Number Theory – group L3

Instructor: Dušan Djukić *Date: 26.11.2021.*

1. Suppose that each of the positive integers a, b, c, d is divisible by $ad - bc$. Find all possible values of $ad - bc$.
2. Find all positive integers n and d with $d \mid n$ such that $n^2 + d^2$ is divisible by $nd + 1$.
3. Let $n > 1$ be an integer and $d > n$ be a divisor of $n^2 + 1$. Prove that $d > n + \sqrt{n}$.

- If p is a prime, x and y integers and $v_p(x) = k < v_p(y) = \ell$, then $v_p(x \pm y) = k$. $\square$

4. Given n positive integers, denote by d_k the greatest common divisor of all product of k of these integers. Prove that $d_k^2 \mid d_{k-1}d_{k+1}$ for $2 \leqslant k \leqslant n - 1$.
5. Suppose that a and b are positive integers such that $a^2 + b^2 + a$ is divisible by ab. Prove that a is a perfect square.
6. Find all pairs (a, b) of positive integers such that $2ab^2 - b^3 + 1$ divides a^2.
7. Find all triples of positive integers (x, y, z) such that each of the numbers $x^2 - 1$, $y^2 - 2$, $z^2 - 4$ is divisible by $x + y + z$.
8. If x, y, z are rational numbers such that $xyz = 1$ and $x + y + z$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ are both integers, prove that $|x| = |y| = |z| = 1$. (HW)
9. If $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n} [\frac{n}{i}]^2 = \sum_{i=1}^{n} (2i - 1)[\frac{n}{i}]$. (HW)
10. Integers $a > b > 1$ are such that $a^2 + b - 1$ is divisible by $b^2 + a - 1$. Prove that $b^2 + a - 1$ cannot a power of a prime. (HW)

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8. If x, y, z are rational numbers such that $xyz = 1$ and $x+y+z$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ are both integers, prove that $|x| = |y| = |z| = 1$.
9. If $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n}[\frac{n}{i}]^2 = \sum_{i=1}^{n}(2i-1)[\frac{n}{i}]$.
10. Integers $a > b > 1$ are such that $a^2 + b - 1$ is divisible by $b^2 + a - 1$. Prove that $b^2 + a - 1$ cannot a power of a prime.
11. Can all integers greater than 10^{100} be written as a sum of a prime and a perfect square?
12. We are given two integers a and b of different parities. Prove that there is an integer c such that $c + a$, $c + b$ and $c + ab$ are all perfect squares.
13. Find the largest positive integer whose digits are all nonzero and distinct, and that is divisible by all its digits.
14. Let n and a be given positive integers. Suppose that for every $m \in \mathbb{N}$ there is an integer x such that $x^n \equiv a \pmod{m}$. Prove that $a = y^n$ for some integer y. (HW)
15. Prove that there exist infinitely many positive integers k for which the equation $\frac{x}{\tau(x)} = k$ has no solutions in $\mathbb{N}$. (HW)
16. Find all pairs of positive integers a and b such that $\operatorname{lcm}(a+1, b+1) = a^2 - b^2$. (HW)
17. Find all positive integers x for which $3x^4 + 10x^2 + 3$ is a square. (HW)

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14. Let n and a be given positive integers. Suppose that for every $m \in \mathbb{N}$ there is an integer x such that $x^n \equiv a \pmod{m}$. Prove that $a = y^n$ for some integer y.
15. Prove that there exist infinitely many positive integers k for which the equation $\frac{x}{\tau(x)} = k$ has no solutions in $\mathbb{N}$.
16. Find all pairs of positive integers a and b such that $\operatorname{lcm}(a+1, b+1) = a^2 - b^2$.
17. Find all positive integers x for which $3x^4 + 10x^2 + 3$ is a square.
18. Find all pairs of positive integers (a, b) for which a is odd, b is a power of 2, and $a^2 - ab + b^2$ is a perfect square. (HW)
19. Determine all prime numbers $p > 2$ such that both $\frac{p+1}{2}$ and $\frac{p^2+1}{2}$ are perfect squares. (HW)
20. Find all pairs (m, n) of positive integers such that $mn - 1$ divides $n^3 + 1$. (HW)
21. Find all $n \in \mathbb{N}$ for which the sum of digits of $n!$ is equal to 9. (HW)
22. Given a positive integer n, we write down the fraction $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$, each in lowest terms. Define $f(n)$ to be the sum of the numerators of these fractions. Find all n for which $f(n)$ and $f(999n)$ have opposite parities. (HW)
23. Prove that there are infinitely many integers $n > 0$ for which $[\sqrt{n}] + [\sqrt{2n}] + [\sqrt{3n}]$ divides n. (HW)

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18. Find all pairs of positive integers (a, b) for which a is odd, b is a power of 2, and $a^2 - ab + b^2$ is a perfect square.

19. Determine all prime numbers $p > 2$ such that both $\frac{p+1}{2}$ and $\frac{p^2+1}{2}$ are perfect squares.

20. Find all pairs (m, n) of positive integers such that $mn - 1$ divides $n^3 + 1$.

21. Find all $n \in \mathbb{N}$ for which the sum of digits of $n!$ is equal to 9.

22. Given a positive integer n, we write down the fraction $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$, each in lowest terms. Define $f(n)$ to be the sum of the numerators of these fractions. Find all n for which $f(n)$ and $f(999n)$ have opposite parities. (HW)

23. Prove that there are infinitely many integers $n > 0$ for which $[\sqrt{n}] + [\sqrt{2n}] + [\sqrt{3n}]$ divides n. (HW)

24. Prove that $2^{58} + 1$ has at least three distinct prime divisors. (HW)

> *Chinese Remainder Theorem.* Let $a_1, a_2, \dots, a_k$ be integers and let $n_1, n_2, \dots, n_k$ be pairwise coprime positive integers. Then the system of congruences
>
> $$x \equiv a_i \pmod{n_i} \quad \text{for } i = 1, 2, \dots, k$$
>
> has a unique solution x modulo $n_1 n_2 \cdots n_k$. $\square$

25. Prove that there exist 201 consecutive positive integers, each of which has a prime divisor not exceeding 103. (HW)

26. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant a(b-1) + b(c-1) + c(a-1) + 1$. Show that equality occurs for infinitely many triples (a, b, c). (HW)

27. If a and b are positive integers such that $\operatorname{lcm}[a, b] + \operatorname{lcm}[a+2, b+2] = 2 \cdot \operatorname{lcm}[a+1, b+1]$, prove that $a \mid b$ or $b \mid a$. (HW)

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22. Given a positive integer n, we write down the fraction $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$, each in lowest terms. Define $f(n)$ to be the sum of the numerators of these fractions. Find all n for which $f(n)$ and $f(999n)$ have opposite parities.

23. Prove that there are infinitely many integers $n > 0$ for which $[\sqrt{n}] + [\sqrt{2n}] + [\sqrt{3n}]$ divides n.

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$$x \equiv a_i \pmod{n_i} \quad \text{for } i = 1, 2, \dots, k$$

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28. If $n > 1$ is an integer, prove that $4^n + 2^n + 1$ cannot be divisible by $3^n - 2^n$.

29. Call an integer an *almost-square* if it is a product of two consecutive integers. Prove that every almost-square can be written as a quotient of two other almost-squares.

30. Find all positive integers n that cannot be written as $n = [a, b] + [b, c] + [c, a]$ for some $a, b, c \in \mathbb{N}$.

31. Let $n > 2$ be a positive integer and let $a_1 < a_2 < \dots < a_k$ be all positive integers less than n and coprime to n. Find all numbers n for which none of the sums $a_1 + a_2$, $a_2 + a_3$, $\dots$, $a_{k-1} + a_k$ is divisible by 3.

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Instructor: Dušan Djukić *Date: 1.12.2021.*

25. Prove that there exist 201 consecutive positive integers, each of which has a prime divisor not exceeding 103.

26. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant a(b-1) + b(c-1) + c(a-1) + 1$. Show that equality occurs for infinitely many triples (a, b, c).

27. If a and b are positive integers such that $\operatorname{lcm}[a, b] + \operatorname{lcm}[a+2, b+2] = 2 \cdot \operatorname{lcm}[a+1, b+1]$, prove that $a \mid b$ or $b \mid a$.

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32. Given a prime number $p > 2$, prove that one can find p positive integers less than $2p^2$ such that all their pairwise sums are distinct. (E.g. the pairwise sums of the numbers $1, 3, 6$ are $2, 4, 6, 7, 9, 12$.)

33. Prove that the equation $x^3 + y^3 + z^3 = 2$ has infinitely many solutions in integers.

34. Find all prime numbers p such that $\frac{p^2-p-2}{2}$ is a perfect cube.

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Instructor: Dušan Djukić *Date: 4.12.2021.*

29. Call an integer an *almost-square* if it is a product of two consecutive integers. Prove that every almost-square can be written as a quotient of two other almost-squares.

30. Find all positive integers n that cannot be written as $n = [a, b] + [b, c] + [c, a]$ for some $a, b, c \in \mathbb{N}$.

31. Let $n > 2$ be a positive integer and let $a_1 < a_2 < \cdots < a_k$ be all positive integers less than n and coprime to n. Find all numbers n for which none of the sums $a_1 + a_2$, $a_2 + a_3$, $\dots$, $a_{k-1} + a_k$ is divisible by 3.

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33. Prove that the equation $x^3 + y^3 + z^3 = 2$ has infinitely many solutions in integers.

34. Find all prime numbers p such that $\frac{p^2-p-2}{2}$ is a perfect cube.

35. Let a, b, c, d be positive integers such that $b < c$ and $a + b + c + d = ab - cd$. Prove that $a + c$ is a composite number.

36. Denote by b_n the number of binary unit digits of a positive integer n. We call n *lively* if $b_n \mid n$. Prove that (a) there are no 5 consecutive positive integers that are all lively; (b) there are infinitely many instances of 4 consecutive positive integers that are all lively.

37. We say that a positive integer n is *friendly* if the equation $(x^2 + y)(y^2 + x) = n(x - y)^3$ has a solution in $\mathbb{N}$. Prove that (a) at least 500 positive integers $n \leqslant 2020$ are friendly, and (b) that $n = 2$ is not friendly.

38. Denote by $\omega(x)$ the number of distinct prime divisors of a positive integer x. Let a, b and c be arbitrary positive integers. Prove that there is a positive integer n such that $\omega(an + c) \geqslant \omega(bn + c)$.

Solutions – group L3

Instructor: Dušan Djukić *Nov.26–Dec.4, 2021*

1. The products $a \cdot d$ and $b \cdot c$ are divisible by $|ad - bc|^2$, so $|ad - bc|^2$ divides $ad - bc$. Therefore $ad - bc = \pm 1$.

2. Let $n = de$. Then $nd + 1 = d^2e + 1 \mid n^2 + d^2 = d^2(e^2 + 1)$. Since $(d^2, d^2e + 1) = 1$, we have $d^2e + 1 \mid e^2 + 1$, so $d^2e + 1 \mid e^2 - d^2e = e(e - d^2)$. Since $(e, d^2e + 1) = 1$, we get $d^2e + 1 \mid e - d^2$, but $|e - d^2| < \max\{e, d^2\} < d^2e + 1$, so we must have $e = d^2$. Hence $(n, d) = (d^3, d)$.

3. Denote $d = n + k$. Then $n + k \mid n^2 - k^2$ by default, so $n + k \mid k^2 + 1$ as well. Hence $n + k \leqslant k^2 + 1$, i.e. $n \leqslant k^2 - k + 1 < k^2$, unless $k = 1$. Note that $k = 1$ would imply $n = 1$.

4. We will prove that $v_p(d_{k-1}d_{k+1}) \geqslant v_p(d_k^2)$ for every prime p. Let the exponents at p in the given numbers be $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n$. Then $v_p(d_i) = r_1 + \cdots + r_i$, so $v_p(d_{k-1}d_{k+1}) = 2(r_1 + \cdots + r_{k-1}) + r_k + r_{k+1}$ and $v_p(d_k^2) = 2(r_1 + \cdots + r_k) \leqslant v_p(d_{k-1}d_{k+1})$.

5. We shall prove that $v_p(a) = k$ is even for any prime p. Assume $k > 0$ and $v_p(b) = \ell$. Then $v_p(ab) = k + \ell$, $v_p(a^2) = 2k$ and $v_p(b^2) = 2\ell$, but if $2\ell \neq k$, we have $v_p(b^2 + a) = \min\{k, 2\ell\} < v_p(a^2)$, so $v_p(a^2 + b^2 + a) = \min\{k, 2\ell\} < k + \ell$, contradicting the divisibility. Therefore $k = 2\ell$ is even, as desired.

6. Write $a^2 = n(2ab^2 - b^3 + 1)$. This allows us to make a quadratic in a: $a^2 - 2nb^2 \cdot a + n(b^3 - 1) = 0$. Its discriminant $D^2 = 4n^2b^4 - 4n(b^3 - 1)$ must be a perfect square. Note that $D^2 - (2nb^2 - b)^2 = 4n - b^2$.

   - If $D = 2nb^2 - b$, we get $4n = b^2$, so $b = 2k$ is even and $n = k^2$, which leads us to $(a, b) = (k, 2k)$ or $(a, b) = (8k^4 - k, 2k)$.
   - If $D > 2nb^2 - b$, then $4n - b^2 \geqslant (2nb^2 - b + 1)^2 - (2nb^2 - b)^2 = 4nb^2 - 2b + 1$, which reduces to $4n(b^2 - 1) = -(b - 1)^2$ and this is only possible if $b = 1$. Then a must be even, so $(a, b) = (2k, 1)$.
   - If $D < 2nb^2 - b$, then $b^2 - 4n \geqslant (2nb^2 - b)^2 - (2nb^2 - b - 1)^2 = 4nb^2 - 2b - 1$, which reduces to $(b + 1)^2 \geqslant 4n(b^2 + 1)$, which is impossible.

7. Denote $s = a + b + c$. Modulo s we have $x^2 \equiv 1$, $y^2 \equiv 2$ and $(x + y)^2 \equiv z^2 \equiv 4$. It follows that $2xy = (x + y)^2 - x^2 - y^2 \equiv 1 \pmod{s}$, so $4 \cdot 1 \cdot 2 \equiv 4x^2y^2 = (2xy)^2 \equiv 1 \pmod{s}$. Therefore $s \mid 7$, and for $s = 7$ we have a unique solution $(x, y, z) = (1, 4, 2)$.

8. Suppose that e.g. $v_p(x) = -k < 0$. Since $v_p(x + y + z) \geqslant 0$, one of $v_p(y), v_p(z)$ is $-k$, so the other one (since $xyz = 1$) is $2k$. Thus two of $v_p(x), v_p(y), v_p(z)$ are negative and one is positive. But then among $v_p(\frac{1}{x}), v_p(\frac{1}{y}), v_p(\frac{1}{z})$ only one is negative, so $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ cannot be an integer.

9. Use induction on n (base $n=1$). For the inductive step, when $n-1$ increases to n, only the summands corresponding to $i \mid n$ change, as then $[\frac{n}{i}] = [\frac{n-1}{i}]+1$. Verify that both sides of the equality get exactly the same increment.

10. Suppose that $b^2+a-1=p^k$ for some prime p. Then $p^k \mid (a^2+b+1)-(b^2+a+1) = (a-b)(a+b-1)$. However, both factors are less than p^k, so both must be divisible by p. Thus $p \mid a-b$, $p \mid a+b-1 \equiv 2a-1$ and $p \mid b^2+a-1 \equiv a^2+a-1 \pmod p$, and hence $p \mid 4(a^2+a-1)-(2a-1)(2a+3) = -1$, a contradiction.

11. Can n^2 with $n > 10^{50}$ always be written in that form? If $n^2 = a^2+p$, then $p = (n+a)(n-a)$, so $a = n-1$ and $p = 2n-1$, so it is possible only when $2n-1$ is a prime, which is not always the case.

12. We will set c so that $c+a = k^2$ and $c+b = (k+1)^2$ are two consecutive squares. Then $b-a = 2k+1$, so the corresponding c will be $c = k^2-a = \frac{a^2+b^2+1-2ab-2a-2b}{4}$. Luckily, then $c+ab = (\frac{a+b-1}{2})^2$.

13. If we include digit 5, then we must exclude all even digits. So if we want more than five digits, we cannot use digit 5. Due to divisibility by 3, one more digit shall be excluded. Let us try to keep 9: then we must exclude digit 4. The available digits are $1,2,3,6,7,8,9$.

    If we start with 987, the divisibility by 8 induces the three-digit ending $136, 216, 312, 632$, but none of the obtained numbers is divisible by 7. However, we can start with 986 and then the number 9867312 works, so this is the answer.

14. Take $m = a^2$. There is $x \in \mathbb{Z}$ such that $a^2 \mid x^n - a$. So if p is any prime divisor of a and $v_p(a) = k$, then $v_p(x^n-a) \geqslant 2k$, which implies $v_p(x^n) = k$. Thus $n \mid k$, and since this holds for all p, number a must be an n-th power.

15. We will prove that if $k = p^{p-1}$, where $p \geqslant 5$ is a prime, the given equation has no solutions. Suppose that $x = p^{p-1}\tau(x)$ for some $x \in \mathbb{N}$. Clearly, $r = v_p(x) \geqslant p-1$. Moreover, $p^{p-1} = \frac{x}{\tau(x)} = \frac{p^r}{\tau(p^r)} \cdot \frac{x/p^r}{\tau(x/p^r)}$, which implies $\frac{x/p^r}{\tau(x/p^r)} = \frac{r+1}{p^{r+1-p}}$. If $r = p-1$, then $\frac{x/p^r}{\tau(x/p^r)} = p$, which is impossible because $p \nmid x/p^r$. On the other hand, if $r \geqslant p+1$, then $\frac{x/p^r}{\tau(x/p^r)} \leqslant \frac{p+2}{p^2} < 1$, which is also impossible. Finally, we have a contradiction for $r = p$ as well: $\frac{x/p^r}{\tau(x/p^r)} = \frac{p+1}{p}$ - indeed, note that $\tau(y) \leqslant \frac{y}{2}+1 < \frac{p}{p+1}y$ for $y \geqslant p+1$.

16. Let $a+1 = dx$ and $b+1 = dy$, where d, x, y are positive integers with $x > y$ and $\gcd(x,y) = 1$. Then $\operatorname{lcm}(a+1, b+1) = dxy = (dx-1)^2-(dy-1)^2 = d(x-y)(dx+dy-2)$ and hence $dx+dy-2 = \frac{xy}{x-y}$. Since $x-y$ is coprime to both x and y, we must have $x-y = 1$. The previous equality becomes $d(2x-1)-2 = x(x-1)$, so $2x-1 \mid x^2-x+2 \mid (2x-1)^2+7$ and consequently $2x-1 \mid 7$, i.e. $x = 1$ or $x = 4$. The first option fails and the second one yields $d = 2$ and $(a,b) = (7,5)$.

17. We have $3x^4+10x^2+3 = (3x^2+1)(x^2+3)$. The GCD of $3x^2+1$ and x^2+3 divides $3(x^2+3)-(3x^2+1) = 8$, so it is 1, 2, 4 or 8. Therefore $3x^2+1$ and x^2+3 are either squares, or squares multiplied by 2. However, $x^2+3 = 2a^2$ is impossible modulo 3, so both factors must be squares. Then $x^2+3 = a^2$, which is only possible for $x = 1$. This is a solution indeed.

18. Let $b = 2^n$ and $a^2 - ab + b^2 = c^2$. We can rewrite this as $3 \cdot 2^{2n-2} = \frac{3}{4}b^2 = c^2 - (a - \frac{b}{2})^2 = (c + a - 2^{n-1})(c - a + 2^{n-1})$.

    We easily check the cases $n \leqslant 2$ and find no solutions. Assume that $n \geqslant 3$. Then the factors $c + a - 2^{n-1}$ and $c - a + 2^{n-1}$ are even and not both multiples of 4, so one of them equals ± 2 or ± 6. Assuming c is positive, both factors are positive as well. Checking all four possibilities we find only two possibilities for $n \geqslant 3$: $a = 2^{2n-4} + 2^{n-1} - 3$ or $a = 3 \cdot 2^{2n-4} + 2^{n-1} - 1$, and in addition, $a = 3$ for $n = 3$.

19. Let $p + 1 = 2a^2$ and $p^2 + 1 = 2b^2$. Then $p \mid p^2 - p = 2(b^2 - a^2) = 2(b - a)(b + a)$, but $0 < a, b < p$, so we must have $a + b = p$; hence $b - a = \frac{p-1}{2}$, so $a = \frac{p+1}{4}$. Solving the equation in p yields $p = 7$.

20. Since $\frac{n^3+1}{mn-1} \equiv \frac{1}{-1} = -1 \pmod{n}$, we have $\frac{n^3+1}{mn-1} \geqslant n - 1$. On the other hand, $mn - 1 \mid n^3 + 1 \Leftrightarrow mn - 1 \mid (m^3n^3 - 1) + n^3 + 1 = n^3(m^3 + 1) \Leftrightarrow mn - 1 \mid m^3 + 1$, we can assume w.l.o.g. that $m \geqslant n$.

    If $n \leqslant 2$, the only pairs (m, n) are $(2, 1)$, $(3, 1)$, $(2, 2)$ and $(5, 2)$.

    Let $n > 2$. Then $mn - 1 \geqslant n^2 - 1 > n^2 - n + 1$ and consequently $\frac{n^3+1}{mn-1} < n + 1 \leqslant 2n - 1$, so in this case we must have $\frac{n^3+1}{mn-1} = n - 1$. Hence $n - 1 \mid n^3 + 1 \equiv 2 \pmod{n-1}$, i.e. $n - 1 \mid 2$, so $n = 3$. This leads to $(m, n) = (5, 3)$.

21. Hint: Can a number whose sum of digits is 9 be divisible by 11?

22. Let $n = 2^k m$ with m odd. The numerator in $\frac{a}{n}$ in lowest terms is even if and only if $2^{k+1} \mid a$, and there are $[\frac{n}{2^{k+1}}] = \frac{m-1}{2}$ such values of a. In the remaining $n - \frac{m-1}{2}$ fractions the numerator is odd, so $f(n) \equiv n - \frac{m-1}{2} \pmod{2}$. Then $f(999n) \equiv 999n - \frac{999m-1}{2} \pmod{2}$, so $f(999n) - f(n) \equiv 499m \equiv 1 \pmod{2}$. Hence $f(n)$ and $F(999n)$ have opposite parities for every n.

23. We will prove that for every positive integer k there exists $n \in \mathbb{N}$ such that $n = k \cdot f(n)$. Fix k and consider $g(n) = n - kf(n)$. Increase n one by one. In each step, $f(n)$ increases by 0 or 1, so $g(n)$ either increases by 1 or decreases by $k - 1$. However, $g(1) < 0$ and $g(n)$ grows to infinity as $n \to \infty$ (note that $f(n) < 5\sqrt{n}$), so it cannot skip zero.

24. $2^{58} + 1 = 4a^4 + 1 = (2a^2 - 2a + 1)(2a^2 + 2a + 1)$, where $a = 2^{14}$. The two factors are coprime and greater than 5. Also, $2a^2 + 2a + 1$ is divisible by 5 but not by 25. Thus $2a^2 + 2a + 1$ gives two distinct prime factors and $2a^2 - 2a + 1$ gives a third one.

25. Take n to be divisible by $100!$. Then each of the numbers $n - 100, n - 99, \dots, n + 100$ has a prime divisor not exceeding 100, except for $n \pm 1$. We cover these two by setting n so that $101 \mid n + 1$ and $103 \mid n - 1$ (and $100! \mid n$). Such an n exists by the Chinese Remainder Theorem.

    By the way, $100! + 1$ is divisible by 101.

26. The given product of GCD's divides both abc and $(b - 1)(c - 1)(a - 1)$, so it divides the difference, which is $a(b - 1) + b(c - 1) + c(a - 1) + 1$.

    We find an equality case by setting $(a, b, c) = (n, n = 1, n + 2)$, with $3 \mid n - 1$.

27. Case $a = b$ is trivial. Assume w.l.o.g. that $a > b$ and let $[a,b] = ka$, $[a+1,b+1] = \ell(a+1)$ and $[a+2,b+2] = m(a+2)$. Reducing the equality $ka + m(a+2) = 2\ell(a+1)$ modulo $a+1$ we get $a+1 \mid m-k$, but $k, m \leqslant b+2 \leqslant a+1$, so we must have $k = m$. Then $\ell = k = m$, and since $k \mid b$ and $\ell \mid b+1$, we obtain $k = 1$, that is, $b \mid a$.

28. If n is even, then $3^n - 2^n$ is divisible by 5, but $4^n + 2^n + 1$ is not. Now assume n is odd. The number $9^n + 3^n + 1 = 4^n + 2^n + 1 + (3^n - 2^n) + (9^n - 4^n)$ is also divisible by $3^n - 2^n$. However, $9^n + 3^n + 1 = (3^n - 1)^2 + (3^{\frac{n+1}{2}})^2$ is a sum of two squares, and $3^n - 2^n \equiv 3 \pmod 4$ for $n > 1$, so this is impossible. This leaves $n = 1$ as the only possibility.

29. Note that $(x-1)x = \frac{(x^2-1)x^2}{x(x+1)}$.

30. Setting $b = c = 1$ we get $n = 2a + 1$. Moreover, setting $b = c = 2^k$ and $a = 2^k t$, we get $n = 2^k(2t+1)$, which covers all positive integers except powers of two.

    Now suppose that $2^k = [a,b] + [b,c] + [c,a]$, where k is smallest possible. Clearly, $k \geqslant 2$. At least two of a, b, c must be even (otherwise $[a,b] + [b,c] + [c,a]$ will be odd), say $a = 2a'$ and $b = 2b'$, whereas c is odd due to minimality of k. But then $2^k = 2[a',b'] + 2[b',c] + 2[a',c]$, yielding a solution for 2^{k-1}, a contradiction.

31. Since $a_1 = 1$, we cannot have $a_2 = 2$, so n is even. Next, $3 \nmid a_{\frac{k}{2}} + a_{\frac{k}{2}+1} = n$. Also, $3 \nmid a_{k-1} + a_k = (n-3) + (n-1) = 2n - 4$, so we must have $n \equiv 1 \pmod 3$. But $(a_k + a_{k+1}) + (a_{n-k} + a_{n-k+1}) = 2n \equiv 2 \pmod 3$, so $a_k + a_{k+1} \equiv a_{n-k} + a_{n-k+1} \equiv 1 \pmod 3$ for all k. Now finish.

32. Try the numbers $2(k-1)p + (k^2 \pmod p)$ for $k = 1, \ldots, p$.

33. Find infinitely many solutions of the form $(x, 1+t, 1-t)$.

34. One solution is $p = 2$, but there is also an unexpected solution.

35. We have $(a-1)(b-1) = (c+1)(d+1) = n$ for some n, so $a-1$ and $c+1$ are divisors of n whose product is greater than n; hence $\gcd(a-1, c+1) > 1$, so $a + c = (a-1) + (c+1)$ cannot be prime.

36. (a) The numbers $4k+1$ and $4k+2$ have the same number of binary unit digits, so they cannot both be lively. However, in every five consecutive numbers one can find $4k+1$ and $4k+2$.

    For (b), set n so that $b_n = 6$, $b_{n+1} = 7$, $b_{n+2} = 4$ and $b_{n+3} = 5$. We can take e.g. $n = 2^a + 2^b + 2^c + 14$ with $a > b > c > 4$. Then see how to make $n, n+1, n+2, n+3$ all lively.