Mock Olympiad

Time Limit: 4 hours

1. Let p be an odd prime number. For each positive integer k with $1 \le k \le p-1$, let a_k be the number of divisors of $kp+1$ strictly between k and p. Find the value of $a_1 + a_2 + \cdots + a_{p-1}$.

2. A quadrilateral $ABCD$ is inscribed in a circle ω. The tangent to ω at A intersects the ray CB at K, and the tangent to ω at B intersects the ray DA at M. Prove that if $AM = AD$ and $BK = BC$, then $ABCD$ is a trapezoid.

3. Let p and q be prime numbers with $q = 2p+1$. Prove that there is some multiple of q such that the sum of its decimal digits is at most 3.

4. Let $a_1, a_2, \ldots, a_m$ be nonzero real numbers satisfying
$$1^k \cdot a_1 + 2^k \cdot a_2 + \cdots + m^k \cdot a_m = 0$$
for each $k = 0, 1, \ldots, n$. Prove that the sequence $a_1, a_2, \ldots, a_m$ changes sign at least $n+1$ times.

5. An $n \times n \times n$ cube is divided into unit cubes. We are given a closed non-self-intersecting polygon in space, the sides of which join centres of two unit cubes sharing a common face. The faces of unit cubes which intersect the polygon are said to be distinguished. Prove that the edges of the unit cubes may be coloured in two colours so that each distinguished face has an odd number of edges of each colour, while each undistinguished face has an even number of edges of each colour.