Warmup Problems

Algebra

1. Consider a composition of functions sin, cos, tan, cot, arcsin, arccos, arctan, arccot, applied to the number 1. Each function may be applied arbitrarily many times and in any order. Can one obtain the number 2010 in this way?

2. At a circular track, 10 cyclists started from some point at the same time in the same direction with different constant speeds. If any two cyclists are at some point at the same time again, we say that they meet. No three or more of them have met at the same time. Prove that by the time every two cyclists have met at least once, each cyclist has had at least 25 meetings.

3. Let a_0, b_0 be positive integers, and define $a_{i+1} = a_i + \lfloor \sqrt{b_i} \rfloor$ and $b_{i+1} = b_i + \lfloor \sqrt{a_i} \rfloor$ for all $i \geq 0$. Show that there exists a positive integer n such that $a_n = b_n$.

4. Determine whether there exist two reals x, y and a sequence $\{a_n\}_{n=0}^{\infty}$ of nonzero reals such that $a_{n+2} = xa_{n+1} + ya_n$ for all $n \geq 0$ and for every positive real number r, there exist positive integers i, j such that $|a_i| < r < |a_j|$.

5. Given a set S of n variables, a binary operation $\times$ on S is called simple if it satisfies $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$ and $x \times y \in \{x, y\}$ for all $x, y \in S$. Given a simple operation $\times$ on S, any string of elements in S can be reduced to a single element, such as $xyz \to x \times (y \times z)$. A string of variables in S is called full if it contains each variable in S at least once, and two strings are equivalent if they evaluate to the same variable regardless of which simple $\times$ is chosen. For example xxx, xx, and x are equivalent, but these are only full if $n = 1$. Suppose T is a set of strings such that any full string is equivalent to exactly one element of T. Determine the number of elements of T.

Combinatorics

1. The doctor instructed a person to take 48 pills for next 30 days. Every day he take at least 1 pill and at most 6 pills. Show that there is some number of conscutive days such that the total numbers of pills he take within those days is equal to 11.

2. On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses. Which of the players, Pete or Basil, has a winning strategy?

3. Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is reachable from a string A if it is possible to insert some amount of $\leftarrow$'s in A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

   Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

4. Let $n \geq 2$ be a given integer. Initially, we write n sets on the blackboard and do a sequence of moves as follows: choose two sets A and B on the blackboard such that none of them is a subset of the other, and replace A and B by $A \cap B$ and $A \cup B$. This is called a *move*. Find the maximum number of moves in a sequence for all possible initial sets.

5. Consider the infinite grid of lattice points in $\mathbb{Z}^3$. Little D and Big Z play a game, where Little D first loses a shoe on an unmunched point in the grid. Then, Big Z munches a shoe-free plane perpendicular to one of the coordinate axes. They continue to alternate turns in this fashion, with Little D's goal to lose a shoe on each of n consecutive lattice points on a line parallel to one of the coordinate axes. Determine all n for which Little D can accomplish his goal.

Geometry

1. Prove that any circumscribed polygon (that is, a polygon with an inscribed circle tangent to all sides of the polygon) has three sides that can form a triangle.

2. In triangle ABC, let the circumcenter, incenter, and orthocenter be O, I, and H respectively. Segments AO, AI, and AH intersect the circumcircle of triangle ABC at D, E, and F. CD intersects AE at M and CE intersects AF at N. Prove that MN is parallel to BC.

3. In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

4. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$. Show that γ is tangent to the both the circumcircle and incircle of $\triangle ABC$.

5. We say a finite set S of points in the plane is very if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

   (a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

   (b) Find the largest possible size of a very set not contained in any line.

   (Here, an inversion with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Number Theory

1. Determine positive integers n, such that $3^n + n^2 + 2019$ is a perfect square.

2. Find the number of polynomials $f(x) = ax^3 + bx$ satisfying both following conditions: (i) $a, b \in \{1, 2, \ldots, 2013\}$; (ii) the difference between any two of $f(1), f(2), \ldots, f(2013)$ is not a multiple of 2013.

3. Let $a > 1$ be a positive integer. Prove that for some nonnegative integer n, the number $2^{2^n} + a$ is not prime.

4. Let P be the set of all primes, and let M be a non-empty subset of P. Suppose that for any non-empty subset $p_1, p_2, ..., p_k$ of M, all prime factors of $p_1 p_2 ... p_k + 1$ are also in M. Prove that $M = P$.

5. Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n.