The Art of Induction

Canada IMO Training

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Induction is simultaneously one of the most fundamental and one of the most powerful tools in mathematics. In this lecture, we will explore the various ways in which induction can be applied to solve problems in Olympiads.

Proof by Induction

Recall that the natural numbers $\mathbb{N}$ are defined essentially as starting with 0 (or 1) and being closed under succession (if a is in $\mathbb{N}$, then so is $a+1$). Logically, this means that if we have a series of statements $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ indexed by the natural numbers, as long as we can prove the **base case** $\mathcal{P}_0$ and the **inductive step** $\mathcal{P}_n \to \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$, then we will have proved all the statements $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ via **induction**.

Let's take a look at an example.

Example 1. [IMO 2013 1 (N2)] *Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that*

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Proof. We proceed by induction on $k \geq 1$. When $k=1$, we choose $m_1 = n$. Suppose that the claim holds for $k=m$; when $k=m+1$, we can write

$$\frac{n+(2^k-1)}{n}=\left(1+\frac{1}{n+(2^k-2)}\right)\cdot\frac{\frac{n}{2}+(2^{k-1}-1)}{\frac{n}{2}}$$

when n is even and

$$\frac{n+(2^k-1)}{n}=\left(1+\frac{1}{n}\right)\cdot\frac{\frac{n+1}{2}+(2^{k-1}-1)}{\frac{n+1}{2}}$$

when n is odd. We can then apply the inductive hypothesis to the second term in both cases to arrive at the desired form, completing the induction. $\square$

Induction can often be the key to solving a problem. At the level of the IMO, finding this key can be extremely tricky. The following few sections will cover some of the patterns that arise in these difficult applications.

Mapping $\mathcal{P} \to \mathbb{N}$

Aside from $\mathcal{P}_0$ and $\mathcal{P}_n \to \mathcal{P}_{n+1}$, another important step in proof by induction is figuring out what the $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ should be. It isn't enough for $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ to imply $\mathcal{P}$, but both $\mathcal{P}_0$ and $\mathcal{P}_n \to \mathcal{P}_{n+1}$ need to be tractable subproblems. For the following two examples, take a few minutes to try to figure out how you should set up the problem as a proof by induction.

Exercise 2. [ISL 2014 C1] *Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.*

Exercise 3. [ISL 2014 N1] *Let $n \geq 2$ be an integer, and let A_n be the set*

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z}, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

Guessing the Pattern

One of the best ways to gain intuition on a problem is to try out some small cases and pick out a pattern. This can be especially true for induction: sometimes the key to figuring out the right $\mathcal{P}_n \to \mathcal{P}_{n+1}$ is to simply guess a formula. For the following two examples, take a few minutes to try to figure out how you should set up the problem as a proof by induction.

Exercise 4. [ISL 2013 A1] *Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$ and*

$$u_{k+1} = u_k + a_k u_{k-1}, \quad v_{k+1} = v_k + a_{n-k} v_{k-1} \quad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

Exercise 5. [ISL 2019 C1] *The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily different) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and*

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$. Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

Generalizing the Problem

Generalizing can be an extremely powerful tool; after all, much of modern mathematics is made possible only by the study of abstract generalizations of common objects. Some problems can be very resistant to a direct application of induction, but with the right generalization, end up being not all that difficult. Unfortunately, finding the right generalization can be elusive. For the following two examples, take a few minutes to try to figure out how you should set up the problem as a proof by induction.

Exercise 6. [ISL 2006 N7] *Prove that, for every positive integer n, there exists an integer m such that $2^m + m$ is divisible by n.*

Exercise 7. [ISL 2019 C9] *For any two different real numbers x and y, we define $D(x, y)$ to be the unique integer d satisfying $2^d \leq |x - y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the* scales *of x in $\mathcal{F}$ are the values of $D(x, y)$ for $y \in \mathcal{F}$ with $x \neq y$. Let k be a given positive integer. Suppose that each member x of $\mathcal{F}$ has at most k different scales in $\mathcal{F}$ (note that these scales may depend on x). What is the maximum possible size of $\mathcal{F}$?*

Strong Induction

We can strengthen the inductive hypothesis from $\mathcal{P}_n \to \mathcal{P}_{n+1}$ to $\{\mathcal{P}_m\}_{m \leq n, \in \mathbb{N}} \to \mathcal{P}_{n+1}$; this is known as **strong induction**. In practice, it is always good to assume the strong inductive hypothesis by default—just keep in mind that most of the time, most of the information to prove $\mathcal{P}_{n+1}$ will come from $\mathcal{P}_n$ and not $\mathcal{P}_k$ for $k < n$.

Let's take a look at an example:

Example 8. [IMO 2009 6 (C7)] *Let $a_1, a_2, \dots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \dots, a_n$ in some order. Prove that the order can be chosen in such a way that they grasshopper never lands on any point in M.*

Proof. We proceed by induction on $n \geq 1$. When $n = 1$, the problem is trivial. Suppose that the claim holds for $n \leq k$; when $n = k+1$, WLOG $a_1 < a_2 < \cdots < a_{k+1}$ and focus on the largest jump a_{k+1}. There are four cases:

- $a_{k+1} \in M, \forall\, i \leq k, a_i \notin M$. Make the jump a_{k+1}. By the inductive hypothesis (k), there exists a sequence of jumps from a_{k+1} to s; switch the first one with a_{k+1}.
- $a_{k+1} \in M, \exists\, i \leq k, a_i \in M$. By pigeonhole, there exists some $i \leq k$ such that $a_i, a_i + a_{k+1} \notin M$. Make the jumps a_i and a_{k+1}. By the inductive hypothesis $(k-1)$, there exists a sequence of jumps from $a_i + a_{k+1}$ to s.
- $a_{k+1} \notin M, \forall\, i \leq k, a_i \notin M$. Make the jump a_{k+1}. By the inductive hypothesis (k), there exists a sequence of jumps from a_{k+1} to s that lands only on the smallest mine m_1. Switch the jump immediately after m_1 with a_{k+1}.
- $a_{k+1} \notin M, \exists\, i \leq k, a_i \in M$. Make the jump a_{k+1}. By the inductive hypothesis (k), there exists a sequence of jumps from a_{k+1} to s.

Thus, the inductive step is complete. □

Though this one might seem a bit easy for its placement, it does prove rather tricky in practice: splitting up the cases and applying induction in a way that works is a bit of a minefield in itself. Note that while this problem only requires $\mathcal{P}_{n-1}, \mathcal{P}_n$ to prove $\mathcal{P}_{n+1}$, others may end up requiring the full strength of $\{P_m\}_{m \leq n, \in \mathbb{N}}$. Here are some strong induction problems, for practice.

1. [ISL 2006 C3] Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. Prove that for every real number x,
$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$
where the sum is taken over all convex polygons with vertices in S.
2. [ISL 2019 N3] We say that a set S of integers is *rootiful* if for any positive integer n and any $a_0, a_1, \dots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that also contain all numbers of the form $2^a - 2^b$ for positive integers a and b.
3. [ISL 2009 C4] For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Multistep Problems

Some of the harder induction problems are difficult because they involve multiple steps; either a fairly substantial lemma needs to be proven before induction is feasible, or a fairly substantial amount of work needs to be done after proving a lemma via induction. Figuring out this intermediate lemma is not easy. Here are some multistep induction problems, for practice.

4. [ISL 2012 C7] There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.
5. [ISL 2013 C7] Let $n \geq 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n-1$ rotations of an arrangement are considered to be equal. A circular arrangement is called beautiful if, for any four distinct numbers $0 \leq a, b, c, d \leq n$ with $a+c = b+d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d. Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x+y \leq n$ and $\gcd(x, y) = 1$. Prove that $M = N+1$.
6. [ISL 2017 A7] Let $a_0, a_1, a_2, \ldots$ be a sequence of integers and $b_0, b_1, b_2, \ldots$ be a sequence of *positive* integers such that $a_0 = 0$, $a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1} & \text{if } b_{n-1} = 1, \\ a_n b_n - a_{n-1} & \text{if } b_{n-1} > 1, \end{cases} \quad \text{for } n = 1, 2, \ldots.$$

Prove that at least one of the two numbers a_{2017} and a_{2018} must be greater than or equal to 2017.

Recursion

Recursion is another form of inductive logic. It is essentially equivalent to induction, but with two key differences: (1) it approaches the problem from a top-down point of view, and (2) recursion is not limited to returning a yes or no statement (e.g. induction might return whether a problem can be solved, whereas recursion might return an algorithm that solves the problem). The change in perspective between induction and recursion can be very helpful at times.

Example 9. [IMO 2017 5 (C4)] *An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold: no one stands between the two tallest players, no one stands between the third and fourth tallest players, ..., no one stands between the two shortest players. Show that this is always possible.*

Proof. We provide a recursive algorithm. Group the players into N contiguous groups of $N+1$ players. Locate the group with the tallest second-tallest player; choose the two tallest players from that group, remove the rest of the group, and remove the tallest player from every other group. There are now $N-1$ contiguous groups of N players left, of which all are shorter than the two chosen players.

This algorithm provides a correct solution because of the remaining $2N$ players, the two tallest are in the first located group, the next two are in the second, and so on. Since the groups were contiguous, the problem condition is satisfied. □

Infinite Descent and Vieta Jumping

Fermat's method of **infinite descent** is also a form of inductive logic, based on the fact that there does not exist a map $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(a) < f(b)$ whenever $a > b$. The proof of the irrationality of $\sqrt{2}$ is the classic example.

We can go a step further and deduce that, for any map $f : S \to \mathbb{N}$, there must exist some $a \in S$ such that $f(a) \leq f(b)$ for all $b \in \mathbb{N}$, a.k.a. a is a minimal solution with respect to f. This technique provides a neat way of proving that solutions cannot exist, and can make the problem a bit easier to think about than if using the original infinite descent. This approach can be useful in Diophantine equations, extremal combinatorics, and more. Let's take a look at an example:

Example 10. [IMO 1988 6] *Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that*

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

Proof. For a fixed $k \in \mathbb{Z}^+$, suppose that a solution $(a, b) \in \mathbb{Z}^+$ to the equation $a^2+b^2 = k\cdot(ab+1)$ exists (WLOG $a \le b$). We consider the solution with a minimal. The quadratic

$$x^2 - kax + a^2 - k = 0$$

has integer roots b and $c := ka - b$. From Vieta's formulas,

$$ac \le bc = a^2 - k < a^2 \quad \longrightarrow c < a,$$

so by the minimality of a, it must be that $c \le 0$. Again, from Vieta's formulas,

$$(b+1)(c+1) = bc + b + c + 1 = a^2 - k + ak + 1 = a^2 + (a-1)k + 1 \ge 1,$$

so it must be that $c = 0$. It follows that $b^2 = k$, thus k must be a perfect square.

The usage of Vieta's formulas here in conjunction with the minimality assumption (a.k.a. infinite descent) is known as **Vieta jumping**. Here are some infinite descent problems, for practice.

7. [IMO 2007 5] Let a and b be positive integers. Show that if $4ab - 1 \mid (4a^2-1)^2$, then $a = b$.
8. [IMO 1977 6] Let $f(n)$ be a function $f : \mathbb{N}^+ \to \mathbb{N}^+$. Prove that if $f(n+1) > f(f(n))$ for each positive integer n, then $f(n) = n$.
9. [IMO 1986 3] To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x+y, -y, z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Practice Problems

Here are twelve problems, taken from some of the more recent ISLs and IMOs, that can be solved in large part by induction. Enjoy!

10. [ISL 2013 C1] Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \ldots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \le a_i \le 1$ for $i = 1, 2, \ldots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.
11. [ISL 2015 A1] Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \ge \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \ge n$ for every $n \ge 2$.

12. [ISL 2007 A2] The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \quad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \text{ for } n \ge 1.$$

Show that $a_n > 0$ for $n \ge 1$.

13. [ISL 2019 C2] You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \le r \le 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r-2$.

14. [ISL 2008 N3] Let $a_0, a_1, a_2, \dots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \ge 2^n$ for all $n \ge 0$.

15. [ISL 2016 A3] Find all integers $n \ge 3$ with the following property: for all real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \le k \le n$, there exist $x_1, x_2, \dots, x_n$ each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \le 1.$$

16. [ISL 2008 N4] Let n be a positive integer. Show that the numbers

$$\binom{2^n-1}{0}, \binom{2^n-1}{1}, \binom{2^n-1}{2}, \dots, \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo 2^n to $1, 3, 5, \dots, 2^n - 1$ in some order.

17. [ISL 2008 G5] Let k and n be integers with $0 \le k \le -2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of interesection points of lines in L. Let O be a point in the plane not lying on any line of L. A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

18. [ISL 2011 A5] Prove that for every positive integer n, the set $\{2, 3, 4, \dots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

19. [IMO 2007 6 (A7)] Let n be a positive integer. Consider

$$S = \{(x, y, z) \ : \ x, y, z \in \{0, 1, \dots, n\}, \ x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contain S but does not include $(0, 0, 0)$.

20. [IMO 2016 3 (N7)] Let $P = A_1 A_2 \cdots A_k$ be a convex polygon in the plane. The vertices $A_1, A_2, \dots, A_k$ have integral coordinates and lie on a circle. Let S be the area of P. An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n. Prove that $2S$ is an integer divisible by n.

21. [ISL 2017 N7] Say that an ordered pair (x, y) of integers is an *irreducible lattice point* if x and y are relatively prime. For any finite set S of irreducible lattice points, show that there is a homogenous polynomial in two variables, $f(x, y)$, with integer coefficients, of degree at least 1, such that $f(x, y) = 1$ for each (x, y) in the set S.

$\{P_n\}_{n\in\mathbb{N}}$ for Exercises

2. Let k denote the number of rectangles. Prove by induction on k that $n \leq k-1$.
3. Let $F_n := (n-2)\cdot 2^n + 1$. Prove by induction on n that any $m \geq F_n$ has a representation.
4. Let $\mathcal{I}_k$ denote the set of sequences $\boldsymbol{i} := i_1, \ldots, i_t$ satisfying $0 < i_1 < \cdots < i_t < k$ and $i_{j+1} - i_j \geq 2$. Prove by induction on k that
$$u_k = \sum_{\boldsymbol{i}\in\mathcal{I}} a_{i_1} \cdot a_{i_t}.$$
5. Prove by induction on k that there exists some $0 \leq 2l \leq k+1$ so that $\{a_1, a_2, \ldots, a_k\} = \{1, 2, \ldots, l-1, 1, 2, \ldots, k-l\}$ as a multiset.
6. Prove by induction on n that for arbitrarily large N there exist positive integers $b_0, b_1, \ldots, b_{d-1}$ satisfying $b_i > N$ and $2^{b_i} + b_i \equiv i \bmod d$.
7. For a set $\mathcal{S}$, denote by $r_{\mathcal{S}}(x)$ the number of different scales of x in $\mathcal{S}$. Prove by induction on $|\mathcal{S}|$ that
$$\sum_{x\in\mathcal{S}} 2^{-r_{\mathcal{S}}(x)} \leq 1.$$

Hints for Problems

1. Consider removing subsets of the convex hull C.
2. How can you use strong induction to show that $S = \mathbb{Z}^+$?
3. Consider the rectangle in a minimal partition that contains a non-diagonal corner cell.
4. First prove by induction on $|G|$ the following lemma: A graph G whose vertices v have degree d_v contains an independent set of size at least
$$\sum_{v \in G} \frac{1}{d_v + 1}.$$
5. First prove by induction on n the following lemma: for any k, the chords whose endpoints add up to k satisfy the property that among any three, one separates the other two.
6. First prove that $a_n \geq 1$.
7. The divisibility condition implies $4ab - 1 \mid (a-b)^2$.
8. Infinite descent can be used to show that $f^{-1}(1) = f(1) = 1$; strong induction can be used to show that $f(n) = n$.
9. Define an integer-valued function that strictly decreases whenever an operation is performed.
10. To show that the desired minimal k works, induct on d.
11. First prove that $a_1 + a_2 + \cdots + a_k \geq \frac{k}{a_{k+1}}$.
12. How can you write a_{k+1} in terms of $a_1, a_2, \ldots, a_k$?
13. Relax the statement so that the total weight is $s \leq 2n$.
14. The tricky case is $a_{n+1} = \frac{3}{2} a_n$.
15. Use induction for odd n.
16. First prove that
$$\binom{2^n - 1}{2k} \equiv (-1)^k \binom{2^{n-1} - 1}{k} \bmod 2^n.$$
17. What can you say about two adjacent points $P, Q \in I$?
18. Use the fact that if $\{a, b, c\}$ is obtuse for $a < b < c$, then so is $\{a, b + x, c + x\}$ for any $x > 0$.
19. These planes determine a polynomial that vanishes on S but not at $(0, 0, 0)$.
20. Induction fails when no diagonals are divisible by n, but is that possible?
21. Recursively define $f_{\{1,2,\ldots,n+1\}}(x, y) = f_{\{1,2,\ldots,n\}}(x, y)^\alpha - C \cdot P(x, y) \cdot f_{\{n+1\}}(x, y)^\beta$ for some appropriately chosen α, β, C, P.