HELLO

OK?

Problem $4ab-1 \mid (4a^2-1)^2 \Rightarrow a=b$. (IMO 2007)

Similar to $mn-1 \mid (n^2-n+1)^2$

HINT: $4ab-1 \mid (a-b)^2$

Short:

$$4ab-1 \mid b^2\underbrace{(4a^2-1)}_{\mid 4ab-1}^2 - \underbrace{(4ab-1)}_{\mid 4ab-1}(4a^3b-2ab+a^2) =$$

$$= (a-b)^2$$

$4ab-1 \mid (a-b)^2$ We solved $\boxed{\text{2 classes ago}}$

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$$4ab-1 \mid (4a^2-1)^2$$

$$4ab-1 \mid (4a^2-1+4ab-1)^2$$

$$4ab-1 \mid \left(4a^2-1-(4ab-1)\right)^2 = (4a^2-4ab)^2 =$$

$$= (4a)^2(a-b)^2$$

AND $(4ab-1,\ 4a^2) = 1$. SO

$$4ab-1 \mid (a-b)^2. \quad :)$$

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$ab-1 \mid a^2+b^2+ab$ positive integer.

(3)

Show: $\dfrac{a^2+b^2+ab}{ab-1} = k$ 4 or 7.

$ab-1 \mid a^2+b^2+ab \Rightarrow ab-1 \mid a^2+b^2+1$

$\dfrac{a^2+b^2+1}{ab-1} = k-1$

Assume $a \geq b$. Case $a=b$. $\Rightarrow \dfrac{2a^2+1}{a^2-1} = 2 + \dfrac{3}{a^2-1}$, so

$a^2-1 \mid 3 \Rightarrow a^2-1 = \underset{a=2}{3}, \cancel{-3}, \cancel{1}, \underset{a\neq}{\cancel{-1}} \Rightarrow a=2 \Rightarrow a=b=2$

$\dfrac{2^2+2^2+1}{2\cdot 2-1} = \dfrac{9}{3} = 3 = k-1 \Rightarrow \boxed{k=4}$

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$\underline{a > b}$ (a, b) minimal sum.

$$\frac{a^2+b^2+1}{ab-1} = k-1 \quad \text{then} \longrightarrow \quad \cancel{a^2+b^2+1} = abk - ab - k + \cancel{1}$$

$$\boxed{a^2 - ab(k-1) + b^2 + 4 = 0}$$

$$a^2 - (k-1)ab + b^2 + k = 0$$

$$x_2 = \underbrace{(k-1)b - a}_{\in \mathbb{Z}} = \frac{b^2+k}{a} > 0 \qquad x_2 \text{ is positive integer,}$$

so (x_2, b) is also a solution. but (a, b) is minimal, so $x_2 + \cancel{b} \geq a + \cancel{b} \Rightarrow \boxed{x_2 \geq a}$

$$x_2 = (k-1)b - a \geq a \Rightarrow \boxed{k-1 \geq \frac{2a}{b}} \quad \text{then}$$

$$\frac{a^2+b^2+1}{ab-1} = k-1 \geq \frac{2a}{b} \Rightarrow b(a^2+b^2+1) \geq 2a(ab-1) \Rightarrow$$

$$\cancel{ba^2} + b^3 + b \geq \cancel{2}a^2b - 2a$$

$$b^3 + b \geq a^2b - 2a = a(ab-2)$$

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$$b^3+b \geq a^2b-2a = a(ab-2) \geq (b+1)((b+1)b-2)$$

$$= (b+1)^2 b - 2(b+1)$$

$$= (b^2+2b+1)b - 2b - 2$$

$$= b^3+2b^2+b-2b-2$$

$$= b^3+2b^2-b-2$$

$a \geq b+1$ (because $a>b$)

$$\cancel{b^3}+b \geq \cancel{b^3}+2b^2-b-2$$

$$2b^2-2b-2 \leq 0 \Rightarrow \boxed{b^2-b-1 \leq 0}$$

$$\Downarrow$$

$$\boxed{b=1}$$

$$\searrow \left(b-\frac{1}{2}\right)^2 \leq 1\frac{1}{4}$$

$$\left|b-\frac{1}{2}\right| \leq \frac{\sqrt{5}}{2}$$

$b \in \mathbb{Z}$. $b=1$

so $\boxed{b=1}$

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$$\frac{a^2+b^2+1}{ab-1} = k-1 \quad , \quad \boxed{b=1} \qquad \underline{k \in \{4,7\}}.$$

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a^2+1^2+1 is divisible by $a \cdot 1 - 1 = a-1$

$$\mathbb{Z} \ni \frac{a^2+2}{a-1} = a+1+\frac{3}{a-1}, \text{ so } a-1 \mid 3 \text{ so}$$

$$a-1 = \underset{a=4}{3}, \underset{a<0}{\cancel{-3}}, \underset{a=2}{1}, \underset{a=0}{\cancel{-1}}$$

so we have pairs $(4,1)$ and $(2,1)$

$$k-1 = \frac{a^2+b^2+1}{ab-1} = \frac{4^2+1^2+1}{4 \cdot 1 - 1} = 6 \Rightarrow \boxed{k=7}$$

$$k-1 = \frac{a^2+b^2+1}{ab-1} = \frac{2^2+1^2+1}{2 \cdot 1 - 1} = 6 \Rightarrow \boxed{k=7}$$

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a, b, c, m pos. int

$$abcm = 1 + a^2 + b^2 + c^2 \Rightarrow \underline{m = 4} \quad \boxed{AMM}$$

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<u>Remark</u> a, b, c — three <u>variables</u>.

Take this equation <u>modulo 4</u> and prove that $\boxed{4 \mid m}$.

<u>hint</u>

- a, b, c odd $\Rightarrow a^2, b^2, c^2 \equiv 1 \pmod 4$ so

$$a^2 + b^2 + c^2 + 1 \equiv 0 \pmod 4$$

$$\| $$

$$abcm \Rightarrow \underline{4 \mid m}.$$

are odd

- a, b are odd $a^2, b^2 \equiv 1 \pmod 4$, so

$$\underbrace{a^2 + b^2 + c^2}_{\underline{2 \mid abcm}} + 1 \equiv 1 + 1 + 0 + 1 = 3 \pmod 4$$

$\lightning$

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• a is odd then $a^2 \equiv 1$ so $a^2+b^2+c^2+1 = 1+0+0+1 = 2 \bmod 4$

$\parallel$

$4 \mid abcm$ ↯ 8

Therefore $a, b, c,$ are odd and so $\boxed{m = 4 \cdot n}$

We need $m=4$ i.e. $\boxed{n=1}$ Suppose $n>1$

$$4abcn = 1+a^2+b^2+c^2$$

WLOG: $a \geq b \geq c$ and $\underline{a+b+c}$ is minimal

Make quadratic eq.:

$$a^2 - a(4bcn) + b^2+c^2+1 = 0$$

$$\boxed{x_2 = 4bcn - a} = \frac{b^2+c^2+1}{a} > 0$$

(x_2, b, c) smaller sum.

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$x_2 = 4bcn - a$ Suppose $n > 1$

If $x_2 \geq a$ then $4bcn - a \geq a \Rightarrow 4bcn \geq 2a$ (9)

$a^2 + b^2 + c^2 + 1 = 4abcn \geq 2a^2 \Rightarrow$

$\boxed{2b^2 + 1 \geq b^2 + c^2 + 1 \geq a^2}$ so

$(b \geq c)$

$a^2 < a^2 + 1 \leq 2b^2 + 1 \leq 4b^2$, so $a^2 < 4b^2 \Rightarrow a < 2b$

But then $\boxed{n > 1}$

$4abcn > 2a^2cn \geq 4a^2 \geq a^2 + b^2 + c^2 + 1$

$a < 2b$ $\boxed{cn \geq 2}$ $b^2 + c^2 + 1 = 4abcn$

$4abcn > a^2 + b^2 + c^2 + 1 = 4abcn$ ↯ so $x_2 < a$

so $x_2 + b + c < a + b + c \rightarrow$ ↯ with minimality of (a, b, c)

$$x_2 = \underset{\in \mathbb{Z}}{4bcn} - a = \frac{b^2+c^2+1}{a} > 0 \;:$$

$$x_2 + \cancel{b} + \cancel{c} \geq a + \cancel{b} + \cancel{c} \Rightarrow x_2 \geq a.$$

so $\dfrac{b^2+c^2+1}{a} \geq a \Rightarrow \boxed{b^2+c^2+1 \geq a^2}$

$4bcn - a \geq a \Rightarrow \boxed{2bcn \geq a} \Rightarrow$

$\Rightarrow$

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0,1 0,2 0,1

$$+\begin{cases} x^2+6y^2=z^2 \\ 6x^2+y^2=t^2 \end{cases}$$

0+0
1+0

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$$7x^2+7y^2=z^2+t^2$$

$7 \mid z^2+t^2 \Rightarrow 7 \mid z,\ 7 \mid t \Rightarrow z=7z_1$
$t=7t_1$

$$7x^2+7y^2=49z_1^2+49t_1^2 \quad |:7$$

$$x^2+y^2=7z_1^2+7t_1^2$$

$7 \mid x^2+y^2 \Rightarrow x=7x_1,\ y=7y_1 \Rightarrow$

$$49x_1^2+49y_1^2=7z_1^2+7t_1^2 \Rightarrow$$

$$7x_1^2+7y_1^2=z_1^2+t_1^2 \Rightarrow$$

By Fermat descent

$$x=y=z=t=0$$

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(Kolmogorov Cup)

$a_1 = 2$

$a_{n+1} = 2^{a_n} + 2 \quad n \geq 1.$

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Prove: $a_n \mid a_{n+1}$.

1) NO HOPE TO PROVE : $a_n \mid a_{n+1}$

$a_{n+1} \mid a_{n+2}$ /.

YOU REMEMBER $\boxed{1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} \leq 2 - \frac{1}{2^n}} < 2$

TRY $\left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}\right) + \frac{1}{2^{n+1}} < 2$

$2 + \frac{1}{2^{n+1}} \overset{?}{<} 2$

Impossible.

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$$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n} \leq \cancel{2-\frac{1}{2^n}}\ 2.$$

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$$1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^{n+1}} = 1+\frac{1}{2}+\dots+\frac{1}{2^n}+\frac{1}{2^{n+1}} \leq$$

$$\leq 2-\frac{1}{2^n}+\frac{1}{2^{n+1}} \leq 2-\frac{1}{2^{n+1}}$$

$$=$$

$a_1 = 2$

$a_n = 2^{a_n} + 2$

Prove

$a_1 = 2 \mid a_2 = 6$

$a_1 - 1 = 1 \mid a_2 - 1 = 5$

$$\boxed{\text{"}\ a_n \mid a_{n+1} \quad \text{AND} \quad a_n - 1 \mid a_{n+1} - 1\ \text{"}}$$

HINT STANDARD INDUCTION WORKS.

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We prove by induction

"$a_n \mid a_{n+1}$ and $a_n - 1 \mid a_{n+1} - 1$" (1)

Base of induction

$a_1 = 2 \mid a_2 = 6$, $a_1 - 1 = 1 \mid a_2 - 1 = 5$ ✓,

Suppose holds for $(n-1)$

$$\underbrace{\text{"}a_{n-1} \mid a_n \text{ and } a_{n-1} - 1 \mid a_n - 1\text{"}}_{\text{We HEVE}}$$

$$\underbrace{\text{"}a_n \mid a_{n+1} \text{ and } a_n - 1 \mid a_{n+1} - 1\text{"}}_{\text{We WANT.}}$$

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We want $a_n \mid a_{n+1}$

$$\parallel \qquad\qquad \parallel$$

$$2^{a_{n-1}}+2 \overset{?}{\mid} 2^{a_n}+2$$

$$\parallel \qquad\qquad \parallel$$

$$\cancel{2}\left(2^{a_{n-1}-1}+1\right) \overset{?}{\mid} \cancel{2}\left(2^{a_n-1}+1\right)$$

$$\boxed{2^{a_{n-1}-1}+1 \mid 2^{a_n-1}+1}$$

But By induction assumption.

$a_{n-1}-1 \mid a_n-1$ so done.

But Remember

$$\boxed{2^a+1 \mid 2^b+1 \iff a \mid b}$$

$$\gcd\left(2^x-1,\ 2^y-1\right) = 2^{\gcd(x,y)}-1.$$

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We need to prove

$$a_n - 1 \mid a_{n+1} - 1$$

$$\parallel$$

$$2^{a_n - a_1} + 2 - 1 \mid 2^{a_n} + 2 - 1$$

$$2^{a_{n-1}} + 1 \mid 2^{a_n} + 1$$

But From Assumption of INDUCTION

$$a_{n-1} \mid a_n$$

$\boxed{\text{So done}}$

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$a_1 = 2 \qquad a_n = 2^{a_{n-1}} + 2.$

Standard induction fails (" $a_n \mid a_{n+1}$ "). We prove sth. stronger.

" $a_n \mid a_{n+1}$ and $a_n - 1 \mid a_{n+1} - 1$ "

(1)

$(n-1 \Rightarrow n)$

$n = 1 \qquad 2 \mid 6 \qquad 1 \mid 5$ ✓

Suppose $a_{n-1} \mid a_n$ and $\boxed{a_{n-1} - 1 \mid a_n - 1}$. We want

$a_n \mid a_{n+1}$ and $a_n - 1 \mid a_{n+1} - 1$.

$a_n \mid a_{n+1}$

$\parallel$

$2^{a_{n-1}} + 2 \mid 2^{a_n} + 2 \rightsquigarrow 2^{a_{n-1}-1} + 1 \mid 2^{a_n - 1} + 1$ but $a_{n-1} - 1 \mid a_n - 1$ so done

$a_n - 1 \mid a_{n+1} - 1$

$\parallel$

$2^{a_{n-1}} + 1 \mid 2^{a_n} + 1$ but $a_{n-1} \mid a_n$ so done.

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$\boxed{\text{Ex}}$

$7 \mid a^2+b^2 \Rightarrow 7 \mid a$ and $7 \mid b$.

$13 \mid a^2+b^2 \Rightarrow 13 \mid a$ and $13 \mid b$.

$p = 4k+3$, $p \mid a^2+b^2 \Rightarrow \boxed{p \mid a,\ p \mid b}$

A

<u>Problem 1</u> Prove that number $4mn - m - n$ is not perfect square (m, n, positive integers).

$$4mn - m - n = x^2$$

$$\underbrace{(4m-1)(4n-1) = 4x^2+1}_{\parallel}$$

$$(2x)^2 + 1^2$$

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$p \mid a^2+b^2$, suppose $p \nmid a$, and so $p \nmid b$

(19)

$$a^2+b^2 \equiv 0 \pmod p$$

$$a^2 \equiv -b^2 \pmod p \quad \wedge \quad \frac{p-1}{2} = \frac{4k+3-1}{2} = 2k+1$$

odd

$$(a^2)^{\frac{p-1}{2}} \equiv (-1)^{\text{odd}} \cdot (b^2)^{\frac{p-1}{2}} \pmod p$$

$$a^{p-1} \equiv -1 \cdot b^{p-1} \pmod p$$

$$a^{p-1} \equiv -b^{p-1} \pmod p$$

LFT $\equiv$ $\qquad$ $\equiv$ LFT

$$1 \qquad -1 \quad \Rightarrow \quad 1 \equiv -1 \pmod p \Rightarrow$$

$$2 \equiv 0 \pmod p \Rightarrow \boxed{p \mid 2}$$

$\lightning$

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$(4m-1)(4n-1) = 4x^2+1 = (2x)^2+1^2$

(2)

4SH1 But $\exists$ $p = 4k+3$ such that $p \mid 4m-1$. Why?

If not, then all prime divisors of $4m-1$ are of the form $4l+1$, but product of numbers of the form $4m+1$ is of the same form:

$$(4a+1)(4b+1) = \underbrace{16ab+4a+4b}_{4l}+1$$

Premise i.e.

$p = 4k+3 \mid (2x)^2+1^2 \implies p \mid 2x, \; p \nmid 1$

$\lightning$

IMO Shortlist 19...

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1) $(a^2+b^2)(c^2+d^2)$ is always sum of two squares. Why?

$$\| $$

$$(ac)^2+(ad)^2+(bc)^2+(bd)^2 = \underline{(ac-bd)^2+(ad+bc)^2}.$$

2) $5 = 1^2+2^2$, $13 = 3^2+2^2$, $17 = 4^2+1^2$, $29 = 5^2+2^2$

$\equiv 1 \bmod 4$, $\equiv 1 \bmod 4$, $\equiv 1 \bmod 4$, $\equiv 1 \bmod 4$

<u>Theorem</u> (Fermat) Any prime of the form $p = 4k+1$ is a sum of two squares.

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First step ✓

Lemma $p = 4k+1$ then there exists n integer such that

$$\underline{p \mid n^2+1}.$$

(2)

Proof $p = 4k+1$, $\frac{p-1}{2} = \underset{\text{even}}{2k}$, take $n := \left(\frac{p-1}{2}\right)!$

then $p \mid \left(\left(\frac{p-1}{2}\right)!\right)^2 + 1$

Wilson says: $p \mid (p-1)! + 1$.

modulo p.

$$-1 \equiv (p-1)! = 1 \cdot 2 \cdot 3 \cdots \frac{p-1}{2} \cdot \left(p - \frac{p-1}{2}\right) \cdots p-1 \equiv$$

$$\equiv 1 \cdot 2 \cdot 3 \cdots \frac{p-1}{2} \cdot \left(-\frac{p-1}{2}\right) \cdot \left(-\frac{p-3}{2}\right) \cdots (-1) =$$

$$= \left(\frac{p-1}{2}\right)! \cdot (-1)^{\frac{p-1}{2}} \cdot \left(\frac{p-1}{2}\right)! = \left(\left(\frac{p-1}{2}\right)!\right)^2 \underbrace{(-1)^{\frac{p-1}{2}}}_{=1} =$$

$$= \left(\left(\frac{p-1}{2}\right)!\right)^2 \text{ (modulo } p)$$

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Another idea "legal form"

$$\left(\frac{-1}{p}\right) = 1$$

Second step

Lemma (Thue) Let n be positive integer and take a integer such that $\gcd(n, a) = 1$. Then there exists $0 < x, y \leq \sqrt{n}$ such that

$$\overset{n}{\overset{\shortparallel}{a}}x \equiv \pm y \pmod{\overset{p}{\overset{\shortparallel}{n}}}$$

Proof

Consider set $S := \{ ax - y : 0 \leq x, y \leq \sqrt{n} \}$

$\# S = (\lfloor \sqrt{n} \rfloor + 1)^2 > n$ $\quad \lfloor \sqrt{n} \rfloor + 1 . \quad \lfloor \sqrt{n} \rfloor + 1$

so by Pigeonhole Principle $ax_1 - y_1$, $ax_2 - y_2$ with the same residues modulo n.

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So $\quad ax_1 - y_1 \equiv ax_2 - y_2 \pmod{n}$

WLOG $\quad x_1 > x_2$ because $x_1 = x_2 \Rightarrow -y_1 \equiv -y_2 \pmod{n} \Rightarrow$

$y_1 = y_2$

$a(x_1 - x_2) \equiv y_1 - y_2 \pmod{n}$

take $x = |x_1 - x_2|$ any $y := \cancel{y_1 - y_2}$

done

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Last step

take prime $p = 4k+1$. From I step we know that $p \mid n^2+1$ for some n. In particular $\gcd(p,n)=1$. From II step (Thue Lemma) applied for p, n; we see that there are

$$0 < x, y < \sqrt{p} \quad \text{such that}$$

$$\overset{a}{\Downarrow}\ nx \equiv \pm y \ (\text{modulo } \overset{n}{\overset{\parallel}{p}})\ \wedge 2$$

$$(nx)^2 \equiv y^2 \pmod p$$

$$-x^2 \equiv n^2x^2 \equiv y^2 \pmod p \Rightarrow -x^2 \equiv y^2 \bmod p \Rightarrow$$

$p \mid x^2+y^2$, but $0 < x^2+y^2 < (\sqrt{p})^2+(\sqrt{p})^2 = 2p$

$(0, 2p)$

so $\boxed{p = x^2+y^2}$:)

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We proved

1) $p = 4k+1$ is sum of 2 squares :

2) $p = 4k+3$ if is sum of 2 squares the $p \mid a$; $p \mid b$ so $p^2 \mid a^2+b^2$

$\overset{\shortparallel}{a^2+b^2}$

$$\boxed{a^2+b^2 = n} = \underbrace{\overbrace{\overset{\alpha}{2} \cdot \underbrace{\overset{\alpha_1}{p_1} \cdots \overset{\alpha_k}{p_k}}_{p_i \text{ are } 4k+1}}^{A^2+B^2}}_{(A^2+B^2)} \cdot \underbrace{\overset{2\beta_1}{q_1} \cdots \overset{\beta_\ell}{q_\ell}}_{4k+3}$$

$2 = (1^2+1^2)(1^2+1^2)$, $p_i \equiv 1 \bmod 4 \Rightarrow p_i = a_1^2 + b_1^2$

$C^2(A^2+B^2)$

$\shortparallel$

$(CA)^2 + (CB)^2$

Always sum of 2 squares.

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What about case when

$$a^2+b^2 = n = q \cdot N$$

$(2k+1)^2$

$q = 4k+3$

$q \mid a, \; q \mid b$

$q^2 \mid a^2+b^2$

$\downarrow$

q^2 always at some point you will get

$$a_s^2+b_s^2 = q \cdot N$$

q^2

$\lightning$

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Theorem

n is sum of 2 squares iff exponent of primes of the form $4k+3$ is $\boxed{\text{even}}$.

For example

$$2020 = 2 \cdot 1010 = 2^2 \cdot 505 = \underset{\downarrow}{2^2} \cdot \underset{4k+1}{5} \cdot \underset{4k+1}{101}$$ it is sum of 2 squares.

$$\underline{2019} = \underset{\underset{4k+3}{\boxed{11}}}{3^1} \cdot 673$$ — no sum of 2 squares.

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