P7 For every integer $n \geqslant 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geqslant 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.

Answer: All integers n for which $n+1$ is a prime.

Solution 1. First, observe that every pair x, y of real numbers for which the sum $x+y$ is integer satisfies

$$\lfloor x \rfloor + \lfloor y \rfloor \geqslant x + y - 1. \tag{1}$$

The inequality is strict if x and y are integers, and it holds with equality otherwise.

We estimate the sum S as follows.

$$2S = \sum_{1 \leqslant i,j \leqslant n} \left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{ij}{n+1} \right\rfloor \right) = \sum_{1 \leqslant i,j \leqslant n} \left(\left\lfloor \frac{ij}{n+1} \right\rfloor + \left\lfloor \frac{(n+1-i)j}{n+1} \right\rfloor \right)$$

$$\geqslant \sum_{1 \leqslant i,j \leqslant n} (j-1) = \frac{(n-1)n^2}{2}.$$

The inequality in the last line follows from (1) by setting $x = ij/(n+1)$ and $y = (n+1-i)j/(n+1)$, so that $x+y=j$ is integral.

Now $S = \frac{1}{4}n^2(n-1)$ if and only if the inequality in the last line holds with equality, which means that none of the values $ij/(n+1)$ with $1 \leqslant i,j \leqslant n$ may be integral.

Hence, if $n+1$ is composite with factorisation $n+1 = ab$ for $2 \leqslant a, b \leqslant n$, one gets a strict inequality for $i = a$ and $j = b$. If $n+1$ is a prime, then $ij/(n+1)$ is never integral and $S = \frac{1}{4}n^2(n-1)$.

Solution 2. To simplify the calculation with indices, extend the table by adding a phantom column of index 0 with zero entries (which will not change the sum of the table). Fix a row i with $1 \leqslant i \leqslant n$, and let $d := \gcd(i, n+1)$ and $k := (n+1)/d$. For columns $j = 0, \ldots, n$, define the remainder $r_j := ij \bmod (n+1)$. We first prove the following

Claim. For every integer g with $1 \leqslant g \leqslant d$, the remainders r_j with indices j in the range

$$(g-1)k \leqslant j \leqslant gk - 1 \tag{2}$$

form a permutation of the k numbers $0 \cdot d,\ 1 \cdot d,\ 2 \cdot d,\ \ldots,\ (k-1) \cdot d$.

Proof. If $r_{j'} = r_j$ holds for two indices j' and j in (2), then $i(j'-j) \equiv 0 \bmod (n+1)$, so that $j'-j$ is a multiple of k; since $|j'-j| \leqslant k-1$, this implies $j' = j$. Hence, the k remainders are pairwise distinct. Moreover, each remainder $r_j = ij \bmod (n+1)$ is a multiple of $d = \gcd(i, n+1)$. This proves the claim.

We then have

$$\sum_{j=0}^{n} r_j = \sum_{g=1}^{d} \sum_{\ell=0}^{(n+1)/d-1} \ell d = d^2 \cdot \frac{1}{2} \left(\frac{n+1}{d} - 1 \right) \frac{n+1}{d} = \frac{(n+1-d)(n+1)}{2}. \tag{3}$$

By using (3), compute the sum S_i of row i as follows:

$$S_i = \sum_{j=0}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \sum_{j=0}^{n} \frac{ij - r_j}{n+1} = \frac{i}{n+1} \sum_{j=0}^{n} j - \frac{1}{n+1} \sum_{j=0}^{n} r_j$$

$$= \frac{i}{n+1} \cdot \frac{n(n+1)}{2} - \frac{1}{n+1} \cdot \frac{(n+1-d)(n+1)}{2} = \frac{(in - n - 1 + d)}{2}. \tag{4}$$

Equation (4) yields the following lower bound on the row sum S_i, which holds with equality if and only if $d = \gcd(i, n+1) = 1$:

$$S_i \geqslant \frac{(in - n - 1 + 1)}{2} = \frac{n(i-1)}{2}. \tag{5}$$

By summing up the bounds (5) for the rows $i = 1, \dots, n$, we get the following lower bound for the sum of all entries in the table

$$\sum_{i=1}^{n} S_i \geqslant \sum_{i=1}^{n} \frac{n}{2}(i-1) = \frac{n^2(n-1)}{4}. \tag{6}$$

In (6) we have equality if and only if equality holds in (5) for each $i = 1, \dots, n$, which happens if and only if $\gcd(i, n+1) = 1$ for each $i = 1, \dots, n$, which is equivalent to the fact that $n+1$ is a prime. Thus the sum of the table entries is $\frac{1}{4}n^2(n-1)$ if and only if $n+1$ is a prime.

Comment. To simplify the answer, in the problem statement one can make a change of variables by introducing $m := n+1$ and writing everything in terms of m. The drawback is that the expression for the sum will then be $\frac{1}{4}(m-1)^2(m-2)$ which seems more artificial.

P8

Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbours*— namely, the k students closest to S on the right, and the k students closest to S on the left.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

Solution. We replace the girls by 1's, and the boys by 0's, getting the numbers $a_1, a_2, \dots, a_{2n+1}$ arranged in a circle. We extend this sequence periodically by letting $a_{2n+1+k} = a_k$ for all $k \in \mathbb{Z}$. We get an infinite periodic sequence

$$\dots, a_1, a_2, \dots, a_{2n+1}, a_1, a_2, \dots, a_{2n+1}, \dots.$$

Consider the numbers $b_i = a_i + a_{i-k-1} - 1 \in \{-1, 0, 1\}$ for all $i \in \mathbb{Z}$. We know that

$$b_{m+1} + b_{m+2} + \dots + b_{m+2n+1} = 1 \qquad (m \in \mathbb{Z}); \tag{1}$$

in particular, this yields that there exists some i_0 with $b_{i_0} = 1$. Now we want to find an index i such that

$$b_i = 1 \quad \text{and} \quad b_{i+1} + b_{i+2} + \dots + b_{i+k} \geqslant 0. \tag{2}$$

This will imply that $a_i = 1$ and

$$(a_{i-k} + a_{i-k+1} + \dots + a_{i-1}) + (a_{i+1} + a_{i+2} + \dots + a_{i+k}) \geqslant k,$$

as desired.

Suppose, to the contrary, that for every index i with $b_i = 1$ the sum $b_{i+1} + b_{i+2} + \dots + b_{i+k}$ is negative. We start from some index i_0 with $b_{i_0} = 1$ and construct a sequence $i_0, i_1, i_2, \dots$, where i_j $(j > 0)$ is the smallest possible index such that $i_j > i_{j-1} + k$ and $b_{i_j} = 1$. We can choose two numbers among $i_0, i_1, \dots, i_{2n+1}$ which are congruent modulo $2n+1$ (without loss of generality, we may assume that these numbers are i_0 and i_T).

On the one hand, for every j with $0 \leqslant j \leqslant T-1$ we have

$$S_j := b_{i_j} + b_{i_j+1} + b_{i_j+2} + \cdots + b_{i_{j+1}-1} \leqslant b_{i_j} + b_{i_j+1} + b_{i_j+2} + \cdots + b_{i_j+k} \leqslant 0$$

since $b_{i_j+k+1}, \ldots, b_{i_{j+1}-1} \leqslant 0$. On the other hand, since $(i_T - i_0) \mid (2n+1)$, from (1) we deduce

$$S_0 + \cdots + S_{T-1} = \sum_{i=i_0}^{i_T-1} b_i = \frac{i_T - i_0}{2n+1} > 0.$$

This contradiction finishes the solution.

Comment 1. After the problem is reduced to finding an index i satisfying (2), one can finish the solution by applying the (existence part of) following statement.

Lemma (Raney). If $\langle x_1, x_2, \ldots, x_m \rangle$ is any sequence of integers whose sum is $+1$, exactly one of the cyclic shifts $\langle x_1, x_2, \ldots, x_m \rangle, \langle x_2, \ldots, x_m, x_1 \rangle, \ldots, \langle x_m, x_1, \ldots, x_{m-1} \rangle$ has all of its partial sums positive.

A (possibly wider known) version of this lemma, which also can be used in order to solve the problem, is the following

Claim (Gas stations problem). Assume that there are several fuel stations located on a circular route which together contain just enough gas to make one trip around. Then one can make it all the way around, starting at the right station with an empty tank.

Both Raney's theorem and the Gas stations problem admit many different (parallel) proofs. Their ideas can be disguised in direct solutions of the problem at hand (as it, in fact, happens in the above solution); such solutions may avoid the introduction of the b_i. Below, in Comment 2 we present a variant of such solution, while in Comment 3 we present an alternative proof of Raney's theorem.

Comment 2. Here is a version of the solution which avoids the use of the b_i.

Suppose the contrary. Introduce the numbers a_i as above. Starting from any index s_0 with $a_{s_0} = 1$, we construct a sequence $s_0, s_1, s_2, \ldots$ by letting s_i to be the smallest index larger than $s_{i-1}+k$ such that $a_{s_i} = 1$, for $i = 1, 2, \ldots$. Choose two indices among $s_1, \ldots, s_{2n+1}$ which are congruent modulo $2n+1$; we assume those two are s_0 and s_T, with $s_T - s_0 = t(2n+1)$. Notice here that $s_{T+1} - s_T = s_1 - s_0$.

For every $i = 0, 1, 2, \ldots, T$, put

$$L_i = s_{i+1} - s_i \quad \text{and} \quad S_i = a_{s_i} + a_{s_i+1} + \cdots + a_{s_{i+1}-1}.$$

Now, by the indirect assumption, for every $i = 1, 2, \ldots, T$, we have

$$a_{s_i-k} + a_{s_i-k+1} + \cdots + a_{s_i+k} \leqslant a_{s_i} + (k-1) = k.$$

Recall that $a_j = 0$ for all j with $s_i + k < j < a_{s_{i+1}}$. Therefore,

$$S_{i-1} + S_i = \sum_{j=s_{i-1}}^{s_i+k} a_j = \sum_{j=s_{i-1}}^{s_i-k-1} a_j + \sum_{j=s_i-k}^{s_i+k} a_j \leqslant (s_i - s_{i-1} - k) + k = L_{i-1}.$$

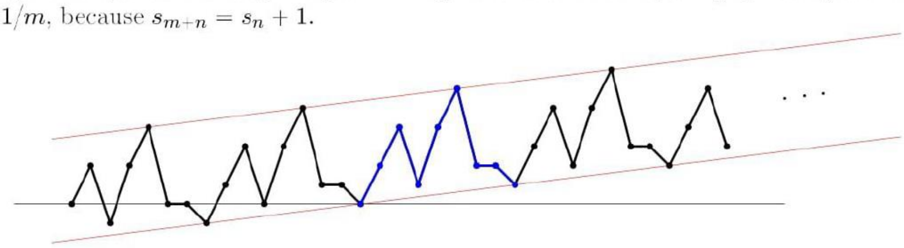
Summing up these equalities over $i = 1, 2, \ldots, T$ we get

$$2t(n+1) = \sum_{i=1}^{T} (S_{i-1} + S_i) \leqslant \sum_{i=1}^{T} L_{i-1} = (2n+1)t,$$

which is a contradiction.

Comment 3. Here we present a proof of Raney's lemma different from the one used above.

If we plot the partial sums $s_n = x_1 + \cdots + x_n$ as a function of n, the graph of s_n has an average slope of $1/m$, because $s_{m+n} = s_n + 1$.



The entire graph can be contained between two lines of slope $1/m$. In general these bounding lines touch the graph just once in each cycle of m points, since lines of slope $1/m$ hit points with integer coordinates only once per m units. The unique (in one cycle) lower point of intersection is the only place in the cycle from which all partial sums will be positive.

Comment 4. The following example shows that for different values of k the required girl may have different positions: 0110 0110 1.

P9 Prove that there are only finitely many quadruples (a, b, c, n) of positive integers such that

$$n! = a^{n-1} + b^{n-1} + c^{n-1}.$$

Solution. For fixed n there are clearly finitely many solutions; we will show that there is no solution with $n > 100$. So, assume $n > 100$. By the AM–GM inequality,

$$\begin{aligned} n! &= 2n(n-1)(n-2)(n-3) \cdot (3 \cdot 4 \cdots (n-4)) \\ &\leqslant 2(n-1)^4 \left(\frac{3 + \cdots + (n-4)}{n-6}\right)^{n-6} = 2(n-1)^4 \left(\frac{n-1}{2}\right)^{n-6} < \left(\frac{n-1}{2}\right)^{n-1}, \end{aligned}$$

thus $a, b, c < (n-1)/2$.

For every prime p and integer $m \neq 0$, let $\nu_p(m)$ denote the p-adic valuation of m; that is, the greatest non-negative integer k for which p^k divides m. Legendre's formula states that

$$\nu_p(n!) = \sum_{s=1}^{\infty} \left\lfloor \frac{n}{p^s} \right\rfloor,$$

and a well-know corollary of this formula is that

$$\nu_p(n!) < \sum_{s=1}^{\infty} \frac{n}{p^s} = \frac{n}{p-1}. \qquad (\heartsuit)$$

If n is odd then $a^{n-1}, b^{n-1}, c^{n-1}$ are squares, and by considering them modulo 4 we conclude that a, b and c must be even. Hence, $2^{n-1} \mid n!$ but that is impossible for odd n because $\nu_2(n!) = \nu_2((n-1)!) < n-1$ by ($\heartsuit$).

From now on we assume that n is even. If all three numbers $a+b, b+c, c+a$ are powers of 2 then a, b, c have the same parity. If they all are odd, then $n! = a^{n-1} + b^{n-1} + c^{n-1}$ is also odd which is absurd. If all a, b, c are divisible by 4, this contradicts $\nu_2(n!) \leqslant n-1$. If, say, a is not divisible by 4, then $2a = (a+b) + (a+c) - (b+c)$ is not divisible by 8, and since all $a+b$, $b+c$, $c+a$ are powers of 2, we get that one of these sums equals 4, so two of the numbers of a, b, c are equal to 2. Say, $a = b = 2$, then $c = 2^r - 2$ and, since $c \mid n!$, we must have $c \mid a^{n-1} + b^{n-1} = 2^n$ implying $r = 2$, and so $c = 2$, which is impossible because $n! \equiv 0 \not\equiv 3 \cdot 2^{n-1} \pmod 5$.

So now we assume that the sum of two numbers among a, b, c, say $a+b$, is not a power of 2, so it is divisible by some odd prime p. Then $p \leqslant a+b < n$ and so $c^{n-1} = n! - (a^{n-1} + b^{n-1})$ is divisible by p. If p divides a and b, we get $p^{n-1} \mid n!$, contradicting ($\heartsuit$). Next, using ($\heartsuit$) and the Lifting the Exponent Lemma we get

$$\nu_p(1)+\nu_p(2)+\cdots+\nu_p(n) = \nu_p(n!) = \nu_p(n!-c^{n-1}) = \nu_p\left(a^{n-1}+b^{n-1}\right) = \nu_p(a+b)+\nu_p(n-1). \quad (\diamondsuit)$$

In view of ($\diamondsuit$), no number of $1, 2, \ldots, n$ can be divisible by p, except $a+b$ and $n-1 > a+b$. On the other hand, $p|c$ implies that $p < n/2$ and so there must be at least two such numbers. Hence, there are two multiples of p among $1, 2, \ldots, n$, namely $a+b = p$ and $n-1 = 2p$. But this is another contradiction because $n-1$ is odd. This final contradiction shows that there is no solution of the equation for $n > 100$.

Comment 1. The original version of the problem asked to find all solutions to the equation. The solution to that version is not much different but is more technical.

Comment 2. To find all solutions we can replace the bound $a, b, c < (n-1)/2$ for all n with a weaker bound $a, b, c \leqslant n/2$ only for even n, which is a trivial application of AM-GM to the tuple $(2, 3, \ldots, n)$. Then we may use the same argument for odd n (it works for $n \geqslant 5$ and does not require any bound on a, b, c), and for even n the same solution works for $n \geqslant 6$ unless we have $a+b = n-1$ and $2\nu_p(n-1) = \nu_p(n!)$. This is only possible for $p = 3$ and $n = 10$ in which case we can consider the original equation modulo 7 to deduce that $7 \mid abc$ which contradicts the fact that $7^9 > 10!$. Looking at $n \leqslant 4$ we find four solutions, namely,

$$(a, b, c, n) = (1, 1, 2, 3), (1, 2, 1, 3), (2, 1, 1, 3), (2, 2, 2, 4).$$

Comment 3. For sufficiently large n, the inequality $a, b, c < (n-1)/2$ also follows from Stirling's formula.