8. Solve in integers the equation

$$(2x+y)(2y+x) = 9\min(x,y).$$

WLOG assume that $x \geq y$

$$(2x+y)\underbrace{(2y+x)}_{m} = 9y$$

$$x = -2y+m \Rightarrow 2x+y = -4y+2m+y = -3y+2m$$

$$m(2m-\underline{3y}) = \underline{9y}$$

$3 \mid m(2m-3y)$, if $\underline{\gcd(m,3)=1} \Rightarrow 3 \mid 2m-3y \Rightarrow 3 \mid 2m$

$\Rightarrow 3 \mid m \Rightarrow \Leftarrow$ Therefore, $3 \mid m$

$m = 3n$

$$3n(6n-3y) = 9y$$

$$n(2n-y) = y$$

$$2n^2 - ny = y$$

$$2n^2 = y(n+1) \Rightarrow y = \frac{2n^2}{n+1} = \frac{2n^2-2}{n+1} + \frac{2}{n+1}$$

$$\Rightarrow y = 2(n-1) + \frac{2}{n+1}$$

However, y is an integer $\Rightarrow n+1 \mid 2$

$\Rightarrow n = 1, 0, -2, -3$

$\Rightarrow y = 1, 0, -8, -9$

$\Rightarrow x = 1, 0, 10, 9$

$\Rightarrow (x,y) = (1,1), (0,0), (10,-8), (9,-9)$

9. Let $n \geq 4$ and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \text{ and } a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that

$$\max\{a_1, a_2, \ldots, a_n\} \geq 2.$$

Assume that $a_i < 2 \quad \forall \quad i = 1, 2, \ldots, n$

Let $b_i = 2 - a_i \Rightarrow b_i > 0 \ (1) \ \forall \ i = 1, 2, \ldots, n$

and $S = b_1 + b_2 + \cdots + b_n \Rightarrow S > 0 \quad (2)$

- $n \leq a_1 + a_2 + \cdots + a_n = 2n - S \Rightarrow n \geq S \ (3)$
- $n^2 \leq a_1^2 + a_2^2 + \cdots + a_n^2 = \sum_{i=1}^{n} (2 - b_i)^2$

$$= \sum_{i=1}^{n} 4 - 4b_i + b_i^2$$

$$= 4n - 4\sum_{i=1}^{n} b_i + \sum_{i=1}^{n} b_i^2$$

$$= 4n - 4S + \sum_{i=1}^{n} b_i^2$$

$$n^2 \leq 4n - 4S + \sum b_i^2 < 4n - 4S + \left(\sum_{i=1}^{n} b_i\right)^2$$

$$n^2 < 4n - 4S + S^2$$

$$n^2 - 4n + 4 < S^2 - 4S + 4 \Rightarrow (n-2)^2 < (S-2)^2$$

* Case 1: $S \geq 2 \Rightarrow S - 2 \geq 0 \Rightarrow S - 2 > n - 2$

$\Rightarrow S > n$ but $n \geq S \Rightarrow \Leftarrow !!$

* Case 2: $S < 2 \Rightarrow 2 - S > 0 \Rightarrow 2 - S > n - 2$

$\Rightarrow 2 > n - 2 \Rightarrow 4 > n$

$\Rightarrow \Leftarrow !!$

$\square$

Another way :

$$n^2 - 4n < s^2 - 4s$$

$$n^2 - s^2 - 4n + 4s < 0$$

$$(n-s)(n+s-4) < 0$$

However, $n \geq s$ and $n \geq 4,\ s > 0$

$\Rightarrow n+s-4 > 0$

which is contradiction

26. Solve in real numbers the system

$$\begin{cases} ab(a+b)+bc(b+c)+ca(c+a)=2 \\ ab+bc+ca=-1 \\ ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=-2. \end{cases}$$

Denote $\begin{cases} x = a+b+c \\ y = ab+bc+ca \\ z = abc \end{cases}$

Then $ab(a+b)+bc(b+c)+ca(c+a)$

$$= \sum_{cyc} a^2b + \sum_{cyc} ab^2 = (ab+bc+ca)(a+b+c) - 3abc$$

$$= xy - 3z$$

$$\begin{cases} xy - 3z = 2 \quad (1) \\ y = -1 \quad (2) \end{cases}$$

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) = -2$$

$$ab(a^2+b^2+c^2-c^2)+bc(a^2+b^2+c^2-a^2)+ca(a^2+b^2+c^2-b^2)=-2$$

$$(ab+bc+ca)(a^2+b^2+c^2) - abc^2 - bca^2 - cab^2 = -2$$

$$\underbrace{(ab+bc+ca)}_{y}\ \times\ \underbrace{(a^2+b^2+c^2)}_{(x^2-2y)} - abc(a+b+c) = -2$$

$$y \times (x^2-2y) - zx = -2$$

$$(-1)(x^2+2) - zx = -2 \Rightarrow x^2 + xz = 0$$

$$x(x+z) = 0 \quad (3)$$

Case 1: $x+z=0$

From (1) $-x-3z=2 \Rightarrow -2z=2 \Rightarrow z=-1,\ x=1$

$$\begin{cases} a+b+c=1 \\ ab+bc+ca=-1 \\ abc=-1 \end{cases} \Rightarrow \text{ a,b,c are roots for } t^3-t^2-t+1=0$$

1 is a root

$$t^3-t^2-(t-1)=(t-1)(t^2-1)=(t-1)^2(t+1)$$

$$\Rightarrow \{a,b,c\}=\{1,1,-1\}$$

$$\Rightarrow (a,b,c)=(1,1,-1),(1,-1,1),(-1,1,1)$$

Case 2: $x=0$

From (1) $-x-3z=2 \Rightarrow -3z=2 \Rightarrow z=-\frac{2}{3}$

$$\begin{cases} a+b+c=0 \\ ab+bc+ca=-1 \\ abc=-\frac{2}{3} \end{cases} \Rightarrow \text{ a,b,c are roots for}$$

$$t^3-0t^2-t+\frac{2}{3}=0$$

$$t^3-t+\frac{2}{3}=0$$

$a+b+c=0 \Rightarrow$ one of them >0. WLOG assume that $a>0$

Note that:

$ab+bc+ca=-1 \Rightarrow a(b+c)+bc=-1$

$\Rightarrow a(-a)+bc=-1 \Rightarrow a^2-bc=1$

$\Rightarrow a^2+\frac{2}{3a}=1$

$$\Rightarrow 1=a^2+\frac{1}{3a}+\frac{1}{3a} \geq 3\sqrt[3]{a^2\frac{1}{3a}\frac{1}{3a}}=3\sqrt[3]{3} \Rightarrow \Leftarrow$$

$\square$

24. Let a, b, c be real numbers greater than $-\frac{1}{2}$. Prove that

$$\frac{a^2+2}{b+c+1}+\frac{b^2+2}{c+a+1}+\frac{c^2+2}{a+b+1} \geq 3.$$

Let $x = a+\frac{1}{2}$, $y = b+\frac{1}{2}$, $z = c+\frac{1}{2}$

$$\text{L.H.S.} = \sum \frac{\left(x-\frac{1}{2}\right)^2+2}{y+z} = \sum_{cyc} \frac{x^2-x+\frac{9}{4}}{y+z}$$

However, $x^2+\frac{9}{4} \geqslant 2\cdot\sqrt{\frac{3}{2}x^2} = 3x$

$$\text{L.H.S} = \sum \frac{x^2+\frac{9}{4}-x}{y+z} \geqslant \sum_{cyc} \frac{3x-x}{y+z}$$

$$\text{L.H.S} \geqslant \frac{2x}{y+z}+\frac{2y}{x+z}+\frac{2z}{x+y}$$ Nesbitt's inequality

$$= 2\left(\frac{x^2}{xy+xz}+\frac{y^2}{xy+yz}+\frac{z^2}{zx+zy}\right)$$

Cauchy's inequality

$$\geqslant 2\left(\frac{(x+y+z)^2}{\sum(xy+xz)}\right) = 2\frac{(\sum x)^2}{2\sum xy}$$

$$= \frac{\sum x^2}{\sum xy} \geqslant 3$$ AM-GM

equality cause holds when

$$x = y = z = \frac{3}{2}$$

$\square$

25. Solve in real numbers the system

$$\begin{cases} 7(a^5+b^5) = 31(a^3+b^3) \\ a^3-b^3 = 3(a-b). \end{cases}$$

We have two cases

Case 1: $a = \pm b$

if $a=b$, then $(a,b) = (0,0), \left(\sqrt{\frac{31}{7}}, \sqrt{\frac{31}{7}}\right), \left(-\sqrt{\frac{31}{7}}, -\sqrt{\frac{31}{7}}\right)$

if $a=-b$, then $(a,b) = (0,0), (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$

Case 2: $a \neq \pm b$

The we can divide by $a+b$ and $a-b$

$$\begin{cases} 7(a^4-a^3b+a^2b^2-ab^3+b^4) = 31(a^2-ab+b^2) \\ a^2+ab+b^2 = 3 \end{cases}$$

$x = a^2+b^2, \quad y = ab$

$$a^4-a^3b+a^2b^2-ab^3+b^4 = (a^2+b^2)^2 - a^2b^2 - ab(a^2+b^2)$$

$$= x^2-y^2-xy$$

$$\begin{cases} 7(x^2-y^2-xy) = 31(x-y) \\ x+y=3 \end{cases}$$

$y = 3-x \Rightarrow 7(x^2-(3-x)^2-x(3-x)) = 31(2x-3)$

$\Rightarrow$ can be solved easily $\Rightarrow x=5, \ x=\frac{6}{7}$

$\Rightarrow y = -2, \frac{15}{7}$

$y=-2 \Rightarrow ab=-2, \ a^2+b^2=5 \Rightarrow (a+b)^2 = 5-2.2 = 1$

$\Rightarrow a+b = 1$ or -1

if $\begin{cases} a+b=1 \\ ab=-2 \end{cases} \Rightarrow a,b$ roots $t^2-t+2=0 \Rightarrow \cdots \Rightarrow (-2,1)$ $(2,-1)$

Case 2 : $a^2+b^2=\frac{6}{7}$, $ab=\frac{15}{7}$

$a^2+b^2 \geq 2ab$

$\frac{6}{7} \geq 2\,\frac{15}{7}$ $\Rightarrow$ contradiction