Number Theory

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Problems – April 20

- If p is a prime and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$. *(Fermat's theorem)*
- If n is a positive integer and a coprime to n, then $a^{\varphi(n)} \equiv 1 \pmod{n}$. *(Euler's theorem)*

1. Find all n such that $5^n - 4^n$ is divisible by 61.
2. Prove that $19 \cdot 8^n + 17$ is composite for all n.
3. Find the last two digits of $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ and $3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}$ (100 twos/threes).
4. Find the largest positive integer n such that, for any primes p and q, n divides $p^8q^4 - p^4q^8$.
5. Let p be a prime and $n = 2^p - 1$. Prove that $2^{n-1} \equiv 1 \pmod{n}$.
6. Find all primes p for which $\frac{2^{p-1}-1}{p}$ is a perfect square.
7. Prove that there also exist composite numbers n such that $a^{n-1} \equiv 1 \pmod{n}$ whenever a is coprime to n.
8. Prove that $7^n - 1$ is never divisible by $6^n - 1$.
9. Can $2^n + 1$ ever be divisible by 247?
10. Define $a_n = 2^n + 3^n + 6^n - 1$ for each positive integer n. Find all primes that do not divide any term of this sequence.
11. If a and b are any coprime positive integers, prove that there are infinitely many exponents n for which $a^n + b$ is composite.
12. Prove that there is an infinite set of pairwise coprime integers of the form $2^n - 3$.
13. Define $a_1 = 2$ and $a_{n+1} = 2^{a_n} + 2$ for all n. Prove that a_m divides a_n whenever $m < n$.