Graph Theory and Extremal Combinatorics
Canada IMO Camp, Winter 2020

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January 9, 2020

1 Introduction

There are two extremely good lectures/handouts about using graph theory in the IMO. I'm not going to try to out-do them, instead we'll use those as references:

`https://mikepawliuk.ca/2018/06/24/imo-resources-for-graph-theory/`

The first two links in there will also be handed out. They contain many good practice problems.

Here I will focus on some techniques and perspectives that those lectures avoided.

2 Basic Concepts and Definitions

- A graph $G = (V, E)$ is a pair where V is a (non-empty) set called the **nodes** or **vertices**, and E is a collection of unordered pairs of V called the **edges**. (Graphs can be infinite or finite, but by convention we will mean *finite* unless we say otherwise.)
- If $\{v, w\} \in E$ then we sometimes use the notation vEw to indicate that there is an edge between v and w. If there is an edge between v and w we say that they are **adjacent**, or they are **neighbours**.
- By convention, we usually assume that there are no edges between any node and itself. i.e. there are **no loops**.
- The **degree** of a vertex v is the number of other vertices it is adjacent to in G. We write $\deg_G(v)$. The **degree sequence** of a graph is a list (potentially with repetitions) of all the degrees of the vertices in G.

- The **neighbourhood** of a vertex v is the collection of all vertices adjecent to v. We write $N_G(v)$. Note $|N_G(v)| = \deg_G(v)$.
- A graph with n nodes and all possible edges is called the **complete graph** or **clique** on n nodes, and is written K_n. (Such a graph must have $\binom{n}{2}$ edges by the handshake lemma.)
- A graph with n nodes and no edges is called the **independent graph** on n nodes, and is written I_n.
- A **path** P is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that $v_i E v_{i+1}$ for all $1 \leq i < n$. (A priori the vertices need not be distinct.)
- A graph is **bipartite** if the vertices can be written as the disjoint union $V = V_1 \cup V_2$ where V_1 is an independent set, v_2 is an independent set, and it can contain edges between V_1 and V_2.
- A **cycle** C is a sequence of vertices $v_1, v_2, \ldots, v_n$ where all vertices are distinct except $v_1 = v_n$ and the only edges are between v_i and v_{i+1} for all i. If we need to refer to a cycle with n nodes (and n edges) we sometimes write it as C_n.
- A **Hamiltonian path** in a graph G is a path that uses every vertex exactly once.
- A **Hamiltonian cycle** in a graph G is a Hamiltonian path that is also a cycle.
- An **Eulerian path** in a graph G is a path that uses every edge exactly once (but may repeat vertices).
- An **Eulerian cycle** in a graph G is an Eulerian path that uses every edge exactly once and starts and ends at the same vertex.
- A graph is **connected** if for every two distinct vertices $v, w \in V$ there is a path from v to w.
- A **tree** is a connected graph with no cycles. (A tree with n nodes must have $n-1$ edges, and any connected graph with n nodes and $n-1$ edges must be a tree.)

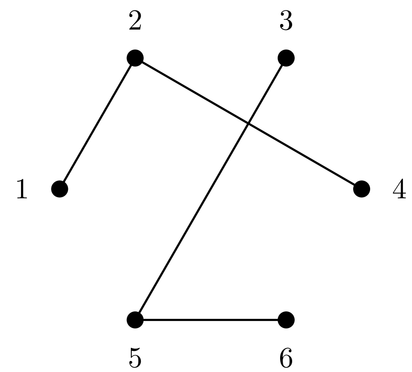


Figure 1: This graph has 6 nodes and 4 edges. $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{\{1, 2\}, \{2, 4\}, \{3, 5\}, \{5, 6\}\}$. Vertex 2 and 5 have degree 2, and the others have degree 1. It is not connected.

3 Most important basic results and techniques

Graph theory has many, many results. Some of them are particularly beautiful or fundamental.

1. $\sum_{v \in V} \deg(v) = 2|E|$. In particular, the sum of all degrees is always even. (**Proof**: If you sum the degrees you count every edge exactly twice.)

2. A graph must always contain an even number of vertices of odd degree.

3. A bipartite graph cannot contain cycles of odd length. (**Proof**: The cycle must contain vertices that alternate between V_1 and v_2.)

4. A graph without odd-cycles is bipartite.

5. Trees are bipartite graphs. (You should see this using the vertex partition definition, and you should see it using the cycle-free equivalence.)

6. Every connected graph G contains a connected subgraph on the same vertex set with a minimal number of edges; this is a tree, and is called a **minimal spanning tree** of G. (**Proof**: If it had a cycle you could remove an edge and maintain connectedness.)

7. If a graph has degree 2 for all of its vertices, then it is a cycle.

4 Important techniques

- **Extreme objects**. Often you will want to start with a maximal or minimal object with respect to some graph property. For example, you might ask for:

  - a vertex with maximal degree,
  - a minimal spanning tree,
  - the largest cycle in the graph,
  - the largest independent subset in the graph,
  - the largest clique in the graph.

  This technique combines well with proof by contradiction (by showing that it wasn't really maximal/minimal.)

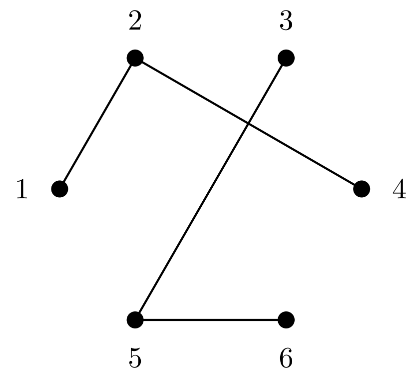
- **Induction and Strong Induction**. This is really the same technique as above, but worded slightly differently. Induction in graph theory proofs often iterates over one of:

  - The number of vertices,
  - The number of edges,
  - The largest degree,
  - The number of connected components.

- **Connected components**. You might not know that a graph is connected, but you can always break it up into its connected components: the maximal subgraphs of the graph that are connected. This works especially well if your connected subgraphs all have some structure (like they are all cycles).

- **The Dual graph**. If you have a map of some countries, you can make a graph where every country is a node, and two countries have an edge if they have touching borders on the map. This is a very general construction and allows you to transfer the geometry of the map into the combinatorics of graph theory. For example, you can use Euler's Formula F+V-E = 2 (about faces, vertices and edges) if you got a graph in this way.

- **Graphs describe relations**. If you are in a scenario where points can either be related or not, then it might be worth describing the



situation in terms of a graph. Get creative! A common (general) way to get an edge relation is to use information about the vertices and whether the vertices are "similar" or not. For example, you might have a set of positive integers and you put an edge between them if their GCD is 1.

4.1 Matrices

The **adjacency matrix** of a graph $G = (V, E)$ is a matrix (a chart) that keeps track of the edge relations in G. If $V = \{v_1, v_2, \ldots, v_n\}$ then the adjacency matrix $A(G)$ is the $n \times n$ matrix where the ij^{th} entry is 1 if there is an edge between v_i and v_j, and the entry is 0 if there is no edge. In particular, the (ii^{th}) entries on the main diagonal are always 0 since we assume our graphs have no loops.

The graph above has adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Big idea. The reason we mention this representation is that in this form you can perform *algebra* on the matrix (that you couldn't have in the nodes-edges version of the graph).

Theorem 4.1. *If A is the adjacency matrix of a graph then the ij^{th} entry of $A^n = A \times A \times \ldots \times A$ is the number of paths of length n from v_i to v_j in the original graph.*

Note that these paths counted above might reuse edges and nodes. For example, using the example graph above:

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The 2 in entry $(5,5)$ corresponds to the paths $5-6-5$ and $5-3-5$.

Corollary 4.2. *If A is the adjacency matrix of a graph G then the number of K_3 (triangles) in G is the sum of the main diagonal entries of A^3, divided by* 6*.*

Proof. The ii^{th} entry of A^3 counts the number of paths of length 3 that start and end at v_i. These paths cannot be degenerate, and must be K_3.

Adding up these entries will count each triangle in the graph 6 times, since each triangle can be thought of as a path starting at each of the 3 nodes, then proceeding clockwise or counterclockwise. □

This result does *not* immediately extend to counting other subgraphs, since the entries of A^n also counts “degenerate” subgraphs that repeat edges.

Fun (non-IMO) fact: This method of counting the triangles in a general graph is computationally efficient. There are no-known ways of efficiently counting the number of K_4 in a general graph.

5 Extremal Combinatorics

We will now get into some results in extremal combinatorics. Broadly speaking, extremal combinatorics looks at questions of “How bad can it get?” or “What’s the largest structure that is guaranteed to exist in a graph with a fixed number of edges/vertices or minimal degree?”. Graph colourings and Ramsey Theory are contained in this area (but we won’t be focusing on them here).

It’s likely that you’ve seen Hall’s Marriage Theorem before. We will see that again here, and see how it’s related to some other seemingly different results.

5.1 How wide is a poset?

Related (sort of) to graphs is the structure of a partially ordered set or **poset** for short. Formally, it is a pair $(X, \leq)$ such that:

- $x \leq x$ for all $x \in X$,
- $x \leq y$ and $y \leq x$ implies $x = y$,
- $x \leq y$ and $y \leq z$ implies $x \leq z$.

Notably, you do not need every pair of elements to be comparable. There can be x, y such that neither $x \leq y$ or $y \leq x$.

Important example 1: If X is a set of integers, then "divides" is a partial order. (Check that "divides" satisfies all three conditions above.)

This example allows us to leverage combinatorial results about posets into results about divisibility and integers.

Important example 2: If X is a family of sets, then $\subseteq$ (subset or equal) is a partial ordering. For example, $X = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\}\}$ is a poset with 5 elements.

Measuring the height of a poset is natural:

- A **chain** $\{c_1, c_2, \ldots, c_n\} \subseteq X$ in a poset $(X, \leq)$ is a set such that $c_1 \leq c_2 \leq \ldots \leq c_n$. (Usually we insist that the elements are all different.)
- The **height** of a finite poset $(X, \leq)$ is the largest length of a chain in $(X, \leq)$.

The width is more subtle. There are two ways to define it:

- The **width** of a poset is the least number of chain required to cover the poset.
- An **anti-chain** $\{a_1, a_2, \ldots, a_n\} \subseteq X$ in a poset $(X, \leq)$ is a set such that $c_i \not\leq c_j$ unless $i = j$.
- The **width** of a finite poset $(X, \leq)$ is the largest size of an antichain in $(X, \leq)$.

Theorem 5.1 (Dilworth, 1950)**.** *These two versions of width are the same number.*

At first this doesn't look helpful! But here's the useful corollary:

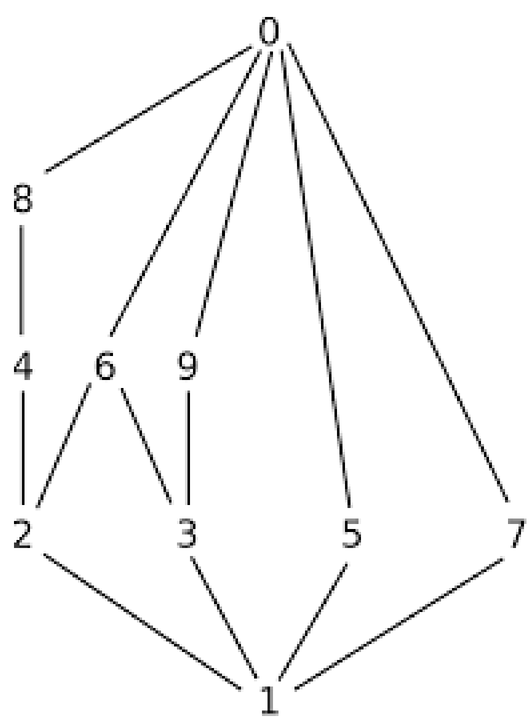


Figure 2: The divisibility poset of the integers $\{0, 1, \ldots, 9\}$. It has height 5 (thanks to the chain of powers of 2), and width 5 from the antichain $\{7, 5, 9, 6, 8\}$. [Picture from Brilliant.org]

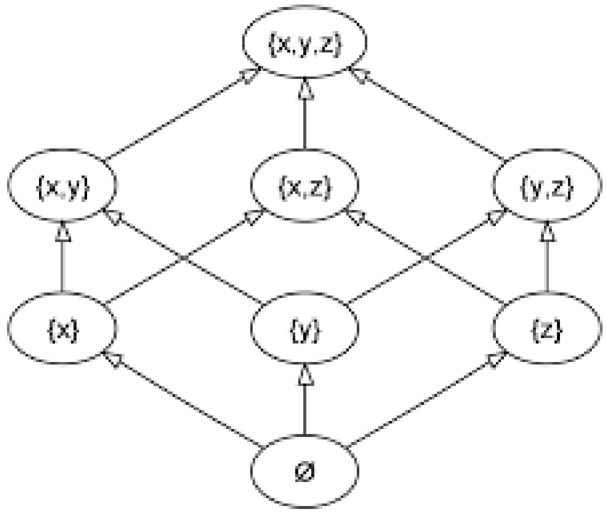


Figure 3: The poset of the subsets of $\{x, y, z\}$ ordered by subset-inclusion. It has height 4 and width 3. [Picture from Brilliant.org]

Theorem 5.2 (Dilworth's bound, 1950)**.** *If a poset has at least $rs + 1$ elements, then it has a chain with $r + 1$ elements or an antichain with $s + 1$ elements.*

This says a poset must either be wide or tall!

Proof. Suppose otherwise, that the largest chain has no more than r elements and the largest antichain has s elements. By Dilworth's theorem, this means that the poset can be covered by s chains. So the poset must have at most rs points. □

This bound is a clear generalization of the following fact:

Theorem 5.3 (Erdös–Szekeres theorem, 1935)**.** *Every list of rs+1 real numbers contains an increasing subsequence of length r+1 or it contains a decreasing subsequence of length s+1.*

This definitely looks like the Dilworth bound! The interesting part of the proof is the "agree/disagree" partial order that we use. This is a general construction that appears in many extremal combinatorics arguments.

Proof. Let $X = \{a_1, a_2, \ldots, a_{rs+1}\}$. Define a partial order $\to$ by $a_i \to a_j$ if and only if $i \leq j$ (as indices) and $a_i \leq a_j$ (as numbers). In other words, we point $a_i \to a_j$ if and only if the number order and index order agree. (You can check this is a poset.)

Now all we have to realize is that a chain (in $\to$) corresponds to an increasing subsequence of X, and an antichain corresponds to a decreasing subsequence of X. So Dilworth's bound gives us the result we wanted. $\square$

Exercise 1, Brilliant.org: Let $I_1, I_2, \ldots, I_{10}$ be intervals on the real number line. Suppose that no four of the intervals are disjoint. Show that there must be four intervals that share a common point.

Exercise 2, Brilliant.org: Let F be the collection of divisors of 2016. Let S be a subset of F such that for all distinct $a, b \in S$ we have that a does not divide b and b does not divide a. How large can $|S|$ be?

For more information read: `https://brilliant.org/wiki/dilworths-theorem/`

For some more (mostly theoretical) applications, see: `http://math.mit.edu/~cb_lee/18.318/lecture8.pdf`

If you like these types of results, then you should read "The Mathematical Coloring book" by Soifer.

Exercise, Helly's Theorem: Let $X_1, X_2, \ldots, X_n$ be convex subsets of $\mathbb{R}^d$ with $n > d$ such that any $d+1$ of these subsets have a common intersection. Show that all the sets have a common point. (**Hint**: Think about convex hulls.)

5.2 Hall's Marriage

In this last section, we connect Dilworth's Theorem with Hall's Marriage Theorem.

For a bipartite graph $G = (V, E)$ with disjoint vertex partition $V = A \cup B$, a **matching** is a collection of edges which have no endpoints in common. We say that A has a perfect matching to B if there is a matching which hits every vertex in A. This can be thought of as an injection from A into B using only some prescribed edges.

Theorem 5.4 (Hall's Marriage Theorem)**.** *For any set $S \subset A$, let $N(S)$ denote the set of vertices (necessarily in B) which are adjacent to at least one vertex in S. Then, A has a perfect matching to B if and only if $|N(S)| \geq |S|$ for every $S \subset A$.*

We will prove Hall's Marriage Theorem from Dilworth's Theorem.

Proof. Define an ordering on V by: $x < y$ if $x \in A$, $y \in B$ and $y \in N(\{x\})$.

The set A forms an antichain, so there is a chain cover of V with $|A|$ many chains. A chain cover is just a collection of edges in the original graph. $\square$

The lesson here is that Dilworth's Theorem can apply in many situations: it can build increasing/decreasing sequences as in the Erdös-Szekeres theorem, it can also build injections.