Problems on graph theory

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1. A simple graph has 160 vertices and 500 edges. Prove that it contains a cycle of length at most 8.
2. If a graph on n vertices contains no triangles, prove that it has at most $n^2/4$ edges.
3. If a graph has $2n$ edges and n^2+1 triangles, prove that it contains at least n triangles.
4. Given a graph G with n vertices, its "closure" is obtained by repeatedly adding edges between any two non-adjacent vertices u and v with $\deg u + \deg v \geqslant n$. Prove that G has a Hamiltonian cycle if and only if so does its closure.
5. In each of three schools there are n students. Every student knows at least $n+1$ students from the other two schools. Prove that there are three students from different schools who know each other.
6. In a simple graph G with 100 vertices, inn every set of four vertices there are at least two edges. Moreover, every two vertices have a non-neighbor in common. Prove that this graph has a Hamiltonian path.
7. Prove that every tournament has a Hamiltonian path.
8. In Puerto Paranoia there are 16 secret agents, each of which follows at least one of the others, but no two follow each other. Assume that every 10 agents can be ordered so as to form a cycle (i.e. the first follows the second, the second follows the third, etc, the tenth follows the first). Prove that every 11 agents can also be so ordered.
9. In a simple directed graph with 102 vertices, every vertex has the indegree and the outdegree both equal to 40. Prove that from every vertex one can reach every other vertex using at most three edges.
10. Prove that every planar graph on n vertices has at most $3n-6$ edges, and that every bipartite planar graph has at most $2n-4$ edges.
11. Every edge of a convex polyhedron is assigned $+$ or $-$. Prove that there is a vertex having at most two facial angles with differently signed edges.
12. Prove that the vertices of a planar graph can be colored with 5 colors so that no edge connects two vertices of the same color.