March Online Camp 2020

Number Theory

Level L3

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Contents

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Covered topics

- Fermat's descend method: 1, 2, 10, 11, 12, 13,20, 24, 26,35,
- Guessing module: 3, 4, 8, 21,
- Square between square: 5, 6, 7, 9, 14
- Induction: 15, 16, 18,19, 36,
- Vieta's Jumping: 17, 22, 23, 25,27, 28, 29, 30,31, 32, 33, 34,
- Sum of two squares, Fermat theorem: 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51
- Quadratic residues: 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62

Classes

Date	Class	Homework
11/03/2020	1, 2, 3, 4, 5, 6, 7	8, 9, 10, 11, 12, 13, 14
12/03/2020	Homework + 15, 16, 17	18, 19, 20, 21, 22, 23, 24
14/03/2020	Homework + 25, 26	27, 28, 29, 30
15/03/2020	Homework + 31	32, 33, 34, 35, 36
16/03/2020	Homework + 37, 38	39, 40, 41, 42, 43, 44
17/03/2020	Homework + 45, 46	47, 48, 49, 50, 51
18/03/2020	Homework	52, 53, 54, 55, 56
19/03/2020	Homework	57, 58, 59, 60, 61, 62
21/03/2020	Homework	

Problems

1. Class 1

Problem 1. Find all integer solutions of

$$x^3 + 3y^3 + 9z^3 - 3xyz = 0.$$

Problem 2. Find all rationals a, b such that

$$a^2 + ab + b^2 = 2.$$

Problem 3. Solve in integers the following equation

$$y^4 = x^3 + 7.$$

Problem 4. Solve in integers the following equation

$$x^5 = y^2 + 4.$$

Problem 5. Find all solutions of the following equation in integers

$$x^2 + x + 1 = y^2.$$

Problem 6. Find all solutions of the following equation in integers

$$x^4 + y = x^3 + y^2.$$

Problem 7. Find all positive integers (a, b) for which $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are perfect cubes.

1.1. **Homework.**

Problem 8. Solve in integers the following equation
$$2x^6 + y^7 = 11.$$

Problem 9. Find all positive integers (k, m) for which $k^2 + 4m$ and $m^2 + 5k$ are perfect squares.

Problem 10. Solve in integers the following equation
$$x^2 + y^2 = 3z^2.$$

Problem 11. Solve in integers the following equation
$$x^2 + y^2 + z^2 - 2xyz = 0.$$

Problem 12. Solve in integers the following equation
$$x^4 + y^4 + z^4 = 9u^4.$$

Problem 13. Solve in integers the following equation
$$x^2 + y^2 + z^2 = x^2y^2.$$

Problem 14. Prove that there are no positive integers a, b such that $2a^2+1$, $2b^2 + 1$, $2(ab)^2 + 1$ are all perfect squares.

2. Class 2

Problem 15. Prove that for all positive integers n, the equation
$$x^2 + y^2 + z^2 = 59^n$$
is solvable in integers.

Problem 16. Prove that for all $n \geq 6$ the equation
$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = 1$$
is solvable in distinct integers.

Problem 17. Let a, b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Prove that
$$\frac{a^2 + b^2}{ab + 1}$$
is a perfect square.

2.1. **Homework.**

Problem 18. Prove that for all positive integers n, the equation
$$x^2 + xy + y^2 = 7^n$$
is solvable in integers.

Problem 19. Prove that for all $n \geq 6$ the equation
$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2} = 1$$
is solvable in integers.

Problem 20. Solve the following equation in positive integers:
$$x^2 - y^2 = 2xyz.$$

Problem 21. Decide whether the equation $x^4 + y^3 = z! + 7$ has an infinite number of positive integer solutions.

Problem 22. Suppose that a, b are positive integers such that $4ab-1$ divides $(a-b)^2$. Prove that $a=b$.

Problem 23. Suppose that a and b are odd positive integers such that $2ab+1 \mid a^2+b^2+1$. Prove that $a=b$.

Problem 24. Prove that the equation

$$6(6a^2+3b^2+c^2)=5n^2$$

has no solutions in integers except $a=b=c=n=0$

3. Class 3

Problem 25. Let a, b be positive integers such that ab divides $a^2+b^2+a+b+1$. Prove that
$$\frac{a^2+b^2+a+b+1}{ab}=5.$$

Problem 26. Let a_1, a_2, ..., a_{2n+1} be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove that $a_1=a_2=\ldots=a_{2n+1}$.

3.1. Homework.

Problem 27. Positive integers a, b satisfy $ab \mid a^2+b^2+1$. Prove that
$$\frac{a^2+b^2+1}{ab}=3.$$

Problem 28. Find all positive integers a, b such that $ab+a+b$ divides a^2+b^2+1.

Problem 29. Positive integers a, b satisfy $ab-1 \mid a^2+b^2$. Prove that
$$\frac{a^2+b^2}{ab-1}=5.$$

Problem 30. Find all positive integers m, n such that $mn-1$ divides $(n^2-n+1)^2$.

4. Class 4

Problem 31. Find all positive integers a, b such that $a \mid b^2+1$ and $b \mid a^2+1$.

4.1. Homework.

Problem 32. Let a and b be positive integers, such that $ab-1$ divides a^2+b^2+ab. Prove that

$$\frac{a^2+b^2+ab}{ab-1} \in \{4, 7\}.$$

Problem 33. Let a and b be positive integers, such that $4ab-1$ divides $(4a^2-1)^2$. Prove that $a=b$.

Problem 34. Let a, b, c and m be positive integers such that

$$abcm = 1 + a^2 + b^2 + c^2.$$

Prove that $m=4$.

Problem 35. Find all integer solutions of the following system of equations

$$\begin{cases} x^2+6y^2 = z^2 \\ 6x^2+y^2 = t^2 \end{cases}$$

Problem 36. Define a sequence $(a_n)_{n\geq 1}$ by setting $a_1 = 2$ and

$$a_{n+1} = 2^{a_n} + 2$$

for $n \geq 1$. Prove that a_n divides a_{n+1} for $n \geq 1$.

Problem 37. Let p be a prime of the form $4k+3$ such that $p \mid a^2+b^2$. Prove that $p \mid a$ and $p \mid b$.

Problem 38. Prove that there are no positive integers m, n such that

$$4mn - m - n$$

is a square.

5.1. Homework.

Problem 39. Solve in integers the following equation

$$x^2 + 4 = y^5.$$

Problem 40. Solve in integers the following equation

$$x^3 + 7 = y^2.$$

Problem 41. Prove that the equation

$$3^k - 1 = m^2 + n^2$$

has infinitely many solutions in positive integers.

Problem 42. Prove that the equation

$$x^4 - 4 = y^2 + z^2$$

does not have integer solutions.

Problem 43. Prove that

$$x^8 + 1 = n!$$

has only finitely many solutions in nonnegative integers.

Problem 44. Find all n-tuples $(a_1, a_2, \ldots, a_n)$ of positive integers such that

$$(a_1! - 1)(a_2! - 1) \ldots (a_n! - 1) - 16$$

is a perfect square.

6. Class 6

Problem 45. Find all pairs (m, n) of positive integers such that
$$m^2 - 1 \mid 3^m + (n! - 1)^m.$$

Problem 46. Solve in integers the equation
$$x^2 = y^7 + 7.$$

6.1. Homework.

Problem 47. Solve in integers the equation
$$y^3 - 9 = x^2.$$

Problem 48. Prove that a positive integer can be written as the sum of two perfect squares if and only if it can be written as the sum of the squares of two rational numbers.

Problem 49. Prove that
$$\frac{x^2+1}{y^2-5}$$
is not an integer for any integers $x, y > 2$.

Problem 50. Prove that each prime p of the form $4k+1$ can be represented in exactly one way as the sum of the squares of two integers, up to the order and signs of the terms.

Problem 51. Prove that there are infinitely many pairs of consecutive numbers, no two of which have any prime factor of the form $4k+3$.

7. Class 7

7.1. Homework.

Problem 52. Compute

$$\left(\frac{600}{953}\right), \quad \left(\frac{2020^3}{953}\right), \quad \left(\frac{-7000}{757}\right).$$

Problem 53. Prove that

- -2 is a quadratic residue modulo a prime $p > 2$ iff $p \equiv 1, 3 \pmod 8$,
- 2 is a quadratic residue modulo a prime $p > 2$ iff $p \equiv \pm 1 \pmod 8$,
- -3 is a quadratic residue modulo a prime $p > 2$ iff $p \equiv 1 \pmod 6$,
- 3 is quadratic residue modulo a prime $p > 2$ iff $p \equiv \pm 1 \pmod{12}$.

Problem 54. Let p be a prime number. Prove that there exists $x \in \mathbb{Z}$ for which $p \mid x^2 - x + 3$ if and only if there exists $y \in \mathbb{Z}$ for which $p \mid y^2 - y + 25$.

Problem 55. Suppose that for some prime p and integers a, b, c the following are true

$$6 \mid p+1, \qquad p \mid a+b+c, \qquad p \mid a^4+b^4+c^4.$$

Prove that $p \mid a$, $p \mid b$ and $p \mid c$.

Problem 56. Prove that number $2^n + 1$ does not have prime divisor of the form $8k - 1$.

8. Class 8

8.1. Homework.

Problem 57. Let $p > 2$ be a prime. Compute

$$\left(\frac{1}{p}\right) + \left(\frac{2}{p}\right) + \ldots + \left(\frac{p-1}{p}\right).$$

Problem 58. Prove that the number $3^n + 1$ have no divisor of the form $12k + 11$.

Problem 59. Let a and b are integers such that a is different from 0 and the number $3 + a + b^2$ is divisible by $6a$. Prove that a is negative.

Problem 60. Let $x_1 = 7$ and

$$x_{n+1} = 2x_n^2 - 1, \quad \text{for } n \geq 1.$$

Prove that 2003 does not divide any term of the sequence.

Problem 61. Let $p > 2$ be a prime such that there exists integers x, y that

$$p = x^2 + xy + y^2.$$

Prove that $p = 3$ or $p \equiv 1 \pmod 3$.

Problem 62. Suppose that $p \equiv 1 \pmod 3$ is a prime. Using Thue's lemma prove that there exists integers $0 \leq x, y < \sqrt{p}$ (not both zero) such that $p \mid 3x^2 + y^2$. Conclude that there are integers a, b such that

$$p = a^2 + ab + b^2.$$

TEST

Problem 63. Compute
$$\left(\frac{-12000}{821}\right), \quad \left(\frac{2^{2019}}{953}\right).$$

Problem 64. Prove that 5 is quadratic residue modulo a prime $p > 2$ iff $p \equiv \pm 1 \pmod{10}$.

Problem 65. Find all integer solutions of the following equation
$$a^2 + b^2 + c^2 = 7d^2.$$

Problem 66. Prove that for any positive integer n every prime divisor p of number
$$n^4 - n^2 + 1$$
is of the form $12k + 1$.

Problem 67. Let a, b be a positive integers such that $a^2 + b^2 + ab$ is divisible by $ab - 2$. Find all possible values of
$$\frac{a^2 + b^2 + ab}{ab - 2}.$$

Problem 68. Find all integers such that
$$x^2 + 5 = y^3.$$

Problem 69. Find all positive integers such that $x^2 + 3y$ and $y^2 + 3x$ are squares.

Solutions

Problem 1. Find all integer solutions of
$$x^3 + 3y^3 + 9z^3 - 3xyz = 0.$$

Proof. From that equation you see that $3 \mid x^3$, so $x = 3x_1$. Putting it into given equation you will get
$$9x_1^3 + y^3 + 3z^3 - 3x_1yz = 0,$$
so $3 \mid y^3$ i.e. $y = 3y_1$. Then
$$3x_1^3 + 9y_1^3 + z^3 - 3x_1y_1z = 0,$$
so $3 \mid z^3$ i.e. $z = 3z_1$. Therefore
$$x_1^3 + 3y_1^3 + 9z_1^3 - 3x_1y_1z_1 = 0$$
which is the same equation as original one. Hence by Fermat's descend we get $x = y = z = 0$. □

Discussion.

Problem 2. Find all rationals a, b such that
$$a^2 + ab + b^2 = 2.$$

Proof. We can find integers x, $y \neq 0$, z such that $a = \frac{x}{y}$, $b = \frac{z}{y}$. Then
$$x^2 + xz + z^2 = 2y^2.$$
Easy to see that $2 \mid x, z$, so $x = 2x_1$ and $z = 2z_1$. Therefore
$$2x_1^2 + 2x_1z_1 + 2z_1^2 = y^2,$$
so $2 \mid y$ i.e. $y = 2y_1$, thus
$$x_1^2 + x_1z_1 + z_1^2 = 2y_1^2,$$
which is the same as original. Fermat's descend gives $x = y = z = 0$ – contradiction since $y \neq 0$. □

Discussion.

Problem 3. Solve in integers the following equation
$$y^4 = x^3 + 7.$$

Proof. Consider all possible residues modulo 13. RHS leads to residues

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$x^3 \pmod{13}$	0	1	8	1	12	8	8	5	5	1	12	5	12
$x^3+7 \pmod{13}$	7	8	12	8	6	2	2	12	12	8	6	12	6

while LHS produces the following residues

y	0	1	2	3	4	5	6	7	8	9	10	11	12
$y^4 \pmod{13}$	0	1	3	3	9	1	9	9	1	9	3	3	1

Both sets of residues are disjoint thus the equation has not integer solutions. $\square$

Discussion.

Problem 4. Solve in integers the following equation

$$x^5 = y^2 + 4.$$

Proof. We have $x^{10} \equiv 0, 1 \pmod{11}$; thus $x^5 \equiv -1, 0, 1 \pmod{11}$. Also, $y^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}$; thus $y^2 + 4 \equiv 2, 4, 5, 7, 8, 9 \pmod{11}$. Hence $y^2 + 4$ and x^{10} are different mod 11. $\square$

Discussion.

Problem 5. Find all solutions of the following equation in integers

$$x^2 + x + 1 = y^2.$$

Proof. If $x > 0$, then

$$(x+1)^2 > x^2 + x + 1 > x^2.$$

Thus $x^2 + x + 1$ lies between squares, hence cannot be a perfect square.

If $x \leq -2$, then

$$x^2 > x^2 + x + 1 > (x+1)^2,$$

and again we get a contradiction.

It remains to check $x = 0, -1$, which lead to solutions

$$(x, y) = (0, 1), (0, -1), (-1, 1), (-1, -1).$$

$\square$

Discussion.

Problem 6. Find all solutions of the following equation in integers

$$x^4 + y = x^3 + y^2.$$

Proof. We see that

$$x^4 + y = x^3 + y^2 \implies x^4 - x^3 = y^2 - y \implies 4x^4 - 4x^3 + 1 = (2y-1)^2.$$

But, whenever $x \geq 2$ or $x \leq -2$, then

$$(2x^2 - x - 1)^2 < 4x^4 - 4x^3 + 1 < (2x^2 - x)^2.$$

Therefore $(2y-1)^2$ it lies between 2 consecutive squares and cannot – contradiction.

So, $x \in \{-1, 0, 1\}$, this gives the solutions

$$(x, y) = (0,0), (0,1), (1,0), (1,1), (-1,2), (-1,-1).$$

□

Discussion.

Problem 7. Find all positive integers (a, b) for which $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are perfect cubes.

Proof. WLOG $a \leq b$, then

$$b^3 < b^3 + 6ab + 1 \leq b^3 + 6b^2 + 1 < b^3 + 6b^2 + 12b + 8 = (b+2)^3.$$

Since $b^3 + 6ab + 1$ is a perfect cube, we must have

$$b^3 + 6ab + 1 = (b+1)^3$$

or equivalently $2ab = b(b+1)$ i.e. $b = 2a - 1$.

It remains to check whether $a^3 + 6ab + 1$ is a cube if $b = 2a - 1$. Thus we need to find all integers a for which $a^3 + 12a^2 - 6a + 1$ is a cube. From the inequality

$$a^3 \leq a^3 + 6a^2 - 6a < a^3 + 12a^2 - 6a + 1 < a^3 + 12a^2 + 48a + 64 = (a+4)^3$$

we get that

$$a^3 + 12a^2 - 6a + 1 \in \{(a+1)^3, (a+2)^3, (a+3)^3\}$$

. Therefore we are left with three cases:

- $a^3 + 12a^2 - 6a + 1 = (a+1)^3$, then $9a^2 - 9a = 0$, so $a = 0$ or $a = 1$.
- $a^3 + 12a^2 - 6a + 1 = (a+2)^3$, then $6a^2 - 18a - 7 = 0$ – no solutions.
- $a^3 + 12a^2 - 6a + 1 = (a+3)^3$, then $3a^2 - 33a - 26 = 0$ – no solutions.

Finally $(a, b) = (1, 1)$ is the only pair satisfying given conditions. □

Discussion.

Problem 8. Solve in integers the following equation

$$2x^6 + y^7 = 11.$$

Proof. If we choose $p = 6 \cdot 7 + 1$ we see that there are only 7 nonzero residues of $y^6 \pmod{43}$ i.e. 1, 4, 11, 16, 21, 35, 41. Analogously, there are 6 nonzero residues of $y^7 \pmod{43}$ i.e. 1, 6, 7, 36, 37, 42. Easy to see that from above sets of residues the number $2x^6 + y^7$ cannot take $11 \pmod{43}$. □

Discussion.

Problem 9. Find all positive integers (k, m) for which $k^2 + 4m$ and $m^2 + 5k$ are perfect squares.

Proof. If $m \geq k$, then

$$(m+3)^2 = m^2 + 6m + 9 > m^2 + 5m \geq m^2 + 5k > m^2,$$

since $m^2 + 5k$ is a perfect square, it follows that $m^2 + 5k = (m+1)^2$ or $m^2 + 5k = (m+2)^2$.

If $m^2 + 5k = (m+1)^2 = m^2 + 2m + 1$, then $2m = 5k - 1$ and form problem condition $k^2 + 4m = k^2 + 2(5k - 1) = k^2 + 10k - 2$ is a perfect square. But $k^2 + 10k - 2 < k^2 + 10k + 25 = (k+5)^2$, so

$$k^2 + 10k - 2 \leq (k+4)^2 = k^2 + 8k + 16.$$

Therefore $2k \leq 18$ and $k \leq 9$. Since $2m = 5k - 1$, k must be odd. Values of $k^2 + 10k - 2$ at $k = 1, 3, 5, 7, 9$ are equal 9, 37, 73, 117, 169, respectively. Thus only $k = 1$ and $k = 9$ provide squares. Respective values of $m = \frac{1}{2}(5k - 1)$ are equal 2 and 22.

If$m^2 + 5k = (m+2)^2 = m^2 + 4m + 4$, then $4m = 5k - 4$, so $k^2 + 4m = k^2 + 5k - 4$ is a perfect square. But

$$k^2 + 5k - 4 < k^2 + 6k + 9 = (k+3)^2,$$

hence $k^2 + 5k - 4 \leq (k+2)^2 = k^2 + 4k + 4$, which gives $k \leq 8$. Moreover $m = \frac{5}{4}k - 1$ is an integer, so $4 \mid k$. Again $k^2 + 5k - 4$ for $k = 4, 8$ equals 32, 100, respectively and only for $k = 8$ we get a square. Also $m = \frac{5}{4}k - 1 = 9$.

It remains to consider the case $m < k$. Then

$$(k+2)^2 = k^2 + 4k + 4 > k^2 + 4k > k^2 + 4m > k^2,$$

and so $k^2 + 4m = (k+1)^2 = k^2 + 2k + 1$, thus $2k = 4m - 1$ – contradiction since $2 \nmid 4m - 1$.

Finally $(k, m) = (1, 1), (9, 22), (8, 9)$ are only pairs satisfying given conditions. □

Discussion.

Problem 10. Solve in integers the following equation

$$x^2 + y^2 = 3z^2.$$

Proof. Since $3 \mid x^2 + y^2$ we see that $3 \mid x$ and $3 \mid y$, so $x = 3x_1$ and $y = 3y_1$. After this substitution our equations is equivalent to $3x_1^2 + 3y_1^2 = z_2$, so $z = 3z_1$ and hence $x_1^2 + y_1^2 = 3z_1^2$. By Fermat descend $x = y = z = 0$. □

Discussion.

Problem 11. Solve in integers the following equation
$$x^2 + y^2 + z^2 - 2xyz = 0.$$

Proof. Since $2 \mid x^2 + y^2 + z^2$ so we have 0 or 2 odd number within x, y, z. If there are 2 odd numbers then we get
$$x^2 + y^2 + z^2 \equiv 2 \pmod 4,$$
but
$$4 \mid 2xyz = x^2 + y^2 + z^2$$
– contradiction. Therefore $2 \mid x$, $2 \mid y$ and $2 \mid z$ and Fermat's descend finishes problem. □

Discussion.

Problem 12. Solve in integers the following equation
$$x^4 + y^4 + z^4 = 9u^4.$$

Proof. If $5 \nmid u$ then
$$x^4 + y^4 + z^4 = 9u^4 \equiv 4 \pmod 5$$
from LFT, but again from LFT
$$x^4 + y^4 + z^4 \leq 3 \pmod 5.$$
Therefore $5 \mid u$. Thus $5 \mid x^4 + y^4 + z^4$ and from LFT we see that $5 \mid x$, $5 \mid y$, $5 \mid z$. By Fermat's descend we are done. □

Discussion.

Problem 13. Solve in integers the following equation
$$x^2 + y^2 + z^2 = x^2y^2.$$

Proof. Note that $x^2 \equiv 0, 1 \pmod 4$. If x, y, z are all odd, then
$$(xy)^2 = x^2 + y^2 + z^2 \equiv 3 \pmod 4$$
– impossible. If two of them are odd then
$$(xy)^2 = x^2 + y^2 + z^2 \equiv 2 \pmod 4$$
– impossible. If one of them is odd then
$$0 \equiv (xy)^2 = x^2 + y^2 + z^2 \equiv 1 \pmod 4$$
– impossible. Therefore $2 \mid x$, $2 \mid y$ and $2 \mid z$, so by Fermat's descend we are done. □

Discussion.

Problem 14. Prove that there are no positive integers a, b such that $2a^2+1$, $2b^2+1$, $2(ab)^2+1$ are all perfect squares.

Proof. Assume that such a, b exist. Clearly $a, b > 1$ and WLOG $a \geq b$. Then

$$4(2a^2+1)(2(ab)^2+1) = (4a^2b+b)^2 + 8a^2 - b^2 + 4$$

is a perfect square. But

$$(4a^2b+b)^2 < (4a^2b+b)^2+8a^2-b^2+4 < (4a^2b+b+1)^2 = (4a^2b+b)^2+8a^2b+2b+1.$$

□

Discussion.

Problem 15. Prove that for all positive integers n, the equation

$$x^2+y^2+z^2 = 59^n$$

is solvable in integers.

Proof. For $n = 1$ we have solution $(1, 3, 7)$, for $n = 2$ triple $(14, 39, 42)$ works. We prove by induction that from triple working for n we can construct triple which works for $n+2$. Let (x, y, z) be triple working for n. Then consider triple $(59x, 59y, 59z)$. Then

$$(59x)^2 + (59y)^2 + (59z)^2 = 59^2 \cdot (x^2+y^2+z^2) = 59^{n+2},$$

so this triple works for $n+2$. □

Discussion.

Problem 16. Prove that for all $n \geq 6$ the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = 1$$

is solvable in distinct integers.

Proof. For $n = 3$ we have

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

Suppose that we have distinct numbers $x_1, \ldots, x_n$ for $n \geq 3$. Then

$$1 = \frac{1}{2} + \frac{1}{2x_1} + \frac{1}{2x_2} + \ldots + \frac{1}{2x_n}$$

is the expression of 1 which uses $n+1$ distinct numbers $2, 2x_1, 2x_2, \ldots, 2x_n$. Done by induction □

Discussion.

Problem 17. Let a, b be positive integers such that $ab+1$ divides a^2+b^2. Prove that

$$\frac{a^2+b^2}{ab+1}$$

is a perfect square.

Proof. Consider the following equation with fixed positive integer k

$$\frac{a^2+b^2}{ab+1}=k. \tag{1}$$

Let $\mathcal{A}$ be a set of all pairs (a,b) of nonegative integers a and b such that (1) holds i.e.

$$\mathcal{A}=\left\{(a,b)\in\mathbb{N}\times\mathbb{N}\colon \frac{a^2+b^2}{ab+1}=k\right\}.$$

Suppose that k is not a perfect square.

Let $(a_0,b_0)\in\mathcal{A}$ be an element of $\mathcal{A}$ with a minimal sum a_0+b_0 among all elements of $\mathcal{A}$. We may assume that $a_0\geq b_0>0$.

The equations

$$\frac{x^2+b_0^2}{xb_0+1}=k,$$

is equivalent to a quadratic equation in x

$$x^2-kb_0x+b_0^2-k=0. \tag{2}$$

Note that $x_1=a_0$ is a root of (2). From *Vieta's formulas* we get another root x_2 of (2) i.e.

$$x_2=kb_0-a_0=\frac{b_0^2-k}{a_0}.$$

From (2) follows that, x_2 is a nonzero integer $x_2\neq 0$, (otherwise $k=b_0^2$ which contradicts to assumption about k.)

Moreover $x_2>0$. Indeed, if $x_2<0$ then

$$0=x_2^2-kb_0x_2+b_0^2-k\geq x_2^2+k+b_0^2-k>0,$$

contradiction. Therefore $x_2\geq 0$, hence $(x_2,b_0)\in\mathcal{A}$. By the formula (2) and inequality $a_0\geq b_0$ we have

$$x_2=\frac{b_0^2-k}{a_0}\leq\frac{a_0^2-k}{a_0}<a_0.$$

It means that $x_2+b_0<a_0+b_0$which contradicts to minimality of a_0+b_0. $\square$

Discussion.

Problem 18. Prove that for all positive integers n, the equation

$$x^2+xy+y^2=7^n$$

is solvable in integers.

Proof. Notice that if (x, y) is solution of the above equation for n, then $(7x, 7y)$ is solution for $n+2$. Indeed

$$(7x)^2 + 7x \cdot 7y + (7y)^2 = 7^2 \cdot (x^2 + xy + y^2) = 7^2 \cdot 7^n = 7^{n+2}.$$

Therefore it is enough to find solutions for $n = 1$, 2 i.e. $(1, 2)$ and $(3, 5)$.

Alternatively we can also notice that if (x, y) is a solution for n, then $(2x-y, x+3y)$ is solution for $n+1$. $\square$

Discussion.

Problem 19. Prove that for all $n \geq 6$ the equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2} = 1$$

is solvable in integers.

Proof. Note that

$$\frac{1}{a^2} = \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2}.$$

Hence in the fixed solution

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2} = 1$$

for n, we can put

$$\frac{1}{x_n^2} = \frac{1}{(2x_n)^2} + \frac{1}{(2x_n)^2} + \frac{1}{(2x_n)^2} + \frac{1}{(2x_n)^2}$$

to obtain solution for $n+3$. By induction, it is enough to find solutions for 6, 7, 9 i.e. respectively

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \frac{1}{36} = 1,$$
$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = 1,$$
$$\frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{36} + \frac{1}{36} = 1.$$

$\square$

Discussion.

Problem 20. Solve the following equation in positive integers:

$$x^2 - y^2 = 2xyz.$$

Proof. If $\gcd(x, y) = 1$, then $x \mid y_2 \implies x = 1$. Similarly $y = 1$.

If $\gcd(x, y) = d > 1$, then $x = dx_1$, $y = dy_1$ and so

$$x_1^2 - y_1^2 = 2x_1y_1z,$$

where $\gcd(x_1, y_1) = 1$. Therefore $x_1 = y_1 = 1$ and so $x = y$. $\square$

Discussion.

Problem 21 (Baltic Way 2013)**.** Decide whether the equation $x^4+y^3=z!+7$ has an infinite number of positive integer solutions.

Proof. We prove that for $z \geq 13$, the given equation has no integer solutions. Indeed, if $z \geq 13$ and $x^4+y^3=z!+7$, then $x^4+y^3 \equiv 7 \pmod{13}$. Consider all possible residues modulo 13 of $7-x^4$ and y^3:

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$x^4 \pmod{13}$	0	1	3	3	9	1	9	9	1	9	3	3	1
$7-x^4$	7	6	4	4	11	6	11	11	6	11	4	4	6

y	0	1	2	3	4	5	6	7	8	9	10	11	12
$y^3 \pmod{13}$	0	1	8	1	12	8	8	5	5	1	12	5	12

From these tables we read that $x^4+y^3 \not\equiv 7 \pmod{13}$.

Therefore the equation $x^4+y^3=z!+7$ forces $z \leq 12$. Thus $x \leq x^4 \leq 12!+7$ and $y \leq y^3 \leq 12!+7$. It means that the number of solution is finite. $\square$

Discussion.

Problem 22 (IMO 2007)**.** Suppose that a, b are positive integers such that $4ab-1$ divides $(a-b)^2$. Prove that $a=b$.

Proof. Consider the following equation with fixed positive integer k

$$\frac{(a-b)^2}{4ab-1}=k. \tag{3}$$

Let $\mathcal{A}$ be a set of all pairs (a,b) of nonegative integers a and b such that (1) holds i.e.

$$\mathcal{A}=\left\{(a,b)\in\mathbb{N}\times\mathbb{N}\colon \frac{(a-b)^2}{4ab-1}=k\right\}.$$

Let $(a_0,b_0)\in\mathcal{A}$ be an element of $\mathcal{A}$ with a minimal sum a_0+b_0 among all elements of $\mathcal{A}$. We may assume that $a_0 \geq b_0 > 0$.

The equations

$$\frac{(x+b_0)^2}{4xb_0-1}=k,$$

is equivalent to a quadratic equation in x

$$x^2-(2b_0+4kb_0)x+b_0^2+k=0. \tag{4}$$

Note that $x_1=a_0$ is a root of (2). From *Vieta's formulas* we get another root x_2 of (4) i.e.

$$x_2=2b_0+4kb_0-a_0=\frac{b_0^2+k}{a_0}. \tag{5}$$

From (5) follows that, x_2 is a positive integer, hence $(x_2,b_0)\in\mathcal{A}$.

Now we have the following inequality

$$k = \frac{(a_0-b_0)^2}{4a_0b_0-1} \le \frac{(a_0-b_0)(a_0+b_0)}{4a_0b_0-1} = \frac{(a_0^2-b_0^2)}{4a_0b_0-1} < a_0^2-b_0^2.$$

Therefore

$$x_2 = \frac{b_0^2+k}{a_0} \le \frac{b_0^2+(a_0^2-b_0^2)}{a_0} = a_0,$$

which means that $x_2+b_0 < a_0+b_0$ – contradiction. Thus $a_0 = b_0$ and $k = 0$ i.e. $a = b$. $\square$

Discussion.

Problem 23 (Iran 2013)**.** Suppose that a, b are two odd positive integers such that $2ab+1 \mid a^2+b^2+1$. Prove that $a = b$.

Proof. Note that $2ab+1 \mid a^2+b^2+1$ implies that $2ab+1 \mid (a-b)^2$.

Now consider the positive integer solution set (a, b) of the equation

$$\frac{(a-b)^2}{2ab+1} = k$$

where k is a fixed positive integer. Let (a_0, b_0) be a solution for which the sum is minimal. Without loss of generality let $a_0 > b_0$. Now we consider another equation

$$\frac{(x-b_0)^2}{2xb_0+1} = k \iff x^2 - 2xb_0(k+1) + {b_0}^2 - k = 0.$$

Obviously one of the roots is a_0. The other root

$$x_2 = 2b_0(k+1) - a_0 = \frac{{b_0}^2-k}{a_0}.$$

Easy to see that x_2 is positive and odd (a_0 is odd).

If $x_2 \le -1$, then

$$\frac{{b_0}^2-k}{a_0} \le -1 \implies k \ge {b_0}^2 + a_0$$

i.e.

$$\frac{(a_0-b_0)^2}{2a_0b_0+1} = k \ge {b_0}^2 + a_0,$$

contradiction. Therefore x_2 is odd positive integer, so (x_2, b_0) is also a solution.

But

$$\frac{{b_0}^2-k}{a_0} < \frac{{b_0}^2}{a_0} < a_0 \Longrightarrow a_0+b_0 > a_0+x_2$$

– contradiction to our assumption. Therefore $a_0 = b_0$, so $k = 0$ i.e. $a = b$. $\square$

Discussion.

Problem 24. Prove that the equation
$$6(6a^2 + 3b^2 + c^2) = 5n^2$$
has no solutions in integers except $a = b = c = n = 0$

Proof. Note that we can assume that $\gcd(a, b, c, n) = 1$. Since $6 \mid n$ and $3 \mid c$, we put $n = 6m$, $c = 3d$ and get
$$2a^2 + b^2 + 3d^2 = 10m^2.$$
Since $x^2 \equiv 0, 1, 4 \pmod 8$ we see that
$$2a^2 + b^2 + 3d^2 = 10m^2 \equiv 0, 2 \pmod 8.$$
It can be possible only if $2 \mid b^2$ and $2 \mid 3d^2$, so b and d are even and hence c is even i.e. $c = 2s$, $b = 2r$. Then from the original equation we get
$$36a^2 + 72r^2 + 24s^2 = 180m^2,$$
so $8 \mid 36a^2$, so a is even therefore a, b, c, n are even – contradiction with $\gcd(a, b, c, n) = 1$. $\square$

Discussion.

Problem 25. Let a, b be positive integers such that ab divides $a^2 + b^2 + a + b + 1$. Prove that
$$\frac{a^2 + b^2 + a + b + 1}{ab} = 5.$$

Proof. $\square$

WLOG we can assume $a \geq b$. If $a = b$, then $a^2 \mid 2a^2 + 2a + 1$, so $a^2 \mid 2a + 1$, so $a = 1$ and hence the quotient is equal to 5.

Therefore take (a, b) solution with $a > b$ and minimal sum Suppose
$$\frac{a^2 + b^2 + a + b + 1}{ab} := k \neq 5.$$
Then we have quadratic equation
$$a^2 - a(kb - 1) + b^2 + b + 1 = 0.$$
From Vieta's formulas we get another root of the above equation:
$$x_2 = kb - 1 - a = \frac{b^2 + b + 1}{a}.$$
Easy to see that x_2 is positive integer. Therefore (x_2, b) is also a solution, so $x_2 + b \geq a + b$, hence $x_2 \geq a$ i.e.
$$\frac{b^2 + b + 1}{a} \geq a \implies b^2 + b + 1 \geq a^2,$$
but $a \geq b + 1$, so
$$b^2 + b + 1 \geq a^2 \geq b^2 + 2b + 1,$$

contradiction.

Discussion.

Problem 26. Let a_1, a_2, $\ldots$, a_{2n+1} be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove that $a_1 = a_2 = \ldots = a_{2n+1}$.

Proof. Let S denotes the sum of all numbers. Then for any i, the number $S - a_i$ is even, so a_i have the same parity for $i \in \{1, 2, \ldots, 2n+1\}$.

If all of them are even, then numbers $\frac{a_1}{2}$, $\frac{a_2}{2}$, $\ldots$, $\frac{a_{2n+1}}{2}$ also satisfies problems conditions sa by descent $a_1 = a_2 = \ldots = a_{2n+1}$.

If all of them are odd then $\frac{a_1+1}{2}$, $\frac{a_2+1}{2}$, $\ldots$, $\frac{a_{2n+1}+1}{2}$ also satisfies problems conditions sa by descent $a_1 = a_2 = \ldots = a_{2n+1}$.

□

Discussion.

Problem 27. Positive integers a, b satisfy $ab \mid a^2 + b^2 + 1$. Prove that

$$\frac{a^2 + b^2 + 1}{ab} = 3.$$

Proof. WLOG we can assume $a \geq b$. If $a = b$, then $a^2 \mid 2a^2 + 1$, so $a^2 \mid 1$, so $a = 1$ and hence the quotient is equal to 3.

Therefore take (a, b) solution with $a > b$ and minimal sum Suppose

$$\frac{a^2 + b^2 + 1}{ab} := k \neq 3.$$

Then we have quadratic equation

$$a^2 - a(kb - 1) + b^2 + 1 = 0.$$

From Vieta's formulas we get another root of the above equation:

$$x_2 = kb - a = \frac{b^2 + 1}{a}.$$

Easy to see that x_2 is positive integer. Therefore (x_2, b) is also a solution, so $x_2 + b \geq a + b$, hence $x_2 \geq a$ i.e.

$$\frac{b^2 + 1}{a} \geq a \implies b^2 + 1 \geq a^2,$$

but $a \geq b + 1$, so

$$b^2 + 1 \geq a^2 \geq b^2 + 2b + 1,$$

contradiction.

□

Discussion.

Problem 28. Find all positive integers a, b such that $ab + a + b$ divides $a^2 + b^2 + 1$.

Proof. Let
$$\frac{a^2+b^2+1}{ab+a+b} := k.$$
If $k = 1$, then we get
$$(a-1)^2 + (b-1)^2 + (a-b)^2 = 0,$$
so $a = b = 1$.

If $k = 2$, then we get $4a = (b-a-1)^2$, so $a = d^2$ and hence $b = (d \pm 1)^2$.

Suppose $k \geq 3$. Consider quadratic equation
$$a^2 - a(kb+1) + b^2 + 1 - kb = 0.$$
Then
$$x_2 = kb + 1 - a = \frac{b^2+1-kb}{a}.$$
If $x_2 \leq -1$, then
$$\frac{b^2+1-kb}{a} \leq -1 \implies k \geq \frac{b^2+a+1}{b}$$
and so
$$\frac{a^2+b^2+1}{ab+a+b} = k \geq \frac{b^2+a+1}{b},$$
contradiction (RHS is too large). Therefore $x_2 \geq 0$. Of course $x_2 \neq 0$, otherwise $b^2 + 1 = kb$ and so $b \mid 1$.

Take (a, b) minimal sum solution of the above equation. Then $x_2 + b \geq a + b$, so $x_2 \geq a$ i.e.
$$\frac{b^2+1-kb}{a} \geq a,$$
so
$$b^2 + 1 - kb \geq a^2 \implies b^2 + 1 - a^2 \geq kb.$$
But $k \geq 3$, so
$$b^2 + 1 - 3b \geq b^2 + 1 - kb \geq a^2 \geq b^2,$$
so $3b \leq 1$, contradiction.

$\square$

Discussion.

Problem 29. Positive integers a, b satisfy $ab - 1 \mid a^2 + b^2$. Prove that
$$\frac{a^2+b^2}{ab-1} = 5.$$

Proof. WLOG we can assume $a \geq b$. If $a = b$, then $a^2 - 1 \mid 2a^2$, so $a^2 - 1 \mid 2$ – contradiction.

If $b = 1$, then $a - 1 \mid a^2 + 1$, so $a - 1 \mid 2$, so $a \in \{2, 3\}$ and corresponding quotient is equal 3.

Therefore take (a, b) solution with $a > b > 1$ and minimal sum. Suppose

$$\frac{a^2 + b^2}{ab - 1} := k \neq 3.$$

Then we have quadratic equation

$$a^2 - a \cdot kb + b^2 + k = 0.$$

From Vieta's formulas we get another root of the above equation:

$$x_2 = kb - a = \frac{b^2 + k}{a}.$$

Easy to see that x_2 is positive integer. Therefore (x_2, b) is also a solution, so $x_2 + b \geq a + b$, hence $x_2 \geq a$ i.e.

$$\frac{b^2 + k}{a} \geq a \implies b^2 + k \geq a^2 \implies k \geq a^2 - b^2.$$

Therefore

$$\frac{a^2 + b^2}{ab - 1} = k \geq a^2 - b^2 \implies a^2 + b^2 \geq a^3 b - ab^3 - a^2 + b^2 \implies 2a \geq b(a^2 - b^2).$$

But $b \leq a - 1$, so

$$2a \geq b(a^2 - b^2) \geq b(a^2 - (a - 1)^2) = b(2a - 1) \geq 2(2a - 1),$$

so $a = 1$ – contradiction.

$\square$

Discussion.

Problem 30. Find all positive integers m, n such that $mn - 1$ divides $(n^2 - n + 1)^2$.

Proof. Note that

$$0 \equiv (n^2 - n + 1)^2 \equiv (n^2 - n + 1 + mn - 1)^2 \equiv (n^2 + mn - n)^2 \equiv n^2(m + n - 1)^2 \pmod{mn - 1},$$

so $mn - 1 \mid (m + n - 1)^2$ (since $\gcd(mn - 1, n^2) = 1$).

WLOG we can assume $a \geq b$. If $a = b$, then $a^2 - 1 \mid 2a^2$, so $a^2 - 1 \mid 2$ – contradiction.

Take (m, n) solution with $m \geq n$ and minimal sum. Suppose

$$\frac{(m + n - 1)^2}{mn - 1} := k.$$

We prove that $k \mid \{3, 4\}$.

We have the following quadratic equation

$$m^2 - m \cdot (kn + 2 - 2n) + (n - 1)^2 + k = 0.$$

From Vieta's formulas we get another root of the above equation:

$$x_2 = kn + 2 - 2n - m = \frac{(n-1)^2 + k}{m}.$$

Easy to see that x_2 is positive integer. Therefore (x_2, b) is also a solution, so $x_2 + n \geq m + n$, hence $x_2 \geq m$ i.e.

$$\frac{(n-1)^2 + k}{m} \geq m \implies (n-1)^2 + k \geq m^2 \implies k \geq (m-n+1)(m+n-1).$$

Suppose that $m > n$. Then

$$k \geq (m-n+1)(m+n-1) \geq 2(m+n-1),$$

so

$$\frac{(m+n-1)^2}{mn-1} = k \geq 2(m+n-1) \implies m+n-1 \geq 2mn-2 \implies (2n-1)(m+1) \leq 2n.$$

But $m > 1$, so

$$3n \geq (2n-1)(m+1) > 4n-2,$$

so $n = 1$. Therefore $m - 1 \mid m^2$ i.e. $m = 2$, and so

$$k = \frac{(2+1-1)^2}{2 \cdot 1 - 1} = 4.$$

If $m = n$, then $m^2 - 1 \mid (2m-1)^2$ i.e. $m^2 - 1 \mid |-4m+5|$, so $m = 2$ and then $m = n = 2$. Therefore

$$k = \frac{(2+2-1)^2}{2 \cdot 2 - 1} = 3.$$

Suppose $k = 3$. Then

$$\frac{(m+n-1)^2}{mn-1} = 3$$

i.e.

$$(m-n)^2 + (m-2)^2 + (n-2)^2 = 0,$$

so $m = n = 2$.

Suppose $k = 4$. Then

$$(m-n)^2 - 2(m+n) + 5 = 0.$$

Take $m - n = 2t - 1$ for some integer t. Then $m + n = 2t^2 - 2t + 3$, so $m = t^2 + 1$, $n = t^2 - 2t + 2$.

□

Discussion.

Problem 31. Find all positive integers a, b such that $a \mid b^2 + 1$ and $b \mid a^2 + 1$.

Proof. Easy to see that $\gcd(a,b) = 1$, so from $a \mid a^2+b^2+1$ and $b \mid a^2+b^2+1$, we see that $ab \mid a^2+b^2+1$. By problem 27 we have $a^2+b^2+1 = 3ab$.

By Vieta we know that if (a,b) is a solution, then $(3b-a,b)$ is also a solution, we can start with the base case, then flip the root (since it's symmetric) and obtain infinitely many solution. the base case is $(1,1)$. so we can do the transformation $(a,b) \to (3b-a,b)$ like the following: $(1,1) \to (2,1)$, then we "flip" $(1,2) \to (5,2)$, flip again $(2,5) \to (13,5)$ and it goes on like this:

$$(1,1) \to (2,1) \to (5,2) \to (13,5) \to (34,13) \to (89,34) \to \dots$$

now consider this sequence, $1,1,2,5,13,34,89,\dots$ as $a_1,a_2,\dots$ then we can see that (a_i,a_{i+1}) will be a solution to the equation above, the sequence is defined by the transformation $(a,b) \to (3b-a,b)$, in terms of sequence, It's defined like this.

$$a_n = 3a_{n-1} - a_{n-2}$$

for $n > 2$ where $a_1 = 1$ and $a_2 = 1$ (base case)

the formula is

$$a_n = \frac{10-4\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{10+4\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^n$$

□

Discussion.

Problem 32. Let a and b be positive integers, such that $ab-1$ divides a^2+b^2+ab. Prove that

$$\frac{a^2+b^2+ab}{ab-1} \in \{4,7\}.$$

Proof. Let

$$\frac{a^2+b^2+ab}{ab-1} = k \implies \frac{a^2+b^2+1}{ab-1} = k-1.$$

WLOG $a \geq b$ and $a+b$ has minimal sum.

If $a = b$ then

$$\frac{2a^2+1}{a^2-1} = k-1 = 2 + \frac{3}{a^2-1} \implies a = 2 \quad \text{and } k-1 = 3 \implies k = 4.$$

Let $a > b$ and consider quadratic equation in a

$$a^2 - (k-1)ab + b^2 + k = 0.$$

Another solution of this equation is

$$x_2 := b(k-1) - a = \frac{b^2+k}{a},$$

so x_2 is positive integer, so $x_2 + b \geq a + b \implies x_2 \geq a$. Therefore

$$a \leq b(k-1) - a \implies k-1 \geq \frac{2a}{b},$$

so

$$\frac{a^2+a^2+1}{ab-1} = k-1 \geq \frac{2a}{b} \implies b^3+b \geq a^2b-2a \geq (b+1)^2b-2(b+1) \implies$$
$$\implies 2b^2-2b-2 \leq 0 \implies b=1.$$

Thus

$$k = \frac{a^2+2}{a-1}+1 = a+2+\frac{3}{a-1} \implies a=2,4 \implies k=7.$$

□

Discussion.

Problem 33 (IMO 2007)**.** Let a and b be positive integers, such that $4ab-1$ divides $(4a^2-1)^2$. Prove that $a=b$.

Proof. Observe that

$$4ab-1 \mid b^2(4a^2-1)^2-(4ab-1)(4a^3b-2ab+a^2) = (a-b)^2.$$

Thus we are done by 22.

□

Discussion.

Problem 34 (AMM)**.** Let a, b, c and m be positive integers such that

$$abcm = 1+a^2+b^2+c^2.$$

Prove that $m=4$.

Proof. Viewing the equation modulo 4 shows that 4 divides m. Let $n=m/4$. Now suppose there is a solution with $n>1$. Let (a,b,c) be such a solution where $a+b+c$ is minimal. Name the values so that $a \geq b \geq c$.

Now a is a solution to the quadratic equation

$$x^2-x(4bcn)+(b^2+c^2+1)=0.$$

By Vietas formula, another solution is

$$x_2 = 4bcn-a.$$

If $x_2 \geq a$, then

$$a^2+b^2+c^2+1 = 4abcn \geq 2a^2,$$

and so

$$a^2 \leq b^2+c^2+1 \leq 2b^2+1.$$

Now

$$a^2 < a^2+1 \leq 2b^2+2 \leq 4b^2,$$

so $a<2b$. This yields

$$4abcn > 2a^2cn \geq 4a^2 \geq a^2+b^2+c^2+1,$$

which contradicts (a,b,c) being a solution.

Thus (x_2, b, c) is a solution that contradicts the minimality of $a+b+c$. We conclude that $n > 1$ is impossible, so $n = 1$ and $m = 4$.

□

Discussion.

Problem 35. Find all integer solutions of the following system of equations

$$\begin{cases} x^2 + 6y^2 = z^2 \\ 6x^2 + y^2 = t^2 \end{cases}$$

Proof. Adding these equations gives

$$7x^2 + 7y^2 = z^2 + t^2.$$

Therefore $7 \mid z^2 + t^2$, so $7 \mid z$ and $7 \mid t$, i.e. $z = 7z_1$ and $t = 7t_1$, so

$$7x^2 + 7y^2 = 49z_1^2 + 49t_1^2 \implies x^2 + y^2 = 7z_1^2 + 7t_1^2.$$

Similarly $x = 7x_1$ and $y = 7y_1$, so

$$49x_1^2 + 49y_1^2 = 7z_1^2 + 7t_1^2 \implies 7x_1^2 + 7y_1^2 = z_1^2 + 7t_1^2.$$

By Fermat's descent we are done with $x = y = z = t = 0$.

□

Discussion.

Problem 36 (Kolmogorov Cup)**.** Define a sequence $(a_n)_{n\geq 1}$ by setting $a_1 = 2$ and

$$a_{n+1} = 2^{a_n} + 2$$

for $n \geq 1$. Prove that a_n divides a_{n+1} for $n \geq 1$.

Proof. By induction prove statement:

$$a_n \mid a_{n+1} \quad \text{and} \quad a_n - 1 \mid a_{n+1} - 1.$$

□

Discussion.

Problem 37. Let p be a prime of the form $4k+3$ such that $p \mid a^2 + b^2$. Prove that $p \mid a$ and $p \mid b$.

Proof. Suppose that $p \nmid a$. Then $p \nmid b$, so using MTF

$$a^2 \equiv -b^2 \pmod{p} \implies a^{p-1} \equiv (-1)^{\frac{p-1}{2}} b^{p-1} \pmod{p} \implies 1 \equiv -1 \pmod{p},$$

contradiction.

□

Discussion.

Problem 38. Prove that there are no positive integers m, n such that

$$4mn - m - n$$

is a square.

Proof. Suppose that

$$4mn - m - n = x^2,$$

then

$$(4m-1)(4n-1) = (2x)^2 + 1.$$

But the number $4m-1$ has a prime divisor p of the form $4\ell+3$, so by 38 we have that $p \mid 2x$ and $p \mid 1$ – contradiction.

$\square$

Discussion.

Problem 39. Solve in integers the following equation

$$x^2 + 4 = y^5.$$

Proof. If x is even, then y too, but then $x^2+4 \equiv 4, 8 \pmod{16}$ and $y^5 \equiv 0 \pmod{16}$ – contradiction. Therefore x is odd, then $x^2+4 \equiv 1 \pmod 4$, so $y \equiv 1 \pmod 4$. Hence

$$x^2 + 6^2 = y^5 + 2^5,$$

and $y+2 \mid y^5+2^5$, so $y+2 \mid x^2+6^2$, but $y+2 \equiv 3 \pmod 4$, so there exists prime $p \equiv 3 \pmod 4$ of odd exponent ($\gcd(y+2, \frac{y^5+2^2}{y+2} = 1)$ which divides x^2+6^2 – contradiction. $\square$

Discussion.

Problem 40. Solve in integers the following equation

$$x^3 + 7 = y^2.$$

Proof. If x is even then y is odd, so $y^2 \equiv 1 \pmod 8$. Therefore $x^3 \equiv 2 \pmod 8$ – contradiction. If x is odd, then y is even, so $y^2 \equiv 0 \pmod 4$, so $x^3 \equiv 1 \pmod 4$ and thus $x \equiv 1 \pmod 4$. But then the number

$$y^2 + 1 = x^3 + 2^3$$

is divisible by $x+2 \equiv 3 \pmod 4$ – contradiction.

$\square$

Discussion.

Problem 41. Prove that the equation

$$3^k - 1 = m^2 + n^2$$

has infinitely many solutions in positive integers.

Proof. Note that

$$3^{2^\ell} - 1 = (1^2 + 1^2) \cdot 2^2 \cdot (3^2 + 1) \cdot \ldots \cdot (3^{2^{\ell-1}} + 1),$$

which is a sum of 2 squares because all factors are sum of two squares.

$\square$

Discussion.

Problem 42. Prove that the equation

$$x^4 - 4 = y^2 + z^2$$

does not have integer solutions.

Proof. Note that

$$x^4 - 4 = (x^2 + 2)(x^2 + 2).$$

If x is odd, then $x^2 + 2 \equiv 3 \pmod 4$, so there exists prime $p \equiv 3 \pmod 4$ with odd exponent in the prime decomposition of $x^2 + 2$. But $p \nmid x^2 - 2$ (otherwise $p = 2$), so p has odd exponent in $x^4 - 4$ – cannot divide sum of two squares.

If $x = 2k$ is even, then

$$x^4 - 4 = 4(2k^2 + 1)(2k^2 - 1).$$

If k is even, then $2k^2 - 1 \equiv 3 \pmod 4$, so the above argument also works, if k is odd then $2k^2 + 1 \equiv 3 \pmod 4$, so the above argument again works.

$\square$

Discussion.

Problem 43. Prove that

$$x^8 + 1 = n!$$

has only finitely many solutions in nonnegative integers.

Proof. Note that

$$n! = (x^4)^2 + 1$$

cannot have divisor of the form $4k + 3$, so $n \leq 3$ and so we have only finitely many solutions.

$\square$

Discussion.

Problem 44. Find all n-tuples $(a_1, a_2, \ldots, a_n)$ of positive integers such that
$$(a_1! - 1)(a_2! - 1)\ldots(a_n! - 1) - 16$$
is a perfect square.

Proof. Suppose
$$(a_1! - 1)(a_2! - 1)\ldots(a_n! - 1) = k^2 + 4^2.$$
Of course $a_i \neq 1$. If $a_i > 3$, then $a_i! - 1 \equiv 3 \pmod 4$, so some $p \equiv 3 \pmod 4$ divides $k^2 + 4^2$ – contradiction. Therefore $a_i \in \{2, 3\}$. Let m be th number of 3's, then the equations can be transform into $5^m - 16 = k^2$. As k is odd, modulo 8 argument shows that m is even. Therefore $m = 2s$, so
$$(5^s - k)(5^s + k) = 16.$$
Easy to see that $s = 1$, so $m = 2$.

□

Discussion.

Problem 45. Find all pairs (m, n) of positive integers such that
$$m^2 - 1 \mid 3^m + (n! - 1)^m.$$

Proof. Assume $n > 2$. If m is odd, then $8 \mid m^2 - 1$, but $3^m + (n! - 1)^m$ is odd. Therefore m is even. But $m^2 - 1 \equiv 3 \pmod 4$. So exists prime $p \equiv 3 \pmod 4$ such that $p \mid 3^m + (n! - 1)^m$ and since m is even, $p \mid 3$ and $n! - 2$, so $3 \mid n! - 2$ – contradiction.

Hence $n \leq 2$. If $n = 1$, then either $m^2 - 1 \mid 3^m + 1$ and m is even or $m^2 - 1 \mid 3^m - 1$ and m is odd. In the first case the same argument as previously gives contradiction since $m^2 - 1 \equiv 3 \pmod 4$. Second case is impossible since $3^m - 1$ is not a multiple of 8 when m is odd.

Thus $m = 2$ and $m^2 - 1 \mid 3^m$. So for some $k \leq m$ we have $(m - 1)(m + 1) = 3^k$, but then $m - 1$ and $m + 1$ are powers of three which differ by 2. Thus $m - 1 = 1$, so $m = 2$.

□

Discussion.

Problem 46. Solve in integers the equation
$$x^2 = y^7 + 7.$$
square.

Proof. We clearly have no solutions for $y < -1$, thus assume $y + 2 > 0$. Looking at the equation $\mod 4$, we get that $y \equiv 1 \mod 4$ (if $y \equiv 0, 2 \mod 4$, then $x^2 \equiv$

$7 \equiv -1 \mod 4$, impossible; if $y \equiv -1 \mod 4$, then $x^2 \equiv -1+7 \equiv 2 \mod 4$, impossible). Rewrite it as $x^2+11^2 = y^7+2^7$ now. Factoring the left hand side gives

$$x^2 + 11^2 = (y+2)(y^6 + 2y^5 + 4y^4 + 8y^3 + 16y^2 + 32y + 64).$$

Note that x^2+11^2 is a sum of two squares, thus every prime factor $p \equiv 3 \mod 4$ of that number must occur an even number of times.

Also note that

$$\gcd(y+2, y^6 + 2y^5 + 4y^4 + 8y^3 + 16y^2 + 32y + 64) = \gcd(y+2, 7\cdot 64) \mid 64$$

cannot contain such a factor $\equiv 3 \mod 4$. But $y \equiv 1 \mod 4$ gives $y+2 \equiv 3 \mod 4$, thus $y+2$ has an odd number of prime factors that are $3 \mod 4$. So one of them, call it q, occurs an odd number of times in $y+2$, and q doesn't occur in $y^6+2y^5+4y^4+8y^3+16y^2+32y+64$. In total, q occurs an odd number of times in $y^7+2^7 = x^2+11^2$, a contradiction.

$\square$

Discussion.

Problem 47. Solve in integers the equation

$$y^3 - 9 = x^2.$$

Proof. If x is odd, then y is even so $x^2+9 \equiv 2 \pmod 4$ and $y^3 \equiv 0 \pmod 4$ – contradiction.

Therefore x is even, so $x^2+9 \equiv 1 \pmod 4$. Hence $y \equiv 1 \pmod 4$, so $x^2+1 = y^3-8$ has divisor $y-2 \equiv 3 \pmod 4$.

$\square$

Discussion.

Problem 48. Prove that a positive integer can be written as the sum of two perfect squares if and only if it can be written as the sum of the squares of two rational numbers.

Proof. One implication is trivial. Suppose that

$$n = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2,$$

hence $a^2+b^2 = nc^2$, so for any $p \equiv 3 \pmod 4$ we have

$$v_p(n) = v_p(a^2+b^2) - 2v_p(c),$$

so is even (since $2 \mid v_p(a^2+b^2)$), therefore n is sum of two squares of integers.

$\square$

Discussion.

Problem 49. Prove that
$$\frac{x^2+1}{y^2-5}$$
is not an integer for any integers $x, y > 2$.

Proof. If y is even, y^2-5 is of the form $4k+3$, and thus cannot divide x^2+1.

If y is odd, then y^2-5 is divisible by 4, while x^2+1 is never a multiple of 4. □

Discussion.

Problem 50. Prove that each prime p of the form $4k+1$ can be represented in exactly one way as the sum of the squares of two integers, up to the order and signs of the terms.

Proof. Suppose there are two solutions $p = a^2+b^2 = c^2+d^2$ for positive integers a, b, c, d. WLOG assume $a > c$. Subtracting from both sides and factoring gives
$$(a-c)(a+c) = (d-b)(d+b).$$
A factoring lemma (4 numbers theorem) says that there exist positive integers w, x, y, z such that
$$a-c = xy$$
$$a+c = wz$$
$$d-b = xw$$
$$d+b = yz$$
Therefore $a = \dfrac{xy+wz}{2}$ and $b = \dfrac{yz-xw}{2}$. Plugging back in gives
$$4p = (x^2+z^2)(y^2+w^2).$$
Of course p must divide one of the sums on the left, so we have two cases

- If $p \mid y^2+w^2$, then $x^2+z^2 \mid 4$, which only has the solution $x = z = 1$. This gives $a = d = \dfrac{y+w}{2}$ and $b = -c = \dfrac{y-w}{2}$.
- If $p \mid x^2+z^2$, then $y^2+w^2 \mid 4$, which only has the solution $y = w = 1$. This gives $a = d = \dfrac{x+z}{2}$ and $b = c = \dfrac{z-x}{2}$

Both cases yield the necessary contradiction.

□

Discussion.

Problem 51. Prove that there are infinitely many pairs of consecutive numbers, no two of which have any prime factor of the form $4k+3$.

Proof. For example:

$$\left((n^2+1)^2, (n^2+1)^2+1\right).$$

Another example:

$$\left(2^{2n}, 2^{2n}+1\right).$$

□

Discussion.

Problem 52. Compute

$$\left(\frac{600}{953}\right), \quad \left(\frac{2020^3}{953}\right), \quad \left(\frac{-7000}{757}\right).$$

Proof. It's obvious. Answers: -1, 1, 1. □

Discussion.

Problem 53. Prove that

- -2 is a quadratic residue modulo a prime $p>2$ iff $p \equiv 1,3 \pmod 8$,
- 2 is a quadratic residue modulo a prime $p>2$ iff $p \equiv \pm 1 \pmod 8$,
- -3 is a quadratic residue modulo a prime $p>2$ iff $p \equiv 1 \pmod 6$,
- 3 is quadratic residue modulo a prime $p>2$ iff $p \equiv \pm 1 \pmod{12}$.

Proof. Since all of the above dots are similar we prove only first and third.

Suppose that $\left(\frac{-2}{p}\right) = 1$ i.e.

$$(-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p^2-1}{8}} = 1.$$

If $p \equiv 5,7 \pmod 8$, then $\frac{p^2-1}{8} \equiv 1,0 \pmod 2$ and $\frac{p-1}{2} \equiv 0,1 \pmod 2$, respectively. Therefore in both cases $\frac{p-1}{2} + \frac{p^2-1}{8}$ is even. Hence $p \equiv 1,3 \pmod 8$.

On the other hand if $p \equiv 1,3 \pmod 8$, then

$$\left(\frac{-2}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p^2-1}{8}} = (-1)^{\frac{p-1}{2}+\frac{p^2-1}{8}} = 1,$$

since $\frac{p-1}{2} + \frac{p^2-1}{8}$ is even.

Suppose that $\left(\frac{-3}{p}\right) = 1$ i.e.

$$(-1)^{\frac{p-1}{2}} \cdot \left(\frac{3}{p}\right) = 1.$$

If $p \equiv 5 \pmod 6$, then by quadratic reciprocity

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}} \left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}} \left(\frac{2}{3}\right) = (-1) \cdot (-1)^{\frac{p-1}{2}},$$

so

$$1 = (-1)^{\frac{p-1}{2}} \cdot \left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot (-1) \cdot (-1)^{\frac{p-1}{2}} = -1,$$

contradiction.

On the other hand if $p \equiv 1 \pmod 6$, then using quadratic reciprocity we get

$$\left(\frac{-3}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}} \cdot \left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p-1}{2}} \cdot \left(\frac{1}{3}\right) = 1.$$

$\square$

Discussion.

Problem 54. Let p be a prime number. Prove that there exists $x \in \mathbb{Z}$ for which $p \mid x^2 - x + 3$ if and only if there exists $y \in \mathbb{Z}$ for which $p \mid y^2 - y + 25$.

Proof. The statement is trivial for $p \leq 3$, so we can assume that $p \geq 5$. Since $p \mid x^2 - x + 3$ is equivalent to

$$p \mid 4(x^2 - x + 3) = (2x - 1)^2 + 11,$$

integer x exists if and only if 11 is a quadratic residue modulo p. Likewise, since

$$4(y^2 - y + 25) = (2y - 1)^2 + 99,$$

y exists if and only if 99 is a quadratic residue modulo p. Now the statement of the problem follows from

$$\left(\frac{-11}{p}\right) = \left(\frac{-11 \cdot 3^2}{p}\right) = \left(\frac{-99}{p}\right).$$

$\square$

Discussion.

Problem 55. Suppose that for some prime p and integers a, b, c the following are true

$$6 \mid p + 1, \qquad p \mid a + b + c, \qquad p \mid a^4 + b^4 + c^4.$$

Prove that $p \mid a$, $p \mid b$ and $p \mid c$.

Proof. Suppose that $p \nmid c$. Then

$$p \mid (b+c)^4 + b^4 + c^4 = 2(b^2+bc+c^2)^2 \Longrightarrow p \mid b^2+bc+c^2 \Longrightarrow p \mid (2b+c)^2+3c^2 \Longrightarrow \left(\frac{-3}{p}\right) = 1.$$

Moreover using reciprocity law and the condition $6 \mid p + 1$ we have that

$$1 = \left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}}\left(\frac{p}{3}\right) = \left(\frac{-1}{3}\right) = -1,$$

contradiction. $\square$

Discussion.

Problem 56. Prove that number $2^n + 1$ does not have prime divisor of the form $8k - 1$.

Proof. Assume that p is a prime of the form $8k-1$ that divides 2^n+1. Of course, if n is even, the contradiction is immediate, since in this case we have

$$-1 \equiv \left(2^{\frac{n}{2}}\right)^2 \pmod{p},$$

so $\left(\frac{-1}{p}\right) = 1$ i.e. $(-1)^{\frac{8k-1-1}{2}} = 1$ – contradiction.

Now, assume that n is odd. Then

$$-2 \equiv \left(2^{\frac{n+1}{2}}\right)^2 \pmod{p},$$

so $\left(\frac{-2}{p}\right) = 1$ – contradiction with problem 53. ☐

Discussion.

Problem 57. Let $p > 2$ be a prime. Compute

$$\left(\frac{1}{p}\right) + \left(\frac{2}{p}\right) + \ldots + \left(\frac{p-1}{p}\right).$$

Proof. There are exactly $\frac{p-1}{2}$ quaratic residues and $\frac{p-1}{2}$ quaratic nonresidues, hence given sum is equal to 0. ☐

Discussion.

Problem 58. Prove that the number 3^n+1 has no prime divisor of the form $12k+11$.

Proof. Assume that $p = 12k+11$ is prime and $\mid 3^n+1$. We have two cases.

- n is odd. Then
$$\left(3^{\frac{n+1}{2}}\right)^2 \equiv -3 \pmod{p} \implies \left(\frac{-3}{p}\right) = 1,$$
contradiction with problem 53.
- n is even. In this case
$$\left(3^{\frac{n}{2}}\right)^2 \equiv -1 \pmod{p} \implies \left(\frac{-1}{p}\right) = 1.$$
On the other hand
$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1,$$
contradiction.

☐

Discussion.

Problem 59. Let a and b are integers such that a is different from 0 and the number $3+a+b^2$ is divisible by $6a$. Prove that a is negative.

Proof. Suppose that $3+a+b^2 = 6ak$, then $3+b^2 = a(6k-1)$. Suppose that a is positive, therefore $6k-1$ is also positive and so k is positive. Then $6k-1$ has a prime divisor p of the form $6\ell+5$. But $p \mid b^2+3$, hence $\left(\dfrac{-3}{p}\right) = 1$ – contradiction. $\square$

Discussion.

Problem 60. Let $x_1 = 7$ and

$$x_{n+1} = 2x_n^2 - 1, \quad \text{for } n \geq 1.$$

Prove that 2003 does not divide any term of the sequence.

Proof. Note that 2003 is prime number. Suppose that $2003 \mid x_{n+1}$ for some n. Then

$$2x_n^2 \equiv 1 \pmod{2003} \implies (2x_n)^2 \equiv -2 \pmod{2003}.$$

Therefore $\left(\dfrac{-2}{2003}\right) = 1$, but

$$\left(\frac{-2}{2003}\right) = (-1)^{\frac{2003^2-1}{8}} = (-1)^{501501} = -1,$$

contradiction. $\square$

Discussion.

Problem 61. Let $p > 2$ be a prime such that there exists integers x, y that

$$p = x^2 + xy + y^2.$$

Prove that $p = 3$ or $p \equiv 1 \pmod 3$.

Proof. Of course $p \nmid x, y$, for $p = 3$ we have $x = y = 1$. Assume $p > 3$. From the given condition we see that

$$p \mid 4(x^2 + xy + y^2) = (2x+y)^2 + 3y^2,$$

so

$$(2x+y)^2 \equiv -3y^2 \pmod p \implies -3 \equiv \left((2x+y) \cdot b^{-1}\right)^2 \pmod p,$$

so $\left(\dfrac{-3}{p}\right) = 1$ i.e. $p \equiv 1 \pmod 3$ by 53. $\square$

Discussion.

Problem 62. Suppose that $p \equiv 1 \pmod 3$ is a prime. Using Thue's lemma prove that there exists integers $0 < x, y < \sqrt{p}$ such that $p \mid 3x^2 + y^2$. Conclude that there are integers a, b such that

$$p = a^2 + ab + b^2.$$

Proof. If $p \equiv 1 \pmod 3$, then $\left(\frac{-3}{p}\right) = 1$, so $-3 \equiv n^2 \pmod p$, for some n. Hence from Thue's lemma we see that there are integers $0 < x, y < \sqrt{p}$ such that $nx \equiv \pm y \pmod p$, so

$$-3x^2 \equiv (nx)^2 \equiv y^2 \pmod p \implies p \mid 3y^2 + x^2.$$

But

$$0 < 3y^2 + x^2 < 3(\sqrt{p})^2 + (\sqrt{p})^2 = 4p,$$

so $3y^2 + x^2 \in \{p, 2p, 3p\}$.

(1) If $p = 3y^2 + x^2$, then
$$p = (y - x)^2 + (y - x) \cdot 2x + (2x)^2,$$
so we can take $a := y - x$ and $y := 2x$.

(2) If $2p = 3y^2 + x^2$, then y and x have the same parity, so $4 \mid 3y^2 + x^2 = 2p$, contradiction.

(3) If $3p = 3y^2 + x^2$, then $3 \mid x$, so $x = 3x_1$ and hence $p = y^2 + 3x_1^2$ – this is the first case (1).

$\square$

Discussion.

Problem 63. Compute

$$\left(\frac{-12000}{821}\right), \quad \left(\frac{2^{2019}}{953}\right).$$

Proof. $\square$

Answers: 1 and 1.

Discussion.

Problem 64. Prove that 5 is quadratic residue modulo a prime $p > 2$ iff $p \equiv \pm 1 \pmod{10}$.

Proof. Suppose that $\left(\frac{5}{p}\right) = 1$. If $p \equiv 3 \pmod{10}$, then by quadratic reciprocity

$$\left(\frac{5}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{5-1}{2}} \left(\frac{p}{5}\right) = (-1)^{p-1} \left(\frac{3}{5}\right) = -1,$$

contradiction. The same with $p \equiv 7 \pmod{10}$.

On the other hand if $p \equiv 1 \pmod{10}$, then using quadratic reciprocity we get

$$\left(\frac{5}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{5-1}{2}} \cdot \left(\frac{p}{5}\right) = \left(\frac{1}{5}\right) = 1.$$

$\square$

Discussion.

Problem 65. Find all integer solutions of the following equation

$$a^2 + b^2 + c^2 = 7d^2.$$

Proof. We will do modulo 4. Since $x^2 \equiv 0, 1 \pmod 4$ we see that if $2 \nmid d$, then a, b, c are odd. Therefore

$$a^2 + b^2 + c^2 \equiv 3 \pmod 8$$

and $7d^2 \equiv 7 \pmod 8$ – contradiction. Therefore $2 \mid d$ and so $2 \mid a, b, c$, so by descent argument we have $a = b = c = d = 0$.

There is theorem due to Euler which says that the number is a sum of three squares iff is not of the form $4^s(8\ell + 7)$, where s, ℓ are non-negative integers.

$\square$

Discussion.

Problem 66. Prove that for any positive integer n every prime divisor p of number

$$n^4 - n^2 + 1$$

is of the form $12k + 1$.

Proof. Observe that

$$n^4 - n^2 + 1 = (n^2 - 1)^2 + n^2 \quad \text{and} \quad n^4 - n^2 + 1 = (n^2 + 1)^2 - 3n^2.$$

First equality gives that $p \equiv 1 \pmod 4$ (because $\left(\frac{-1}{p}\right) = 1$) and second one gives $p \equiv \pm 1 \pmod{12}$, since $\left(\frac{3}{p}\right) = 1$. $\square$

Discussion.

Problem 67. Let a, b be a positive integers such that $a^2 + b^2 + ab$ is divisible by $ab - 2$. Find all possible values of

$$\frac{a^2 + b^2 + ab}{ab - 2}.$$

Proof. Let
$$\frac{a^2+b^2+ab}{ab-2} = k \implies \frac{a^2+b^2+2}{ab-2} = k-1.$$
WLOG $a \geq b$ and $a+b$ has minimal sum.

If $a = b$ then
$$\frac{2a^2+2}{a^2-2} = k-2 = 2+\frac{6}{a^2-2},$$
so $a = 1, 2$ and hence $k = -3, 6$.

Let $a > b$ and consider quadratic equation in a
$$a^2 - (k-1)ab + b^2 + 2k = 0.$$
Another solution of this equation is
$$x_2 := b(k-1) - a = \frac{b^2+2k}{a},$$
so x_2 is positive integer, so $x_2 + b \geq a + b \implies x_2 \geq a$. Therefore
$$a \leq b(k-1) - a \implies k-1 \geq \frac{2a}{b},$$
so
$$\frac{a^2+b^2+2}{ab-2} = k-1 \geq \frac{2a}{b} \implies b^3 + 2b \geq a^2 b - 4a \geq (b+1)^2 b - 4(b+1) \implies$$
$$\implies 2b^2 - 5b - 4 \leq 0 \implies b = 1, 2, 3.$$

If $b = 1$ then
$$k = \frac{a^2+3}{a-2} + 1 = a + 2 + \frac{7}{a-2} + 1 \implies a = 1, 3, 9 \implies k = 13.$$

If $b = 2$ then $2 \mid a$ and so $a = 2a_1$, so
$$k = \frac{a^2+6}{2a-2} + 1 = a_1 + \frac{a_1+3}{2a_1-1} + 1 \implies a = 2, 8 \implies k = 6.$$

If $b = 2$ then $3a-2 \mid 99+6a = 2(3a-2)+103$, so $3a-2 \mid 102$ thus $a = 1, 35 \implies k = 13$. $\square$

Discussion.

Problem 68. Find all integers such that
$$x^2 + 5 = y^3.$$

Proof. If x is odd, then y is even, but $x^2 + 5 \equiv 2 \pmod 4$ and $y^3 \equiv 0 \pmod 4$ – contradiction. Therefore x is even. Hence $y^3 \equiv 1 \pmod 4$, so $y \equiv 1 \pmod 4$. We have
$$x^2 + 4 = y^3 - 1 = (y-1)(y^2+y+1),$$
but $y^2 + y + 1 \equiv 4 \pmod 4$ and divides $x^2 + 2^2$, contradiction. $\square$

Discussion.

Problem 69. Find all positive integers such that $x^2 + 3y$ and $y^2 + 3x$ are squares.

Proof. Suppose that $x \geq y$. Then
$$x^2 < x^2 + 3y \leq x^2 + 3x < x^2 + 4x + 4,$$
so $x^2 + 3y = (x+1)^2$, i.e. $3y = 2x + 1$.

Since $y^2 + 3x$ is square it follows that $4y^2 + 12x = 4y^2 + 18y - 6$ is square. For $y \geq 3$ we have
$$(2y+3)^2 = 4y^2 + 12y + 9 < 4y^2 + 18y - 6 < (2y+5)^2,$$
so
$$4y^2 + 18y - 6 = (2y+4)^2 \implies y = 11$$
and hence $x = 16$.

If $y \leq 3$, then easy to get solution $(1, 1)$. $\square$

Discussion.

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