Sequences

Winter Camp 2020

1 Example Problems

Problem 1. *(IMO Shortlist 2015) Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by*

$$a_0 = M + \frac{1}{2} \qquad \textit{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \textit{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

Proof. We again start with small cases. We note that for $M = 1$, we have $a_0 = 1.5, a_1 = 1.5, \ldots,$ which is never an integer, but for $M = 2$, we have $a_1 = 5$; for $M = 3$, we have $a_2 = 110$; for $M = 4$, we have $a_1 = 18$, for $M = 5$, we have $a_2 = 742.5$ and hence $a_3 = 371 * 1485$. In particular, we note that whenever a_i is of the form $2K + 0.5$ for some integer K, then a_{i+1} is an integer. This begs the question: what conditions on a_{i-1} will force $a_i = 2K + 0.5$? Well, since we know that a_{i-1} is a half-integer, let's let $a_{i-1} = x + 0.5$, and then we have $a_i = x^2 + 0.5x = x^2 + 0.5(x-1) + 0.5$. Thus, we have our desired condition when x is odd and $x^2 + 0.5(x-1)$ is even; that is, when $2x^2 + x - 1$ is divisible by 4. Since the first term is always $2 \mod 4$, this occurs when x is $3 \mod 4$. If we take this one step further, we can see that if $a_i = (4K + 3) + 0.5$ for some positive integer K, then $a_{i-1} = x + 0.5$ when x is $5 \mod 8$. This suggests us to consider the number of powers of 2 dividing $M - 1$.

More formally, assume $M - 1 = 2^N \cdot P$, where N is a nonnegative integer and P is odd. Then,

$$a_1 = M(M + \frac{1}{2}) = \frac{1}{2} + 2^{2N} \cdot P^2 + 5 \cdot 2^{N-1} \cdot P + 1 = \frac{1}{2} + 1 + (2^{N-1})(2^{N+1}P^2 + P),$$

and we note that the number of powers of 2 dividing $2^{N-1}(2^{N+1}P^2 + P)$ is precisely $N - 1$ because P is odd by assumption. This means that we have reduced the number of powers of 2 dividing $M - 1$ after going one step forward, which means that as long as $M \neq 1$, we will eventually reach an integer. □

We work through a few example problems about sequences, which are typically problems where a sequence satisfies some set of conditions, and for which one desires to prove another condition about the sequence.

Problem 2. *(IMO Shortlist 2001) Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}(1 + n^{-1})$ holds for infinitely many positive integers n.*

Proof. For this problem, the sequence does not have any conditions at all, so it is definitely impossible to force the inequality to hold for any specific number n. Indeed, given any finite subset S

of the natural numbers, it is also impossible to force the inequality to hold for at least one of the indices in S. This motivates an approach by contradiction: we assume that there is some sequence a_i where the inequality only holds for finitely many integers n, and then attempt to arrive at a contradiction.

This is now equivalent to asserting that the inequality is eventually false; that is, there is some N for which the inequality is false for $n > N$. Thus, we have reduced the problem to showing that there exists no positive sequence for which

$$1 + a_n \leq a_{n-1}(1 + n^{-1})$$

for all $n > N$. Now note that given any such sequence a_i, we may consider the sequence b_i for which $a_N = b_N$ and $1 + b_n = b_{n-1}(1 + n^{-1})$ for all $n > N$. The motivation here is that we are no worse off than before, because we can show that $b_i > a_i$ for all $i > N$ inductively, and so if we can force a to be eventually negative, we must be able to do the same for b.

We have now reduced the problem to analyzing the recursion $b_n = b_{n-1}(1 + n^{-1}) - 1$ starting with some initial condition for b_N and showing that the terms are eventually negative. We now simply write out the closed form for b_{N+k}:

$$\begin{aligned} b_{N+k} &= \prod_{i=1}^{k}(1 + (N+i)^{-1})b_N - \sum_{i=1}^{k}\sum_{j=i}^{k}(1 + (N+j)^{-1}) \\ &= \frac{N+1+k}{N+1}b_N - \sum_{i=1}^{k}\frac{N+1+k}{N+j}. \end{aligned}$$

We will now arrive at a contradiction if we can find some k for which b_{N+k} is negative. But this intuitively should happen because the first term in the equation above increases linearly in k and the second term increases by an arbitrarily large amount when k increases by 1. $\square$

Problem 3. *(Turkey TST 2016) A sequence of real numbers $a_0, a_1, \ldots$ satisfies the condition*

$$\sum_{n=0}^{m} a_n(-1)^n \binom{m}{n} = 0$$

for all large enough positive integers m. Prove that there exists a polynomial P such that $a_n = P(n)$ for all $n \geq 0$.

Proof. We start by defining N such that

$$\sum_{n=0}^{m} a_n(-1)^n \binom{m}{n} = 0$$

for all $m > N$. Now, we know that these equations determine a_m inductively for all $m > N$ given $a_0, a_1, \ldots, a_N$. Intuitively, this means that we may start with a polynomial P such that $a_n = P(n)$ for $n \leq N$ and hope that this polynomial also satisfies the condition for $n > N$. However, we know that we need some more constraints on the polynomial P: for instance, you can always find a polynomial P that takes on any finite number of values. So we have to figure out what the other constraints are.

If we try doing the problem for $N = 1$, then we have

$$a_2 = 2a_1 - a_0$$

$$a_3 = 3a_2 - 3a_1 + a_0 = 3a_1 - 2a_0,$$

$$a_4 = 4a_3 - 6a_2 + 4a_1 - a_0 = 4a_1 - 3a_0,$$

which suggests that the correct polynomial is linear: $P(n) = n(a_1 - a_0) + a_0$. Similarly, for $N = 2$, we observe

$$a_3 = 3a_2 - 3a_1 + a_0,$$

$$a_4 = 4a_3 - 6a_2 + 4a_1 - a_0 = 6a_2 - 8a_1 + 3a_0,$$

$$a_5 = 5a_4 - 10a_3 + 10a_2 - 5a_1 + a_0 = 10a_2 - 15a_1 + 6a_0,$$

suggesting that the correct polynomial is quadratic:

$$P(n) = \frac{(n-1)(n-2)}{2}a_0 - n(n-2)a_1 + \frac{n(n-1)}{2}a_2,$$

where I have written the formula in the above form to more clearly satisfy that $P(n) = a_n$ for $n = 0, 1, 2$.

These two small cases suggest that we want our polynomial P to be of degree at most N in the general case. It turns out that there is a unique polynomial of degree at most N such that $P(n) = a_n$ for $n \leq N$ (going through $N + 1$ points), but we only need the existence. Call such a polynomial P. We now wish to show that $P(N + 1) = a_{N+1}$, and this would finish the problem because we now have a polynomial with degree at most $N + 1$ going through the first $N + 2$ points and we can apply the same procedure again.

Put more precisely, we wish to show that whenever P is a polynomial of degree at most N, we have

$$\sum_{n=0}^{N+1} P(n)(-1)^n \binom{N+1}{n} = 0.$$

In essence, we have reduced the problem from being one about sequences with lots of conditions to a problem about polynomials. We can now finish the proof by induction on N: for instance, we can rewrite

$$\sum_{n=0}^{N+1} P(n)(-1)^n \binom{N+1}{n} = \sum_{n=0}^{N} (P(n+1) - P(n))(-1)^{n+1} \binom{N}{n}$$

by noting that $\binom{N+1}{n} = \binom{N}{n-1} + \binom{N}{n}$, and then noting that $P(n + 1) - P(n)$ is a polynomial of degree at most $N - 1$ in n.

$\square$

2 Olympiad Problems

1. (Belarus TST 2019) Let the sequence (a_n) be constructed in the following way:

$$a_1 = 1,\ a_2 = 1,\ a_{n+2} = a_{n+1} + \frac{1}{a_n},\ n = 1, 2, \ldots.$$

Prove that $a_{180} > 19$.

2. (Vietnam 2018) The sequence (x_n) is defined as follows:

$$x_1 = 2,\ x_{n+1} = \sqrt{x_n + 8} - \sqrt{x_n + 3}$$

for all $n \geq 1$. For every $n \geq 1$, prove that

$$n \leq x_1 + x_2 + \cdots + x_n \leq n + 1.$$

3. (IMO Shortlist 2004) Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$\left(f(m)^2 + f(n)\right) \mid \left(m^2 + n\right)^2$$

for any two positive integers m and n.

4. (Taiwan 2000) Define a function $f : \mathbb{N} \to \mathbb{N}_0$ by $f(1) = 0$ and

$$f(n) = \max_j \{f(j) + f(n - j) + j\} \quad \forall\, n \geq 2$$

Determine $f(2000)$.

5. (Belarus 2017) Find all positive real numbers α such that there exists an infinite sequence of positive real numbers $x_1, x_2, \ldots$, such that

$$x_{n+2} = \sqrt{\alpha x_{n+1} - x_n}$$

for all $n \geq 1$.

6. (IMO 2014) Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

7. (Turkey 2019) Let $f : \{1, 2, \ldots, 2019\} \to \{-1, 1\}$ be a function, such that for every $k \in \{1, 2, \ldots, 2019\}$, there exists an $\ell \in \{1, 2, \ldots, 2019\}$ such that

$$\sum_{i \in \mathbb{Z} : (\ell - i)(i - k) \geqslant 0} f(i) \leqslant 0.$$

Determine the maximum possible value of

$$\sum_{i \in \mathbb{Z} : 1 \leqslant i \leqslant 2019} f(i).$$

8. (Taiwan 1999) Let $a_1, a_2, ..., a_{1999}$ be a sequence of nonnegative integers such that for any i, j with $i + j \leq 1999$, $a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$. Prove that there exists a real number x such that $a_n = [nx] \forall n$.

9. (IMO Shortlist 2008) Let a_1, a_2, $\ldots$, a_n be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1$, $3a_2$, $\ldots$, $3a_n$.

10. (USA 2007) Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k - 1$ for which $a_1 + a_2 + ... + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, ...$ Prove that for any n the sequence $a_1, a_2, ...$ eventually becomes constant.

11. (IMO Shortlist 2015) Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

12. (IMO Shortlist 2004) Let a_0, a_1, a_2, ... be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where a_0 and a_1 are two different positive reals. Prove that the sequence is unbounded.

13. (Turkey 2005) Suppose that a sequence $(a_n)_{n=1}^{\infty}$ of integers has the following property: For all sufficiently large n, a_n equals the number of indices i, $1 \leq i < n$, such that $a_i + i \geq n$. Find the maximum possible number of integers which occur infinitely many times in the sequence.

14. (IMO 2015) The sequence $a_1, a_2, \ldots$ of integers satisfies the conditions:

    (i) $1 \leq a_j \leq 2015$ for all $j \geq 1$, (ii) $k + a_k \neq \ell + a_\ell$ for all $1 \leq k < \ell$.

    Prove that there exist two positive integers b and N for which

$$\left| \sum_{j=m+1}^{n} (a_j - b) \right| \leq 1007^2$$

for all integers m and n such that $n > m \geq N$.

15. (ISL 2012) Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

16. (China 2011) Given integer $n \geq 2$ and real numbers $x_1, x_2, \cdots, x_n$ in the interval $[0, 1]$. Prove that there exist real numbers $a_0, a_1, \cdots, a_n$ satisfying the following conditions:

    - $a_0 + a_n = 0$;
    - $|a_i| \leq 1$, for $i = 0, 1, \cdots, n$;
    - $|a_i - a_{i-1}| = x_i$, for $i = 1, 2, \cdots, n$.

3 One-variable Linear Recurrences

A special setup is the one-variable linear recurrence, which has a closed form solution. The following few problems are designed to derive the full solution for a one-variable linear recurrence.

Definition 1. *A one-variable linear recurrence of degree j is a sequence $a_1, a_2, \ldots,$ satisfying the following property for $n \geq j$:*

$$a_n = c_{j-1}a_{n-1} + c_{j-2}x_{n-2} + \cdots + c_0 a_{n-j},$$

such that $c_0, c_1, \ldots, c_{j-1}$ are constants and $c_0 \neq 0$.

Our goal is to determine a_n in terms of $a_j, a_{j-1}, \ldots, a_1$ in a simple manner. We will work through some simple examples and gradually increase the generality until we can solve the problem in the generality of Definition 1.

Exercise 1. *You already know how to solve one-variable linear recurrences of degree 1. Similarly, you know how to solve a special type of linear recurrence of degree 2: when we have the recurrence $a_n = c_0 a_{n-2}$. Now, let's consider the recurrence*

$$a_n = 3a_{n-1} - 2a_{n-2}, \tag{1}$$

and let's try to do the same thing. We add the equation

$$ca_{n-1} = 3ca_{n-2} - 2ca_{n-3},$$

to (1) *get the equation*

$$a_n - (3-c)a_{n-1} = (3c-2)a_{n-2} - 2ca_{n-3}. \tag{2}$$

When $c = 2$, can we separate the recurrence (2) *into two second-order recurrences? In particular, what if we start with the initial values $a_1 = 1, a_2 = 2$?*

Exercise 2. *Is there another value of c for which we can do the same thing? What initial values are needed there?*

Exercise 3. *The word* linear *in the phrase* one-variable linear recurrence *comes from the fact that the recurrence relations are linear equations in the a_i. In particular, this means that given two solutions $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ satisfying the same one-variable linear recurrence, for any constants c and d, we have that the sequence $ca_1 + db_1, ca_2 + db_2, \ldots$ satisfies the same recurrence. There is also nothing special about there being only two solutions; the same property holds for any* linear combination *of solutions of the same one-variable linear recurrence.*

Going back to (2)*, let's try to solve the equation given general a_1 and a_2. What if we let $a_1 = (2a_1 - a_2) + (a_2 - a_1)$ and $a_2 = (2a_1 - a_2) + 2(a_2 - a_1)$? Can you find recurrences b_i and c_i that you know how to solve, and that add up to a_i?*

Exercise 4. *The Fibonacci sequence is given by the recurrence*

$$a_n = a_{n-1} + a_{n-2},$$

with initial values $a_1 = 1, a_2 = 2$. Use the same procedure as Exercises 1 - 3 to get the closed form for the Fibonacci sequence.

Exercise 5. *There is another way to derive the same closed form. Given the recurrence relation in* (1)*, we may rewrite it as*

$$a_n - 2a_{n-1} = a_{n-1} - 2a_{n-2}.$$

This suggests the substitution $b_{n-1} = a_n - 2a_{n-1}$*, such that the recurrence relation for the* b_i *is* $b_n = b_{n-1}$*, and the initial value is* $b_1 = a_2 - 2a_1$*. We can then get a closed form for the* b_i*. From here, how do we get the closed form for the* a_i*?*

Does anything change if we rewrite Equation (1) *as*

$$a_n - a_{n-1} = 2(a_{n-1} - a_{n-2}), \tag{3}$$

and substitute $b_{n-1} = a_n - a_{n-1}$*?*

Exercise 6. *What happens when we have the recurrence* $a_n = 4a_{n-1} - 4a_{n-2}$*? Note that the procedure in Exercises 1 - 3 do not generalize to this case, but we can still do it with a procedure similar to that of 5. What is the closed form of the recurrence?*

Exercise 7. *To summarize the intuition from the previous exercises, we note that in Equation* (1)*, we have closed forms* $b_i = 1$ *and* $c_i = 2^{i-1}$ *for initial conditions* $(b_1, b_2) = (1, 1)$ *and* $(c_1, c_2) = (1, 2)$*. Using linearity, we can extend these two special solutions to a solution for all such initial conditions. But for* (3)*, we do not have these closed forms. Can you come up with other simple closed forms for special initial values?*

Exercise 8. *Now let's take the general recursion*

$$a_n = c_{j-1}a_{n-1} + c_{j-2}x_{n-2} + \cdots + c_0 a_{n-j},$$

and define the characteristic polynomial *of this recurrence as the polynomial*

$$p(x) = x^j - c_{j-1}x^{n-1} - c_{j-2}x^{n-2} - \cdots - c_0.$$

We let this polynomial have roots $r_1, r_2, \ldots, r_k$ *with multiplicity* $m_1, m_2, \ldots, m_k$*, so that*

$$p(x) = \prod_{i=1}^{k}(x - r_i)^{m_i}.$$

Can you find j *distinct* j*-tuples of values* $(a_1, a_2, \ldots, a_j)$ *for which the recurrence admits a closed form? For example, for any* r_i*, the* j*-tuple* $(1, r_i, r_i^2, \ldots, r_i^{j-1})$ *is one such tuple.*

Exercise 9. *It remains to show that for any tuple of initial values* $(a_1, a_2, \ldots, a_j)$*, we can find some linear combination of the* j j*-tuples of initial values above that sum up to* $(a_1, a_2, \ldots, a_j)$*. Prove this.*

Exercise 10. *Given a one-variable linear recurrence, when can you say that it is bounded? (that is, there exists a value* M *such that each of the elements of the sequence has absolute value less than* M*)*