Warmup Problems

Algebra

1. Let a, b, c, x, y, z be real numbers such that

   $$a^2 + x^2 = b^2 + y^2 = c^2 + z^2 = (a+b)^2 + (x+y)^2 = (b+c)^2 + (y+z)^2 = (c+a)^2 + (z+x)^2$$

   Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

2. I have a circular cake, initially with icing on the top but not the bottom. Fix an angle $\theta < 180$. I cut out a slice with angle θ out of the cake and flip it upside down, inserting it back into the cake. I then rotate the cake by an angle θ counterclockwise and cut another slice with angle θ, flipping it upside down. If I repeat this process, will I eventually have a cake where the icing is eventually all on the top again? (in terms of θ)

3. Let $a_1 < a_2 < a_3 < a_4 < \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Show that there exists a number that occurs exactly once in the sequence

   $$\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots.$$

4. Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

   $$\sum_{1 \le i < j \le n} \frac{1}{n - a_i a_j} \le \frac{n}{2}$$

   must hold.

Combinatorics

1. How many ways can the nine cells of a 3×3 grid be colored red, blue, and green, so that four cells are colored red, four cells are colored blue, and one cell is colored green, so that the set of red cells and the set of blue cells are congruent figures?

2. Let G be the set of points (x, y) such that x and y are positive integers less than or equal to 6. A magic grid is an assignment of an integer to each point in G such that, for every square with horizontal and vertical sides and all four vertices in G, the sum of the integers assigned to the four vertices is the same as the corresponding sum for any other such square. A magic grid is formed so that the product of all 36 integers is the smallest possible value greater than 1. What is this product?

3. A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.

   All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

4. Consider a computer network consisting of servers and bi-directional communication channels among them. Unfortunately, not all channels operate. Each direction of each channel fails with probability p and operates otherwise. (All of these stochastic events are mutually independent, and $0 \leq p \leq 1$.) There is a root server, denoted by r. We call the network operational, if all servers can reach r using only operating channels. Note that we do not require r to be able to reach any servers.

   Show that the probability of the network to be operational does not depend on the choice of r. (In other words, for any two distinct root servers r_1 and r_2, the operational probability is the same.)

Geometry

1. Let G be the centroid of a triangle ABC, and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and YB and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

2. There are 300 points in space. Four planes A, B, C, and D each have the property that they split the 300 points into two equal sets. (No plane contains one of the 300 points.) What is the maximum number of points that can be found inside the tetrahedron whose faces are on A, B, C, and D?

3. Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Points X and Y lie on sides AB and CD, respectively. Suppose the circumcircles of ADX and BCY meet line XY again at P and Q, respectively. Show that $OP = OQ$.

4. Let $ABCD$ be a cyclic quadrilateral that inscribed in the circle ω.Let I_1, I_2 and r_1, r_2 be incenters and radii of incircles of triangles ACD and ABC,respectively.assume that $r_1 = r_2$. let ω' be a circle that touches AB, AD and touches ω at T. tangents from A, T to ω meet at the point K.prove that I_1, I_2, K lie on a line.

Number Theory

1. Find a positive integer N such that the seventh-to-last (millions) digit of each of the numbers N, $18N$, 18^2N, 18^3N, and 18^4N is equal to 1.

2. Let a positive integer n have at least four positive divisors. Let the least four positive divisors be $1 = d_1 < d_2 < d_3 < d_4$. Find, with proof, all solutions to $n^2 = d_1^3 + d_2^3 + d_4^3$.

3. Let $F_0, F_1, \ldots$ be the sequence of Fibonacci numbers, with $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $m > 2$, let R_m be the remainder when the product $\prod_{k=1}^{F_m - 1} k^k$ is divided by F_m. Prove that R_m is also a Fibonacci number.

4. Prove that for every prime $p > 100$ and every integer r, there exist two integers a and b such that p divides $a^2 + b^5 - r$.