Set 2. Orthocenter

Omar Habibullah

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1. Let D, E, F be the projections of A, B, C on BC, CA, AB on the triangle ABC respectively. Let H be the orthocenter of ABC. Prove that H is the incenter and A, B, C are excenters of $\triangle DEF$.

2. Let X be the reflection of H over BC in the triangle ABC. (H is the orthocenter of $\triangle ABC$). Prove that A, B, C, X are in one circle.

3. H, O are the orthocenter, circumcenter of $\triangle ABC$. Prove that AO, AH are isogonal.

4. (Euler circle) $\triangle ABC$ is given. M_a, M_b, M_c are the midpoints of BC, CA, AB respectively. H, H_a, H_b, H_c are the orthocenter, the altitudes of A, B, C on the sides BC, CA, AB respectively. N_a, N_b, N_c are the midpoints of HA, HB, HC Prove that $M_a, M_b, M_c, N_a, N_b, N_c, H_a, H_b, H_c$ are in one circle. Find the center of that circle.

5. M is the midpoint of BC and H is the orthocenter of $\triangle ABC$. E, F are the altitudes of B, C on the lines CA, AB. Prove that $MH, (ABC), (AEF)$ intersect at one point.

6.H, H_a, H_b, H_c are the orthocenter, the altitudes of A, B, C on the sides BC, CA, AB respectively on $\triangle ABC$. K is the second intersection of $(ABC), (AH_bH_c)$. Prove that AK, H_bH_c, BC are concurrent.

7. (Simson line) $\triangle ABC$ is Given. H is the orthocenter. P is a point on (ABC). X, Y, Z are the altitudes of P on BC, CA, AB respectively. M is the midpoint of PH. Prove that X, Y, Z, M are collinear.

8. (steiner line) $\triangle ABC$ is Given. H is the orthocenter. P is a point on (ABC). P_a, P_b, P_c are the reflections of P over BC, CA, AB. Prove that P_a, P_b, P_c, H are collinear.

9. Prove that $AH = 2OM$ if M is the midpoint of BC and H and O are the orthocenter and the circumcenter of $\triangle ABC$

10. Let ABC be triangle in which $AB = AC$. Suppose the orthocentr of the triangle lies on the in-circle. Find the ratio $\frac{AB}{BC}$.

11.(ELMO 2012) In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

12. (IMO Shortlist 2010) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

13. (Iran 2011) Let ABC be a triangle with altitudes AD, BE, and CF. Let M be the midpoint of BC, and let X and Y be the midpoints of ME and MF, respectively. Let Z be the point on line XY such that $ZA||BC$. Show that $ZA = ZM$

14. (IMO 2013) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

15. (USAMO 1990) An acute-angled triangle ABC is given in the plane. The circle with diameterAB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

16. (TSTST 2011) Acute triangle ABC is inscribed in circle . Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet at P and Q, respectively. Lines MN and PQ meet at R. Prove that OA is perpendicular to RA.

17. (TSTST 2017) Let ABC be a triangle with circumcircle Ω, circumcenter O, and orthocenter H. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Ω at A. Let Q be the intersection point, other than A, of Ω with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that PR is perpendicular to OH.