Email training, N5

Level 4, October 11-17

Problem 5.1. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.

Solution 5.1. We consider the sequence

$$a_1 = 1, a_2 = 2,$$
$$a_{2n+1} = 2a_{2n},$$
$$a_{2n+2} = a_{2n+1} + r_n,$$

where r_n is the smallest natural number that cannot be written in the form $a_i - a_j$, with $i, j =\leq 2n+1$. It satisfies to the conditions of the problem

Problem 5.2. Prove the identity

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \ldots + (-1)^n \frac{\binom{n}{n}}{x+n}.$$

Solution 5.2. By applying the identity

$$\frac{1}{(x+a)(x+b)} = \frac{1}{a-b}\Big(\frac{1}{x+b} - \frac{1}{x+a}\Big)$$

multiple times one may get the following relation

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \sum_{k=0}^{n} \frac{A_k}{x+k}.$$

By multiplying both sides by $x(x+1)(x+2)\ldots(x+n)$ and by putting $n = -k$ one gets

$$n! = A_k \cdot (-k) \cdot (-k+1) \cdot (-k+2) \cdot \ldots \cdot (-1) \cdot 1 \cdot 2 \cdot \ldots \cdot (n-k)$$

so

$$A_k = \frac{(-1)^k A_k}{k!(n-k)!} = (-1)^k \binom{n}{k}.$$

Problem 5.3. Prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Solution 5.3. Let's apply Bernoulli inequality.

$$\Big(1 + \frac{5}{n}\Big)^n > 1 + n \cdot \frac{5}{n} = 6,$$

therefore

$$1 + \frac{5}{n} > 6^{1/n}.$$

Also

$$\Big(1 + \frac{-5}{6n}\Big)^n > 1 + n \cdot \frac{-5}{6n} = \frac{1}{6},$$
$$\Big(\frac{6n-5}{6n}\Big)^n > \frac{1}{6},$$
$$6 > \Big(\frac{6n}{6n-5}\Big)^n,$$
$$6^{1/n} > \frac{6n}{6n-5} = 1 + \frac{5}{6n-5}.$$

Problem 5.4. Let $x, y, z \geq 0$ and $x+y+z=3$. Prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx.$$

Solution 5.4. One has

$$3(x+y+z)=(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx).$$

Hence it follows that

$$xy+yz+zx=\frac{1}{2}(3x-x^2+3y-y^2+3z-z^2).$$

Then

$$\sqrt{x}+\sqrt{y}+\sqrt{z}-(xy+yz+zx)=$$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\frac{1}{2}(x^2-3x+y^2-3y+z^2-3z)$$

$$=\frac{1}{2}\sum_{cyc}(x^2-3x+2\sqrt{x})=\frac{1}{2}\sum_{cyc}\sqrt{x}(\sqrt{x}-1)^2(\sqrt{x}+2) \geq 0.$$

Problem 5.5. Let $a, b, c > 0$. Prove that

$$\frac{a+b}{a^2+b^2}+\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution 5.5. By applying the AM-GM for the denominator one gets

$$\frac{a+b}{a^2+b^2} \leq \frac{a+b}{2ab}=\frac{1}{2}\Big(\frac{1}{a}+\frac{1}{b}\Big).$$

By applying the same estimation for 2 other expressions of the left side and by taking the sum we get the desired statement.

Problem 5.6. Let $n > 3$, $x_1, x_2, \ldots, x_n > 0$ and $x_1x_2\ldots x_n = 1$. Prove that

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\ldots+\frac{1}{1+x_n+x_nx_1} > 1.$$

Solution 5.6.

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\ldots+\frac{1}{1+x_n+x_nx_1} >$$

$$\frac{1}{1+x_1+x_1x_2+x_1x_2x_3+\ldots+x_1x_2\ldots x_{n-1}}+$$

$$\frac{1}{1+x_2+x_2x_3+x_2x_3x_4+\ldots+x_2x_3\ldots x_n}+\ldots+$$

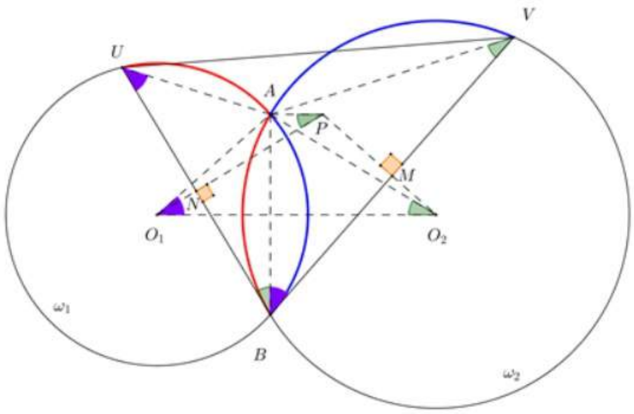
$$\frac{1}{1+x_n+x_nx_1+x_nx_1x_2+\ldots+x_nx_1\ldots x_{n-2}}.$$

Denote $S=1+x_1+x_1x_2+\ldots+x_1x_2\ldots x_{n-1}$. By multiplying the nominator and denominator of second term by x_1, of the third term by x_1x_2 and son on in n-th term by $x_1x_2\ldots x_{n-1}$ and by taking into account that $x_1x_2\ldots x_n = 1$ one gets

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\ldots+\frac{1}{1+x_n+x_nx_1} >$$

$$\frac{1}{S}+\frac{x_1}{S}+\frac{x_1x_2}{S}+\ldots+\frac{x_1x_2\ldots x_{n-1}}{S}=1.$$

Solution 5.7. Let ω_1 and ω_2 are the two circles and O_1 and O_2 are their their centres. We try to catch the fixed point P that satisfies the condition of the problem. We are trying to activate the role of common point B which seems to have the password.



When the first particle moves from A to B on ω_1, let the second particle move from A to V on ω_2 in the same time, so the arcs AB in ω_1 and AV in ω_2 are equal in size. Let when the second particle reaches to B, the first one reaches to U. Therefore $\angle AUB = \angle ABV$, hence BV is tangent to ω_1, and similarly BU is tangent to ω_2. But our point P is equidistant from B, U, V, hence P is the circumecenter of the triangle BUV. Hence $PO_1 \perp UB$, and N their intersection, and $PO_2 \perp VB$, and M their intersection.

we have $\angle AUB = \angle ABV = \angle ABM = \angle AO_1O_2 = \alpha$,

and $\angle AVB = \angle ABU = \angle ABN = \angle AO_2O_1 = \beta$.

Now $\angle PNB + \angle PMB = 90° + 90° = 180°$, Hence $PNBM$ is cyclic.

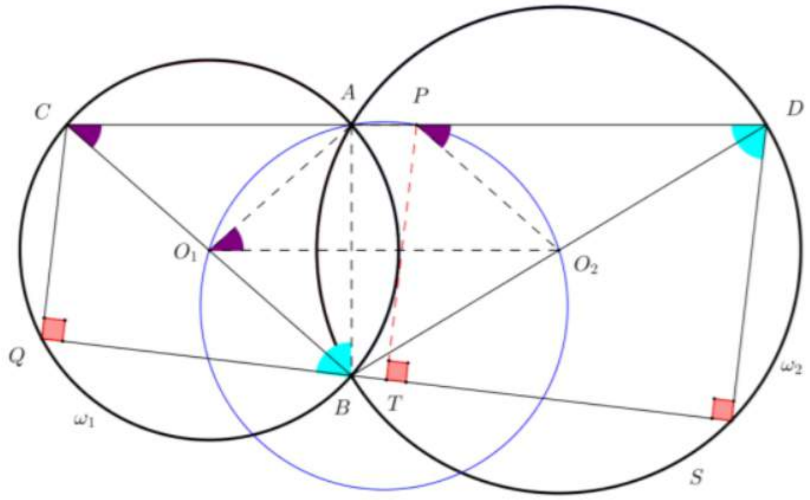
Therefore $\angle NPM (= \angle O_1PO_2) + \angle NBM = 180°$, so $\angle O_1PO_2 + \alpha + \beta = 180°$.

But from the triangle AO_1O_2 we have $\angle O_1AO_2 + \alpha + \beta = 180°$, hence $\angle O_1AO_2 = \angle O_1PO_2$, and $\angle O_1APO_2$ is cyclic.

Hence $\angle APN = \angle APO_1 = \angle AO_2O_1 = \beta = \angle ABN$. Hence $APBN$ is cyclic.

Therefore $\angle PAB = \angle PNB = 90°$, And $AP \perp AB$.

Now let AP intersects ω_1, ω_2 at C, D respectively. Since $\angle PAB = \angle DAB = 90°$, hence BD is a diameter in ω_2, and similarly CB is a diameter in ω_1.



Since APO_2O_1 is cyclic, hence $\angle DPO_2 = \angle AO_1O_2 = \angle ACB = \alpha$, hence $PO_2 \parallel CB$. In the triangle *CDB*, O_2 is the midpoint of *BD*, hence *P* is the midpoint of *CD*.

Now we have to prove that *P* is equidistant for any other corresponding positions to the two particles. Let *Q* and *S* are two positions like this, so $\angle ADS = \angle ABQ$, and so $\angle ABS + \angle ABQ = \angle ABS + \angle ADS = 180°$ (since *ADSB* is cyclic). Hence *Q,B,S* in the same line. Let *T* is the midpoint of *QS*, since $CQ \parallel DS$ (each of them is perpendicular *QS* , since *CB, BD* are diameters in ω_1, ω_2 respectively), hence *CQSD* is trapezoid, and $PT \parallel DS$. Hence *PT* is the perpendicular bisector of *QS*, therefore $PQ = PS$.

Finally if the point *B,Q,S* on a line in that order, also *CQSD* will be a trapezoid , the difference that *P,T* will be midpoints of its diagonals *CD,QS* respectively, but it is easy to see that *PT* is parallel to *CQ,DS*. So *PT* will still the perpendicular bisector of *QS*. And we are done.