May Online Camp 2021

Number Theory

Level L4

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(31-05-2021, 11:08)

Problems

Problem 1. Let p_i for $i = 1, 2, \ldots, k$ be a sequence of consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 5 \ldots$). Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_k$. Prove that in a set $\{1, 2, \ldots, N\}$ there are exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

Problem 2. Find all sets of positive integers $\{x_1, x_2, \ldots, x_{20}\}$ such that

$$x_{i+2}^2 = \operatorname{lcm}(x_{i+1}, x_i) + \operatorname{lcm}(x_i, x_{i-1})$$

for $i = 1, 2, \ldots, 20$ where $x_0 = x_{20}$, $x_{21} = x_1$, $x_{22} = x_2$.

Problem 3. Let $n > 1$ be odd integer. Consider numbers n, $n+1$, $n+2$, $\ldots$, $2n-1$ written on the blackboard. Prove that we can erase one number, such that the sum of all numbers will be not divided any number on the blackboard.

Problem 4. Let $n > 20$ and $k > 1$ be integers such that k^2 divides n. Prove that there exist positive integers a, b, c, such that

$$n = ab + bc + ca.$$

Problem 5. For the triple (a, b, c) of positive integers we say it is *interesting* if $c^2 + 1 \mid (a^2 + 1)(b^2 + 1)$ but none of the $a^2 + 1$, $b^2 + 1$ are divisible by $c^2 + 1$. Let (a, b, c) be an interesting triple, prove that there are positive integers u, v such that (u, v, c) is interesting and $uv < c^3$.

Pro☠lem 6. Consider a square-free even integer n and a prime p, such that

(1) $(n, p) = 1$;
(2) $p \leqslant 2\sqrt{n}$;
(3) There exists an integer k such that $p \mid n + k^2$.

Prove that there exists pairwise distinct positive integers a, b, c such that $n = ab + bc + ca$.

Problem 7. Let p, q be primes such that $p < q < 2p$. Prove that there are two consecutive positive integers, such that largest prime divisor of first number is p, and the largest prime divisor of second number is q.

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that

$$a = \frac{x+y}{z} \quad \text{and} \quad b = \frac{xy}{z}.$$

Problem 9. Let S be the set of all positive integers that are not perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \ldots < a_r$ and $n \cdot a_1 \cdot a_2 \ldots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Problem 10. Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n - a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Amy prevent Bob win?

Problem 11. Let $n \geqslant 50$ be a natural number. Prove that n is expressible as sum of two natural numbers $n = x + y$, so that for every prime number p such that $p \mid x$ or $p \mid y$ we have $\sqrt{n} \geqslant p$.

Problem 12. Each cell of a $3 \times n$ table was filled by a number. In each of three rows, the number $1, 2, \ldots, n$ appear in some order. It is know that for each column, the sum of pairwise product of three numbers in it is a multiple of n. Find all possible value of n.

Problem 13. Let $p > 3$ be a prime. Prove that there is a positive integer $y < \frac{p}{2}$ and such that $py+1$ cannot be represented as a product of two integers, each of which is greater than y.

Problem 14. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a+b+c+d$ and s divides $abc+abd+acd+bcd$.

Problem 15. Let a, b, c, d be positive integers such that $ad \neq bc$ and $\gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an + b, cn + d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Problem 16. Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Hamza may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Hamza can write 1 on the board in finitely many steps.

Problem 17. Let k, n be a positive integers such that $k > n!$. Prove that there exist distinct prime numbers $p_1, p_2, \ldots, p_n$ such that $p_i \mid k + i$ for all $i = 1, 2, \ldots, n$.

Problem 18. For any integer $N \geq 2$, let $f(N)$ denotes sum of N and the greatest divisor of N (other than N). Prove that for any integer $A \geq 2$, by iterating f on A we can get a number divisible by 3^{2021}.

Problem 19. Let m, n be a positive integers such that set $\{1, 2, \ldots, n\}$ contains exactly m different prime numbers. Prove that if we choose any $m + 1$ different numbers from $\{1, 2, \ldots, n\}$ then we can find number from $m + 1$ choosen numbers, which divide product of other m numbers.

Problem 20. Integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \ldots a_n$, then prove that

$$(a_1 a_2 \ldots a_n)^{m-1} \geqslant (n!)^m.$$

Solutions

Problem 1. Let p_i for $i = 1, 2, \dots, k$ be a sequence of consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 5 \dots$). Let $N = p_1 \cdot p_2 \cdot \dots \cdot p_k$. Prove that in a set $\{1, 2, \dots, N\}$ there are exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

Solution. Let's call the numbers which are in $\{1, 2, \dots, N\}$ and divisible by odd number of p_i's nice. We claim that: If $1 \le n \le \frac{N}{2}$, then exactly one of the numbers $\{n, n + \frac{N}{2}\}$ is lucky.

Indeed: let $n = p_{i_1}^{r_1} \cdot p_{i_2}^{r_2} \dots p_{i_m}^{r_m}$. Then

$$n + \frac{N}{2} = p_{i_1}^{r_1} \cdot p_{i_2}^{r_2} \dots p_{i_m}^{r_m} + p_2 \cdot p_3 \dots p_k.$$

Note that n and $n + \frac{N}{2}$ have the same set of prime divisors among $\{p_2, p_3, \dots, p_k\}$. Notice also that the parity of n and $n + \frac{N}{2}$ are different. So one of them is lucky and other is not, as desired. $\square$

Discussion.

Problem 2. Find all sets of positive integers $\{x_1, x_2, \dots, x_{20}\}$ such that

$$x_{i+2}^2 = \operatorname{lcm}(x_{i+1}, x_i) + \operatorname{lcm}(x_i, x_{i-1})$$

for $i = 1, 2, \dots, 20$ where $x_0 = x_{20}$, $x_{21} = x_1$, $x_{22} = x_2$.

Solution. Firstly, notice that for any i, $\gcd(x_{i+1}, x_i) \ge 2$. Indeed, there exists i so that x_i is divisible by prime p, since otherwise $x_i = 1$ for all i, which does not satisfy the given. Since

$$x_{i+3}^2 = \operatorname{lcm}(x_{i+2}, x_{i+1}) + \operatorname{lcm}(x_{i+1}, x_i) \quad \text{and} \quad x_{i+2}^2 = \operatorname{lcm}(x_{i+1}, x_i) + \operatorname{lcm}(x_i, x_{i-1}),$$

thus

$$(x_{i+3} - x_{i+2})(x_{i+3} + x_{i+2}) = \operatorname{lcm}(x_{i+2}, x_{i+1}) - \operatorname{lcm}(x_i, x_{i-1}).$$

As $p \mid x_i$, we get $p \mid x_{i+2}$, and therefore from the above equality $p \mid x_{i+3}$, inducing all up, we have every x_i divisible by p. Therefore, for every i, $\gcd(x_{i+1}, x_i) \ge 2$.

We sum all the equations up and obtain that

$$2 \sum_{i=1}^{20} \operatorname{lcm}(x_{i+1}, x_i) = \sum_{i=1}^{20} x_i^2 \ge \sum_{i=1}^{20} x_i x_{i+1},$$

where equality holds if and only if $x_i = x_{i+1}$ for all i. Now we rewrite

$$\operatorname{lcm}(x_{i+1}, x_i) = \frac{x_i x_{i+1}}{\gcd(x_{i+1}, x_i)}$$

and as $\gcd(x_{i+1}, x_i) \geq 2$, we conclude that the inequality must hold, therefore all integers are equal. Now, we must have $x_i^2 = 2x_i \implies x_i = 2$.

$\square$

Discussion.

Problem 3. Let $n > 1$ be odd integer. Consider numbers n, $n+1$, $n+2$, $\ldots$, $2n-1$ written on the blackboard. Prove that we can erase one number, such that the sum of all numbers will be not divided any number on the blackboard.

Solution. Let S be the sum of all numbers on the blackboard ,

$$S = n^2 + \frac{n(n-1)}{2} = \frac{n(3n-1)}{2}.$$

If we erase any number x the sum will be $S - x$. If two of $S - x$'s are divided by the same number then

$$S - x \equiv S - y \pmod{n+i} \iff x \equiv y \pmod{n+i}$$

which is absurd, so all of $n-1$, $S-x$'s have different divisors from the set of written numbers on blackboard.

Assume that all $S - x$'s are divided by a written number, then all of them must be divided by exactly one of the written numbers (because there are $n-1$ numbers and $n-1$ sums). But taking $x = n$ the sum is $\frac{n(3n-3)}{2}$ which is divided by both n and $\frac{3n-3}{2}$ (because n is odd and $n < \frac{3n-3}{2} < 2n-1$ is a written integer – contradiction. $\square$

Discussion.

Problem 4. Let $n > 20$ and $k > 1$ be integers such that k^2 divides n. Prove that there exist positive integers a, b, c, such that

$$n = ab + bc + ca.$$

Solution. So note that if $n = ab + bc + ca$, then $n + a^2 = (a+b)(a+c)$, so we have to construct a for each non-squarefree n, such that $n + a^2$ is representable as the product of two numbers, bigger than a.

Consider prime p, such that $n = p^2 l$. So firstly consider whether we can take $a = p$. We want $(l+1)p^2$ to represented as a product in the above way. If $l + 1 > p$, we have It. If $l+1$ is composite, then it is st, and take ps and pt. So we are left with the case when it's prime q, so $n = (q-1)p^2$.

Now take $p = mq + r$, where r is the remainder (now we look in the case where r is positive integer), and choose $a = r$ and the rest is to choose the one number to be $q > r$, the other is bashed to be bigger than r. We are only left to consider $q = p$, then choose $a = 6$ and note that $p^3 - p^2 + 36 = (p+3)(p^2 - 4p + 12)$. Since $n > 20$, then $p > 3$ and we are done.

$\square$

Discussion.

Problem 5. For the triple (a,b,c) of positive integers we say it is *interesting* if $c^2+1 \mid (a^2+1)(b^2+1)$ but none of the a^2+1, b^2+1 are divisible by c^2+1. Let (a,b,c) be an interesting triple, prove that there are positive integers u, v such that (u,v,c) is interesting and $uv < c^3$.

Solution. If the product $(a^2+1)(b^2+1)$ is divisible by c^2+1, then c^2+1 can be decomposed into the product of two factors X and Y such that a^2+1 is a multiple of X and b^2+1 is a multiple of Y. (For this, it suffices, for example, to set $X = \gcd(a^2+1, c^2+1)$). We assume, without loss of generality, that $X \geq Y$, then $Y \leq c$.

Let $u = a \pmod{c^2+1}$ and $v = b \pmod{Y}$. Then $u \leq c^2$, $v < Y \leq c$, whence $uv < c^3$. It is easy to see that u^2+1 is divisible by X but not divisible by c^2+1, and v^2+1 is divisible by Y but not divisible by c^2+1. Therefore, the product $(u^2+1)(v^2+1)$ is divisible by $XY = c^2+1$. Thus, (u,v,c) is interesting triple. □

Discussion.

Pro☠lem 6. Consider a square-free even integer n and a prime p, such that

(1) $(n,p) = 1$;
(2) $p \leqslant 2\sqrt{n}$;
(3) There exists an integer k such that $p \mid n + k^2$.

Prove that there exists pairwise distinct positive integers a, b, c such that $n = ab + bc + ca$.

Solution. Let $k \equiv m \pmod{p}$ where $0 \leqslant m < p$. Note that $m > 0$ since $p \mid k$ would imply that $p \mid n$ which is a contradiction. Since $\gcd(n,p) = 1$ and n is even, it follows that p is odd. Hence $p - m$ and m are of different parity. Let c be the odd positive integer in the set $\{m, p-m\}$. Since $0 < m < p$ and $p \mid n + k^2$, it follows that $c > 0$ and that $p \mid n + c^2$. Now let $pq = n + c^2$ where $q \in \mathbb{N}$. Now note that by AM-GM,

$$q = \frac{n+c^2}{p} \geqslant \frac{2c\sqrt{n}}{p} \geqslant c.$$

However, since n is square-free, it cannot follow that $n = c^2$ and therefore the inequality is strict and $q > c$. Now let $a = q - c$ and $b = p - c$ and note that

$$n + c^2 = pq = (a+c)(b+c) = ab + ac + bc + c^2.$$

This implies that $n = ab + bc + ca$ where $a, b, c > 0$ since $c > 0$, $q > c$ and $p > c$. Now it remains to show that a, b and c are pairwise distinct. If $b = c$, then $p = 2c$ which contradicts the fact that p is odd. If $a = b$, then $p = q$ which implies that $n = p^2 - c^2 = (p-c)(p+c)$. However, since p and c are both odd, $2 \mid p - c$ and

$2 \mid p+c$ and hence $4 \mid n$ which contradicts the fact that n is square-free. If $a=c$, then $q=2c$ and $2pc-n=c^2$ which is a contradiction since n is even and c is odd. Hence a, b and c are pairwise distinct as desired. $\square$

Discussion.

Problem 7. Let p, q be primes such that $p<q<2p$. Prove that there are two consecutive positive integers, such that largest prime divisor of first number is p, and the largest prime divisor of second number is q.

Solution. We know $qb-pa=1$ for some positive integers a,b with $1 \leq b \leq p$, $1 \leq a \leq q$; this is straightforward Bézout and noticing that if (a,b) is solution, then so is $(a-q, b-p)$.

If $a \leq \frac{q}{2}$ then $b \leq \frac{p}{2}$, and the largest prime divisor of qb is q, and that of pa is p since $p > \frac{q}{2}$. If $a > \frac{q}{2}$ then $(q-a, p-b)$ satisfies $px-qy=1$; and repeat same argument, since $q-a<\frac{q}{2}$, $p-b<\frac{p}{2}$.

$\square$

Discussion.

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that

$$a=\frac{x+y}{z} \quad \text{and} \quad b=\frac{xy}{z}.$$

Solution. Take

$$x=\frac{b+1}{a}, \quad y=\frac{b(b+1)}{a}, \quad z=\frac{(b+1)^2}{a^2}.$$

$\square$

Discussion.

Problem 9. Let S be the set of all positive integers that are not perfect squares. For n in S, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n<a_1<a_2<\ldots<a_r$ and $n \cdot a_1 \cdot a_2 \ldots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5, 2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2)=6$. Show that the function f from S to the integers is one-to-one.

Solution. Assume for contradiction that f is not one-to-one. There exist integers n and m, both in S, such that $n<m$ and $f(n)=f(m)$. There exist a_1, a_2, …, a_r such that $n<a_1<a_2<\ldots<a_r$ and $n \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_r$, where a_r is the smallest integer for which this is true. There exist b_1, b_2, …, b_k such that $m<b_1<b_2<\ldots<b_k$ and $m \cdot b_1 \cdot b_2 \cdot \ldots \cdot b_k$, where b_k is minimized as before. We have $a_r=b_k$.

Let $P_1 = \{n, a_1, a_2, \ldots, a_r\}$ and $P_2 = \{m, b_1, b_2, \ldots, b_k\}$. We know that $\prod_{p \in P_1} p$ and $\prod_{p \in P_2} p$ are both perfect squares. Let

$$P = (P_1 \cup P_2) \setminus (P_1 \cap P_2)$$

and $Q = P_1 \cap P_2$.

Observe that if $a \mid b^2$ and $a \mid c^2$, then $\frac{b^2c^2}{a^2}$ is a perfect square. Now it is clear that

$$\prod_{p \in P} p = \frac{\left(\prod_{p \in P_1} p\right)\left(\prod_{p \in P_2} p\right)}{\left(\prod_{q \in Q} q\right)^2}$$

is a perfect square. Notice that $\min P = n$ and $\max P < a_r$. Therefore, the new set $P = \{n, c_1, c_2, \ldots, c_t\}$ is a set such that $n < c_1 < c_2 < \ldots < c_t$ and $n \cdot c_1 \cdot c_2 \cdot \ldots \cdot c_t$ is a perfect square and $c_t < a_r$. This contradicts that $f(n) = a_r$, and hence f is one-to-one. $\square$

Discussion.

Problem 10. Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n - a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Amy prevent Bob win?

Solution. For each positive integer n, there are unique $k(n), r(n)$ such that $n = k(n)^2 r(n)$, where $r(n)$ is a squarefree positive integer. If Bob has a positive integer n, he'll replace it by $n - [k(n)]^2$. Thus Bob can always strictly decrease $r(n)$. Clearly Amy's move cannot increase $r(n)$, so Bob eventually wins. $\square$

Discussion.

Problem 11. Let $n \geqslant 50$ be a natural number. Prove that n is expressible as sum of two natural numbers $n = x + y$, so that for every prime number p such that $p \mid x$ or $p \mid y$ we have $\sqrt{n} \geqslant p$.

Solution. Let $n = t^2 + s$, where $0 \leqslant s \leqslant 2t$, that is $t \leqslant \sqrt{n} < t + 1$.

For $s = 0$ we can take a representation $n = t(t-1) + t$. For $1 \leqslant s \leqslant t$, the representation $n = t^2 + s$ works.

Suppose $t+1$ is not a prime. Then, for $t < s \leqslant 2t$ we can take $n = t(t+1) + (s-t)$.

Let us suppose $t+1$ is a prime number. In that case, $t+2$ is not a prime. For $t < s \leqslant 2t-2$, we have $n = (t-1)(t+2) + (s-t+2)$. When $s = 2t$, we have $n = t^2 + 2t$.

It remains to find representation for $s = 2t-1$. But in that case $n = (t-2)(t+4)+7$ and since $7 < \sqrt{n}$ such expression works. □

Discussion.

Problem 12. Each cell of a $3 \times n$ table was filled by a number. In each of three rows, the number $1, 2, \ldots, n$ appear in some order. It is know that for each column, the sum of pairwise product of three numbers in it is a multiple of n. Find all possible value of n.

Solution. It is possible for odd n.

Suppose $2 \mid n$ and take column a, b, c. Then $n \mid ab + bc + ca$, so $2 \mid ab + bc + ca$, thus among a, b, c there is at most one odd number. Therefore any column of the table $3 \times n$ contains at most one odd number – so in the whole table there is no more than n of them. On the other hand any row contains $n/2$ odd numbers; therefore in total we get $3n/2 > n$ odd numbers – contradiction.

Suppose now $2 \nmid n$. Fill the table in the following form:

- I, II row: in the i-th cell write $2i \pmod n$; residue 0 in the last cell we replace by n.
- III row: in the i-th cell write $n - i$; residue 0 in the last cell we replace by n.

Since $2 \nmid n$, in any row we have different numbers. Moreover for any column a, b, c we get:

$$ab + bc + ca \equiv 2i \cdot 2i + 2i \cdot (-i) + 2i \cdot (-i) \equiv 0 \pmod n.$$

□

Discussion.

Problem 13. Let $p > 3$ be a prime. Prove that there is a positive integer $y < \frac{p}{2}$ and such that $py+1$ cannot be represented as a product of two integers, each of which is greater than y.

Solution. We put $p = 2k+1$. Suppose the opposite: for each of the numbers $y = 1, 2, \ldots, k$ there is a decomposition $py + 1 = a_y b_y$, where $a_y > y$, $b_y > y$. Note that each of the numbers a_y and b_y is strictly greater than 1, and also that $a_y < p$, $b_y < p$, otherwise $a_y b_y \geq p(y+1) > py + 1$.

Hence, each of the $p-1$ numbers in the set $a_1, b_1, a_2, b_2, \ldots a_k, b_k$ lies in the set of $p-2$ numbers $\{2, 3, \ldots, p-1\}$. Thus, this set contains two equal numbers. Let each of these two numbers be equal to d. Let these equal numbers have equal indices in the set, that is, $a_y = b_y = d$ for some y. Then $py + 1 = d^2$, so the number $d^2 - 1 = (d-1)(d+1) = py$ is divisible by prime p. Since $1 \leq d-1 < d+1 \leq p$, this can only be for $d+1 = p$. Then the corresponding value of y is $d-1 = p-2 = 2k-1$, which is greater than k for $p > 3$. A contradiction (since $y \leq k$).

Otherwise, there exist indices $y_1 < y_2$ such that $1 \le y_1 < y_2 < d$, for which the numbers py_1+1 and py_2+1 are divisible by d. Then $p(y_2-y_1) = (py_2+1)-(py_1+1)$ is also divisible by d. Since d and p are coprime, we find that $y_2 - y_1$ is divisible by d, but this is impossible, since $0 < y_2 - y_1 < y_2 < d$.

Thus, in each case, a contradiction is obtained and, therefore, the number y indicated in the problem statement will always be found. $\square$

Discussion.

Problem 14. Determine all integers $s \ge 4$ for which there exist positive integers a, b, c, d such that $s = a+b+c+d$ and s divides $abc+abd+acd+bcd$.

Solution. Observe that $a+b+c+d \mid abc+abd+acd+bcd$ is equivalent to

$$\begin{aligned} 0 &\equiv abc + (ab+bc+ca)d \\ &\equiv abc - (a+b+c)(ab+bc+ca) \\ &\equiv -(a+b)(b+c)(c+a) \pmod{a+b+c+d}. \end{aligned}$$

Note that $a+b$, $b+c$, $c+a$ are each less than $a+b+c+d$, so the condition cannot hold if $s = a+b+c+d$ is prime. Moreover, each non-prime $s = mn$ can be attained by taking $a = 1$, $b = m-1$, $c = n-1$, and $d = (m-1)(n-1)$, so the answer follows. $\square$

Discussion.

Problem 15. Let a, b, c, d be positive integers such that $ad \ne bc$ and $\gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an+b, cn+d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Solution. We extend the problem statement by allowing a and c take non-negative integer values, and allowing b and d to take arbitrary integer values. (As usual, the greatest common divisor of two integers is non-negative.) Without loss of generality, we assume $0 \le a \le c$. Let

$$S(a, b, c, d) = \{\gcd(an+b, cn+d) \colon n \in \mathbb{Z}_+\}.$$

Now we induct on a. We first deal with the inductive step, leaving the base case $a = 0$ to the end of the solution. So, assume that $a > 0$; we intend to find a 4-tuple (a', b', c', d') satisfying the requirements of the extended problem, such that $S(a', b', c', d') = S(a, b, c, d)$ and $0 \le a' \le a$, which will allow us to apply the induction hypothesis.

The construction of this 4-tuple is provided by the step of the Euclidean algorithm. Write $c = aq + r$, where q and r are both integers and $0 \le r < a$. Then for every n we have

$$\gcd(an+b, cn+d) = \gcd(an+b, q(an+b) + rn + d?qb) = \gcd(an+b, rn + (d?qb)),$$

so a natural intention is to define $a' = r$, $b' = d?qb$, $c' = a$, and $d' = b$ (which are already shown to satisfy $S(a', b', c', d') = S(a, b, c, d)$). The check of the problem requirements is straightforward: indeed,

$$a'd'?b'c' = (c?qa)b?(d?qb)a =?(ad?bc) \neq 0$$

and

$$\gcd(a', b', c', d') = \gcd(c?qa, b?qd, a, b) = \gcd(c, d, a, b) = 1.$$

It remains to deal with the base case $a = 0$, i.e., to examine the set $S(0, b, c, d)$ with $bc \neq 0$ and $\gcd(b, c, d) = 1$. Let b' be the integer obtained from b by ignoring all primes b and c share (none of them divides $cn + d$ for any integer n, otherwise $\gcd(b, c, d) > 1$). We thus get $\gcd(b', c) = 1$ and $S(0, b', c, d) = S(0, b, c, d)$.

Finally, it is easily seen that $S(0, b', c, d)$ is the set of all positive divisors of b'. Each member of $S(0, b', c, d)$ is clearly a divisor of b'. Conversely, if δ is a positive divisor of b', then $cn + d \equiv \delta \pmod{b'}$ for some n, since b' and c are coprime, so δ is indeed a member of $S(0, b', c, d)$. □

Discussion.

Problem 16. Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Hamza may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Hamza can write 1 on the board in finitely many steps.

Solution. I claim the answer is all $m + n = 2^k$ for some $k \in \mathbb{N}$. First, we prove that it works, letting $m + n = 2^k$. Then, we can take the following weighted arithmetic mean

$$\frac{1}{2^k}\left(m\left(\frac{n}{m}\right) + n\left(\frac{m}{n}\right)\right).$$

If $m + n$ is divisible by an odd prime p then we have $m/n \equiv n/m \equiv -1 \pmod{p}$. So all numbers that can ever appear on the blackboard will be congruent to -1 modulo p because if $\frac{a}{b}$ and $\frac{c}{d}$ are congruent to -1 modulo p then

$$\frac{\frac{a}{b} + \frac{c}{d}}{2} \equiv \frac{-2}{2} \equiv -1 \pmod{p}, \qquad \frac{2\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)}{\frac{a}{b} + \frac{c}{d}} = \frac{2}{\frac{b}{a} + \frac{d}{c}} \equiv \frac{2}{-2} \equiv -1 \pmod{p}$$

So 1 never appears since it isn't congruent to -1 mod p. □

Discussion.

Problem 17. Let k, n be a positive integers such that $k > n!$. Prove that there exist distinct prime numbers $p_1, p_2, \ldots, p_n$ such that $p_i \mid k + i$ for all $i = 1, 2, \ldots, n$.

Solution. For $i = 1, 2, \dots, n$ let
$$a_i = \operatorname{lcm}(\text{divisors of } k+i \text{ which not exceed } n).$$
Then $a_i \leqslant n! < k$. Moreover $a_i \mid k+i$, thus
$$\frac{k+1}{a_1}, \quad \frac{k+2}{a_2}, \dots, \frac{k+n}{a_n}$$
are integers greater than 1.

No we prove that these numbers are coprime. Take any $1 \leqslant i, j \leqslant n$. Since $(k+i) - (k+j) < n$, than $d := \gcd(k+i, k+j) \leqslant n$, so $d \mid a_i$ and $d \mid a_j$. It means that $\frac{k+i}{a_i}$ and $\frac{k+j}{a_j}$ are divisors of $\frac{k+i}{d}$ and $\frac{k+j}{d}$, respectively. But the letter numbers are coprime, so $\frac{k+i}{a_i}$ and $\frac{k+j}{a_j}$ are coprime too.

Finally easy to observe that these numbers satisfy problem statement. □

Discussion.

Problem 18. For any integer $N \geq 2$, let $f(N)$ denotes sum of N and the greatest divisor of N (other than N). Prove that for any integer $A \geq 2$, by iterating f on A we can get a number divisible by 3^{2021}.

Solution. Note that f takes even values for odd arguments. Moreover taking even number of the form $2^k a$, where $k \geq 1$ and $2 \nmid a$, we see that
$$2^k a \xrightarrow{f} 2^{k-1} \cdot 3a \xrightarrow{f} 2^{k-2} \cdot 3^2 a \xrightarrow{f} \dots \xrightarrow{f} 3^k a.$$

We will prove inductively, that for any natural n by iterating f, from any integer (≥ 2) we can made odd number divisible by 3^n.

Base case of an induction was at the beginning, since we made from any number, the odd number divisible by 3. Suppose that by iterating f we obtained number of the form $3^n a$, where a is odd number. Then
$$3^n a \xrightarrow{f} 2^2 \cdot 3^{n-1} a \xrightarrow{f} 2 \cdot 3^n a \xrightarrow{f} 3^{n+1} a,$$
which ends inductive step. □

Discussion.

Problem 19. Let m, n be a positive integers such that set $\{1, 2, \dots, n\}$ contains exactly m different prime numbers. Prove that if we choose any $m+1$ different numbers from $\{1, 2, \dots, n\}$ then we can find number from $m+1$ choosen numbers, which divide product of other m numbers.

Solution. Suppose that problem statement doesn't hold. Then there exists $(m+1)$-elements set $A \subset \{1, 2, \dots, n\}$, such that no $x \in A$ which divide product of remaining elements in A. Therefore any $x \in A$ has a prime divisor p, whose exponent is greater then exponent of p in a product of numbers in $A \setminus \{x\}$.

Thus to any $x \in A$ we associate a prime number from $\{1, 2, \dots, n\}$. Since A consists of $m+1$ elements, then by the Pigeonhole Principle some prime p is associated for two different elements $x, y \in A$. Denote by w the product $m-1$ elements of the set $A \setminus \{x, y\}$. There exists non-negative integers k and l such that $p^k \mid x$, $p^k \nmid wy$, $p^l \mid y$ and $p^l \nmid wx$. Then exponent of p in $wy \cdot wx$ is smaller then $k+l$, and simultaneously $p^{k+l} \mid xy \mid wy \cdot wx$ – contradiction. □

Discussion.

Problem 20. Integers $a_1, a_2, \dots, a_n$ satisfy

$$1 < a_1 < a_2 < \dots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \dots a_n$, then prove that

$$(a_1 a_2 \dots a_n)^{m-1} \geqslant (n!)^m.$$

Solution. Let us write $a_i = p^{k_i} \cdot b_i$, where $p \nmid b_i$ for a prime divisor p of $a_1 a_2 \dots a_n$. Then, due to $a_1 < a_2 < \dots < a_n < 2a_1$ we get that b_i are pairwise distinct. Indeed, if $b_i = b_j$ for some $i < j$ then

$$\frac{a_j}{a_i} = \frac{p^{k_j} \cdot b_i}{p^{k_i} \cdot b_i} = p^{k_j - k_i} \geqslant 2.$$

Thus

$$b_1 b_2 \dots b_n \geqslant n!.$$

Multiplying such inequalities for each $p \mid a_1 a_2 \dots a_n$ we get the result. □

References

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