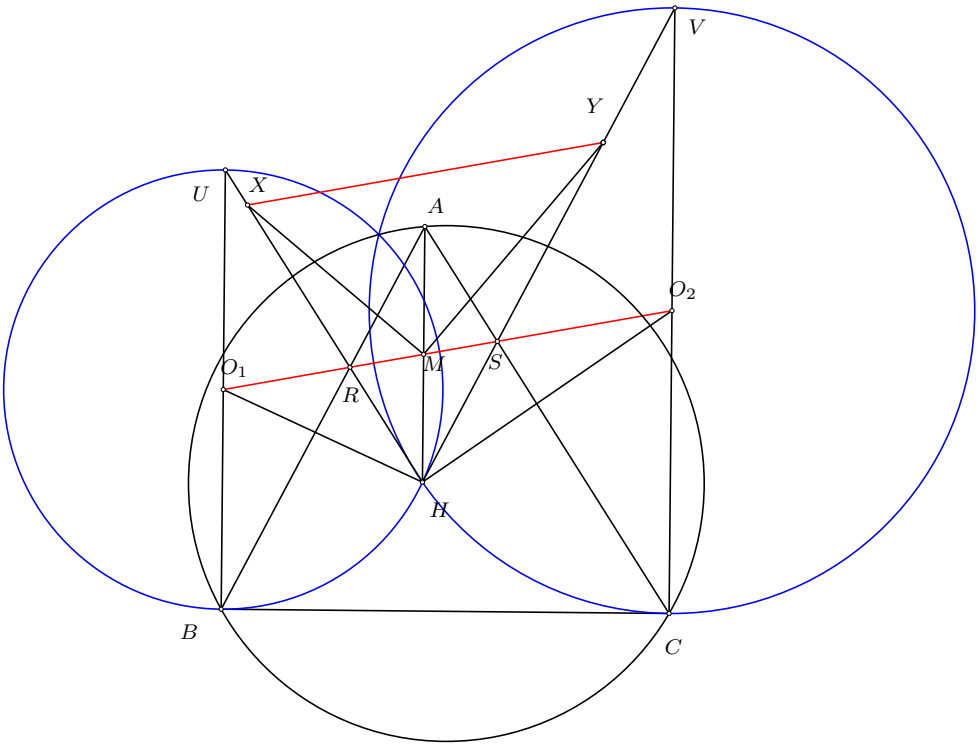
Due to Newton binomial formula $(1\pm x)^n$ have coefficient equal n, also all coefficients of

$$(1+x)^n(1+x^2)^{n-1}\dots(1+x^{2^{n-2}})^2(1+x^{2^{n-1}})$$

are positive and bigger than coefficients of $(1+x)^n$. □

Problem 2. *Let ABC be an acute, non-isosceles triangle with H is the orthocenter and M is the midpoint of AH. Denote O_1, O_2 as the centers of circles pass through H and respectively tangent to BC at B, C. Let X, Y be the ex-centers which respect to angle H in triangles HMO_1, HMO_2. Prove that XY is parallel to O_1O_2.*

Solution. Let BU, CV be the diameter of circles $(O_1), (O_2)$ and denote $R = HU \cap AB, S = HV \cap AC$. We have $BH \perp HU$, but $BH \perp AC$ then $HR \parallel AC$. Similarly $HS \parallel AB$ implying that $ARHS$ is a parallelogram. Hence, M is the midpont of the segment RS.



In the other hand, O_1 is the midpoint of BU and $BU \parallel AH$ thus, by the property of trapezoid, we get O_1, R, M are collinear. Similarly, O_2, S, M are also collinear. Hence, five points O_1, O_2, R, S, M are collinear.

Since X is the excenter of triangle HMO_1 then MX is the external angle bisector of $\angle O_1MH$, thus $\frac{XR}{XH} = \frac{MR}{MH}$. Similarly, $\frac{YS}{YH} = \frac{MS}{MH}$, but $MR = MS$ then $\frac{XR}{XH} = \frac{YS}{YH}$, this implies that $XY \parallel RS$ or $XY \parallel O_1O_2$.

□

Problem 3. *Let x, y and z be odd positive integers such that $\gcd(x, y, z) = 1$ and the sum $x^2 + y^2 + z^2$ is divisible by $x + y + z$. Prove that $x + y + z - 2$ is not divisible by 3.*

Solution. Suppose there exists a prime divisor $p \equiv 2 \pmod 3$ of $x + y + z$. Since $z \equiv -(x + y) \pmod p$ we have $2(x^2 + y^2 + xy) \equiv 0 \pmod p$. Multiplying by $x - y$ we get $x^3 \equiv y^3 \pmod 3$ but this yields $x \equiv y \pmod p$, because $\gcd(3, p - 1) = 1$. Similarly $x \equiv z \pmod p$ whence $3x \equiv 0 \pmod p$ which means that x, y and z are divisible by p which contradicts $\gcd(x, y, z) = 1$. Hence $x + y + z$ have no prime divisors which have a remainder 2 modulo 3 so $x + y + z - 2$ is not divisible by 3.

□

Problem 4. *In the popular game of Minesweeper, some fields of an $a \times b$ board are marked with a mine and on all the remaining fields the number of adjacent fields that contain a mine is recorded. Two fields are considered adjacent if they share a common vertex. For which $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is it possible for some a and b , $ab > 2021$, to create a board whose fields are covered in mines, except for 2021 fields who are all marked with k?*

Solution. For $k = 0$ this is impossible, since at least one non-mine field will be adjacent to the bombs unless the entire field is covered with bombs.

- For $k = 1$ we take $a = 1$, $b = 3 \cdot 1011 - 1$, and 1011 mines spaced two-spaces apart, with one on one of the endpoints, so the total number of fields marked 1 is 2021.
- For $k = 2$ we again take $a = 1$, b large enough, and place 2021 empty fields in isolation of each other, and not on any of the endpoints.
- For $k = 3$, we take $a = 2$, b large enough, and 2019/3 mutually isolated L-triminoes as empty spaces, with two corner points on opposite sides of the longer edge also empty.
- For $k = 4$, we take $a = 3$, b large enough, and 1008 pairs of adjacent empty fields on the edge, with one empty field on the middle row surrounded on all diagonals by 4 empty fields on the edge, resembling an X shape.
- For $k = 5$ we take $a = 3$, b large enough, and place 2021 isolated empty fields on the edge.
- For $k = 6$ we take a and b large enough, and place 2013/3 isolated empty L-triminoes and 2 diamond-shaped regions of 4 fields each.
- For $k = 7$ it is impossible to place an empty field on the edge and since each empty field in the interior is adjacent to exactly one other empty field it follows that the empty fields are paired and thus their total number cannot be 2021, an odd number.
- For $k = 8$ we simply take 2021 isolated empty fields in the interior.

Thus, the set of solutions is: $\{1, 2, 3, 4, 5, 6, 8\}$.

□

2. Day 2

Problem 5. *There are $n \geq 2$ positive integers written on the whiteboard. A move consists of three steps: calculate the least common multiple N of all numbers then choose any number a and replace a by $\frac{N}{a}$.*

Prove that, using a finite number of moves, you can always make all the numbers on the whiteboard equal to 1.

Solution. Clearly a move cannot add any new prime to the set $\mathcal{P}$ of all prime divisors of numbers written on the whiteboard. Let's fix any $p \in \mathcal{P}$ and arrange all numbers in descending order of their exponents of p, so that $v_p(a_1) \geq v_p(a_2) \geq \ldots \geq v_p(a_n)$. Note that $v_p(a_1) = v_p(lcm(a_1, a_2, \ldots, a_n))$ whence $v_p(\frac{N}{a_1}) = 1$. Therefore after we choose as a consecutively numbers a_1, a_2, $\ldots$, a_n, at the end none of the numbers on the board will be a multiple of p. Hence we can always decrease $|\mathcal{P}|$ by at least one, and at some moment we will reach $|\mathcal{P}| = 0$ which means that all numbers on the board will be equal to 1.

□

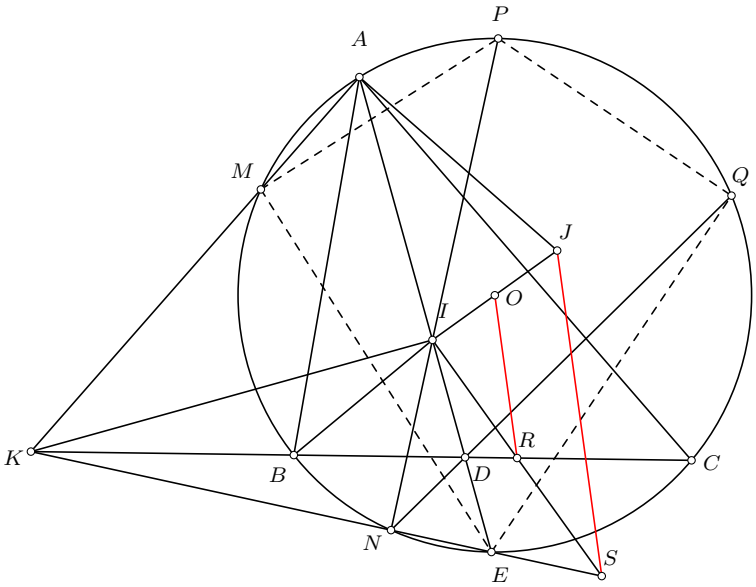
Problem 6. *Let ABC be an acute triangle with $AB < AC$ and inscribed in the circle (O). Denote I as the incenter of ABC and D, E as the intersections of AI with $BC, (O)$ respectively. Take a point K on BC such that $\angle AIK = 90^\circ$ and KA, KE meet (O) again at M, N respectively. The rays ND, NI meet the circle (O) at Q, P.*

1. *Prove that the quadrilateral $MPQE$ is a kite.*
2. *Take J on IO such that $AK \perp AJ$. The line through I and perpendicular to OI cuts BC at R, cuts EK at S. Prove that $OR \parallel JS$.*

Solution. First, we know that $EI = EB = EC$ then E is the circumcenter of triangle BIC. Thus KI is the tangent line of circle (BIC). Hence,

$$KI^2 = KB \cdot KC = KE \cdot KN.$$

By combining with KIE is right triangle, we get $IN \perp KE$.



So PE is the diameter of circle (O). We have $\Delta BDE \sim \Delta ABE$ so

$$\frac{BE}{AE} = \frac{DE}{BE} \Rightarrow EA \cdot ED = EB^2 = EI^2 = EN \cdot EK$$

Implies that $ADNK$ is cyclic. Thus $\angle MAE = \angle DNE$ which leads to EM, EQ are equal. Therefore, $MPQE$ is a kite.

Since $KI^2 = KM \cdot KA$, we get $IM \perp AK$. Construct the diameter AA' of (O) then M, I, A' are collinear. Thus $AJ \parallel IM$ implies that $AIA'J$ is a parallelogram and then O is the midpoint of IJ.

Since IO is the median of triangle IPE so Ix, IO, IP, IE form a harmonic quartet with $Ix \parallel PE$. Denote Ky as the ray that perpendicular to IO then notice that $KE \perp IP, KI \perp IE, KD \perp Ix, K, Ky \perp IO$ so KD, Ky, KI, KE form another harmonic quartet. Since $Ky \parallel RS$ then R is the midpoint of OS. Therefore OR is the midline of triangle IJS which implies that $OR \parallel JS$. $\square$

Problem 7. *Let a, b, and c be positive real numbers. Prove that*

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

Solution. Since $x^2 - 1$ and $x^3 - 1$ have the same sign, $0 \leq (x^2 - 1)(x^3 - 1) = x^5 - x^3 - x^2 + 1$, with equality when $x = 1$. So, it is sufficient to prove that

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3.$$

Let $x = \sqrt{a}$, $y = \sqrt{b}$, and $z = \sqrt{c}$. Then some two of x, y, and z are both at least 1 or both at most 1. Without loss of generality, say these are x and y. Then the sequences $(x, 1, 1)$ and $(1, 1, y)$ are oppositely sorted, yielding

$$(x^6 + 1 + 1)(1 + 1 + y^6) \geq 3(x^6 + 1 + y^6)$$

by Chebyshev's Inequality. By the Cauchy-Schwarz Inequality we have

$$(x^6 + 1 + y^6)(1 + z^6 + 1) \geq (x^3 + y^3 + z^3)^2.$$

Applying Chebyshev's and the Cauchy-Schwarz Inequalities each once more, we get

$$3(x^3 + y^3 + z^3) \geq (x^2 + y^2 + z^2)(x + y + z),$$

and

$$(x^3 + y^3 + z^3)(x + y + z) \geq (x^2 + y^2 + z^2)^2.$$

Multiplying the above four inequalities together yields

$$(x^6 + 2)(y^6 + 2)(z^6 + 2) \geq (x^2 + y^2 + z^2)^3,$$

as desired, with equality if and only if $x = y = z = 1$. □

Problem 8. *A set of n points in space is given, no three of which are collinear and no four of which are co-planar (on a single plane), and each pair of points is connected by a line segment. Initially, all the line segments are colorless. A positive integer b is given and Alice and Bob play the following game. In each turn Alice colors one segment red and then Bob colors up to b segments blue. This is repeated until there are no more colorless segments left. If Alice colors a red triangle, Alice wins. If there are no more colorless segments and Alice hasn't succeeded in coloring a red triangle, Bob wins. Neither player is allowed to color over an already colored line segment.*

1. *Prove that if $b < \sqrt{2n - 2} - \frac{3}{2}$, then Alice has a winning strategy.*
2. *Prove that if $b \geq 2\sqrt{n}$, then Bob has a winning strategy.*

Solution.

1. We will call a *threat* an uncolored segment whose coloring of red would produce a triangle. We will show that Alice can use the following strategy to win: Alice will keep coloring segments whose one end-point is A unless she can complete a triangle. We will call the other endpoints *opposite ends*. Let us assume that Bob can stop this strategy. Bob's optimal strategy to counter Alice's is first eliminating all threats in his move and then using his remaining moves to color segments coming out of A. Note that after the k-th red line is drawn, Bob must color $k-1$ threats blue, which are the lines connecting the new opposite end with all the previous opposite ends. Thus in the k-th pair of moves of Alice and Bob, the number of colored edges coming out of A has increased by $b+1-(k-1) = b-k+2$. Assume there exists a l such that after l moves all segments out of A have been colored before Alice could draw a triangle. Then we have

$$(b+2)l - \frac{l(l+1)}{2} \geq n-1 \Rightarrow l^2 + 2l(b+\frac{3}{2}) + 2n - 2 \leq 0,$$

therefore the discriminant of the quadratic equation is non-negative, i.e. $4(b+\frac{3}{2})^2 - 4(2n-2) \geq 0$ and therefore $b \geq \sqrt{2n-2} - \frac{3}{2}$. Since this inequality is violated, it follows that Alice's winning strategy works.

2. We will show that Bob can use the following strategy to win: Whenever Alice colors a line AB, Bob will color $\lfloor \frac{b}{2} \rfloor$ edges out of A blue and $\lceil \frac{b}{2} \rceil$ edges out of B. This ensures at most $\frac{n}{\lfloor \frac{b}{2} \rfloor + 1} + 1$ red edges out of any point, which is not larger than $\lfloor \frac{b}{2} \rfloor + 1$ for $b \geq 2\sqrt{n}$. Assuming Alice completes a triangle ABC and WLOG moves AB, AC and BC in sequence, it follows that before placing BC there were at least $\lfloor \frac{b}{2} \rfloor + 1$ threats out of C. However this means, since placing AC created these threats, that, not counting AC, there were at least $\lfloor \frac{b}{2} \rfloor + 1$ red lines out of A and thus the total number of red lines coming out of A is larger than $\lfloor \frac{b}{2} \rfloor + 1$, a contradiction.

$\square$

Solution to JBMO tests

1. Test 1

Problem 1. *Find all positive integers a, b, c and prime number p such that*

$$73p^2 + 6 = 9a^2 + 17b^2 + 17c^2 \tag{1}$$

Solution. Suppose that $b \leq c$, we distinguish two cases regarding the value of p

1. if $p \neq 2$ implies p is odd, so $p^2 \equiv 1 \pmod 8$. It follows that

$$a^2 + b^2 + c^2 \equiv 9a^2 + 17b^2 + 17c^2 \equiv 73p^2 + 6 \equiv 7 \pmod 8. \tag{2}$$

   Otherwise, $a^2 + b^2 + c^2 \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 8$, which is contracts (2).

2. if $p = 2$ then $9a^2 + 17b^2 + 17c^2 = 73p^2 + 6 = 298$. Since $a, b, c \geq 1$, we have

$$298 = 9a^2 + 17b^2 + 17c^2 \geq 26 + 17c^2$$

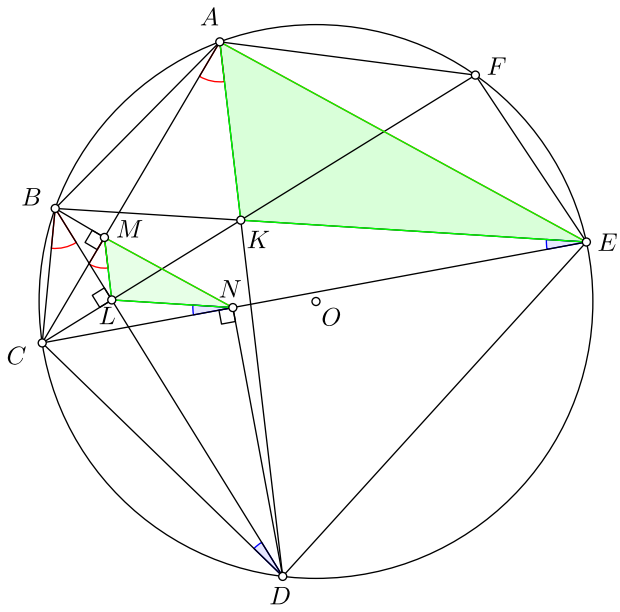
   , which means $c \leq 4$. We investigate two cases

   - if $b \leq c = 4$ then $9a^2 + 17b^2 = 26$, which means $a = b = 1$.
   - if $b \leq c \leq 3$ then it is easy to check that there is no solutions satisfied.

Hence, the only solution satisfied (1) is $(a, b, c, p) = (1, 1, 2, 2)$. □

Problem 2. *In a circle O, there are six points, A, B, C, D, E, F in a counterclockwise order. $BD \perp CF$, and CF, BE, AD are concurrent. Let the perpendicular from B to AC be M, and the perpendicular from D to CE be N. Prove that $AE \| MN$.*

Solution. Let K be the concurrent point of CF, BE, AD, and let CF intersects BD at L.



We have $\angle BMC = 90^\circ = \angle BLC$ since BM is perpendicular to AC and BL is perpendicular to CF, so B, M, L, C lie on the same circle with diameter BC. Hence,

$$\angle CML = \angle CBL = \angle CBD = \angle CAD$$

implying that ML is parallel to AK. Similarly, we also have LN is parallel to EK. Thus, we have

$$\frac{CM}{CA} = \frac{CL}{CK} = \frac{CN}{CE}.$$

This implies that MN is parallel to AE, as desired. □

Problem 3. *Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 9$ and*

$$a_{n+1} = \frac{(n+5)a_n + 22}{n+3}$$

for $n \geq 1$. Find all positive integers n for which a_n is a perfect square.

Solution. First of all, we will prove by induction on n that $a_n = n^2 + 7n + 1$. The base case is trivial. Assume that $a_k = k^2 + 7k + 1$, so

$$a_{k+1} = \frac{(k+5)a_k + 22}{k+3} = \frac{(k+5)(k^2+7k+1) + 22}{k+3} = (k+1)^2 + 7(k+1) + 1$$

We distinguish two cases regarding the value of n

- if $n \geq 9$ then $(n+3)^2 < n^2 + 7n + 1 < (n+4)^2$, so a_n is not a perfect square.
- if $n \leq 8$ then we have

$a_1 = 9$	$a_2 = 31$	$a_5 = 61$	$a_7 = 99$
$a_2 = 19$	$a_4 = 45$	$a_6 = 79$	$a_8 = 121$

  Hence, a_n is a perfect square if and only if $n = 1, 8$.

□

Problem 4. *Let F is the set of all sequences $\{(a_1, a_2, \ldots, a_{2020})$ with $a_i \in \{-1, 1\}\}$ for all $i = 1, 2, \ldots, 2020$. Prove that there exists a set S, such that $S \subset F, |S| = 2020$ and for any $(a_1, a_2, \ldots, a_{2020}) \in F$ there exists $(b_1, b_2, \ldots, b_{2020}) \in S$, such that $\sum_{i=1}^{2020} a_i b_i = 0$.*

Solution. For each $i \in \{1, 2, \ldots, 2021\}$, let $e_i = (\underbrace{1, \ldots, 1}_{i-1}, \underbrace{-1, \ldots, -1}_{2021-i})$.

For two sequences $a = (a_1, a_2, \ldots, a_{2020}), b = (b_1, b_2, \ldots, b_{2020})$, we define

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_{2020} b_{2020}.$$

It's easy to verify that $a \cdot b$ is even for each $a, b \in F$. Let $a = (a_1, a_2, \ldots, a_{2020}) \in F$ and denote $b_i = a \cdot e_i$ for each $i \in \{1, 2, \ldots, 2020\}$, then

$$b_{i+1} - b_i = a \cdot e_{i+1} - a \cdot e_i = 2a_i \Rightarrow |b_{i+1} - b_i| = 2, \forall i \in \{1, 2 \ldots, 2020\}.$$

Let $S = \{e_1, e_2, \ldots, e_{2020}\}$. It is clear that $b_{2021} = -b_1$ and b_i is even for each $i \in \{1, 2, \ldots, 2020\}$. If $b_1 \neq 0$, there exists an integer k such that $1 < k < 2021$ and $b_k = 0$. Hence, S satisfies the problem's requirements. □

2. Test 2

Problem 5. *Let $(a_n)_{n \geq 1}$ be a sequence given by $a_1 = 45$ and $a_n = a_{n-1}^2 + 15a_{n-1}$ for $n > 1$. Prove that the sequence contains no perfect squares.*

Solution. By induction we can show that $a_n > 0$ for all positive integer n.
Suppose that there is exist positive integer n such that a_n is perfect square.

$$a_n = a_{n-1}^2 + 15a_{n_1} = a_{n-1}(a_{n-1} + 15)$$

so $a_{n-1} \mid a_n$ for all positive integers n. Thus, it follows that

$$a_1 = 45 \mid a_n, \qquad \text{for all positive integers } n.$$

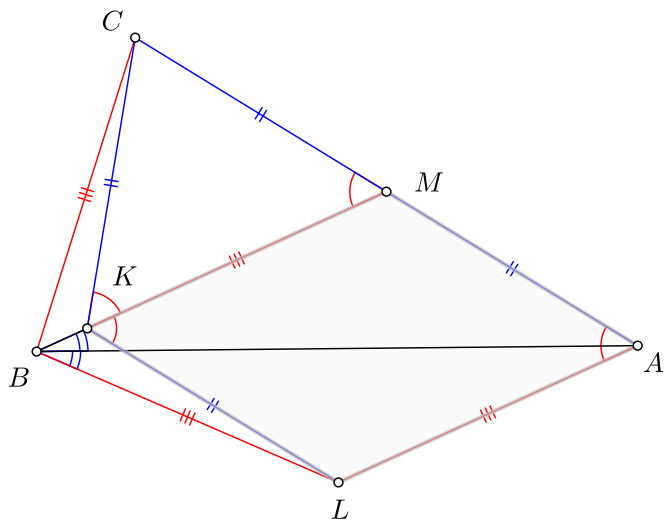
Let $k > 1$ is the smallest positive integer satisfied a_k is perfect square and $a_{k-1} = 45x$ with x is positive integer. We have

$$a_k = 45x(45x + 15) = 15^2 \cdot 3x(3x + 1)$$

is a perfect square. On the other hand, $(3x, 3x + 1) = 1$ implies $3x$ and $3x + 1$ are perfect square, so $3x = 0$ and $3x + 1 = 1$. It is impossible since $a_n > 0$ for all n. □

Problem 6. *In a triangle ABC, let K be a point on the median BM such that $CM = CK$. It turned out that $\angle CBM = 2\angle ABM$. Show that $BC = KM$.*

Solution. Let L be the reflection of C through BM. Since $CK = CM$, we have $\angle CKM = \angle CMK$. On the other hand, since L is the reflection of C through KL, we have $\angle LKM = \angle CKM = \angle CMK$, implying that KL is parallel to CM.



Also, we have $KL = KC = CM = MK$, and KL is parallel to AM. Hence, $KMAL$ is a parallelogram, implying that $KM = AL$ and KM is parallel to AL. We have $\angle LBM = \angle CBM = 2\angle ABM$, hence $\angle KBA = \angle ABL$. Since AL is parallel to MB, we have $\angle BAL = \angle MBA = \angle LBA$, implying that $LB = LA$. Hence, we have

$$KM = LA = LB = BC$$

as desired. □

Problem 7. *We have $n > 2$ nonzero integers such that everyone of them is divisible by the sum of the other $n-1$ numbers, Show that the sum of the n numbers is precisely* 0.

Solution. Let these numbers be $a_1, a_2, \ldots, a_n$ and S is the sum of these numbers. For every $i \in \{1, 2, \ldots, n\}$, we have $S - a_i \mid a_i$, which means $S - a_i \mid S$.
Suppose that $S \neq 0$, without the lost of generality, $S > 0$. We investigate two cases

- If $\exists i \in \{1, 2, \ldots, n\}$ such that $a_i < 0$ then $|S - a_i| > |S|$ so $S - a_i \nmid S$.
- $a_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. Let a_k be the smallest of $a_1, a_2, \ldots, a_n$. We have $S - a_m > 2a_m - a_m = a_m$, so $S - a_m \nmid a_m$.

Hence, $S \neq 0$ is impossible, which means $S = 0$. □

Problem 8. *Let us call a set of positive integers nice if the number of its elements equals to the average of its numbers. Call a positive integer n an amazing number if the set $\{1, 2, , \ldots, n\}$ can be partitioned into nice subsets.*

a) Prove that every perfect square is amazing

b) Show that there are infinitely many positive integers which are not amazing.

Solution. a) Let $A = \{1, 2, \ldots, n\}$ and $n = k^2$ for some positive integer k. Denote B as the set of first m odd positive integers. It's clear that B is *nice*.
Assume that $n \geq 2m - 1$, let $C = A \setminus B$. C is *nice* if and only if

$$\frac{(1 + 2 + \cdots + n) - m^2}{n - m} = n - m \Longleftrightarrow n = n^2 - 4mn + 4m^2 \Longleftrightarrow m = \frac{n - k}{2}.$$

Therefore, we can choose m so that C is *nice*. Hence, n is *amazing*.
b) We prove that if $n = 4k + 2$ for some positive integer k, then n is not *amazing*.
Assume that $A = \{1, 2, \ldots, n\}$ can be partitioned into nice subsets $A_1, A_2, \ldots, A_m$ with cardinality $a_1, a_2, \ldots, a_m$ respectively. Therefore,

$$\frac{n(n+1)}{2} = \sum_{x \in A} x = \sum_{i=1}^{m} \sum_{x \in A_i} x = \sum_{i=1}^{m} a_i^2 \equiv \sum_{i=1}^{m} a_i = n \pmod 2,$$

that yields contradiction. $\square$

Solution to IMO Team selection tests

1. Day 1

Problem 1. *For a non-empty set $\mathcal{T}$ denote by $p(\mathcal{T})$ the product of all elements of $\mathcal{T}$. Does there exist a set $\mathcal{T}$ of 2021 elements such that for any $a \in \mathcal{T}$ one has that $p(\mathcal{T}) - a$ is an odd integer. Consider two cases:*

1. *All elements of $\mathcal{T}$ are irrational numbers?*
2. *At least one element of $\mathcal{T}$ is a rational number?*

Solution.

1. Consider the polynomial
$$f(x) = x(x+2)(x+4)\dots(x+4040) - x - (2m-1)$$
for some $m \in \mathbb{Z}^+$ then $\lim\limits_{x\to+\infty} f(x) = +\infty$ and $f(0) = 2^{2020}\cdot 2020! - (2m-1)$.Take m big enough to get $f(0) < 0$ then by the continuous property of $f(x)$, there exists some real number x_0 such that $f(x_0) = 0$. Then take the set T as follows
$$T = \{x_0, x_0+2, x_0+4, \dots, x_0+4040\}$$
Then $p(T) = x_0(x_0+2)(x_0+4)\dots(x_0+4040)$ and $p(T) - x_0 = 2m-1$ is an odd integer. Since every element in T has form $x_0 + 2k$ with $k \in \mathbb{Z}$ then $p(T) - a$ is an odd integer for all $a \in T$. Finally, take $m = 2^{2019}\cdot 2020!$ then $f(0) = 1$, suppose on the contrary that $x_0 \in \mathbb{Q}$. Note that $f(x)$ is monic then $x_0 \in \mathbb{Z}$ and $x_0 \mid f(0) = 1$, implies that $x_0 \in \{-1; 1\}$, but this is contradiction since it is easy to check that $f(x)$ cannot have odd integer root.

2. We will proof that the answer is negative. Note that for all $x, y \in T$ then $x - y$ is even. Suppose that T contains some rational number b. Then we can list elements of T as follows $b, b+a_1, b+a_2, \dots, b+a_{2020}$ in which $a_1, a_2, \dots, a_k$ are even.

   Put $c = b(b+a_1)(b+a_2)\dots(b+a_k) - b$ then c is odd. Consider polynomial
$$g(x) = x(x+a_1)(x+a_2)\dots(x+a_k) - x - c$$
then $g(b) = 0$ and $x = b$ is the rational root of $g(x)$. Note that g is monic then $b \in \mathbb{Z}$. From this we conclude that all the elements in T are integers that share the same parity, so does $p(T)$. Thus $p(T) - b$ is even, contradiction.

□

Problem 2. *Find all positive integers n, such that n is a perfect number and $\varphi(n)$ is power of 2. Note: a positive integer n is called perfect if the sum of all its positive divisors is equal to $2n$.*

Solution. Suppose that $\varphi(n) = 2^m$ for some $m \in \mathbb{Z}^+$. Base on the formula of $\varphi(n)$, we have

$$\varphi(n) = \prod_{p|n} p^{t-1}(p-1) = 2^m$$

with $v_p(n) = t$. This implies that $t = 1$ for all odd prime divisor, since otherwise, $p \mid 2^m$, a contradiction. Thus $p - 1$ is the power of 2 .

With odd prime divisor of n, put $p = 2^s + 1$ for some $s \in \mathbb{Z}^+$. If $s = 1$ then $p = 3$, otherwise s must be even since $p \equiv (-1)^s + 1 \equiv 0 (\bmod 3)$ when s is the odd number, contradiction. In case s has some proper odd prime divisor q then put $s = qt$ with $t > 1$ then $2^s + 1 = (2^t)^q + 1$ is divisible by $2^t + 1$ contradiction also.

These imply that s is also a power of 2 , so by putting $s = 2^k$, we get $p = 2^{2^k} + 1$. Now let $n = 2^a p_1 p_2 \ldots p_l$ with $a \geq 0, l \geq 0$ and $p_1, p_2, \ldots, p_l$ are odd primes in ascending order, in which $p_1 = 2^{2^k} + 1$. So

$$2n = \sigma(n) = \left(2^{a+1} - 1\right)(p_1 + 1)(p_2 + 1) \ldots (p_l + 1)$$

is the sum of divisors of n. Note that $l > 0$, otherwise, $\sigma(n) = 2^{a+1} - 1$, contradiction. If $p_1 > 3$ then $3 \nmid 2n$ and

$$p_1 + 1 = 2^{2^k} + 2 = 2\left(2^{2^k - 1} + 1\right) \equiv 2\left((-1)^{2^k - 1} + 1\right) \equiv 0 \pmod 3$$

implying that $3 \mid 2n$, contradiction. Thus $p_1 = 3$ and $v_3(2n) = 1$ leads to $l \leq 2$. There are two cases

- If $l = 2$ then $2^{a+1} \cdot 3 \cdot p_2 = (2^{a+1} - 1) \cdot 4 \cdot (p_2 + 1)$, note that $v_2(p_2 + 1) = 1$ so comparing the exponent of 2 in both sides to get $a = 2$. Thus $6p_2 = 7(p_2 + 1)$, contradiction.
- If $l = 1$ then $2^{a+1} \cdot 3 = (2^{a+1} - 1) \cdot 4$, then $a = 1$ implies that $n = 6$ is a perfect square.

Therefore, $n = 6$ is only solution of this problem. □

Problem 3. *Let I and I_A be the incenter and the A-excenter of an acute-angled triangle ABC with $AB < AC$. Let the incircle meets BC at D, and the line AD meets BI_A and CI_A at E and F, respectively. Prove that the circumcircles of triangles AID and I_AEF are tangent to each other.*

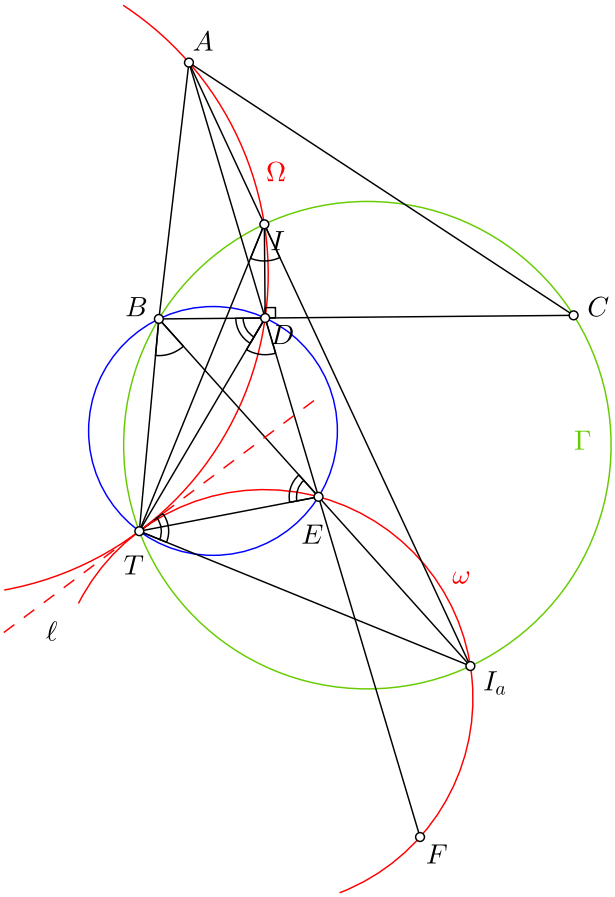
Solution. Let $\sphericalangle(p, q)$ denote the directed angle between lines p and q.

The points B, C, I, and I_A lie on the circle Γ with diameter II_A. Let ω and Ω denote the circles (I_AEF) and (AID), respectively. Let T be the second intersection point of ω and Γ.

Then T is the Miquel point of the complete quadrilateral formed by the lines BC, BI_A, CI_A, and DEF, so T also lies on circle (BDE) (as well as on circle (CDF)). We claim that T is desired tangency point of ω and Ω.

In order to show that T lies on Ω, use cyclic quadrilaterals $BDET$ and BII_AT to write

$$\measuredangle(DT, DA) = \measuredangle(DT, DE) = \measuredangle(BT, BE) = \measuredangle(BT, BI_A) = \measuredangle(IT, II_A) = \measuredangle(IT, IA)$$



To show that ω and Ω are tangent at T, let ℓ be the tangent to ω at T, so that $\measuredangle(TI_A, \ell) = \measuredangle(EI_A, ET)$. Using circles $(BDET)$ and $(BICI_A)$, we get

$$\measuredangle(EI_A, ET) = \measuredangle(EB, ET) = \measuredangle(DB, DT)$$

Therefore,

$$\measuredangle(TI, \ell) = 90^\circ + \measuredangle(TI_A, \ell) = 90^\circ + \measuredangle(DB, DT) = \measuredangle(DI, DT)$$

which shows that ℓ is tangent to Ω at T. □

2. Day 2

Problem 4. *In a regular 100-gon, 41 vertices are colored black and the remaining 59 vertices are colored white. Prove that there exist 24 convex quadrilaterals $Q_1, Q_2, \ldots, Q_{24}$ whose corners (vertices of the quadrilateral) are vertices of the 100 -gon, so that*

- *the quadrilaterals $Q_1, Q_2, \ldots, Q_{24}$ are pairwise disjoint, and*
- *every quadrilateral Q_i has three corners of one color and one corner of the other color.*

Solution. Call a quadrilateral skew-colored, if it has three corners of one color and one corner of the other color. We will prove the following

Claim. If the vertices of a convex $(4k+1)$ -gon P are colored black and white such that each color is used at least k times, then there exist k pairwise disjoint skew-colored quadrilaterals whose vertices are vertices of P. (One vertex of P remains unused.)The problem statement follows by removing 3 arbitrary vertices of the 100 -gon and applying the Claim to the remaining 97 vertices with $k = 24$.

Proof of the Claim. We prove by induction. For $k = 1$ we have a pentagon with at least one black and at least one white vertex. If the number of black vertices is even then remove a black vertex; otherwise remove a white vertex. In the remaining quadrilateral, there are an odd number of black and an odd number of white vertices, so the quadrilateral is skew-colored.

For the induction step, assume $k \geqslant 2$. Let b and w be the numbers of black and white vertices, respectively; then $b, w \geqslant k$ and $b + w = 4k + 1$. Without loss of generality we may assume $w \geqslant b$, so $k \leqslant b \leqslant 2k$ and $2k + 1 \leqslant w \leqslant 3k + 1$.

We want to find four consecutive vertices such that three of them are white, the fourth one is black. Denote the vertices by $V_1, V_2, \ldots, V_{4k+1}$ in counterclockwise order, such that V_{4k+1} is black, and consider the following k groups of vertices:

$$(V_1, V_2, V_3, V_4), (V_5, V_6, V_7, V_8), \ldots, (V_{4k-3}, V_{4k-2}, V_{4k-1}, V_{4k})$$

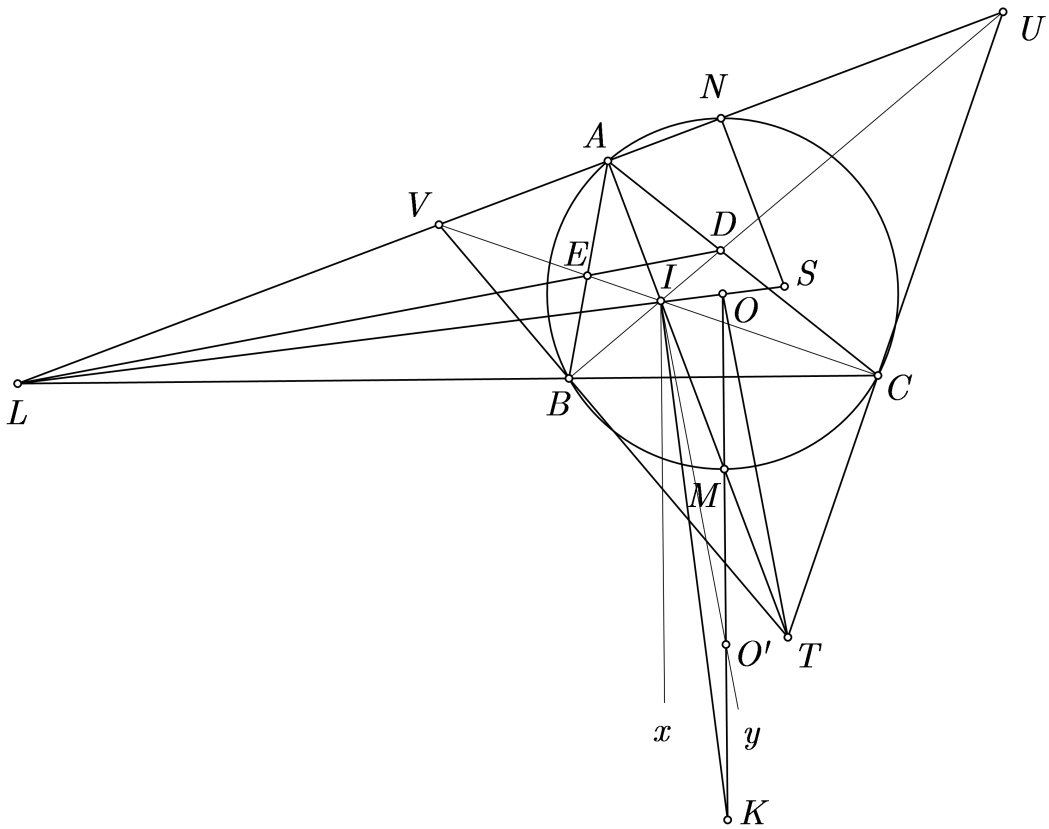
In these groups there are w white and $b-1$ black vertices. Since $w > b-1$, there is a group, $(V_i, V_{i+1}, V_{i+2}, V_{i+3})$ that contains more white than black vertices. If three are white and one is black in that group, we are done. Otherwise, if $V_i, V_{i+1}, V_{i+2}, V_{i+3}$ are all white then let V_j be the first black vertex among $V_{i+4}, \ldots, V_{4k+1}$ (recall that V_{4k+1} is black); then V_{j-3}, V_{j-2} and V_{j-1} are white and V_j is black.

Now we have four consecutive vertices $V_i, V_{i+1}, V_{i+2}, V_{i+3}$ that form a skew-colored quadrilateral. The remaining vertices form a convex $(4k-3)$ -gon; $w-3$ of them are white and $b-1$ are black. Since $b-1 \geqslant k-1$ and $w-3 \geqslant (2k+1)-3 > k-1$, we can apply the Claim with $k-1$. □

Problem 5. *Let ABC be a non isosceles triangle with incenter I and let the circumcircle of the triangle ABC has radius R. Let AL is the external angle bisector of $\angle BAC$ with $L \in BC$. Let K is the point on the perpendicular bisector of BC such that $IL \perp IK$. Prove that $OK = 3R$.*

Solution. Denote M as the midpoint of the arc BC not containing A of (O), we need to prove $MK = 2OM$. Denote BD, CE as the internal bisectors of ABC then D, E, L are collinear. Note that $L(AI, DB) = -1$ and take the orthogonal projection from I to get $(Ix, Iy, IK, IM) = -1$ in which

$$IM \perp LA, IK \perp LI, Iy \perp LD, Ix \perp LB$$



Suppose that Iy cuts MK at O', combining with $Ix \| MK$, then O' is the midpoint of MK. Thus it remains to prove M is the midpoint of OO'.

Denote T as the ex-center respect to angle A of triangle ABC then M is the midpoint of IT. Then $MO = MO'$ if and only if $IO' \| OT$, or $OT \perp DE$.

Now take U, V as the ex-centers respect to angles B, C of triangle ABC then A, B, C are the foot of altitudes in triangle TUV and I, O are the orthocenter and nine-point center of that triangle, thus the reflection S of I over O is the circumcenter. Note that B, D, U and C, E, V are collinear then

$$\mathcal{P}_{D/(O)} = \overline{DA} \cdot \overline{DC} = \overline{DU} \cdot \overline{DI} = \mathcal{P}_{D/(IUV)}$$

and similarly, $\mathcal{P}_{E/(O)} = \mathcal{P}_{E/(IUV)}$ so DE is the radical axis of (O) and (IUV) which implies that DE is perpendicular to the line that joining centers of (O) and (IUV). We know that center Z of (IUV) is the reflection of S over UV, and since $IT = 2NS$, it is easy to check that $Z \in TO$. Thus, we have $DE \perp OT$ which finishes the proof. $\square$

Problem 6. *Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that*

$$f^{\left(a^2+b^2\right)}(a+b) = af(a) + bf(b)$$

for every $a, b \in \mathbb{Z}$. Here, f^n denotes the nth iteration of f, i. e. $f^{(0)}(x) = x$ and $f^{(n+1)}(x) = f\left(f^{(n)}(x)\right)$ for all $n \geq 0$.

Solution 1. Refer to the main equation as $E(a,b)$.
$E(0,b)$ reads as $f^{b^2}(b) = bf(b)$. For $b=-1$ this gives $f(-1)=0$.
Now $E(a,-1)$ reads as

$$f^{a^2+1}(a-1) = af(a) = f^{a^2}(a).$$

For $x \in \mathbb{Z}$ define the orbit of x by $\mathcal{O}(x) = \{x, f(x), f(f(x)), \ldots\} \subseteq \mathbb{Z}$. We see that the orbits $\mathcal{O}(a-1)$ and $\mathcal{O}(a)$ differ by finitely many terms. Hence, any two orbits differ by finitely many terms. In particular, this implies that either all orbits are finite or all orbits are infinite.

Case 1: All orbits are finite. Then $\mathcal{O}(0)$ is finite. Using $E(a,-a)$ we get

$$a(f(a)-f(-a)) = af(a) - af(-a) = f^{2a^2}(0) \in \mathcal{O}(0) \tag{1}$$

For $|a| > \max\limits_{z\in\mathcal{O}(0)} |z|$, this yields $f(a) = f(-a)$ and $f^{2a^2}(0) = 0$. Therefore, the sequence $\left(f^k(0) : k = 0, 1, \ldots\right)$ is purely periodic with a minimal period T which divides $2a^2$. Analogously, T divides $2(a+1)^2$, therefore, $T \mid \gcd(2a^2, 2(a+1)^2) = 2$, i.e., $f(f(0)) = 0$ and $a(f(a)-f(-a)) = f^{2a^2}(0) = 0$ for all a. Thus,

$$f(a) = f(-a) \quad \text{for all } a \neq 0; \tag{2}$$

$$\text{in particular,} \quad f(1) = f(-1) = 0 \tag{3}$$

Next, for each $n \in \mathbb{Z}$, by $E(n, 1-n)$ we get

$$nf(n) + (1-n)f(1-n) = f^{n^2+(1-n)^2}(1) = f^{2n^2-2n}(0) = 0 \tag{4}$$

Assume that there exists some $m \neq 0$ such that $f(m) \neq 0$. Choose such an m for which $|m|$ is minimal possible. Then $|m| > 1$ due to (2);$f(|m|) \neq 0$ due to (3); and $f(1-|m|) \neq 0$ due to (4) for $n = |m|$. This contradicts to the minimality assumption.

So, $f(n) = 0$ for $n \neq 0$. Finally, $f(0) = f^3(0) = f^4(2) = 2f(2) = 0$. Clearly, the function $f(x) \equiv 0$ satisfies the problem condition, which provides the first of the two answers.

Case 2: All orbits are infinite.
Since the orbits $\mathcal{O}(a)$ and $\mathcal{O}(a-1)$ differ by finitely many terms for all $a \in \mathbb{Z}$, each two orbits $\mathcal{O}(a)$ and $\mathcal{O}(b)$ have infinitely many common terms for arbitrary $a, b \in \mathbb{Z}$.

For a minute, fix any $a, b \in \mathbb{Z}$. We claim that all pairs (n,m) of nonnegative integers such that $f^n(a) = f^m(b)$ have the same difference $n-m$. Arguing indirectly, we have $f^n(a) = f^m(b)$ and $f^p(a) = f^q(b)$ with, say, $n-m > p-q$, then $f^{p+m+k}(b) = f^{p+n+k}(a) = f^{q+n+k}(b)$, for all nonnegative integers k. This means that $f^{\ell+(n-m)-(p-q)}(b) = f^{\ell}(b)$ for all sufficiently large ℓ, i.e., that the sequence $(f^n(b))$ is eventually periodic, so $\mathcal{O}(b)$ is finite, which is impossible.

Now, for every $a, b \in \mathbb{Z}$, denote the common difference $n-m$ defined above by $X(a,b)$. We have $X(a-1,a) = 1$ by (1). Trivially, $X(a,b) + X(b,c) = X(a,c)$, as if $f^n(a) = f^m(b)$ and $f^p(b) = f^q(c)$, then $f^{p+n}(a) = f^{p+m}(b) = f^{q+m}(c)$. These two properties imply that $X(a,b) = b-a$ for all $a, b \in \mathbb{Z}$.

But (1) yields $f^{a^2+1}(f(a-1)) = f^{a^2}(f(a))$, so

$$1 = X(f(a-1), f(a)) = f(a) - f(a-1) \text{ for all } a \in \mathbb{Z}$$

Recalling that $f(-1) = 0$, we conclude by (two-sided) induction on x that $f(x) = x + 1$ for all $x \in \mathbb{Z}$. Finally, the obtained function also satisfies the assumption. Indeed, $f^n(x) = x + n$ for all $n \geqslant 0$, so

$$f^{a^2+b^2}(a+b) = a + b + a^2 + b^2 = af(a) + bf(b).$$

□

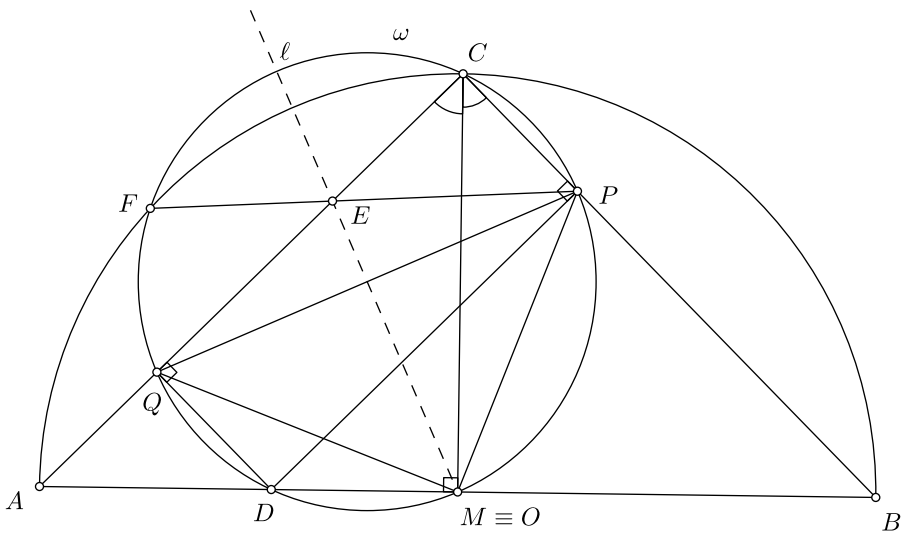
3. Day 3

Problem 7. *Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside the side AB such that $AD < DB$. Let P and Q be two points inside the sides BC and CA respectively, such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ meets the line segment CQ at E and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E and F are collinear. Prove that $\angle ACB = 90°$.*

Solution. Let ℓ be the perpendicular bisector of PQ, and denote by ω the circle $CFPQ$. By $DP \perp BC$ and $DQ \perp AC$, the circle ω passes through D; moreover, CD is a diameter of ω.

The lines QE and PE are symmetric about ℓ, and ℓ is a symmetry axis of ω as well; it follows that the chords CQ and FP are symmetric about ℓ, hence C and F are symmetric about ℓ. Therefore, the perpendicular bisector of CF coincides with ℓ. Thus ℓ passes through the circumcenter O of ABC.

Let M be the midpoint of AB. Since $CM \perp DM, M$ also lies on ω. By $\angle ACM = \angle BCM$, the chords MP and MQ of ω are equal. Then, from $MP = MQ$ it follows that ℓ passes through M.



Finally, both O and M lie on lines ℓ anh CM, therefore $O = M$, and $\angle ACB = 90°$ follows. $\square$

Problem 8. *The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist $x, y \in S$, such that $x - y = F_k$.*

Solution. First we show that if a set $S \subset \mathbb{Z}$ satisfies the conditions then $|S| \geqslant \frac{n}{2} + 1$.

Let $d = [n/2]$, so $n \leqslant 2d \leqslant n + 1$. In order to prove that $|S| \geqslant d + 1$, construct a graph as follows. Let the vertices of the graph be the elements of S. For each $1 \leqslant k \leqslant d$, choose two elements $x, y \in S$ such that $x - y = F_{2k-1}$, and add the pair (x, y) to the graph as edge. (Note that by the problem's constraints, there must be a pair (x, y) with $x - y = F_{2k-1}$ for every $3 \leqslant 2k - 1 \leqslant 2d - 1 \leqslant n$; moreover, due to $F_1 = F_2$ we have a pair with $x - y = F_1$ as well.) We will say that the length of the edge (x, y) is $|x - y|$.

We claim that the graph contains no cycle. For the sake of contradiction, suppose that the graph contains a cycle $(x_1, \ldots, x_\ell)$, and let the longest edge in the cycle be (x_1, x_ℓ) with length F_{2m+1}. The other edges $(x_1, x_2), \ldots, (x_{\ell-1}, x_\ell)$ in the cycle are shorter than F_{2m+1} and distinct, their lengths form a subset of $\{F_1, F_3, \ldots, F_{2m-1}\}$. But this is not possible because

$$\begin{aligned} F_{2m+1} = |x_\ell - x_1| &\leqslant \sum_{i=1}^{\ell-1} |x_{i+1} - x_i| \leqslant F_1 + F_3 + F_5 + \ldots + F_{2m-1} \\ &= F_2 + (F_4 - F_2) + (F_6 - F_4) + \ldots + (F_{2m} - F_{2m-2}) = F_{2m} < F_{2m+1} \end{aligned}$$

Hence, the graph has d edges and cannot contain a cycle, therefore it must contain at least $d+1$ vertices, so $|S| \geqslant d+1$. Now we show a suitable set with $d+1$ elements. Let

$$S = \{F_0, F_2, F_4, F_5, \ldots, F_{2d}\}$$

For $1 \leqslant k \leqslant d$ we have $F_0, F_{2k-2}, F_{2k} \in S$ with differences $F_{2k} - F_{2k-2} = F_{2k-1}$ and $F_{2k} - F_0 = F_{2k}$, so each of $F_1, F_2, \ldots, F_{2d}$ occurs as difference between two elements in S. So this set containing $d + 1$ numbers is suitable. $\square$

Problem 9. *For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime with n. Prove that that for any number C there exists an integer n for which*

$$\frac{\varphi(d(n))}{d(\varphi(n))} > C.$$

Solution. Fix $N > 1$, let $p_1, \ldots, p_k$ be all primes between 1 and N and $p_{k+1}, \ldots, p_{k+s}$ be all primes between $N + 1$ and $2N$. Since for $j \leqslant k + s$ all prime divisors of $p_j - 1$ do not exceed N, we have

$$\prod_{j=1}^{k+s} (p_j - 1) = \prod_{i=1}^{k} p_i^{c_i}$$

with some fixed exponents $c_1, \ldots, c_k$. Choose a huge prime number q and consider a number

$$n = (p_1 \cdot \ldots \cdot p_k)^{q-1} \cdot (p_{k+1} \cdot \ldots \cdot p_{k+s})$$

Then

$$\varphi(d(n)) = \varphi\left(q^k \cdot 2^s\right) = q^{k-1}(q-1)2^{s-1}$$

and

$$d(\varphi(n)) = d\left((p_1 \cdot \ldots \cdot p_k)^{q-2} \prod_{i=1}^{k+s} (p_i - 1)\right) = d\left(\prod_{i=1}^{k} p_i^{q-2+c_i}\right) = \prod_{i=1}^{k} (q - 1 + c_i)$$

So

$$\frac{\varphi(d(n))}{d(\varphi(n))} = \frac{q^{k-1}(q-1)2^{s-1}}{\prod_{i=1}^{k}(q-1+c_i)} = 2^{s-1} \cdot \frac{q-1}{q} \cdot \prod_{i=1}^{k} \frac{q}{q-1+c_i}$$

which can be made arbitrarily close to 2^{s-1} by choosing q large enough. It remains to show that s can be arbitrarily large, i.e. that there can be arbitrarily many primes between N and $2N$.

This follows, for instance, from the well-known fact that $\sum \frac{1}{p} = \infty$, where the sum is taken over the set $\mathbb{P}$ of prime numbers. Indeed, if, for some constant C, there were always at most C primes between 2^ℓ and $2^{\ell+1}$, we would have

$$\sum_{p \in \mathbb{P}} \frac{1}{p} = \sum_{\ell=0}^{\infty} \sum_{\substack{p \in \mathbb{P} \\ p \in [2^\ell, 2^{\ell+1})}} \frac{1}{p} \leqslant \sum_{\ell=0}^{\infty} \frac{C}{2^\ell} < \infty$$

which is a contradiction. □

4. Day 4

Problem 10. *Given a positive integer k, show that there exists a prime p such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots, p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, 3, \ldots, k$.*

Solution. First we choose distinct positive rational numbers $r_1, \ldots, r_{k+3}$ such that

$$r_i r_{i+1} r_{i+2} r_{i+3} = i \quad \text{for } 1 \leqslant i \leqslant k$$

Let $r_1 = x, r_2 = y, r_3 = z$ be some distinct primes greater than k; the remaining terms satisfy $r_4 = \frac{1}{r_1 r_2 r_3}$ and $r_{i+4} = \frac{i+1}{i} r_i$. It follows that if r_i are represented as irreducible fractions, the numerators are divisible by x for $i \equiv 1 \pmod 4$, by y for $i \equiv 2 \pmod 4$, by z for $i \equiv 3 \pmod 4$ and by none for $i \equiv 0 \pmod 4$. Notice that $r_i < r_{i+4}$; thus the sequences

$$r_1 < r_5 < r_9 < \ldots, r_2 < r_6 < r_{10} < \ldots, r_3 < r_7 < r_{11} < \ldots, r_4 < r_8 < r_{12} < \ldots$$

are increasing and have no common terms, that is, all r_i are distinct.

If each r_i is represented by an irreducible fraction $\frac{u_i}{v_i}$, choose a prime p which divides neither $v_i, 1 \leqslant i \leqslant k+1$, nor $v_i v_j (r_i - r_j) = v_j u_i - v_i u_j$ for $i < j$, and define a_i by the congruence $a_i v_i \equiv u_i (\mathrm{mod} p)$. Since $r_i r_{i+1} r_{i+2} r_{i+3} = i$, we have

$$\begin{aligned} i v_i v_{i+1} v_{i+2} v_{i+3} &= r_i v_i r_{i+1} v_{i+1} r_{i+2} v_{i+2} r_{i+3} v_{i+3} \\ = u_i u_{i+1} u_{i+2} u_{i+3} &\equiv a_i v_i a_{i+1} v_{i+1} a_{i+2} v_{i+2} a_{i+3} v_{i+3} \qquad (\mathrm{mod}\ p) \end{aligned}$$

and therefore $a_i a_{i+1} a_{i+2} a_{i+3} \equiv i (\mathrm{mod} p)$ for $1 \leqslant i \leqslant k$.

If $a_i \equiv a_j (\mathrm{mod} p)$, then $u_i v_j \equiv a_i v_i v_j \equiv u_j v_i (\mathrm{mod} p)$, a contradiction. □

Problem 11. *Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac + bd$. Find the smallest possible value of*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$$

Solution. First of all, apply the AM-GM inequality as follows:

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right) \geqslant 2\sqrt{\frac{ac}{bd}} + 2\sqrt{\frac{bd}{ac}} = \frac{2(ac+bd)}{\sqrt{abcd}}$$

Continuing to apply the AM-GM inequality, then

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right) \geqslant \frac{2(a+c)(b+d)}{\sqrt{abcd}} \geqslant 2 \cdot \frac{2\sqrt{ac} \cdot 2\sqrt{bd}}{\sqrt{abcd}} = 8$$

The above inequalities turn into equalities when $a = c$ and $b = d$. Then the condition $(a+c)(b+d) = ac + bd$ can be rewritten as $4ab = a^2 + b^2$. This is equivalent to $a/b = 2 \pm \sqrt{3}$.

Hence, S attains value 8, e.g., when $a = c = 1$ and $b = d = 2 + \sqrt{3}$. □

Problem 12. *Let p be an odd prime and let $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leq N$, denote by $r(n)$ the fraction of integers in $\{1, 2, \ldots, n\}$ that are red (number of red numbers divided by n). Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq \frac{a}{p}$ for all $n = 1, 2, \ldots, N$*

Solution. Denote by $R(n)$ the number of red numbers in $\{1, 2, \ldots, n\}$, i.e., $R(n) = nr(n)$. Similarly, denote by $B(n)$ and $b(n) = B(n)/n$ the number and proportion of blue numbers in $\{1, 2, \ldots, n\}$, respectively. Notice that $B(n) + R(n) = n$ and $b(n) + r(n) = 1$. Therefore, the statement of the problem does not change after swapping the colors.

Arguing indirectly, for every $a \in \{1, 2, \ldots, p-1\}$ choose some positive integer n_a such that $r(n_a) = a/p$ and, hence, $R(n_a) = an_a/p$. Clearly, $p \mid n_a$, so that $n_a = pm_a$

for some positive integer m_a, and $R(n_a) = am_a$. Without loss of generality, we assume that $m_1 < m_{p-1}$, as otherwise one may swap the colors. Notice that

$$m_a \leqslant \frac{N}{p} < \frac{p^2-1}{4} \quad \text{for all } a = 1, 2, \ldots, p-1 \ .$$

The solution is based on a repeated application of the following simple observation.
Claim. Assume that $m_a < m_b$ for some $a, b \in \{1, 2, \ldots, p-1\}$. Then

$$m_b \geqslant \frac{a}{b} m_a \quad \text{and} \quad m_b \geqslant \frac{p-a}{p-b} m_a.$$

Proof. The first inequality follows from $bm_b = R(n_b) \geqslant R(n_a) = am_a$. The second inequality is obtained by swapping colors. Let $q = (p-1)/2$. We distinguish two cases.

Case 1: All q numbers $m_1, m_2, \ldots, m_q$ are smaller than m_{p-1}.
. Let m_a be the maximal number among $m_1, m_2, \ldots, m_q$; then $m_a \geqslant q \geqslant a$. Applying the Claim, we get

$$m_{p-1} \geqslant \frac{p-a}{p-(p-1)} m_a \geqslant (p-q)q = \frac{p^2-1}{4}$$

which contradicts (1).

Case 2: There exists $k \leqslant q$ such that $m_k > m_{p-1}$. Choose k to be the smallest index satisfying $m_k > m_{p-1}$; by our assumptions, we have $1 < k \leqslant q < p-1$

Let m_a be the maximal number among $m_1, m_2, \ldots, m_{k-1}$; then $a \leqslant k-1 \leqslant m_a < m_{p-1}$. Applying the Claim, we get

$$\begin{aligned} m_k \geqslant \frac{p-1}{k} m_{p-1} &\geqslant \frac{p-1}{k} \cdot \frac{p-a}{p-(p-1)} m_a \\ &\geqslant \frac{p-1}{k} \cdot (p-k+1)(k-1) \\ &\geqslant \frac{k-1}{k} \cdot (p-1)(p-q) \\ &\geqslant \frac{1}{2} \cdot \frac{p^2-1}{2} \end{aligned}$$

which contradicts (1) again. □

Problems without solution

Problem 1. *Let ABC be an acute, non-isosceles triangle with AD, BE, CF are altitudes and d is the tangent line of the circumcircle of triangle ABC at A. The line throught H and parallel to EF cuts DE, DF at Q, P respectively. Prove that d is tangent to the ex-circle respect to vertex D of triangle DPQ.*

Problem 2. *Let ABC be an acute, non isosceles triangle with the orthocenter H, circumcenter O and AD is the diameter of (O). Suppose that the circle (AHD) meets the lines AB, AC at F, Erespectively. Denote J, K as orthocenter and nine-point center of AEF. Prove that HJ parallel to BC and $KO = KH$.*

Problem 3. *Let ABC be an acute, non-isosceles triangle inscribed in (O) and BB', CC' are altitudes. Denote E, F as the intersections of BB', CC' with (O) and D, P, Q are projections of A on BC, CE, BF. Prove that the perpendicular bisectors of PQ bisects two segments AO, BC.*

Problem 4. *Let ABC be a triangle with incircle (I), tangent to BC, CA, AB at D, E, F respectively. On the line DF, take points M, P such that $CM \parallel AB$, $AP \parallel BC$. On the line DE, take points N, Q such that $BN \parallel AC$, $AQ \parallel BC$. Denote X as intersection of PE, QF and K as the midpoint of BC. Prove that if $AX = IK$ then $\angle BAC \leq 60°$.*

Problem 5. *(Tran Quang Hung, Vietnam) Let $ABCD$ be a rectangle with P lies on the segment AC. Denote Q as a point on minor arc PB of (PAB) such that $QB = QC$. Denote R as a point on minor arc PD of (PAD) such that $RC = RD$. The lines CB, CD meet (CQR) again at M, N respectively. Prove that $BM = DN$.*

Problem 6. *Let A be a point lies outside circle (O) and tangent lines $AB, AC$of (O). Consider points D, E, M on (O) such that $MD = ME$. The line DE cuts MB, MC at R, S. Take $X \in OB, Y \in OC$ such that $RX, RY \perp DE$. Prove that $XY \perp AM$.*

Problem 7. *Let ABC be a triangle with $AB = AC$ and M is the midpoint of the altitude AD. Consider (ω) as the circle of center M and tangent to AB, AC. From some point Ton the line BC (outside triangle ABC), construct two tangents of (ω) and they cut AB at P, Q, cut AC at R, S. Prove that $PQ = RS$.*

Problem 8. *Let ABC be an non-isosceles triangle with incenter I, circumcenter O and a point D on segment BC such that (BID) cut segments AB at $E \neq B$ and (CID) cuts segmentAC at $F \neq C$. Circle (DEF) cuts segments AB, AC again at M, N. Let $P = IB \cap DE$ and $Q = IC \cap DF$. Prove that EN, FM, PQ are parallel and the median of vertex I in triangle IPQ bisects the arc BAC of (O).*

Problem 9. *Let ABC be a triangle inscribed in circle (O) with diamter KL passes through the midpoint M of AB such that L, C lie on the different sides respect to AB. A circle passes through M, K cuts LC at P, Q (point P lies between Q, C). The line KQ cuts (LMQ) at R. Prove that $ARBP$ is cyclic and AB is the symmedian of triangle APR.*

Problem 10. *Let AB be a chord of the circle (O). Denote M as the midpoint of the minor arc AB. A circle (O') tangent to segment AB and internally tangent to (O). A line passes through M, perpendicular to $O'A, O'B$ and cuts AB respectively at C, D. Prove that $AB = 2CD$.*

Problem 11. *Consider circles $(O_1), (O_2), (O_3)$ are tangent to d at A, B, C and (O_2) is the biggest circle, externally tangent to $(O_1), (O_3)$. Let BD be the diameter of (O_2). The external tangent line (differs from d) of $(O_1), (O_3)$ cuts (O_2) at X, Y. Let K be the midpoint of the arc XBY of (O_2). Prove that the circle of diameter AC touches DX, DY.*

Problem 12. *Let ABC be a triangle with circumcenter O and incenter I, ex-center in angle A is J. Denote D as the tangent point of là (I) on BC and the angle bisector of angle A cuts $BC, (O)$ respectively at E, F. The circle (DEF) meeds (O) again at T. Prove that AT passes through an intersection of (J) and (DEF).*

Problem 13. *Let $ABCD$ be a quadrilteral with $\angle A = \angle B = 90°$, $AB = AD$. Denote E as the midpoint of AD, suppose that $CD = BC + AD, AD > BC$. Prove that $\angle ADC = 2.\angle ABE$.*

Problem 14. *Let ABC be a triangle inscribed in a fix circle (O) with BC is fix and A vary on (O). Denote H as the orthocenter of triangle ABC and take D, E on AB, AC respectively such that H is the midpoint of DE. Prove that when A moves on (O), the center of (ADE) belongs a fixed circle.*

Problem 15. *Let ABC be convex quadrilateral and X lying inside it such that $XA \cdot XC^2 = XB \cdot XD^2$ and $\angle AXD + \angle BXC = \angle CXD$. Prove that $\angle XAD + \angle XCD = \angle XBC + \angle XDC$.*

Problem 16. *Let ABC be an acute, non-isosceles triangle with circumcenter O, incenter I and (I) tangent to BC, CA, AB at D, E, F respectively. Suppose that EF cuts (O) at P, Q. Prove that (PQD) bisects segment BC.*

Problem 17. *Let ABC be an acute, non-isosceles triangle with circumcenter O. Tangent lines to (O) at B, C meet at T. A line passes through T cuts segments AB at D and cuts ray CA at E. Take M as midpoint of DE and suppose that MA cuts (O) again at K. Prove that (MKT) is tangent to (O).*

Problem 18. *Let ABC be a triangle with $AB < AC$ and incircle (I) tangent to BC at D. Take K on AD such that $CD = CK$. Suppose that AD cuts (I) at G and BG cuts CK at L. Prove that K is the midpoint of CL.*

Problem 19. *Let ABC be a triangle with $AB < AC$ inscribed in (O). Tangent line at A of (O) cuts BC at D. Take H as the projection of A on OD and E, F as projections of H on AB, AC. Suppose that EF cuts (O) at R, S. Prove that (HRS) is tangent to OD.*

Problem 20. *Let ABC be an acute, non-isosceles triangle with altitude AD ($D \in BC$), M is the midpoint of AD and O is the circumcenter. Line AO meets BC at K and circle of center K, radius KA cuts AB, AC at E, F respectively. Prove that AO bisects EF.*

Problem 21. *Let $ABCD$ be a cyclic quadrilateral with O is circumcenter and AC meets BD at I. Suppose that rays DA, CD meet at E and rays BA, CD meet at F. The Gauss line of $ABCD$ meets AB, BC, CD, DA at points M, N, P, Q respectively. Prove that the circle of diameter OI is tangent to two circles $(ENQ), (FMP)$.*

Problem 22. *Let ABC be a non-isosceles triangle with altitudes AD, BE, CF with orthocenter H. Suppose that $DF \cap HB = M, DE \cap HC = N$ and T is the circumcenter of triangle HBC. Prove that $AT \perp MN$.*

Problem 23. *Let ABC be triangle with the symmedian point L and circumradius R. Construct parallelograms $ADLE, BHLK, CILJ$ such that $D, H \in AB; K, I \in BC; J, E \in CA$. Suppose that DE, HK, IJ pairwise intersect at X, Y, Z. Prove that inradius of XYZ is $\frac{R}{2}$.*

Problem 24. *Let ABC be triangle with M is the midpoint of BC and X, Y are excenters with respect to angle B, C. Prove that MX, MY intersect AB, AC at four points are vertices of circumscribe quadrilateral.*

Problem 25. *The Magician and his Assistant show trick. The Viewer writes on the board the sequence of N digits. Then the Assitant covers some pair of adjacent digits so that they become invisible. Finally, the Magician enters the show, looks at the board and guesses the covered digits and their order. Find the minimal N such that the Magician and his Assistant can agree in advance so that the Magician always guesses right.*

Problem 26. *Given an infinite sequence of numbers $a_1, a_2, a_3, ...$ such that for each positive integer k, there exists positive integer t for which $a_k = a_{k+t} = a_{k+2t} = ...$ Does this sequences must be periodic?*

Problem 27. *Each of N people have chosen some 5 elements from a $23-$element set so that any two people share at most 3 chosen elements. Does this mean that $N \leq 2020$? Answer the same question with 25 instead of 23.*

Problem 28. *Find all positive integer $n \geq 3$ such that it is possible to mark the vertices of a regular $n-$ gon with the number from 1 to n so that for any three vertices A, B and C with $AB = AC$, the number in A is greater or smaller than both numbers in B, C.*

Problem 29. *Prove that it is impossible to fill the cells of an 8×8 table with the numbers from 1 to 64 (each number must be used once) so that for each 2×2 square, the difference between products of the numbers on it's diagonals will be equal to 1.*

Problem 30. *For a positive integer k, denote by $f(k)$ the number of positive integer m such that the remainder of km modulo 2019^3 is greater than m. Find the amount of different numbers among $f(1), f(2), ..., f(2019^3)$.*

Problem 31. *Let n be a positive integer. What is the smallest value of m with $m > n$ such that the set $M = \{n, n+1, ..., m\}$ can be partitioned into subsets so that in each subset, there is a number which equals to the sum of all other numbers of this subset?*

Problem 32. *Let N be a positive integer. Consider the sequence $a_1, a_2, ..., a_N$ of positive integers, none of which is a multiple of 2^{N+1}. For $n \geq N+1$, the number a_n is defined as follows: choose k to be the number among $1, 2, ..., n-1$ for which the remainder obtained when a_k is divided by 2^n is the smallest, and define $a_n = 2a_k$ (if there are more than one such k, choose the largeast such k). Prove that there exist M for which $a_n = a_M$ holds for every $n \geq M$.*

Problem 33. *Call a positive integer x to be remote from a square cubes if each integer k satisfies both $|x - k^2| > 10^6$ and $|x - k^3| > 10^6$. Prove that there exist infinitely many positive integer n such that 2^n is remote from squares and cubes.*

Problem 34. *Let coefficients of the polynomial $P(x) = a_d x^d + ... + a_2 x^2 + a_0$ where $d \geq 2$, are positive integers. The sequences (b_n) is defined by $b_1 = a_0$ and $b_{n+1} = P(b_n)$ for $n \geq 1$. Prove that for any $n \geq 2$, there exists a prime number p such that $p|b_n$ but it does not divide $b_1, b_2, ..., b_{n-1}$.*

Problem 35. *Let $P(x)$ be a nonconstant integer polynomial and positive integer n. The sequence $a_0, a_1, ...$ is defined by $a_0 = n$ and $a_k = P(a_{k-1})$ for $k \geq 1$. Given that for each positive integer b, the sequence contains a $b-th$ power of some positive integer greater than 1. Prove that $degP = 1$.*

Problem 36. *There are 330 seats in the first row of the auditorium. Some of these seats are occupied by 25 viewers. Prove that among the pairwise distances between these viewers, there are two equal.*

Problem 37. *Given $n \geq$ distinct positve integers $a_1, a_2, ..., a_n$ none of which is a perfect cube. Find the maximal possible number of perfect cubes among their pairwise products.*

Problem 38. *Prove that the set of all divisors of a positive integer which is not a perfect square can be divided into pairs so that in each pair, one number is divided by another.*

Problem 39. *Determine if there exists pairwise distinct positive integers*

$$a_1, a_2, ..., a_{101}, b_1, b_2, ..., b_{101}$$

satisfying the following property: for each non-empty subset S of $\{1, 2, ..., 101\}$ the sum $\sum_{i \in S} a_i$ divides $100! + \sum_{i \in S} b_i$.

Problem 40. *Given m, n such that $m > n^{n-1}$ and the number $m+1, m+2, ..., m+n$ are composite. Prove that there exist distinct primes $p_1, p_2, ..., p_n$ such that $m + k$ is divisible by p_k for each $k = 1, 2, ...$*