49. Let a and b be nonnegative real numbers such that

$$2a^2 + 3ab + 2b^2 \leq 7.$$

Prove that $\max(2a+b, 2b+a) \leq 4$.

Assume that $a \geqslant b$, $2a+b = \max(2a+b, 2b+a)$

We want to show that

$$\frac{7}{16}(2a+b)^2 \leqslant 2a^2+3ab+2b^2$$

Proof:

$$7(2a+b)^2 \leqslant 32a^2+48ab+32b^2$$

$$7(4a^2+4ab+b^2) \leqslant 32a^2+48ab+32b^2$$

$$\Longleftrightarrow \quad 8a^2+28ab+7b^2 \leqslant 32a^2+48ab+32b^2$$

$$\Longleftrightarrow \quad 24a^2+20ab+25b^2 \geqslant 0$$

$$\Longleftrightarrow \quad (4a^2+20ab+25b^2)+20a^2 \geqslant 0$$

$$\Longleftrightarrow \quad (2a+5b)^2+20a^2 \geqslant 0$$

which is true for any $a, b \in \mathbb{R}$

As a result, $\frac{7}{16}(2a+b)^2 \leqslant 2a^2+3ab+2b^2 \leqslant 7$

$$\Longrightarrow \quad (2a+b)^2 \leqslant 16 \Rightarrow -4 \leqslant 2a+b \leqslant 4$$

50. Let a, b, c be positive real numbers. Prove that

$$\frac{1+a(b+c)}{(1+b+c)^2}+\frac{1+b(c+a)}{(1+c+a)^2}+\frac{1+c(a+b)}{(1+a+b)^2} \geq 1.$$

We want to make $\frac{1+a(b+c)}{(1+b+c)^2} \geqslant$ some expression in a, b, c. Then we will add the three inequalites

$$\left.\begin{array}{l} \frac{1+a(b+c)}{(1+b+c)^2} \geqslant \boxed{a,b,c} \\ \qquad\qquad\qquad \geqslant \boxed{} \\ \qquad\qquad\qquad \geqslant \boxed{} \end{array}\right\} + \quad \geqslant 1$$

$$=$$

For example, $\frac{1+a(b+c)}{(1+b+c)^2} \geqslant \frac{1+a(b+c)}{3(b^2+c^2+1)}$

because $(1+b+c)^2 \leqslant (b^2+c^2+1)(1+1+1)$

However, we can use Cauchy-Shwarz with $1+ab+ac$:

$$\frac{1+ab+ac}{(1+b+c)^2} \geqslant f(a,b,c)$$

$$\Longleftrightarrow \quad (1+ab+ac)\left(1+\frac{b}{a}+\frac{c}{a}\right) \geqslant (1+b+c)^2$$

$$\Longleftrightarrow \quad \frac{(1+a(b+c))}{(1+b+c)^2} \geqslant \frac{a}{a+b+c}$$

$$\Longleftrightarrow \quad \sum_{cyc} \frac{1+a(b+c)}{(1+b+c)^2} \geqslant \sum_{cyc} \frac{a}{a+b+c} = 1$$

53. Let a, b, c, d be real numbers greater than 0 satisfying $abcd = 1$. Prove that

$$\frac{1}{a+b+2} + \frac{1}{b+c+2} + \frac{1}{c+d+2} + \frac{1}{d+a+2} \leq 1.$$

Our goal is $\dfrac{1}{a+b+2} \leq$ something in a, b, c, d

* What if we use AM-GM directly? won't work

$a, b, c, d = x^4, y^4, z^4, t^4$ s.t. $xyzt = 1$

$a+b+2 = x^4+y^4+1+1 \geq 4\sqrt[4]{x^4y^4} = 4xy$

$\sum_{cyc} \frac{1}{a+b+2} \leq \frac{1}{4xy} + \frac{1}{4yz} + \frac{1}{4zt} + \frac{1}{4tx} \leq 1$

can be very large

example $x, y = 500$, $z, t = \frac{1}{500}$

In other words $a+b+1+1 \geq 4\sqrt[4]{ab}$

$\sum \frac{1}{a+b+2} \leq \sum \frac{1}{4\sqrt[4]{ab}}$ won't work because $\frac{1}{x} \longrightarrow \infty$ as $x \to 0$

However, note that $\frac{1}{1+x} \leq 1$ for any $x \geq 0$

* idea 2: Keep the constant term

Take $a+b+2 \geq 2\sqrt{ab}+2 \Rightarrow \frac{1}{a+b+2} \leq \frac{1}{2\sqrt{ab}+2}$

$\Rightarrow \sum \frac{1}{a+b+2} \leq \frac{1}{2}\left(\frac{1}{\sqrt{ab}+1} + \frac{1}{\sqrt{bc}+1} + \frac{1}{\sqrt{cd}+1} + \frac{1}{\sqrt{da}+1}\right)$

let $u = ab$, $v = bc$ $\Rightarrow cd = \frac{1}{u}$, $da = \frac{1}{v}$

$\text{R.H.S.} = \frac{1}{2}\left(\frac{1}{u+1} + \frac{1}{v+1} + \frac{1 \times u}{\left(\frac{1}{u}+1\right) \times u} + \frac{1 \times v}{\left(\frac{1}{v}+1\right) \times v}\right)$

$= \frac{1}{2}\left(\frac{1}{u+1} + \frac{u}{1+u} + \frac{1}{v+1} + \frac{v}{1+v}\right) = \frac{1}{2}(1+1) = 1$

Example 9.9. Find the minimum possible value of

$$\max\{a+b+c, b+c+d, c+d+e, d+e+f, e+f+g\}.$$

over all nonnegative real numbers a, b, c, d, e, f, g such that

$$a+b+c+d+e+f+g=1.$$

Let $M = \max\{a+b+c,\ b+c+d,\ c+d+e,\ d+e+f,\ e+f+g\}$

$$\left.\begin{array}{l} M \geq a+b+c \\ M \geq c+d+e \\ M \geq e+f+g \end{array}\right\} \Rightarrow \begin{array}{l} 3M \geq \sum_{cyc} a + c + e \\ \\ 3M \geq 1 + c + e \end{array}$$

Therefore, $3M \geq 1$ and $3M = 1 \iff c = e = 0$

Let $a = d = g = \frac{1}{3}$ $\quad\left(\frac{1}{3}, 0, 0, \frac{1}{3}, 0, 0, \frac{1}{3}\right)$

$b = c = e = f = 0$

in this example $M = \frac{1}{3} \Rightarrow \frac{1}{3}$ is achievable

and it's the minimum $\square$

37. Determine the maximum value attained by

$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$

over all real numbers $x > 1$.

Note that $\dfrac{x^4 - x^2}{x^6+2x^3-1} = \dfrac{x - \frac{1}{x}}{x^3 + 2 - \frac{1}{x^3}}$

let $u = x - \frac{1}{x} > 0$

Note that $\left(x - \frac{1}{x}\right)^3 = x^3 - 3x^2\frac{1}{x} + 3x\frac{1}{x^2} - \frac{1}{x^3}$

$$= x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$$

$$\Rightarrow x^3 - \frac{1}{x^3} + 2 = u^3 + 3u + 2$$

$$\Rightarrow \frac{x - \frac{1}{x}}{x^3 + 2 - \frac{1}{x^3}} = \frac{u}{u^3 + 3u + 2}$$

We want to maximize

$\dfrac{u}{u^3 + 3u + 2}$ where $u > 0$

$$\frac{u}{u^3+3u+2} = \frac{u}{(u^3+1+1)+3u} \underset{\text{AM-GM}}{\leq} \frac{u}{3u+3u} = \frac{1}{6}$$

equality holds when $u^3 = 1 = 1 \iff x - \frac{1}{x} = 1$

$\iff x^2 - x - 1 = 0$

$x = \dfrac{1 \pm \sqrt{5}}{2}$ $\square$

39. Solve the following equation in integers:

$$3x^3 - x^2y - xy^2 + 3y^3 = 2013.$$

$$3x^3 - x^2y - xy^2 + 3y^3 = 3(x^3+y^3) - xy(x+y)$$

$$= 3(x+y)(x^2-xy+y^2) - xy(x+y)$$

$$= (x+y)(3x^2-4xy+3y^2)$$

Therefore, $(x+y)(3x^2-4xy+3y^2) = 2013 = 3 \cdot 11 \cdot 61$

$2x^2+2y^2-4xy = 2(x-y)^2 \Rightarrow (3x^2-4xy+3y^2) \geq x^2+y^2 \geq \frac{1}{2}(x+y)^2$

$(3x^2-4xy+3y^2) = \frac{5}{2}(x-y)^2 + \frac{1}{2}(x+y)^2 \geq \frac{1}{2}(x+y)^2$

Since $3x^2-4xy+3y^2 \geq 0 \Rightarrow x+y \geq 0$

$\Rightarrow 2013 \geq (x+y)\ \frac{1}{2}(x+y)^2$

$\Rightarrow 4026 \geq (x+y)^3 \Rightarrow (x+y) < 16$

$\Rightarrow x+y \in \{1, 3, 11\}$

1) $x+y=1$

$\frac{5}{2}(x-y)^2 + \frac{1}{2}(x+y)^2 = 2013$

$\Rightarrow \frac{5}{2}(x-y)^2 + \frac{1}{2} \cdot 1 = 2013$

$\Rightarrow (x-y)^2 = 805$

2) $x+y=3$

$\frac{5}{2}(x-y)^2 + \frac{1}{2}(x+y)^2 = 671 \Rightarrow (x-y)^2 = \frac{1333}{5} \notin \mathbb{Z}$

3) $x+y=11$

$\frac{5}{2}(x-y) + \frac{1}{2}\overset{11^2}{(x+y)^2} = 183 \Rightarrow (x-y)^2 = 49$

$\Rightarrow x-y = \pm 7$

$\Rightarrow (x, y) = (9, 2), (2, 9)$ □

40. Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 1$. Prove that

$$\frac{a+b}{\sqrt{ab+c}}+\frac{b+c}{\sqrt{bc+a}}+\frac{c+a}{\sqrt{ca+b}} \geq 3\sqrt[6]{abc}.$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant 1 \implies ab+bc+ca \geqslant abc$$

Hint:

$$\sqrt{abc+c^2} \leqslant \sqrt{ab+bc+ca+c^2}$$
$$= \sqrt{(c+a)(c+b)} \implies \sqrt{ab+c} \leqslant \frac{\sqrt{(c+a)(c+b)}}{\sqrt{c}}$$

$$\frac{a+b}{\sqrt{ab+c}} \geqslant \frac{(a+b)\sqrt{c}}{\sqrt{(c+a)(c+b)}}$$

$$\sum_{cyc} \frac{a+b}{\sqrt{ab+c}} \geqslant \sum_{cyc} \frac{(a+b)\sqrt{c}}{\sqrt{(c+a)(c+b)}}$$

By applying AM-GM

$$R.H.S \geqslant 3 \sqrt[3]{\frac{(a+b)\sqrt{c}}{\sqrt{(c+a)(c+b)}} \frac{(b+c)\sqrt{a}}{\sqrt{(a+b)(a+c)}} \frac{(c+a)\sqrt{b}}{\sqrt{(b+a)(b+c)}}}$$

$$= 3\sqrt[3]{\sqrt{a}\sqrt{b}\sqrt{c}} = 3\sqrt[6]{abc} \quad \square$$

Homework:

42. Let x and y be real numbers such that

$$x^3 + y^3 + (x + y)^3 + 30xy = 2000.$$

Prove that $x + y = 10$.

45. The real numbers a, b, c, d, e, and f satisfy the conditions

$$a + b + c + d + e + f = 10$$

and

$$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 + (d - 1)^2 + (e - 1)^2 + (f - 1)^2 = 6.$$

Determine the greatest possible value of f.