Ex 2

$ab+bc+ca=3$

$$(1+a^2)(1+b^2)(1+c^2) \geq 8.$$

Try to square both sides

$$(1+a^2)^2(1+b^2)^2(1+c^2)^2 \geq 4^4 = 4\cdot 8^2$$

$\cancel{1+a^2+b^2}$ $\left[(1+a^2)(1+b^2)\right]\cdot\left[(1+b^2)(1+c^2)\right]\left[(1+c^2)(1+a^2)\right]$

$$\left[\boxed{1}+\boxed{1}+1+1\right]\left[1+\boxed{a^2}+\boxed{b^2}+a^2b^2\right]\left[1+b^2+c^2+\boxed{b^2c^2}\right]\left[1+\boxed{c^2}+a^2+c^2a^2\right] \geq \ldots$$

$$\ldots \geq \left(\sqrt[4]{1\cdot a^2b^2\cdot b^2\cdot a^2}+\sqrt[4]{1\cdot b^2\cdot b^2c^2\cdot c^2}+\sqrt[4]{1\cdot a^2\cdot c^2\cdot c^2a^2}+\sqrt[4]{1\cdot 1\cdot 1\cdot 1}\right)^4 =$$

$$= (ab+bc+ca+1)^4 = 4^4 \quad \text{so}$$

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Problem 27

Find all positive integers s.t

$$\boxed{2(x^3+y^3+z^3) = 3(x+y+z)^2}$$

$2 \mid x+y+z$

I 1° $\sqrt[3]{\dfrac{x^3+y^3+z^3}{3}} \geq \dfrac{x+y+z}{3} \implies x+y+z \leq 13$

II 2° Holder.

$$(1^3+1^3+1^3)(1^3+1^3+1^3)(x^3+y^3+z^3) \geq (x+y+z)^3.$$

$$PM_{t>2} = \left(\frac{x^t+y^t+z^t}{3}\right)^{\frac{1}{t}}$$

$x+y+z \leq 13.$

$PM_{t>2} \geq AM = PM_{t=1}$

$x^3+y^3+z^3 \equiv x+y+z \mod 3$

$0 \equiv 3(x+y+z)^2 \equiv 2(x+y+z) \mod 3$

$PM_t \geq PM_s$

$t \geq s.$

so $3 \mid x+y+z$ so

$6 \mid x+y+z$

$$\boxed{x+y+z=6, \quad \text{or} \quad x+y+z=12}$$

$$\Downarrow$$

$$(3,4,5)$$

Problem 28

$$\frac{a}{\sqrt{b^2+2c}}+\frac{b}{\sqrt{c^2+2a}}+\frac{c}{\sqrt{a^2+2b}} \geq \sqrt{3} \quad \text{if } abc=1,$$

Trick

$$\left(\sum_{cyc}\frac{a}{\sqrt{b^2+2c}}\right)\left(\sum_{cyc}\frac{a}{\sqrt{b^2+2c}}\right)\left(\sum_{cyc}(b^2+2c)\cdot a\right) \geq$$

$$\geq \left(\underbrace{\sqrt[3]{\underbrace{\frac{a}{\sqrt{b^2+2c}}\cdot\frac{a}{\sqrt{b^2+2c}}}\cdot(b^2+2c)\cdot a}_{a}}_{a^3}+b+c\right)^3 = (a+b+c)^3$$

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$$\underbrace{\frac{X}{\sqrt{A}} + \frac{Y}{\sqrt{B}} + \frac{Z}{\sqrt{C}}}_{S} \geqslant \ldots$$

$$\left(\frac{X}{\sqrt{A}} + \frac{Y}{\sqrt{B}} + \frac{Z}{\sqrt{C}}\right)\left(\frac{X}{\sqrt{A}} + \frac{Y}{\sqrt{B}} + \frac{Z}{\sqrt{C}}\right)\left(X \cdot A + Y \cdot B + Z \cdot C\right)$$

Hölder.

$$\geq \left(\sqrt[3]{\frac{X}{\sqrt{A}} \cdot \frac{X}{\sqrt{A}} \cdot X \cdot A} + \sqrt[3]{\frac{Y}{\sqrt{B}} \cdot \frac{Y}{\sqrt{B}} \cdot YB} + \sqrt[3]{\frac{Z}{\sqrt{C}} \cdot \frac{Z}{\sqrt{C}} \cdot Z \cdot C}\right)^3$$

$$= \left(X + Y + Z\right)^3$$

$$S^2 \geq \frac{(x+y+z)^3}{XA + YB + ZC}$$

TRICK

$$\underbrace{\left(\sum_{cyc}' \frac{a}{\sqrt{b^2+2c}} \right)^2}_{\text{we want } \geq 3} \geq \frac{(a+b+c)^3}{\sum_{cyc}' (b^2+2c)a}$$

so enough to prove

$$\frac{(a+b+c)^3}{\sum_{cyc}' (b^2+2c)a} \geq 3 \quad \text{or}$$

$$(a+b+c)^3 \geq \cancel{3(ab^2+bc^2+ca^2)} + 6(ab+bc+ca)$$

$$\| $$

$$a^3+b^3+c^3+\cancel{3(ab^2+bc^2+ca^2)} + 3(a^2b+b^2c+c^2a) + 6abc$$

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$$a^3+b^3+c^3+3(a^2b+b^2c+c^2a)+6abc \geq 6(ab+bc+ca)$$

$abc=1$

$$a^3+b^3+c^3+3(a^2b+b^2c+c^2a)+6 \geq 6(ab+bc+ca)$$

Now AM – GM

$$a^3+c^2a+c^2a+c^2a+1+1 \geq 6\cdot\sqrt[6]{a^6c^6} = 6ac$$

Next problem do the same

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$$

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$$\underbrace{\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}}}_{S} \geq 1$$

$$S^2 \cdot \left(a\cdot(a^2+8bc) + b\cdot(b^2+8ca) + c(c^2+8ab) \right) \geq$$

$$\geq (a+b+c)^3$$

It is enough to prove:

$$(a+b+c)^3 \geq a^3+b^3+c^3 + 8abc + 8abc + 8abc$$

$$\|$$

$$\cancel{6abc} + \cancel{3}(a^2b+bc^2+ca^2+ab^2+b^2c+c^2a) \geq \cancel{24}\,abc = 6abc$$

$$a^2b+bc^2+ca^2+ab^2+b^2c+c^2a \geq 6abc$$

AM-GM

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$a, b, c > 0$ reals

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a+b+c)^3$$

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$a_1, a_2, \ldots, a_4 \in \mathbb{R}_+$

$$\sum_{i=1}^{4} a_i^3 = 3$$

$$\sum_{i=1}^{4} a_i^5 = 5$$

Prove:

$$\sum_{i=1}^{n} a_i > \frac{3}{2}$$

$a^5 - a^2 + 3 \geq a^3 + 2$ then

$$\geq (a^3+2)(b^3+2)(c^3+2) \overset{\text{Holder}}{\geq} (a^3+1+1)(b^3+1+1)(c^3+1+1) \geq$$

$$\geq (a+b+c)^3$$

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$\underbrace{a^{\alpha_1} b^{\alpha_2} c^{\alpha_3}}$ 3 varelile $\underline{a, b, c}$

$\deg\left(a^{\alpha_1} \cdot b^{\alpha_2} \cancel{b} c^{\alpha_3}\right) = (\alpha_1, \alpha_2, \alpha_3)$

$\deg\left(a^{2}\right) = (2, 0, 0)$

$\deg\left(a^{2} b^{3} c\right) = (2, 3, 1)$

$\deg\left(a^{2020}\right) = (2, 0, 0)$

$\sum' \deg(\alpha_1, \alpha_2, \alpha_3) = \sum'_{\substack{a,b,c \\ \text{permuteol}}} a^{\alpha_1} b^{\alpha_2} c^{\alpha_3}$

$\sum' \deg(2, \cancel{1}, 0) = \sum'_{\substack{a,b,c \\ \text{perm}}} \cancel{2} a^2 b^1 c^0 = a^2 b^1 c^0 + a^2 \cancel{c}^1 b^0 +$
$+ b^2 c^1 a^0 + b^2 c^0 a^1 +$
$+ c^2 a^1 b^0 + c^2 a^0 b^1 =$

$= \underline{a^2 b + ab^2 + ac^2 + ca^2 + bc^2 + b^2 c}$

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$$a^5 - a^2 - a^3 + 1 \geq 0$$

$$a^2(a^3-1) - (a^3-1) \geq 0 \iff (a^3-1)(a^2-1) \geq 0 \iff$$

$$\iff (a-1)^2(a+1)(a^2+a+1) \geq 0$$

$\boxed{a > 0}$

Muirhead Inequality

$\boxed{a, b, c}$

$$\sum ! f(a_1, \ldots, a_n)$$

$$\sum{}' f(2,0,0) = a^2b^0c^0 + \cancel{a} a^2c^0b^0 + b^2c^0a^0 + b^2a^0c^0$$

$$c^2a^0b^0 + c^2b^0a^0 =$$

a, b, c

$$= a^2 + a^2 + b^2 + b^2 + c^2 + c^2 = 2(a^2 + b^2 + c^2)$$

$$\sum{}'_1 f(2,1,0) = a^2b + b^2a + c^2a + ca^2 + cb^2 + c^2b$$

$$\sum{}'_1 f(3,0,0) = 2(a^3 + b^3 + c^3)$$

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$$\sum_{a,b,c}' \deg(2,0,0) = a^2b^0c^0 + a^2c^0b^0 + b^2c^0a^0 + b^2a^0c^0 + c^2a^0b^0 + c^2b^0a^0$$
$$= 2(a^2+b^2+c^2)$$

$$\sum_{a,b,c}' \deg(k,0,0) = 2(a^k+b^k+c^k)$$

$$\sum_{a,b,c,d}' \deg(2,0,0,0) = 6(a^2+b^2+c^2+d^2)$$

$$\sum_{a_1,a_2,\dots,a_n}' \deg(2,\underbrace{0,0,\dots,0}_{n-1}) = (n-1)!\,(a_1^2+a_2^2+\dots+a_n^2).$$

$$\sum_{a,b,c}' \deg(3,2,2) = 2\left(a^3b^2c^2 + a^2b^3c^2 + a^2b^2c^3\right)$$

$$\sum_{a,b,c,d}' \deg(3,1,1,0) =$$

$$\sum_{a,b,c}' f(4,1,0)$$

$a^4 b^1 c^0$

$a^4 c^1 b^0$

$b^4 a^1 c^0$

$b^4 c^1 a^0$

$c^4 a^1 b^0$

$c^4 b^1 a^0$

$$= \sum_{cy}' \left(a^4 b + a b^4 + \right)$$

$$\sum_{a_1, a_2, \ldots, a_n}' \deg(\underbrace{1, 1 \ldots 1}_{n})$$

$$\|$$

$$a_1^1 a_1^1 \ldots a_4^1$$

$$\|$$

$$\boxed{n!\, a_1 a_2 \ldots a_n}$$

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$$\sum_{a_1 b_2}' \deg(1,1,1) = 6abc \qquad (2,1,0) \prec (1,1,1)$$

$2 \geq 1 \qquad 2+1 \geq 1+1$

$$\sum_{a_1 b_2}' \deg(2,1,0) = a^2b + ab^2 + bc^2 + b^2c + ca^2 + c^2a.$$

$$\boxed{\sum_{1}' \deg(2,1,0) \geq \sum_{1}' \deg(1,1,1)}$$

Suppose we have ponite integers. $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, $b_1 \geq b_2 \geq \dots b_n \geq 0$

$A = (a_1, a_2, a_3, \dots, a_n)$, $B = (b_1, b_2, \dots b_n)$

A majorize B ($\cancel{A}$ $A \prec B$) iff

1) $a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_n$

2) $a_1 \geq b_1$, $a_1 + a_2 \geq b_1 + b_2$, $a_1 + a_2 + a_3 \geq b_1 + b_2 + b_3$ ...

$a_1 + a_2 + \dots + a_i \geq b_1 + b_2 + \dots + b_i$ for any i.

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[Muirhead] : if $A \prec B$ than

$$\sum_{x_1 \ldots x_n}' \deg A \geq \sum_{x_1 \ldots x_n}' \deg B$$

Example

$A = (2,1,0) \qquad B = (1,1,1)$

$2 \geq 1 \geq 0 \qquad 1 \geq 1 \geq 1$

$$\left.\begin{aligned} 2+1+0 &= 1+1+1 \\ 2 &\geq 1 \\ 2+1 &\geq 1+1 \\ 2+1+0 &\geq 1+1+1 \end{aligned}\right\} A \prec B \text{ . so}$$

$$\sum_{a,b,c}' \deg (2,1,0) \geq \sum_{a,b,c}' \deg (1,1,1)$$

$$\| $$

$$a^2b+ab^2+bc^2+b^2c+ca^2+c^2a \geq 6abc$$

Example

$A = (3,0,0)$ $\qquad$ $B = (2,1,0)$

$3+0+0 = 2+1+0$

$3 \geq 2$ $\qquad$ $A \succ B$ so

$3+0 \geq 2+1$

$3 \geq 0 \geq 0$

$2 \geq 1 \geq 0$

$$\sum{}' \deg A \geq \sum{}' \deg B$$

$$\sum{}' \deg(3,0,0) \geq \sum{}' \deg(2,1,0)$$

$$\parallel$$

$$2(x^3+y^3+z^3) \geq (x^2y+xy^2)+(y^2z+yz^2)+(z^2x+zx^2)$$

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Ex.

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \le \frac{1}{abc}$$ multiply everything.

$$\sum_{cyc} (b^3+c^3+abc)(c^3+a^3+abc)\,abc \le (a^3+b^3+abc)(b^3+c^3+abc)\cdot(c^3+a^3+abc)$$

$$\parallel\; b^4c^4a \quad c^4a^4b$$

$\boxed{\text{PLAN B}}$

INVOLVES TIPS

LHS

$$\frac{3}{2}\sum{}'(\cancel{4},4,1) + 2\sum{}'(5,2,2) + \frac{1}{2}\sum{}'(\cancel{7},1,1) + \frac{1}{2}\sum{}'(\cancel{3},3,3)$$

RHS

$$\frac{1}{2}\sum{}'(3,\cancel{3},3) + \sum{}'(6,3,0) + \frac{3}{2}\sum{}'(\cancel{4},4,1) + \frac{1}{2}\sum{}'(\cancel{7},1,1) + \sum{}'(\cancel{5},2,2)$$

Exercise $\quad \sum{}'(6,3,0) \ge \sum{}'(5,2,2)$ Muirhead

a^4b^4c $\quad (6,3,0) \succ (5,2,2)$

$$\frac{3}{2}\sum{}' (4,4,1) = a^4b^4c^1\ a^4c^4b^1\ b^4a^4c^1\ b^4c^4a^1\ c^4a^4b^1\ c^4b^4a^1 =$$

$$= \frac{3}{2}\cdot 2\left(a^4b^4c + b^4c^4a + c^4a^4b\right) =$$

$$= 3\left(a^4b^4c + b^4c^4a + c^4a^4b\right)$$

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