Training Session in Mathematics
November-December 2021
Level 3
Nikola Petrović

Lesson 1
Tilings and partitions

Problems:

1. A unit square is tiled into rectangles. For each proper rectangle in the partition the shorter side is selected and for each square one of the sides is selected. Show that the length of the selected sides is not smaller than 1. When does equality hold?

2. A unit square is tiled into triangles. Show that the sum of the perimeters of all the triangles is at least $4 + 2\sqrt{2}$. When does equality hold?

3. A rectangular board is tiled with 1×2 dominos (which may be placed both horizontally and vertically). Show that there exists a coloring of the fields of the board in two colors such that each domino covers two fields of different color and in any other tiling of the board with dominos there exists a domino which covers two tiles of the same color.

4. Is it possible to partition a circle into mutually congruent shapes such that some of the shapes do not contain the center of the circle (neither as an interior point nor as a boundary point)?

5. A regular $4n$-gon is partitioned into parallelograms. Prove that there are at least n rectangles among these parallelograms.

6. A rectangle of dimensions $a \times b$, where $a \geq b$, is partitioned into right-angled triangles such that each two adjacent triangles shares a common side and this side is the hypothenuse for one of the triangles and one of the legs (catheti) for the other triangle. Prove that $a \geq 2b$.

7. Find the minimum number of colors needed to color an 8×8 chessboard so that no 2 squares of an L-tetramino have the same color.

8. An isosceles trapezoid is called *proper* if exactly one pair of its sides is parallel (thus, parallelograms and rectangles are not proper). A partition of a rectangle into n proper isosceles trapezoids is called strict if the union of any $i\,(2 \leq i \leq n)$ trapezoids in the partition does not form a proper isosceles trapezoid. Prove that for any $n, n \geq 9$ there is a strict partition of a 2020×2021 rectangle into n proper isosceles trapezoids.

9. An $n \times n$ board is filled in with integers such that each pair of integers on adjacent fields differ by at most 1. Prove that there is an integer which appears at least n times on the board. Two fields are considered adjacent if they share a common side.

10. An L-tromino is a 2×2 square with one of its units removed. For which values of M, N can an $M \times N$ rectangle be tiled by L-trominos such that there is no smaller rectangle which is tiled by a subset of these L-trominos?

11. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be partitioned into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to partition R into at least $n + 1$ smaller rectangles.

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Lesson 2
Graph theory

Problems:

1. Let A and B be two countries such that each city is connected to exactly $k \geq 2$ cities in the other country via an interstate road. Interstate roads connect a city in country A with a city in country B with no stops in-between and no pair of cities is connected by more than one interstate. Assuming each city is reachable from every other city using the interstate roads is it possible to destroy a single road so that this no longer holds?

2. In each of three schools there are n students. Each student knows at least $n+1$ students from the other two schools. Prove that there are three students from three schools who know each other.

3. Let a k-group be a collection of k people who all know each other (knowing someone is a symmetric relation). Assume at a party that there are no 5-groups and that no two 3-groups are disjoint, in other words every pair of three groups has at least one common person. Prove that there are two people whose departure would result in there being no 3-groups at the party.

4. Let n be a complete graph, i.e. a graph whose every pair of vertices is connected. In each move, we select a cycle of length 4, i.e. $A_1A_2A_3A_4$ such that A_1A_2, A_2A_3, A_3A_4 and A_4A_1 all belong to the graph and remove one of the edges, say A_1A_2. We continue making moves until there are no more cycles of length 4 in the graph. Find the lowest number of edges we can obtain.

5. On the continent of Graphia there are $n \geq 3$ countries. On Year 1 of the New Mathematical Era we find that some pairs of countries are at war. Each country is at war with at least one other country and it is impossible to divide the country into two groups such that no two countries in opposing groups are at war. Starting from Year 2, each year two countries that are at war sign a peace treaty. Unfortunately, each country that is at war with exactly one of the two countries that signed the peace treaty will also declare war on the other country. Additionally, if at any point in time we divide the countries into two arbitrary non-empty groups, after a finite amount of time two countries in the opposite groups will sign a peace treaty. Prove that after a finite amount of time there will be a country that is at war with all other countries.

6. Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \dots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\dots$; c_{m-1} and c_m; c_m and c_1. Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

7. A graph is called good if among any three vertices there are at least two which are connected with each other. A complete graph of order 3 will be called a triangle and also a windmill of order 1, and a windmill of order $n > 1$ is defined to be a collection of n triangles that only intersect in a unique vertex common to all the n triangles (in other words, a collection of $2n+1$ distinct vertices $a, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n$ such that ab_i, ac_i and b_ic_i are all edges in the graph for every $i = 1, 2, \dots, n$). Find $f(n)$, the smallest number such that every good graph with $f(n)$ vertices contains a windmill of order n.

8. Let n and k be given natural numbers. In a candy shop one morning there were $2n$ pieces of each flavor of candy. That day a total of m children bought candies at the shop and each of the m children bought 2 pieces candy of (mutually) different flavor. It is known that in each subset of $k+1$ children that bought candy that day there are 2 children who bought at least one pair of candies of the same flavor. Find the largest possible value of m. The store is not replenished with new candy during the day.

9. In the country of Graphia there are 100 towns, each numbered from 1 to 100. Some pairs of towns may be connected by a (direct) road and we call such pairs of towns adjacent. No two roads connect the same pair of towns.

   Peter, a foreign tourist, plans to visit Graphia 100 times. For each i, $i = 1, 2, \ldots, 100$, Peter starts his i-th trip by arriving in the town numbered i and then each following day Peter travels from the town he is currently in to an adjacent town with the lowest assigned number, assuming such that a town exists and that he hasn't visited it already on the i-th trip. Otherwise, Peter deems his i-th trip to be complete and returns home.

   It turns out that after all 100 trips, Peter has visited each town in Graphia the same number of times. Find the largest possible number of roads in Graphia.

10. Turan's theorem: Let G be a graph on n vertices and m is a positive integer with $2 \le m \le n$. Suppose G does not contain a complete graph of order m in G. Prove that the number of edges in G is at most $\frac{n^2}{2}\left(1 - \frac{1}{m-1}\right)$.

11. A graph has 2017 vertices. Some pairs of vertices are connected with either blue or red edges. In every subset of 4 vertices there is an even number of edges, at least one edge is red and if there are blue edges then there are more blue edges than red edges. Prove that there are 673 vertices such that each pair of vertices is connected with a red edge.

12. In a country called Graphia each city is connected with three other cities via a road. A single road starts and ends in a city without passing through any other city. Prove that it is possible to color all the roads in four colors so that no two roads of the same color share a common endpoint city.

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Lesson 3
Miscellaneous combinatorics

Problems:

1. An $n \times n$ grid is the collection of edges of all squares on an $n \times n$ board. Let an $n \times n$ grid be tiled by unit angles oriented up-left, up-right, down-left or down-right (thus, a unit square is tiled by two unit angles). Prove that the number of up-left unit angles is equal to the number of down-right unit angles.

2. Let n and k be positive integers such that $1 < k \leq n^2$. Assuming we label the centers of squares with integer coordinates, let an n-diamond be a collection of squares for which $|x| + |y| \leq n$ (Thus the X pentomino is a 1-diamond). Show that the number of ways of placing k dominoes (2×1 or 1×2) on an n-diamond is larger than the number of ways of placing k 2×2 squares on a $(2n+1) \times (2n+1)$ square, in both cases without any overlap.

3. On a $1 \times (m+1)$ board the squares are labeled from 0 to m. Initially there are n stones at position zero. In each move we can stone to the right up to as many squares as the number of stones on the square the stone is currently occupying (e.g. a lone stone can only be moved one square). Show that the number of moves needed for all stones to reach square m is at least $\lceil \frac{m}{1} \rceil + \lceil \frac{m}{2} \rceil + \cdots + \lceil \frac{m}{n} \rceil$.

4. Let n be a positive integer, where $2n$ people are in a spacecraft, and any two of them are friends or enemies (the relationship is reciprocal). Two aliens play the following game: Alternately, each player chooses one person, such that the chosen one (of each round) is friend of the person chosen (by the other player) in the previous round. In the first round, the player can choose any person, each person can be chosen in at most once and the player who can not play anymore loses the game. Prove that the second player has the winning strategy, if and only if, the $2n$ people can be split into n pairs, such that two people on the same pair are friends.

5. Players A and B play the following game. Initially, they have a large piece of plastic. In his move, A cuts one of the existing pieces into three pieces of arbitrary sizes and B glues two existing pieces into one. Then they repeat this procedure until player A wins if at some point there are among the cut pieces 100 pieces of equal weight. Can player B prevent A from winning?

6. Two players play the following game. At the outset there are two piles, containing 10,000 and 20,000 tokens, respectively . A move consists of removing any positive number of tokens from a single pile or removing $x > 0$ tokens from one pile and $y > 0$ tokens from the other, where $x + y$ is divisible by 2015. The player who can not make a move loses. Which player has a winning strategy

7. On each square of a $3 \times n$ board a black and white coin is placed with the black side facing up. In each move we select a square and flip all its adjacent squares. A square is adjacent to another square if it has at least one common vertex with that square. A square is not adjacent to itself. For which n is it possible to end up with all coins having the white side facing up.

8. Let $n \geq 3$ be an integer and let f be a real valued function on a plane such that $f(A_1) + f(A_2) + \cdots + f(A_n) = 0$ for every regular n-gon $A_1 A_2 \ldots A_n$. Prove that f is zero everywhere on the plane.

9. The faces of a polyhedron are colored in either black or white so that there are more black faces than white faces and no two black faces share an edge. Prove that it is impossible to inscribe a sphere inside the polyhedron.

10. On a 100×100 field we have 99 squares that have been overrun by weeds. If a square is adjacent to two squares with weeds it too will grow weeds. Is it possible for all squares to have weeds? Two squares are considered adjacent if they share a common side.

11. On a single pile there are 1001 stones. In a single move one is allowed to remove one stone from a pile and split the remaining stones on the pile into two non-zero piles of arbitrary amount. Is it possible to end up only with piles containing three stones?

12. A car races along a circular track along which there are several gas stations, each with a fixed (non-replenishable) amount of gas. The total amount of gas on the stations is just large enough to complete one round trip along the track. Prove that there exists a gas station from which the car can start with an empty (and large enough) tank and complete a round trip (i.e. return to the starting station without running out of gas). Assume that the car consumes a fixed amount of gas per unit length.