My plan

Levels ~ JBMO, BMO, P1/P4 IMO | EGMO

- Algebra manipulations
  - system of eq.
  - min/max
  - equality
  - inequalities
  - AM–GM
  - C–S
- Induction
  - $f: \mathbb{Z} \to \mathbb{Z}$, $\mathbb{N} - \mathbb{R}$
  - sequences
  - domain $\mathbb{N}, \mathbb{Q}, \mathbb{Z}$.
- Polynomials
  - $a - b \mid f(a) - f(b)$
  - poly equations
  - number of roots
- Functional equations
  - $\mathbb{R} \to \mathbb{R}$

$\boxed{P1}$ In $\mathbb{R}$ solve / Polis MO 1 stage

$$\begin{cases} (x+y)^3 = 8z \\ (y+z)^3 = 8x \\ (z+x)^3 = 8y \end{cases}$$

$\boxed{P2}$ <u>Pole MO P4 on 2 stage</u>

$\Downarrow$

$xy = 0$

$x = 0$

$\boxed{y=0}$ $a^3 = a \rightsquigarrow a = 1,\ a = -1$

≥ 3 pts

$$\begin{cases} (x+y)^3 = 8z & (1) \\ (y+z)^3 = 8x & (2) \\ (z+x)^3 = 8y & (3) \end{cases}$$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^2-b^2=(a-b)(a+b)$

$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\dots+b^{n-1})$

$(a+b)^2=a^2+2ab+b^2$

$(a-b)^2=a^2-2ab+b^2$

<u>Idea</u>

$(1)-(2)$

$$(\underbrace{x+y}_{A})^3-(\underbrace{y+z}_{B})^3=8(z-x)$$

$$A^3-B^3$$

$$\parallel$$

$$(A-B)(A^2+AB+B^2)=8(z-x)$$

$A-B$?

$$\parallel$$

$$x+y-(y+z)=x-z$$

$$\cancel{(x-z)}(A^2+AB+B^2)=\cancel{8(z-x)}$$

NO!

$$(x-2)(A^2+AB+B^2+8)=0$$

$$\boxed{\text{YES}}$$

$$\Downarrow$$

1° : $x=2$

2° $\boxed{A^2+AB+B^2+8}=0$

$$A^2+2AB+B^2 \geq 0$$

AB

$$(A+B)^2 \geq AB$$

$$2A^2+2AB+2B^2 \geq 0$$

$$(A+B)^2+A^2+B^2 \geq 0$$

$$\Downarrow$$

$$\boxed{x=y=z}$$

$$(x+y)^3 = 8z \Rightarrow (2x)^3 = 8x$$

$$8x^3 = 8x$$

$$\boxed{x^3 = x}$$

$(0,0,0)$, $(1,1,1)$, $(-1,-1,-1)$

$$\begin{cases} (x+y)^3 = 8z \\ (y+z)^3 = 8x \\ (z+x)^3 = 8y \end{cases}$$

$$S = x+y+z$$

$$\begin{cases} (S-z)^3 = 8z \\ (S-x)^3 = 8x \\ (S-y)^3 = 8y \end{cases}$$

cyclic + symmetric

$$x \geq y \geq z$$

$$\boxed{x \geq y}$$

$$(S-x)^3 \leq (S-y)^3$$

$$\cancel{8}x \leq \cancel{8}y$$

$$\boxed{x \leq y}$$

$$\boxed{x = y}$$

$y = z$

$z = x$

IN REALS

$$\boxed{(x^4+3y^2)}\sqrt{|x+2|+|y|} = 4|xy^2|$$

$$\boxed{(x^4+y^2+y^2+y^2)}$$

$$|x+2|+|y| \geq |y|$$

$\boxed{|x+2|=0}$ or $x^4+3y^2=0$ $\rightarrow$ $(x,y)=(0,0)$

$x=-2$

$$(x^4+3y^2)\sqrt{|x+2|+|y|} \geq (x^4+3y^2)\cdot\sqrt{|y|}$$

$$\parallel$$

AM-GM

$$x^4+y^2+y^2+y^2 \geq 4\cdot\sqrt[4]{x^4y^2y^2y^2} =$$

$$= 4\sqrt[4]{x^4y^6} =$$

$$= 4|xy|\cdot\sqrt{|y|}.$$

$$\boxed{\begin{aligned}(x^4+3y^2)\sqrt{|x+2|+|y|} &\geq 4|xy|\cdot\sqrt{|y|}\cdot\sqrt{|y|}\\ &= 4|xy^2|\end{aligned}}$$

$x^4=y^2 \rightsquigarrow x^2=|y|$

$\underline{x=-2}$

$4=|y|$

$y=-4, 4$

$(0,0)$

$(-2,4)$

$(-2,-4)$

$(-2,0)$

$\boxed{y=0} \rightsquigarrow \boxed{x=-2}$

- $|x+2|+|y| \geq |y|$

$$(x^4+3y^2)\sqrt{|x+2|+|y|} \geq (x^4+3y^2)\sqrt{|y|}$$

$$=$$

iff

$$x^4+3y^2=0 \text{ or } |x+2|=0$$

$$\parallel$$

$$(0,0) \text{ or } \boxed{x=-2}$$

AM-GM

- $x^4+3y^2 \geq 4|xy|\sqrt{|y|}$

$$x^4=y^2 \rightarrow x^2=|y|$$

$$(x^4+3y^2)\sqrt{|y|} \geq 4|xy^2|$$

$$=$$

$$x^2=|y| \text{ or } y=0$$

$(0,0)$
$(-2,0)$
$(-2,4)$

$(-7,-4)$

P3 | Solve in $\mathbb{R}_+$

$$\begin{cases} a^3+b^3+c^3 = 3d^3 \\ b^4+c^4+d^4 = 3a^4 \\ c^5+d^5+a^5 = 3b^5 \end{cases}$$

$M = \max\{a,b,c,d\}$

$m = \min\{a,b,c,d\}$

Claim One of the numbe a, b or d is equal to m or M

if $c = m = M \Rightarrow$

$a = b = c = d = m = M$

1° case $\boxed{b = m}$

$$3b^5 = c^5 + d^5 + a^5 \geq b^5 + b^5 + b^5$$
$$= 3b^5$$

$c = b$, $d = b$ $a = b$

$b = M$

$$3b^5 = c^5 + d^5 + a^5 \leq b^5 + b^5 + b^5$$
$$= 3b^5$$

2° $d = m/M$

$$3d^3 \leq a^3 + b^3 + c^3 = 3d^3$$

PMO 2 st. $\boxed{P1}$

4) For any $n \geq 3$ find all sequences

$$(x_1, x_2, \ldots, x_n):$$

$$\sum_{i=1}^{n} x_i = n, \qquad \sum_{i=1}^{n} (x_{i-1} - x_i + x_{i+1})^2 = n$$

$$x_0 = x_n$$

$$x_{n+1} = x_1$$

A-Q,

C-S

(4, 5) $\boxed{\text{at home}}$