Examples

1. A sequence $a_1, a_2, a_3, \ldots$ of positive integers satisfies $a_{n+1} = a_n + 2d(n)$ for every $n \geqslant 1$, where $d(n)$ is the number of different positive divisors of n. Is it possible that two consecutive terms of this sequence are perfect squares?
2. Let $P(x)$ be the polynomial with real coefficients. Given that there exists an infinite sequence (a_n) of pairwise distinct positive integers such that $P(a_1) = 0$, $P(a_2) = a_1$, $P(a_3) = a_2$, $\ldots$, find all possible values of $\deg P$.
3. Prove that for some positive integer n the remainder of 3^n when divided by 2^n is greater than 10^{2021}.
4. Prove that the set of primes dividing all numbers $x_n = a^n - b^n$, where $\gcd(a, b) = 1$, is infinite.

Exercises

5. Prove that the sequence $a_n = d(n^2 + 1)$ is not strictly increasing from any moment.
6. Denote by $S(n)$ the sum of the digits in the decimal representation of a positive integer n. Does there exist $m \in \mathbb{N}$ such that $S(2^{n+1}) \geqslant S(2^n)$ for all $n \geqslant m$?
7. Let $P(x)$ be a monic (i. e. leading coefficient is 1) polynomial such that for any positive integer m there exists $k \in \mathbb{N}$ such that $P(k) = 2^m$. Prove that $\deg P = 1$.
8. Let c be a positive integer and (a_n) be a sequence of integers satisfying the inequality $a_n < a_{n+1} < a_n + c$. Prove that the set of prime numbers dividing the elements of the sequence (a_n) is infinite.

Challenging problems (HOMEWORK)

9. **a)** Do there exist positive integers a and b such that $a \cdot 2^n + b \cdot 5^n$ is a square for each positive integer n? **b)** Do there exist positive integers a, b and c such that $a \cdot 2^n + b \cdot 5^n + c$ is a square for each positive integer n?
10. Call a positive integer x to be *remote from squares and cubes* if each integer k satisfies both $|x - k^2| > 10^6$ and $|x - k^3| > 10^6$. Prove that there exist infinitely many $n \in \mathbb{N}$ such that 2^n is remote from squares and cubes.
11. Let a_1, a_2, $\ldots$, a_n be positive integers and let $a > 1$ be a multiple of $a_1 a_2 \ldots a_n$. Prove that $a^{n+1} + a - 1$ is not divisible by $(a + a_1 - 1)(a + a_2 - 1) \ldots (a + a_n - 1)$.
12. Let $x_n = a^n + b^{n+1}$ where a and b are distinct positive integers. Prove that the set of prime numbers dividing at least one element of the sequence (x_n) is infinite.
13. The coefficients of the polynomial $P(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_3 x^3 + a_2 x^2 + a_0$, where $d \geqslant 2$, are positive integers. The sequence (b_n) is defined by $b_1 = a_0$ and $b_{n+1} = P(b_n)$ for $n \geqslant 1$. Prove that for any $n \geqslant 2$ there exists a prime number p such that $p \mid b_n$ but it doesn't divide b_1, b_2, $\ldots$, b_{n-1}.
14. Let $P(x)$ be a nonconstant integer polynomial and $n \in \mathbb{N}$. The sequence a_0, a_1, $\ldots$ is defined by $a_0 = n$ and $a_k = P(a_{k-1})$ for $k \geqslant 1$. Given that for each $b \in \mathbb{N}$ the sequence contains a b-th power of a positive integer greater than 1. Prove that $\deg P = 1$.

1. **Answer: no.** Suppose the contrary: $a_n = x^2$ and $a_{n+1} = y^2$ for some positive integers x and y. Note that $a_{n+1} > a_n$ and they have the same parity, hence $a_n \geqslant n$ and $y \geqslant x + 2$. Therefore $2d(n) = a_{n+1} - a_n = y^2 - x^2 \geqslant (x+2)^2 - x^2 = 4x + 4$. This implies $d(n) \geqslant 2x + 2 = 2\sqrt{n} + 1$ which is impossible since $d(n) \leqslant 2\sqrt{n}$ for any integer $n \geqslant 1$.
2. **Hint:** note that if $\deg P > 1$ then $|P(x)| > |x|$ for large enough x.
3. Suppose that $3^n \bmod 2^n$ is bounded and consider maximal remainder r among all remainders that occur infinitely many times. Choose n such that $3^n \bmod 2^n = r$ and $3^m \bmod 2^m \leqslant r$ for all $m \geqslant n$. Then $3^n = k2^n + r$ where $k < (3/2)^n$. Hence for large enough n we obtain $3^{2n} \bmod 2^{2n} = 2kr2^n + r^2 > r$.
4. Assume that the set of these primes is finite: $\{p_1, \dots, p_k\}$. Consider a fixed integer n, and $a^n - b^n = p_1^{a_1} \dots p_k^{a_k}$. Then for any $m = n + s \cdot \varphi(p_1^{a_1+1} \dots p_k^{a_k+1})$ we will get $a^m - b^m \equiv a^n - b^n \mod p_1^{a_1+1} \dots p_k^{a_k+1}$ whence $a^m - b^m = a^n - b^n$, absurd.
5. Note that for even n holds $d(n^2+1) < n$. Suppose $d((n+1)^2+1) > d(n^2)$ for each $n \geqslant N$ and let $\delta = d(N^2+1)$. Then $d((n+k)^2+1) \geqslant d(n^2) + 2k$, take $s > N - \delta$ such that $2|N + s$ and take $n = N$ and $k = s$.
6. **Answer: no.** Notice that the remainders of $S(2^n)$ modulo 9 are periodic with the period 2, 4, 8, 7, 5, 1 of length 6. Suppose such m exists then for any $k \in \mathbb{N}$ such that $6k > n$ the next inequalities hold:

$$\begin{aligned} &S(2^{6k+1}) \geqslant S(2^{6k}) + 1, && S(2^{6k+2}) \geqslant S(2^{6k+1}) + 2, && S(2^{6k+3}) \geqslant S(2^{6k+2}) + 4, \\ &S(2^{6k+4}) \geqslant S(2^{6k+3}) + 8, && S(2^{6k+5}) \geqslant S(2^{6k+4}) + 7, && S(2^{6k+6}) \geqslant S(2^{6k+5}) + 5. \end{aligned}$$

   Therefore, $S(2^{6k+6}) \geqslant S(2^{6k}) + 27$, so $S(2^{6k+6\ell}) \geqslant S(2^{6k}) + 27\ell$ for all $\ell \in \mathbb{N}$. On the other hand, the number of non-vanishing digits of $2^{6m+6\ell}$ does not exceed $2m + 2\ell$, since $2^3 < 10$. Hence $18m + 18\ell \geqslant S(2^{6m+6\ell}) \geqslant S(2^{6k}) + 27\ell$. Now, fixing m, we will get a contradiction for sufficiently large ℓ.
7. Suppose $\deg P = k > 1$ and let $P(x) = x^k + R(x)$. If $R(x) \equiv 0$ then $P(n) = 2^m$ impossible if $k \not| m$. If $R(x)$ is nonconstant then by shifting $P'(x) = P(x+t)$ we can get a new polynomial $P'(x) = x^k + R'(x)$ where the leading coefficient of R' is between 0 and $k-1$. But $P'(2^m - 1) < 2^{km} < P'(2^m + 1)$ for sufficiently large m, whence $2^{km} = P(2^m)$ and the polynomial R' has infinitely many roots of the form 2^m which lead us to the first case $R'(x) \equiv 0$.
8. Suppose contrary and take different primes $q_1, q_2 \dots, q_c$ which are not divisors of these elements. Using CRT choose $a > a_1$ such that $a \equiv i \mod q_i$. Then the sequence (a_n) has an element from $a+1, a+2, \dots, a+c$ which is divisible by at least one q_i.

Hints and solutions

9. **Hints: a)** Let $v_5(a) = k$ then for each $n > k$ holds $\nu_5(x_n^2) = k$ hence $\frac{x_n}{5^{k/2}}$ is an integer. Consider consecutive $(\frac{x_n}{5^{k/2}})^2 \equiv 2^n c \pmod 5$. **b)** If $x_n^2 = 2^n a + 5^n b + c$ then $25x_n^2 > x_{n+2}^2$ whence $x_{n+2} \leqslant 5x_n - 1$. Square it to obtain $10x_n \leqslant 21 \cdot 2^n a + 24c + 1$ and show that it's impossible.
10. **Hint:** First choose large enough N so that the difference between squares and cubes of integers after N is big enough. If $N = 2^{300}$ then these differences are greater than 2^{100}. Now choose big power of 2 with exponent divisible by 6, for example 2^{30}. Finally choose any $n \geqslant N$ and consider 2^n, 2^{n+30}, 2^{60} and prove that at most one of them is **not** remote from squares and at most one is **not** remote from cubes.
11. Assume the contrary. Clearly all $a_i > 1$. Let $a = ba_1a_2\dots a_n$ and $a^{n+1} + a - 1 = c(a + a_1 - 1)(a + a_2 - 1)\dots(a + a_n - 1)$. Note that $\gcd(b, c) = 1$ and both b and c are less than a. Hence $ba_1\dots a_n \equiv ca_1\dots a_n \mod a - 1$ implies $b = c = 1$. It's easy to obtain $a^{n+1} + a - 1 < a(a/2 + 1)^n \leqslant a^{n+1}$ which is impossible.
12. Consider separately primes which divides $d = \gcd(a, b)$ and which doesn't. If $p|d$ then $v_p(a^n + b^{n+1}) \leqslant v_p(d^{n+1})$ for all large enough n. If $p\nmid d$ then wlog $\gcd(p, ab) = 1$. Let $v_p(b + 1) = k$ and choose m such that $a^m \equiv b^m \equiv 1 \pmod{p^{k+1}}$. For each $n = mt$ we have $a^n + b^{n+1} \equiv b + 1 \pmod{p^k + 1}$ whence $v_p(a^n + b^{n+1}) = k$. Now suppose that that the number of prime divisors is finite and for each $p_i\nmid d$ choose appropriate n_i. Now for any $N = An_1\dots n_s$ we get that $a^N + b^{N+1} \leqslant d^{N+1}(b + 1)$.
13. Assume contrary: each prime divisor p of b_n is a divisor of some b_i, $i < n$. If $v_p(b) = r$ then the sequence (b_j) is periodic modulo p^{r+1} whence $b_n \equiv b_{kn} \pmod{p^{r+1}}$. Similarly $v_p(b_i) \equiv b_{\ell i} \pmod{p^{r+1}}$, hence substitution $k = i$ give us $v_p(b_n) = v_p(b_i)$. So b_n divides $b_1b_2\dots b_{n-1}$, in particular $b_n \leqslant b_1b_2\dots b_{n-1}$. On the other hand for each j holds $b_j = P(b_{j-1}) > b_{j-1}$, therefore $b_1b_2\dots b_{n-1} < b_n^{2^{-1}+2^{-2}+\dots+2^{-(n-1)}} < b_n$.
14. **Hints:** Note that for each b the sequence contains infinitely many b-th powers. Let $d_k = a_{k+1} - a_k$ then $(a_n)_{n\geqslant k}$ is constant modulo d_k. For $\ell = v_p(d_k)$ take $b = p^\ell(p-1)\ell$ and prove that $\ell \leqslant v_p(a_k(a_k - 1))$, whence $d_k|a_k(a_k - 1)$. Therefore for infinitely many integers x holds $|x(x - 1)| \geqslant |P(x) - x|$ so $\deg P \leqslant 2$ and if $\deg P = 2$ then the leading coefficient of P equals ± 1. Consider separately these cases.

Examples

15. Prove that $\binom{k+n}{n}$ can be represented as the product of n pairwise coprime factors: $\binom{k+n}{n} = a_1a_2 \ldots a_n$ where a_j divides $k + j$ for each $j = 1, 2, \ldots, n$.
16. Given two positive integers a and b. Prove that there exist infinitely many positive integers n such that the number $n^b + 1$ does **not** divide $a^n + 1$.
17. Prove that for each pair (a, b) of positive integers there exists a positive integer n such that $n^2 + an + b$ has at least 2021 distinct prime divisors.
18. Prove that for any pair (m, n) of an odd digit $m \neq 5$ and positive integer n there exists a positive integer number N such that the last n digits in the decimal representation of N^3 are equal to m.

Exercises

19. Does there exist a sequence $a_1, a_2, a_3, \ldots$ of positive integers such that each positive integer appears there exactly once and for each $n \geqslant 2$ the product $a_1a_2 \ldots a_n$ is an n-power of a positive integer?
20. Prove that for each positive integer m one can find m consecutive positive integers n such that $(1^3 + 2018^3)(2^3 + 2018^m) \ldots (n^3 + 2018^3)$ is not a perfect power.
21. Show that for each square-free integer $n > 1$ there exist prime divisor $p \mid n$ and integer number m such that $n \mid p^2 + p \cdot m^p$.
22. Prove that there exist infinitely pairs (m, n) such that $m + n \mid (m!)^n + (n!)^m + 1$.

Challenging problems (HOMEWORK)

23. Find all positive integers $n > 1$ such that there exist positive integers b_1, b_2, $\ldots$, b_n (some of them can be equal but not all) such that for all positive integers k the product $(b_1 + k)(b_2 + k) \ldots (b_n + k)$ is a power of positive integer? (The exponent may depend on k but must exceed 1.)
24. Does there exist a sequence (a_n) of positive integers such that each positive integer appears there exactly once and the number $d(na_{n+1}^n + (n + 1)a_n^{n+1})$ is divisible by n for all $n \in \mathbb{N}$?
25. The polynomial $P(n) \in \mathbb{Z}[n]$ for any positive integer n is divisible by at least one number of the set $\{a_1, \ldots, a_m\}$. Prove that there exists i such that for each n the number $P(n)$ is divisible by a_i.
26. Prove that there exists a positive integer n and n^9 positive integers less than n^{10} such that no three of them form an arithmetic progression.
27. Determine if there exist pairwise distinct positive integers a_1, a_2, $\ldots$, a_{10}, b_1, b_2, $\ldots$, b_{10} satisfying the following property: for each non-empty subset S of $\{1, 2, \ldots, 10\}$ the sum $\sum_{i \in S} a_i$ divides $12 + \sum_{i \in S} b_i$.
28. Determine if there exist pairwise distinct positive integers a_1, a_2, $\ldots$, a_{101}, b_1, b_2, $\ldots$, b_{101} satisfying the following property: for each non-empty subset S of $\{1, 2, \ldots, 101\}$ the sum $\sum_{i \in S} a_i$ divides $100! + \sum_{i \in S} b_i$.

Classwork solutions

15. **Hint:** for each prime divisor p of $\binom{k+n}{n}$ chose j with maximal $v_p(k+j)$ and show that $v_p(\binom{k+n}{n}) \leqslant v_p(k+j)$.
16. **Hints:** check that all these construction works: 1) if n is even then choose n or n^3; 2) choose any number 2^k with odd k, except at most one of them; 3) choose a^k where $k > a$ is prime; 4) let p be a prime divisor of $(a(a-1))^b + 1$ and $n_i = a(a-1) + ip$, then n_i or n_{i+1} works.
17. **Hint:** prove by induction. Suppose $m_k = n_k^2 + an_k + b$ has k distinct prime divisors $p_1, p_2, \ldots, p_k$. Consider $n_{k+1} = n_k(m_k^2 + 1)$.
18. If the last n digits in the decimal representation of N^3 are equal to m then for some positive integer k, $N^3 = 10^n k + \frac{10^n-1}{9}m$, hence 10^n divides $9N^3 + m$. Conversely, let $9N^3 + m = 10^n M$, and $M = 9q + r$, $0, \leqslant r \leqslant 9$, then $m \equiv s \pmod 9$. Thus, if $m = s$ or $m = 9$ and $s = 0$, then $N^3 = 10^n r + \frac{10^n-1}{9}m$ or $N^3 = 10^n r - 1$, respectively. Therefore, the last n digits in the decimal representation of N^3 are equal to m if and only if 10^n divides $9N^3 + m$. We will prove the problem statement by induction on the variable n. First, observe that $1^3 = 1$, $7^3 = 3$, $3^3 = 27$ and $9^3 = 729$. Fix any odd digit $m \neq 5$ and suppose that for some $n \geqslant 1$ there exists positive integers N_n and k_n such that $9N^3 + m = 10^n k$. Since the last digit of N_n^3 is m, the last digit of N is an odd digit not equal to 5. Thus, the last digit of $27N_n^2$ is 3 or 7. It is easy to see that there exists an integer α_n such that the number $27N_n^2\alpha_n + k_n$ is divisible by 10. Let $N_{n+1} = 10^n\alpha_n + N_n$, then $9N_{n+1}^3 + m = 9(10^n\alpha_n + N_n)^3 \equiv 10^n(27N_n^2\alpha_n + k_n) \equiv 0 \pmod{10^{n+1}}$, which completes the step of induction and the proof of the problem.
19. **Answer: yes.** Take $a_1 = 1$, $a_2 = 4$ and suppose we have already defined $a_1, \ldots, a_{n-1}$, where $n \geqslant 3$. Let $a_1a_2\ldots a_{n-1} = a$ and b is the minimal positive integer distinct from all $a_1, a_2, \ldots, a_{n-1}$. Choose $a_{n+1} = b$ and $a_n = a^{n^2+n-1}b^n$, clearly $a_n > a_1, \ldots, a_{n-1}$, and a_{n+1}. Wherein $a_1a_2\ldots a_n = (a^{n+1}b)^n$ and $a_1a_2\ldots a_{n+1} = (a^n b)^{n+1}$.
20. **Hint:** Let $p = 3k + 1$ be a prime such that $p > \max(m, 2018)$. Prove that each $n = p - 2019 + i$, $1 \leqslant i \leqslant m$, satisfies $v_p((1^3+2018^3)(2^3+2018^m)\ldots(n^3+2018^3)) = 1$.
21. Let p be the largest prime divisor of n and $q_1, q_2, \ldots, q_k$ — all other prime divisors. Using CRT we will chose m such that $m^p \equiv -p \pmod{q_i}$, $1 \leqslant i \leqslant k$. It's enough to check that at least one such congruence has a solution. For $s \equiv 1/p \pmod{q_i - 1}$ take $m \equiv (-p)^s \pmod{q_i}$.
22. **Hint:** $m = p - 1$ and $n = (p-1)! - p + 2$ works for all sufficiently large primes p.

Hints and solutions

23. **Answer: all composite numbers.** If $n = rs$ where $r > 1$ and $s > 1$, we can choose $b_1 = \ldots = b_r = 1$ and $b_{r+1} = \ldots = b_n = 2$. Now suppose n is prime and $(b_1, \ldots, b_n)$ satisfy the problem conditions. Wlog let $b_1, \ldots, b_q$ be all pairwise different numbers while any number from $b_{q+1}, \ldots, b_n$ is equal to one of them. For each $i = 1, \ldots, q$, by s_i denote the multiplicity of b_i. Take q different primes $p_1, \ldots, p_q$ greater than all b_i. Using CRT choose $m \equiv p_i - b_i \pmod{p_i^2}$, $1 \leqslant i \leqslant q$ and suppose $(b_1 + k)(b_2 + k)\ldots(b_n + k) = u^v$. Then $v_{p_i}(b_1 + k)(b_2 + k)\ldots(b_n + k) = s_i$, $1 \leqslant i \leqslant q$, whence all s_i are divisible by v but their sum is a prime number n, absurd.
24. Choose $a_1 = 1$ and suppose we have already defined $a_1, \ldots, a_n$ where n is odd. Take $a_{n+2} = y$ — the smallest positive integer not chosen yet and denote $a_n = x$. By Dirichlet's theorem there exist primes p and q such that $p \equiv 2 \pmod n$ and $q \equiv 2 \pmod{n+2}$. Using CRT find a_{n+1} such that $na_{n+1}^n + (n+1)x^{n+1} \equiv p^n \pmod{p^{n+1}}$ and $(n+1)y^{n+1} + (n+2)a_{n+1}^{n+2} \equiv q^{n+1} \pmod{q^{n+2}}$.
25. Suppose contrary then for each a_i there exist an integer number x_i and prime p_i such that $v_{p_i}(P(x_i)) \leqslant v_{p_i}(a_i)$. For each prime p_j from $\{p_1, \ldots, p_m\}$ choose the minimal $v_{p_i}(P(x_i))$ over all i such that $p_i = p_j$, let it be $d_j = v_{p_{i(j)}}(P(x_{i(j)}))$. Now solve by CRT the system of $x \equiv x_{i(j)} \pmod{p_j^{d_j+1}}$ for all j.
26. Let $k, p \in \mathbb{N}$. Consider k-digit numbers in base $(2p + 1)$ such that all digits are not bigger than p. The sum of squares of digits of such a number doesn't exceed kp^2 whence at least $q = \frac{(p+1)^k - 1}{kp^2}$ of them have the same sum r. Choose these q numbers: neither three of them form an arithmetic progression (**why?**). So there exist q positive integers less than $(2p + 1)^k$ which satisfy problem conditions. For $p = 2^k - 1$ we obtain more than $\frac{2^{k(k-2)}}{k}$ numbers which are less than $2^{k(k+1)}$. It's enough to take $n = \lceil \sqrt[10]{2^{k(k+1)}} \rceil$ with large enough k.
27. **Hint.** Recall that any set of integers $\{a_1, \ldots, a_n\}$ contains a subset with the sum divisible by n (**why?**). Apply this lemma for $\{a_1, \ldots, a_5\}$ and for $\{a_6, \ldots, a_{10}\}$.

Examples

29. Prove that there is no set of 2021 different positive integers such that the sum of any 2019 of them is divisible by the sum of the rest two numbers.
30. Let A_1, A_2, …, A_k be subsets of $\{1, \dots, n\}$ such that $A_i \cap A_j \neq \varnothing$ for all $1 \leqslant i, j \leqslant k$. Prove that there exist n distinct positive integers x_1, x_2, …, x_n such that for each $1 \leqslant j \leqslant k$ holds $\mathrm{lcm}_{i \in A_j}\{x_i\} > \mathrm{lcm}_{i \notin A_j}\{x_i\}$.
31. Let us call a set of positive integers *nice*, if its number of elements is equal to the average of all its elements. Call a number n *amazing*, if one can partition the set $\{1, 2, \dots, n\}$ into nice subsets. **a)** Prove that any perfect square is amazing. **b)** Prove that there exist infinitely many positive integers which are not amazing.

Exercises

32. Prove that all positive integers can be colored in two colors such that the following conditions hold: 1) for every prime p and every $n \in \mathbb{N}$ the numbers p^n, p^{n+1} and p^{n+2} are not all in the same color; 2) there doesn't exist an infinite geometric progression of numbers in the same color.
33. Prove that there exists a 10×10 table of different positive integers such that, if we define r_i to be the product of elements in the i-th row, and s_i to be the product of elements in the i-th column, then r_1, r_2, …, r_{10} form a nonconstant arithmetic progression, and also s_1, s_2, …, s_{10} form a nonconstant arithmetic progression.

Homework

34. Prove that the vertices of any polyhedron can be labeled with positive integers, so that the labels of every two vertices connected by an edge are not coprime, and the labels of every two vertices not connected by an edge are coprime.
35. Prove that the set of all divisors of a positive integer which is not a perfect square can be divided into pairs so that in each pair one number is divided by another.
36. Dos there exist an infinite sequence of positive integers such that for ever positive integers k , the sum of every k consecutive terms of this sequence is divisible by $k+1$?
37. Given $n \geqslant 2$ distinct positive integers a_1, a_2, …, a_n none of which is a perfect cube. Find the maximal possible number of perfect cubes among their pairwise products.
38. An immortal flea jumps along the coordinate line landing at integer points. It starts from 0, first jump has length 3, the second — length 5, the third — length 9 and so on (the k^{th} jump has length $2^k + 1$). Flea chooses the direction of each jump independently. May it happen that a flea eventually visits all the points with integer coordinates? (The flea can visit the points more than once.)
39. Is it possible to arrange 100 positive integers (not all equal to 1) along the circle so that for any three consecutive numbers one of them is the product of two others?

Test discussion

Problem. Let x, y and z be odd positive integers such that $\gcd(x,y,z)=1$ and the sum $x^2+y^2+z^2$ is divisible by $x+y+z$. Prove that $x+y+z-2$ is not divisible by 3.

Solution. Suppose there exists a prime divisor $p \equiv 2 \pmod 3$ of $x+y+z$. Since $z \equiv -(x+y) \pmod p$ we have $2(x^2+y^2+xy) \equiv 0 \pmod p$. Multiplying by $x-y$ we get $x^3 \equiv y^3 \pmod 3$ but this yields $x \equiv y \pmod p$, because $\gcd(3, p-1)=1$. Similarly $x \equiv z \pmod p$ whence $3x \equiv 0 \pmod p$ which means that x, y and z are divisible by p which contradicts $\gcd(x,y,z)=1$. Hence $x+y+z$ have no prime divisors which have a remainder 2 modulo 3 so $x+y+z-2$ is not divisible by 3.

Homework discussion

28. **Hints.** Choose numbers in order a_1, b_1, a_2, b_2, $\ldots$, starting from $a_1=1$ and $b_1=100!$. Show that it's always possible to satisfy the next conditions: 1) all b_i are divisible by $100!$; 2) all $\sum a_i$ are distinct; 3) sums $\sum a_i$ do not share any prime greater than 2^{101}; 4) $a_i \equiv 1 \pmod p)^2$ for each prime $100 < p < 2^{101}$; 5) $a_i \equiv 1 \pmod p)^{v_p(100!)+1}$ for each prime $p < 101$.

Hints and solutions

29. Suppose such numbers $a_1 > a_2 > \ldots > a_{2021}$ exist and denote their sum by S. Note that $a_1 > S/2021$ whence 2020 numbers: a_1+a_2, a_1+a_3,$\ldots$, a_1+a_{2021} are different divisors of S all of which are bigger than $S/2021$. But there are only 2019 possible divisors: $S/2$, $S/3$, $\ldots$, $S/2021$.
30. **Hint.** Let p_1, p_2, $\ldots$, p_k be pairwise distinct primes and for each $i=1,\ldots,n$ define $x_i = \prod_{i \in A_j} p_j$, then $\operatorname{lcm}_{i\in A_j}\{x_i\} = \prod p_s > \prod_{s\neq j} p_s \geqslant \operatorname{lcm}_{i\notin A_j}\{x_i\}$.
31. **Hints.** Let a_1, a_2, $\ldots$, a_k be the numbers of elements of all nice subsets of a partition then $\sum a_i = n$ and $\sum a_i^2 = n^2$. For part **b)** show that if n is congruent to 2 or 3 modulo 4 these two conditions are impossible. For part **a)** try $k=2$.
32. Color all nonnegative integers in two colors (white and red) so that in each group from 2^i-1 to $2^{i+1}-2$ colors alternate and for even i first number is white while for odd i first number is odd. Now there is no three consecutive numbers of the same color and no infinite arithmetic progression of numbers of the same color (**why?**). It was preliminary coloring, let's introduce final coloring: for a number $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ we use the preliminary color of $\alpha_1+\ldots+\alpha_k$ as the final color of n.
33. First fill cells of a 9×9 square with 81 different primes p_1, p_2, $\ldots$, p_{81} greater than 7 and denote $d = p_1p_2\ldots p_{81}$. Then multiply by d all numbers in cells $A_{i,j}$ where $i \neq j$. Fill the rest 19 cells so that all products r_i and s_i become d^9. Now it's enough to multiply by i the number in each cell $A_{i,i}$.

Exercises

40. **a)** Let (a_n) be an increasing sequence of positive integers. The number a_k is said to be *funny* if it can be represented as the sum of other (not necessarily distinct) terms of the sequence. Prove that only finite number of terms of the sequence can be not funny. **b)** Is the same statement true for the sequence of positive rationals?
41. The sum of several (not necessary different) positive integers not exceeding 10 is equal to S. Find all possible values of S such that these numbers can always be partitioned into two groups with the sum of the numbers in each group not exceeding 70.
42. The numbers 1, 2, $\ldots$, 20 are written on the whiteboard. Bob can choose any two numbers if they differ at least by 2, increase smaller number by 1 and decrease bigger by 1. Find the maximal number of operations which Bob can perform.

Homework

43. There are 330 seats in the first row of the auditorium. Some of these seats are occupied by 25 viewers. Prove that among the pairwise distances between these viewers there are two equal.
44. For a set M of positive integers with n elements, where n is odd, a nonempty subset T of M is called *good*, if the product of the elements of T is divisible by the sum of the elements of M, but not divisible by its square. If M is good, find the maximum possible number of the good subsets of M.
45. Given m, n such that $m > n^{n-1}$ and the numbers $m+1$, $m+2$, $\ldots$, $m+n$ are composite. Prove that there exist distinct primes p_1, p_2, $\ldots$, p_n such that $m+k$ is divisible by p_k for each $k = 1, 2, \ldots, n$.
46. Let p be the an odd prime. For an integer k, $1 \leqslant k \leqslant p-1$, let a_k be the number of divisors of $kp+1$, greater than or equal to k and less than p. Determine the value of $a_1 + a_2 + \ldots + a_{p-1}$.

Homework discussion

34. Label each edge with prime number so that all primes are pairwise distinct. Then label each vertex with the product of labels of incident edges.
35. Let n be the given number. Since n is not a perfect square, $v_p(n)$ is odd for some prime p. Consider the next bijection: for $d \mid n$ where $v_p(d)$ is even, correspond pd.
36. Suppose such sequence exists. Consider $a_1, a_2, \ldots, a_{2k-1}$. The sum $a_1 + \ldots + a_{2k-1}$ is divisible by $2k$ and the sums $a_2 + \ldots + a_k$ and $a_{k+1} + a_{2k-1}$ are both divisible by k. Hence $k \mid a_1$ for all $k \in \mathbb{N}$.
37. Note that if a, b and c are not cubes then at most two numbers among ab, ac and bc are cubes. Consider the graph G with vertices a_i and edges corresponding to products which are cubes. Since G doesn't have 3-cliques, it has at most $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil$ edges (and G must be bipartite with parts of sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$). To obtain an example label vertices of one part with different numbers of the form 2^{3k+1} and vertices of one part — with different numbers of the form 2^{3k+2}.
38. It's enough to show that the flea can always change it's position by one. Suppose it have jumped $k-1$ times then the next jump will have length 2^k+1. Denote $m = 2^k$ and consider the equality $(2^k+1) + (2^{k+1}+1) + \ldots + (2^{k+m-1}+1) = (2^{k+m}+1) - 1$.
39. Suppose such numbers exist: $a_1, a_2, \ldots, a_{100}$ and let p be a prime divisor of their product. The exponents $v_p(a_i)$ are nonnegative integers and for any three consecutive exponents one of them is the sum of two others. Since we can divide all these exponents by their gcd, assume that at least one exponent, lets' say, $v_p(a_1)$ is odd. Consider all exponents modulo 2 and denote $v_p(a_i) \equiv r_i \pmod 2$. Note that after $r_1 = 1$ there must be $\{r_2, r_3\} = \{0, 1\}$ and among any three consecutive numbers r_i, r_{i+1}, r_{i+2} exactly one is 0 and two are 1. Whence the sequence (r_i) is periodic with the length 3 of the period. Therefore $r_{100} = r_1 = 1$ and $r_2 = r_{202} = r_1 = 1$ — a contradiction.

Hints and solutions

40. **a)** For each remainder $r \in \{0, 1, \ldots, a_1 - 1\}$ denote by n_r the minimal index such that $a_{n_r} \equiv r \pmod{a_1}$ if such index exists, otherwise put $n_r = 0$. Then all numbers a_n with $n > \max(n_0, n_1, \ldots, n_{a_1-1})$ are funny. **b)** Consider the sequence $a_n = n + 10^{-n}$.
41. **Answer:** $S \leqslant 133$. Clearly $S \leqslant 140$. Suppose $S \geqslant 134$ and let $S = 134 + m$, $0 \leqslant m \leqslant 6$. Consider the next collection: one number equals 8, m numbers equal 10 and $14 - m$ numbers equal 9. At least 8 of them will be in the same group, but the sum of the smallest eight numbers is not less than 72. Suppose $S \leqslant 133$. We will successive put numbers to the first group and at some moment the sum of all numbers of this group will satisfy the next two conditions: 1) $60 < A \leqslant 70$; and 2) $A + a > 70$ for any remaining number a. If $A \geqslant 63$, then $S - A \leqslant 70$ and we can form the second group from all remaining numbers. If $A \leqslant 62$ then $A = 62 - x$, where x is 0 or 1. For any remaining number a the inequality $A + a > 70$ is equivalent to $a \geqslant 9 + x$. If there are not more than 7 numbers left, their sum does not exceed 70 and we can form the second group from them. Otherwise the sum $S - A$ of the remaining numbers is not less than $8 \cdot (9 + x) = 72 + 8x$, which is impossible since then $S \geqslant 134 + 7x > 133$.
42. **Hint.** The sum of squares of all numbers is decreasing by at least 2. Prove that Bob can obtain a unique final set of numbers decreasing the sum of squares by exactly 2 each turn.

Homework discussion

43. Let the numbers of viewers seats be $a_1 < a_2 < \ldots < a_{24} < a_{25}$. Clearly $a_{25} - a_1 \leqslant 329$, consider the equalities: $a_{25} - a_1 = (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_{24} - a_{23}) + (a_{25} - a_{24})$ and $a_{25} - a_1 = (a_3 - a_1) + (a_5 - a_3) + \ldots + (a_{23} - a_{21}) + (a_{25} - a_{23})$. If all 36 differences are pairwise distinct then $2(a_{25} - a_1) \geqslant 1 + 2 + \ldots + 36$, whence $a_{25} - a_1 \geqslant 333$.
44. Clearly both a subset or it's complementary cannot be good hence the total number of good subsets is not greater than 2^{n-1}. We will provide an example that this bound is sharp. Let $n = 2k + 1$, choose prime $p > \frac{n(n-1)}{2}$ and set $a_i = i$ for $1 \leqslant i \leqslant n - 1$ and $a_n = p^k - \frac{n(n-1)}{2}$. Finally consider the set $M = \{pa_i \mid i = 1, 2, \ldots, n\}$.
45. Note that it's enough to consider only numbers $m + k$ which have less than n prime divisors. For each such number choose $p_k \mid m + k$ such that $p_k^{v_{p_k}(m+k)} \geqslant n$. It's easy to see that such p_k exist and are pairwise distinct.
46. Each r, $1 \leqslant r \leqslant p - 1$, is counted exactly once which whence the answer is $p - 1$: since a_k counts only divisors which are greater or equal to k, it's enough to consider $p + 1, \ldots, rp + 1$ and they cover all remainders modulo r exactly once.

Harder exercise

47. Let N be a positive integer. Consider a sequence $a_1, a_2, \ldots, a_N$ of positive integers, none of which is a multiple of 2^{N+1}. For $n \geqslant N + 1$ the number a_n is defined as follows: Choose k to be the number among $1, 2, \ldots, n - 1$ for which the remainder obtained when a_k is divided by 2^n is the smallest, and define $a_n = 2a_k$ (if there are more than one such k, choose the largest such k). Prove that there exist M for which $a_n = a_M$ holds for every $n \geqslant M$.

Solution

47. Let $m = \min\{a_1, \ldots, a_N\}$ and $L_n = \max\{a_1, \ldots, a_{n-1}\}$, clearly $m = \min\{a_1, \ldots, a_n\}$ for each $n \geqslant N$. If $L_M < 2^M$ for some $M \geqslant N$ then $L_n < 2^n$ for all $n \geqslant M$ whence $a_n = 2m$ for all $n \geqslant M$. Suppose $L_n \geqslant 2^n$ for all $n \geqslant N + 1$ and let A be the largest nonnegative integer such that $L_n \geqslant 2^{n+A}$ for all $n \geqslant N+1$. Then $2^{\ell+A} \leqslant L_n < 2^{\ell+A+1}$ for some $\ell \geqslant N + 1$. Since $L_{\ell+1} \geqslant 2^{n+A+1}$ we know that $a_\ell = L_{\ell+1}$, and since $L_{\ell+2} \geqslant 2^{n+A+2}$ it follows that $a_{\ell+1} = 2a_\ell$. Now $2^{\ell+A+1} \leqslant L_{n+1} < 2^{\ell+A+2}$ and by induction $a_n = 2^{n-\ell}a_\ell$ for every $n \geqslant \ell$. From the problem statement the remainder of a_n modulo 2^{n+1} is less or equal than m for each $n \geqslant \ell$ but this remainder doubles starting from $n = \ell$, therefore it must be 0. This implies that a_n is a multiple of 2^{n+1} for each $n \geqslant \ell$ which is impossible.

Harder exercises

48. Peter wrote 100 distinct integers on a board. Basil needs to fill the cells of a table 100×100 with integers so that the sum in each rectangle 1×3 (either vertical, or horizontal) is equal to one of the numbers written by Peter. Find the greatest n such that Basil can always fill the table so that it would contain each of 1, 2, $\ldots$, n at least once (and possibly some other integers).
49. Let n be a positive integer and S be the set of $2n$ positive integers. A pairing of elements of S is called *squarefree* if none of the products of numbers in pairs is a perfect square. Suppose S has a squarefree pairing. Prove that it has at least $n!$ squarefree pairings.
50. Let n be a positive integer. What is the smallest value of m, $m > n$, such that the set $M = \{n, n+1, \ldots, m\}$ can be partitioned into subsets so that in each subset there is a number which equals to the sum of all other numbers of this subset?

Hints and solutions.

48. **Answer:** 6. Basil can choose any number $a \geqslant 9$ written on the board and fill the 3×3 table with the numbers 1, 5, $a_1 - 6$, 6, $a_1 - 8$, 2, $a_1 - 7$, 3, 4 and then fill all table repeating this pattern. Assume that $n \geqslant 7$ and let all Peters numbers be divisible by 19. Color the cells of a 3×3 table with 9 colors and fill all table repeating this pattern. Then all number of the same color are congruent modulo 19. Since we have 7 remainders from 1 to 7, at least 3 of them will be in the same rectangle 1×3 but it's impossible: their sum is a most $5 + 6 + 7 < 19$.
49. For each $m \in S$ by $P(m)$ denote the set of all prime divisors of m such that $v_p(m)$ is odd. Since there exists a squarefree pairing, for each set of primes P there is at most n numbers of S such that $P = P(m)$. Let's call sets of primes *colors* and color each $m \in S$ with the color $P(m)$. Prove by induction the next Lemma: if $2n$ elements are colored so that there is at most n elements of the same color, then there exist at least $n!$ pairings such that elements of each pair have different colors.
50. **Hints.** Suppose the set is partitioned into k subsets. Note that each subset has at least 3 elements, this gives an upper bound for k. The sum of all numbers is twice the sum of maximal numbers of all subsets, this gives an upper bound for the sum of all numbers. Deduce that $m \geqslant 7n - 4$ and show that this value is always possible.

Challenging problems

51. Find all positive integers $n \geqslant 3$ such that it is possible to mark the vertices of a regular n-gon with the numbers from 1 to n so that for any three vertices A, B and C with $AB = AC$ the number in A is greater or smaller than both numbers in B and C.
52. **(5-min exercise)** Find all positive integers n which can be represented in the form $n = \operatorname{lcm}[a, b] + \operatorname{lcm}[b, c] + \operatorname{lcm}[c, a]$ where $a, b, c \in \mathbb{N}$.
53. For a positive integer k denote by $f(k)$ the number of positive integers m such that the remainder of km modulo 2019^3 is greater than m. Find the amount of different numbers among $f(1)$, $f(2)$, $\ldots$, $f(2019^3)$.
54. Prove that it's impossible to fill the cells of an 8×8 table with the numbers from 1 to 64 (each number must be used once) so that for each 2×2 square the difference between products of the numbers on it's diagonals will be equal 1.

Hints and solutions.

51. **Answer:** n is a power of two. First consider only triples of consecutive vertices. The largest and the smallest vertices must alternate whence n is even. Note that if $m \mid n$ and n satisfies the problem then m satisfies the problem, therefore n must be a power of 2. For powers of 2 construct an example by induction: take two examples for $n = 2^k$, increase by 2^k all numbers in one of them, and combine them alternating the vertices through one.
52. **Answer:** n is not a power of two. First note that $(a, b, c) = (k, 1, 1)$ gives $n = 2k+1$ whence all odd integers can be represented. Then show that if n can be represented then any multiple of n can be represented, whence all numbers except the powers of two can be represented. Finally show that if $n = 2^k$ can be represented then 2^{k-1} can be represented, while 2^1 cannot, and use the descending induction.
53. Let $2019^3 = N$, $k, m \in \mathbb{N}$ and $1 \leqslant k \leqslant N$. When we calculate $f(k)$ it's enough to consider $m < N$. Denote $km \mod N$ by $r_k(m)$. Since $N \mid km + k(N-m)$, we know that $r_k(m) = 0$ or m iff $r_k(N-m) = 0$ or $N-m$. If $r_k(m) \neq 0$ or m, then $r_k(N-m) = N - r_k(m)$ and $r_k(N-m) - (N-m) = -(r_k(m) - m)$ whence exactly one of $r_k(m)$ and $r_k(N-m)$ is counted. Let $g(k)$ be the number of all m such that $r_k(m) = 0$ or m then $f(k) = (N - 1 - g(k))/2$ so it's enough to count distinct numbers among $g(1)$, $g(2)$, $\ldots$, $g(N-1)$. Since $r_k(m) = 0$ or m iff $N \mid km$ or $N \mid (k-1)m$. These amounts equal to $\gcd(k, N) - 1$ and $\gcd(k-1, N) - 1$ and these two sets do not overlap, hence $g(k) = \gcd(k, N) - 1 + \gcd(k-1, N) - 2$. Note that $g(k) + 2$ is a sum of two relatively prime divisors of N. If one of $\gcd(k, N)$ and $\gcd(k-1, N)$ is a multiple of 2019 then the other is 1, whence if $2019 \mid k$ and $k \mid 2019^3$ then $g(k) = k - 1$. When neither $\gcd(k, N)$ nor $\gcd(k-1, N)$ is a multiple of N then one of them is a power of 3 and another is a power of 673. The first case give us $3 \cdot 3$ numbers and the second — $4 \cdot 4$ all these numbers are different and exists since CRT. So the sum of $g(k)$ is 25.
54. **Hint:** note that in each 2×2 square the number on two diagonals have different parity: the two odd numbers a, d on one and two even numbers b and c on another diagonal. Prove that $\frac{bc}{ad} \leqslant \frac{(a+1)(d+1)}{ad}$ and multiply all such inequalities.

Homework discussion.

55. Let S be a set of positive integers with 100 digits. A number from S is called *bad* if it is not divisible by the sum of nay two (not necessarily distinct) numbers from S. Find the maximal possible number of elements of S if it is known that S contains at most 10 bad numbers.
56. An infinite table whose rows and columns are numbered with positive integers, is given. For a sequence of functions $f_1(x)$, $f_2(x)$, ... let us place the number $f_i(j)$ into the cell (i, j) of the table (for all $i, j \in \mathbb{N}$). A sequence $f_1(x)$, $f_2(x)$ ... is said to be *nice* if all the numbers in the table are positive integers, and each positive integer appears there exactly once. Determine if there exists a nice sequence of integer monic polynomials $f_1(x)$, $f_2(x)$, ... of degree 101.

Numbers in combinatorics.

57.[1] The Magician and his Assistant show trick. The Viewer writes on the board the sequence of N digits. Then the Assistant covers some pair of adjacent digits so that they become invisible. Finally, the Magician enters the show, looks at the board and guesses the covered digits and their order. Find minimal N such that the Magician and his Assistant can agree in advance so that the Magician always guesses right.
58. Given an infinite sequence of numbers a_1, a_2, a_3, ... such that for each $k \in \mathbb{N}$ there exists $t \in \mathbb{N}$ for which $a_k = a_{k+t} = a_{k+2t} = \ldots$. Does this sequence must be periodic (i. e. $a_k = a_{k+T}$ for some fixed T and each k)?
59. Find the minimal possible number of colors which is required to color all vertices, edges and diagonals of a convex n-gon so that: 1) if two segments share an endpoint they has different color, 2) the color of every vertex differs from the color of any adjacent segment.
60. **a)** Each of N people have chosen some 5 elements from a 23-element set so that any two people share at most 3 chosen elements. Does this mean that $N \leqslant 2020$? **b)** Answer the same question with 25 instead of 23.

[1] Problem from the Final Prep in September 2020. Remember the explicit construction for $N = 101$.