Mock Olympiad Solutions

1. Let d be any positive integer with $2 \leq d \leq p-1$. Since p is coprime to d, it follows that $ip \equiv jp \pmod{d}$ implies that d divides $p(i-j)$ and thus that $i \equiv j \pmod{d}$. Therefore $0, p, 2p, \dots, (d-1)p$ are distinct modulo d and there is a unique k with $1 \leq k < d$ and $kp \equiv -1 \pmod{d}$. This implies that d divides $ip+1$ and is counted in a_i if and only if $i = k$. Therefore $a_1 + a_2 + \cdots + a_{p-1} = p - 2$.

   Source: Japan 2016

2. By power of the point M with respect to ω, it holds that $MB^2 = MA \cdot MD = 2 \cdot MA^2$ and thus $MB = MA \cdot \sqrt{2}$. Similarly, $KA = KB \cdot \sqrt{2}$. Now note that since KA and MB are tangent to ω, it follows that $\angle MBA = \frac{1}{2}\widehat{AB} = \angle KAB$. Let B' be the other point on AB with $KB' = KB$. It follows by SSA similarity that MAB is similar to either KBA or $KB'A$. This implies that either $\angle MAB = \angle KBA$ or $\angle MAB = \angle KB'A$. Therefore either $AD \| BC$ or $\angle BAD = \angle ABC$, which in either case implies that $ABCD$ is a trapezoid.

   Source: Russia 2002

3. If $p = 2$ and $q = 5$, then take the number 10. If $q = 11$, then take 11. Now assume $p > 2$ and consider the set A of residues of the form $10^k \pmod{q}$. The order $|A|$ of 10 modulo q divides $2p$. Note that $|A| = 1, 2$ is not possible since $q \neq 3, 11$. If $-1 \in A$ then there is some t such that $10^t \equiv -1 \pmod{q}$. Therefore q divides $10^t + 1$ and we are done. If $|A| = 2p$, then A consists of all nonzero residues modulo q and thus $-1 \in A$. It suffices to consider the case in which $-1 \notin A$ and $|A| = p$. Let B be the set of residues of the form $-10^k \pmod{q}$ and note that $|A| = |B| = p$. If A and B have a common element then since 10 is coprime to q, it follows that $-1 \in A$. Thus $|A| = |B| = p$ implies that $A \cup B$ is all of the nonzero residues modulo q. If b is the smallest element of B, then $b - 1 \in A$ since $1 \in A$. This yields some k, h such that $10^k + 1 \equiv -10^h \pmod{q}$ and thus q divides $10^k + 10^h + 1$, proving the result.

   Source: Brazil 2009

4. Assume for contradiction that $a_1, a_2, \dots, a_m$ changes sign at most n times. Consider the polynomial $P(x)$ with root $i + \frac{1}{2}$ if a_i and a_{i+1} have different signs. Now choose the leading coefficient of $P(x)$ so that $P(i)$ has the same sign as a_i for each i. Note that $P(x)$ has at most n roots and thus has degree at most n. Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$.

Now note that

$$\begin{aligned} 0 &= \sum_{k=0}^{n} c_k \left(1^k \cdot a_1 + 2^k \cdot a_2 + \cdots + m^k \cdot a_m\right) \\ &= \sum_{i=1}^{m} a_i \left(c_0 + c_1 i + \cdots + c_n i^n\right) \\ &= \sum_{i=1}^{m} a_i P(i) \end{aligned}$$

However, the last sum is positive since a_i has the same sign as $P(i)$ for each i. This is a contradiction and thus $a_1, a_2, \ldots, a_m$ changes sign at least $n+1$ times.

Source: Russia 1996

5. An $n \times n \times n$ cube is divided into unit cubes. We are given a closed non-self-intersecting polygon in space, the sides of which join centres of two unit cubes sharing a common face. The faces of unit cubes which intersect the polygon are said to be distinguished. Prove that the edges of the unit cubes may be coloured in two colours so that each distinguished face has an odd number of edges of each colour, while each undistinguished face has an even number of edges of each colour.

6. First we prove the following lemma.

   Lemma 1.

   Proof. $\square$

   Source: Russia 1997