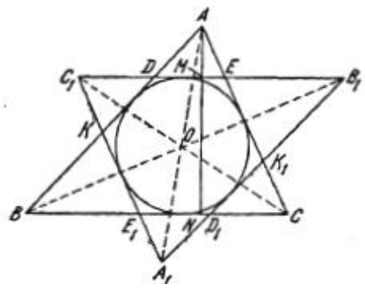
Problem 1.4. Two triangles ABC and $A_1B_1C_1$ are symmetric about the center of their common incircle of radius r. Prove that the product of the areas of the triangles ABC, $A_1B_1C_1$ and the six other triangles formed by the intersecting sides of the triangles ABC and $A_1B_1C_1$ is equal to r^{16}.

Sol. The symmetry of ABC and $A_1B_1C_1$ about the centre O of the inscribed circle implies that the corresponding points of $\triangle ABC$ and $\triangle A_1B_1C_1$ lie on a straight line passing through O and are equidistant from this point.



In particular, $OC = OC_1$, $OB = OB_1$ and BCB_1C_1 is a parallelogram; hence, $BC = B_1C_1$. Analogously, $AC = A_1C_1$, $AB = A_1B_1$ and $\triangle ABC = \triangle A_1B_1C_1$. Considering the parallelograms ABA_1B_1, BDB_1D_1, ACA_1C_1 and ECE_1C_1 we conclude that $AD = A_1D_1$, $AE = A_1E_1$, and, since $\angle A = \angle A_1$, we see that $\triangle ADE = \triangle A_1D_1E_1$. Similarly, $\triangle B_1EK_1 = \triangle BE_1K$ and $\triangle DC_1K = \triangle D_1CK_1$.

Let us denote by S the area of $\triangle ABC$, by S_1 the area of $\triangle ADE$, by S_2 the area of $\triangle DC_1K$, by S_3 the area of $\triangle KBE_1$. Put $AB = c$, $BC = a$ and $AC = b$, and let h_A, h_B and h_C be the altitudes drawn from the vertices A, B and C, respectively. Then we have

$$S = pr = \frac{ah_A}{2} = \frac{bh_B}{2} = \frac{ch_C}{2},$$

Let AM (AN) be the altitude in $\triangle ADE$ (in $\triangle ABC$). Then

$$S_1 = \frac{DE \cdot AM}{2}.$$

The similarity of the triangles ABC and ADE implies that

$$DE = \frac{a(h_A - 2r)}{h_A}.$$

Hence,

$$S_1 = \frac{a(h_A - 2r)^2}{2h_A} = \frac{a\left(\frac{2pr}{a} - 2r\right)^2}{2h_A} = \frac{r^2(p-a)^2}{S}.$$

Analogously,

$$S_2 = \frac{r^2(p-c)^2}{S}, \qquad S_3 = \frac{r^2(p-b)^2}{S}.$$

Using Heron's formula we obtain

$$S^2S_1^2S_2^2S_3^2 = \frac{r^{12}(p-a)^4(p-b)^4(p-c)^4 S^2}{S^6} = r^{12}\frac{S^4}{p^4} = r^{16}.$$