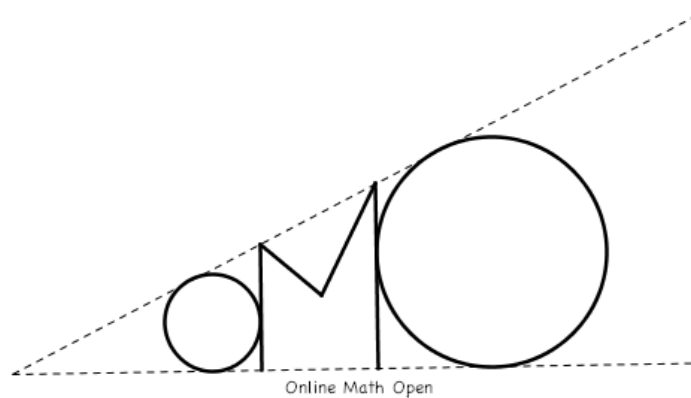


The Online Math Open Spring Contest

April 3 - 14, 2015



Acknowledgements

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at OnlineMathOpenTeam@gmail.com.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student can belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.

Only one member on each team needs to register an account on the website. Please check the website, http://internetolympiad.org/pages/14-omo_info, for registration instructions.

Note: when we say “up to four”, we really do mean “up to”! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2015 Spring Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

April 3 - 14, 2015

from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs is prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.
2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.
3. Members of different teams cannot communicate with each other about the contest while the contest is running.
4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the “hardest” problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)
5. *Participation in the Online Math Open is free.*

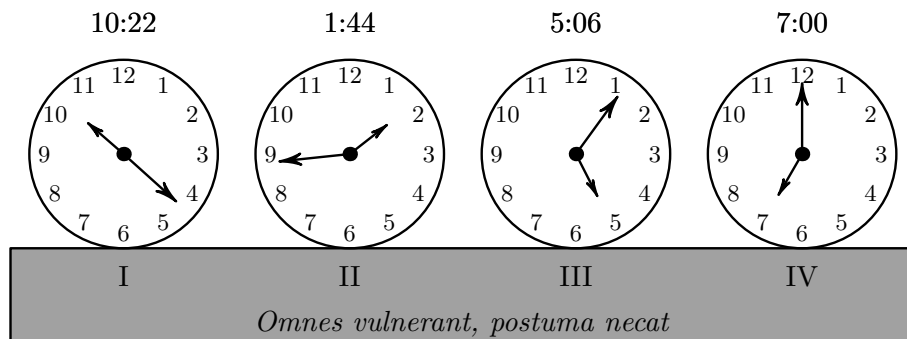
Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at http://internetolympiad.org/pages/n/omo_problems. If you have a question about problem wording, please email OnlineMathOpenTeam@gmail.com with “Clarification” in the subject. We have the right to deny clarification requests that we feel we cannot answer.

After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to OnlineMathOpenTeam@gmail.com. (Include “Protest” in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

OMO Spring 2015
April 3 - 14, 2015

1. What is the largest positive integer which is equal to the sum of its digits?
2. A classroom has 30 students, each of whom is either male or female. For every student S , we define his or her *ratio* to be the number of students of the opposite gender as S divided by the number of students of the same gender as S (including S). Let Σ denote the sum of the ratios of all 30 students. Find the number of possible values of Σ .
3. On a large wooden block there are four twelve-hour analog clocks of varying accuracy. At 7PM on April 3, 2015, they all correctly displayed the time. The first clock is accurate, the second clock is two times as fast as the first clock, the third clock is three times as fast as the first clock, and the last clock doesn't move at all. How many hours must elapse (from 7PM) before the times displayed on the clocks coincide again? (The clocks do not distinguish between AM and PM.)



4. Find the sum of all distinct possible values of $x^2 - 4x + 100$, where x is an integer between 1 and 100, inclusive.
5. Let ABC be an isosceles triangle with $\angle A = 90^\circ$. Points D and E are selected on sides AB and AC , and points X and Y are the feet of the altitudes from D and E to side BC . Given that $AD = 48\sqrt{2}$ and $AE = 52\sqrt{2}$, compute XY .
6. We delete the four corners of a 8×8 chessboard. How many ways are there to place eight non-attacking rooks on the remaining squares?
7. A geometric progression of positive integers has n terms; the first term is 10^{2015} and the last term is an odd positive integer. How many possible values of n are there?
8. Determine the number of sequences of positive integers $1 = x_0 < x_1 < \dots < x_{10} = 10^5$ with the property that for each $m = 0, \dots, 9$ the number $\frac{x_{m+1}}{x_m}$ is a prime number.
9. Find the sum of the decimal digits of the number

$$5 \sum_{k=1}^{99} k(k+1)(k^2+k+1).$$

10. Nicky has a circle. To make his circle look more interesting, he draws a regular 15-gon, 21-gon, and 35-gon such that all vertices of all three polygons lie on the circle. Let n be the number of distinct vertices on the circle. Find the sum of the possible values of n .
11. Let S be a set. We say S is D^* -finite if there exists a function $f : S \rightarrow S$ such that for every nonempty proper subset $Y \subsetneq S$, there exists a $y \in Y$ such that $f(y) \notin Y$. The function f is called a *witness* of S . How many witnesses does $\{0, 1, \dots, 5\}$ have?
12. At the Intergalactic Math Olympiad held in the year 9001, there are 6 problems, and on each problem you can earn an integer score from 0 to 7. The contestant's score is the *product* of the scores on the 6 problems, and ties are broken by the sum of the 6 problems. If 2 contestants are still tied after this, their ranks are equal. In this olympiad, there are $8^6 = 262144$ participants, and no two get the same score on every problem. Find the score of the participant whose rank was $7^6 = 117649$.

OMO Spring 2015
April 3 - 14, 2015

13. Let ABC be a scalene triangle whose side lengths are positive integers. It is called *stable* if its three side lengths are multiples of 5, 80, and 112, respectively. What is the smallest possible side length that can appear in any stable triangle?
14. Let $ABCD$ be a square with side length 2015. A disk with unit radius is packed neatly inside corner A (i.e. tangent to both \overline{AB} and \overline{AD}). Alice kicks the disk, which bounces off \overline{CD} , \overline{BC} , \overline{AB} , \overline{DA} , \overline{DC} in that order, before landing neatly into corner B . What is the total distance the center of the disk travelled?
15. Let a, b, c , and d be positive real numbers such that

$$a^2 + b^2 - c^2 - d^2 = 0 \quad \text{and} \quad a^2 - b^2 - c^2 + d^2 = \frac{56}{53}(bc + ad).$$

Let M be the maximum possible value of $\frac{ab+cd}{bc+ad}$. If M can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $100m + n$.

16. Joe is given a permutation $p = (a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$. A *swap* is an ordered pair (i, j) with $1 \leq i < j \leq 5$, and this allows Joe to swap the positions i and j in the permutation. For example, if Joe starts with the permutation $(1, 2, 3, 4, 5)$, and uses the swaps $(1, 2)$ and $(1, 3)$, the permutation becomes

$$(1, 2, 3, 4, 5) \rightarrow (2, 1, 3, 4, 5) \rightarrow (3, 1, 2, 4, 5).$$

Out of all $\binom{5}{2} = 10$ swaps, Joe chooses 4 of them to be in a set of swaps S . Joe notices that from p he could reach any permutation of $(1, 2, 3, 4, 5)$ using only the swaps in S . How many different sets are possible?

17. Let A, B, M, C, D be distinct points on a line such that $AB = BM = MC = CD = 6$. Circles ω_1 and ω_2 with centers O_1 and O_2 and radius 4 and 9 are tangent to line AD at A and D respectively such that O_1, O_2 lie on the same side of line AD . Let P be the point such that $PB \perp O_1M$ and $PC \perp O_2M$. Determine the value of $PO_2^2 - PO_1^2$.
18. Alex starts with a rooted tree with one vertex (the root). For a vertex v , let the size of the subtree of v be $S(v)$. Alex plays a game that lasts nine turns. At each turn, he randomly selects a vertex in the tree, and adds a child vertex to that vertex. After nine turns, he has ten total vertices. Alex selects one of these vertices at random (call the vertex v_1). The expected value of $S(v_1)$ is of the form $\frac{m}{n}$ for relatively prime positive integers m, n . Find $m + n$.

Note: In a rooted tree, the subtree of v consists of its indirect or direct descendants (including v itself).

19. Let ABC be a triangle with $AB = 80, BC = 100, AC = 60$. Let D, E, F lie on BC, AC, AB such that $CD = 10, AE = 45, BF = 60$. Let P be a point in the plane of triangle ABC . The minimum possible value of $AP + BP + CP + DP + EP + FP$ can be expressed in the form $\sqrt{x} + \sqrt{y} + \sqrt{z}$ for integers x, y, z . Find $x + y + z$.
20. Consider polynomials P of degree 2015, all of whose coefficients are in the set $\{0, 1, \dots, 2010\}$. Call such a polynomial *good* if for every integer m , one of the numbers $P(m) - 20, P(m) - 15, P(m) - 1234$ is divisible by 2011, and there exist integers m_{20}, m_{15}, m_{1234} such that $P(m_{20}) - 20, P(m_{15}) - 15, P(m_{1234}) - 1234$ are all multiples of 2011. Let N be the number of good polynomials. Find the remainder when N is divided by 1000.
21. Let $A_1 A_2 A_3 A_4 A_5$ be a regular pentagon inscribed in a circle with area $\frac{5+\sqrt{5}}{10}\pi$. For each $i = 1, 2, \dots, 5$, points B_i and C_i lie on ray $\overrightarrow{A_i A_{i+1}}$ such that

$$B_i A_i \cdot B_i A_{i+1} = B_i A_{i+2} \quad \text{and} \quad C_i A_i \cdot C_i A_{i+1} = C_i A_{i+2}^2$$

where indices are taken modulo 5. The value of $\frac{[B_1 B_2 B_3 B_4 B_5]}{[C_1 C_2 C_3 C_4 C_5]}$ (where $[P]$ denotes the area of polygon P) can be expressed as $\frac{a+b\sqrt{5}}{c}$, where a, b , and c are integers, and $c > 0$ is as small as possible. Find $100a + 10b + c$.

OMO Spring 2015
April 3 - 14, 2015

22. For a positive integer n let $n\#$ denote the product of all primes less than or equal to n (or 1 if there are no such primes), and let $F(n)$ denote the largest integer k for which $k\#$ divides n . Find the remainder when $F(1) + F(2) + F(3) + \cdots + F(2015\# - 1) + F(2015\#)$ is divided by 3999991.
23. Let $N = 12!$ and denote by X the set of positive divisors of N other than 1. An *pseudo-ultrafilter* U is a nonempty subset of X such that for any $a, b \in X$:
- If a divides b and $a \in U$ then $b \in U$.
 - If $a, b \in U$ then $\gcd(a, b) \in U$.
 - If $a, b \notin U$ then $\text{lcm}(a, b) \notin U$.

How many such pseudo-ultrafilters are there?

24. Suppose we have 10 balls and 10 colors. For each ball, we (independently) color it one of the 10 colors, then group the balls together by color at the end. If S is the expected value of the square of the number of distinct colors used on the balls, find the sum of the digits of S written as a decimal.
25. Let $V_0 = \emptyset$ be the empty set and recursively define V_{n+1} to be the set of all $2^{|V_n|}$ subsets of V_n for each $n = 0, 1, \dots$. For example

$$V_2 = \{\emptyset, \{\emptyset\}\} \quad \text{and} \quad V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

A set $x \in V_5$ is called *transitive* if each element of x is a subset of x . How many such transitive sets are there?

26. Consider a sequence T_0, T_1, \dots of polynomials defined recursively by $T_0(x) = 2$, $T_1(x) = x$, and $T_{n+2}(x) = xT_{n+1}(x) - T_n(x)$ for each nonnegative integer n . Let L_n be the sequence of Lucas Numbers, defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_n + L_{n+1}$ for every nonnegative integer n .

Find the remainder when $T_0(L_0) + T_1(L_2) + T_2(L_4) + \cdots + T_{359}(L_{718})$ is divided by 359.

27. Let $ABCD$ be a quadrilateral satisfying $\angle BCD = \angle CDA$. Suppose rays AD and BC meet at E , and let Γ be the circumcircle of ABE . Let Γ_1 be a circle tangent to ray CD past D at W , segment AD at X , and internally tangent to Γ . Similarly, let Γ_2 be a circle tangent to ray DC past C at Y , segment BC at Z , and internally tangent to Γ . Let P be the intersection of WX and YZ , and suppose P lies on Γ . If F is the E -excenter of triangle ABE , and $AB = 544$, $AE = 2197$, $BE = 2299$, then find $m + n$, where $FP = \frac{m}{n}$ with m, n relatively prime positive integers.
28. Find the number of ordered pairs $(P(x), Q(x))$ of polynomials with integer coefficients such that

$$P(x)^2 + Q(x)^2 = (x^{4096} - 1)^2.$$

29. Let ABC be an acute scalene triangle with incenter I , and let M be the circumcenter of triangle BIC . Points D , B' , and C' lie on side BC so that $\angle BIB' = \angle CIC' = \angle IDB = \angle IDC = 90^\circ$. Define $P = \overline{AB} \cap \overline{MC'}$, $Q = \overline{AC} \cap \overline{MB'}$, $S = \overline{MD} \cap \overline{PQ}$, and $K = \overline{SI} \cap \overline{DF}$, where segment EF is a diameter of the incircle selected so that S lies in the interior of segment AE . It is known that $KI = 15x$, $SI = 20x + 15$, $BC = 20x^{5/2}$, and $DI = 20x^{3/2}$, where $x = \frac{a}{b}(n + \sqrt{p})$ for some positive integers a, b, n, p , with p prime and $\gcd(a, b) = 1$. Compute $a + b + n + p$.

30. Let S be the value of

$$\sum_{n=1}^{\infty} \frac{d(n) + \sum_{m=1}^{\nu_2(n)} (m-3)d\left(\frac{n}{2^m}\right)}{n},$$

where $d(n)$ is the number of divisors of n and $\nu_2(n)$ is the exponent of 2 in the prime factorization of n . If S can be expressed as $(\ln m)^n$ for positive integers m and n , find $1000n + m$.