June camp 2019 – Geometry – Level IMO

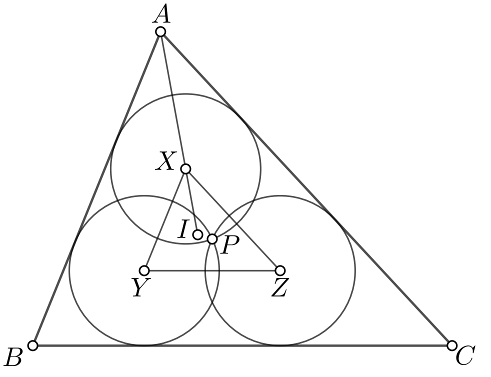
Dominik Burek

Contents

1. Homothety

Problem 1. Given is an arbitrary triangle ABC. Three circles with the same radius r have a common point P and each of them is tangent to the two sides of the triangle ABC. Prove that the incenter of triangle ABC, point P and the circumcenter of triangle ABC lie on one straight line.

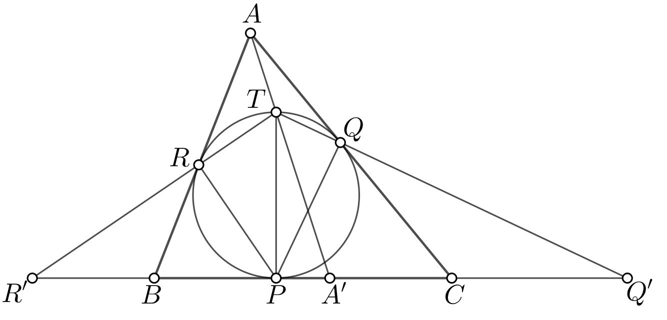
Proof. Denote centres of these circles by X, Y, Z. From the given equal radii condition we may observe that triangles XYZ and ABC are homothetic, since their respective sides are parallel. Of course the centre of this homothety is I - incenter of triangle ABC.



Moreover $XP = YP = ZP = r$, so P is the circumcenter of XYZ. Eventually, from the definition of homothety, we conclude that the centre of homothety that maps circumcircles of triangle ABC and XYZ - I - lies on a line joining these circumcenters i.e. P and O, which finishes the proof. $\square$

Problem 2. Let ω be an incircle of triangle ABC which is tangent to BC, CA, AB at P, Q, R, respectively. The segment PT is a diameter of ω. Lines AT, QT, RT intersect BC at A', Q', R', respectively. Prove that $A'Q' = Q'R'$.

Proof. From the well known lemma $BP = A'C$ (One just needs to consider homothety centred at A which transform ω to A-excircle ω_A of triangle ABC. Then A' is an image of T under this homothety and so A' is touching point of ω_A with BC.) Moreover $\sphericalangle R'RP = 90°$, so $R'B = BP = BR$. Using analogues equality of segments on line BC we easily get statement.



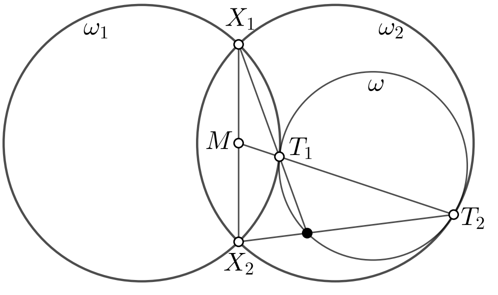
□

Problem 3. Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that the lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

Proof. Consider the composition of homotheties

$$\omega_1 \xrightarrow{T_1} \omega \xrightarrow{T_2} \omega_2.$$

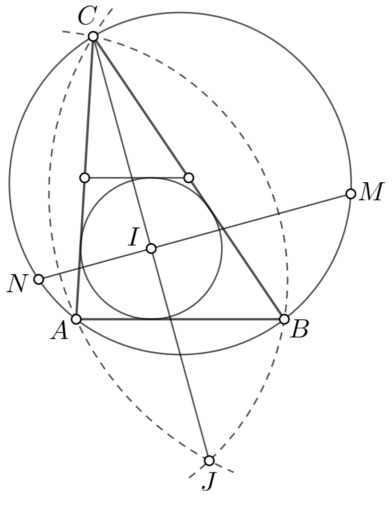
This composition is a negative homothety, and since ω_1 and ω_2 have equal radius, it follows that consider composition is just a reflection about the midpoint of segment X_1X_2. In particular, it sends X_1 to X_2. Thus we may conclude that the point $X_1T_1 \cap X_2T_2$ is the image of X_1 under the first homothety ($\omega_1 \xrightarrow{T_1} \omega$).



$\square$

Problem 4. Let I be the incenter of triangle ABC, and M, N be the midpoints of arcs ABC and BAC of its circumcircle. Prove that points M, I, N are collinear if and only if $AC + BC = 3AB$.

Proof. Suppose that points M, I, N are collinear. Let J be the centre of the excircle touching side AB. Then, by simple angle chasing, M and N are the centers of circumcircles of triangles ACJ and BCJ. Therefore MN is the perpendicular bisector of the segment CJ, i.e. I is the midpoint of CJ.

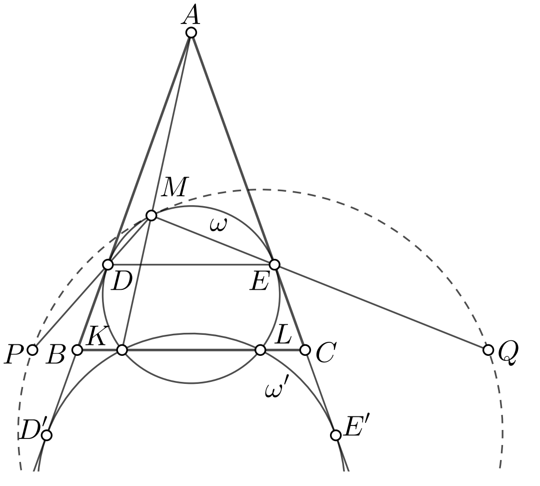


Applying the homothety centred at C with ratio $\frac{1}{2}$ we obtain that the incircle touches the medial line parallel to AB. The trapezoid formed by this medial line and the sidelines of ABC is circumscribed if and only if the sought equality is correct, which finishes the proof. $\square$

Problem 5. The circle ω is tangent to the equal sides AB and AC of the isosceles triangle ABC and intersects the side BC at the points K and L. The segment AK intersects ω in a second time at point M. Points P and Q are symmetric to point K with respect to points B and C, respectively. Prove that the circumcircle of triangle PMQ tangent to the circle ω.

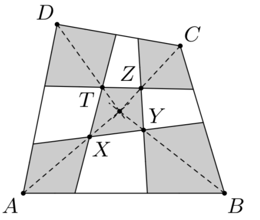
Proof. Denote by D and E tangent points of ω with sides AB and AC. From symmetry wrt bisector of the angle BAC, it follows that $DE \parallel BC$. Consider a homothety with centre A and scale $\dfrac{AK}{AM}$, it maps ω to circle ω'.

The circle ω' passes through the point K, and hence through L (from symmetry wrt the bisector of the angle BAC). Moreover ω' touches the rays AB and AC at some points D' and E'.

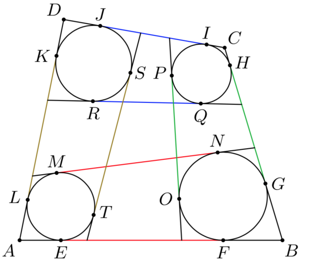


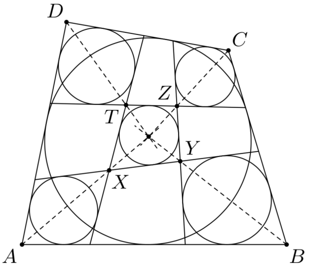
It follows from the properties of the homothety that $MD \parallel KD'$. Further, according to the power of a point we see that $BD^2 = BK \cdot BL = BD'^2$, whence $BD = BD'$. It follows that the points M, D, P lie on one line. Similarly, the points M, E and Q lie on the same line. The triangles MDE and MPQ are homothetic with centre M, therefore, their circumscribed circles are also homothetic, i.e. tangent at point M. $\square$

Problem 6. A convex quadrilateral $ABCD$ is cut into 9 convex quadrilaterals, as shown on the figure. Prove that if there exist incircles of the shaded quadrilaterals, then AX, BY, CZ and DT intersect at one point.



Proof. Denote the points of tangency as shown on figure. Observe that there exists an incircle of the central quadrilateral if and only if $MN+RQ = ST+PO$. Using this equality one can show that $AB+CD = BC+DA$. Therefore $ABCD$ is circumscribed.





Now apply the Monge theorem to the incircle of $XYZT$, incircle of $ABCD$ and any corner circle (4 times). We get that lines AX, BY, CZ and DT intersect at insimilarity centre of small central circle and incircle of $ABCD$. $\square$

Problem 7. On the sides AB and AC of an arbitrary triangle ABC, points P and Q are chosen, respectively, so that $PQ \parallel BC$. Lines BQ and CP intersect at point O. Denote by A' point symmetric to A with respect to line BC. The line $A'O$ intersects the circumcircle ω of triangle APQ at point S. Prove that the circumcircle of BSC is tangent to ω.

Proof. The case $AB = AC$ follows from symmetry; without loss of generality, we assume that $AC > AB$. We choose on point X so that $PAXQ$ is an isosceles trapezoid. Then

$$\sphericalangle XQP = \sphericalangle APQ = \sphericalangle ABC \sphericalangle CBA'$$

and, similarly, $\sphericalangle XPQ = \sphericalangle BCA'$. So $XQ \parallel BA'$ and $XP \parallel CA'$. Therefore, a homothety with center O which transforms the segment PQ to CB transforms the triangle XPQ into ACB; consequently, the point O (and therefore also the point S) lies on AX.

Let M be the center of the circumcircle of triangle ASA'. Then

$$\sphericalangle MAA' = 90^\circ - \sphericalangle ASX.$$

Since $XA \parallel BC \perp AA'$, we get

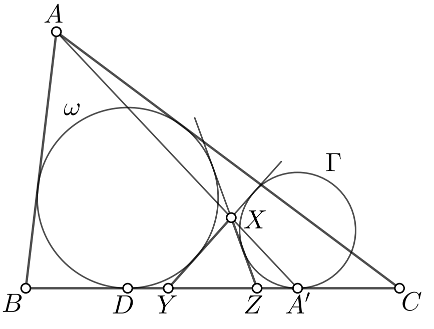
$$\sphericalangle MAX = \sphericalangle MAA' + 90^\circ = 180^\circ - \sphericalangle ASX,$$

i.e. MA is tangent at point A to ω. Since $MA = MS$, MS is tangent to ω.

Let Ω be the circumcircle of triangle ABC; then ω and Ω are also homothetic with center A, since $PQ \parallel BC$. So MA is also tangent to Ω. In addition, M lies on the perpendicular bisector BC of the segment AA'; therefore $MA^2 = MB \cdot MC$. So, $MS^2 = MA^2 = MB \cdot MC$, i.e. MS touches the circumcircle of the triangle BSC and at point S. $\square$

Problem 8. Given is a triangle ABC, in which $AB < AC$ and ω is its incirle. The A-excircle is tangent to BC at A'. Point X lies on AA' such that segment $A'X$ does not intersect with ω. The tangents from X to ω intersect with BC at Y, Z. Prove that the sum $XY + XZ$ does not depends of the position of point X.

Proof. Denote by Ω the A-excircle of triangle ABC. Let Γ the Y-excircle of triangle XYZ and D be the tangency point of ω to segment BC. Suppose that points D, Y, Z lie on BC in this order.



The insimilicenter of ω and Γ is X; the insimilicenter of Ω and Γ lie on BC (because BC is their common tangent); the exsimilicenter of Ω and ω is A. Therefore by Monge's theorem we may conclude that the insimilicenter of Ω and Γ is A'. It means that Γ is tangent to BC at A'. Hence $ZA' = YD$ (recall that points A' and D are symmetric with respect to the midpoint of segment BC). It easily implies that $XZ + XY = A'D$. Length of segment $A'D$ is fixed, since points A' and D are fixed. $\square$

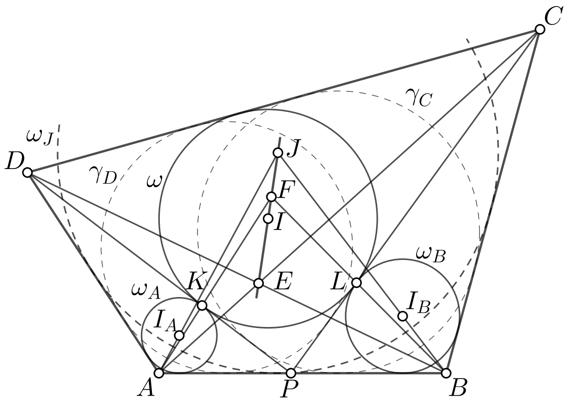
Problem 9. Point P lies on the side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I and F are collinear.

Proof. Denote by ω_A and ω_B the incircles of triangles APD and BPC, respectively (existence of these circles is a well known fact which can be proved by tangentail segment calculations) and let I_A and I_B be their centres. Suppose that lines AI_A and BI_B meet at J. Then let ω_J be the circle centred at J tangent to lines AB, AD, and BC (such circle exists since lines AI_A and BI_B are angle bisectors). We will prove that points E, F, I, J are collinear.

First, notice that

$$AP - AD = KP - KD = CP - CD.$$

Thus we may conclude that $APCD$ has an incircle γ_D. Define γ_C similarly.



By Monge's theorem applied to circles $\gamma_C, \omega, \omega_J$, the exsimilicenter of ω and ω_J lies on line AC; similarly it lies on line BD. Hence it is E.

On the other hand, Monge's theorem applied to $\omega_A, \omega, \omega_J$ yields that the insimilicenter of ω and ω_J belongs to AK; similarly it belongs to BL, thus this point is F. Hence points E, F, I, J are collinear and we are done. $\square$

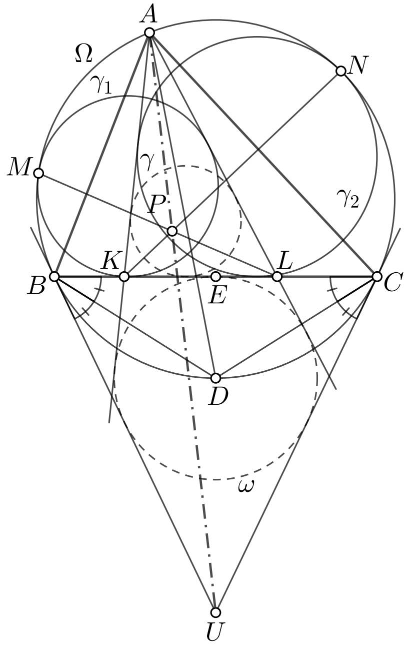
Problem 10. Given is an acute-angled triangle ABC inscribed in the circle Ω. Point D is the midpoint of minor arc BC of circle Ω. Circle ω centred D is tangent to segment BC at E. Tangents to ω passing through point A intersect line BC at K, L and points B, K, L, C lie in this order on line BC. Circle γ_1 is tangent to segments AL and BL and to Ω at point M. Similarly circle γ_2 is tangent to segments AK and CK and to Ω at point N. Lines KN and LM intersect at P. Prove that $\sphericalangle KAP = \sphericalangle EAL$.

Proof. Denote by U the intersection point of tangents to Ω at points B and C. First notice that

$$\sphericalangle UBD = \sphericalangle BCD = \sphericalangle DBC = \sphericalangle DCU.$$

It means that lines BD and CD are angle bisectors of UBC and BCU, respectively, which means that circle ω is inscribed in triangle UBC.

According to symmedian's theorem line AU is A-symmedian in triangle ABC. Since AE is A-median in triangle ABC, it implies that AU and AE are symmetric with respect to the angle bisector AD of angle BAC. Moreover lines AK and AL are symmetric with respect to AD too, because point D is center of circle ω inscribed in angle KAL. Therefore in order to prove that angles KAP and EAL are equal, it is sufficient to show that points A, P and U are collinear.



Let γ be the incircle of triangle AKL. Denote by P' centre of homothety with ratio $k > 0$, which maps Ω to γ. We will prove that $P = P'$. By Monge's theorem applied to circles Ω, γ_1 and γ, we may obtain that points M, P' and L are collinear.

Analogously by Monge's theorem applied to circles Ω, γ_2 and γ, we may obtain that points N, P' and L are collinear. Thus indeed $P = P'$.

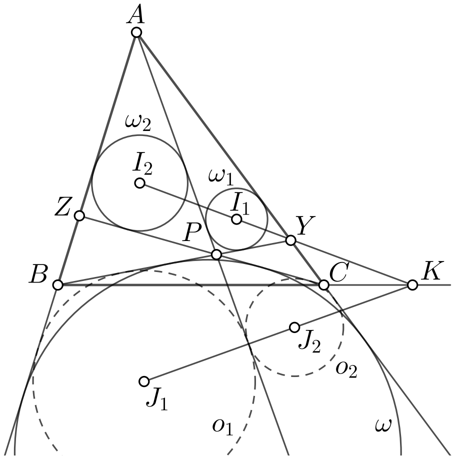
Once again applying Monge's theorem, this time to circles Ω, ω and γ, we are finally able to conclude that points A, P, U are collinear (A is the center of homothety which sends γ to ω; P is the center of homothety which sends Ω to γ; U is the center of homothety which sends ω to Ω). Hence we are done. $\square$

Problem 11. Let ABC is an acute triangle and let P be a point such that

$$AB + BP = AC + CP.$$

Suppose that BP intersects AC at Y and CP intersects AB at Z. Denote by ω_1 and ω_2 incircles of APY and APZ. Prove that the external common tangents of ω_1, ω_2 and BC are concurrent.

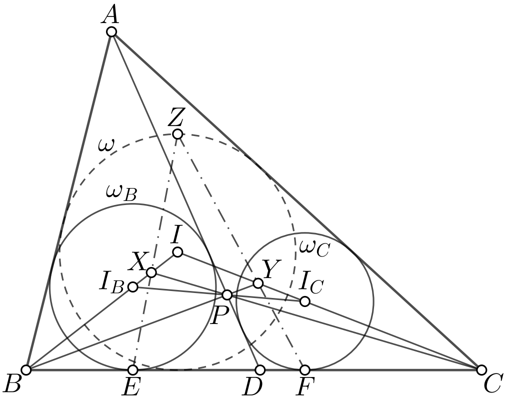
Proof. Denote by I_1 and I_2 the incentres of circles ω_1 and ω_2. Then $J_1 \equiv PI_1 \cap AI_2$ and $J_2 \equiv PI_2 \cap AI_1$ are then the A-excenters of triangles APB and APC, respectively. Denote these A-excircles of triangles APB and APC with centers J_1 and J_2 o_1 and o_2, respectively.



Since $AB+BP = AC+CP$, we may conclude that there exists a circle ω tangent to AB and AC and the rays PB and PC. Hence B is the exsimilicenter of ω and o_1 and similarly C is the exsimilicenter of ω and o_2. Thus by Monge's theorem, $K \equiv BC \cap J_1J_2$ is the exsimilicenter of o_1 and o_2. Because circles o_1 and o_2 are tangent to line AP at the same point (simple segment-chasing), we obtain that $A(J_1, J_2; P, K) = -1$ (well known lemma!). Therefore, from the fact that $I_1J_1J_2I_2$ is a complete quadrangle, it follows that $K \in I_1I_2$, thus K is also exsimilicenter of ω_1 and ω_2, i.e. external common tangents of ω_1 and ω_2 meet on BC. $\square$

Problem 12. Let ABC be a triangle with incenter I. Let D be a point on side BC and let ω_B and ω_C be the incircles of triangles ABD and ACD, respectively. Suppose that ω_B and ω_C are tangent to segment BC at points E and F, respectively. Let P be the intersection of segment AD with the line joining the centrers of ω_B and ω_C. Let X be the intersection point of lines BI and CP and let Y be the intersection point of lines CI and BP. Prove that lines EX and FY meet on the incircle of triangle ABC.

Proof. Let Z be the diametrically opposite point on the incircle. We claim this is the desired intersection.

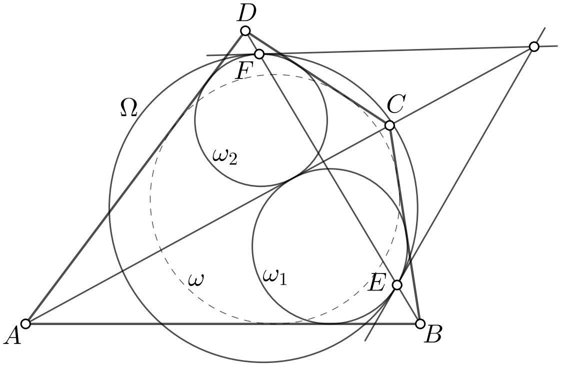


Note that P is the insimilicenter of ω_B and ω_C (because it lies on their common tangent and line connecting centers of these circles) and C is the exsimilicenter of ω and ω_C. Thus by Monge's theorem, the insimilicenter of ω_B and ω lies on line CP.

This insimilicenter should also lie on the line joining the centers of ω and ω_B, which is BI, hence it coincides with the point X. So X lies on EZ (since we deal with *insimilicenter*) as desired. Analogously Y lies on ZF. $\square$

Problem 13. Let $ABCD$ be a circumscribed quadrilateral. The circles ω_1 and ω_2 are inscribed in triangles ABC and ADC, respectively. Let BD intersects the circle ω_1 at points E and P, and the circle ω_2 at the points F and Q so that the points P and Q lie on the segment EF. Prove that the tangent to ω_1 at the point E, and the tangent to ω_2 at F, intersect on the line AC or are parallel.

Proof. It is enough to prove that there is a circle Ω, which touches the circles ω_1 and ω_2 internally at the points E and F, respectively. Indeed, then the tangent to ω_1 at the point E, the tangent to ω_2 at the point F, and the line AC are pairwise common tangents to the circles ω_1, ω_2 and Ω, and, therefore by radical axis theorem, they intersect at one point or are parallel (recall that circles ω_1 and ω_2 are tangent to AC at the same point; it means that they are tangent to each other at common point X.).

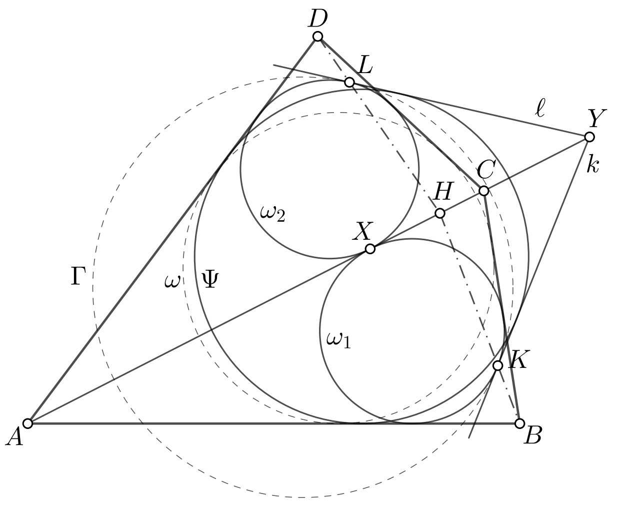


Consider the circle Ω tangent to ω_1 at E and tangent to ω_2 at F'. By the Monge theorem applied for ω_1, ω_2 and Ω we see that line EF' contains exsimilarity centre of ω_1 and ω_2. Again, using the Monge theorem for circles ω_1, ω_2 and ω we see that exsimilarity centre of ω_1 and ω_2 lies on BD, therefore $F = F'$. $\square$

Problem 14. Let $ABCD$ be a circumscribed quadrilateral. The circles ω_1 and ω_2 are inscribed in triangles ABC and ADC, respectively. The circle Γ touches the circles ω_1 and ω_2 internally at the points K and L, respectively. Prove that the lines BK and DL intersect on the line AC.

Proof. As in the problem 9 we can see that ω_1 and ω_2 are tangent to each other at the common point X. Let H be the centre of homothety h with a positive coefficient that maps ω to Γ. Homothety h is a composition of homothety with centre B transforming ω to ω_1 and the homothety with centre K taking ω_1 to Γ, so by Monge's theorem H lies on the line BK. Similarly, H lies on the line DL. Thus, it is enough to prove that H lies on the line AC.

Let k be the tangent to ω_1 drawn at the point K, and let ℓ be a tangent to ω_2, drawn at a point L. Then k, ℓ and AC are pairwise common tangents to circles ω_1, ω_2 and Γ, which means that k and l intersect at some point Y lying on the line AC or $\ell \parallel l \parallel AC$ (we are using radical axis theorem).



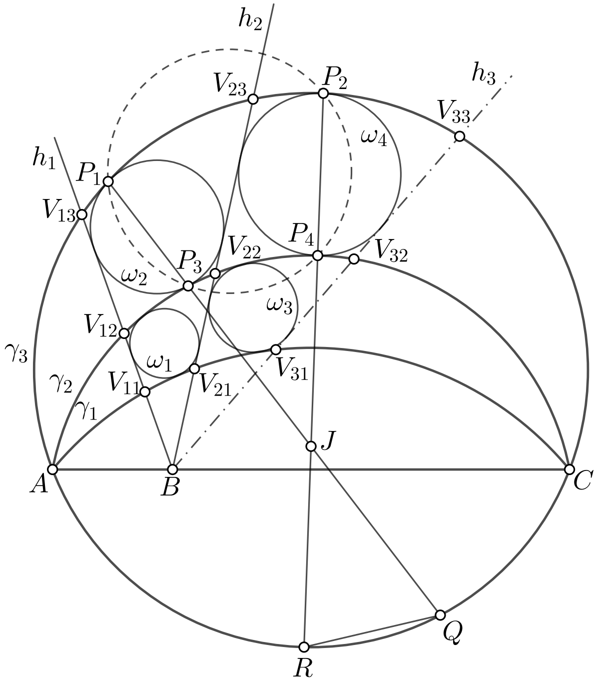
Let us consider the first case. According to a well-known lemma, a circle Ψ can be inscribed into a convex quadrilateral formed by straight lines AB, AD, k and ℓ. We obtain that h is a composition of a homothety with center A, transforming ω into Ψ, and the homothety with center Y, translating Ψ into Γ, which means (by Monge's theorem) that H lies on the straight line AY (which coincides with AC).

It remains to consider the particular case, when $k \parallel l \parallel AC$. In this case, the homothety with centre B, which takes ω_1 to the excircle γ_1 of triangle ABC, translates K to the tangency point Z of circle γ_1 and AC. We obtain that the straight line BK passes through the point Z, which is symmetrical to the point X with respect to the midpoint of AC. Similarly, we show that the straight line DL passes through the same point Z.

□

Problem 15. Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j. Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

Proof. Denote the circle tangent to γ_1, γ_2 and h_1, h_2 by ω_1; the circle tangent to γ_2, γ_3 and h_1, h_2 by ω. The two other circles tangent to h_2, γ_1, γ_2 and h_2, γ_2, γ_3 will be called ω_3 and ω_4, respectively. Moreover denote the points of tangency of ω_2 and ω_4 with γ_3 by P_1, P_2, respectively. Call the tangency points of ω_2 and ω_4 with γ_2 as P_3, P_4, respectively. Let P_1P_3 and P_2P_4 intersect γ_3 again at Q and R, respectively. It is enough to prove that h_3 is tangent to ω_3 and ω_4.



First we will prove that quadrilateral $P_1P_2P_4P_3$ is cyclic. According to Monge's theorem, applied twice - once to the circles γ_2, γ_3, ω_2 and once to the circles γ_2, γ_3 and ω_4, we may obtain that P_1P_3 and P_2P_4 both pass through the insimilicenter J of

γ_2 and γ_3, so they both intersect at the insimilicenter of those two circles. Therefore QR is parallel to P_3P_4. But QR is anti-parallel to P_1P_2, so P_3P_4 is also anti-parallel to P_1P_2. Hence $P_1P_2P_4P_3$ is indeed cyclic.

Now we will show that the exsimilicenter of ω_2 and ω_4 lies on AC. We apply Monge's theorem once again - this time to circles ω_2, ω_4, γ_3 and ω_2, ω_4, γ_2, to find that P_1P_2 and P_3P_4 intersect at the exsimilicenter of ω_2 and ω_4. On the other hand we also know that the intersection of P_1P_2 and P_3P_4 must lie on AC since $P_1P_2P_4P_3$ is cyclic and AC is the radical axis of γ_2 and γ_3 (we used radical axis theorem).

Using the same reasoning as in the two previous paragraphs, but applying it to the circles ω_1 and ω_3, we may conclude that the exsimilicenter of ω_1 and ω_3 also lies on AC.

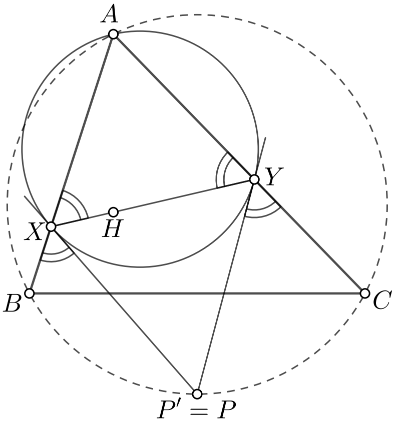
Now applying Monge's theorem to the circles ω_2, ω_1, and ω_4 and using fact that the exsimilicenter of ω_2 and ω_4 lies on AC, we obtain that the exsimilicenter of ω_4 and ω_1 lies on AC.

We can also apply Monge's theorem to ω_1, ω_3, and ω_4. We proved that the exsimilicenters of ω_1, ω_3 and ω_4, ω_1 both lie on AC, so the exsimilicenter of ω_3 and ω_4 lies on AC as well. Thus their exsimilicenter is B (since h_2 is tangent to circles ω_3 and ω_4). Hence the line h_3 is tangent to both circles. $\square$

2. Simson and Steiner lines

Problem 16. Points X and Y lie on segments AB and AC of acute-angled triangle ABC, such that $AX = AY$ and the orthocenter of triangle ABC lies on line XY. Tangents to circumcircle of triangle AXY at X and Y intersect at P. Show that points A, B, C, P are concyclic.

Proof. From the Steiner theorem we know that the reflections of line XY across sides AB and AC intersect at some point P', lying on circumcircle of triangle ABC. But since $AX = AY$, thus these reflections are also tangents to circumcircle of triangle AXY at X and$Y$. Hence $P = P'$ and therefore we are done.



□

Problem 17. Point O is the circumcenter of triangle ABC. Circle passing through points A, O intersects lines AB, AC at P, Q, respectively. Prove that the orthocenter of triangle OPQ lies on BC.

Proof. Let us consider situation shown on the picture. Other possibilities are consider analogously. Denote by E second intersection of circumcircle of triangle OPQ with circumcircle of triangle ABC. We will show that BC is Steiner line of point E with respect to triangle OPQ.

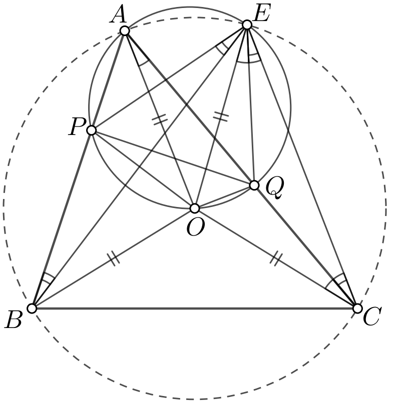
First notice that

$$\sphericalangle PBE = \sphericalangle ABE = \sphericalangle ACE = \sphericalangle QCE$$

and

$$\sphericalangle BPE = 180^\circ - \sphericalangle APE = 180^\circ - \sphericalangle AQE = \sphericalangle CQE.$$

It implies that $\sphericalangle PEB = \sphericalangle QEC$. Now we will prove that $\sphericalangle QEC = \sphericalangle QCE$. It will finish the proof, because then $QE = QC$ and since $OE = OC$, thus OQ will be the perpendicular bisector of segment EC. Analogously we will be able to show that OP is the perpendicular bisector of segment EB. It will yield to the conclusion that BC is actually Steiner line of point E with respect to triangle POQ.



It's easy to note that $\sphericalangle OCA = \sphericalangle OAC = \sphericalangle OEQ$. But $\sphericalangle OCE = \sphericalangle OEC$ and the conclusion follows. $\square$

Problem 18. Quadrilateral $ABCD$ is circumscribed about a circle. Line ℓ passing through point A intersects BC at M and ray DC at N. Points I_1, I_2 and I_3 are incenters of triangles ABM, MNC i NDA, respectively. Prove that the orthocenter of triangle $I_1I_2I_3$ lies on ℓ.

Proof. First notice that triples of points I_3, I_2, N and I_2, M, I_1 are collinear, since they lie on respective angle bisectors.

Consider second tangent passing through point C to the incircle of triangle ABM. Denote by E the intersection of this tangent with line AM. Then the incircle centred at I_1 of triangle ABM is also the incircle of quadrilateral $ABCE$. Therefore $AE + BC = CE + AB$.

Because quadrilateral $ABCD$ is circumscribed about the circle, we may write that $AB + CD = BC + DA$.

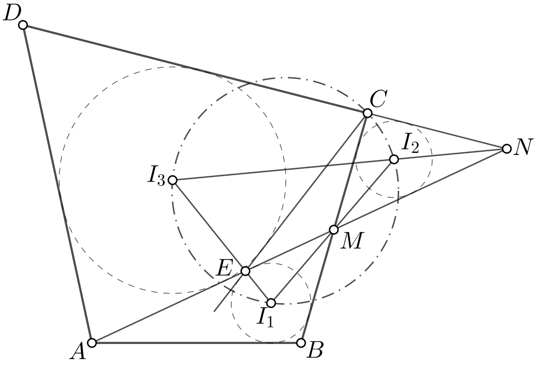
Thus

$$AE + CD = (CE + AB - BC) + (BC + DA - AB) = CE + DA.$$

It means that this tangent is also tangent to the incircle of triangle NDA centred at I_3. Thus both quadrilaterals $ABCE$ and $AECD$ are circumscribed about circles, which implies that points I_1, E, I_3 lie on one line. Now we are able to compute:

$$\sphericalangle I_3CI_1 = \frac{1}{2}\sphericalangle DCB = \frac{1}{2}(180^\circ - \sphericalangle MCN) = \sphericalangle NMI_2 + \sphericalangle MNI_2 = \sphericalangle I_1I_2I_3.$$

It yields that quadrilateral $I_1I_2I_3C$ is cyclic.



Note that the reflections of point C about lines I_1I_2, I_1I_3 lie on AN. Hence line AN is Steiner line of point C with respect to triangle $I_1I_2I_3$, which finishes the proof. $\square$

☠**Problem 19.** Given is a triangle ABC. Lines k_a, k_b, k_c passing through vertex A, B, C, respectively, are pairwise parallel. Let ℓ_a, ℓ_b, ℓ_c be the reflection of k_a, k_b, k_c across lines BC, CA and AB, respectively. Denote by XYZ the triangle formed by lines ℓ_a, ℓ_b, ℓ_c. Find the geometrical locus of the incenters of triangles XYZ.

Proof. We claim that the locus of such incenters is the circle centred at O with radius $2R$, where O denotes circumcenter of triangle ABC and R is the length of circumradius.

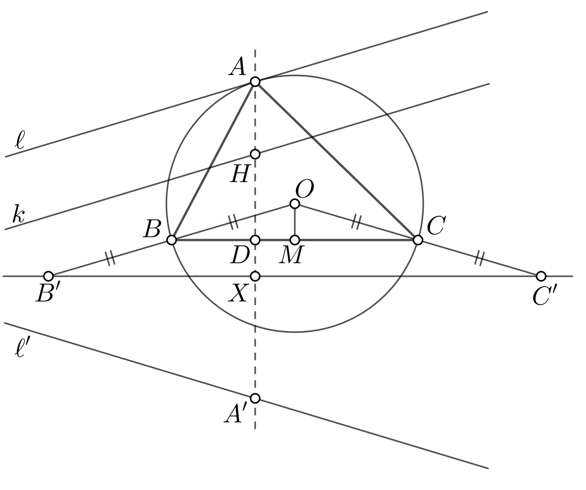
Let us first prove two following lemmas.

Lemma. *Given is a triangle ABC with orthocenter H and circumcenter O. Let ℓ be a line passing through A and ℓ' be the reflection of ℓ across BC. Denote by B', C' the reflection of O in B and C, respectively . Moreover let k be a line passing through H and parallel to ℓ. Then k and ℓ' are symmetric with respect to line $B'C'$.*

Proof. Let A' be the reflection of A across BC; M be the midpoint of segment BC and $D = AH \cap BC$, $X = AH \cap B'C'$. It is obvious that A' lies on ℓ'. It is well-known that $AH = 2OM$. Moreover $OM = DX$, because $OM \parallel DX$ and $BC \parallel B'C'$. Therefore

$$HA' = AA' - AH = 2(AD - OM) = 2HX.$$

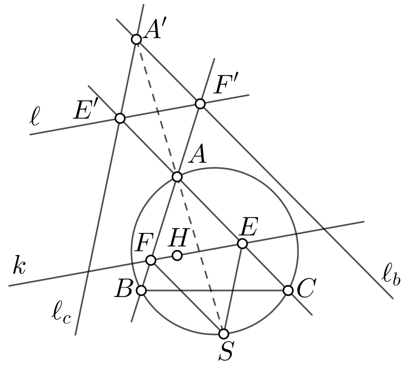
Hence we may obtain that H and A' are symmetric with respect to $B'C'$. In addition ℓ and k are parallel, so we may now conclude that k and ℓ' are symmetric with respect to $B'C'$, which finishes the proof of first lemma.



□

Lemma. *Given is a triangle ABC with orthocenter H. Let ℓ be an arbitrary line on the plane and k be the line parallel to ℓ, passing through H. Denote by ℓ_a, ℓ_b, ℓ_c the reflections of ℓ across BC, CA and AB, respectively. Triangle $A'B'C'$ is a triangle* *determined by lines ℓ_a, ℓ_b, ℓ_c. Then the anti-Steiner point S of k with respect to triangle ABC is the incenter of triangle $A'B'C'$.*

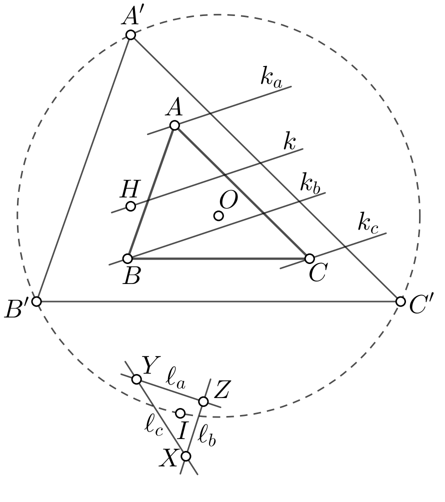
Proof. Denote by E and F the intersections of k with AC and AB, respectively. Let E' and F' be intersections of ℓ with AC and AB, respectively (notice that lines ℓ_c and ℓ_b pass through points E' and F').



According to definition of reflection, we may obtain that $EF \parallel E'F'$, $ES \parallel E'A'$, $FS \parallel F'A'$, which means that triangles $A'E'F'$ and SEF are homthetic. Therefore points A, A', S are collinear. Since AE', AF' are the external angle bisectors in triangle $A'E'F'$ (from reflection), A is the A'-excenter of triangle $A'E'F'$, i.e. $A'A$ bisect angle $E'A'F'$. By similar reasoning, we may prove that lines $B'S$, $C'S$ bisect angles $C'B'A'$ and $A'C'B'$, respectively. Thus indeed S is the incenter of triangle $A'B'C'$. □

Let's come back to the main problem.

Denote by O the circumcenter of triangle ABC. Let A', B', C' be the reflection of O in A, B, C, respectively; H be the orthocenter of triangle $A'B'C'$. Consider line k passing through H and parallel to lines k_a, k_b, k_c.



From 2 we obtain that lines ℓ_a, ℓ_b, ℓ_c are the reflections of k in $B'C'$, $C'A'$, $A'B'$, respectively. Combining this fact with 2 it yields that the incenter I of triangle XYZ is the anti-Steiner point of k with respect to triangle $A'B'C'$. Now we may conclude that the locus of I is the circumcircle of triangle $A'B'C'$, i.e. the locus is a circle concentric with circumcircle of triangle ABC and radius $2R$. $\square$

3. Miscellaneous

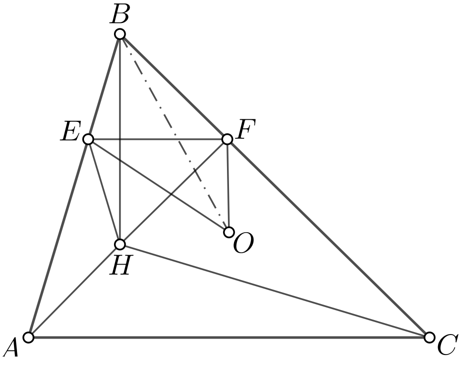
3.1. IMO 1&4.

Problem 20. Let O be the circumcenter of triangle ABC and H its orthocenter. The perpendicular bisector of BH intersects BA and BC at E and F, respectively. Prove that OB is bisector of angle AOF.

Proof. First notice that

$$\measuredangle EHF + \measuredangle AHC = \measuredangle ABC + \measuredangle AHC = 180^\circ.$$

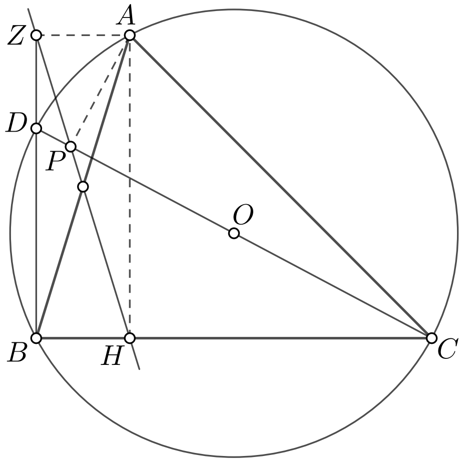
According to the theorem about existence of isogonal conjugates points in quadrilaterals, we may observe that H has an isogonal conjugate in quadrilateral $ACEF$. Since O and H are isogonal with respect to angles BAC and ACB, we conclude that O and H are isogonal conjugated in quadrilateral $ACEF$. According to this fact, the statement easily follows from trivial angle-chasing.



□

Problem 21. Let O be the circumcenter of triangle ABC. Denote by P and H orthogonal projections of point A onto CO and CB, respectively. Prove that PH passes through the midpoint of segment AB.

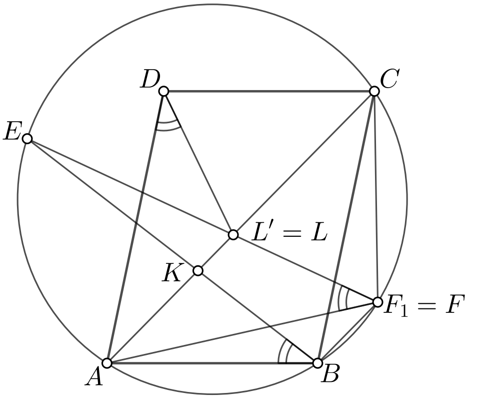
Proof. Notice that PH is the Simson line for points A with respect to triangle BCD, where CD is the diameter of the circumcircle of triangle ABC. Then line PH passes through the projection Z of point A onto the line BD. Thus, the segment PH is the diagonal of the rectangle $AHBZ$, which means that it halves its other diagonal AB.



□

Problem 22. Let $ABCD$ be a parallelogram with an obtuse angle B, in which $AD > AB$. On the diagonal AC, the points K and L are chosen, such that $\sphericalangle ABK = \sphericalangle ADL$ (the points A, K, L, C are different, and K lies between A and L). The line BK intersects the circumcircle ω of the triangle ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.

Proof. Consider point F_1 symmetric to point D with respect to line AC. Note that $F_1 \in \omega$ because $\sphericalangle AF_1C = \sphericalangle CDA = \sphericalangle ABC$.



Since the segments AB and CD are of equal lengths (by the parallelogram property), the segments CD and CF are of equal length too. Thus we may conclude that the quadrialteral ABF_1C is an isosceles trapezoid and $BF_1 \parallel AC$.

Let EF_1 intersects AC at L'. Then

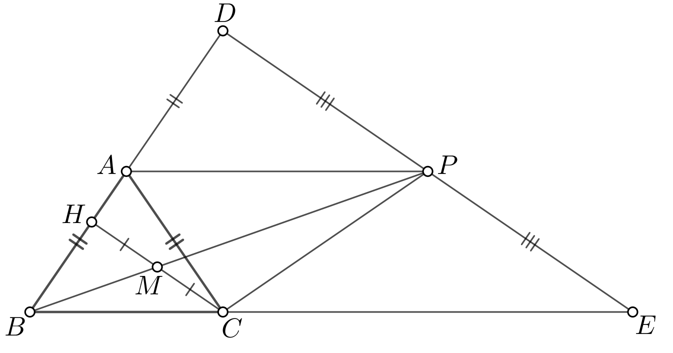
$$\sphericalangle L'DA = \sphericalangle AF_1L' = \sphericalangle AF_1E = \sphericalangle ABK = \sphericalangle LDA,$$

so $L = L'$ and $F_1 = F$, which finishes the proof. $\square$

Problem 23. In an isosceles triangle ABC, where $AB = AC$, point M is the midpoint of altitude CH. The line passing through C and perpendicular to line AC intersects the line passing through A and parallel to line BC at point P. Prove that points B, M and P are collinear.

Proof. Construct the right-angled triangle APD, symmetrical to the triangle APC with respect to line AP. Since AP is the external bisector of the angle BAC, the point D lies on the line AB, and $DA = AC = AB$.

Let the lines DP and BC intersect at the point E. As $BA = AD$ and $AP \parallel BE$, then AP is the middle line of the triangle BDE, which means $DP = PE$. The right-angled triangles BHC and BDE are homothetic with the center B, therefore the midpoints of segments CH and ED lie on the same line with point B, that is, the points M, P, and B are collinear, as required.



□

Problem 24. In the acute triangle ABC, the side AB is smaller than the side BC, BH_b is the height, the point O is the circumcenter. The line passing through H_b parallel to the line CO intersects the line BO at the point X. Prove that the point X and the midpoints of the sides AB and AC lie on one line.

Proof. Let M be the midpoint of AB. We have

$$\sphericalangle OBC = \sphericalangle OCB = 90^\circ - \sphericalangle A.$$

It is enough to prove that $MX \parallel BC$, or that $\sphericalangle MXB = 90^\circ - \sphericalangle A$. Notice, that

$$\sphericalangle H_bXB = \sphericalangle XOC = 2\sphericalangle OBC = 180^\circ - 2\sphericalangle A.$$

From the right triangle ABH_b we see that

$$MA = MB = MH_b \quad \text{and} \quad \sphericalangle BMH_b = \sphericalangle MAH_b + \sphericalangle MH_bA = 2\sphericalangle A.$$

In the quadrilateral MH_bXB, the sum of the opposite angles is 180°, which means that it is cyclic. The inscribed angles BXM and MXH_b are based on equal chords, therefore, they are equal. From here

$$\sphericalangle MXB = \frac{1}{2}BXH_b = 90^\circ - \sphericalangle A.$$

□

Problem 25. In the triangle ABC, the sides AB and BC are equal. The point D lies inside the triangle such that $\sphericalangle ADC = 2\sphericalangle ABC$. Prove that the double distance from point B to the external bisector of the angle ADC, is equal to $AD + DC$.

Proof. Let the circumcircle of triangle ADC intersects the height of triangle ABC, drawn from point B, at points O and P, and assume O lies inside triangle ABC. Of course O and P are the midpoints of the arcs AC (and OP is the diameter), therefore, $\sphericalangle ADP = \sphericalangle CDP$, i.e. DP is the bisector of the angle AD and DO is the external bisector of the angle ADC.

From problems conditions O is the centre of the circumscribed circle of the triangle ABC. Assume $AD \geq CD$. Let K be the projection of point O onto AD, and let L be the projection of point B onto OD, C' is a point symmetric to C with respect to OD. Since OD is an external bisector, C' lies on the extension of segment AD beyond point D.

Moreover $AC' = AD + DC' = AD + CD$ and $OC' = OC$. But since $OA = OC$, the triangle OAC' is isosceles, so $AK = AC' = AD + CD$. Now, to complete the solution, it suffices to prove the equality of the segments BL and AK. We have

$$\sphericalangle AOK = 90^\circ - \sphericalangle OAD = 90^\circ - \sphericalangle OPD = \sphericalangle POD = \sphericalangle BOL.$$

We obtain that the right triangles AOK and BOL are congruent ($AO = OB$, since O is the center of the circumscribed circle of the triangle ABC). Therefore, $BL = AK$, as required. □

3.2. Spiral similarity.

Problem 26. Let A_1 and B_1 be points on sides CA and CB in triangle ABC such that AA_1B_1B is cyclic. Denote by S the intersection point of AA_1 and BB_1. Points X and Y are reflections of S wih respect to CB and CA, respectively. Circumcircles of triangles CA_1B_1 and CAB intersects at point $P \neq A$. Prove that $XPCY$ is cyclic.

Proof. Easy to see that PA_1B and PB_1C are spiral similar, so in particular kites $PXBA_1$ and $PSAB_1$ are spiral similar. Therefore

$$\measuredangle XAS = \measuredangle A_1PB_1 = \measuredangle ACB$$

and similarly $\measuredangle SPY = \measuredangle BCA$, so $\measuredangle XPY = 2\measuredangle BCA$.

Therefore

$$\measuredangle XCY = \measuredangle XCA + \measuredangle BCA + \measuredangle BCY = \measuredangle ACS + \measuredangle ACB + \measuredangle SCB = 2\measuredangle ACB = \measuredangle XPY.$$

□

3.3. Pascal theorem.

3.4. Nice lemma.

3.5. Equal Tangents Theorem.

Problem 27. The extension of median CM of the triangle ABC intersects its circumcircle ω at N. Let P and Q be the points on the rays CA and CB respectively such that $PM \parallel BN$ and $QM \parallel AN$. Let X and Y be the points on the segments PM and QM respectively such that PY and QX are tangent to ω. The segments PY and QX intersect at Z. Prove that the quadrilateral $MXZY$ is circumscribed.

Proof. Let PY and QX touch ω at Y_1 and X_1 respectively. Since $ACBN$ is cyclic and $PM \parallel BN$ we have $\measuredangle ACN = \measuredangle ABN = \measuredangle AMP$, i. e. the circumcircle of triangle AMC is tangent to the line PM. Thus $PM^2 = PA \cdot PC$. But $PA \cdot PC = PY_1^2$, and therefore $PM = PY_1$. In the same way we have $QM = QX_1$. Obviously $ZX_1 = ZY_1$. It remains to note that the desired result follows from the concave version of inscribed quadraliteral theorem:

$$PM - QM = PY_1 - QX_1 = (PZ + ZY1) - (QZ + ZX1) = PZ - QZ,$$

and so $PM - QM = PZ - QZ$. □

Problem 28. Let $ABCDEF$ be bicentric convex hexagon. Denote by ω_A, ω_B, ω_C, ω_D, ω_E and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF and EFA, respectively. Let l_{AB}, be the external common tangent of ω_A and ω_B different from line AB; lines l_{BC}, l_{CD}, l_{DE}, l_{EF}, l_{FA} are analogously defined. Let A_1 be the intersection point of the lines l_{FA} and l_{AB}. Define B_1, C_1, D_1, E_1, F_1 analogously. Prove that A_1D_1, B_1E_1, C_1F_1 are concurrent.

Proof. First note that from simple segment computations (using Two Tangent Theorem and some obvious facts) we get that

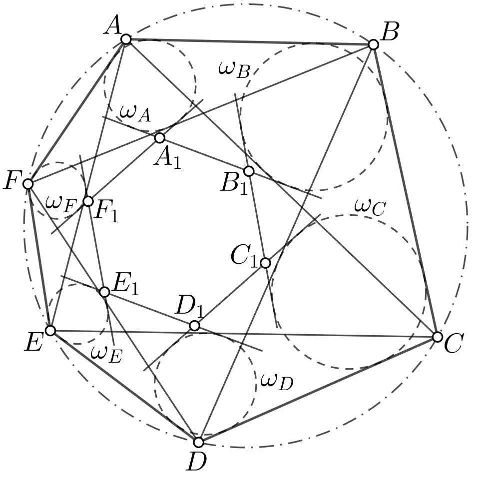
$$AB - BC + CD - DE + EF - FA = A_1B_1 - B_1C_1 + C_1D_1 - D_1E_1 + E_1F_1 - F_1A_1.$$

Since $ABCDEF$ is bicentric hexagon, then

$$AB + CD + EF = BC + DE + FA \Longrightarrow 0 = AB - BC + CD - DE + EF - FA =$$
$$= A_1B_1 - B_1C_1 + C_1D_1 - D_1E_1 + E_1F_1 - F_1A_1.$$

and hence

$$A_1B_1 + C_1D_1 + E_1F_1 = B_1C_1 + D_1E_1 + F_1A_1.$$



It's easy to notice that $B_1C_1 \parallel AD$ (because BC is antiparallel to AD - $ADBC$ is cyclic and B_1C_1 is antiparallel to BC - B_1BCC_1 is cyclic too - and therefore $B_1C_1 \parallel AD$). It means that $C_1B_1 \parallel AD \parallel E_1F_1$. Analogously $A_1B_1 \parallel D_1E_1$ and $A_1F_1 \parallel C_1D_1$.

Now it is enough to prove lemma:

Lemma. *Let $ABCDEF$ be a hexagon in which $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$ and $AB + CD + EF = BC + DE + FA$. Show that AD, BE, CF are concurrent.*

Proof. Consider midpoints M_A, M_B, M_C of diagonals AD, BE and CF, respectively. Its easy to notice that $M_AM_B \parallel AB \parallel DE$, $M_BM_C \parallel BC \parallel EF$ and $M_AM_C \parallel AF \parallel CD$.

Since $ABDE$, $BCEF$ and $CDFA$ are trapezoid, it's well known that

$$M_AM_B = |AB - DE|,$$

$$M_BM_C = |BC - EF|$$

and

$$M_AM_C = |AF - CD|.$$

It's really easy to check that then (without loss of generality let $AB > DE$, then $BC - EF > CD - FA$)

$$M_AM_B + M_BM_C + M_CM_A = 0.$$

Thus M_A, M_B, M_C have to be collinear. And so $M_A = M_B = M_C$, hence AD, BE, CF are concurrent. $\square$

$\square$

3.6. **Big picture.**

Problem 29. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of ABC intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of AOB and BOC intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.

Proof. Let T be the midpoint of BM and S the point such that $SA = SC$, $BS \parallel AC$. Let ω be the circumcircle of $BSOT$, which passes through the midpoints of AB and AC. Since $TP = TQ$ and OT is the external angle bisector of $\measuredangle POQ$, the quadrilateral $TOQP$ is cyclic as well (midarc lemma), and so by radical axis lines PQ, BS and OT intersect at point R (TO is perpendicular bisector of BM). $\square$

Problem 30. Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, and let NP, NT be the tangents to the incircle of this triangle. The lines BP and BT meet the circumcircle for the second time at points P_1 and T_1 respectively. Prove that $PP_1 = TT_1$.

Proof. Note that

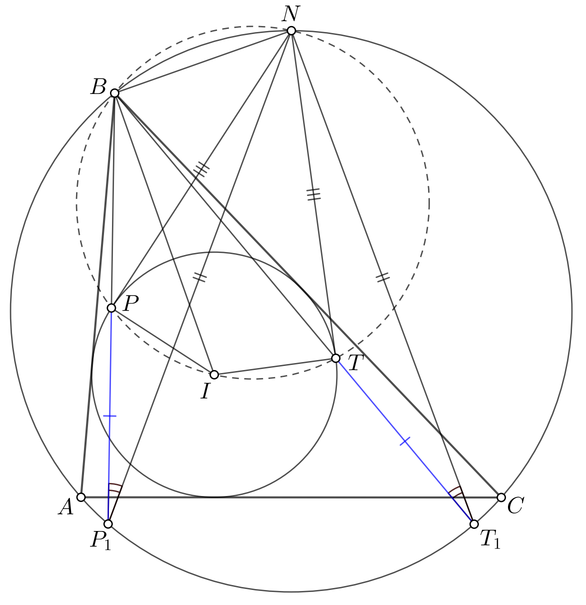
$$\measuredangle IBN = \measuredangle IPN = \measuredangle ITN = 90^\circ.$$

Thus $BPITN$ is cyclic. Moreover $IP = IT$, which means that BP and BT are isogonal. Therefore from symmetry, $NP_1 = NT_1$.

Notice that $NP = NT$ and

$$\measuredangle PNT = \measuredangle PBT = \measuredangle P_1BT_1 = \measuredangle P_1NT_1$$

so $\measuredangle PNP_1 = \measuredangle TNT_1$. Thus triangles NPP_1 NTT_1. Hence $PP_1 = TT_1$ and we are done.



□

Problem 31. On the sides AB, BC and CA of the triangle ABC, respectively, the points N, K and L are given such that $AL = BK$ and CN is the bisector of the angle ACB. The segments AK and BL intersect at point P. Denote by I and J the centres of the inscribed circles of triangles APL and BPK, respectively. Let Q be the intersection point of lines CN and IJ. Prove that $IP = JQ$.

Proof. If $CA = CB$, then the problem is obvious. If $CA \neq CB$, then without loss of generality, we can assume that CN intersects the segment PK. Let the circumscribed circles ω_1 and ω_2 of the triangles APL and BPK, intersect at the point $T \neq P$, respectively. Then

$$\sphericalangle LAT = \sphericalangle TPB = \sphericalangle TKB \quad \text{and} \quad \sphericalangle ALT = \sphericalangle APT = \sphericalangle TBK, \tag{3.1}$$

i.e. triangles ALT and KBT are congruent, so $AT = TK$. From 3.1 it also follows that the quadrilateral $ACKT$ is cyclic and so T lies on bisector CN.

Let IJ intersects ω_1 and ω_2 again at I_1 and J_1, respectively. Since triangles AI_1L and KJ_1B are similar nd $AL = BK$, they are in fact congruent. From trillium lemma

$$II_1 = I_1L = I_1A = KJ_1 = J_1B = JJ_1,$$

so $I_1I = JJ_1$. Moreover $I_1T = J_1T$, because AI_1LT is congruent to KJ_1BT. Thus, T lies on the perpendicular bisector of the segment IJ,. It remains to prove that T lies on the perpedicular bisector of PQ. Let $R = AK \cap CT$. Then

$$\begin{aligned} &\sphericalangle PRT = \sphericalangle ART = \sphericalangle RAC + \sphericalangle ACR = \\ &= \sphericalangle RAC + \sphericalangle AKT = \sphericalangle RAC + \sphericalangle KAT = \sphericalangle LAT = \sphericalangle BPT. \end{aligned}$$

Since PQ divides the angle RPB angle in half, we see that

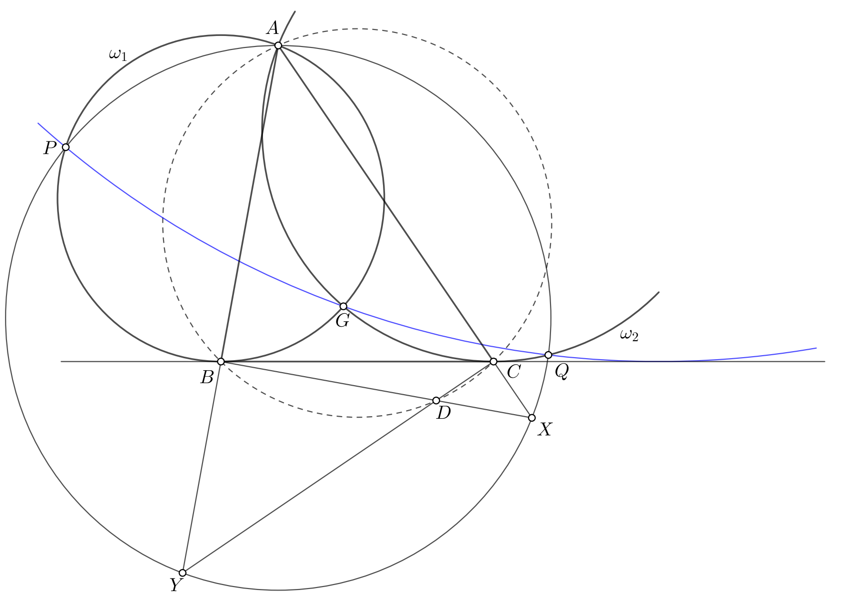
$$\sphericalangle PQT = \sphericalangle PRT + \sphericalangle RPQ = \sphericalangle BPT + \sphericalangle RPQ = \sphericalangle BPT + \sphericalangle QPB = \sphericalangle QPT,$$

therefore, T lies on the perpendicular bisector of the segment PQ. $\square$

3.7. Inversion $\sqrt{bc}$.

☠**Problem 32.** Consider point D on circumcircle Ω of ABC such that AD is diameter of this circle. Let DB, DC intersect AC, AB at X, Y, respectively. ω_1 is circle passing through A, B and tangent to BC; analogously ω_2 passes through A, C and is tangent to BC. Let ω_1 and ω_2 intersect at G; circumcircle of AXY intersect ω_1, ω_2 at P, Q, respectively. Prove that circumcircle of triangle GPQ is tangents to BC.

Proof. Invert about A with radius $\sqrt{AB \cdot AC}$ followed by reflection in the angle bisector of angle BAC. It's not hard to notice that $\phi(\omega_1) = \ell_C$, where ℓ_C stands for tangent to Ω at point C. Analogously $\phi(\omega_2) = \ell_B$, where ℓ_B stands for tangent to Ω at point B. If $\ell_B \cap \ell_C = K$, then $\phi(G)$ is obviously K.



Denote by H_A, H_B, H_C bases of altitudes in triangle ABC from vertexes A, B, C, respectively. It is clear that $\phi(D) = H_A$ and therefore by basic properties of inversion $\sqrt{bc}$ we may conclude that $\phi(X) = H_C$ and $\phi(Y) = H_B$. Hence circumcircle of triangle AXY is sent to line H_BH_C. If $\ell_B \cap H_BH_C = Q'$, $\ell_C \cap H_BH_C = P'$, then $\phi(P) = P'$ and $\phi(Q) = Q'$.

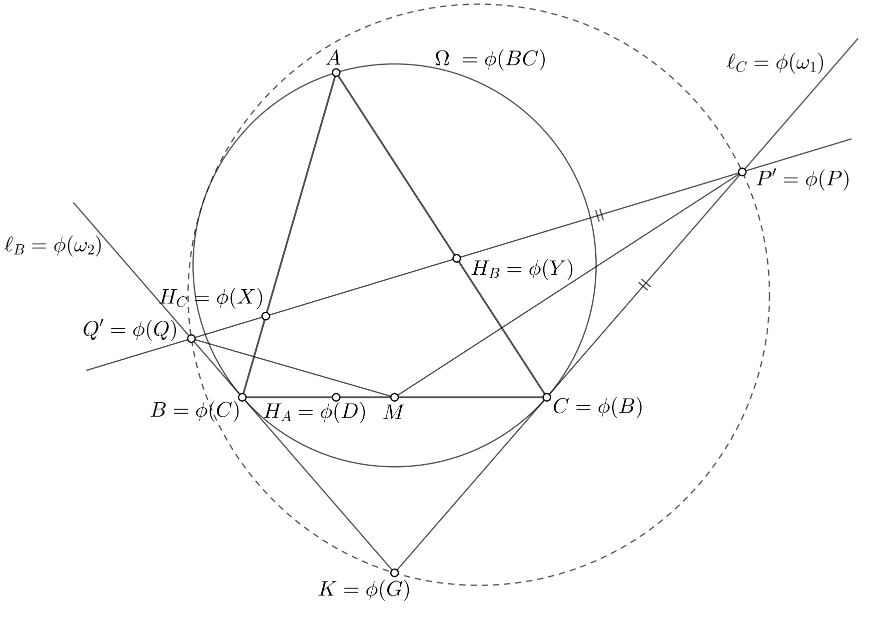
Since $\phi(BC) = \Omega$, thus we want to prove that circumcirle of triangle $KP'Q'$ is tangent to Ω.

Let M be the segment BC. Then we have $MH_B = MC$, which means that M lies on the perpendicular bisector of CH_B. Moreover

$$\measuredangle P'CH_B = \measuredangle ABC = \measuredangle AH_BH_C = \measuredangle P'H_BC \Longrightarrow P'C = P'H_B.$$

This means that $P'M$ is the perpendicular bisector of CH_B, or equivalently that PM is the internal angle bisector of angle $Q'P'K$. Similarly, we can show that QM is the internal angle bisector of angle $P'Q'K$. It means that M is in fact the incenter of triangle $KP'Q'$.

Since $KM \perp BC$, thus points B and C must be the points where the K-mixtilinear incircle [1] of $KP'Q'$ touches KP' and KQ' (by Sawayama Theorem). However, such a circle is Ω only, and so Ω is the K-mixtilinear incircle of KPQ, which is tangent to circumcircle of $KP'Q'$ by definition. Hence we are done.



$\square$

[1] A circle that in internally tangent to two sides of a triangle and to the circumcircle is called a mixtilinear incircle. There are three mixtilinear incircles, one corresponding to each angle of the triangle.

3.8. **The Gauss line.**

Problem 33. In triangle ABC the incircle ω with centre I touches BC at D. Let AH is altitude of triangle ABC, K is symmetric to H with respect to D. Draw tangent KL to ω where L lies on AC. Prove ID bisects BL.

Proof. Let M and N be midpoints of BL and AK, respectively. Points N, I, D lies on the middle line of triangle AHK ($AH \perp BC$ and $ID \perp BC$). Since M, N and I are collinear, so I, M, D too. □

Problem 34. Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

Proof. For simplicity, we consider only the case when X lies inside segment BA_0. All other cases are similar.

Let B_0 and C_0 be the midpoints of segments AC and AB, respectively. Let one tangent meet segment AC_0 at P and let the other tangent meet segment CB_0 at Q.

By the Gauss-Newton Theorem for circumscribed quadrilateral $APXQ$, the midpoints of segments AA_0, AX, and PQ are collinear. Therefore, the midpoint R of segment PQ lies on the midline of triangle ABC opposite to vertex A.

Let S be the reflection of point A about point R. Then S lies on line BC, and quadrilateral $APSQ$ is a parallelogram. Therefore,

$$\frac{C_0P}{A_0S} = \frac{B_0Q}{A_0S}$$

and so $C_0P = B_0Q$. □

3.9. **Butterfly theorem.**

Problem 35. Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of BC. The points D, E, F are selected on sides BC, CA, AB such that $ID \perp BC$, $IE \perp AI$, and $IF \perp AI$. Suppose that the circumcircle of the triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Proof. Letting D_1 be the extouch point, we observe that

$$\measuredangle(XM, AD_1) = \measuredangle XMI = \measuredangle XBF = \measuredangle XBA$$

since $IM \parallel AD_1$, and triangles XBF and XMI are spiral similar.

Thus XM and AD_1 meet on Γ. Now by the Butterfly Theorem, XD and AM meet on Γ as well. $\square$

Problem 36. Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\measuredangle FHP = \measuredangle BAC$.

Proof. $\square$

3.10. **Brianchon.**

Problem 37. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\sphericalangle EAB = \sphericalangle BCD$, and $\sphericalangle EDC = \sphericalangle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

Proof. Let I be the intersection of bisectors of angles CBA and DCB. From given conditions we see that triangles AIB, ICB and IDC are congruent. Hence

$$\sphericalangle IAB = \sphericalangle ICB = \frac{1}{2}\sphericalangle DCB = \frac{1}{2}\sphericalangle EAB,$$

so AI bisects angle EAB. Similarly DI bisects angle EDC and so EI bisects angle AED. Therefore the pentagon $ABCDE$ is circumscribed. Denote by T the intersection point of BC with incircle. Applying Brianchon therem to degenerated hexagon $ABTCDE$ we get the statement. $\square$

3.11. **Combinatorial geometry.**

Problem 38. Prove that in convex polygon with area S and perimeter L, we can fit a circle with radius $\frac{S}{L}$.

Proof. □

Problem 39. On the plane there are 25 points. Let D be the maximal distance between two points and d minimal one. Prove that $D > 2d$.

Proof. □

Problem 40. Prove that arbitrary triangle with area 1 we cat cut by a line and then bend along this line to obtain and region with area at least $\frac{\sqrt{5}-1}{2}$.

Proof. □

Problem 41. Prove that in every convex $(2n)$-gon, of area S, there is a side and a vertex that jointly span a triangle of area not less than $\frac{S}{n}$.

Proof. By main diagonals of the $(2n)$-gon we shall mean those which partition the $(2n)$-gon into two polygons with equally many sides. For any side b of the $(2n)$-gon denote by $\triangle_b$ the triangle ABP where A, B are the endpoints of b and P is the intersection point of the main diagonals AA', BB'. We claim that the union of triangles $\triangle_b$, taken over all sides, covers the whole polygon.

To show this, choose any side AB and consider the main diagonal AA' as a directed segment. Let X be any point in the polygon, not on any main diagonal. For definiteness, let X lie on the left side of the ray AA'. Consider the sequence of main diagonals AA', BB', CC', ... , where A, B, C, ... are consecutive vertices, situated right to AA'.

The n-th item in this sequence is the diagonal $A'A$ (i.e. AA' reversed), having X on its right side. So there are two successive vertices K, L in the sequence A, B, C, ... before A' such that X still lies to the left of KK' but to the right of LL'. And this means that X is in the triangle $\triangle_{l'}'$, where $l' = K'L'$. Analogous reasoning applies to points X on the right of AA' (points lying on main diagonals can be safely ignored). Thus indeed the triangles $\triangle_b$ jointly cover the whole polygon.

The sum of their areas is no less than S. So we can find two opposite sides, say $b = AB$ and $b' = A'B'$ (with AA', BB' main diagonals) such that

$$[\triangle_b] + [\triangle_{b'}] \geq \frac{S}{n},$$

where $\mathcal{F}$ stands for the area of a region $\mathcal{F}$. Let AA', BB' intersect at P; assume without loss of generality that $PB \geq PB'$. Then

$$[ABA'] = [ABP] + [PBA'] \geq [ABP] + [PA'B'] = [\triangle_b] + [\triangle_{b'}] \geq \frac{S}{n}.$$

$\square$

3.12. **3D-vision.**

Problem 42. Let ABC be a regular triangle. Line ℓ intersects BC, CA and AB at points K, L, M, (other then vertices) respectively. Prove that, there exists point P such that

$$PK = AK, \quad PL = BL, \quad PM = CM.$$

Proof. Take regular tetrahedron $ABCP'$. We see that P' satisfies given condition. Now rotate plane spanned by P' and ℓ P' along ℓ to plane ABC. We get point P which satisfies problem assumptions. □

Problem 43. Let X be a point inside triangle ABC such that

$$XA \cdot BC = XB \cdot AC = XC \cdot AC.$$

Let I_1, I_2, I_3 be the incenters of XBC, XCA, XAB. Prove that AI_1, BI_2, CI_3 are concurrent.

Proof. Let $SABC$ be a tetrahedron such that:

$$SA \cdot BC = SB \cdot AC = SC \cdot AB.$$

If points A', B', C' are incenters of triangles SBC, SAC and SAB respectively, then by the condition lines AA', BB' and CC' concur at one point. Now if Appolonius spheres pairs (A, B), (B, C) and (C, A) intersect at circle ω, we can tending S to X along the ω and we have configuration in our problem. □

3.13. **Constructions.**

Problem 44. Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (vertices maybe ordered clockwise or counter-clockwise) such that $A \neq A'$, $B \neq B'$, $C \neq C'$, $D \neq D'$ and AA', BB', CC', DD' are concurrent.

Proof. If $AC \perp BD$, take $P = AC \cap BD$, and it is trivial to construct a square $A'B'C'D'$ with centre P such that AA', BB', CC', DD' are concurrent at P. Thus, suppose that $AC \not\perp BD$. We will construct a point $P \in BD$ distinct from B, D such that $\measuredangle APB = \measuredangle CPB$.

Let the perpendicular bisector of AC meet BD at Q and let P be the second intersection of BD with circumcircle ω of triangle ACQ. Note that Q is a midpoint of arc AC of ω because $QA = QC$. Therefore, BD bisects $\measuredangle APC$. If P coincides with B, repeat above construction about vertex D instead of vertex B. If P then coincides with D, we have $\measuredangle ABD = \measuredangle CBD$ and $\measuredangle ADB = \measuredangle ACB$. Therefore, $ABCD$ is a kite, implying that $AC \perp BD$, impossible. Thus, we may assume assume that $P \neq B$.

Now, choose A_1, C_1 on PA, PC such that

$$\measuredangle A_1BP = \measuredangle C_1BP = 45^\circ$$

and let D_1 be the reflection of B in A_1C_1. Since $\measuredangle A_1PB = \measuredangle C_1PB$ it follows that triangles PBA_1 an PCA_1 are congruent. Hence, $BA_1 = BC_1$. Together with $\measuredangle A_1BC_1 = 90^\circ$ it follows that $A_1BC_1D_1$ is a square. Moreover, $D_1 \in BD$ by symmetry. To finish, we dilate $A_1BC_1D_1$ with centre P to a new square $A'B'C'D'$ with all vertices distinct from $ABCD$. Then AA', BB', CC', DD' are concurrent at P, as desired.

□

☠**Problem 45.** Convex quadrilaterals $ABCD$ and $PQRS$ have equal areas. Moreover

$$AB = PQ, \quad BC = QR, \quad CD = RS, \quad DA = SP.$$

Prove that there exist points P', Q', R', S' which lie on a plane of quadrilateral $ABCD$, such that

$$AP' = BQ' = CR' = DS'$$

and quadrilaterals $PQRS$ and $P'Q'R'S'$ are congruent.

Proof. □

3.14. **Poncelet Porism.**

Problem 46. Equilateral triangle ABC is inscribed in circle Γ and described around circle ω. On the sides AC and AB, points P and Q are chosen, respectively, so that the segment PQ is tangent to ω. The circle b with center P passes through B, and the circle c with center Q passes through C. Prove that the circles, b, c and Ω have a common point.

Proof. Consider the regular triangle XYZ, described around ω and inscribed in Γ such that the points P and Q lie on its side YZ (Poncelet porism). Then the point X is the required point; This follows from the fact that the triangles ABC and XYZ are symmetric to each other with respect to PO and QO, where O is the center of Ω. □

Problem 47. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

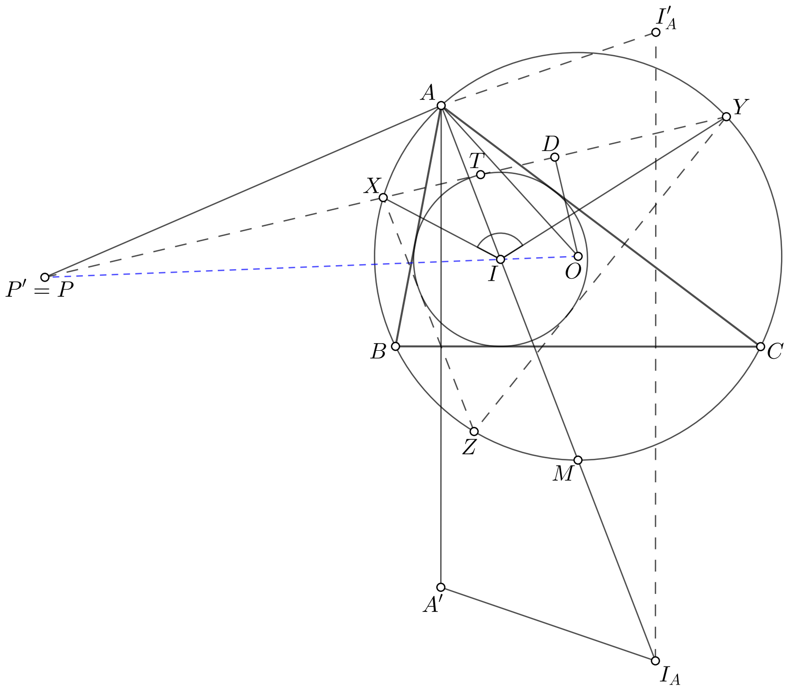
(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.
(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\measuredangle XIY = 120°$.

Proof.

(a) Let A' be the reflection of A in BC and let M be the second intersection of line AI and the circumcircle Γ of triangle ABC. As triangles ABA' and AOC are isosceles with $\measuredangle ABA' = 2\measuredangle ABC = \measuredangle AOC$, they are similar to each other. Also, triangles ABI_A and AIC are similar. Therefore we have

$$\frac{AA'}{AI_A} = \frac{AA'}{AB} \cdot \frac{AB}{AI_A} = \frac{AC}{AO} \cdot \frac{AI}{AC} = \frac{AI}{AO}.$$

Together with $\measuredangle A'AI_A = \measuredangle IAO$ (we are using the fact that O and H are isogonal conjugated), we find that triangles $AA'I_A$ and AIO are similar.



Denote by P' the intersection of line AP and line OI. Using directed angles, we obtain:

$$\sphericalangle MAP = \sphericalangle I_A^{'} AI_A = \sphericalangle I_A^{'} AI'_A - \sphericalangle I_A AA' = \sphericalangle AA'I_A - \sphericalangle (AM, OM) =$$
$$= \sphericalangle AIO - \sphericalangle AMO = \sphericalangle MOP'.$$

This shows that points M, O, A, P' are concyclic.

Denote by R and r the circumradius and inradius of triangle ABC. Then

$$IP' = \frac{IA \cdot IM}{IO} = \frac{IO^2 - R^2}{IO}$$

is independent of A. Hence, BP also meets line OI at the same point P' so that $P' = P$, and P lies on OI.

(b) By Poncelets Porism, the other tangents to the incircle of triangle ABC from X and Y meet at a point Z on Γ. Let T be the touching point of the incircle to XY, and let D be the midpoint of XY . We have

$$OD = IT \cdot \frac{OP}{IP} = r\left(1 + \frac{OI}{IP}\right) = r\left(1 + \frac{OI^2}{OI \cdot IP}\right) = r\left(1 + \frac{R^2 - 2Rr}{R^2 - IO^2}\right) =$$
$$= r\left(1 + \frac{R^2 - 2Rr}{2Rr}\right) = \frac{R}{2} = \frac{OX}{2}.$$

This shows $\sphericalangle XZY = 60°$ and hence $\sphericalangle XIY = 120°$.

□

4. JBMO TESTS

4.1. **Test 1 – 16-06-2019 (JBMO).**

Problem 48. Let ABC be a triangle such that $3AC = AB + BC$. The C-excircle is tangent to AB and AC at P and Q, respectively. Prove that $\sphericalangle CPQ = 90°$.

Proof. From the problem conditions $BP = BR$, $AP = AQ$ and $CR = CQ$, so

$$\begin{aligned} 3AC &= AB + BC = AP + PB + BC = AP + RB + BC = \\ &= CR + AP = CQ + AP = AC + AQ + AP = AC + 2AP, \end{aligned}$$

thus $AC = AP = AQ$ i.e. $\sphericalangle CPQ = 90°$. □

Problem 49. Point P lies inside triangle ABC such that

$$\sphericalangle PBA = \sphericalangle PCA = \frac{1}{3}(\sphericalangle ABC + \sphericalangle ACB).$$

Prove that

$$\frac{AC}{AB + PC} = \frac{AB}{AC + PB}.$$

Proof. Let $K = BP \cap AC$ and $L = CP \cap AB$. We see that

$$\begin{aligned} \sphericalangle KPC &= \sphericalangle PBC + \sphericalangle PCB = \sphericalangle ABC - \sphericalangle ABP + \sphericalangle ACB - \sphericalangle ACP = \\ &= \sphericalangle ABC + \sphericalangle ACB - 2 \cdot \frac{1}{3}(\sphericalangle ABC + \sphericalangle ACB) = \\ &= \frac{1}{2}(\sphericalangle ABC + \sphericalangle ACB) = \sphericalangle PCK, \end{aligned}$$

which means that $KP = KC$. Similarly $LP = LB$. Triangles ABK and ACL are similar, so

$$\frac{AB}{AC} = \frac{AK + KB}{AL + LC} = \frac{AC - KC + KP + PB}{AB - LB + LP + PC} = \frac{AC + PB}{AB + PC}.$$

□

Problem 50. In an isosceles triangle ABC ($AB = AC$), the point M is the midpoint of height CH. The line passing through C and perpendicular to line AC, intersects the line passing through A and parallel to line BC at point P. Prove that points B, M and P lie on one line.

Proof. Construct the right-angled triangle APD, symmetrical to the triangle APC with respect to AP. Since AP is the external bisector of the angle BAC, the point D lies on the line AB, and $DA = AC = AB$. Let the lines DP and BC intersect at the point E. As $BA = AD$ and $AP \parallel BE$, then AP is the middle line of the triangle BDE, which means $DP = PE$.

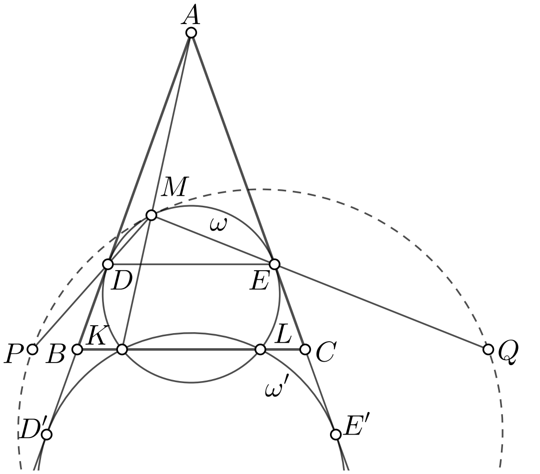
The right-angled triangles BHC and BDE are homothetic with the center B, therefore the midpoints of segments CH and ED lie on the same line with point B, that is, the points M, P, and B are collinear, as required.

□

Problem 51. The circle ω is tangent to the equal sides AB and AC of the isosceles triangle ABC and intersects the side BC at the points K and L. The segment AK intersects ω in a second time at point M. Points P and Q are symmetric to point K with respect to points B and C, respectively. Prove that the circumcircle of triangle PMQ is tangent to the circle ω.

Proof. Denote by D and E tangent points of ω with sides AB and AC. From symmetry wrt bisector of the angle BAC, it follows that $DE \parallel BC$. Consider a homothety with centre A and scale $\dfrac{AK}{AM}$, it maps ω to circle ω'.

The circle ω' passes through the point K, and hence through L (from symmetry wrt the bisector of the angle BAC). Moreover ω' touches the rays AB and AC at some points D' and E'.



It follows from the properties of the homothety that $MD \parallel KD'$. Further, according to the power of a point we see that $BD^2 = BK \cdot BL = BD'^2$, whence $BD = BD'$. It follows that the points M, D, P lie on one line. Similarly, the points M, E and Q lie on the same line. The triangles MDE and MPQ are homothetic with centre M, therefore, their circumscribed circles are also homothetic, i.e. tangent at point M. □