

# Ratio Chasing

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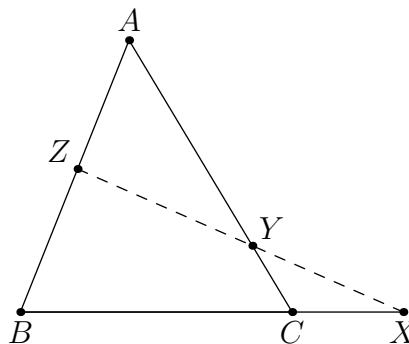
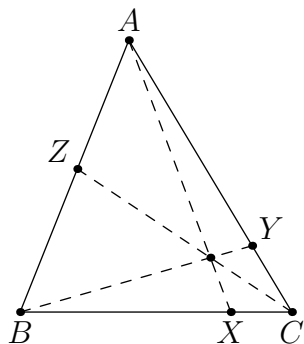
## 1 Concurrency and Collinearity

In a triangle  $\triangle ABC$ , a *cevian* is a line segment connecting one of the vertices  $A$ ,  $B$ , or  $C$  to a point lying on the side opposite to the vertex. Ceva's Theorem relates the concurrency of three such cevians to length ratios.

### Theorem 1: Ceva's Theorem

Given  $\triangle ABC$ , let  $X$ ,  $Y$ , and  $Z$  be points on lines  $BC$ ,  $CA$ , and  $AB$  respectively. Then  $AX$ ,  $BY$ , and  $CZ$  are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

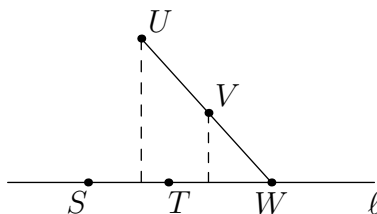


Before proving this theorem, let us consider the following lemma:

### Lemma 1

Let  $\ell$  be a line and  $U, V$  points in the plane. Let  $S$  and  $T$  be points on  $\ell$ . Line  $UV$  intersects  $\ell$  at point  $W$ . Then

$$\frac{[\triangle UST]}{[\triangle VST]} = \frac{d(U, \ell)}{d(V, \ell)} = \frac{UW}{VW}.$$



*Proof.* We will first show that if  $AX$ ,  $BY$ , and  $CZ$  concur, the given equation is true. Let  $P$  be the point of intersection of the three lines. From Lemma 1,  $\frac{[PAB]}{[PCA]} = \frac{BX}{XC}$ . Similarly,  $\frac{[PBC]}{[PAB]} = \frac{CY}{YA}$  and  $\frac{[PCA]}{[PBC]} = \frac{AZ}{ZB}$ . Multiplying the three inequalities gives

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{[PAB]}{[PCA]} \cdot \frac{[PBC]}{[PAB]} \cdot \frac{[PCA]}{[PBC]} = 1.$$

To prove the other direction, let  $P$  be the intersection of  $BY$  and  $CZ$ . Instead of directly proving that  $P$  lies on  $AX$ , we will define  $X'$  to be the intersection of  $AP$  and  $BC$  and show that  $X = X'$ . From the direction we have shown,

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

But we also have the assumption that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

Dividing the two equations gives

$$\frac{BX'}{X'C} = \frac{BX}{XC}.$$

It is easy to see that this implies  $X = X'$  and we are done.  $\square$

### Theorem 2: Menelaus's Theorem

Given  $\triangle ABC$ , let  $X$ ,  $Y$ , and  $Z$  be points on lines  $BC$ ,  $CA$ , and  $AB$  respectively. Then  $X$ ,  $Y$ , and  $Z$  are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

The two theorems are almost the same save for the  $-1$  in the right-hand side of Menelaus's Theorem and collinearity rather than concurrency. The negative sign arises from directed lengths. It is generally unnecessary to pay much attention when using directed lengths; they tend to work out without much trouble.

## 2 Angles

The Law of Sines provides a way of moving between ratios involving lengths and ratios involving angles.

### Theorem 3: Extended Law of Sines

Let  $\triangle ABC$  be a triangle with circumradius  $R$ . Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

**Theorem 4: Ratio Lemma**

Given  $\triangle ABC$ , let  $X$  be a point on  $BC$ . Then

$$\frac{BX}{XC} = \frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{AB}{AC}.$$

*Proof.* Applying the Law of Sines to  $\triangle ABX$  and  $\triangle ACX$ , we get

$$\frac{\sin \angle BAX}{BX} = \frac{\sin \angle BXA}{AB}$$

and

$$\frac{\sin \angle XAC}{XC} = \frac{\sin \angle CXA}{AC}.$$

Dividing these two equations and rearranging gives

$$\frac{\sin \angle BXA}{\sin \angle CXA} \cdot \frac{BX}{XC} = \frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{AB}{AC}.$$

Since  $\angle BXA$  and  $\angle CXA$  are supplementary,  $\sin \angle BXA = \sin \angle CXA$ . So we have

$$\frac{BX}{XC} = \frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{AB}{AC}$$

as expected. □

We can apply the Ratio Lemma to Ceva's Theorem and Menelaus's Theorem to obtain new formulations:

**Theorem 5: Trigonometric Form of Ceva's Theorem**

Given  $\triangle ABC$ , let  $X$ ,  $Y$ , and  $Z$  be points on lines  $BC$ ,  $CA$ , and  $AB$  respectively. Then  $AX$ ,  $BY$ , and  $CZ$  are concurrent if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = 1.$$

**Theorem 6: Trigonometric Form of Menelaus's Theorem**

Given  $\triangle ABC$ , let  $X$ ,  $Y$ , and  $Z$  be points on lines  $BC$ ,  $CA$ , and  $AB$  respectively. Then  $X$ ,  $Y$ , and  $Z$  are collinear if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = -1.$$

When working with these forms, we often obtain results about the ratio of sines of angles. The following lemma allows these results to cleanly translate into results about the angles themselves.

**Lemma 2**

Let  $x_1, y_1, x_2, y_2$  be real numbers in  $[0, \pi)$  such that  $x_1 + y_1 = x_2 + y_2$ ,  $x_1 + y_1 < \pi$  and  $\frac{\sin x_1}{\sin y_1} = \frac{\sin x_2}{\sin y_2}$ . Then  $x_1 = x_2, y_1 = y_2$ .

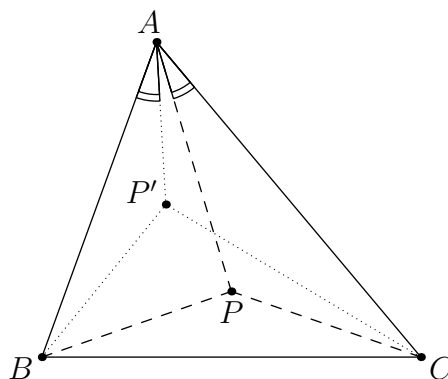
Note the bounds on  $x_1 + y_1$ ; the lemma is not true when the angles sum to  $\pi$  and needs alterations when the angles sum to greater than  $\pi$ .

**3 Isogonal Lines**

Consider a point  $P$  and rays  $PU$  and  $PV$ . Two lines  $\ell_1$  and  $\ell_2$  passing through  $P$  are called *isogonal* lines if  $\ell_1$  and  $\ell_2$  are reflections over the bisector of  $\angle UPV$ . In other words,  $\angle(\ell_1, PU) = \angle(\ell_2, PV)$  and  $\angle(\ell_1, PV) = \angle(\ell_2, PU)$  (the two are equivalent).

**Lemma 3: Existence of Isogonal Conjugate**

Let  $\triangle ABC$  be a triangle and  $P$  be some point. Then the isogonal of  $AP$  with respect to  $\angle CAB$ , the isogonal of  $BP$  with respect to  $\angle ABC$ , and the isogonal of  $CP$  with respect to  $\angle BCA$  intersect at a point. This point is called the *isogonal conjugate* of  $P$ .



As an aside, the isogonal conjugate is what you get by flipping all the angles in trigonometric Ceva's. If you instead flipped all the side lengths, that is what's called the *isotomic conjugate*.

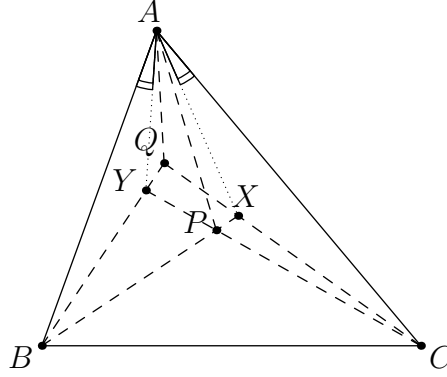
**Lemma 4**

Let  $\triangle ABC$  be a triangle with points  $D$  and  $E$  on  $BC$ . Prove that  $AD$  and  $AE$  are isogonal with respect to  $\angle CAB$  if and only if

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left( \frac{AB}{AC} \right)^2.$$

**Lemma 5: Isogonal Line Lemma**

Given triangle  $ABC$ , let  $P$  and  $Q$  be points such that lines  $AP$  and  $AQ$  are isogonal lines with respect to  $\angle BAC$ . Define  $X$  as the intersection of lines  $BP$  and  $CQ$  and  $Y$  as the intersection of lines  $BQ$  and  $CP$ . Then  $AX$  and  $AY$  are isogonal lines with respect to  $\angle BAC$ .



*Proof.* We first apply the Ratio Lemma four times: on  $\triangle AYQ$  with point  $B$ , on  $\triangle AYP$  with point  $C$ , on  $\triangle AXQ$  with point  $C$ , and on  $\triangle AXP$  with point  $B$ . These give

$$\begin{aligned} \frac{AY}{AQ} \cdot \frac{\sin \angle YAB}{\sin \angle BAQ} &= \frac{YB}{BQ} = \frac{[YBC]}{[QBC]}, \\ \frac{AY}{AP} \cdot \frac{\sin \angle YAC}{\sin \angle CAP} &= \frac{YC}{CP} = \frac{[YBC]}{[PBC]}, \\ \frac{AX}{AQ} \cdot \frac{\sin \angle XAC}{\sin \angle CAQ} &= \frac{XC}{CQ} = \frac{[XBC]}{[QBC]}, \\ \frac{AX}{AP} \cdot \frac{\sin \angle XAB}{\sin \angle BAP} &= \frac{XB}{BP} = \frac{[XBC]}{[PBC]}. \end{aligned}$$

Now consider the product of the first and fourth equation, and the product of the second and third equations. These share many terms and we can divide the two, leaving

$$\frac{\sin \angle YAB}{\sin \angle BAQ} \cdot \frac{\sin \angle XAB}{\sin \angle BAP} = \frac{\sin \angle YAC}{\sin \angle CAP} \cdot \frac{\sin \angle XAC}{\sin \angle CAQ}.$$

We can now use the isogonality condition on  $AP$  and  $AQ$  to cancel out the denominators. Rearranging, we have

$$\frac{\sin \angle YAB}{\sin \angle YAC} = \frac{\sin \angle XAC}{\sin \angle XAB}.$$

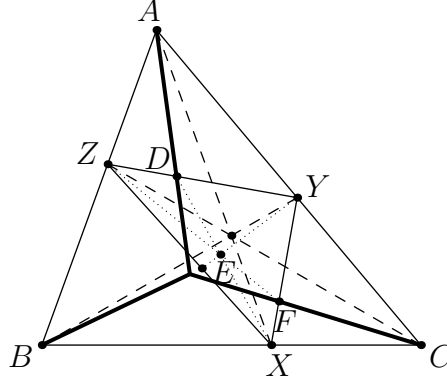
By Lemma 2, this implies that  $\angle YAB = \angle XAC$  and we are done.

□

## 4 Examples

### Example 1: Cevian Nest Theorem

Let  $AX$ ,  $BY$ , and  $CZ$  be concurrent cevians of  $\triangle ABC$ . Let  $XD$ ,  $YE$ , and  $ZF$  be concurrent cevians in  $\triangle XYZ$ . Prove that  $AD$ ,  $BE$ , and  $CF$  are concurrent.



*Proof.* By the Ratio Lemma on  $\triangle AYZ$  with cevian  $AD$ ,

$$\frac{YD}{DZ} = \frac{\sin \angle YAD}{\sin \angle DAZ} \cdot \frac{AY}{AZ}.$$

Similarly,

$$\frac{ZE}{EX} = \frac{\sin \angle ZBE}{\sin \angle EBX} \cdot \frac{BZ}{BX} \quad \text{and} \quad \frac{XF}{FY} = \frac{\sin \angle XCF}{\sin \angle FCY} \cdot \frac{CX}{CY}.$$

Multiplying the three equations yields

$$\left( \frac{YD}{DZ} \cdot \frac{ZE}{EX} \cdot \frac{XF}{FY} \right) = \left( \frac{\sin \angle YAD}{\sin \angle DAZ} \cdot \frac{\sin \angle ZBE}{\sin \angle EBX} \cdot \frac{\sin \angle XCF}{\sin \angle FCY} \right) \cdot \left( \frac{AY}{AZ} \cdot \frac{BZ}{BX} \cdot \frac{CX}{CY} \right)$$

But from Ceva's Theorem on  $\triangle XYZ$  with points  $D$ ,  $E$ ,  $F$  and on  $\triangle ABC$  with points  $X$ ,  $Y$ ,  $Z$ , we know that

$$\frac{YD}{DZ} \cdot \frac{ZE}{EX} \cdot \frac{XF}{FY} = 1 \quad \text{and} \quad \frac{AY}{AZ} \cdot \frac{BZ}{BX} \cdot \frac{CX}{CY} = \left( \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} \right)^{-1} = 1.$$

Thus,

$$\frac{\sin \angle CAD}{\sin \angle DAB} \cdot \frac{\sin \angle ABE}{\sin \angle EBC} \cdot \frac{\sin \angle BCF}{\sin \angle FCA} = \frac{\sin \angle YAD}{\sin \angle DAZ} \cdot \frac{\sin \angle ZBE}{\sin \angle EBX} \cdot \frac{\sin \angle XCF}{\sin \angle FCY} = 1$$

and we are done. □

**Example 2**

Let  $\triangle ABC$  be a triangle with incenter  $I$  and contact triangle  $\triangle DEF$ . If  $M$  is the midpoint of  $BC$ , prove that  $EF$ ,  $AM$  and  $DI$  concur.

*Proof.* Let  $X_1$  be the intersection of  $AM$  and  $EF$  and let  $X_2$  be the intersection of  $DI$  and  $EF$ . We wish to prove that  $X_1 = X_2$ . It suffices to show  $\frac{EX_1}{X_1F} = \frac{EX_2}{X_2F}$ .

Let us first compute  $\frac{EX_1}{X_1F}$ . From the Ratio Lemma on  $\triangle AEF$  and on  $\triangle ACB$  with cevian  $AX_1$  and  $AM$  respectively, we obtain

$$\frac{EX_1}{X_1F} = \frac{\sin \angle EAX_1}{\sin \angle X_1AF} \cdot \frac{AE}{AF} \quad \text{and} \quad \frac{CM}{MB} = \frac{\sin \angle CAM}{\sin \angle MAB} \cdot \frac{AC}{AB}.$$

Dividing the two equations and using  $\angle EAX_1 = \angle CAM$ ,  $\angle X_1AF = \angle MAB$  gives

$$\begin{aligned} \frac{EX_1}{X_1F} \cdot \frac{MB}{CM} &= \frac{AE}{AF} \cdot \frac{AB}{AC} \\ \frac{EX_1}{X_1F} &= \frac{AB}{AC}. \end{aligned}$$

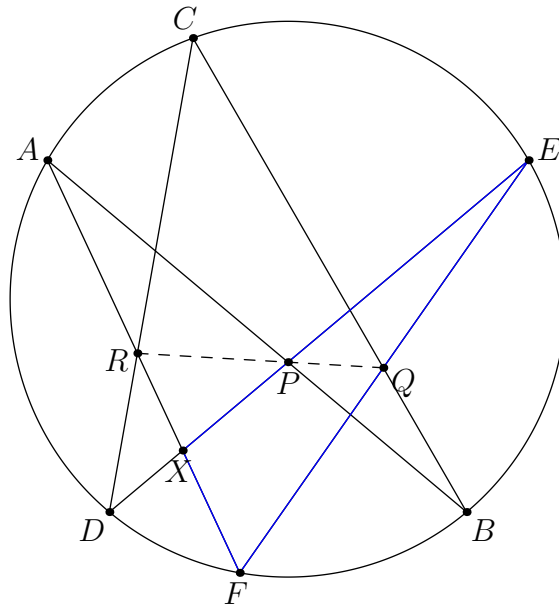
To compute  $\frac{EX_2}{X_2F}$ , we will apply the Ratio Lemma to  $\triangle IEF$  with cevian  $IX_2$ . From angle chasing, we can see that  $\angle EIX_2 = \angle BCA$  and  $\angle X_2IF = \angle ABC$ .

$$\begin{aligned} \frac{EX_2}{X_2F} &= \frac{\sin \angle EIX_2}{\sin \angle X_2IF} \cdot \frac{IE}{IF} \\ &= \frac{\sin \angle EIX_2}{\sin \angle X_2IF} \\ &= \frac{\sin \angle BCA}{\sin \angle ABC} \\ &= \frac{AB}{AC}. \end{aligned}$$

Thus,  $X_1 = X_2$  and  $EF$ ,  $AM$ , and  $DI$  concur. □

**Example 3: Pascal's Theorem**

Let points  $A, B, C, D, E, F$  lie on a circle. Let  $P := AB \cap DE$ ,  $Q := BC \cap EF$ ,  $R := CD \cap FA$ . Prove that  $P, Q, R$  are collinear.



*Proof.* Define  $X$  to be the intersection of  $DE$  and  $AF$ . By Menelaus's Theorem on  $\triangle EFX$ , it suffices to show

$$\frac{EQ}{QF} \cdot \frac{FR}{RX} \cdot \frac{XP}{PE} = -1.$$

We can compute  $\frac{EQ}{QF}$  by using the Ratio Lemma on  $\triangle CEF$ . In particular, we get

$$\frac{EQ}{QF} = \frac{CE}{CF} \cdot \frac{\sin \angle CEB}{\sin \angle BCF} = \frac{CE}{CF} \cdot \frac{EB}{BF}.$$

Similarly, by Ratio Lemma on  $\triangle DFX$  we get  $\frac{FR}{RX} = \frac{DF}{DX} \cdot \frac{FC}{CE}$  and by Ratio Lemma on  $\triangle AXE$  we get  $\frac{AX}{AE} \cdot \frac{FB}{BE}$ . Putting this all together, it suffices to show

$$\frac{CE}{CF} \cdot \frac{EB}{BF} \cdot \frac{DF}{DX} \cdot \frac{FC}{CE} \cdot \frac{AX}{AE} \cdot \frac{FB}{BE} = -1.$$

Most of these terms cancel out leaving us with

$$\frac{DF}{DX} \cdot \frac{AX}{AE} \stackrel{?}{=} 1$$

which is true as  $\triangle DFX \sim \triangle AEX$ . □



## 5 Problems

### A Problems

**A1.** (Gergonne point) Let  $\triangle DEF$  be the contact triangle of  $\triangle ABC$ . Prove that  $AD$ ,  $BE$ , and  $CF$  are concurrent.

**A2.** Let  $AD$ ,  $BE$ , and  $CF$  be cevians in  $\triangle ABC$  which all intersect at a point  $P$ . Prove that

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1.$$

**A3.** Let  $M$ ,  $E$ , and  $F$  be points on  $BC$ ,  $CA$ , and  $AB$ . Prove that  $EF \parallel BC$  if and only if  $M$  is the midpoint of  $BC$ .

**A4.** Let  $\triangle ABC$  be a triangle and let  $\alpha$  be some fixed angle in  $[0, \pi)$ . Let  $A_1$  be the point such that  $\angle BA_1C = \alpha$ ,  $A_1B = A_1C$ , and  $A_1$  is on the opposite of  $BC$  as  $A$ . Define  $B_1$  and  $C_1$  similarly. Prove that  $AA_1$ ,  $BB_1$ , and  $CC_1$  are concurrent.

**A5.** Let  $AD$ ,  $BE$ , and  $CF$  be cevians in  $\triangle ABC$  which are concurrent. The circumcircle of  $\triangle DEF$  intersects  $BC$ ,  $CA$ , and  $AB$  again at  $D'$ ,  $E'$  and  $F'$  respectively. Prove that  $AD'$ ,  $BE'$ , and  $CF'$  are concurrent.

**A6.** (CMO 2008)  $ABCD$  is a convex quadrilateral for which  $AB$  is the longest side. Points  $M$  and  $N$  are located on sides  $AB$  and  $BC$  respectively, so that each of the segments  $AN$  and  $CM$  divides the quadrilateral into two parts of equal area. Prove that the segment  $MN$  bisects the diagonal  $BD$ .

**A7.** (Euclid 2012) Let  $ABCD$  be a convex quadrilateral and let  $M$  and  $N$  be points on segments  $AB$  and  $CD$  respectively such that  $\frac{AM}{AB} = \frac{CN}{CD}$ . Line segments  $AN$  and  $MD$  intersect at point  $P$  and  $BN$  and  $CM$  intersect at point  $Q$ . Prove that  $[PMQN] = [BQC] + [APD]$ .

**A8.** (Lemoine axis) Let  $\Gamma$  be the circumcircle of  $\triangle ABC$ . The tangents to  $\Gamma$  at  $A$ ,  $B$ , and  $C$  intersect  $BC$ ,  $CA$ , and  $AB$  at points  $D$ ,  $E$ , and  $F$  respectively. Prove that  $D$ ,  $E$ , and  $F$  are collinear.

**A9.** (CMOQR 2013) In triangle  $ABC$ ,  $\angle CAB = 90^\circ$  and  $\angle BCA = 70^\circ$ . Let  $F$  be the point on  $AB$  such that  $\angle ACF = 30^\circ$  and let  $E$  be the point on  $CA$  such that  $\angle CFE = 20^\circ$ . Prove that  $BE$  bisects  $\angle ABC$ .

**A10.** (2016 AIME) Triangle  $ABC$  is inscribed in circle  $\omega$ . Points  $P$  and  $Q$  are on side  $\overline{AB}$  with  $AP < AQ$ . Rays  $CP$  and  $CQ$  meet  $\omega$  again at  $S$  and  $T$  (other than  $C$ ), respectively. If  $AP = 4$ ,  $PQ = 3$ ,  $QB = 6$ ,  $BT = 5$ , and  $AS = 7$ , find  $ST$ .

**A11.** (Symmedians) Let  $X$  be the intersection of the tangents to the circumcircle of  $\triangle ABC$  at  $B$  and  $C$ . Prove that  $AX$  is isogonal to the  $A$ -median with respect to  $\angle CAB$ .

**A12.** (Monge's Theorem) Let  $\Gamma_1$ ,  $\Gamma_2$ , and  $\Gamma_3$  be three circles in the plane such that none of the circles contains another. For each pair, construct the intersection of their common external tangents. Prove that these three intersections are collinear.

**A13.** (Exeter point) Let  $\triangle ABC$  be a triangle. Let the medians through  $A$ ,  $B$ , and  $C$  meet the circumcircle of  $\triangle ABC$  at  $A'$ ,  $B'$ , and  $C'$  respectively. Let  $\triangle DEF$  be the triangle formed by the tangents at  $A$ ,  $B$ , and  $C$  to the circumcircle of  $\triangle ABC$ . Prove that  $DA'$ ,  $EB'$ , and  $FC'$  are concurrent.

**A14.** Let  $ABCD$  be a cyclic quadrilateral. Prove that

$$AB \cdot BD \cdot DA + BC \cdot CD \cdot DB = AC \cdot CD \cdot DA + AB \cdot BC \cdot CA.$$

## B Problems

**B1.** (CMO 1994) Let  $ABC$  be an acute triangle and let  $AD$  be the altitude on  $BC$  ( $D$  on  $BC$ ). Let  $H$  be any point on  $AD$ . Lines  $BH$  and  $CH$ , when extended, intersect  $AC$  and  $AB$  at  $E$  and  $F$ , respectively. Prove that  $\angle EDH = \angle FDH$ .

**B2.** Let the incircle ( $I$ ) of a triangle  $ABC$  touch  $BC, CA$  and  $AB$  at  $D, E$  and  $F$  respectively. Denote by  $M, N$  and  $P$  the  $A$ -excenter,  $B$ -excenter and  $C$ -excenter of the triangle  $ABC$  respectively. Prove that the circles  $(IDM), (IEN)$  and  $(IFP)$  are coaxial.

**B3.** Let  $ABC$  be right-angled at  $A$ , and let  $D$  be a point lying on the side  $AC$ . Denote by  $E$  the reflection of  $A$  over line  $BD$ , and by  $F$  the intersection of  $CE$  with the perpendicular through  $D$  to the line  $BC$ . Prove that  $AF, DE$ , and  $BC$  are concurrent.

**B4.** (ISL 2006) Let  $ABCDE$  be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals  $BD$  and  $CE$  meet at  $P$ . Prove that the line  $AP$  bisects the side  $CD$ .

**B5.** (ISL 2000) Let  $O$  be the circumcenter and  $H$  the orthocenter of an acute triangle  $ABC$ . Show that there exist points  $D, E$ , and  $F$  on sides  $BC, CA$ , and  $AB$  respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines  $AD, BE$ , and  $CF$  are concurrent.

**B6.** (USAMO 2012) Let  $P$  be a point in the plane of  $\triangle ABC$ , and  $\gamma$  a line passing through  $P$ . Let  $A', B', C'$  be the points where the reflections of lines  $PA, PB, PC$  with respect to  $\gamma$  intersect lines  $BC, AC, AB$  respectively. Prove that  $A', B', C'$  are collinear.

**B7.** (Newton-Gauss line) Let  $ABCD$  be a convex quadrilateral. Let  $E$  be the intersection of  $AB$  with  $CD$  and  $F$  be the intersection of  $AD$  with  $BC$ . Prove that the midpoints of  $AC, BD$ , and  $EF$  are collinear.

**B8.** (USATSTST 2017) Let  $ABC$  be a triangle with incenter  $I$ . Let  $D$  be a point on side  $BC$  and let  $\omega_B$  and  $\omega_C$  be the incircles of  $\triangle ABD$  and  $\triangle ACD$ , respectively. Suppose that  $\omega_B$  and  $\omega_C$  are tangent to segment  $BC$  at points  $E$  and  $F$ , respectively. Let  $P$  be the intersection of segment  $AD$  with the line joining the centers of  $\omega_B$  and  $\omega_C$ . Let  $X$  be the intersection point of lines  $BI$  and  $CP$  and let  $Y$  be the intersection point of lines  $CI$  and  $BP$ . Prove that lines  $EX$  and  $FY$  meet on the incircle of  $\triangle ABC$ .

**B9.** (APMO 2016) Let  $AB$  and  $AC$  be two distinct rays not lying on the same line, and let  $\omega$  be a circle with center  $O$  that is tangent to ray  $AC$  at  $E$  and ray  $AB$  at  $F$ . Let  $R$  be a point on segment  $EF$ . The line through  $O$  parallel to  $EF$  intersects line  $AB$  at  $P$ . Let  $N$  be the intersection of lines  $PR$  and  $AC$ , and let  $M$  be the intersection of line  $AB$  and the line through  $R$  parallel to  $AC$ . Prove that line  $MN$  is tangent to  $\omega$ .

**B10.** (USATST 2012) In cyclic quadrilateral  $ABCD$ , diagonals  $AC$  and  $BD$  intersect at  $P$ . Let  $E$  and  $F$  be the respective feet of the perpendiculars from  $P$  to lines  $AB$  and  $CD$ . Segments  $BF$  and  $CE$  meet at  $Q$ . Prove that lines  $PQ$  and  $EF$  are perpendicular to each other.

**B11.** (China 2002) Let  $E$  and  $F$  be the intersections of opposite sides of a convex quadrilateral  $ABCD$ . The two diagonals meet at  $P$ . Let  $O$  be the foot of the perpendicular from  $P$  to  $EF$ . Show that  $\angle BOC = \angle AOD$ .

## C Problems

**C1.** (USATST 2021) Let  $A, B, C, D$  be four points such that no three are collinear and  $D$  is not the orthocenter of  $ABC$ . Let  $P, Q, R$  be the orthocenters of  $\triangle BCD, \triangle CAD, \triangle ABD$ , respectively. Suppose that the lines  $AP, BQ, CR$  are pairwise distinct and are concurrent. Show that the four points  $A, B, C, D$  lie on a circle.

**C2.** (USATST 2020) Two circles  $\Gamma_1$  and  $\Gamma_2$  have common external tangents  $\ell_1$  and  $\ell_2$  meeting at  $T$ . Suppose  $\ell_1$  touches  $\Gamma_1$  at  $A$  and  $\ell_2$  touches  $\Gamma_2$  at  $B$ . A circle  $\Omega$  through  $A$  and  $B$  intersects  $\Gamma_1$  again at  $C$  and  $\Gamma_2$  again at  $D$ , such that quadrilateral  $ABCD$  is convex.

Suppose lines  $AC$  and  $BD$  meet at point  $X$ , while lines  $AD$  and  $BC$  meet at point  $Y$ . Show that  $T, X, Y$  are collinear.

**C3.** (ISL 2011) Let  $ABC$  be a triangle with  $AB = AC$  and let  $D$  be the midpoint of  $AC$ . The angle bisector of  $\angle BAC$  intersects the circle through  $D, B$  and  $C$  at the point  $E$  inside the triangle  $ABC$ . The line  $BD$  intersects the circle through  $A, E$  and  $B$  in two points  $B$  and  $F$ . The lines  $AF$  and  $BE$  meet at a point  $I$ , and the lines  $CI$  and  $BD$  meet at a point  $K$ . Show that  $I$  is the incentre of triangle  $KAB$ .

**C4.** (USAMO 2017) In convex cyclic quadrilateral  $ABCD$ , we know that lines  $AC$  and  $BD$  intersect at  $E$ , lines  $AB$  and  $CD$  intersect at  $F$ , and lines  $BC$  and  $DA$  intersect at  $G$ . Suppose that the circumcircle of  $\triangle ABE$  intersects line  $CB$  at  $B$  and  $P$ , and the circumcircle of  $\triangle ADE$  intersects line  $CD$  at  $D$  and  $Q$ , where  $C, B, P, G$  and  $C, Q, D, F$  are collinear in that order. Prove that if lines  $FP$  and  $GQ$  intersect at  $M$ , then  $\angle MAC = 90^\circ$ .

**C5.** (China 2015) Let  $A, B, D, E, F, C$  be six points lie on a circle (in order) such that  $AB = AC$ . Let  $P = AD \cap BE, R = AF \cap CE, Q = BF \cap CD, S = AD \cap BF, T = AF \cap CD$ . Let  $K$  be a point lie on  $ST$  satisfy  $\angle QKS = \angle ECA$ . Prove that  $\frac{SK}{KT} = \frac{PQ}{QR}$ .

**C6.** (USAMO 2016) Let  $\triangle ABC$  be an acute triangle, and let  $I_B, I_C$ , and  $O$  denote its  $B$ -excenter,  $C$ -excenter, and circumcenter, respectively. Points  $E$  and  $Y$  are selected on  $\overline{AC}$  such that  $\angle ABY = \angle CBY$  and  $\overline{BE} \perp \overline{AC}$ . Similarly, points  $F$  and  $Z$  are selected on  $\overline{AB}$  such that  $\angle ACZ = \angle BCZ$  and  $\overline{CF} \perp \overline{AB}$ .

Lines  $\overleftrightarrow{I_B F}$  and  $\overleftrightarrow{I_C E}$  meet at  $P$ . Prove that  $\overline{PO}$  and  $\overline{YZ}$  are perpendicular.

**C7.** (USATST 2015) Let  $ABC$  be a non-equilateral triangle and let  $M_a, M_b, M_c$  be the midpoints of the sides  $BC, CA, AB$ , respectively. Let  $S$  be a point lying on the Euler line. Denote by  $X, Y, Z$  the second intersections of  $M_a S, M_b S, M_c S$  with the nine-point circle. Prove that  $AX, BY, CZ$  are concurrent.