1

$+(k+1)$

$\leq +(n+1)$

1 line $=2$

40 lines $\leq 2+2+3+\cdots+41$

$= 821$ $\boxed{821}$

2

$40081 = 4\cdot 10^4 + 3^4$

$= (10^2+7^2)(10^2+17^2)$

$= 149 \times 269$ $\boxed{269}$

3

$A = \{a, a-1, \cdots, a-(m-1)\}$

$B = \{b, b-1, \ldots, b-(2m-1)\}$

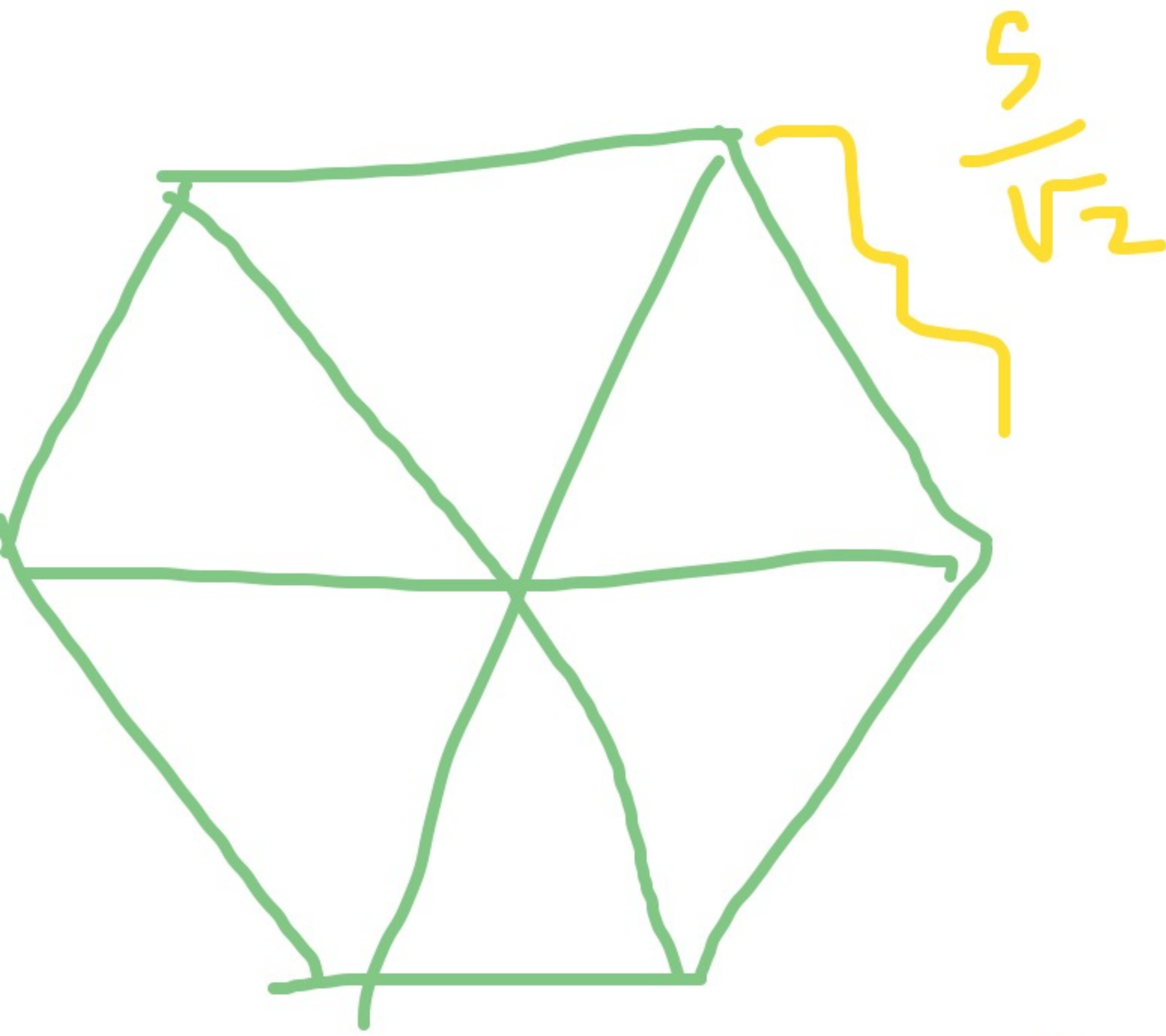
$$\sum(A) = 2m \Rightarrow a = \frac{m+3}{2}$$

$$\sum(B) = m \Rightarrow b = m$$

$$b - a = 100 \Leftrightarrow m = 203$$

$\boxed{203}$

4



$$[IJKLMN] = 6 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{5}{\sqrt{2}}\right)^2$$

$$= \frac{75\sqrt{3}}{4} \rightarrow \boxed{082}$$

5

$\{x\} + \{\frac{1}{x}\} = 1$

$\Rightarrow \quad x + \frac{1}{x} \in \mathbb{Z} \quad , \{x, \frac{1}{x}\} \notin \mathbb{Z}$

$x + \frac{1}{x} = 3, 4, 5, \ldots$

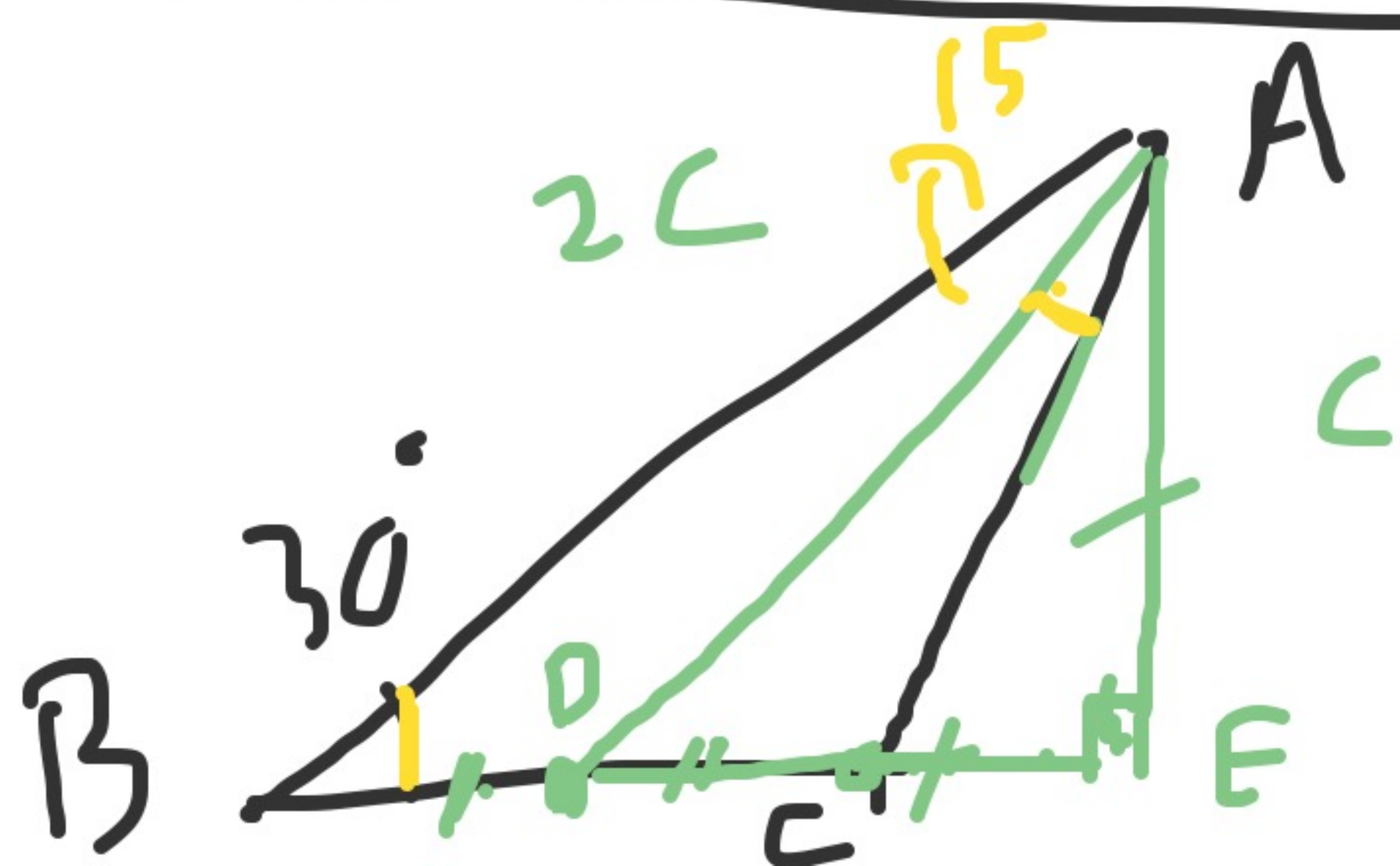
$x > 1 \quad f(x) = x + \frac{1}{x}$

is increasing or $(1, \infty)$

$x + \frac{1}{x} = 3$ has the smallest sol >1

which is $\frac{3+\sqrt{3}}{2} \rightarrow \boxed{011}$

6



$ED = c$

$EB = \sqrt{3}\, c$

$CE = (2-\sqrt{3})\, c$

$CA^2 = CD \cdot CB$

$\Longrightarrow \quad \angle C = \boxed{105}$

7

$111111111 \cdots\cdots 1110$

$$\begin{array}{r} 999- \\ \hline \cdots\cdots 110111 \end{array} \longrightarrow \boxed{999}$$

8

$1 \quad \{53\}$ ✓

$2^4 - 1 \quad \{2, 4, \cancel{6}, 8\} \geq 1$

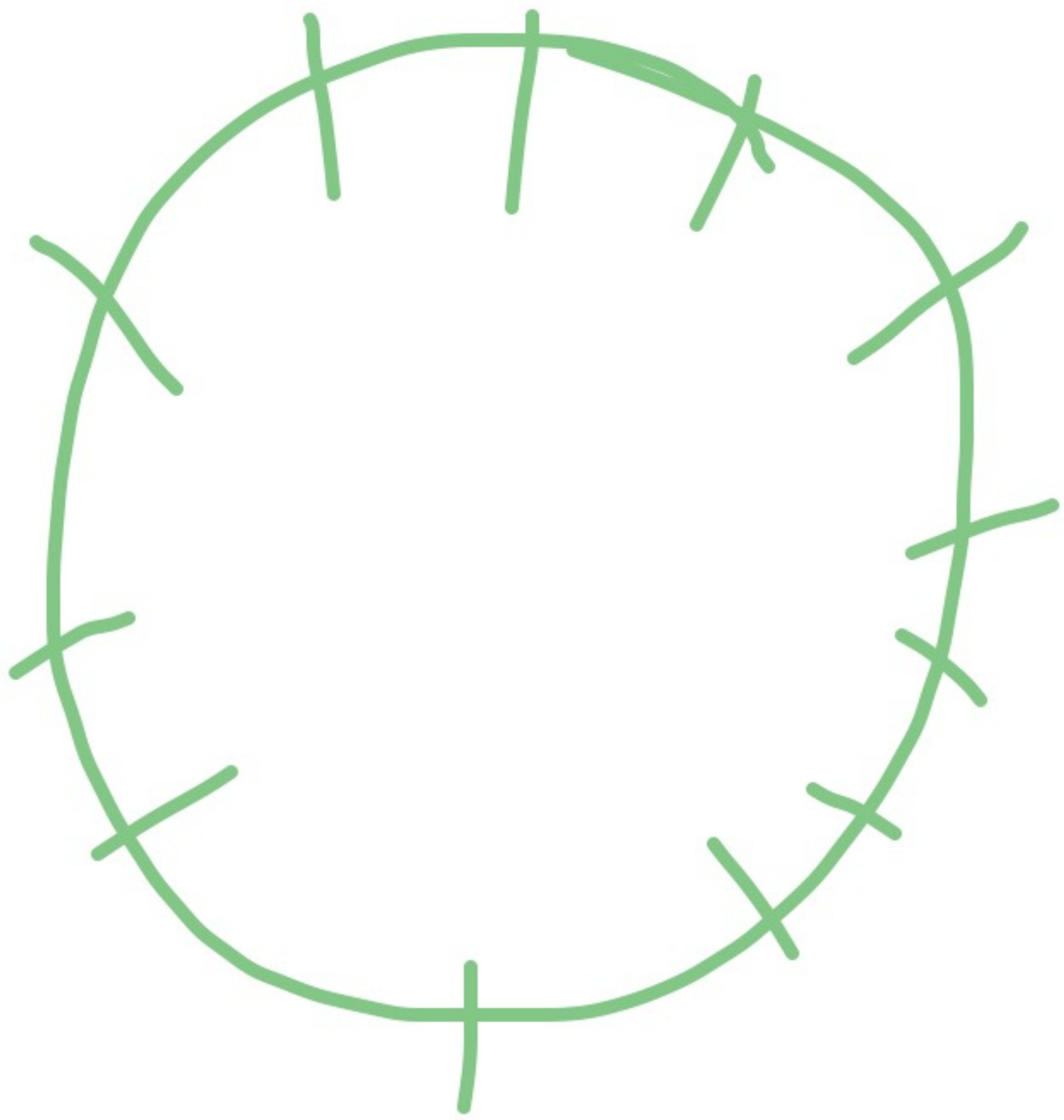
$2^4 \quad \{1, 3, 7, 9\}$

$$\frac{2^4(2^{11}-1)}{2^9-1} \longleftarrow$$

$$\longrightarrow \frac{240}{511} \longrightarrow \boxed{751}$$

If we had 2^n prisoners, 0

will survive (namely, the one holding the sword)



$\boxed{9}$

$\Rightarrow$ when 512 prisoners remain, the one holding the sword will survive

$0, 2, 4, \ldots, \boxed{976} \rightarrow \boxed{976}$

$0, 1, 2, \ldots, \boxed{488}$

$$1 \leq n \leq 256^2$$

$$n^2 \equiv 256 \; [1000]$$

4
16^2

$$2^3 \cdot 5^3 \mid (n-16)(n+16)$$

$$d = \gcd(n-16, n+16) \mid 32$$

$$(d, 5^3) = 1$$

$\Rightarrow \quad 5^3 \mid n-16$ or $5^3 \mid n+16$

clearly, n even, if $n \equiv 2 \, (4)$

$\Rightarrow 2^3 \nmid n^2 - 256$ X $\Rightarrow 4 \mid n$

$\Rightarrow \quad 4 \mid n-16$ and $4 \mid n+16$

$\Rightarrow 500 \mid n-16$ or $500 \mid n+16$

$\Rightarrow \quad n \equiv \pm 16 \; [500]$

$n \equiv 016, 484, 516, 984 \; [1000]$

$1 \leq n \leq 256^2 = 65536$

$$\left.\begin{array}{l} 0 \xrightarrow[4]{} 1000 \\ 1000 \xrightarrow[4]{} 2000 \\ \vdots \\ 64000 \xrightarrow[4]{} 65000 \\ 65000 \xrightarrow[3]{} 65536 \end{array}\right\} \quad 4 \times 65 + 3 = \boxed{263}$$

- Each match adds one win and one lose

$\Rightarrow$ # of matches

$= \binom{n}{2} = w_1 + w_2 + \cdots + w_n$

$= l_1 + l_2 + \cdots + l_n$

- $w_i + l_i =$ # matches P_i played

$= n - 1$

$\Rightarrow \sum w_i^2 - \sum l_i^2 =$

$\sum (w_i + l_i)(w_i - l_i) = (n-1) \sum (w_i - l_i)$

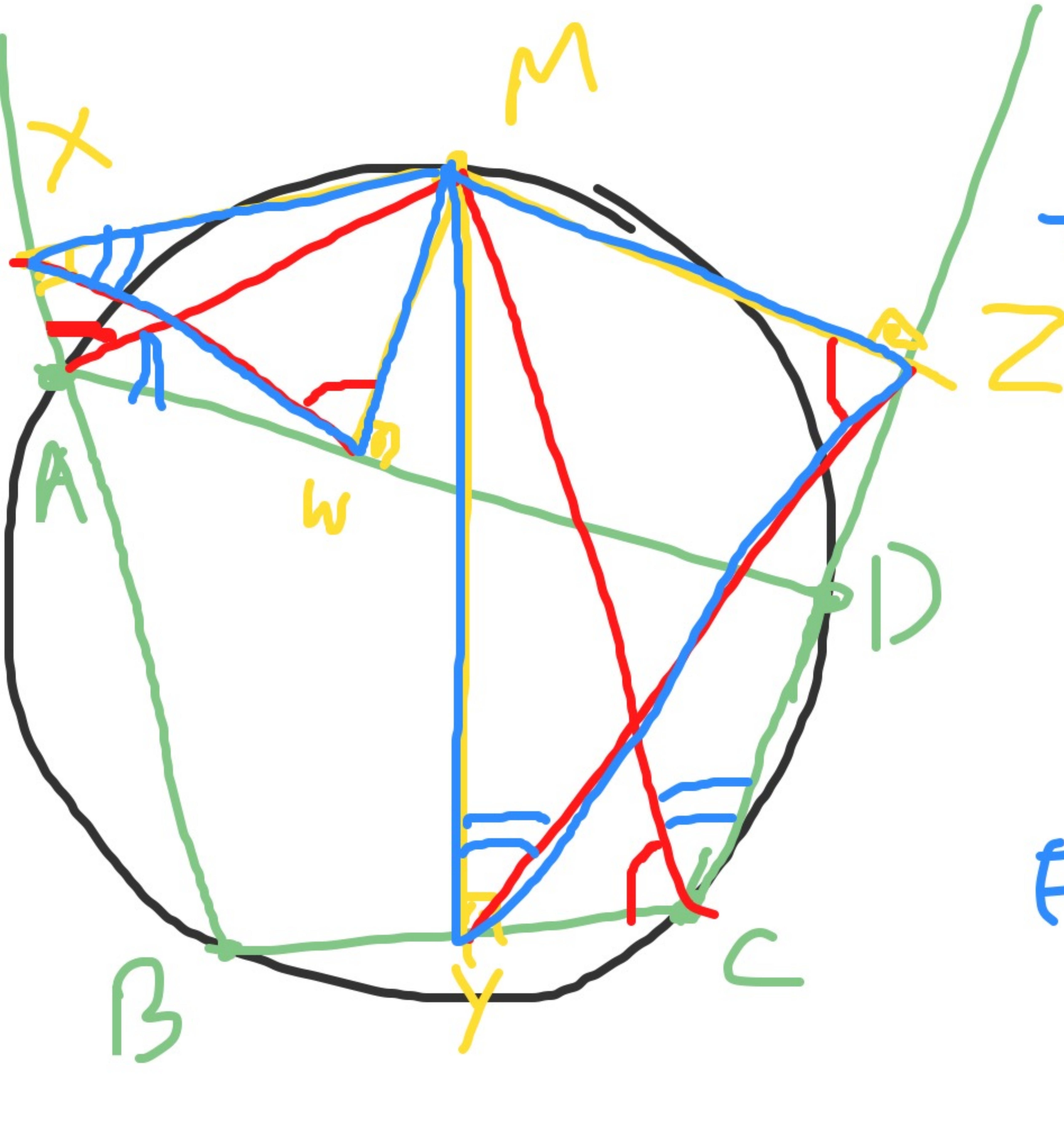
$= 0 \Longrightarrow \sum w_i^2 = \sum l_i^2$

$$\Longrightarrow \sum \binom{w_i}{2} = \frac{\sum w_i^2 - \sum w_i}{2}$$

$$= \frac{\sum l_i^2 - \sum l_i}{2} = \sum \binom{l_i}{2}$$

■

12



$$\Longrightarrow \Delta MXW \sim \Delta MYZ$$

$$\Longrightarrow \frac{MX}{MW} = \frac{MY}{MZ}$$

$$\Longleftrightarrow MX \cdot MZ = MY \cdot MW$$

■

$\boxed{13}$

$R_n = \underbrace{11\cdots 1}_{2^n}$

(enough to verify for R_n)

$9R_n = 10^{2^n} - 1$

$= 9(10+1)(10^2+1)\cdots(10^{2^{n-1}}+1)$

$\Leftrightarrow R_n = (10+1)(10^2+1)\cdots(10^{2^{n-1}}+1)$

<u>Lemma</u>: $(10^{2^k}+1, 10^{2^l}+1) = 1$

$\forall\, 0 \leq k < l$

<u>Proof</u>: Let $p \in \mathbb{P}$, $p \mid 10^{2^k}+1$

then $10^{2^k} \equiv -1 \, [p]$

$\Rightarrow 10^{2^{k+1}} \equiv 1 \, [p]$,

now, square the last congruence till we get

$10^{2^l} \equiv 1 \, [p]$

$\Rightarrow p \mid 10^{2^l} - 1$, clearly $p > 2$

$\Rightarrow p \nmid 10^{2^l} + 1$ $\square$

Now, the result immediately follows from the Lemma $\blacksquare$