JMO 2010 Solution Notes

Compiled by Evan Chen

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This is an compilation of solutions for the 2010 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found on the Art of Problem Solving forums.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $P(n)$ be the number of permutations $(a_1, \dots, a_n)$ of the numbers $(1, 2, \dots, n)$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

2. Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \dots, x_{n-1}$ of positive integers with the following three properties:

(a) $x_1 < x_2 < \dots < x_{n-1}$;

(b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \dots, n-1$;

(c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

3. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

4. A triangle is called a *parabolic* triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

5. Two permutations $a_1, a_2, \dots, a_{2010}$ and $b_1, b_2, \dots, b_{2010}$ of the numbers $1, 2, \dots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \le k \le 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \dots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

6. Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

§1 JMO 2010/1, proposed by Andy Niedermier

Let $P(n)$ be the number of permutations $(a_1, \dots, a_n)$ of the numbers $(1, 2, \dots, n)$ for which ka_k is a perfect square for all $1 \le k \le n$. Find with proof the smallest n such that $P(n)$ is a multiple of 2010.

The answer is $n = 4489$.

We begin by giving a complete description of $P(n)$:

Claim — We have
$$P(n) = \prod_{c \text{ squarefree}} \left\lfloor \sqrt{\frac{n}{c}} \right\rfloor !$$

Proof. Every positive integer can be uniquely expressed in the form $c \cdot m^2$ where c is a squarefree integer and m is a perfect square. So we may, for each squarefree positive integer c, define the set
$$S_c = \left\{c \cdot 1^2, c \cdot 2^2, c \cdot 3^2, \dots\right\} \cap \{1, 2, \dots, n\}$$
and each integer from 1 through n will be in exactly one S_c. Note also that
$$|S_c| = \left\lfloor \sqrt{\frac{n}{c}} \right\rfloor .$$

Then, the permutations in the problem are exactly those which send elements of S_c to elements of S_c. In other words,
$$P(n) = \prod_{c \text{ squarefree}} |S_c|! = \prod_{c \text{ squarefree}} \left\lfloor \sqrt{\frac{n}{c}} \right\rfloor ! \qquad \square$$

We want the smallest n such that 2010 divides $P(n)$.

- Note that $P(67^2)$ contains $67!$ as a term, which is divisible by 2010, so 67^2 is a candidate.
- On the other hand, if $n < 67^2$, then no term in the product for $P(n)$ is divisible by the prime 67.

So $n = 67^2 = 4489$ is indeed the minimum.

§2 JMO 2010/2, proposed by Răzvan Gelca

Let $n > 1$ be an integer. Find, with proof, all sequences $x_1, x_2, \ldots, x_{n-1}$ of positive integers with the following three properties:

(a) $x_1 < x_2 < \cdots < x_{n-1}$;

(b) $x_i + x_{n-i} = 2n$ for all $i = 1, 2, \ldots, n-1$;

(c) given any two indices i and j (not necessarily distinct) for which $x_i + x_j < 2n$, there is an index k such that $x_i + x_j = x_k$.

The answer is $x_k = 2k$ only, which obviously work, so we prove they are the only ones. Let $x_1 < x_2 < \ldots < x_n$ be any sequence satisfying the conditions. Consider:

$$x_1 + x_1 < x_1 + x_2 < x_1 + x_3 < \cdots < x_1 + x_{n-2}.$$

All these are results of condition (c), since $x_1 + x_{n-2} < x_1 + x_{n-1} = 2n$. So each of these must be a member of the sequence.

However, there are $n-2$ of these terms, and there are exactly $n-2$ terms greater than x_1 in our sequence. Therefore, we get the one-to-one correspondence below:

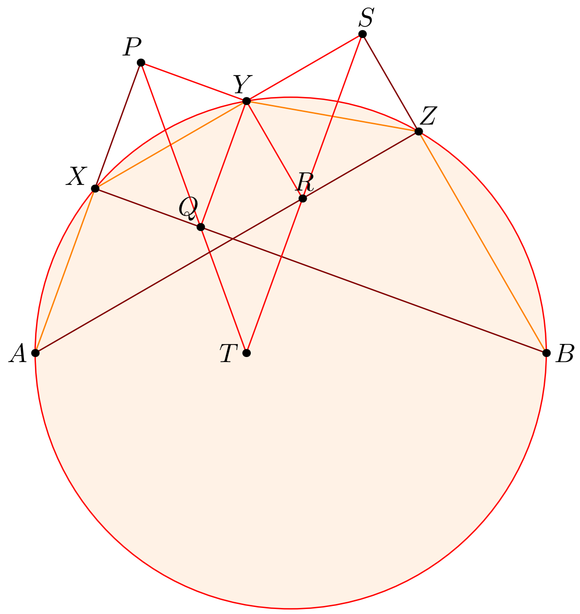
$$\begin{aligned} x_2 &= x_1 + x_1 \\ x_3 &= x_1 + x_2 \\ &\vdots \\ x_{n-1} &= x_1 + x_{n-2} \end{aligned}$$

It follows that $x_2 = 2x_1$, so that $x_3 = 3x_1$ and so on. Therefore, $x_m = mx_1$. We now solve for x_1 in condition (b) to find that $x_1 = 2$ is the only solution, and the desired conclusion follows.

§3 JMO 2010/3, proposed by Titu Andreescu

Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

Let T be the foot from Y to $\overline{AB}$. Then the Simson line implies that lines PQ and RS meet at T.



Now it's straightforward to see $APYRT$ is cyclic (in the circle with diameter $\overline{AY}$), and therefore

$$\angle RTY = \angle RAY = \angle ZAY.$$

Similarly,

$$\angle YTQ = \angle YBQ = \angle YBX.$$

Summing these gives $\angle RTQ$ is equal to half the measure of arc $\widehat{XZ}$ as needed.

(Of course, one can also just angle chase; the Simson line is not so necessary.)

§4 JMO 2010/4, proposed by Zuming Feng

A triangle is called a *parabolic* triangle if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

For $n = 0$, take instead $(a, b) = (1, 0)$.

For $n > 0$, consider a triangle with vertices at (a, a^2), $(-a, a^2)$ and (b, b^2). Then the area of this triangle was equal to

$$\frac{1}{2}(2a)\left(b^2 - a^2\right) = a(b^2 - a^2).$$

To make this equal $2^{2n}m^2$, simply pick $a = 2^{2n}$, and then pick b such that $b^2 - m^2 = 2^{4n}$, for example $m = 2^{4n-2} - 1$ and $b = 2^{4n-2} + 1$.

§5 JMO 2010/5, proposed by Gregory Galperin

Two permutations $a_1, a_2, \dots, a_{2010}$ and $b_1, b_2, \dots, b_{2010}$ of the numbers $1, 2, \dots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \le k \le 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \dots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

A valid choice is the following 1006 permutations:

$$\begin{array}{ccccccc|ccccc}
1 & 2 & 3 & \cdots & 1004 & 1005 & 1006 & 1007 & 1008 & \cdots & 2009 & 2010 \\
2 & 3 & 4 & \cdots & 1005 & 1006 & 1 & 1007 & 1008 & \cdots & 2009 & 2010 \\
3 & 4 & 5 & \cdots & 1006 & 1 & 2 & 1007 & 1008 & \cdots & 2009 & 2010 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1004 & 1005 & 1006 & \cdots & 1001 & 1002 & 1003 & 1007 & 1008 & \cdots & 2009 & 2010 \\
1005 & 1006 & 1 & \cdots & 1002 & 1003 & 1004 & 1007 & 1008 & \cdots & 2009 & 2010 \\
1006 & 1 & 2 & \cdots & 1003 & 1004 & 1005 & 1007 & 1008 & \cdots & 2009 & 2010
\end{array}$$

This works. Indeed, any permutation should have one of $\{1, 2, \dots, 1006\}$ somewhere in the first 1006 positions, so one will get an intersection.

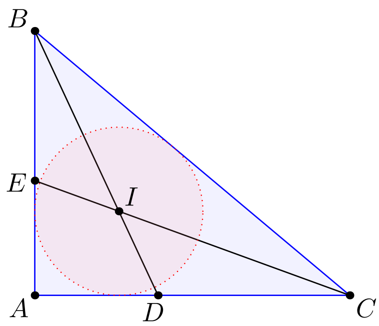
> **Remark.** In fact, the last 1004 entries do not matter with this construction, and we chose to leave them as $1007, 1008, \dots, 2010$ only for concreteness.

> **Remark.** Using Hall's marriage lemma one may prove that the result becomes false with 1006 replaced by 1005.

§6 JMO 2010/6, proposed by Zuming Feng

Let ABC be a triangle with $\angle A = 90^\circ$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

The answer is no. We prove that it is not even possible that AB, AC, CI, IB are all integers.



First, we claim that $\angle BIC = 135^\circ$. To see why, note that

$$\angle IBC + \angle ICB = \frac{\angle B}{2} + \frac{\angle C}{2} = \frac{90^\circ}{2} = 45^\circ.$$

So, $\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 135^\circ$, as desired.

We now proceed by contradiction. The Pythagorean theorem implies

$$BC^2 = AB^2 + AC^2$$

and so BC^2 is an integer. However, the law of cosines gives

$$\begin{aligned} BC^2 &= BI^2 + CI^2 - 2BI \cdot CI \cos \angle BIC \\ &= BI^2 + CI^2 + BI \cdot CI \cdot \sqrt{2}. \end{aligned}$$

which is irrational, and this produces the desired contradiction.