Invariants

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Problems – June 13

1. A square table 3×3 is filled with numbers $0, 1, \ldots, 8$ in this order. In every step we can choose two adjacent cells and the numbers in them both decrease by the smaller of them, without changing the others. Can we obtain the zero-table after several steps?

2. Several real numbers are written on the board. Whenever there are two numbers a and b with $a - b > 100$, we can switch them to $a - 1$ and $b + 1$. Prove that the process will stop after finitely many steps.

3. We start with n numbers equal to 1. At a time, we choose two numbers x and y and change them by the single number $\frac{x+y}{4}$. After $n-1$ steps there is only one number left. Prove that this number is not less than $\frac{1}{n}$.

4. Initially, the numbers $0, 1, \sqrt{2}$ are written on the board. In each step we increase or decrease one of these numbers by the difference of the other two, multiplied by a rational number. Can we obtain the numbers $0, 2, \sqrt{2}$ after several steps?

5. On the board are written 2^m units. In one step we choose two numbers a and b and change both to their sum. Prove that after $m \cdot 2^{m-1}$ steps the sum of all numbers will be at least 4^m.

6. At each vertex of a pentagon there is an integer, and the sum of all five integers is positive. As long as there are negative numbers, we may choose three adjacent numbers x, y, z $(y < 0)$ and switch them to $x + y, -y, z + y$. Prove that this process will complete after finitely many steps.

7. We are initially given the polynomials $x, x^3, \ldots, x^{2k+1}, \ldots$ on the board. If in some moment we have $f(x)$ and $g(x)$, we may append the list with any of the polynomials $af(x) + b$ $(a, b \in \mathbb{R},\ a \geq 0)$, $f(x) + g(x)$ or $f(g(x))$. Can we obtain the polynomial $x^{2011} - 20x + 11$?

8. There are 101^2 light bulbs arranged in a square lattice 101×101. Initially, k bulbs are on. In each step we choose four bulbs forming a unit square, and if three of the bulbs are on, we turn the fourth on as well. For which smallest k is there an initial arrangement for which we can finally turn all the bulbs on?

9. The numbers $1, 2, \ldots, n$ are arranged on a circle in some order. Whenever b is the right neighbor of a and $b \leqslant a - 2$, we can switch their positions. Prove that after at most $\binom{n}{3}$ switches no more moves will be possible.

10. There are 100 pirates sitting at a round table, with the captain holding all n gold coins. At each step, if some pirates hold more than one coin, one of them will give a coin to each of his two neighbors.
    (a) If $n \geqslant 100$, prove that the pirates will never stop.
    (b) If $n < 100$, prove that they must stop at some moment.

11. On a rectangular board $m \times n$ we play a game with markers having one side black and the other side white. Initially, on each of the mn squares there is a marker with the white side up, except for one marker at a corner with the black side up. A move consists of picking a black marker and turning all adjacent markers on the other side. For which m and n can we remove all markers?

12. There are $2n - 1$ light bulbs in a line. Initially the one in the middle is on and the others are off. At each step we can choose two nonadjacent bulbs that are off, under the condition that the bulbs between them must all be on, and change the state of all of them (thus, e.g. the configuration $\bullet \circ \circ \circ \bullet$ changes to $\circ \bullet \bullet \bullet \circ$). At most how many steps can we perform?