Some questions about sequences!

1. (Really a functional equation question, but oh well) Let $a_0, a_1, a_2, \cdots$ be a sequence such that $a_1 = 1$ and $a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$ whenever $m \geq n \geq 0$. Determine a_{2021}.

2. (Australia 1976) Let $a_1, a_2 \in \mathbb{Z}$ such that $0 < a_1, a_2 < 10000$. For $k \geq 3$, (recursively) define $a_k = \min_{1 \leq i < j < k} |a_i - a_j|$. Find the smallest positive integer N such that $x_N = 0$, regardless of a_1, a_2.

3. Fix $0 \leq a_0 \leq 1$. (Recursively) define $a_{n+1} = 1 - |1 - 2a_n|$ whenever $n \geq 0$. Characterize those a_0 such that the sequence $a_0, a_1, a_2, \cdots$ is periodic.

4. (IMO 1994 Shortlist) Let $a_0 = 1994$, and for $n \geq 0$ define $a_{n+1} = \dfrac{a_n^2}{a_n + 1}$. Prove that, for $0 \leq k \leq 998$, we have that the floor of a_k is equal to $1994 - n$.

5. (Found this last minute, no idea how to do it but haven't tried) Fix $a \in (0, 1)$ and fix n to be a positive integer. Recursively define $a_0, a_1, \cdots, a_n$ as follows:

$$
\begin{aligned}
a_0 &= a \\
a_{k+1} &= a_k + \frac{1}{n} a_k^2 \qquad \text{for } 0 \leq k \leq n-1
\end{aligned}
$$

   Prove there exists a number A, which may depend on a but not on n, such that $0 < n(A - a_n) < A^3$.

6. (IMO 1995 (in Canada!)) We are given $x_0, x_1, \cdots, x_{1994}, x_{1995} \in \mathbb{R}$ such that $x_0 = x_{1995}$ and $x_{i-1} + \dfrac{2}{x_{i-1}} = 2x_i + \dfrac{1}{x_i}$ for all $1 \leq i \leq 1995$. Determine the maximal possible value of x_0.

7. (IMO 1982, # 3) We are given a sequence $x_0, x_1, x_2, \cdots$ of positive real numbers such that $x_0 = 1$ and $x_{i+1} \leq x_i$ for all $i \geq 0$. Prove there exists $N \in \mathbb{N}$ such that $\dfrac{x_0^2}{x_1} + \dfrac{x_1^2}{x_2} + \cdots + \dfrac{x_{N-1}^2}{x_N} \geq 3.999$.