Bounding primes

Lesson for the group L4

Problem 1. Without appealing to Bertrand's postulate, prove that there is a prime number between n and $n!$ for any positive integer n.

Problem 2. Let p_n be $n-$th prime number. Can you come up with a function $f(n)$ such that $p_n < f(n)$ for all n? (this $f(n)$ should of course only depend on n, e.g something like $n^{100} + 100n$. Not on say p_n or something like this)

Problem 3. Prove that there is no non-constant polynomial P with integer coefficients such that all of the numbers $P(1), P(2), P(3)....$ are prime. Unfortunately. But quite expected.

Remark: But there does exist some formula that produce p_n (i.e $n-$th prime numbers) for any positive integer n. This formula is called Willan's formula and it is a bit too ugly to be in the problem sheet... o_O

> Prime numbers are mysterious and many simple-sounding facts are still open problems. However, the humanity did manage to get a good estimate on p_n, i.e $n-$th prime number, namely $p_n \approx n{\cdot}ln(n)$ where ln stands for "natural logarithm" (if you do not know what this is, you can think of it of something proportional to log_2). Obviously, this result is totally non-trivial as getting any nice formula for p_n sounds like something difficult. However, it is rather elementary to get to a result about p_n that is quite close to that non-trivial fact! This is rather cool and we are gonna do it now...

Problem 4. Let p be prime and $p^a | \binom{n}{k}$. Prove that $p^a \leq n$.

Problem 5. Let p be $k - th$ largest prime number and let $p = 2m + 1$

i) Prove that $\binom{p}{m} > \frac{2^p}{p}$;

ii) Prove that $\binom{p}{m} \leq p^k$;

Thus we can conclude that $p_k^{k+1} > 2^{p_k}$ where $p_k = p$ (just a different notation). Finally, prove that

$$p_k < 4 \cdot k \cdot log_2(k).$$

Problem 6. Recall why the product of all primes between 1 and N is at most 4^N. Now try showing that

$$p_k > {}^1/_4 \cdot k \cdot log_2(k).$$

> Hence we showed that
> $$\frac{1}{4} \cdot k \cdot log_2(k) < p_k < 4 \cdot k \cdot log_2(k)$$
> for any positive integer k. If you think about it, this result is rather close to the one mentioned in the box above. I.e we are not that far from having the best possible approximation for the $n-$th prime number currently known!
>
> Finally, let's prove some bounds on consequentive primes (i.e p_i, p_{i+1}).

Problem 7. As before let p_k be the $k-$th prime number. Prove that are infinitely many k such that

a) $2p_k < p_{k+1} + p_{k-1}$

b) $2p_k \geq p_{k+1} + p_{k-1}$

c) $p_k - p_{k-1} < \sqrt{k-1}$