TEAM SELECTION TEST
INTRERNATIONAL MATHEMATICAL OLYMPIAD
Day 3, May 5, 2021

Problem 1. Let ABC be an isosceles triangle with $BC = CA$, and let D be a point inside the side AB such that $AD < DB$. Let P and Q be two points inside the sides BC and CA respectively, sich that $\angle DPB = \angle DQA = 90^o$. Let the perpendicular bisector of PQ meets the line segment CQ at E and let the circumcircles of triangles ABC and CPQ meet again at point F, different from C. Suppose that P, E and F are collinear. Prove that $\angle ACB = 90^o$.

Problem 2. The Fibonacci numbers F_0, F_1, F_2, ... are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exist $x, y \in S$, such thath $x - y = F_k$.

Problem 3. For a positive integer n, let $d(n)$ be the number of positive divisors of n, and let $\varphi(n)$ be the number of positive integers not exceeding n which are coprime with n. Prove that that for any number C there exists an integer n for which

$$\frac{\varphi(d(n))}{d(\varphi(n))} > C.$$