April Camp – Level 3,4.

Topic 10.

NUMBER THEORY 1

Phi-function $\varphi(n)=$ number of positive integers not bigger than n and coprime to n.

If $n=p$ is a prime then $\varphi(p^a)=p^a-p^{a-1}$ and $\varphi(mn)=\varphi(m)\varphi(n)$ for $\gcd(m,n)=1$. So

$$\varphi(p_1^{a_1}p_2^{a_2}\dots p_k^{a_k})=(p_1^{a_1}-p_1^{a_1-1})\cdots(p_k^{a_k}-p_k^{a_k-1}).$$

Euler theorem: $a^{\varphi(n)}\equiv 1 \pmod n$ for integer $n>1$ and $\gcd(a,n)=1$.

Problem 1.

a) Find all positive integers n such that $\varphi(n)=2$.

b) Find all positive integers n such that $\varphi(n)=2p$ for odd prime $p>3$.

Problem 2.

a) Prove that the equation $\varphi(x)=2\cdot 3^{6k+1}$ for $k\in\mathbb{Z}^+$ has exactly two solutions x.

b) Prove that $\varphi(a^n+b^n)$ is divisible by $2n$ for any two coprime numbers $a,b\in\mathbb{Z}^+$.

Problem 3.

For a,b are two coprime integers and $a>b>1$, consider the sequence

$$u_n=\varphi\left(a^{2n-1}+b^{2n-1}\right) \text{ for } n=1,2,3,\dots$$

a) Prove that the number $u_1u_2\dots u_{1009}$ is divisible by $\dfrac{2018!}{1009!}$.

b) Suppose that for some fix prime $p>3$, the number $2p$ appears in this sequence. Find all possible values of the sum $a+b$.

Problem 4.

For positive integer $n>1$, suppose that $\varphi(n)\mid n-1$ and n is composite. Prove that n is square-free number and it has at least 3 distinct prime divisor.

Problem 5.

Suppose that for some $n>1$, the number $1^{\varphi(n)}+2^{\varphi(n)}+\dots+n^{\varphi(n)}$ is divisible by n. Prove that n is square-free and find all such number n with no more than 3 prime divisors.

Problem 6.

Find all positive integers n such that n^2+3 is divisible by $\varphi(n)$.