Test 3 - Sols

Problem 1

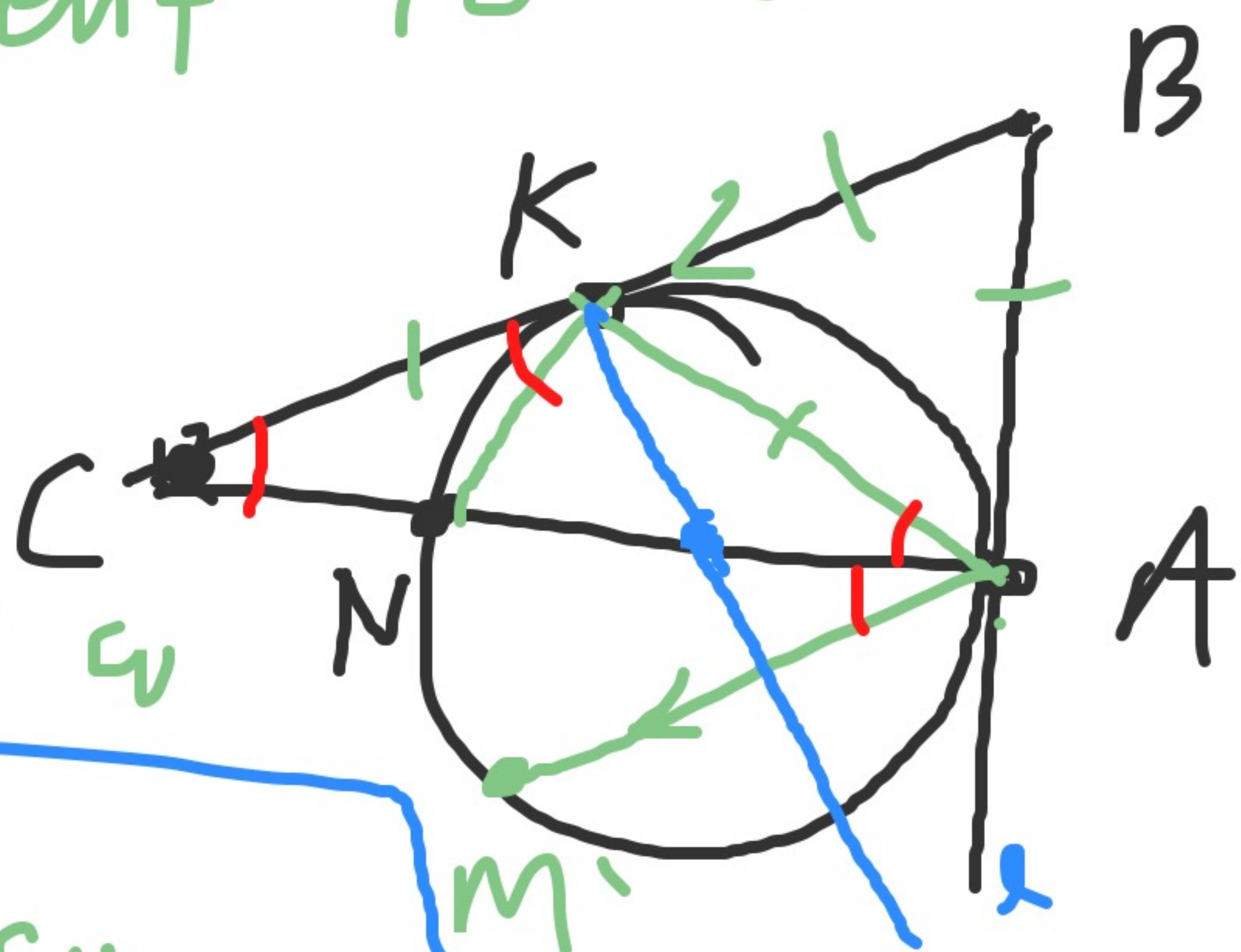
$M' \in \omega$ s.t $AM' \parallel BC$

$\boxed{NC = NK}$

since $BA = BK$

BA tangent to ω

$\Rightarrow AN$ diameter of ω



$\Rightarrow A, M'$ symmetric about l

but B, C are symmetric as well

and $AB = AK \Rightarrow \boxed{M'K = M'C}$ ∴ ■

$\hookrightarrow M' \equiv M$ (∵ $AM'CB$ cyclic)

Problem 2-test 7

$$n \neq \frac{2^a - 2^b}{2^c - 2^d}$$

solution if n even

then $\frac{n}{v_2(n)}$ can't be written

as $\frac{2^a - 2^b}{2^c - 2^d}$ ↯ $\Rightarrow$ n odd

w.l.o.g $b = d = 0$

$1 = \frac{2-1}{2-1}$, $3 = \frac{2^2-1}{2-1}$, $5 = \frac{2^4-1}{2^2-1}$,

$$7 = \frac{2^3 - 1}{2 - 1} \; ; \; 9 = \frac{2^6 - 1}{2^3 - 1}$$

$$11 = \frac{2^a - 1}{2^c - 1} \Rightarrow 11 \mid 2^a - 1$$

$$\Rightarrow 10 \mid a \quad ; \text{ if } c = 1$$

$$\Rightarrow 11 = 2^a - 1 \quad \lightning$$

$$\Rightarrow c > 1 \Rightarrow 11 = \frac{2^a - 1}{2^c - 1} \equiv \frac{-1}{-1}$$

$$\equiv 1 \, [4] \quad \lightning$$

Thus, $\boxed{n = 11}$

$\blacksquare$

Problem 3 - test 3 | $f: \mathbb{N} \to \mathbb{N}$

$$n^2 - 1 \leq f(n)f(f(n)) \leq n^2 + n \quad \forall n$$

$n = 1$: $\quad 0 \leq f(1)f(f(1)) \leq 2$

$\Rightarrow f(1)f(f(1)) = 1$ or 2

if $f(1) = 2 \Rightarrow f(f(1)) = 1$

$\Rightarrow f(2) = 1$

$\Rightarrow n = 2$: $\quad 3 \leq \underbrace{f(2)}_{1}\underbrace{f(f(2))}_{2} \leq 6$

↯

$\Rightarrow \boxed{f(1) = 1}$

if $f(a)=f(b)$ for some

$a<b$ then for

$$T=f(a)f(f(a))=f(b)f(f(b))$$

we have

$$T\leq a^2+a<a^2+2a=(a+1)^2-1$$
$$\leq b^2-1\leq T \ \lightning$$

$\Rightarrow \boxed{f \text{ injective}}$

Now we prove by induction that $f(n)=n \ \forall n$.

the base case $n=1$ is proved

suppose $f(k)=k \quad \forall k<n$

$$\Rightarrow f(m) \geq n \quad \forall m \geq n$$

$$\left[\because \; f(m) \notin \{f(1), \cdots, f(n-1)\} = \{1, 2, \cdots, n-1\} \text{ by injectivity} \right]$$

Now $f(n) \geq n \Rightarrow f(f(n)) \geq n$

but $f(n) f(f(n)) \leq n(n+1)$

$\Rightarrow \quad f(n) \leq n+1 \;;\; f(f(n)) \leq n+1$

if $f(n)=n+1 \Rightarrow f(f(n))=n$

$\Rightarrow f(n+1)=n$

$\Rightarrow$

$$n^2+2n \le f(n+1)f(f(n+1))$$
$$= n^2+n < n^2+2n \ \lightning$$

$\Rightarrow f(n)=n$ as claimed

So, the only solution is

$$\boxed{f(n)=n \quad \forall n\in\mathbb{N}} \quad \blacksquare$$

Problem 4