Competition Preparation for Saudi Arabia Team 2021: Level 4

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Lesson 4

Problems in graph theory

Problems:

1. Let a graph be given such that there are no closed paths and you can reach any vertex from any other going along the edges of the graph. Prove that you can color the vertices of the grap in 2 colors so that no two vertices of the same color are connected. Such a graph is called a bipartite graph.

2. Prove that a graph is bipartite if and only if it contains no odd cycles.

3. In each of three schools there are n students. Each student knows at least $n+1$ students from the other two schools. Prove that there are three students from three schools who know each other.

4. Let a k-group be a collection of k people who all know each other (knowing someone is a symmetric relation). Assume at a party that there are no 5-groups and that no two 3-groups are disjoint, in other words every pair of three groups has at least one common person. Prove that there are two people whose departure would result in there being no 3-groups at the party.

5. Let A and B be two countries such that each city is connected to exactly $k \geq 2$ cities in the other country via an interstate road. Interstate roads connect a city in country A with a city in country B with no stops in-between and no pair of cities is connected by more than one interstate. Assuming each city is reachable from every other city using the interstate roads is it possible to destroy a single road so that this no longer holds?

6. Let n be a complete graph, i.e. a graph whose every pair of vertices is connected. In each move, we select a cycle of length 4, i.e. $A_1A_2A_3A_4$ such that A_1A_2, A_2A_3, A_3A_4 and A_4A_1 all belong to the graph and remove one of the edges, say A_1A_2. We continue making moves until there are no more cycles of length 4 in the graph. Find the lowest number of edges we can obtain.

7. Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\ldots$; c_{m-1} and c_m; c_m and c_1. Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

8. On the continent of Graphia there are $n \geq 3$ countries. On Year 1 of the New Mathematical Era we find that some pairs of countries are at war. Each country is at war with at least one other country and it is impossible to divide the country into two groups such that no two countries in opposing groups are at war. Starting from Year 2, each year two countries that are at war sign a peace treaty. Unfortunately, each country that is at war with exactly one of the two countries that signed the peace treaty will also declare war on the other country. Additionally, if at any point in time we divide the countries into two arbitrary non-empty groups, after a finite amount of time two countries in the opposite groups will sign a peace treaty. Prove that after a finite amount of time there will be a country that is at war with all other countries.

9. In a country called Graphia each city is connected with three other cities via a road. A single road starts and ends in a city without passing through any other city. Prove that it is possible to color all the roads in four colors so that no two roads of the same color share a common endpoint city.