$\boxed{52}$ Determine $f : \mathbb{R} \to \mathbb{R}$

- $f(x) + f(y) \geq xy \quad \forall x, y$
- $\forall x \in \mathbb{R} \ \exists y \in \mathbb{R} : f(x) + f(y) = xy$

Proof $x := y$ in 1 dot

$$f(x) + f(x) \geq x^2 \Longrightarrow f(x) \geq \frac{x^2}{2} \quad \forall x,$$

Fix $x \in \mathbb{R}$. Then there is y:

$$\boxed{f(x) + f(y) = xy}$$

$$+\begin{cases} f(x) \geq \frac{x^2}{2} \\ f(y) \geq \frac{y^2}{2} \end{cases}$$

$$xy = f(x) + f(y) \geq \frac{x^2 + y^2}{2}$$

$$2xy \geq x^2 + y^2$$

$$\Downarrow$$

$$(x-y)^2 \leq 0 \Rightarrow \boxed{x = y}$$

$$0 = f(x) + f(y) - xy \geq \frac{x^2}{2} + \frac{y^2}{2} - xy \geq 0$$

$\Downarrow$ $\Downarrow$

$$\boxed{f(x) = \frac{x^2}{2}}$$

done.

- $f(x) + f(y) \geq xy$

$\frac{x^2}{2} + \frac{y^2}{2} \geq xy$

$$\frac{x^2}{2} + \frac{y^2}{2} = xy$$

put $y = x$

$$\boxed{\frac{x^2}{2} + \frac{x^2}{2} = x^2}$$

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$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right) \leq km - 2$$

$\forall k, m \in \mathbb{R}_+$
$k \neq m$.

use this

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n)$$

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$$\left(k-\frac{1}{k}\right)\left(m-\frac{1}{m}\right) \leq km-2 \quad \mid \cdot km$$

$$(k^2-1)(m^2-1) \leq k^2m^2-2km$$

$$\cancel{k^2m^2}-k^2-m^2+1 \leq \cancel{k^2m^2}-2km$$

$$\boxed{(k-m)^2 \geq 1} \quad \checkmark \quad k-m \geq 1$$

$$\left(k-\frac{1}{k}\right)\boxed{\left(m-\frac{1}{m}\right)\left(n-\frac{1}{n}\right)} \leq$$

WLOG $k = \max\{k,m,n\}$

$$\leq \left(k-\frac{1}{k}\right)(mn-2) \overset{?}{\leq} kmn-(k+m+n)$$

$$\cancel{kmn} - k - \frac{mn}{k} + \frac{2}{k} \overset{?}{\leq} \cancel{kmn} - \cancel{k} - m - n$$

$$\frac{2}{k} \overset{?}{\leq} \frac{mn}{k} + k - m - n = \frac{(k-m)(k-n)}{k}$$

$$\Downarrow$$

$$2 \overset{?}{\leq} (k-m)(k-n) \qquad \checkmark$$

$$(k-m)(k-n) \geq 1 \cdot 2$$

$$\Longleftrightarrow \quad k = n+2, \quad m = n+1, \quad n$$

$$\Longleftrightarrow \quad \{k, m, n\} = \{n+2, n+1, n\} \qquad \square$$

$\boxed{Q:}$ what about 4 expressions?

$\boxed{53}$

$0 < a_0 < a_1 < \dots$ ← seq (infinite) of $\mathbb{Z}_+$

unique

$\exists! \; n \geq 1$:

$$a_n < \frac{a_0 + \dots + a_n}{n} \leq a_{n+1}$$

$$b_n = (a_0 + a_1 + \dots + a_n) - n a_n$$

$$\boxed{b_1 = a_0 + a_1 - a_1 = a_0 > 0}$$

$$b_{n+1} - b_n = n(a_n - a_{n+1}) < 0$$

b_n is decreasing

$$\underset{\wedge}{0}$$

$$b_1 > b_2 > b_3 > b_4 > \dots > b_i$$

$$10 > 6 > 2 > 1 > -2 > \dots$$

$$a_n < \frac{a_0 + a_1 + \dots + a_n}{n} \leq a_{n+1}$$

$$b_n = (a_0 + a_1 + \dots + a_n) - n a_n > 0$$

$$b_n > 0 \geq b_{n+1}$$

$$b_{n+1} = (a_0 + a_1 + \dots + a_{n+1}) - (n+1) a_{n+1} \leq 0$$

$$(a_0 + \dots + a_n) - n a_{n+1} \leq 0$$

$$a_{n+1} \geq \frac{a_0 + \dots + a_n}{n}$$

We need to prov $\exists$ unique n.

$b_1 > 0$

$$\boxed{b_n > 0 \geq b_{n+1}}$$

$$b_1 > b_2 > b_3 > \dots > \dots$$

0

b_2

if a_n are not integ.,

$$a_n = 2 - \frac{1}{2^n}$$

$$2 - \frac{1}{2^n} < \frac{2 - \frac{1}{2^0} + \dots + 2 - \frac{1}{2^n}}{n} \leq 2 - \frac{1}{2^{n+1}}$$

$$\parallel$$

$$\frac{2(n+1) - \left(2 - \frac{1}{2^n}\right)}{n}$$

$$1 + \frac{1}{2} + \dots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

$$\frac{2n + \frac{1}{2^n}}{n}$$

$$\cancel{2} - \frac{1}{2^n} < \cancel{2} + \frac{1}{n2^n} \leq \cancel{2} - \frac{1}{2^{n+1}}$$

a. . .

$\boxed{55}$ Hint $\qquad a, b \in \mathbb{Z}, \quad f \in \mathbb{Z}[X]$

$$a-b \mid f(a)-f(b)$$

$$X-Y \mid f(X)-f(Y)$$

$$a_n X^n + \dots + a_1 X + a_0 - (a_n Y^n + \dots + a_1 Y + a_0) =$$

$$= a_n (X^n - Y^n) + \dots + a_1 (X-Y)$$

$$\boxed{X-Y \mid X^n - Y^n}$$

$\boxed{55}$ $n=2$

$\Downarrow$ inductive.

$$P(Q(x)) = Q(P(x))$$

$$\Downarrow$$

$$P(x)-Q(x) \mid P(P(x)) - Q(Q(x))$$

$$= \underbrace{P(P(x)) - P(\cancel{Q(x)})}_{|\, P(x)-Q(x)} + \underbrace{Q(\cancel{P(x)}) - Q(Q(x))}_{|\, P(x)-Q(x)}$$

$\Downarrow$ induct.

Homework Induction in 55 and's solution.

Lucy, f. eq. Induction poly alg manip.

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