Winter Camp 2021

Algebra Level 2

Day 4 (3)
3 January 2021

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^4 + f(y)) = y + f^4(x) \qquad \forall x, y \in \mathbb{R}$$

$x=0$: $\boxed{f(f(y)) = y + f^4(0)}$ $\to$ f bijective

$\boxed{\exists c : f(c) = 0}$

$x=y=c$: $f(c^4) = c$ $\to$ $f(f(c^4)) = 0$
$= c^4 + f^4(0)$ $\Big\}$ $\Rightarrow$ $c = 0$ $\to$ $f(0) = 0$

$\Rightarrow$ $\boxed{f(f(y)) = y}$

$y \to f(y)$: $f(x^4 + y) = f(y) + f^4(x)$

$y=0$: $\boxed{f(x^4) = f^4(x)}$ $\longrightarrow$ $x \to -x$: $f(x^4) = f^4(-x)$ $\Rightarrow$ $\begin{cases} f(x) = f(-x) \\ f(x) = -f(-x) \end{cases}$

injective $\to$ $f(x) = -f(-x)$
f odd

$\Rightarrow$ $\begin{cases} f(x^4 + y) = f(y) + f(x^4) \\ f(-x^4 - y) = f(-y) + f(-x^4) \end{cases}$

$\Rightarrow f(a+b) = f(a) + f(b)$
$\forall a, b \in \mathbb{R}$

$\Rightarrow$ $\boxed{f \text{ additive}}$

$t = x^4$ $\quad t \geq 0$

$f(t) = f^4(t^{\frac{1}{4}}) \geq 0$

$x = y + t, \boxed{t \geq 0} \xrightarrow{?} f(x) \geq f(y)$

$\qquad\qquad\qquad

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(2x^2 + y + f(y)) = 2y + 2f^2(x) \qquad \forall x, y \in \mathbb{R}$$

$y=0$: $f(2x^2+f(0)) = 2f^2(x)$

$x=0$: $f(y+f(y)) = 2y + 2f^2(0) \longrightarrow f$ surjective

$\Rightarrow \exists c : f(c)=0$

$x=y=0$: $f(f(0)) = 2f^2(0)$

$y=c$: $f(2x^2+c) = 2c+2f^2(x)$

$(0,c)$: $f(c)=0=2c+2f^2(0) \rightarrow c=-f^2(0)$

$(c,0)$: $f(2c^2+f(0))=0 \Rightarrow 2c^2+f(0)=c=-f^2(0)$

$2f^4(0)+f(0)=-f^2(0)$

if $f(0)\neq 0$: $2f^3(0)+f(0)+1=0$

$(-c,0)$: $f(2c^2+f(0)) = 2f^2(-c) \Rightarrow f(-c)=0$

$\Rightarrow -c = c \rightarrow c=0$

$\Rightarrow \boxed{f(x)=0 \Leftrightarrow x=0}$

$y=0$: $\boxed{f(2x^2) = 2f^2(x)}$

$x \to -x$: $f(2x^2) = 2f^2(-x)$

$\Rightarrow f(x) = f(-x)$ or $f(x) = -f(-x)$

$y = -f^2(x)$: $2x^2 - f^2(x) + f(-f^2(x)) = 0$

$f(y+f(y)) = 2y$

- if $f(a) = f(b) \Rightarrow a^2 = b^2$

$\Rightarrow \boxed{-f^2(x) + f(-f^2(x)) = -2x^2}$

$f(-f^2(x) + f(-f^2(x))) = \boxed{-2f^2(x) = f(-2x^2)}$

$\Rightarrow f(2x^2) + f(-2x^2) = 0$

$a = 2x^2 \geq 0 \Rightarrow f(a) + f(-a) = 0$

$\to \boxed{f \text{ odd}}$

$\Rightarrow f$ injective

$y \to -f^2(y)$:

$f(2x^2 - f^2(y) + f(-f^2(y))) = -2f^2(y) + 2f^2(x)$

$= f(2x^2 - 2y^2) = f(-2y^2) + f(2x^2)$

$f(a+b) = f(a) + f(b)$ $\quad a \geq 0 \geq b$

$y \to f^2(y)$: $f(2x^2 + 2y^2) = f(2y^2) + f(2x^2)$

$\to f(a+b) = f(a) + f(b)$ $\quad a, b \geq 0$

$f(-a-b) = f(-a) + f(-b)$ $\quad -a, -b \leq 0$

$\to f$ additive

$f(2x^2) = 2f^2(x) \geq 0 \Rightarrow f$ increasing

$\Rightarrow f(x) = kx \to f(x) = x$

Problem 3. Find all functions $f : \mathbb{R}/\{0\} \to \mathbb{R}/\{0\}$ such that

$$f(x+y) = x^2 f(\frac{1}{x}) + y^2 f(\frac{1}{y})$$

$x=1$: $f(1+y) = f(1) + y^2 f(\frac{1}{y})$

$x=y=1$: $f(2) = 2f(1)$

$y \to \frac{1}{y}$: $f(x+\frac{1}{y}) = x^2 f(\frac{1}{x}) + \frac{1}{y^2} f(y)$

$x=y$: $f(2x) = 2x^2 f(\frac{1}{x}) \longrightarrow 2f(x) = 2x^2 f(\frac{1}{x}) \Leftrightarrow \boxed{f(x) = x^2 f(\frac{1}{x})}$

$f(\frac{1}{x+1}) = f(1) - f(\frac{x}{x+1})$

$\Rightarrow f(x+y) = \dfrac{f(2x)+f(2y)}{2}$

$\forall\, y \in \mathbb{R},\ z \in \mathbb{R}$

$$f(x+y+z) = \frac{1}{2}\left(f(2x+2y) + f(2z)\right)$$

$$= \frac{f(4x)+f(4y)}{4} + \frac{f(2z)}{2}$$

$$= \frac{1}{2}\left(f(2x+2z) + f(2y)\right)$$

$$= \frac{f(4x)+f(4z)}{4} + \frac{f(2y)}{2}$$

$\rightarrow 2f(2y) - f(4y) = 2f(2z) - f(4z)$

$2f(x) - f(2x) = C$

$x=1$: $2f(1) - f(2) = 0$

$\Rightarrow C = 0$

$\Rightarrow \boxed{f(2x) = 2f(x)}$

$\Rightarrow f(x+y) = f(x) + f(y) \longrightarrow f$ additive

$a = b + t$

$t \geq 0$

$$f(a) = f(b+t) = b^2 f\left(\frac{1}{b}\right) + t^2 f\left(\frac{1}{t}\right)$$

$$= f(b) + t^2 f\left(\frac{1}{t}\right) \overset{?}{\geq} f(b)$$

في الغالب لا

$$\Rightarrow f(x) = kx$$

$$\begin{cases} f \text{ additive} \\ f(x) = x^2 f\left(\frac{1}{x}\right) \end{cases}$$

$x \to x+1$:

$$f(x+1) = (x+1)^2 f\left(\frac{1}{x+1}\right)$$

$$= (x+1)^2 \left(f(1) - f\left(\frac{x}{x+1}\right)\right)$$

Problem 4. $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y + xy) = f(x) + f(y) + f(xy)$$

Prove that f is additive.

$x = y = 0$: $f(0) = 0$

$y = -x$: $\cancel{f(-x^2)} = f(x) + f(-x) + \cancel{f(-x^2)}$

$f(x) = -f(-x) \longrightarrow \boxed{f \text{ odd}}$

$y = 1$: $f(2x+1) = 2f(x) + f(1)$

$x \to -x$: $f(-x + y - xy) = -f(x) + f(y) - f(xy)$

$\Rightarrow f(x+y+xy) + f(y-x-xy) = 2f(y)$

$a = x + xy = x(y+1)$ $\quad y \neq -1 \longrightarrow \boxed{a} \in \mathbb{R}$

$$\boxed{f(y+a) + f(y-a) = 2f(y)}$$

$\forall y \in \mathbb{R}, a \in \mathbb{R}$

$y \to \frac{y}{x}$:

$f(x + \frac{y}{x} + y) = f(x) + f(\frac{y}{x}) + f(y)$

$y \to \frac{-y}{x}$:

$f(x - \frac{y}{x} - y) = f(x) - f(\frac{y}{x}) - f(y)$

$y \to \frac{y}{x+1}$:

$f(x + \frac{y}{x+1} + \frac{xy}{x+1}) = f(x) + f(\frac{y}{x+1}) + f(\frac{xy}{x+1})$

$= f(x+y)$

$y \to \frac{-y}{x+1}$: $x \neq -1$

$f(x-y) = f(x) - f(\frac{y}{x+1}) - f(\frac{xy}{x+1})$

$\Rightarrow f(x+y) + f(x-y) = 2f(x)$

$y=1: \quad f(1+a) + f(1-a) = 2f(1)$

$\rightarrow \quad f(-1-a) + f(-1+a) = 2f(-1)$

$y=a: \quad f(2a) = 2f(a)$

$$f(y+a) + f(y-a) = f(2y)$$

$u = y+a$

$v = y-a$

$\rightarrow \quad f(u) + f(v) = f(u+v)$

Problem 5. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y) \qquad \forall x, y \in \mathbb{R}$$

Problem 6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}$$

Problem 7. Find all $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m) + f(n) | m + n \qquad \forall m, n \in \mathbb{N}$$

$m=n=1$: $\quad 2f(1) \mid 2 \to f(1) \mid 1 \to f(1)=1$

$m=n$: $\quad 2f(m) \mid 2m \to f(m) \mid m. \to f(m) \leq m$

$n=1$: $\quad f(m)+1 \mid m+1$

$m=p-1$: $\quad f(p-1)+1 \mid p \Rightarrow \boxed{f(p-1)=p-1}$

الفكرة الأولى: محاول نثبت ال injectivity

أوبي

$f(a)=f(b)$

$f(m)+f(a) \mid m+a \quad , \quad f(m)+f(b) \mid m+b$

$\Rightarrow f(m)+f(a) \mid m+b \Rightarrow f(m)+f(a) \mid a-b$

$m=p-1$: $\quad p-1+f(a) \mid a-b$

$p > |a-b| \Rightarrow a-b=0$

$\Rightarrow f$ injective

$$f(m) \le m$$

استقراء 1 $\quad f(n)=n \quad$ لکل $\quad n \in \{1,2,\dots,m\}$

$$f(m+1) = m+1$$

$$f(m+1) \le m+1 \quad , f(m+1) \notin \{1,2,\dots,m\}$$

$$= \{f(1), f(2), \dots, f(m)\}$$

$m = p-1$:

$$p-1 + f(n) \mid n+p-1$$

$$p-1 + f(n) \mid \underline{n - f(n)}$$

$$p > |n - f(n)|$$

ثبت

$$\Rightarrow \quad n - f(n) = 0 \quad \rightarrow \quad \boxed{f(n) = n}$$