Preparation for Saudi Arabia Team 2021

May/June Session: Level 3

Nikola Petrović

Lesson 3

Inversion in geometry

Problems:

1. Four circles are given such that each is externally tangent to two of the remaining three circles. Prove that the four points of contact are concyclic.

2. Four circles, k_1, k_2, k_3 and k_4 are given. The pairs of circles k_1 and k_2, k_2 and k_3, k_3 and k_4, and k_4 and k_1 respectively intersect in points A and A', B and B', C and C', and D and D'. Prove that if $ABCD$ is a cyclic quadrilateral, then so is $A'B'C'D'$.

3. Let A be an intersection point of circles k and l. Circle Γ_1 is externally tangent to k and l (respectively) in points B_1 and C_1, while circle Γ_1 is externally tangent to k and l in points B_2 and C_2. Prove that the circumcircles of $\triangle AB_1C_1$ and $\triangle AB_2C_2$ are tangent to each other.

4. Let PQ be the diameter of k and let B be a point on the diameter. A line perpendicular to PQ from B intersects k at A. A circle l is internally tangent to k and is also tangent to segment AB and to segment PB at point C. Prove that AC is the symmetry line of $\measuredangle PAB$.

5. Let k_1 and k_2 be circles with radii $r_1 < r_2$ respectively and let AD be a common tangent, d_1, where $A \in k_1$ and $D \in k_2$. Let d_2 be a (different) line parallel to d_1 which is tangent to k_1. A line through D intersects d_2 in B and k_2 the second time in C, where $B \neq C$. Prove that the circumcircle of $\triangle ABC$ is tangent to d_1.

6. A diameter AB of circle k is given and on it point C. Let l be a circle of diameter AC and let circle t be externally tangent to l at T, internally tangent to k and tangent to a line through C perpendicular to AB. Prove that BT is a tangent to l.

7. In triangle ABC a point M is given such that $\measuredangle AMB - \measuredangle ACB = \measuredangle AMC - \measuredangle ABC$. Prove that $AB \cdot CM = AC \cdot BM$.

8. An isosceles triangle ABC, with $AB = AC$, is given and let k be its circumcircle. Let D and E be points on k, let F be the intersection of AD and BC and let G be the second intersection of AE and the circumcircle of $\triangle DEF$. Prove that the circumcircle of $\triangle CGE$ is tangent to AC.

9. Let A_1, B_1 and C_1 be the respective midpoints of the sides of $\triangle ABC$. Let O be the circumcenter of $\triangle ABC$. Circumcenters of BOC and $A_1B_1C_1$ intersect in X and Y. Prove that $\measuredangle BAX = \measuredangle CAY$.

10. Let D be the midpoint of BC of $\triangle ABC$. Let k be the circumcircle of ABD. On the arc AB of k not containing D we notice a point E so that $\measuredangle EDB = \measuredangle DAC$. Let a perpendicular line from A to AD intersect BC in F. Let G be the second intersection point of FE with k unless FE is a tangent of k in which case we define $G \equiv E$. Prove that $DG = DB$.