2001 Winter Camp

Circle Problems from past CMO's

1. (1, 1971): DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the centre of the circle. Join OE and extend OE to cut the circle at C. (See diagram.) Given $EC = 1$, find the radius of the circle.

2. (1, 1972): Given three distinct unit circles, each of which is tangent to the other two, find the radii of the circles which are tangent to all three circles.

3. (6, 1973): If A and B are fixed points on a given circle not collinear with centre O of the circle, and if XY is a variable diameter, find the locus of P (the intersection of the line through A and X and the line through B and Y).

4. (5, 1974): Given a circle with diameter AB and a point X on the circle different from A and B, let t_a, t_b and t_x be the tangents to the circle at A, B and X respectively. Let Z be the point where line AX meets t_b and Y the point where line BX meets t_a. Show that the three lines YZ, t_x and AB are either concurrent (i.e., all pass through the same point) or parallel.

5. (5, 1975): A, B, C, D are four "consecutive" points on the circumference of a circle and P, Q, R, S are points on the circumference which are respectively the midpoints of the arcs AB, BC, CD, DA. Prove that PR is perpendicular to QS.

6. (4, 1976): Let AB be a diameter of a circle, C be any fixed point between A and B on this diameter, and Q be a variable point on the circumference of the circle. Let P be the point on the line determined by Q and C for which $\dfrac{AC}{CB} = \dfrac{QC}{CP}$. Describe, with proof, the locus of the point P.

7. (2, 1977): Let O be the centre of a circle and A a fixed interior point of the circle different from O. Determine all points P on the circumference of the circle such that the angle OPA is a maximum.

8. (4, 1979): A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

9. (4, 1984): An acute-angled triangle has unit area. Show that there is a point inside the triangle whose distance from each of the vertices is at least $\dfrac{2}{\sqrt[4]{27}}$.

10. (3, 1985): Let P_1 and P_2 be regular polygons of 1985 sides and perimeters x and y respectively. Each side of P_1 is tangent to a given circle of circumference c and this circle passes through each vertex of P_2. Prove $x + y \geq 2c$. (You may assume that $\tan\theta \geq \theta$ for $0 \leq \theta < \frac{\pi}{2}$).

11. (3, 1986): A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

12. (3, 1990): Let $ABCD$ be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D' respectively. Prove that

$$|A'B'| + |C'D'| = |A'D'| + |B'C'|.$$

($|A'B'|$ is the length of line segment $A'B'$, etc.)

13. (4, 1994) Let AB be a diameter of a circle Ω and P be any point *not* on the line through A and B. Suppose the line through P and A cuts Ω again in U, and the line through P and B cuts Ω again in V. (Note that in case of tangency U may coincide with A or V may coincide with B. Also, if P is on Ω then $P = U = V$.) Suppose that $|PU| = s|PA|$ and $|PV| = t|PB|$ for some nonnegative real numbers s and t. Determine the cosine of the angle APB in terms of s and t.

14. (4, 1997): The point O is situated inside the parallelogram $ABCD$ so that

$$\angle AOB + \angle COD = 180^\circ\ .$$

Prove that $\angle OBC = \angle ODC$.

15. (2,1999) Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

16. (4B, COMC, 1999) Triangle ABC is any one of the set of triangles having base BC equal to a and height from A to BC equal to h, with $h < \frac{\sqrt{3}}{2}a$. P is a point inside the triangle such that the value of

$$\measuredangle PAB = \measuredangle PBA = \measuredangle PCB = \alpha.$$

Show that the measure of α is the same for every triangle in the set.

17. (3, 1994) Twenty-five men sit around a circular table. Every hour there is a vote, and each must respond *yes* or *no*. Each man behaves as follows: on the n^{th} vote, if his response is the same as the response of at least one of the two people he sits between, then he will respond the same way on the $(n+1)^{th}$ vote as on the n^{th} vote; but if his response is different from that of both his neighbours on the n-th vote, then his response on the $(n+1)$-th vote will be different from his response on the n^{th} vote. Prove that, however everybody responded on the first vote, there will be a time after which nobody's response will ever change.

18. (1, 2000) At 12:00 noon, Anne, Beth and Carmen begin running around a circular track of length three hundred meters, all starting from the same point on the track. Each jogger maintains a constant speed in one of two possible directions for an indefinite period of time. Show that if Anne's speed is different from the other two speeds, then at some later time Anne will be at least one hundred meters from each of the other runners. (Here, distance is measured along the shorter of the two arcs separating the runners.)