موهبة

IMO Booklet

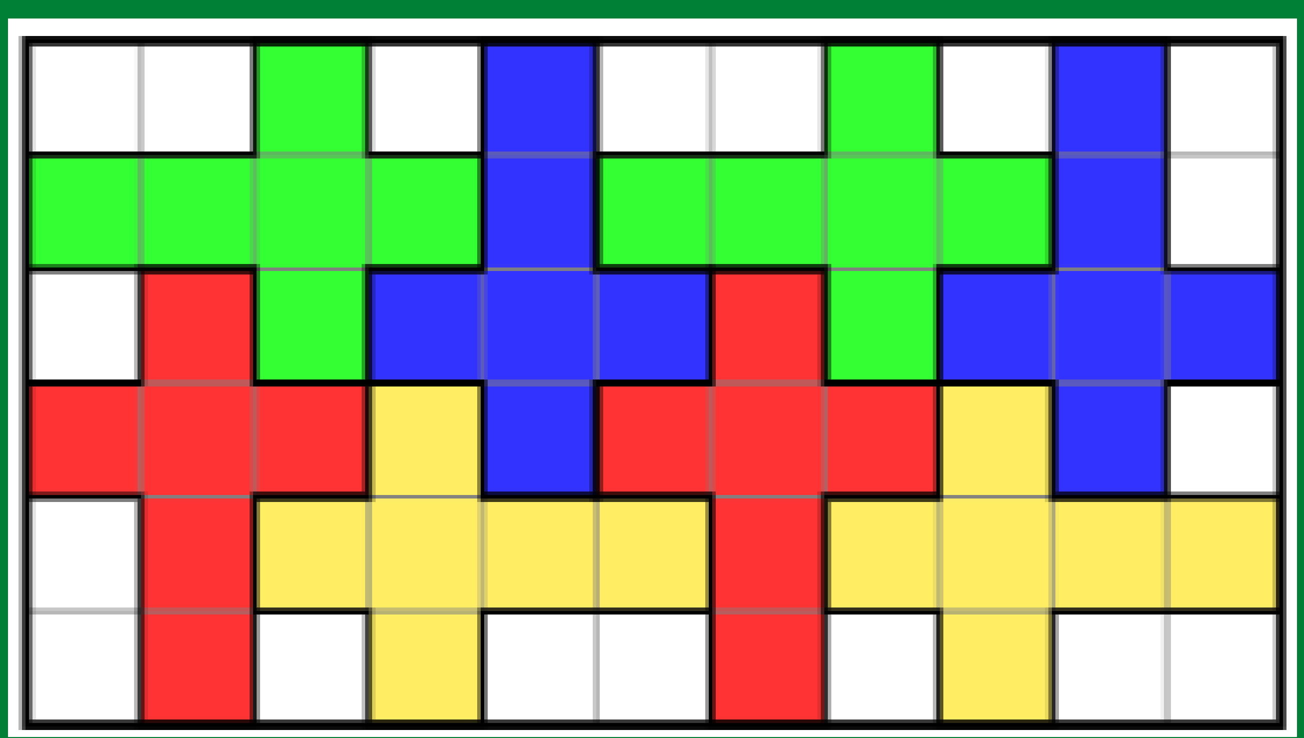
SAUDI ARABIAN MATHEMATICAL COMPETITIONS

2022

$a^2 + b^2 = c^2$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$ax^2 + bx + c = 0$



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SAUDI ARABIAN

MATHEMATICAL COMPETITIONS 2022

The King Abdulaziz and His Companions Foundation for Giftedness and Creativity organization, Saudi Arabia.

www.mawhiba.org.sa

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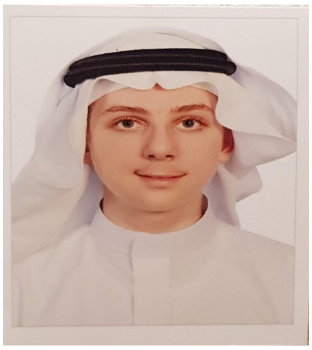
The Saudi Arabian team at **IMO 2022**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Introduction

This booklet contains the Team Selection Tests of the Saudi teams to the Balkan Mathematics Olympiad, Balkan Junior Mathematics Olympiad, and the International Mathematics Olympiad.

The training was supported by the Ministry of Education, which commissioned Mawhiba, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. We would like to express our gratitude to King Abdullah University of Science and Technology KAUST for making its facilities on its beautiful campus available to us for our training.

The Saudi team had three main training camps during the academic year 2021-2022. In addition, the team had an intensive training period from March to the end of June 2022. During this academic year, the selected students participated in the following contests: The Asia Pacific Mathematics Olympiad, the European Girls Mathematics Olympiad in Hungary, Balkan Mathematics Olympiad in Cyprus and the Junior Balkan mathematics Olympiad in Bosnia and Herzegovina.

It is our pleasure to share the selection tests problems with other IMO teams, hoping it will contribute to future cooperation.

Dr. Sultan Albarakati
Leader of the Saudi team in IMO 2022

مقدمة

يحوي هذا الكتيب على مسائل التصفيات لمسابقة البلقان ومسابقة البلقان للناشئين وتصفيات الاولمبياد الدولي للرياضيات ٢٠٢٠.

ان تدريب الفريق كان بدعم من وزارة التعليم بالتعاون مع مؤسسة الملك عبد العزيز ورجاله للموهبة والابداع " موهبة "

وتجدر الاشارة الى التعاون والاسهام الفعّال من جامعة الملك عبدالله للعلوم و التقنية، حيث وفرت لنا كل الامكانات التي احتجنا لها في التدريب في حرمها الجامعي الجميل.

تم عقد ثلاث ملتقيات تدريبية خلال العام الدراسي ٢٠٢١-٢٠٢٢ بالإضافة الى فترة التدريب المكثف التي بدأت في شهر مارس ٢٠١٩ الى نهاية شهر يونيو. كما شارك الطلبة المتميزون في العديد من المسابقات الإقليمية ومنها: اولمبياد الرياضيات لدول آسيا والباسيفيك، اولمبياد الطالبات للدول الأوربية في هنجاريا، اولمبياد البلقان في قبرص وأولمبياد الناشئين لدول البلقان في قبرص.

نأمل ان يكون محتوى هذا الكتيب إسهاماً منا لتقوية اواصر التعاون وتبادل الخبرات بيننا والدول المشاركة في الاولمبياد الدولي.

د. سلطان سعود البركاتي

رئيس الفريق السعودي للأولمبياد الدولي. ٢٠٢٢

Acknowledgement

We extend our sincere thanks and appreciation to Dr. Fawzi Al-Thukair, founder of the Saudi Mathematical Olympiad team, and head of the team from 2010 to 2020.

نتقدم بخالص الشكر والتقدير للدكتور/ فوزي الذكير مؤسس فريق أولمبياد الرياضيات
السعودي، ورئيس الفريق من عام 2010 إلى 2020 .

Selected problems from camps

1. November camp

Test 1

Problem 1. Find all positive integers k such that the product of the first k primes increased by 1 is a power of an integer (with an exponent greater than 1).

Problem 2. Consider the polynomial $f(x) = cx(x-2)$ where c is a positive real number. For any $n \in \mathbb{Z}^+$, the notation $g_n(x)$ is a composite function n times of f and assume that the equation $g_n(x) = 0$ has all of the 2^n solutions are real numbers.

1. For $c = 5$, find in terms of n, the sum of all the solutions of $g_n(x)$, of which each multiple (if any) is counted only once.

2. Prove that $c \geq 1$.

Problem 3. Given is triangle ABC with $AB > AC$. Circles o_B, o_C are inscribed in angle BAC with o_B tangent to AB at B and o_C tangent to AC at C. Tangent to o_B from C different than AC intersects AB at K, and tangent to o_C from B different than AB intersects AC at L. Line KL and the angle bisector of BAC intersect BC at points P and M, respectively. Prove that $BP = CM$.

Problem 4. At a gala banquet, $12n + 6$ chairs, where $n \in N$, are equally arranged around a large round table. A seating will be called a proper seating of rank n if a gathering of $6n + 3$ married couples sit around this table such that each seated person also has exactly one sibling (brother/sister) of the opposite gender present (siblings cannot be married to each other) and each man is seated closer to his wife than his sister. Among all proper seats of rank n find the maximum possible number of women seated closer to their brother than their husband. (The maximum is taken not only across all possible seating arrangements for a given gathering, but also across all possible gatherings.)

Test 2

Problem 5. Define $a_0 = 2$ and $a_{n+1} = a_n^2 + a_n - 1$ for $n \geq 0$. Prove that a_n is coprime to $2n + 1$ for all $n \in \mathbb{N}$.

Problem 6. Given is an acute triangle ABC with $BC < CA < AB$. Points K and L lie on segments AC and AB and satisfy $AK = AL = BC$. Perpendicular bisectors of segments CK and BL intersect line BC at points P and Q, respectively. Segments KP and LQ intersect at M. Prove that $CK + KM = BL + LM$.

Problem 7. A rectangle R is partitioned into smaller rectangles whose sides are parallel with the sides of R. Let B be the set of all boundary points of all the rectangles in the partition, including the boundary of R. Let S be the set of all (closed) segments whose points belong to B. Let a maximal segment be a segment in S which is not a proper subset of any other segment in S. Let an intersection point be a point in which 4 rectangles of the partition meet. Let m be the number od maximal segments, i the number od intersection points and r the number of rectangles. Prove that $m+i=r+3$.

Problem 8. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$2f(x)f(x+y)-f(x^2)=\frac{x}{2}(f(2x)+4f(f(y)))$$

for all $x,y\in\mathbb{R}$.

2. January camp

Test 3

Problem 1. For each non-constant integer polynomial $P(x)$, let's define

$$M_{P(x)}=\max_{x\in[0;2021]}|P(x)|.$$

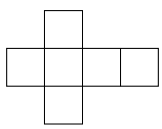
1. Find the minimum value of $M_{P(x)}$ when $\deg P(x)=1$.

2. Suppose that $P(x)\in\mathbb{Z}[x]$ when $\deg P(x)=n$ and $2\leq n\leq 2022$. Prove that $M_{P(x)}\geq 1011$.

Problem 2. Point M on side AB of quadrilateral $ABCD$ is such that quadrilaterals $AMCD$ and $BMDC$ are circumscribed around circles centered at O_1 and O_2 respectively. Line O_1O_2 cuts an isosceles triangle with vertex M from angle CMD. Prove that $ABCD$ is a cyclic quadrilateral.

Problem 3. Let p be a prime number and let m, n be integers greater than 1 such that $n\mid m^{p(n-1)}-1$. Prove that $\gcd(m^{n-1}-1,n)>1$.

Problem 4. The *sword* is a figure consisting of 6 unit squares presented in the picture below (and any other figure obtained from it by rotation).



Determine the largest number of swords that can be cut from a 6×11 piece of paper divided into unit squares (each sword should consist of six such squares).

Test 4

Problem 5. Let ABC be an acute-angled triangle. Point P is such that $AP = AB$ and $PB \parallel AC$. Point Q is such that $AQ = AC$ and $CQ \parallel AB$. Segments CP and BQ meet at point X. Prove that the circumcenter of triangle ABC lies on the circumcircle of triangle PXQ.

Problem 6. Find all positive integers n that have precisely $\sqrt{n+1}$ natural divisors.

Problem 7. Let n be an *even* positive integer. On a board n real numbers are written. In a single move we can erase any two numbers from the board and replace *each of them* with their product. Prove that for every n initial numbers one can in finite number of moves obtain n equal numbers on the board.

Problem 8. Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and satisfying

$$f(x + 2y + f(x + y)) = f(2x) + f(3y),\ \forall x, y > 0.$$

1. Find all functions $f(x)$ that satisfy the given condition.
2. Suppose that $f(4\sin^4 x)f(4\cos^4 x) \geq f^2(1)$ for all $x \in \left(0; \frac{\pi}{2}\right)$. Find the minimum value of $f(2022)$.

3. March camp

Test 5

Problem 1. By $\operatorname{rad}(x)$ we denote the product of all distinct prime factors of a positive integer n. Given $a \in \mathbb{N}$, a sequence (a_n) is defined by $a_0 = a$ and $a_{n+1} = a_n + \operatorname{rad}(a_n)$ for all $n \geq 0$. Prove that there exists an index n for which $\frac{a_n}{\operatorname{rad}(a_n)} = 2022$.

Problem 2. Determine if there exist functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying for every $x \in \mathbb{R}$ the following equations

$$f(g(x)) = x^3 \quad \text{and} \quad g(f(x)) = x^2.$$

Problem 3. We consider all partitions of a positive integer n into a sum of (non-negative integer) exponents of 2 (i.e. $1, 2, 4, 8, \ldots$). A number in the sum is allowed to repeat an arbitrary number of times (e.g. $7 = 2 + 2 + 1 + 1 + 1$) and two partitions differing only in the order of summands are considered to be equal (e.g. $8 = 4 + 2 + 1 + 1$ and $8 = 1 + 2 + 1 + 4$ are regarded to be the same partition). Let $E(n)$ be the number of partitions in which an even number of exponents appear an odd number of times and $O(n)$ the number of partitions in which an odd number of exponents appear an odd number of times. For example, for $n = 5$ partitions counted in $E(n)$ are $5 = 4 + 1$ and $5 = 2 + 1 + 1 + 1$, whereas partitions counted in $O(n)$ are $5 = 2 + 2 + 1$ and $5 = 1 + 1 + 1 + 1 + 1$, hence $E(5) = O(5) = 2$. Find $E(n) - O(n)$ as a function of n.

4. June camp

JBMO Test 1

Problem 1. The positive $n > 3$ called 'nice' if and only if $n+1$ and $8n+1$ are both perfect squares. How many positive integers $k \leq 15$ such that $4n+k$ are composites for all nice numbers n?

Problem 2. Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

Problem 3. 2000 consecutive integers (not necessarily positive) are written on the board. A student takes several turns. On each turn, he partitions the 2000 integers into 1000 pairs, and substitutes each pair by the difference and the sum of that pair (note that the difference does not need to be positive as the student may choose to subtract the greater number from the smaller one; in addition, all the operations are carried simultaneously). Prove that the student will never again write 2000 consecutive integers on the board.

Problem 4. Determine the smallest positive integer a for which there exist a prime number p and a positive integer $b \geq 2$ such that

$$\frac{a^p - a}{p} = b^2.$$

JBMO Test 2

Problem 5. Find all pairs of positive prime numbers (p, q) such that

$$p^5 + p^3 + 2 = q^2 - q.$$

Problem 6. Consider non-negative real numbers a, b, c satisfying the condition $a^2 + b^2 + c^2 = 2$. Find the maximum value of the following expression

$$P = \frac{\sqrt{b^2+c^2}}{3-a} + \frac{\sqrt{c^2+a^2}}{3-b} + a + b - 2022c.$$

Problem 7. Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 is the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.

Problem 8. You plan to organize your birthday party, which will be attended either by exactly m persons or by exactly n persons (you are not sure at the moment). You have a big birthday cake and you want to divide it into several parts (not necessarily equal), so that you are able to distribute the whole cake among the people attending the party with everybody getting cake of equal mass (however, one may get one big slice, while others several small slices - the sizes of slices may differ). What is the minimal number of parts you need to divide the cake, so that it is possible, regardless of the number of guests.

5. May camp

IMO Team Selection Test 1

Problem 1. Let (a_n) be the integer sequence which is defined by $a_1 = 1$ and

$$a_{n+1} = a_n^2 + n \cdot a_n, \ \forall n \geq 1.$$

Let S be the set of all primes p such that there exists an index i such that $p|a_i$. Prove that the set S is an infinite set and it is not equal to the set of all primes.

Problem 2. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Problem 3. Find all non-constant functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the equation

$$f(ab + bc + ca) = f(a)f(b) + f(b)f(c) + f(c)f(a)$$

for all $a, b, c \in \mathbb{Q}^+$.

IMO Team Selection Test 2

Problem 4. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ and BR are concurrent.

Problem 5. Given a positive integer n, find the smallest value of

$$\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_n}{n} \right\rfloor$$

over all permutations $(a_1, a_2, \ldots, a_n)$ of $(1, 2, \ldots, n)$.

Problem 6. The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a path from X to Y is a sequence of reaods such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called diverse if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

IMO Team Selection Test 3

Problem 7. For every integer $n \geq 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{i \cdot j}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Determine all integers $n \geq 1$ for which the sum of the n^2 entries in the table is equal to $\frac{n^2(n-1)}{4}$.

Problem 8. Let n and k be two integers with $n > k \geq 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbours - namely, the k students closest to S on the right and the k students closest to S on the left. Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.

Problem 9. Prove that there are only finitely many quadruples (a, b, c, n) of positive integers such that

$$n! = a^{n-1} + b^{n-1} + c^{n-1}.$$

IMO Team Selection Test 4

Problem 10. There are a) 2022, b) 2023 plates placed around a round table and on each of them there is one coin. Alice and Bob are playing a game that proceeds in rounds indefinitely as follows. In each round, Alice first chooses a plate on which there is at least one coin. Then Bob moves one coin from this plate to one of the two adjacent plates, chosen by him. Determine whether it is possible for Bob to select his moves so that, no matter how Alice selects her moves, there are never more than two coins on any plate.

Problem 11. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$ the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

Problem 12. Let A, B, C, D be points on the line d in that order and $AB = CD$. Denote (P) as some circle that passes through A, B with its tangent lines at A, B are a, b. Denote (Q) as some circle that passes through C, D with its tangent lines at C, D are c, d. Suppose that a cuts c, d at K, L respectively; and b cuts c, d at M, N respectively. Prove that four points K, L, M, N belong to a same circle (ω) and the common external tangent lines of circles $(P), (Q)$ meet on (ω).

Solution of BMO & EGMO TST

Test 1

Problem 1. *Find all positive integers k such that the product of the first k primes increased by 1 is a power of an integer (with an exponent greater than 1).*

Solution. Denote the first n primes as

$$p_1 = 2 < p_2 = 3 < \cdots < p_n$$

Suppose that $p_1 p_2 \cdots p_n + 1 = x^k$ for some integers $x, k \geqslant 2$. We can assume WLOG that k is prime since $x^{kt} = (x^t)^k$. Obviously, x has no prime factors not exceeding p_n, so $x > p_n$ and consequently $k < n < p_n$ is one of the first n primes. Now $x^k \equiv 1 \pmod{k}$, which implies $x \equiv 1 \pmod{k}$, but then by Lifting the Exponent Lemma

$$x^k \equiv 1 \pmod{k^2}.$$

This is a contradiction, as $x^k - 1 = p_1 \cdots p_n$ is not divisible by the square of any prime. Thus, there does not exist any k satisfying the problem. $\square$

Problem 2. *Consider the polynomial $f(x) = cx(x-2)$ where c is a positive real number. For any $n \in \mathbb{Z}^+$, the notation $g_n(x)$ is a composite function n times of f and assume that the equation $g_n(x) = 0$ has all of the 2^n solutions are real numbers.*

1) For $c = 5$, find in terms of n, the sum of all the solutions of $g_n(x)$, of which each multiple (if any) is counted only once.

2) Prove that $c \geq 1$.

Solution.

1) We will prove by induction on n that the solutions of $g_n(x)$ are all distinct. With $n = 1$, we have $g_1(x) = f(x)$ which has two solutions, $x = 0, x = 2$. Suppose the polynomial $g_n(x)$ has all 2^n solutions that are distinct, set as $x_1, x_2, \ldots, x_{2^n}$, we write $g_n(x) = k(x - x_1)(x - x_2) \ldots (x - x_{2^n})$. Hence,

$$g_{n+1}(x) = g_n(f(x)) = k(f(x) - x_1)(f(x) - x_2) \ldots (f(x) - x_{2^n})$$

so with $i \neq j$, obviously the solution set of $f(x) - x_i$ and $f(x) - x_j$ will be disjoint. Also, if the equation $f(x) - x_i = 0$ has a double root, then that solution must be 1. Next, we have

$$g_1(1) = f(1) = -5, \; g_2(1) = f(-5) = 5(-5)(-7) = 175$$

so $g_n(1) > 0, \forall n \geq 2$. Hence, $x = 1$ cannot be a solution of $g_n(x), \forall n$. This shows that $g_{n+1}(x)$ also has a distinct solution 2^{n+1}. The induction is completed.

Finally, from the above analysis, the solutions of $g_n(x)$ can be divided into 2^{n-1} disjoint pairs having sum equal 2 so the sum of all the solutions will be 2^n.

2) Note that for some parameter $d \in \mathbb{R}$, the equation

$$cx(x-2) = d \Longleftrightarrow x^2 - 2x = \frac{d}{c}$$

has two distinct solutions if and only if $\Delta' = 1 + \frac{d}{c} > 0$ or $c > -d$. ($\heartsuit$) It will then have the solutions

$$x = 1 + \sqrt{1 + \frac{d}{c}} \text{ and } x = 1 - \sqrt{1 + \frac{d}{c}}.$$

In order for $g_n(x)$ to have 2^n, $g_{n-1}(x)$ must have enough 2^{n-1} distinct roots. Denote r as one of those roots, we investigation the equation $f(x) = r$.

- If $r < 0$ then the equation has two positive solutions.
- If $r > 0$ then the equation has two solutions with opposite signs.

We see that ($\heartsuit$) generates a constraint for c only when the parameter $d < 0$. Therefore, we are interested in pairs of opposite solutions generated from $r > 0$. Suppose that pair of solutions is (r_1, r_2) with $r_1 < 0 < r_2$. We need $c > -r_1 = -(2 - r_2) = r_2 - 2$ so we take it back to the survey positive solution r_2 in that pair. Starting from an initial solution $r = 2$, we can build a sequence of numbers as follows: $u_1 = 2$ and $u_{n+1} = 1 + \sqrt{1 + \frac{u_n}{c}}$ with $n \geq 1$.

It is easy to see that u_n is the solution of $g_n(x)$ and $u_2 > u_1$ so by induction, we can prove that the sequence (u_n) increases strictly. Let L be the solution of

$$L = 1 + \sqrt{1 + \frac{L}{c}} \text{ or } L = 2 + \frac{1}{c}.$$

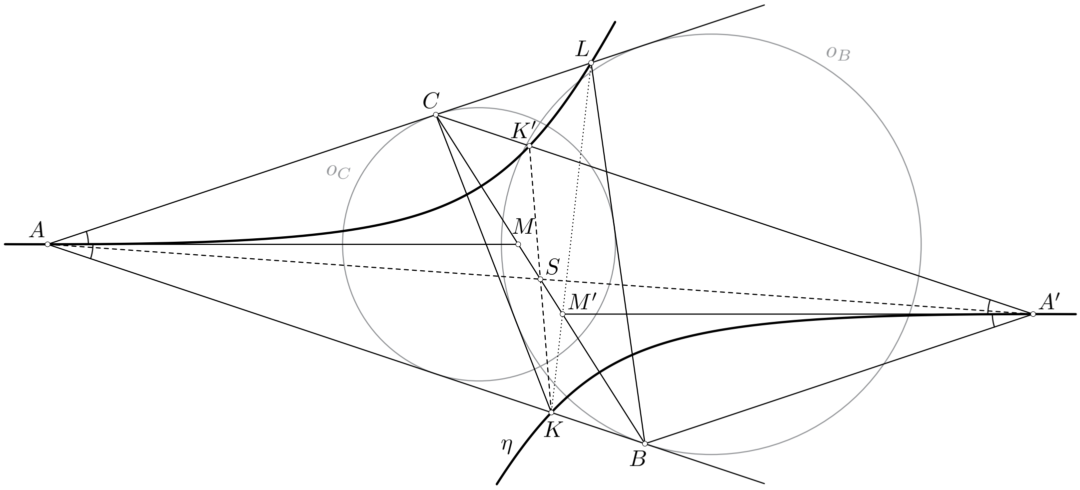
By induction, we can also prove that $u_n < L, \forall n$, so it is clear that the sequence will converge to L. Note that we must always have $c > u_n - 2, \forall n$ so let $n \to +\infty$, we get $c \geq 2 + \frac{1}{c} - 2 = \frac{1}{c}$ or $c \geq 1$. $\square$

Problem 3. *Given is triangle ABC with $AB > AC$. Circles o_B, o_C are inscribed in angle BAC with o_B tangent to AB at B and o_C tangent to AC at C. Tangent to o_B from C different than AC intersects AB at K, and tangent to o_C from B different than AB intersects AC at L. Line KL and the angle bisector of BAC intersect BC at points P and M, respectively. Prove that $BP = CM$.*

Solution. Note that the length of the segment of the common tangent to o_B and o_C joining the tangency points is equal to

$$AB - AC = LB - LC = KC - KB,$$

which means that points A, L lie on the one, and point K on the other leg of some hyperbola η with foci B, C.
Denote by S the midpoint of the segment BC (the center of symmetry of η) and denote by A', K', M' the central reflections in S of A, K, M, respectively. Then $A', K' \in \eta$ and A', K', C are collinear.



From the optical property of a hyperbola follows that $A'M'$ is tangent to η (as it is the bisector of $BA'C$). Therefore Pascal's theorem applied to the degenerate hexagon $AA'A'K'KL$ inscribed in η gives the collinearity of points

$$AA' \cap KK' = S, \quad A'A' \cap KL, \quad A'K' \cap LA = C.$$

Therefore the three lines $A'A'$ (tangent to η in A'), KL, BC are concurrent, which means that $M' = A'A' \cap BC = KL \cap BC = P$ and in consequence $CM = BP$. $\square$

Problem 4. *At a gala banquet, $12n+6$ chairs, where $n \in N$, are equally arranged around a large round table. A seating will be called a proper seating of rank n if a gathering of $6n+3$ married couples sit around this table such that each seated person also has exactly one sibling (brother/sister) of the opposite gender present (siblings cannot be married to each other) and each man is seated closer to his wife than his sister. Among all proper seats of rank n find the maximum possible number of women seated closer to their brother than their husband. (The maximum is taken not only across all possible seating arrangements for a given gathering, but also across all possible gatherings.)*

Solution. We will call a woman unusual if she sits closer to her husband than her brother. Our goal is to find the smallest possible number of unusual women. Let us call this number k. We note that going from each man to his sister and from each woman to her husband we obtain an oriented graph which breaks up into oriented cycles of even length. We also note that within an oriented cycle the sequence of lengths between consecutive members increases unless we encounter an unusual woman. Thus, each cycle must have at least one unusual woman. We also note that since the maximum length between two seats is $6n+3$, this is also the maximum distance within a cycle we can go without encountering an unusual woman. Thus, we can have neither $k = 0$ nor $k = 1$, since by the first requirement we would have only one cycle, but this cycle would then have to have more than $6n+3$ people. We will show that $k=2$ is also impossible. The only options for $k=2$ are to have either one cycle with lengths $1,2,\ldots,6n+3,1,2,\ldots,6n+3$ or two cycles with lengths $1,2,\ldots,6n+3$. We note that the second option does not work because the cycles have odd length, while the first option does not work because the two locations where the unusual women are supposed to be are at an odd distance from each other along the cycle and therefore those positions cannot be occupied by two people of the same gender.

We now give an example for $k=3$. Arrange the three unusual women in an equilateral triangle and place their brothers diametrically opposite of them. Each unusual woman is part of a cycle of length $4n+2$. If we label the members of one of these cycles in order as

$$W_1, M_1, W_2, M_2, \ldots, W_{2n+1}, M_{2n+1},$$

where W_1 is the unusual woman, then if we place W_1 at position 0, we can place each M_k, $k=1,\ldots,2n$ at position k and each W_k at position $k=1,\ldots,2n+1$ at position $-k+1$. Finally, we place M_{2n+1}, the brother of W_1, at position $6n+3$.

The other two unusual women are placed symmetrically at positions $4n+2$ and $-(4n+2)$. We note that all the conditions of the problem are satisfied for this arrangement. Thus, the maximum possible number of women seated closer to their brother than their husband is $6n+3-3=6n$. $\square$

Test 2

Problem 1. *Define $a_0=2$ and $a_{n+1}=a_n^2+a_n-1$ for $n \geq 0$. Prove that a_n is coprime to $2n+1$ for all $n \in \mathbb{N}$.*

Solution. Consider any prime divisor p of a_n. Consider a directed graph with p edges on $\{0,1,\ldots,p-1\} \pmod p$ connecting $x \to y$ if and only if

$$x^2+x-1 \equiv y \pmod p.$$

Observe that $\frac{p-1}{2}$ is connected to $b=\frac{p^2-5}{4} \pmod p$, every element has out-degree 1, and every element other than b has in-degree 2 or 0, elements ± 1 form loops, $-2 \to 1$ and $0 \to -1$. If we have a path $a_0 \to a_1 \to \cdots \to a_n$, then each of the elements has in-degree two, with at least one having in-degree one, giving $2n-1$ edges in total. Counting the four extra edges above, we deduce that $p \geqslant 2n+3$, which implies that $\gcd(a_n, 2n+1)=1$. $\square$

Problem 2. *Given is an acute triangle ABC with $BC<CA<AB$. Points K and L lie on segments AC and AB and satisfy $AK=AL=BC$. Perpendicular bisectors of segments CK and BL intersect line BC at points P and Q, respectively. Segments KP and LQ intersect at M. Prove that $CK+KM=BL+LM$.*

Solution. Let D and E be points of rays $ML^{\to}$ and $MK^{\to}$, respectively, such that $LD=AB$ and $KE=AC$. As $\angle DLA=\angle BLM=\angle LBC$, $LD=AB$ and

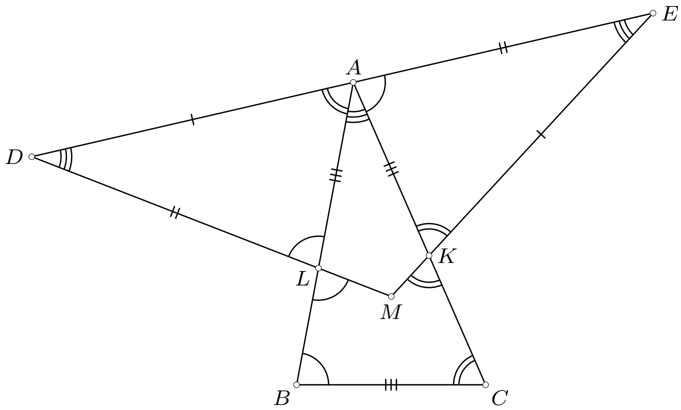
$AL = BC$, triangles DLA and ABC congruent. Analogously EAK and ABC are congruent. From

$$\angle DAL + \angle LAK + \angle KAE = \angle BCA + \angle CAB + \angle ABC = 180°,$$

follows that A belong to the segment DE. Therefore, in the light of $\angle LDA = \angle BAC = \angle AEK$, triangle MDE is isosceles with $MD = ME$. Note that

$$BL + LM = AB + LM - AL = DL + LM - AL = DM - AL,$$
$$CK + KM = AC + KM - AK = EK + KM - AK = EM - AK.$$

From $DM = EM$ and $AL = AK$ follows that the right-hand sides of the two equalities above are equal. Hence the left-hand sides are equal as well, i.e. $CK + KM = BL + LM$.



□

Problem 3. *A rectangle R is partitioned into smaller rectangles whose sides are parallel with the sides of R. Let B be the set of all boundary points of all the rectangles in the partition, including the boundary of R. Let S be the set of all (closed) segments whose points belong to B. Let a maximal segment be a segment in S which is not a proper subset of any other segment in S. Let an intersection point be a point in which 4 rectangles of the partition meet. Let m be the number od maximal segments, i the number od intersection points and r the number of rectangles. Prove that $m + i = r + 3$.*

Solution. Let a minor intersection be a point in S where exactly three rectangles meet and let the number of minor intersections be j. Let side segments be segments corresponding to a side of a rectangle in the partition and let proper segments be segments into which intersection points cut up maximal segments.

Let the number of side segments be s and the number of proper segments be p. If we start from maximal segments, we note that each addition of an intersection point forms two new segments. Ultimately, when all the intersection points are included, only proper segments remain and therefore $p = m + 2i$. We now multiply all proper segments by 2 to account for both sides of a proper segment and subtract 4 to account for the fact that the sides of the rectangle R (which are by definition proper segments) are counted only once. Thereafter, each addition of a minor intersection increases the number of segments by 1 until we get only side segments and therefore $s = 2p + j - 4$. Combining the two equations, we obtain

$$s = 2m + 4i + j - 4.$$

Now, counting rectangles by side segments we obtain $s = 4r$ and counting rectangles by their angles we obtain $4r = 4i + 2j + 4$, the final term accounting for the four corners of R. We transform the equation into $2r - 6 = 2i + j - 4$. Hamza #1 Combining all the obtained equations we get $4r = s = 2m + 2i + 2r - 6$ which gives us $2r + 6 = 2m + 2i$, i.e. $m + i = r + 3$. □

Problem 4. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$2f(x)f(x+y) - f(x^2) = \frac{x}{2}(f(2x) + 4f(f(y)))$$

for all $x, y \in \mathbb{R}$.

Solution. Put $x = 0$, we get $2f(0)f(y) - f(0) = 0$, if $f(0) \neq 0$ then $f(y) = 1/2$ for all y, which does not satisfy. Thus $f(0) = 0$.

Put $y = 0$, we get $2f(x)^2 - f(x^2) = x/2 \cdot f(2x)$ then plugging back $2f(x) \cdot f(x+y) = 2f(x)^2 + 2x \cdot f(f(y))$ so

$$f(x) \cdot f(x+y) = f(x)^2 + x \cdot f(f(y)). \qquad (\diamondsuit)$$

Note that $f(-x)^2 = x \cdot f(f(x))$. In $(\diamondsuit)$, take $y = -2x$ then

$$f(x) \cdot f(-x) = f(x)^2 + x.f(f(-2x)).$$

Note that $-2x \cdot f(f(-2x)) = f(2x)^2$, so $x.f(f(-2x)) = -f(2x)^2/2$. Thus

$$f(x) \cdot f(-x) = f(x)^2 - f(2x)^2/2$$

or

$$-2f(x)f(-x) + 2f(x)^2 = f(2x)^2.$$

In $(\diamondsuit)$, take $x = y \neq 0$ then

$$f(x) \cdot f(2x) = f(x)^2 + x \cdot f(f(x))$$

or

$$f(2x) = f(x) + x \cdot f(f(x))/f(x) = f(x) + f(-x)^2/f(x).$$

So $(f(x) + f(-x)^2/f(x))^2 = 2f(x)^2 - 2f(x) \cdot f(-x)$. Dividing both sides by $f(-x)^2$ and put $k = f(x)/f(-x)$ then

$$(k + 1/k)^2 = 2k^2 - 2k \longleftrightarrow k^2 - 2k = 2 + 1/k^2.$$

Changing the sign $x \to -x$ then we get $(k + 1/k)^2 = 2/k^2 - 2/k$ combining with above, we get $k = -1$. Then $f(x)$ is odd. Now we get $f(x)^2 = x \cdot f(f(x))$. Put $y \to -y$, we get

$$f(x) \cdot f(x-y) = f(x)^2 + x \cdot f(f(-y)),$$

take the sum of these two equations, side by side,

$$f(x)[f(x+y)+f(x-y)] = 2f(x)^2.$$

Consider $x \neq 0$ then $f(x+y)+f(x-y) = 2f(x)$ using $f(0)=0$, this implies that $f(x)$ is additive. Note that

$$2f(x)^2 - f(x^2) = x/2 \cdot f(2x) = x \cdot f(x) \Longleftrightarrow 2f(x)^2 = f(x^2) + xf(x).$$

Plugging $x \to x+1$, it is easily to get $f(x) = ax$ and putting back to the original, $a = 1$. There are two solutions: $f(x) = 0, f(x) = x$. $\square$

Test 3

Problem 1. *For each non-constant integer polynomial $P(x)$, let's define*

$$M_{P(x)} = \max_{x\in[0;2021]} |P(x)|.$$

1) Find the minimum value of $M_{P(x)}$ when $\deg P(x) = 1$.

2) Suppose that $P(x) \in \mathbb{Z}[x]$ when $\deg P(x) = n$ and $2 \leq n \leq 2022$. Prove that $M_{P(x)} \geq 1011$.

Solution.

1) Since $\deg P = 1$, put $P(x) = ax + b$ with $a, b \in \mathbb{Z}$ và $a \neq 0$. Note that

$$|P(2021) - P(0)| = |2021a| \geq 2021 \Rightarrow \max\{|P(2021)|, |P(0)|\} \geq \frac{2021}{2}.$$

Thus, $|P(2021) - P(0)| = |2021a| \geq 2021$ which implies that

$$\max\{|P(2021)|, |P(0)|\} \geq \frac{2021}{2} \Rightarrow \max\{|P(2021)|, |P(0)|\} \geq 1011.$$

On the other hand, $T(x) = x - 1011$ satisfying condition $|T(x)| \leq 1011, \forall x \in [0; 2021]$. Hence, the minimum value of $M_{P(x)}$ is 1011, the equality holds when $P(x) = x - 1011$.

2) Suppose that there is some integer polynomial $P(x)$ with

$$2 \leq \deg P \leq 2022 \text{ and } M_{P(x)} < 1011.$$

One can see that $|P(x)| < 1011, \forall x \in [0; 2021]$ thus $|P(x)| \leq 1010$, $\forall x \in [0; 2021] \cap \mathbb{Z}$. By the property of integer polynomial, $2021 | P(2021) - P(0)$ so $P(2021) = P(0)$ which leads to

$$|P(2021) - P(0)| \leq |P(0)| + |P(2021)| \leq 2020.$$

From that, let's put $P(x) = x(x-2021)Q(x) + c$ with $c \in \mathbb{Z}$ và $Q(x) \in \mathbb{Z}[x]$. So $x(x-2021) \geq 2022, \forall x \in \{2, 3, \ldots, 2019\}$. If there exists $x_0 \in \{2, 3, \ldots, 2019\}$ such that $Q(x_0) \neq 0$ then

$$\max |x_0(x_0 - 2021)Q(x_0) - c| \geq \frac{1}{2} |x_0(x_0 - 2021)Q(x_0)| \geq 1011.$$

From this, we can conclude that $Q(2) = Q(3) = \cdots = Q(2019) = 0$ so let's write

$$P(x) = x(x-2)(x-3)\ldots(x-2019)(x-2021)H(x) + c$$

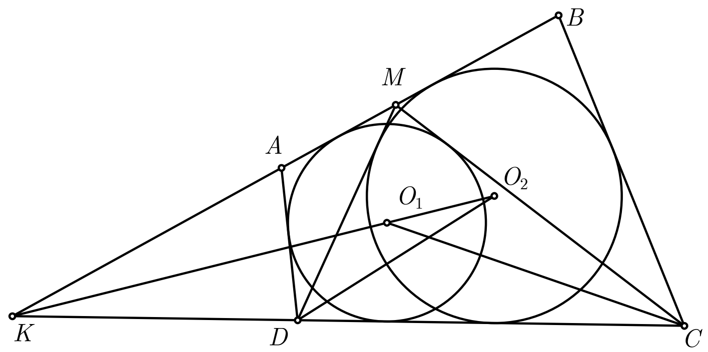
with $H(x) \in \mathbb{Z}[x]$ then $P(1) = -2020 \cdot 2018!H(1) + c$. Note that if $H(1) \neq 0$ then $M_{P(x)} > 1011$. Similarly, $H(2020) \neq 0$ also leads to another contradiction, so $H(1) = H(2020) = 0$ then there is some $R(x) \in \mathbb{Z}[x]$ for with

$$Q(x) = (x-1)(x-2)\ldots(x-2020)R(x).$$

Since $\deg P \leq 2022$ then $R(x) \equiv c \in \mathbb{Z}$. On the other hand, $P\left(\frac{1}{2}\right) > 1011$ so $c = 0$, contradicts to $P(x)$ is non-constant. Therefore, the contrary hypothesis is false and it follows that $M_{P(x)} \geq 1011$. $\square$

Problem 2. *Point M on side AB of quadrilateral $ABCD$ is such that quadrilaterals $AMCD$ and $BMDC$ are circumscribed around circles centered at O_1 and O_2 respectively. Line O_1O_2 cuts an isosceles triangle with vertex M from angle CMD. Prove that $ABCD$ is a cyclic quadrilateral.*

Solution. If $AB \parallel CD$ then the incircles of $AMCD$ and $BMDC$ have equal radii; now the problem conditions imply that the whole picture is symmetric about the perpendicular from M to O_1O_2, and hence $ABCD$ is an isosceles trapezoid (or a rectangle). The conclusion in this case is true.



Now suppose that the lines AB and CD meet at a point K; we may assume that A lies between K and B. The points O_1 and O_2 lie on the bisector of the angle BKC. By the problem condition, this angle bisector forms equal angles with the lines CM and DM; this yields $\angle DMK = \angle KCM$. Since O_1 and O_2 are the incenter of KMC and an excenter of KDM, respectively, we have

$$\angle DO_2K = \frac{1}{2}\angle DMK = \frac{1}{2}\angle KCM = \angle DCO_1,$$

so the quadrilateral CDO_1O_2 is cyclic. Next, the same points are an excenter of AKD and the incenter of KBC, respectively, so

$$\angle KAD = 2\angle KO_1D = 2\angle DCO_2 = \angle KCB;$$

this implies the desired cyclicity of the quadrilateral $ABCD$. $\square$

Problem 3. *Let p be a prime number and let m, n be integers greater than 1 such that $n \mid m^{p(n-1)} - 1$. Prove that* $\gcd(m^{n-1} - 1, n) > 1$.

Solution. Set $\alpha = v_p(n-1)$ and write $n - 1 = r \cdot p^{\alpha}$. Let q be an arbitrary prime divisor of n, and set $d = \operatorname{ord}_q(m)$. Since

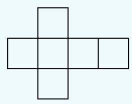
$$m^{p(n-1)} = m^{r \cdot p^{\alpha+1}} \equiv 1 \pmod{q},$$

it follows that $d \mid r \cdot p^{\alpha+1}$. If $v_p(d) \neq \alpha + 1$, then clearly $d \mid n - 1$, and therefore q is a common divisor of n and $m^{n-1} - 1$. Otherwise, suppose that $v_p(d) = \alpha + 1$ for all prime divisors of n. Together with $m^{q-1} \equiv 1 \pmod{q}$ by Fermat's Little Theorem, we have

$$d \mid q - 1 \implies v_p(q-1) \geq v_p(d) = \alpha + 1.$$

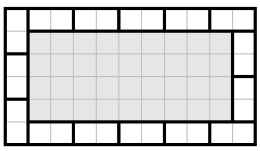
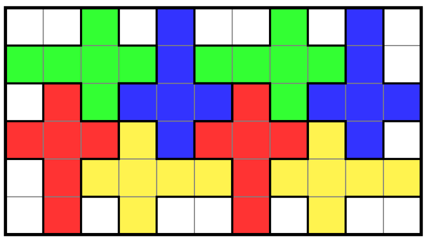
Thus, any prime divisor q of n satisfies $q \equiv 1 \pmod{p^{\alpha+1}}$. Because n is a product of these prime divisors, we deduce that $n \equiv 1 \pmod{p^{\alpha+1}}$. However, this contradicts to $v_p(n-1) = \alpha$. $\square$

Problem 4. *The* sword *is a figure consisting of* 6 *unit squares presented in the picture below (and any other figure obtained from it by rotation).*



Determine the largest number of swords that can be cut from a 6×11 *piece of paper divided into unit squares (each sword should consist of six such squares).*

Solution. Let us cover boundary cells with 15 dominoes, as shown in the left picture In each such domino there is at most one cell belonging to some sword. Therefore, all swords can together contain at most $6 \cdot 11 - 15 = 51$ cells, which means that there are at most 8 swords. Eight is actually possible (see the right picture).

$\square$

Test 4

Problem 1. *Let ABC be an acute-angled triangle. Point P is such that $AP = AB$ and $PB \parallel AC$. Point Q is such that $AQ = AC$ and $CQ \parallel AB$. Segments CP and BQ meet at point X. Prove that the circumcenter of triangle ABC lies on the circumcircle of triangle PXQ.*

Solution. Let D be the vertex of parallelogram $ABDC$. Then $APDC$ and $AQDB$ are isosceles trapezoids. Therefore the perpendicular bisectors to segments PD and QD coincide with the perpendicular bisectors to AC and AB respectively, the circumcenter O of triangle ABC is also the circumcenter of DPQ and $\angle POQ = 2\angle A$. Also since

$$\angle XPD = \angle ADP,\ \angle XQD = \angle ADQ$$

we obtain that $\angle PXQ = 2\angle A$. Thus O, P, Q, X are concyclic. □

Problem 2. *Find all positive integers n that have precisely $\sqrt{n+1}$ natural divisors.*

Solution. First observe that $n = k^2 - 1$ for some positive integer k. As n is not a perfect square, $\tau(n) = k$ must be even, ergo n is odd.
We now establish a bound on $\tau(n)$. Recall that, since n is not a perfect square, there exists a bijective correspondence between factors of n greater than $\lfloor\sqrt{n}\rfloor = k - 1$ and factors of n at most $\lfloor\sqrt{n}\rfloor$. Furthermore, since n is odd, all divisors of n must also be odd. Therefore

$$k = \tau(n) \leq 2 \cdot |\{1, 3, \ldots, k-1\}| = 2 \cdot \tfrac{k}{2} = k.$$

Hence equality must hold, and so every odd number between 1 and $k-1$ must divide $n = k^2 - 1$.
In particular, $k - 3$ must divide $k^2 - 1$; but

$$\frac{k^2-1}{k-3} = k + 3 + \frac{8}{k-3}.$$

Therefore, $k - 3$ is either -1 or 1, so k equals either 2 or 4 and $n \in \{3, 15\}$, both of these work. □

Problem 3. *Let n be an even positive integer. On a board n real numbers are written. In a single move we can erase any two numbers from the board and replace each of them with their product. Prove that for every n initial numbers one can in finite number of moves obtain n equal numbers on the board.*

Solution. We proceed by induction on n. For $n = 2$ one move is obviously enough: $(a, b) \to (ab, ab)$. For $n \geq 4$ we can by inductive assumption have $n - 2$ numbers equal to a and the remaining two equal to b. Now for $n = 4$ we have

$$(a, a, b, b) \to (ab, a, ab, b) \to (ab, ab, ab, ab),$$

and for $n \geq 6$ we have

$$\begin{aligned}
&(a, a, a, a, \ldots, a, a, b, b) \to (a, a, ab, a, \ldots, a, a, ab, b) \\
&\to (a, a, ab, a^2b, \ldots, a, a, a^2b, b) \to \ldots \to (a, a, ab, a^2b, \ldots, a^{n-5}b, a^{n-4}b, a^{n-4}b, b) \\
&\to (a, a, ab, a^2b, \ldots, a^{n-5}b, a^{n-4}b, a^{n-4}b^2, a^{n-4}b^2)
\end{aligned}$$

and then we can pair i-th number in the sequence with the $(n+1-i)$-th one for $i = 1, 2, \ldots, n/2$ to get $a^{n-3}b^2$ everywhere. □

Problem 4. *Consider the function $f:\mathbb{R}^+\to\mathbb{R}^+$ and satisfying*

$$f(x+2y+f(x+y))=f(2x)+f(3y),\ \forall x,y>0.$$

1) Find all functions f that satisfy the problem.

2) Suppose $f(4\sin^4 x)f(4\cos^4 x)\geq f^2(1)$ with $x\in\left(0;\frac{\pi}{2}\right)$. Find the minimum value of $f(2022)$.

Solution.

1) In the given condition, replace $x\to 3x, y\to 2y$, we have

$$f(3x+4y+f(3x+2y))=f(6x)+f(6y).$$

By swapping x,y and comparing the two left sides, we have

$$f(3x+4y+f(3x+2y))=f(4x+3y+f(2x+3y)).$$

From injectivity, we have

$$3x+4y+f(3x+2y)=4x+3y+f(2x+3y)$$

or

$$f(3x+2y)-(3x+2y)=f(2x+3y)-(2x+3y).$$

Set $g(x)=f(x)-x$ then $g(3x+2y)=g(2x+3y),\forall x,y\in\mathbb{R}^+$. Put $a=3x+2y, b=2x+3y$ then

$$x=\frac{3a-2b}{5},y=\frac{3b-2a}{5}$$

so the condition of the pair of numbers (a,b) for the existence of x,y the above relation is $3a-2b>0$ and $3b-2a>0$, equivalent to $\frac{2}{3}<\frac{a}{b}<\frac{3}{2}$.

Hence we have $g(a)=g(b)$ for all $\frac{a}{b}\in\left(\frac{2}{3};\frac{3}{2}\right)$. From here we will show that g is a constant function. Indeed, we have $g(1)=g(x),\forall x\in\left[1;\frac{3}{2}\right)$, from here inductively

$$g(1)=g(x),\forall x\in\left[\left(\frac{3}{2}\right)^k;\left(\frac{3}{2}\right)^{k+1}\right)$$

so $g(1)=g(x),\forall x>1$. Similarly

$$g(1)=g(x),\forall x\in\left(\left(\frac{2}{3}\right)^{k+1};\left(\frac{2}{3}\right)^{k}\right]$$

so there is also $g(1)=g(x),\forall x\in(0;1)$. Therefore, $g(x)$ is a constant function.. Instead we have $f(x)=x+c$ with $c\geq 0$, try again and we are satisfied.

2) Since $f(x)=x+c$, instead of the problem condition, we have

$$\left(4\sin^4 x+c\right)\left(4\cos^4 x+c\right)\geq(1+c)^2 \text{ or } \sin^4(2x)+4(\sin^4 x+\cos^4 x)c\geq 1+2c.$$

Notice that $\sin^4 x+\cos^4 x=1-\frac{1}{2}\sin^2 2x$ and put $t=\sin^2(2x)\in(0;1]$, we rewrite the above inequality as $t^4+(4-2t^2)c\geq 1+2c$ or

$$2(1-t^2)c\geq(1-t^2)(t^2+1).$$

Since $1-t^2>0$ so $2c\geq t^2+1$, which $\max\limits_{(0;1]}\{t^2+1\}=2$ so $2c\geq 2\Leftrightarrow c\geq 1$. So the minimum value of $f(2022)$ is 2023. $\square$

Test 5

Problem 1. *On the sides of triangle ABC, points $P, Q \in AB$ (P is between A and Q) and $R \in BC$ are chosen. The points M and N are defined as the intersection point of AR with the segments CP and CQ, respectively. If $BC = BQ, CP = AP, CR = CN$ and $\angle BPC = \angle CRA$, prove that $MP + NQ = BR$.*

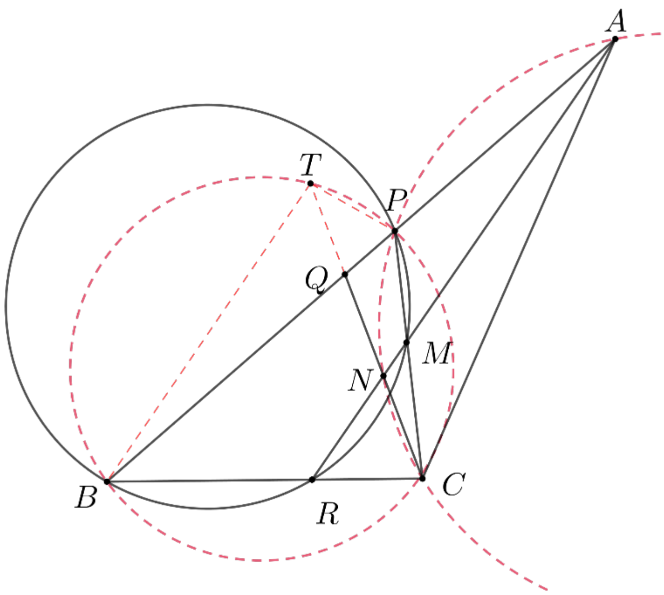
Solution. To prove that $MP + NQ = BR$, by adding RC to both sides it is equivalent to prove $CQ + MP = BC$. We start by defining point T on the segment CQ such that $BC = CT$, then it is sufficient to prove that $QT = PM$, since triangles BTC and NRC are isosceles. One can get that BT is parallel to NR so

$$\angle BTC = \angle RNC = \angle NRC = \angle BPC,$$

thus $BTPC$ is cyclic. Also we have that

$$\angle CNA = 180^\circ - \angle NRC = 180^\circ - \angle BPC = \angle CPA,$$

this implies that $PNCA$ is cyclic.



So we have that

$$\angle TPQ = \angle TCB = \angle BQC = \angle TQP,$$

thus $TQ = TP$. Now by angle chasing,

$$\angle CPA = 180^\circ - \angle BPC = 180^\circ - \angle BTC = 180^\circ - \angle TBC = \angle TPC,$$

thus we have $\angle CPQ = \angle TPC$ and $\angle PCT = \angle PAM$, which implies that TPC and MPA are congruent, since $PC = AP$. In conclusion, $TP = MP = TQ$, which finishes the proof. □

Problem 2. *By $rad(x)$ we denote the product of all distinct prime factors of a positive integer n. Given $a \in \mathbb{N}$, a sequence (a_n) is defined by $a_0 = a$ and $a_{n+1} = a_n + rad(a_n)$ for $n \geq 0$. Prove that there exists an index n for which $\frac{a_n}{rad(a_n)} = 2022$.*

Solution. Denote $b_n = \frac{a_n}{\operatorname{rad}(a_n)}$. Since $\operatorname{rad}(a_n)$ divides $\operatorname{rad}(a_{n+1})$ we have $b_{n+1} \mid b_n + 1$. If there are indices $i < j$ with $b_i < 2022 < b_{i+1}$, we will be done by "continuity". If, to the contrary, this does not happen, there are two possible cases.

- $b_n < 2022$ for all n big enough. Since a_n increases indefinitely, then so does $\operatorname{rad}(a_n)$, so at some moment $\operatorname{rad}(a_n)$ receives a new prime $p > 2022$. This means that $p \nmid a_n$ and $p \mid a_{n+1} = \frac{b_n+1}{b_n} a_n$, so $p \mid b_n + 1$ and hence $b_n \geq 2022$, a contradiction.

- $b_n > 2022$ for all n. We can assume WLOG that b_0 is the smallest term of the sequence (b_n). Suppose that $b_{i+1} = b_i + 1$ for all $0 \leq i < n$. Then

$$\operatorname{rad}(a_0) = \cdots = \operatorname{rad}(a_{n-1}) = R.$$

  But for every prime $p \leq n$ there is a multiple of p among us sus $b_0, \ldots, b_{n-1}$, so $p \mid a_k$ for some k and consequently $p \mid R$. Since not every prime divides R, there must be an index n such that $b_n < b_{n-1} + 1$, i.e. $b_{n-1} + 1 = db_n$ for some $d > 1$ and $rad(a_{n+1}) = dR$, so $\gcd(d, R) = 1$.

  Recall that $b_0 \leq b_n = \frac{b_0+n}{d}$, which reduces to $n \geq (d-1)b_0$. By above, this means that all primes up to $(d-1)b_0$ divide R, but d does not divide R, so $d > (d-1)b_0$, which is impossible.

□

Problem 3. *Determine if there exist functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying for every $x \in \mathbb{R}$ the following equations $f(g(x)) = x^3 \quad$ and $\quad g(f(x)) = x^2$.*

Solution. We will prove that such functions do not exist.
Suppose they do and note that

$$g(x^3) = g(f(g(x)) = (g(x))^2,$$

which means that among the three numbers $g(-1)$, $g(0)$, $g(1)$ at least two are equal (as each of these numbers is either 0 or 1). This means that g is not injective.

On the other hand if $x, y \in \mathbb{R}$ satisfy $g(x) = g(y)$, then

$$x^3 = f(g(x)) = f(g(y)) = y^3,$$

so $x = y$, which means that g is injective, a contradiction. □

Problem 4. *We consider all partitions of a positive integer n into a sum of (non-negative integer) exponents of 2 (i.e. $1, 2, 4, 8, \dots$). A number in the sum is allowed to repeat an arbitrary number of times (e.g. $7 = 2+2+1+1+1$) and two partitions differing only in the order of summands are considered to be equal (e.g. $8 = 4+2+1+1$ and $8 = 1+2+1+4$ are regarded to be the same partition). Let $E(n)$ be the number of partitions in which an even number of exponents appear an odd number of times and $O(n)$ the number of partitions in which an odd number of exponents appear an odd number of times. For example, for $n = 5$ partitions counted in $E(n)$ are $5 = 4+1$ and $5 = 2+1+1+1$, whereas partitions counted in $O(n)$ are $5 = 2+2+1$ and $5 = 1+1+1+1+1$, hence $E(5) = O(5) = 2$. Find $E(n) - O(n)$ as a function of n.*

Solution 1. Let $D(n) = E(n) - O(n)$. We trivially have $O(1) = 1$ and $E(1) = 0$, thus $D(1) = -1$, and $E(2) = O(2) = 1$ (respectively $2 = 1+1$ and $2 = 2$), hence $D(2) = 0$. We will show by total induction that $D(n) = 0$ for all $n > 2$. Assume it holds for all numbers from 2 to $n-1$. If n is odd, a partition must contain at least one 1. Since the addition of 1 changes the parity of the number of one's it follows that $E(n) = O(n-1)$ and $O(n) = E(n-1)$, hence since $D(n) = -D(n-1) = 0$. If n is even the partiton must contain an even number $2k$ of 1s. If it contains $2k = n$ or $2k = n-2$, then we have the unique solutions

$$1 + 1 + \cdots + 1 \text{ and } 2 + 1 + 1 + \cdots + 1,$$

the first adding to $E(n)$, the second to $O(n)$. For other, smaller, values of k we note that the remaining exponents are all even, and we can thus apply the inductive hypothesis to $\frac{n-2k}{2}$. Thus, it follows that for even n we will also have $D(n) = 0$. This completes the proof. □

Solution 2. Let $G(x) = 1 + \sum_{i=1}^{\infty} D(i)x^i$ be the generating function for the sequence we desire. To count partitions with an odd number of odd-appearing exponents as negative, we will multiply those choices as negative. Thus, we have that

$$G(x) = (1-x+x^2-x^3+x^4-\dots)(1-x^2+x^4-x^6+x^8-\dots)(1-x^4+x^8-x^{12}+x^{16}-\dots)\dots,$$

where the i-term corresponds to choosing the number of times 2^{i-1} is represented in the partition. Taking the expression for each power series, one now finds

$$G(x) = \frac{1}{1+x}\frac{1}{1+x^2}\frac{1}{1+x^4}\frac{1}{1+x^8}\dots$$

However,

$$\begin{aligned}\frac{G(x)}{1-x} &= \frac{1}{1-x}\frac{1}{1+x}\frac{1}{1+x^2}\dots = \frac{1}{1-x^2}\frac{1}{1+x^2}\frac{1}{1+x^4}\dots \\ &= \frac{1}{1-x^4}\frac{1}{1+x^4}\frac{1}{1+x^8}\dots = \dots = 1.\end{aligned}$$

Thus, $G(x) = 1 - x$ from which it follows that $D(1) = -1$ and $D(n) = 0$ for all $n > 1$. □

Solution of JBMO TST

Test 1

Problem 1. *The positive $n > 3$ called 'nice' if and only if $n+1$ and $8n+1$ are both perfect squares. How many positive integers $k \leq 15$ such that $4n+k$ are composites for all nice numbers n?*

Solution. First, note that every perfect squares when divide by 3 will give the remainder are 0 or 1. Thus, $n+1$ is perfect square implying that $n \equiv 0, 2 \pmod 3$, and $8n+1$ is perfect square implying that $n \equiv 0, 1 \pmod 3$. Hence, if n is nice, then n is divisible by 3. One can check that the minimum nice number is $n = 15$. Now for all even numbers k will satisfy the condition since $4n + k \geq 4 \cdot 15 - 20 > 2$ and $2|4n+k$.

Similarly, all numbers k that divisible by 3 also satisfy the condition since $3|4n+k$. It is remain to check $k \in \{1, 5, 7, 11, 13\}$. Substitute $n = 15$, it is required $60 + k$ is composite. But $60+1 = 61,\ 60+7 = 67,\ 60+11 = 71,\ 60+13 = 73$ are primes, which implies that $k = 1, 7, 11, 13$ do not meet the condition.

Finally, we will prove that $k = 5$ satisfies the condition. One can check that

$$x(n+1) + y(8n+1) = 4n+5,\ \forall n \longleftrightarrow x = \frac{36}{7}, y = -\frac{1}{7},$$

thus

$$7(4n+5) = 36(n+1) - (8n+1) = 36a^2 - b^2 = (6a-b)(6a+b)$$

for some positive integers a, b. Note that

$$6a - b = 6\sqrt{n+1} - \sqrt{8n+1} \geq 3\sqrt{n+1} \geq 12.$$

Thus if $4n+5 > 7$ is prime, then we must have $6a - b = 7,\ 6a+b = 4n+5$, impossible. This means that $4n+7$ is composite.

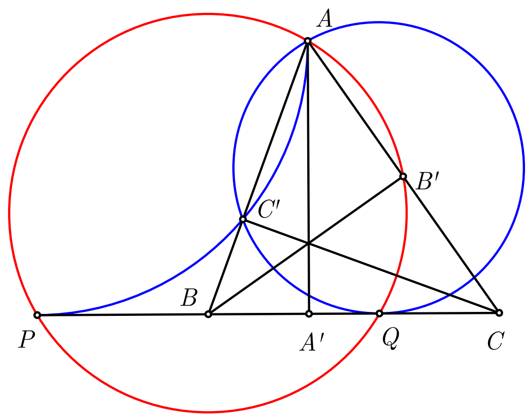
Therefore, there are 11 desired numbers $k \in \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15\}$. $\square$

Problem 2. *Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.*

Solution. Since $BP^2 = BQ^2 = BA \cdot BC'$ and the quadrilaterals $AC'A'C$, $AB'A'B$ are cyclic (AA' is the altitude) we have

$$CP \cdot CQ = CB^2 - BP^2 = CB^2 - BA \cdot BC' = BC^2 - BC \cdot BA' = BC \cdot CA' = CA \cdot CB'.$$

Clearly this is equivalent to the required assertion. $\square$



Problem 3. 2000 *consecutive integers (not necessarily positive) are written on the board. A student takes several turns. On each turn, he partitions the* 2000 *integers into* 1000 *pairs, and substitutes each pair by the difference and the sum of that pair (note that the difference does not need to be positive as the student may choose to subtract the greater number from the smaller one; in addition, all the operations are carried simultaneously). Prove that the student will never again write* 2000 *consecutive integers on the board.*

Solution. Note that $(a-b)^2+(a+b)^2=2(a^2+b^2)$, so the sum of the squares of the numbers written on the board doubles on each turn. Note that

$$n^2+(n+1)^2+\ldots+(n+1999)^2=2000n^2+1999\cdot 2000n+\frac{1999\cdot 2000\cdot 3999}{6},$$

which is congruent to 8 modulo 16. Obviously, when we multiply this sum by 2, we will obtain a number divisible by 16, thus, we will never have consecutive numbers again, which is what we need to show. □

Problem 4. *Determine the smallest positive integer a for which there exist a prime number p and a positive integer $b \geq 2$ such that $\frac{a^p-a}{p}=b^2$.*

Solution. If $p=2$, our equation becomes $a(a-1)=2b^2$, whose smallest solution in $\mathbb{N}$ is $a=9$.

Now let $p \geqslant 3$. Since a and $a^{p-1}-1$ are coprime and $a(a^{p-1}-1)=pb^2$, either a or $a^{p-1}-1$ must be a square, and it is obviously not the latter; hence a is a square. Assume that $a=4$. Then

$$\frac{4^{p-1}-1}{p}=\frac{(2^{p-1}-1)(2^{p-1}+1)}{p}$$

is a square, so either $2^{p-1}-1$ or $2^{p-1}+1$ is a square, but the former is $3 \pmod 4$, so the latter is the square: $2^{p-1}+1=c^2$. Then $(c+1)(c-1)=2^{p-1}$, so both $c-1$ and $c+1$ are powers of 2 and they must be 2 and 4, but then $p=4$, a contradiction. In conclusion, $a=9$ is the answer. □

Test 2

Problem 1. *Find all pairs of positive prime numbers (p, q) such that $p^5+p^3+2 = q^2 - q$.*

Solution. We are given $p^3(p^2+1) = (q+1)(q-2)$. If p divides both $q+1$ and $q-2$, then $p = 3$, giving us a solution $(p, q) = (3, 17)$.

Now suppose that $p \neq 3$. Then either $q+1$ or $q-2$ is a multiple of p^3, whereas the other divides p^2+1. It follows that $q+1 \geqslant p^3$ and $p^2+1 \geqslant q-2$, so $p^3 \geqslant p^2+4$, which holds only for $p = 2$. Then $(p, q) = (2, 7)$. □

Problem 2. *Consider non-negative real numbers a, b, c satisfying the condition $a^2+b^2+c^2 = 2$. Find the maximum value of the following expression*

$$P = \frac{\sqrt{b^2+c^2}}{3-a} + \frac{\sqrt{c^2+a^2}}{3-b} + a + b - 2022c.$$

Solution. First, we will show that $4\sqrt{b^2+c^2} \leq (3-a)^2$. Notice that $b^2+c^2 = 2-a^2 \geq 0$, so we need to prove $4\sqrt{2-a^2} \leq (3-a)^2$. According to the AM-GM inequality, we have

$$4\sqrt{2-a^2} = 4\sqrt{1\cdot(2-a^2)} \leq 4\cdot\frac{1+2-a^2}{2} = 2(3-a^2).$$

We need to prove

$$2(3-a^2) \leq (3-a)^2 \longleftrightarrow 3(a^2-2a+1) \geq 0 \longleftrightarrow 3(a-1)^2 \geq 0,$$

which is true. From this, one can conclude that, $\frac{\sqrt{b^2+c^2}}{3-a} \leq \frac{3-a}{4}$. Similarly, $\frac{\sqrt{c^2+a^2}}{3-b} \leq \frac{3-b}{4}$. Therefore

$$P \leq \frac{3-a}{4} + \frac{3-b}{4} + a + b - 2022c \leq \frac{6+3(a+b)}{4}.$$

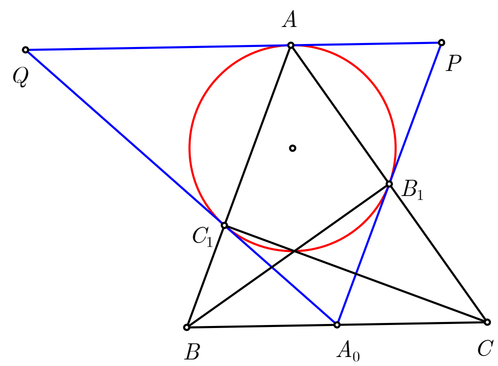
We have the familiar inequality

$$(a+b)^2 \leq 2(a^2+b^2) \leq 2(a^2+b^2+c^2) = 4$$

so $a+b \leq 2$. From this it follows that $P \leq \frac{6+3\cdot 2}{4} = 3$. So the maximum value of P of 3, is obtained when $a = b = 1,\ c = 0$. □

Problem 3. *Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 is the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.*

Solution. Since triangles BCB_1 and BCC_1 are right-angled, their medians B_1A_0, B_1C_0 are equal to the half of hypotenuse $B_1A_0 = A_0C = A_0B = C1A_0$.



Now

$$\angle PB_1A = \angle CB_1A_0 = \angle B_1CA_0 = \angle PAC,$$

thus $PA = PB_1$. Similarly, $QA = QC_1$. Then the incircle of triangle A_0PQ touches its sides in points A, B_1, C_1, which yields the assertion of the problem. $\square$

Problem 4. *You plan to organize your birthday party, which will be attended either by exactly m persons or by exactly n persons (you are not sure at the moment). You have a big birthday cake and you want to divide it into several parts (not necessarily equal), so that you are able to distribute the whole cake among the people attending the party with everybody getting cake of equal mass (however, one may get one big slice, while others several small slices - the sizes of slices may differ). What is the minimal number of parts you need to divide the cake, so that it is possible, regardless of the number of guests.*

Solution. We claim that the answer is $m + n - \gcd(m, n)$. Firstly, note that if we consider the cake as the interval $[0, 1]$ and make cuts at points with coordinates $\frac{k}{m}$ $(0 < k < m)$ and $\frac{l}{n}$ $(0 < l < n)$, then we will be able to satisfy the condition of the problem. Moreover, there will be $m-1$ cuts with $\frac{k}{m}$, $n-1$ cuts with $\frac{l}{n}$, $\gcd(m, n)-1$ of which coincide. Therefore we will have

$$m + n - \gcd(m, n) - 1$$

cuts, so $m + n - gcd(m, n)$ parts. Let us show that this estimate is sharp. For that purpose, consider a bipartite graph, with the vertices of one side corresponding to the m persons on the party, and the vertices of the other side corresponding to the n persons on the party.

We connect the vertices v and u by an edge, corresponding to the piece of cake, if the piece of cake will be given to the person v, if exactly m persons attend, and to the person u, if exactly n persons attend. Consider a component of connectivity of the graph. Then, if it contains m_1 edges in one side and n_1 edges in the other side, then the edges of it correspond to a cake with weight $\frac{m_1}{m} = \frac{n_1}{n}$. Therefore, $m_1 \geq \frac{m}{\gcd(m,n)}$.

Thus, the number of components of connectivity is at most $\gcd(m, n)$. The graph has $m+n$ vertices and at most $\gcd(m, n)$ components of connectivity, so the number of edges is at least $m + n - \gcd(m, n)$ (the equality is obtained, when each of the components is a tree). $\square$

Solution of IMO Team selection tests

Problem 1. *Let (a_n) be the integer sequence which is defined by $a_1 = 1$ and*

$$a_{n+1} = a_n^2 + n \cdot a_n, \ \forall n \geq 1.$$

Let S be the set of all primes p such that there exists an index i such that $p|a_i$. Prove that the set S is an infinite set and it is not equal to the set of all primes.

Solution. First, we shall show that $3 \notin \mathcal{S}$ by proving that

$$a_{3k-2} \equiv 2 \pmod 3,\ a_{3k-1} \equiv 2 \pmod 3 \text{ and } a_{3k} \equiv 2 \pmod 3.$$

Since $a_1 = 1, a_2 = 2, a_3 = 2^2 + 2 \cdot 2 = 8 \equiv 2 \pmod 3$ so the claim is true for $k = 1$. Suppose that the claim holds for $k = n$. We have

$$\begin{aligned}
a_{3n+1} &= a_{3n}(a_{3n} + 3n) \equiv 2 \cdot 2 \equiv 1 \pmod 3, \\
a_{3n+2} &= a_{3n+1}(a_{3n+1} + 3n + 1) \equiv 1(1 + 1) \equiv 2 \pmod 3, \\
a_{3n+3} &= a_{3n+2}(a_{3n+2} + 3n + 2) \equiv 2(2 + 2) \equiv 2 \pmod 3.
\end{aligned}$$

Thus, the claim is also true for $k = n + 1$. So by induction, the claim is proved.

Now suppose on the contrary that $\mathcal{S}$ is finite, denote $\mathcal{S} = \{p_1, p_2, \ldots, p_k\}$. Note that $a_n|a_{n+1}$ so $a_n|a_m$ for all $m \geq n$. By the definition of $\mathcal{S}$, there are some index t such that $p_1|a_t$, thus $p_1|a_{t'}$ for all $t' \geq t$. Similarly for $p_2, p_3, \ldots, p_k$ so there exist N big enough such that $p_1 p_2 \cdots p_k | a_n$ for all $n \geq N$. Taking the integer $\ell > N + 1$ such that $\ell \equiv 2 \pmod{p_1 p_2 \cdots p_k}$. Since $a_\ell = a_{\ell-1}(a_{\ell-1} + \ell - 1)$, we get

$$a_{\ell-1} + \ell - 1 \equiv 1 \pmod{p_1 p_2 \cdots p_k}$$

so $a_{\ell-1} + \ell - 1$ is coprime to all primes in $\mathcal{S}$, which implies that is has some prime divisor that not belong to $\mathcal{S}$, a contradiction. Hence, $\mathcal{S}$ is an infinite set. $\square$

Problem 2. *(IMO SL 2021, G4) Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \parallel CD$ and $TF \parallel AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.*

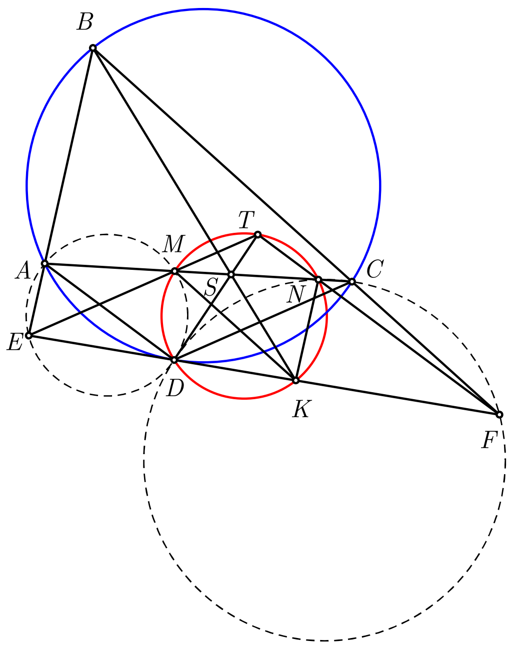
Solution 1. (IMO 2021 Shortlist G4, solved by Mohamed Aldubaisi) Denote by M, N the intersections of AC with TE and TF respectively. By the assumption, it obtains that

$$\angle TNM = \angle DAC = \angle CDF = \angle TEF,$$

which implies that MN is antiparallel to EF with respect to $\angle ETF$. Apply Reim's theorem, since $AD \parallel TF$, it deduces that $AMDE$ is a cyclic quadrilateral. Similarly, $CNDF$ is a cyclic quadrilateral. Then by angle chasing,

$$\begin{aligned}\angle MDN &= 180^\circ - \angle MDE - \angle NDF = 180^\circ - \angle BAC - \angle BCA \\ &= \angle ABC = 180^\circ - \angle ADC = 180^\circ - \angle TMN.\end{aligned}$$

Hence, $TMDN$ is a cyclic quadrilateral. Let ℓ be the altitude from T of triangle TEF, then D and K are symmetric with respect to ℓ. On the other hand, since MN is anti parallel to EF with respect to $\angle ETF$, one gets ℓ pass through the circumcenter of triangle TMN, hence K lies on (TMN).



By angle chasing,

$$\angle KNM = \angle MDE = \angle BAM$$

thus $BA \parallel NK$, and similarly, $BC \parallel MK$. Let S be the intersection of BK and MN. It suffices to show that S lies on TD. Since

$$\angle TMN = \angle DCA,\ \angle TNM = \angle DAC,\ \angle KNM = \angle BAC,\ \angle KMN = \angle BCA,$$

we conclude that $\triangle KNM \sim \triangle BCA, \triangle TMN \sim \triangle DCA$ and combining with Thales's theorem, it implies that

$$\frac{SM}{SC} = \frac{KM}{BC} = \frac{MN}{AC} = \frac{TM}{DC}$$

which means $\triangle SMT \sim \triangle SCD$. Then

$$\angle TSD = \angle TSM + \angle MSD = \angle DSC + \angle MSD = 180^\circ$$

as desired. $\square$

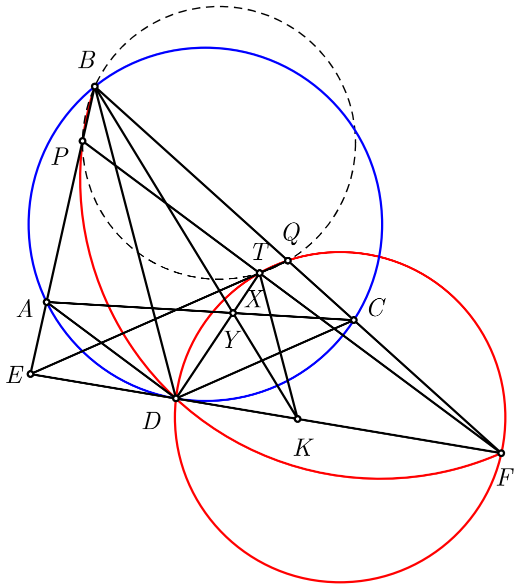
Solution 2. (Solved Marwan Alkhayat) We state the main lemma in this problem.

Lemma. Given a triangle ABC, point X is on the line BC. Then

$$\frac{XB}{XC} = \frac{AB}{AC} \cdot \frac{\sin XAB}{\sin XAC}.$$

This is directly followed by the Law of Sines.

Back to the problem,



Let P be the intersection of AB and FT, and Q be the intersection of BC and ET. By angle chasing,

$$\angle TPB = \angle DAB = \angle DCF = \angle TQF$$

hence $BPTQ$ is a cyclic quadrilateral. Furthermore,

$$\angle BPF = \angle BAD = \angle BDF$$

hence $BPDF$ is a cyclic quadrilateral. By power of a point,

$$\overline{ET} \cdot \overline{EQ} = \overline{EP} \cdot \overline{EB} = \overline{ED} \cdot \overline{EF}$$

thus $TQDF$ is a cyclic quadrilateral. Similarly, one can get $DTPE$ and $BQDE$ are cyclic quadrilateral. This leads to

$$\angle TDA = \angle DTF = \angle DQF = \angle BEF,$$
$$\angle TDC = \angle ETD = \angle DPE = \angle BFE.$$

Let X be the intersection of AC respectively with DT. The lemma leads to

$$\frac{XA}{XC} = \frac{DA}{DC} \cdot \frac{\sin TDA}{\sin TDC} = \frac{DA}{DC} \cdot \frac{\sin BEF}{\sin BFE} = \frac{DA}{DC} \cdot \frac{BF}{BE}.$$

On the other hand, since

$$\angle TKD = \angle TDK = \angle BQE = \angle BCD, \angle TEK = \angle TPD = \angle DBC$$

thus $\triangle BDC \sim \triangle ETK$. Similarly, $\triangle BDA \sim \triangle FTK$. Furthermore,

$$\angle TEF = \angle CDF = \angle DAC, \angle TFE = \angle ADE = \angle DCA$$

thus $\triangle TEF \sim \triangle DAC$. Let Y be the intersection of BK and AC, applying the lemma combining with three pairs of similar triangles that haved proved, we have

$$\begin{aligned}\frac{YA}{YC} = \frac{BA}{BC} \cdot \frac{\sin KBE}{\sin KBF} &= \frac{BA}{BD} \cdot \frac{BD}{BC} \cdot \frac{KE}{KF} \cdot \frac{BF}{BE} \\ &= \frac{BF}{BE} \cdot \frac{FK}{FT} \cdot \frac{ET}{EK}\frac{KE}{KF} = \frac{BF}{BE} \cdot \frac{TE}{TF} = \frac{BF}{BE} \cdot \frac{DA}{DC}.\end{aligned}$$

In conclusion,

$$\frac{XA}{XC} = \frac{BF}{BE} \cdot \frac{DA}{DC} = \frac{YA}{YC}$$

hence X is coincident with Y, thus AC, DT and BK concur.

□

Problem 3. *Find all non-constant functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the equation*

$$f(ab + bc + ca) = f(a)f(b) + f(b)f(c) + f(c)f(a)$$

for all $a, b, c \in \mathbb{Q}^+$.

Solution. Put $c = 1$ in the given condition, we have

$$f(ab + a + b) = f(a)f(b) + f(a)f(1) + f(b)f(1); \quad \forall a, b \in \mathbb{R}^+, \quad (1)$$

Put $b = 3$ into (1), we have

$$f(4a + 3) = f(a)f(3) + f(a)f(1) + f(3)f(1); \quad \forall a \in \mathbb{R}^+.$$

Put $b = 1$ into (1), we have

$$f(2a + 1) = 2f(a)f(1) + f(1)^2; \quad \forall a \in \mathbb{R}^+.$$

Thus

$$f(4a + 3) = 2f(2a + 1)f(1) + f(1)^2 = 4f(1)^2 f(a) + 2f(1)^3 + f(1)^2; \quad \forall a \in \mathbb{R}^+.$$

From these, we can conclude that

$$[f(3) + f(1)]\, f(a) + f(3)f(1) = 4f(1)^2 f(a) + 2f(1)^3 + f(1)^2; \ \forall a \in \mathbb{R}^+.$$

If $f(3) + f(1) \neq 4f(1)^2$ then f is constant. Thus $f(3) + f(1) = 4f(1)^2$, otherwise f will be constant. So we must have

$$f(3) + f(1) = 4f(1)^2 \text{ and } f(3)f(1) = 2f(1)^3 + f(1)^2.$$

Thus $f(3), f(1)$ are solutions of the quadratic equation $t^2 - 2f(1)t + 2f(1)^3 + f(1)^2 = 0$, thus

$$f(1)^2 - 4f(1)^2 + 2f(1)^3 + f(1)^2 = 0 \Leftrightarrow f(1)^2(f(1) - 1) = 0.$$

So we must have $f(1) = 1$ and then $f(3) = 3$. Put $c = 1$ into (1), we have

$$f(ab + a + b) = f(a)f(b) + f(a) + f(b); \quad \forall a, b \in \mathbb{R}^+ \quad (2)$$

Continue to put $b = 1$ and $b = 3$, we get

$$f(4a + 3) = 4f(a) + 3 \text{ and } f(2a + 1) = 2f(a) + 1.$$

Put $a = b = c = \frac{1}{3}$ into the given condition, $f\left(\frac{1}{3}\right) = 3f\left(\frac{1}{3}\right)^2$ so $f\left(\frac{1}{3}\right) = \frac{1}{3}$.
Put $a = 2$ and $b = \frac{1}{3}$ into (2), we have $f(3) = f(2)f\left(\frac{1}{3}\right) + f\left(\frac{1}{3}\right) + f(2)$, thus $f(2) = 2$.

Put $b = c = 2$ into the given condition, $f(4a + 4) = 4f(a) + 4; \quad \forall a \in \mathbb{R}^+$ thus

$$4f(a) + 4 = f(4a + 4) = f\left(4\left(a + \frac{1}{4}\right) + 3\right) = 4f\left(a + \frac{1}{4}\right) + 3.$$

From these, we can conclude that $f\left(a + \frac{1}{4}\right) = f(a) + \frac{1}{4}$, thus

$$f(4a + 4) = 4f(a) + 4; \quad \forall a \in \mathbb{R}^+.$$

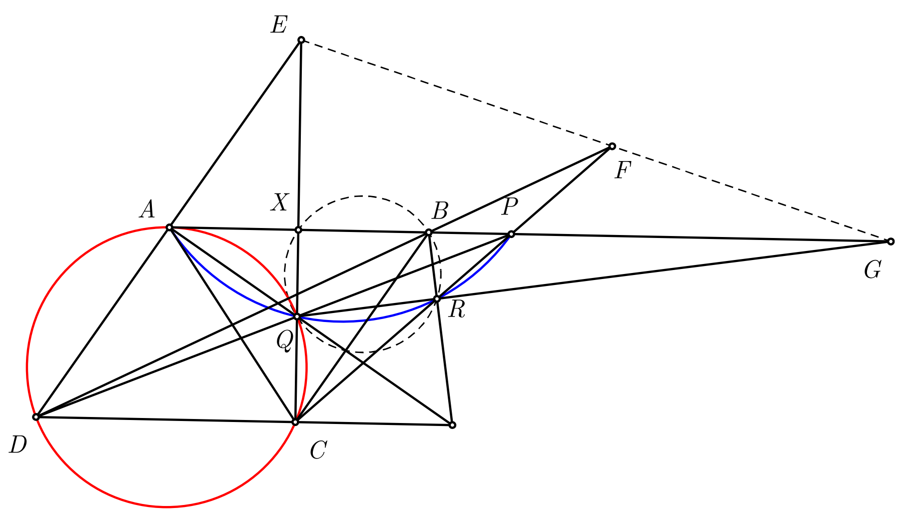
Hence, by induction, one can show that $f(x + n) = f(x) + n$ for all positive integer n and positive real number x; thus $f(n) = n, \quad \forall n \in \mathbb{Z}^+$.
Finally, put $b \to n$ and $a \to \frac{m}{n+1}$ for some $m, n \in \mathbb{Z}^+$ into (2), we get

$$f(m + n) = f(n)f\left(\frac{m}{n + 1}\right) + f\left(\frac{m}{n + 1}\right) + f(n) \to f\left(\frac{m}{n + 1}\right) = \frac{m}{n + 1}.$$

Thus $f(x) = x$ for all $x \in \mathbb{Q}^+$. It is easy to check this function satisfies the condition. □

Problem 4. *Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ and BR are concurrent.*

Solution. (*IMO 2021 Shortlist G1, solved by Marwan Alkhayat*)



Let E be the intersection of AD and CQ, F be the intersection of BD and CP and X be the intersection of CQ and AB. Apply Desargues's theorem on triangles ABD and QRC, one can see that AQ, BR and CD concur if and only if E, F and G are collinear. Apply Menelaus's theorem on triangle ABD it obtains that E, F and G are collinear if and only if

$$\frac{AE}{ED} \cdot \frac{DF}{FB} \cdot \frac{BG}{GA} = 1.$$

On the other hand, by Thales's theorem, we observe that

$$\frac{AE}{ED} = \frac{AX}{CD}, \frac{DF}{FB} = \frac{DC}{BP}.$$

It suffices to prove that

$$\frac{AX}{BP} = \frac{AG}{BG}.$$

Now, we show that $BXQR$ is a cyclic quadrilateral. Since

$$\begin{aligned}\angle ARC &= \angle ARQ + \angle QRC = \angle APQ + \angle QAP \\ &= \angle AQD = \angle ACD = \angle ADC = \angle ABC\end{aligned}$$

since $AC = BC = AD$, thus $ABRC$ is a cyclic quadrilateral. Hence

$$\begin{aligned}\angle BRQ &= \angle BRC - \angle QRC = 180^\circ - \angle CAB - \angle QAP \\ &= 180^\circ - \angle ADC - \angle(AQ, CD) = \angle QAD = 180^\circ - \angle QCD = 180^\circ - \angle QXR.\end{aligned}$$

Then $BXQR$ is a cyclic quadrilateral. Then we have

$$GB \cdot GX = GR \cdot GQ = GP \cdot GA \Leftrightarrow \frac{GA}{GB} = \frac{GX}{GP} = \frac{AX}{BP}$$

as desired.

$\square$

Problem 5. *Given a positive integer n, find the smallest value of*

$$\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \dots + \left\lfloor \frac{a_n}{n} \right\rfloor$$

over all permutations $(a_1, a_2, \dots, a_n)$ of $(1, 2, \dots, n)$.

Solution. (IMO Shortlist 2021, solved by Abdulelah Altaf)

The answer is $\lfloor \log_2 n \rfloor + 1$.

Bijection: We will solve a new problem that is equivalent to the original problem. Draw an $n \times n$ grid and write inside of each cell (in column x and row y) $\lfloor \frac{x}{y} \rfloor$.

Place exactly n rooks such that no two of them attack each other (in the same row or column). Find the least possible sum of all the integers written under the rooks.

Construction: We will provide an example for $\lfloor \log_2 n \rfloor + 1$.

Draw $\lfloor \log_2 n \rfloor + 1$ "red squares" such that each square i contains all the cells from $(2^{i-1}, 2^{i-1})$ (top-left) to $(2^i - 1, 2^i - 1)$ (bottom-right), as shown in the diagram bellow. Note that the last "red square" might be cut-off because of the size of the $n \times n$ grid. Now color the top-right cell of each "red square" purple, as shown in the diagram bellow. And color all the cells inside a "red square" and that are in column x and row $x+1$ yellow. Finally, Place the rooks on the purple and yellow cells.

Notice that in each "red square" the total sum of the chosen cells is exactly 1 (each purple cell is 1 and each yellow cell is 0) making the total sum equal to $\lfloor \log_2 n \rfloor + 1$.

AoMP	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	0	1	1	2	2	3	3	4	4	5	5	6
3	0	0	1	1	1	2	2	2	3	3	3	4
4	0	0	0	1	1	1	1	2	2	2	2	3
5	0	0	0	0	1	1	1	1	1	2	2	2
6	0	0	0	0	0	1	1	1	1	1	1	2
7	0	0	0	0	0	0	1	1	1	1	1	1
8	0	0	0	0	0	0	0	1	1	1	1	1
9	0	0	0	0	0	0	0	0	1	1	1	1
10	0	0	0	0	0	0	0	0	0	1	1	1
11	0	0	0	0	0	0	0	0	0	0	1	1
12	0	0	0	0	0	0	0	0	0	0	0	1

We will prove that the sum of integers chosen in the first 2^m rows is at least $m+1$. $m = \lfloor \log_2 n \rfloor$ (when proven, it obviously follows that our answer is correct).

Base case: When $m = 0$ this is obvious.

Induction $m \to m+1$: *(as shown in figure 1)*

1. **Case 1:** There is at least 1 rook in the blue triangle (as shown in figure 2). This is obvious because - from our inductive claim - the sum of the first 2^m rows is at least $m+1$. Thus, the total sum of chosen squares is at least $m+2$.

2. **Case 2:** All rooks in the last 2^m rows are in the red trapezoid (as shown in figure 2). That means that there is a rook placed in one of the 2^m "big circles" in the last column (as shown in figure 2).

   - Notice that for each "big circle" its value is bigger than the "small circle" (in the same row) by at least 1 (this is obvious).

   - From our inductive claim, the sum of the chosen cells in the "green square" was at least $m+1$.

Combining **(i)** and **(ii)** it follows that the total sum of chosen squares is at least $m+2$, which finishes the proof.

Figure 1

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	0	1	1	2	2	3	3	4
3	0	0	1	1	1	2	2	2
4	0	0	0	1	1	1	1	2
5	0	0	0	0	1	1	1	1
6	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	1

Figure 2

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	0	1	1	2	2	3	3	4
3	0	0	1	1	1	2	2	2
4	0	0	0	1	1	1	1	2
5	0	0	0	0	1	1	1	1
6	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	1

$\square$

Problem 6. *The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a path from X to Y is a sequence of reaods such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called diverse if no road belongs to two or more paths in the collection.*
Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Solution. (*IMO Shortlist 2021, C4*)

We write $X \to Y$ or $Y \leftarrow X$ if the road between X and Y goes from X to Y. Notice that, if there is any route moving from X to Y (possibly passing through some cities more than once), then there is a path from X to Y consisting of some roads in the route. Indeed, any cycle in the route may be removed harmlessly; after some removals 1 obtains a path. Say a path is *short* if it consists of 1 or 2 roads.
Partition all cities different from A and B into four groups, $\mathcal{I}, \mathcal{O}, \mathcal{A}, \mathcal{B}$ according to the following rules: for each city C,

$$C \in \mathcal{I} \iff A \to C \leftarrow B; \qquad C \in \mathcal{O} \iff A \leftarrow C \leftarrow B;$$
$$C \in \mathcal{A} \iff A \to C \to B; \qquad C \in \mathcal{B} \iff A \leftarrow C \leftarrow B.$$

Lemma. Let $\mathcal{P}$ be a diverse collection consisting of p paths from A to B. Then there exists a diverse collection consisting of at least p paths from A to B and containing all short paths from A.

Solution. In order to obtain the desired collection, modify $\mathcal{P}$ as follows.
If there is a direct road $A \to B$ and the path consisting of this single road is not in $\mathcal{P}$, merely add it to $\mathcal{P}$.
Now consider any city $c \in \mathcal{A}$ such that the path $A \to C \to B$ is not in $\mathcal{P}$. If $\mathcal{P}$ contains at most one path containing a road $A \to C$ or $C \to B$, remove that path (if it exists), and add the path $A \to C \to B$ to $\mathcal{P}$ instead. Otherwise, $\mathcal{P}$ contains two paths of the forms $A \to C \dashrightarrow B$ and $A \dashrightarrow C \to B$, where $C \dashrightarrow B$ and $A \dashrightarrow C$ are some paths. In this case, we recombine the edges to form to new paths $A \to C \to B$ and $A \dashrightarrow C \dashrightarrow B$ (removing cycles from the latter if needed). Now we replace the old two paths in $\mathcal{P}$ with the two new ones.
After any operation described above, the number of paths in the collection does not decrease, and the collection remains diverse. Applying such operation to each $C \in \mathcal{A}$, we obtain the desired collection. $\square$

Back to the problem, assume, without loss of generality, that there is a road $A \to B$, and let a and b denote the numbers of roads going out from A to B. We will transform it into a diverse collection $\mathcal{Q}$ consisting of at least $N_{AB} + (b - a)$ paths from A to B. This construction yields

$$N_{BA} \geq N_{AB} + (b - a); \qquad \text{similarly, we get } N_{AB} \geq N_{BA} + (b - a),$$

whence $N_{BA} - N_{AB} = b - a$. This yields the desired equivalence.
Apply the lemma to get a diverse collection $\mathcal{P}'$ of at least N_{AB} paths containing all $|\mathcal{A}| + 1$ short paths from A to B. Notice that the paths in $\mathcal{P}'$ contain no edge of a short part from B to A. Each non-short path in $\mathcal{P}'$ has the form $A \to C \dashrightarrow D \to B$, where $C \dashrightarrow D$ is a path from some city $C \in \mathcal{I}$ to some city $D \in \mathcal{O}$. For each such path, put into $\mathcal{Q}$ the path $B \to C \dashrightarrow D \to A$; also put into $\mathcal{Q}$ all short paths from B to A. Clearly, the collection $\mathcal{Q}$ is diverse.
Now, all roads going out from A end in the cities from $I \cup \mathcal{A} \cup \{B\}$, while all roads going out from B end in the cities from $I \cup \mathcal{B}$. Therefore

$$a = |\mathcal{I} + \mathcal{A} + 1|, \qquad b = |\mathcal{I}| + |\mathcal{B}|, \qquad \text{and hence } a - b = |\mathcal{A}| - |\mathcal{B}| + 1.$$

On the other hand, since there are $|\mathcal{A}| + 1$ short paths from A to B (including $A \to B$) and $|\mathcal{B}|$ short paths from B to A, we infer

$$|\mathcal{Q}| = |\mathcal{P}'| - (|\mathcal{A}| + 1) + |\mathcal{B}| \geq N_{AB} + (b - a),$$

as desired. $\square$

Problem 7. *For every integer $n \geq 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{i \cdot j}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Determine all integers $n \geq 1$ for which the sum of the n^2 entries in the table is equal to $\frac{n^2(n-1)}{4}$.*

Solution. (IMO Shortlist 2021, A2)

First, observe that every pair x, y of real numbers for which the sum $x + y$ is an integer satisfies

$$\lfloor x \rfloor + \lfloor y \rfloor \geq x + y - 1 \qquad (\Diamond)$$

The inequality is strict if x and y are integers and holds with equality otherwise. We estimate the sum S as follows.

$$\begin{aligned} 2S &= \sum_{1\le i,j\le n}\left(\left\lfloor\frac{ij}{n+1}\right\rfloor+\left\lfloor\frac{ij}{n+1}\right\rfloor\right)=\sum_{1\le i,j\le n}\left(\left\lfloor\frac{ij}{n+1}\right\rfloor+\left\lfloor\frac{(n+1-i)j}{n+1}\right\rfloor\right)\\ &\ge \sum_{1\le i,j\le n} j-1=\frac{n^2(n-1)}{2}. \end{aligned}$$

The inequality in the last line follows from $(\lozenge)$ by setting $x=\dfrac{ij}{n+1}$ and $y=\dfrac{(n+1-i)j}{n+1}$ then $x+y=j$. Therefore $S=\dfrac{n^2(n-1)}{4}$ if and only if the inequality in the last line holds with equality which means none of the value $\dfrac{ij}{n+1}$ is an integer where $1\le x,y\le n$.

If $n+1$ is a composite, for example $n+1=ab$ where $1<a,b$, one gets an inequality for $i=a$ and $j=b$. Otherwise, if $n+1$ is prime then $\gcd(n+1,ij)=1$ for every $1\le i,j\le n$ and $S=\dfrac{n^2(n-1)}{4}$. □

Problem 8. *Let n and k be two integers with $n>k\ge 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ neighbours - namely, the k students closest to S on the right and the k students closest to S on the left. Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbours.*

Solution. We replace the girls by $1's$, and the boys by $0's$, getting the numbers $a_1,a_2,\ldots,a_{2n+1}$ arranged in a circle. We extend this sequence periodically by letting $a_{2n+k+1}=a_k$ for all $k\in\mathbb{Z}$. We get an infinite periodic sequence

$$\ldots,a_1,a_2,\ldots,a_{2n+1},a_1,a_2,\ldots,a_{2n+1},\ldots$$

Consider the numbers $b_i=a_i-a_{i-k-1}-1\in\{-1,0,1\}$ for all $i\in\mathbb{Z}$. We know that

$$b_{m+1}+b_{m+2}+\cdots+b_{m+2n+1}=1 \qquad (m\in\mathbb{Z}) \tag{$\lozenge$}$$

in particular, this yields that there exists some i_0 with $b_{i_0}=1$. Now we want to find an index i such that

$$b_i=1 \qquad \text{and} \qquad b_{i+1}+b_{i+2}+\cdots+b_{i+k}\ge 0.$$

This will imply that $a_i=1$ and

$$(a_{i-k}+a_{i-k+1}+\cdots+a_{i-1})+(a_{i+1}+a_{i+2}+\cdots+a_{i+k})\ge k,$$

as desired. Suppose, to the contrary, that for every index i with $b_i=1$ the sum $b_{i+1}+b_{i+2}+\cdots+b_{i+k}$ is negative. We start from some index i_0 with $b_{i_0}=1$ and construct a sequence $i_0,i_1,\ldots$ where i_j is the smallest possible index such that $i_j>i_{j-1}+k$ and $b_{i_j}=1$. We can choose two numbers among $i_0,i_1,\ldots,i_{2n+1}$ which

are congruent modulo $2n+1$ (without loss of generality, we may assume that these numbers are i_0 and i_T).
On the other hand, for every j with $0 \le j \le T-1$ we have

$$S_j := b_{i_j} + b_{i_j+1} + \cdots + b_{i_{j+1}-1} \le b_{i_j} + b_{i_{j+1}} + \cdots + b_{ij+k} \le 0$$

since $b_{i_j+k+1}, \ldots, b_{i_{j+1}-1} \le 0$. On the other hand, since $(i_T - i_0) \mid (2n+1)$, from $(\diamondsuit)$ we deduce

$$S_0 + S_1 + \cdots + S_{T-1} = \sum_{i=i_0}^{i_T-1} b_i = \frac{i_T - i_0}{2n+1} > 0.$$

This contradiction finishes the solution. $\square$

Problem 9. *Prove that there are only finitely many quadruples (a, b, c, n) of positive integers such that $n! = a^{n-1} + b^{n-1} + c^{n-1}$.*

Solution. For fixed n there are clearly finitely many solutions we will show that there is no solution with $n > 100$. So assume $n > 100$. By the AM-GM inequality,

$$\begin{aligned} n! &= 2n(n-1)(n-2)(n-3)\cdot(3\cdot 4\cdots(n-4)) \\ &\le 2(n-1)^4\left(\frac{3+4+\cdots+(n-4)}{n-6}\right)^{n-6} = 2(n-1)^4\left(\frac{n-1}{2}\right)^{n-6} < \left(\frac{n-1}{2}\right)^{n=1}, \end{aligned}$$

thus $a, b, c < (n-1)/2$.
For every prime p and integer $m \ne 0$, let $v_p(m)$ denote the p-adic valuation of m. Legendre's formula states that

$$v_p(n!) = \sum_{s=1}^{\infty} \left\lfloor \frac{n}{p^s} \right\rfloor,$$

and a well-known corollary of this formular is that

$$v_p(n!) < \frac{n}{p-1}. \qquad (\heartsuit)$$

If n is odd then $a^{n-1}, b^{n-1}, c^{n-1}$ are squares, and by considering them modulo 4 we conclude that a, b, and c must be even. Hence $2^{n-1} \mid n!$ but that is impossible for odd n because $v_2(n!) = v_2((n-1)!) < n-1$ by $(\heartsuit)$.

From now on we assume that n is even. If all three numbers $a+b, b+c$ and $c+a$ are powers of 2 then a, b and c have the same parity. If they are odd, then $n! \equiv a+b+c \pmod 2$ is also odd which is absurd. If all a, b, c are divisible by 4, this contradicts $v_2(n!) \le n-1$. If, say, a is not divisible by 4, then $2a = (a+b)+(a+c)-(b+c)$ is not divisible by 8, and since all $a+b, b+c, c+a$ are powers of 2, we get that one of these sums equal to 4, so two of the numbers of a, b, c are equal to 2. Say $a = b = 2$, then $c = 2^r - 2$ and since $c \mid n!$ we must have $c \mid a^{n-1} + b^{n-1} = 2^n$ implying $r = 2$, and so $c = 2$, which is impossible because

$$n! \equiv 0 \not\equiv 3 \cdot 2^{n-1} \pmod 5.$$

So now we assume that the sum of two numbers among a, b, c say $a+b$ is not a power of 2, so it is divisible by some odd number p. Then $p \leq a+b < n$ so $c^{n-1} = n! - (a^{n-1}+b^{n-1})$ is divisible by p. If p divides a and b hence $p^{n-1} \mid n$, contradicting ($\heartsuit$). Next, using ($\heartsuit$) and the LTE Lemma, we obtain

$$\begin{aligned} v_p(1)+v_p(2)+\cdots+v_p(n) &= v_p(n!) = v_p\left(n! - c^{n-1}\right) \\ &= v_p\left(a^{n-1}+b^{n-1}\right) = v_p(a+b) + v_p(n-1). \end{aligned}$$

Now, no number of $1, 2, \ldots, n$ can be divisible by p, except for $a+b$ and $n-1 > a+b$. On the other hand, $p \mid c$ implies that $p < n/2$ and so there must be at least two such numbers. Hence, there are two multiple of p among $1, 2, \ldots, n$, namely $a+b=p$ and $n-1=2p$. But this is another contradiction because $n-1$ is odd. This final contradiction shows that there is no solution of the equation for $n > 100$. $\square$

Problem 10. *There are a) 2022, b) 2023 plates placed around a round table and on each of them there is one coin. Alice and Bob are playing a game that proceeds in rounds indefinitely as follows. In each round, Alice first chooses a plate on which there is at least one coin. Then Bob moves one coin from this plate to one of the two adjacent plates, chosen by him. Determine whether it is possible for Bob to select his moves so that, no matter how Alice selects her moves, there are never more than two coins on any plate.*

Solution. (By Marwan Alkhayat) We will prove that no matter what the number of plates n is, Bob can always guarantee that any k consecutive plates contain at most $k+1$ coins at any time We define the plates periodically since they are on a circle ($P_{n+i} = P_i$).

We will prove our statement inductively, making our moves one by one and making sure the condition is met in each of them The base case is trivial as any k consecutive plates contain exactly k coins. Now let's say that plate P_i is chosen and we can't make a move without breaking the condition.

That means moving a coin to the right (clockwise) or to the left (counter clockwise) is not possible. Moving a coin to the right only effects strings (sequences of plates) that contain P_{i+1} but not P_i (assuming plates are numbered clockwise). That means a string $P_{i+1}, P_{i+2}, \ldots, P_{i+a}$ cannot contain any more coins, so it has exactly $a+1$ coins. Using the same argument, there is a string to the left of P_i, let it be $P_{i-1}, P_{i-2}, \ldots, P_{i-b}$ which has $b+1$ coins.

Since P_i is chosen, it has by default at least 1 coin. That means the string

$$P_{i-b}, P_{i-b+1}, \ldots, , P_{i-1}, P_i, P_{i+1}, \ldots, P_{i+a}$$

before making our move had at least $a+1+b+1+1 = a+b+3$ coins, but that string is of length $a+b+1$. Since we assumed the inductive hypothesis still holds we can say that this case is not possible and we can always make a move without breaking the condition.

Note that we can choose k to be larger than n since nothing bounds us from doing so, and because we can delete the first n plates of the string in that case anyways. Putting $k=1$ gives us what we want. $\square$

Solution 2. (*By Muath Alghamdi*) We prove that Bob has a strategy that always allows him to have at most 2 coins in one plate no matter the number of plates.

Indeed, assume the number of plates is K. Number the plates as P_i, and the coins C_i such that at the start each C_i lies on Pi for all $i = 1, 2, 3, \ldots, k$. If Alice picks plate P_i, Bob does the following:

- A: If P_i only has coin C_i: then Bob moves C_i to P_{i+1}. ($P_{k+1} = P_1$ because the plates are on a circle).
- B: If P_i has coin C_{i-1} (not necessarily alone) : Bob takes C_{i-1} and moves it to $P_{i-1.}$ ($P_k = P_0$ because the plates are on a circle).

Using this method, coin C_i has two ways of moving, either from P_i to P_{i+1} (from A) or from P_{i+1} back to P_i (from B). This means each C_i belongs to P_i or P_{i+1}, so each P_i contains C_i or C_{i-1} so there are at most two coins in P_i for all K. ☐

Problem 11. *Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for every $i = 1, 2, \ldots, k$ the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.*

Solution. (*IMO Shortlist 2021, N3*) For $i = 1, 2, \ldots, k$, let $d_1 + \ldots + d_i = s_i^2$ and define $s_0 = 0$ as well. Obviously $0 = s_0 < s_1 < \ldots < s_k$, so

$$s_i \leq i \quad \text{and} \quad d_i = s_i^2 - s_{i-1}^2 = (s_i + s_{i-1})(s_i - s_{i-1}) \geq s_i + s_{i-1} \geq 2i - 1.$$

The number 1 is one of the divisors $d_1, d_2, \ldots, d_k$ but, due to $d_i \leq 2i - 1$, the only possibility is $d_1 = 1$. Now consider d_2 and $s_2 \geq 2$. By definition, $d_2 = s_2^2 - 1 = (s_2 - 1)(s_2 + 1)$, so the number $s_2 - 1$ and $s_2 + 1$ are divisors of n. In particular, there is some index j such that $d_j = s_2 + 1$.
Notice that

$$s_2 + s_1 = s_2 + 1 = d_j \leq s_j + sj - 1;$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, the index j cannot be greater than 2. Hence, the divisors $s_2 - 1$ and $s_2 + 1$ are listed among d_1 and d_2. That means $s_2 - 1 = d_1 = 1$ and $s_2 + 1 = d_2$; therefore $s_2 = 2$ and $d_2 = 3$.
We can repeat the above process in general.
Claim. $d_i = 2i - 1$ and $s_i = i$ for $i = 1, 2, \ldots, k$.

Solution. Apply induction on i. The Claim has been proved for $i = 1, 2$. Suppose that we have already proved $d_1 = 1, d_2 = 3, \ldots, d_i = 2i - 1$, and consider the next divisor d_{i+1};

$$d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2 = (s_{i+1} - i)(s_{i+1} + i).$$

The number $s_{i+1} + i$ is a divisor of n, so there is some index j such that $d_j = s_{i+1} + i$. Similarly to above arguments, we have

$$s_{i+1} + s_i = s_{i+1} + i = d_j \leq s_j + s_{j-1};$$

since the sequence $s_0 < s_1 < \ldots < s_k$ increases, it easy to check that $j \leq i + 1$. On the other hand, $d_j = s_{i+1} + i > 2i > d_i > \ldots > d_1$, so $j \leq i$ is not possible. The only possibility is $j = i + 1$.

Hence,

$$s_{i+1} + i = d_{i+1} = s_{i+1}^2 - s_i^2 = s_{i+1}^2 - i^2;$$
$$s_{i+1}^2 - s_i = i(i+1).$$

By solving this equation we get $s_{i+1} = i + 1$ and $d_{i+1} = 2i + 1$, that finishes the proof. □

Now we know that the positive divisors of the number n are $1, 3, 5, \ldots, n-2, n$. The greatest divisor is $d_k = 2k - 1 = n$ itself, so n must be odd. The second greatest divisor is $d_{k-1} = n - 2$; then $n - 2 \mid n$ therefore n must be 1 or 3.
Obviously, these values satisfy the given conditions. □

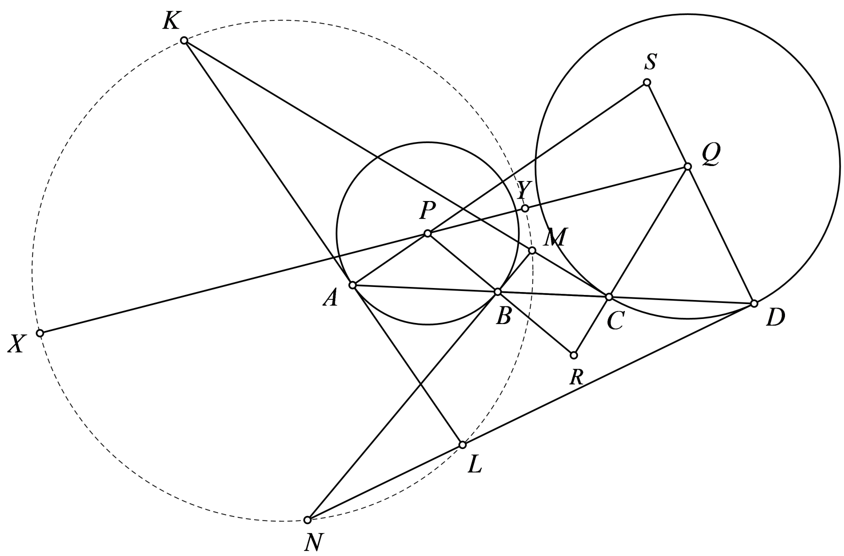
Problem 12. *Let A, B, C, D be points on the line d in that order and $AB = CD$. Denote (P) as some circle that passes through A, B with its tangent lines at A, B are a, b. Denote (Q) as some circle that passes through C, D with its tangent lines at C, D are c, d. Suppose that a cuts c, d at K, L respectively; and b cuts c, d at M, N respectively. Prove that four points K, L, M, N belong to a same circle (ω) and the common external tangent lines of circles $(P), (Q)$ meet on (ω).*

Solution. Consider the points that arranged as following figure, the other cases will be proved similarly.

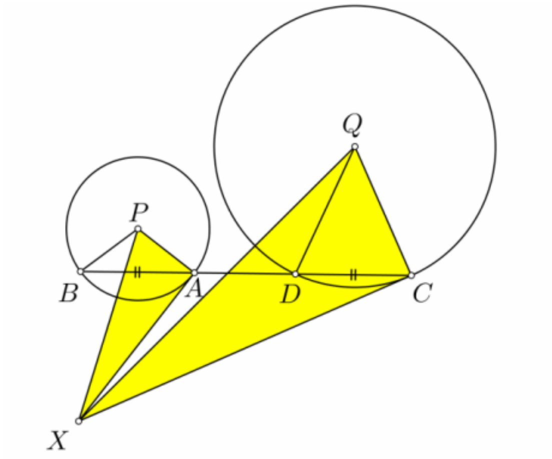
Denote R, S as intersection of the pairs of lines PB, QC and PA, QD. Note that $BMCR$ and $ALDS$ are cyclic quadrilateral. Thus

$$\begin{aligned}\angle KMN = \angle BRC &= 180^\circ - (\angle BCR + \angle CBR)\\ &= 180^\circ - (\angle PBA + \angle QDC)\\ &= 180^\circ - (\angle PAD + \angle QDA) = \angle ASD = \angle KLN.\end{aligned}$$

Hence, K, L, M, N belong to the same circle.



Now, consider the following claim: Let be given two circles $(P, R), (Q, R')$ and some line cuts them at B, A, D, C such that $AB = CD$ (see the figure). The tangent lines at A, C respectively of $(P), (Q)$ meet at X then $\frac{XP}{XQ} = \frac{R}{R'}$.



Indeed, by applying the sine law for triangle XAC, we get

$$\frac{XA}{XC} = \frac{\sin XCA}{\sin XAC} = \frac{\sin \frac{CQD}{2}}{\sin \frac{APB}{2}} = \frac{CD}{AB} \cdot \frac{R}{R'} = \frac{R}{R'} = \frac{PA}{QC}.$$

Thus two triangles XPA and XQC are similar, which implies that $\frac{XP}{XQ} = \frac{R}{R'}$. The claim is proved.

Back to the problem, denote X, Y as the external and the internal homothety centers of $(P), (Q)$ then $\frac{XP}{XQ} = \frac{YP}{YQ} = k$ with k is the ratio of radius of $(P), (Q)$. It is easy to check that these radiuses are different, otherwise the the tangent lines of $(P), (Q)$ will be parallel and points K, L, M, N will not exist, thus $k \neq 1$. In the other hand, by applying the above claim, we get

$$\frac{MP}{MQ} = \frac{NP}{NQ} = \frac{KP}{KQ} = \frac{LP}{LQ} = k.$$

Hence, six points X, Y, M, N, K, L are all belong to the Apollonius circle with ratio k constructing on the segment PQ. Thus, the point X, which also is the intersection of two common external tangent lines of $(P), (Q)$, is on (ω). $\square$

Selected problems from training

Problem 1. *Prove that for every polynomial $P(x)$ there exist polynomials $Q(x)$ and $R(x)$ such that $P(x) = Q(x^2) + R((x+1)^2)$.*

Problem 2. *Find all quadruples (p, a, b, x) that satisfy the equality $p^2 + 4^a 9^b = x^2$, where p is a prime and a, b, x nonnegative integers.*

Problem 3. *Given is an equilatereal triangle ABC. Points D, E, F lie on sides BC, CA, AB, respectively, and satisfy $AF = BD$ and $DF = EF \neq DE$. Prove that $\angle CDE = 90°$.*

Problem 4. *Two players, A and B play the following game. On a $1 \times n$ board, where fields are labeled in order from 1 to n, a coin is placed at position k. Players take turns moving the coin, with player A starting first. Each player can move a coin one or two fields in either direction, with the restriction that the coin cannot move onto a field it had already occupied. The player unable to make a move loses. For which values of (n, k) does which player have a winning strategy?*

Problem 5. *Find the number of integer solutions of the equation*

$$\left[\frac{x}{7}\right] = \left[\frac{x}{12}\right] + \left[\frac{x}{17}\right].$$

$[x]$ is the largest integer not exceeding x.

Problem 6. *Point P lies inside parallelogram $ABCD$ and satisfies $PC = BC$. Prove that the line joining midpoints of segments AP and CD is perpendicular to BP.*

Problem 7. *Let n be a natural number. Find the number of permutations of the set $\{1, 2, \ldots, n\}$ such that for each $i = 1, 2, \ldots, n$, the first i numbers in the permutation are not larger than $i + 1$. For example, there are 4 such permutations for $n = 3$: $\{1, 2, 3, \}$, $\{2, 1, 3\}$, $\{1, 3, 2\}$ and $\{2, 3, 1\}$.*

Problem 8. *The sequence (a_k) is given by $a_1 = \frac{1}{2}$ and $a_{n+1} = 1 - a_1 a_2 \cdots a_n$ for every $n \geq 1$. Prove that $a_{100} > 0.99$.*

Problem 9. *Find all integer triples (a, b, c) satisfying the equation*

$$5a^2 + 9b^2 = 13c^2.$$

Problem 10. *Let I be the incenter of the triangle ABC. Let X lies on segment AB, such that $\angle AIX = 90°$. The circumcircle of triangle BIX intersects circumcircle of triangle ABC at point $Y \neq B$ lying on the same side of AB as point C. Prove that YX is bisector of angle AYB.*

Problem 11. *Let $x, y \in \mathbb{R}$ be such that $x = y(3-y)^2$ and $y = x(3-x)^2$. Find all possible values of $x + y$.*

Problem 12. *In each cell of a 10×10 board one arrow is placed. Each arrow is pointing in one of the four directions $\{\uparrow, \rightarrow, \downarrow, \leftarrow\}$. Find the smallest number n with the following property: it is always (regardless of the initial placement of the arrows) possible to remove at most n arrows from the board in such a way that among the remaining ones no two are pointing at each other. Note: arrows are pointing at each other also if there are some other arrows or empty cells between them.*

Problem 13. *Different positive a, b, c are such that $a^{239} = ac - 1$ and $b^{239} = bc - 1$.Prove that $238^2(ab)^{239} < 1$.*

Problem 14. *What is the maximum number of $2 \times 3 \times 3$ bricks that can be fit inside an $8 \times 8 \times 9$ box?*

Problem 15. *Let O be the circumcenter of triangle ABC. Points X and Y on side BC are such that $AX = BX$ and $AY = CY$. Prove that the circumcircle of triangle AXY passes through the circumcenters of triangles AOB and AOC.*

Problem 16. *You are given n different primes $p_1, p_2, ..., p_n$. Consider the polynomial*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n,$$

where a_i is the product of the first i given prime numbers. For what n can it have an integer root?

Problem 17. *The perpendicular bisector to the side AC of triangle ABC meets BC and AB at points A_1 and C_1 respectively. Let points O and O_1 are the circumcenters of triangles ABC and A_1BC_1 respectively. Prove that C_1O_1 is a tangents the circumcircle of the triangle ABC.*

Problem 18. *We call a positive integer n* venerable *if all its positive divisors less than n (but including 1) add up to $n - 1$. Find all venerable numbers whose some power (with the exponent at least 2) is also venerable.*

Problem 19. *Find all triples (a, b, c) of real numbers satisfying*

$$a + b + c = 1 \quad \text{and} \quad 3(a + bc) = 4(b + ca) = 5(c + ab).$$

Problem 20. *A pile of $2^n - 1$ coins is placed on the player. Alice is allowed to make the following moves: If a pile has an even number of coins, she can throw half of those coins away. If a pile has an odd number of coins k such that $k + 1$ is properly divisible by 2^l (i.e. $2^l | k + 1$ and $2^l < k + 1$) for some positive integer l, Alice can take away $2^l - 1$ coins from this pile and form a new pile with those coins. Alice continues making moves until no more moves can be made. What are the possible values for the number of coins that will remain on the table?*

Problem 21. *Characterize all positive integers $n > 1$ for which the expression*

$$(n-1)! \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right)$$

is not divisible by n.

Problem 22. *Let $x(n)$ be the biggest prime divisor of n. Prove that there exist infinitely many number n such that $x(n) < x(n+1) < x(n+2)$.*

Problem 23. *Let the sequence a_1, a_2, …, a_{20} is the permutation of integers 1, 2, …, 20. Find the maximum possible value of*

$$\min\{|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_{20} - a_{19}|, |a_1 - a_{20}|\}.$$

Problem 24. *Two triangles ABC and $A_1B_1C_1$ are symmetric about the center of their common incircle of radius r. Prove that the product of the areas of the triangles ABC, $A_1B_1C_1$ and the six other triangles formed by the intersecting sides of the triangles ABC and $A_1B_1C_1$ is equal to r^{16}.*

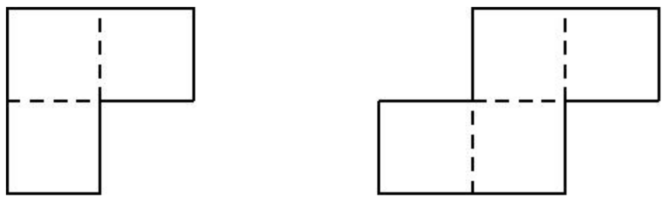
Problem 25. *Does there exist a polynomial $P(x)$ of second degree with integer coefficients such that the leading coefficient is not divisible by 2022 and all numbers $P(1)$, $P(2)$, …, $P(2022)$ give different residues mode 2022.*

Problem 26. *Prove that for any 2 positive integers m and n with $(m, n) \neq (1, 1)$ the value of expression*

$$\frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{m+n-1}$$

is not an integer.

Problem 27. *Let m and n are positive integers greater than 3. Determine the least number of figures (see picture below) needed to cover the rectangle of size $(2n-1) \times (2m-1)$.*



Problem 28. *Let $ABCDE$ is a pentagon such that $BC \parallel AE$, $AB = BC + AE$ and $\angle B = \angle D$. Let M be the midpoint of CE and O be the circumcenter of ΔBCD. Show that if $OM \perp MD$ then $\angle CDE = 2\angle ADB$.*

Problem 29. *Prove that for any positive integer n at least on coefficient of the polynomial*

$$\left(x^4 + x^3 - 3x^2 + x + 2\right)^n$$

is negative.

Problem 30. *Let polynomial*

$$P(x) = \underbrace{((\ldots((x-2)^2 - 2)^2 - \ldots)^2 - 2)^2}_{k \text{ times}}$$

is given. Find coefficient at x^2.

Problem 31. *Let* $f(x) = x^2 - 6x + 5$. *Draw on the plane the set of pairs* (x, y) *that satisfy to the following system on inequalities*

$$\begin{cases} f(x) + f(y) \leq 0 \\ f(x) - f(y) \geq 0 \end{cases}.$$

Problem 32. *Prove that for any* 2 *positive integers* m *and* n *with* $m > n$ *holds the following inequality*

$$lcm(m, n) + lcm(m + 1, n + 1) > \frac{2mn}{\sqrt{m - n}}.$$

Problem 33. *Let convex* s*-gon is divided to* q *quadrilaterals such that* b *of them are not convex. Prove that*

$$q \geq b + \frac{s - 2}{2}.$$

Problem 34. *Let positive numbers are written along the circle, such that all of them are less than* 1. *Prove that one can split the circle to* 3 *parts such that for each two arcs the sums of numbers written on them differs by at most* 1.

Problem 35. *Let the triangle* ABC *is given and* D, E, F *are on sides* BC, AC, AB, *respectively, such that*

$$\frac{BD}{CD} = \frac{CE}{AE} = \frac{AF}{BF}.$$

Show that if the circumcircle of ABC *and* DEF *coincide, then* ABC *is equilateral.*

Problem 36. *Let the sequence* a_1, a_2, $\ldots$, a_n *is such that* $a_1 = 0$, $|a_2| = |a_1 + 1|$, $|a_3| = |a_2 + 1|$, $\ldots$, $|a_n| = |a_{n-1} + 1|$: *Prove that*

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq -\frac{1}{2}.$$

Problem 37. *Do there exist an infinite sequence* p_1, p_2, p_3, $\ldots$ *of prime numbers such that for any positive integer* n *the following condition holds*

$$|p_{n+1} - 2p_n| = 1.$$

Problem 38. *Let incircle of triangle* ABC *has center* I *and touches sides* BC, AC *and* AB *at points* D, E, F *respectively. Let* J_1, J_2, J_3 *be te ex-centres opposite* A, B, C *respectively. Let* J_2F *and* J_3E *intersect at* P, J_3D *and* J_1F *intersect at* Q, J_1E *and* J_2D *intersect at* R. *Show that* I *is the circumcenter of* PQR.

Problem 39. *Find all positive integers* n *such that the products of its digits is equal* $n^2 - 10n - 22$.

Problem 40. *Prove there exist infinitely many positive integers divisible by* 2021 *and each of them containing the same number of digits* 0, 1, $\ldots$, 9.

Problem 41. *Find all values of* a *for which the equation* $x^3 + ax^2 + 56x - 4 = 0$ *has* 3 *roots forming consecutive terms of a geometric progression.*

Problem 42. *Let* $f(x) = \frac{9^x}{9^x+3}$. *Evaluate the sum*

$$\sum_{k=0}^{2021} f\left(\frac{k}{2021}\right).$$

Problem 43. *One cuts a grid of size* 8×8 *by a straight line. Find the maximal possible number of cells that are cut by the line.*

Problem 44. *In the cells of the grid* 10×10 *are written positive integers, all of them less than* 11. *It is known that the sum of* 2 *numbers written in the cells having common vertex is a prime number. Prove that there are* 17 *cells containing the same number.*

Problem 45. *Given a non-isosceles acute angled triangle* ΔABC *where* O *is the midpoint of* BC. *Let the circle with diameter* BC, *intersects* AB, AC *at* D, E *respectively. Let the angle bisectors of* $\angle A$ *and* $\angle DOE$ *intersect at* P. *If the circumcircles of* ΔBPD *and* ΔCPE *intersect at* P *and* Q, *show that* Q *lies on* BC.

Problem 46. *Prove there exist infinitely many positive integers divisible by* 2021 *and each of them containing the same number of digits* $0, 1, \ldots, 9$.

Problem 47. *Do there exist positive integers* m *and* n *such that the decimal representation of* 5^m *starts with* 2^n *and the decimal representation of* 2^m *starts with* 5^n*?*

Problem 48. *Let the sequence* a_i *is defined in the following way:* $a_1 = m \in Z_+$ *and inductively* $a_{i+1} = a_i + \left[\sqrt{a_i}\right]$. *Prove that the sequence* a_i *contains infinitely many perfect square.*

Problem 49. *In each cell of a chessboard (sizes* 8×8*) is put a rock. At each step one can remove from the board one rock which beats an odd number of other rocks (for example in initial configuration top-left rock beats* 2 *rocks). Find the maximal possible number of rocks one can remove from the board.*

Problem 50. *Given* ΔABC *where* $AB < AC$, M *is the midpoint of* BC. *The circle* O *passes through* A *and is tangent to* BC *at* B, *intersecting the lines* AM, AC *at* D, E *respectively. Let* $CF \parallel BE$, *intersecting* BD *extended at* F. *Let the lines* BC *and* EF *intersect at* G. *Show that* $AG = DG$.

Problem 51. *Let* a, b, c, d *be real numbers such that* $a^4 + b^4 + c^4 + d^4 = 16$. *Prove the inequality* $a^5 + b^5 + c^5 + d^5 \leq 32$.

Problem 52. *Let* $S(n)$ *be the sum of divisors of* n *(for example* $S(6) = 1+2+3+6 = 12$*). Find all* n *for which* $S(2n) = 3S(n)$.

Problem 53. *Find all pairs of possitive integers* (x, y) *such that*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{lcm(x,y)} + \frac{1}{gcd(x,y)} = \frac{1}{2}.$$

Problem 54. *Marwan has choosen 8 cells of the chessboard 8×8 such that no any two lie on the same line or in the same row (we call it general configuration). On each step hamza chooses 8 cells in general configuration and puts coins on them. Then Marwan shows all coins that are out of cells chosen by Marwan. If Marwan shows even number of coins then Hamza wins, otherwise Hamza removes all coins and makes the next move. Find the minimal number of moves that Hamza needs to guarantee the win (Hamza>Marwan).*

Problem 55. *Let $1 \leq r \leq n$. We consider all r-element subsets of $(1, 2, \ldots, , n)$. Each of them has a minimum. Prove that the average of these minima is $\frac{n+1}{r+1}$.*

Problem 56. *Given an acute angled triangle ΔABC and its circumcenter O. Let AD, BE, CF are altitudes of the triangle. Show that the line segments OA, OF,OB, OD, OC, OE dissect ΔABC into three pairs of triangles that have equal areas.*

Problem 57. *Consider the positive numbers $x_1, x_2, \ldots, x_n$ such that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{x_i}$. Prove that $\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1$.*

Problem 58. *Find all integer numbers m and n such that*

$$(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n.$$

Problem 59. *Twenty children are queueing for ice cream that is sold at SR5 per cone. Ten of the children have a SR5 coin, the others want to pay with a R10 bill. At the beginning, the ice cream man does not have any change. How many possible arrangements of the twenty kids in a queue are there so that the ice cream man will never run out of change?*

Problem 60. *Given ΔABC, D is on BC and P is on AD. A line ℓ is passing through D intersects AB, PB at M, E respectively, and intersects AC extended and PC extended at F, N respectively. Let $DE = DF$. Prove that $DM = DN$.*

Problem 61. *Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.*

Problem 62. *Prove that for $n \geq 1$ the following inequality holds*

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Problem 63. *Let $x, y, z \geq 0$ and $x + y + z = 3$. Prove that*

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + xz + zx.$$

Problem 64. *Let $a, b, c > 0$. Prove that*

$$\frac{a+b}{a^2+b^2} + \frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Problem 65. *Let $n > 3$, $x_1, x_2, \ldots, x_n > 0$ and $x_1 x_2 \ldots x_n = 1$. Prove that*

$$\frac{1}{1 + x_1 + x_1 x_2} + \frac{1}{1 + x_2 + x_2 x_3} + \ldots + \frac{1}{1 + x_n + x_n x_1} > 1.$$

Problem 66. *Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.*

Problem 67. *Prove the identity*

$$\frac{n!}{x(x+1)(x+2)\dots(x+n)} = \frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \dots + (-1)^n \frac{\binom{n}{n}}{x+n}.$$

Problem 68. *Let A be one of two points of intersection of two unequal circles in the same plane. If two particles from A move each on a circle in a clockwise direction with two uniform velocities until they return to point A at the same instant. Prove that there is always a point in the plane that is equidistant from the two particles at any moment while they are in motion.*

Problem 69. *Let the parabola $y = x^2 + px + q$ is given, which intersects coordinate axes in 3 different points. Consider the circumcircle of the triangle having vertices these 3 points. Prove that there is a point that belongs to that circle, regardless of values p and q. Find that point AoMP.*

Problem 70. *Let ABC be a right-angled triangle with $\angle A = 90^\circ$ and let AD is an altitude of the triangle ABC. Let J, K be the incenters of the triangles ABD, ACD respectively. Let JK intersects AB, AC at E, F respectively. Prove that $AE = AF$.*

Problem 71. *Find all integer polynomials P for which $(x^2 + 6x + 10)P^2(x) - 1$ is the square of an integer polynomial.*

Problem 72.

1. *Find the minimum number of elements that must be deleted from the set $\{1, 2 \dots, 2018\}$ such that the set of the remaining elements does not contain two elements together with their product.*

2. *Does there exist, for any k, an arithmetic progression with k terms in the infinite sequence*
$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots.$$

Problem 73. *Prove that not all zeros of a polynomial of the form $x^n + 2nx^{n-1} + 2n^2x^{n-2} + \dots$ can be real.*

Problem 74. *Let the polynomial $P(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_1x + a_0$ with all coefficients $a_i \in [100, 101]$ is given. Find the minimal possible value of n for which $P(x)$ has a root.*

Problem 75. *Let AD is the altitude of the triangle ABC. Let J, K be the incenters of the triangles ABD, ACD respectively. Let JK intersects AB, AC at E, F respectively. Prove that $AE = AF$ if and only if $AB = AC$ or $\angle A = 90^\circ$.*

Problem 76. *Find all positive integers n such that*

$$3^{n-1} + 5^{n-1} | 3^n + 5^n.$$

Problem 77. *Let ABC be a triangle with midpoints K, M, N of BC , CA , AB respectively. Let AD, BE, CF are the altitudes of the triangle ABC and let U, V, W are the midpoints of FD, DE , EF respectively. Prove that KW, MV, NU intersect at one point.*

Problem 78. *Consider the sequence $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.*

Problem 79. *Prove that every integer can be written as a sum of 5 perfect cubes (may be negative).*

Problem 80. *Let $\phi_n(m) = \phi(\phi_{n-1}(m))$, where $\phi_1(m) = \phi(m)$ is the Euler totient function, and set $\omega(m)$ the smallest number n such that $\phi_n(m) = 1$. If $m < 2^\alpha$, then prove that $\omega(m) \leq \alpha$.*

Problem 81. *Consider the lattice in the plane, from which we may cut rectangles, but only by making cuts along the lines of the lattice. Prove that for any integer $m > 12$ one may cut a rectangle o area grater than m such, that from that rectangle one can't cut a rectangle of area m. The Yellow Fish*

Problem 82. *A quadrilateral $ABCD$ is inscribed inside a circle and $AD \perp CD$. Draw $BE \perp AC$ at E and $BF \perp AD$ at F. Show that the line EF passes through the midpoint of the line segment BD.*

Problem 83. *Find all pairs of integers (m, n) such that $\binom{n}{m} = 1984$.*

Problem 84. *Let u_n be the least common multiple of the first n terms of a strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots, a_{1000}$. Prove that*

$$\sum_{k=1}^{1000} \frac{1}{u_k} \leq 2.$$

Problem 85. *Let $\sigma(n)$ denote the sum of the divisors of n. Prove that there exist infinitely many integers n such that $\sigma(n) > 3n$. Prove also that $\sigma(n) < n(1+\log_2 n)$.*

Problem 86. *Let $\sigma(n)$ denote the sum of divisors of n. Show that $\sigma(n) = 2^k$ if and only if n is a product of Mersenne primes, i.e., primes of the form $2^k - 1$.*

Problem 87. *Let $a_1 = 1$, $a_{n+1} = a_n + [\sqrt{a_n}]$. Find all n that a_n is a perfect square.*

Problem 88. *In an acute angled triangle ΔABC, let D is on BC such that $AD \perp BC$. Let O and H be the circumcenter and orthocenter of ΔABC respectively. The perpendicular bisector of AO intersects BC extended at E. Show that the midpoint of OH is on the circumcircle of ΔADE.*

Problem 89. *Prove that for any natural number $n > 1$ then $2^n - 1 \nmid 3^n - 1$.*

Problem 90. *Given a quadrilateral $ABCD$, the external angle bisectors of $\angle CAD$, $\angle CBD$ intersect at P. Show that if $AD + AC = BC + BD$, then $\angle APD = \angle BPC$.*

Problem 91. *On a 9×9 board, several cells are shaded in such a way that from any shaded cell you can get to any other shaded cell, visiting only the shaded cells and moving only between cells neighboring with a side. Determine the largest possible perimeter of the shaded region.*

Problem 92. *Let ABC be an acute, non isosceles triangle which circumcircle (O). Take M on segment BC and J on (O) such that $\angle BAJ = \angle CAM$. Take a point D on the segment AM and denote O_1, O_2 as the circumcenters of triangles ABD and ACD. The line O_1O_2 meets BC at T and G is the second intersection of circumcircles of triangles TBO_1 and TCO_2. Prove that OA, JA respectively divide the segments O_1O_2, BC by the same ratio and A, G, O, J are concyclic.*

Problem 93. *Let ABC be an acute, non isosceles triangle with orthocenter H and circumcenter O. Denote D, E as midpoints of segments AB, AC. Take M, N on BC such that $MB = BC = CN$ (B is between M, C and C is between N, B). Denote P, Q as the projections of H onto the lines BE, CD. The circumcircles of triangle ABN, ACM respectively meets AQ, AP again at Y, X. Suppose that the line XY cuts BC at K. Prove that AK is perpendicular to OH.*

Problem 94. *From a point O lying outside the line d, draw the projection A of O onto d. Take some point M on d differs from A and H is the projection of A onto OM. Denote D as the midpoint of HM and take N on OA such that $NH \perp AD$. Suppose that two circumcircles of triangles HMN and OAH are tangent, calculate the ratio $\frac{AM}{AO}$.*

Problem 95. *From a point A lying outside the circle (O), draw two tangent lines AB, AC of (O) with B, C are tangent points. A line passes through A, lies inside the angle OAC, cuts (O) at R, S (R is between A and S). The segments BR, BS cut the ray AO respectively at D, E. Denote H as orthocenter and I as circumcenter of triangle BDE. Let BT be the diameter of circumcircle of (I). Prove that $\Delta DHT \sim \Delta RBS$ and calculate the ratio that OI divides BC.*

Problem 96. *Each pair of vertices of a regular $1001-$gon is joined with a segment, which is either red, or blue, or green. Prove that one can choose 11 vertices of this $1001-$gon in such a way that they form a convex $11-$gon, in which at least 10 sides have the same color.*

Problem 97. *Given is an odd integer $n \geq 1$. Let S be the set of all points in the three dimensional space, whose all coordinates belong to the set $\{0, 1, ..., n\}$. Determine the maximum size of a subset $A \subset S$ with the following property: For every two distinct points $(x_1, x_2, x_3), (y_1, y_2, y_3) \in A$ among the three numbers $x_1 - y_1, x_2 - y_2, x_3 - y_3$ there is at least one positive number and at least one negative number.*

Problem 98. *Given is a convex n-gon with no four vertices concyclic. A triple of vertices is called 'round' if the circle passing through these points covers the entire polygon. Determine, in terms of n, all possible values of the number of round triples.*

Problem 99. *Five positive reals a, b, c, d and e having product equal to 1 are given. Prove that*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{d^2} + \frac{d^2}{e^2} + \frac{e^2}{a^2} \geq a + b + c + d + e.$$

Problem 100. *Let a finite set of integers be given, such that each of its elements can be written as a sum of some two elements (not necessarily distinct) from the same set. For such a set, we say that it is of 'safety of order n' if it does not contain a subset of n or less elements whose sum is 0. Prove that there exist sets of arbitrarily large safety order.*