Email training, N2

Level 2, September 20-26

Problem 2.1. Let x_1 and x_2 are the roots of the equation $x^2 + 5x - 11$. Find a quadratic polynomial which roots are x_1x_2 and $x_1^2x_2^2$.

Solution 2.1. Since $x_1$and x_2 are solution of $x^2+5x-11=0$ the, according to Vieta theorem one has $x_1 + x_2 = -5$ and $x_1x_2 = -11$. If x_1x_2 and $x_1^2x_2^2$ are solution of $x^2 + ax + b = 0$ then $-a = x_1x_2 + x_1^2x_2^2 = x_1x_2(x_1x_2 + 1) = -110$ and $b = x_1^3x_2^3 = -1331$. So

$$x^2 - 110x - 1331 = 0$$

Answer: $x^2 - 110x - 1331 = 0$.

Problem 2.2. Simplify

$$\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}.$$

Solution 2.2.

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}} =& \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2+2\sqrt{3}}{\sqrt{4+2\sqrt{3}}}\\ =& \frac{2(1+\sqrt{3})}{\sqrt{1+2\sqrt{3}+3}} = \frac{2(1+\sqrt{3})}{1+\sqrt{3}} = 2:\end{aligned}$$

Problem 2.3. Find all positive integers n for which $n^2 + 3n$ is perfect square.

Solution 2.3. Note that

$$n^2 < n^2 + 3n < n^2 + 4n + 4 = (n+2)^2.$$

Since $n^2 + 3n$ is perfect square, then it is equal to $(n+1)^2$.

$$n^2 + 3n = n^2 + 2n + 1,$$

$$n = 1.$$

Answer: $n = 1$.

Problem 2.4. Find all integer solutions to the equation

$$x^2 - 6xy + 13y^2 = 100.$$

Solution 2.4. Rewrite the equation in the following form

$$x^2 - 6xy + 13y^2 = (x-3y)^2 + (2y)^2 = 100.$$

Since 100 can be written as sum of squares in the following ways

$$100 = 8^2 + 6^2 = 10^2 + 0^2$$

, therefore we have the following options

$$x - 3y = 8, 2y = 6$$

$$x - 3y = -8, 2y = 6$$

$$x - 3y = 6, 2y = 8$$

$$x - 3y = -6, 2y = -8$$

$$x - 3y = 10, 2y = 0$$

$$x - 3y = -10, 2y = 0$$

$$x - 3y = 0, 2y = 10$$

$$x - 3y = 0, 2y = -10$$

From these cases we get solutions

Answer: $(17, 3), (1, 3), (18, 4), (-18, -4), (10, 0), (-10, 0), (15, 5), (-15, -5)$.

Problem 2.5. Find the number of 7-digit positive integers that all digits are ordered in
a) strictly increasing order,
b) strictly decreasing order.

Solution 2.5. a) part is equivalent to write expression 123456789 and remove any 2 digits. It can be done in $\binom{9}{2}$ ways.

b) part is equivalent to write expression 9876543210 and remove any 3 digits. It can be done in $\binom{10}{3}$ ways.

Answer: a) $\binom{9}{2}$, b) $\binom{10}{3}$.

Problem 2.6. A triple $(1, 1, 1)$ is given. On each step one chooses 2 of them and increases by 1. Is it possible after some steps get numbers $(2022, 2022, 2022)$.

Solution 2.6. Note that the sum of written numbers is equal $1 + 1 + 1 = 3$. After first step the total sum will be equal $3 + 2 = 5$, after second step $5 + 2 = 7$ and so on, after nth step the total sum will be $3 + 2n$ which is odd. However, at the end we want to have 3 numbers which equal 2016, so their sum must be $3 \cdot 2022 = 6066$ which is even. We have already shown that the sum must be always odd.

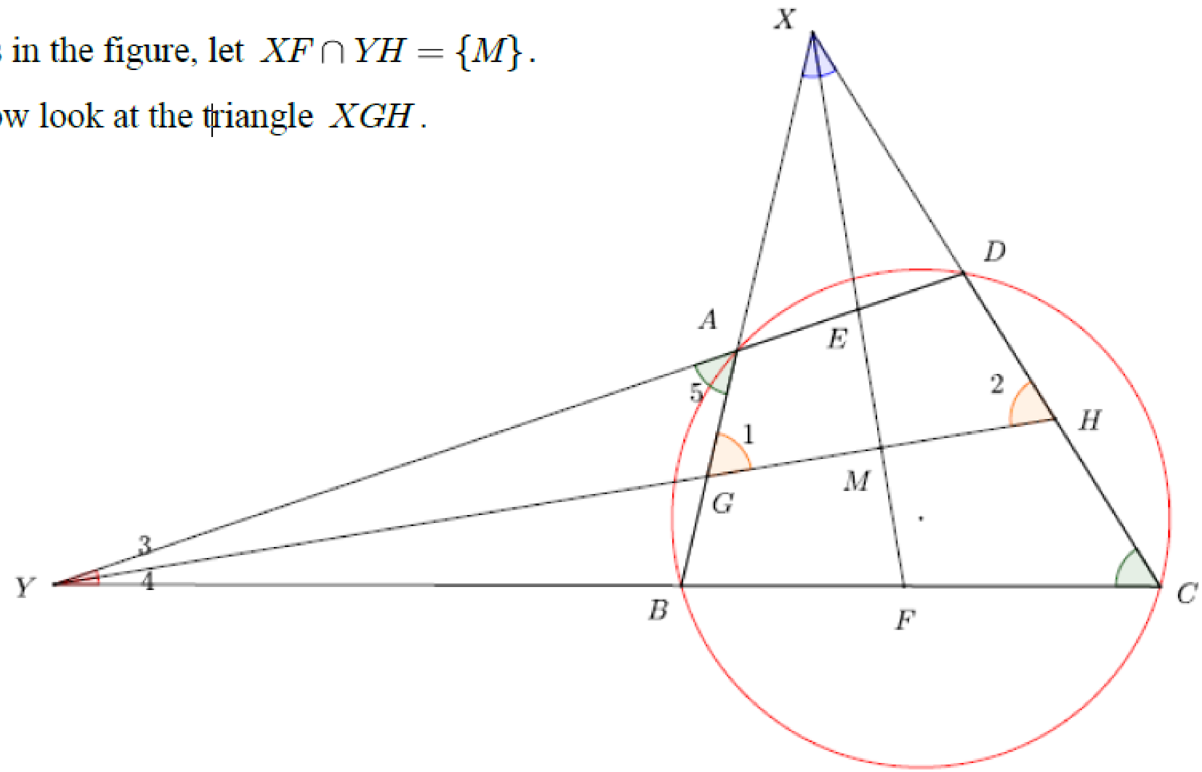
Answer: Not possible.

Problem 2.7. Let $ABCD$ be a cyclic quadrilateral. Let extensions of BA and CD intersect at X, extensions of AD and BC intersect at Y. Let the angle bisector of $\angle X$ intersects AD and BC at E and F, respectively, the angle bisector of $\angle Y$ intersects AB and CD at G and H respectively. Prove that $EGFH$ is a rhombus.

Solution 2.7. -

As in the figure, let $XF \cap YH = \{M\}$.

Now look at the triangle XGH.



$\angle 1$ is exterior angle of the triangle AGY, hence $\angle 1 = \angle 3 + \angle 5$. Also $\angle 2$ is exterior angle of the triangle HCY, hence $\angle 2 = \angle 4 + \angle C$. But $\angle 3 = \angle 4$ and $\angle 5 = \angle C$ (scince $ABCD$ is cyclic), therefore $\angle 1 = \angle 2$. Hence the triangle XGH is isosceles with $XG = XH$, but XM is the bisector of the GXH ,hence XM is the perpendicular bisector of GH. Similarly YM is the perpendicular bisector of EF. So the quadrilateral $EGFH$ have two diagonals bisects each other moreover they are perpendiculars, hence $EGFH$ is a rhombus.