Email training, N2

August 31 - September 7, 2019

Problem 2.1. For a positive integer n, let $f(n)$ denote the greatest odd divisor of n. For example, $f(11) = 11$ and $f(12) = 3$. Prove that for any positive integer n,

$$f(n+1) + f(n+2) + \ldots + f(2n) = n^2.$$

Problem 2.2. Find the smallest positive integer m for which there exists a positive integer n such that

$$\left|\frac{n}{m} - \frac{2}{5}\right| \leq \frac{1}{100}.$$

Problem 2.3. Find the greatest common divisor of $5^{300} - 1$ and $5^{200} + 6$.

Problem 2.4. Find all pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{lcm(x,y)} + \frac{1}{gcd(x,y)} = \frac{1}{2}.$$

Problem 2.5. Ali has choosen 8 cells of the chessboard 8×8 such that no any two lie on the same line or in the same row (we call it general configuration). On each step Baba chooses 8 cells in general configuration and puts coins on them. Then Ali shows all coins that are out of cells chosen by Ali. If Ali shows even number of coins then Baba wins, otherwise Baba removes all coins and makes the next move. Find the minimal number of moves that Baba needs to guarantee the win.

Problem 2.6. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the number of functions $f : S \to S$ such that $f(f(x)) = x$ for all $x \in S$.

Problem 2.7. In the triangle ABC the median AM is drawn. Is it possible that the radius of the circle inscribed to the triangle ABM be twice bigger than the radius of the circle inscribed to the triangle ACM?

Problem 2.8. Let a point M is chosen inside the square $ABCD$ such that $\angle MAC = \angle MCD$. Find $\angle ABM$.

Solution submission deadline September 7, 2019