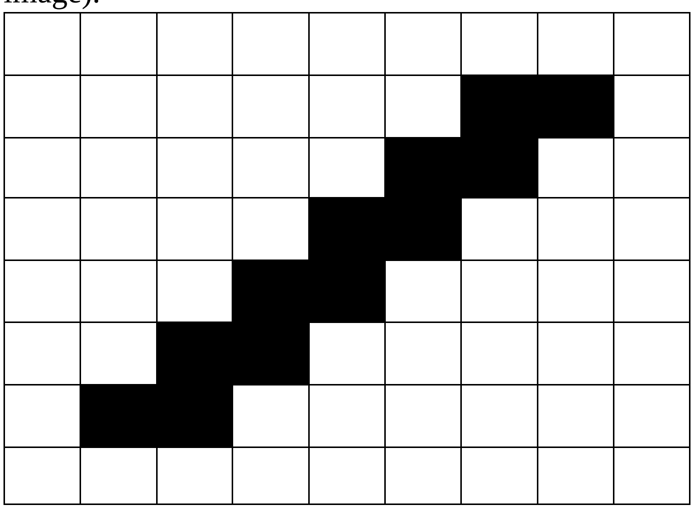
Invariants and monovariants

1. Let $p_1, p_2, \dots, p_n, \dots$ be the prime number in the ascending order and let $0 < x_0 < 1$. Define the sequence x_k:
$$x_k = \begin{cases} 0, & if\ x_{k-1} = 0 \\ \left\{\dfrac{p_k}{x_{k-1}}\right\}, & otherwise \end{cases}$$
Describe all positive $0 < x_k < 1$ such that the sequence is eventually equal to 0.
(USAMO 1997)
2. What is the maximal number of non-overlapping 2×1 dominoes (horizontally or vertically), that can be put on a 8×9 board, if 6 of them are already there (see the image).

(CMO 2007)
3. On an $n \times n$ board, $n - 1$ cells are infected. Each second, a cell, adjacent to two infected cells, may get infected. Show that in the end, at least one cell is left uninfected.
(Stanford Putnam trainings 2007)
4. A computer shows $n \times n$ board, each cell of which is colored either black or white. At each turn, one may choose a rectangle, with sides parallel to the sides of the board, and change colors of all cells in that rectangle. Let X denote the minimal number of turns needed to make the whole board white and Y denote the number of board vertices, adjacent to odd number of black squares. Prove that $\frac{Y}{4} \le X \le \frac{Y}{2}$.
(IOI 2002)
5. Alice and Bob play a game on a 6×6 grid. On his or her turn, a player chooses a rational number not appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a path from the top of the grid to the bottom of the grid that stays in

black squares, and Bob wins if she cannot. (If two squares share a vertex, Alice can draw a path from one to the other that stays in those two squares.) Find a winning strategy for one of the two players.

(USAMO 2004)

6. Define a domino to be an ordered pair of distinct positive integers. A proper sequence of dominoes is a list of distinct dominoes in which the first coordinate of each pair after the first equals the second coordinate of the immediately preceding pair, and in which (i, j) and (j, i) do not both appear for any i and j. Let D_{40} be the set of all dominoes whose coordinates are no larger than 40. Find the length of the longest proper sequence of dominoes that can be formed using the dominoes of D_{40}.

(AIME 1998)

7. The numbers from 1 through 2008 are written on a blackboard. Every second, Dr. Math erases four numbers of the form $a, b, c, a + b + c$, and replaces them with the numbers $a + b, b + c, c + a$. Prove that this can continue for at most 10 minutes.

8. A regular (5 × 5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially, all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

(APMO 2007)

9. Several stones are placed on an infinite (in both directions) strip of squares, indexed by the integers. We perform a sequence of moves, each move being one of the following two types:
(a) Remove one stone from each of the squares $n - 1$ and n and place one stone on square $n + 1$.
(b) Remove two stones from square n and place one stone on each of the squares $n - 2$ and $n + 1$.
Prove that any sequence of such moves will lead to a position in which no further moves can be made, and moreover that this position is independent of the sequence of moves.

10. How many distinct acute angles α satisfy to the condition $\cos(\alpha)\cos(2\alpha)\cos(4\alpha) = \frac{1}{8}$

11. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation: in every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(IMO 2014 shortlist)

12. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them Sand Z-tetrominoes, respectively. S-tetrominoes Z-tetrominoes Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

(IMO 2014 shortlist)

13. Several checkers are placed on a board (see the table). Each turn, a checker may jump diagonally over an adjacent piece if the opposite square is empty. If a checker is jumped over in this way, it is removed from the board. Is it possible to make a sequence of such jumps to remove all but one checker from the board shown below?

O		O		O		O	
	O		O		O		O
O						O	
	O						O
O						O	
	O						O
O		O		O		O	
	O		O		O		O

14. In a country there are two parties – Democrats and Republicans. Country consists of several states, each of which has either Democratic or Republican government. Each month, an election happens in exactly one state. During an election, the government of the state may change, if the majority of the neighbors of the state have different government. Prove, that starting from some point, the elections will not change any government.

15. You have a stack of $2n + 1$ cards, which you can shuffle using the two following operations:

1. Cut: Remove any number of cards from the top of the pile and put them on the bottom.

2. Perfect riffle shuffle: Remove the top n cards from the deck and place them in order in the spaces between the other $n + 1$ cards.

Prove that, no matter how many operations you perform, you can reorder the cards in at most $2n(2n + 1)$ different ways.