June Camp 2022
Problems

Geometry – L2

Tangent segments

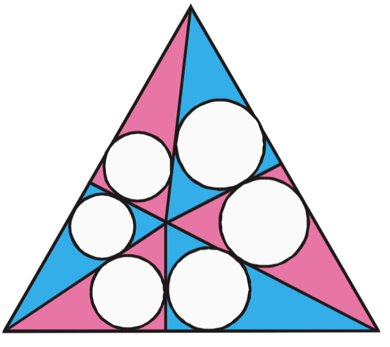
Problem 1. Two excircles of triangle ABC are tangent to the sides BC and AC of triangle ABC at points D and E, respectively. Prove that $AE = BD$.

Problem 2. Given two circles ω_1 and ω_2. A transversal common tangent ℓ touches the circles ω_1 and ω_2 at points B and C, respectively. The line ℓ intersects the external common tangents to the circles ω_1 and ω_2 at points A and D. Prove that $AB = CD$.

Problem 3. A variable point D lies on side AB of a fixed triangle ABC. The external common tangent of the incircles of triangles ADC and BDC (different from the line AB) intersects the line CD at point E. Find the locus of the points E, as D varies on the line segment AB.

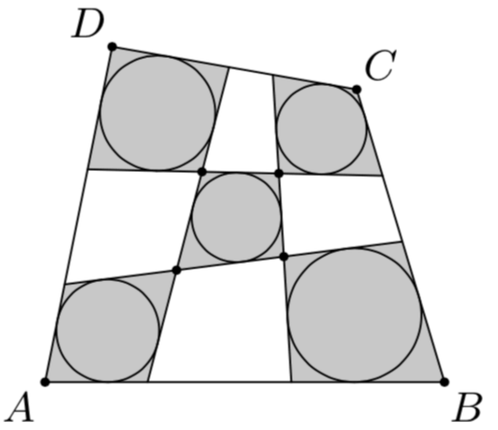
Problem 4. Let ABC be a triangle with inradius r and let ω be a circle of radius $a < r$ inscribed in angle BAC. Tangents from B and C to ω (different from triangle sides) intersect at point X. Show that the incricle of triangle BCX is tangent to incircle of triangle ABC.

Problem 5. Choose point inside a regular triangle a draw perpedicular to sides of this triangle. They divide triangle into 6 triangles (see picture below). Prove that sum of inradius of red triangles is equal to sum of inradius of blue triangles.



Problem 6. In trapezoid $ABCD$ bisectors of the angles A and D intersect at point laying on BC. These bisectors split trapezoid into three triangles, in which we inscribed circle. One of these circles touches base AB at K, while the other two are tangent to the bisector DE at points M and N. Prove that $BK = MN$.

Problem 7. A convex quadrilateral $ABCD$ is cut into 9 convex quadrilaterals, as shown on the figure below. Prove that if there exist incircles of the shaded quadrilaterals, then $ABCD$ is circumscribed.



Problem 8. Prove that there exists an incircle of quadrilateral $ABCD$ if and only if the incircles of triangles ABD and BCD are tangent.

Problem 9. Let $ABCD$ be a circumscribed quadrilateral. Point P lies on the side CD. Prove that there exists a common tangent to the incircles of triangles ABP, BCP and ADP.

Problem 10. Points P and Q lie on sides AB and AD of a convex quadrilateral $ABCD$. The lines DP and BQ meet at S. Prove that if there exist incircles of quadrilaterals $APSQ$ and $BCDS$, then there exists an incircle of quadrilateral $ABCD$.

Menelaus

Problem 11. Let the external angle bisector of $\sphericalangle BAC$ intersect BC at A'. Define B', C' analogously. Prove that A', B', C' are collinear.

Problem 12. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let X be a point on segment AB. Put $P = BC \cap AD$, $Y = CD \cap PX$, $R = AY \cap BD$ and $T = PR \cap AB$. Prove that

$$\frac{1}{AT} = \frac{1}{AX} + \frac{1}{AB}.$$

Problem 13. In triangle ABC let D be the point on the segment BC, and E on the segment CA, for which $BD = CE = AB$. Let ℓ be the line through D that is parallel to AB. If $M = \ell \cap BE$ and $F = CM \cap AB$ prove that

$$AE \cdot BF \cdot CD = (AB)^3.$$

Problem 14. In triangle ABC internal angle bisectors t_a, t_b, t_c meet BC, CA, AB at U, V, W, respectively; and medians m_a, m_b, m_c intersect BC, CA, AB at L, M, N, respectively. Let $m_a \cap t_b = P$, $m_b \cap t_c = Q$, $m_c \cap t_a = R$. Prove that

$$\frac{AR}{RU} \cdot \frac{BP}{PV} \cdot \frac{CQ}{QW} \geq 8.$$

Problem 15. Let D and E be points on sides AB and AC of a triangle ABC such that $DE \parallel BC$. Let P be an interior point of triangle ADE. Lines PB and PC intersect DE at F, G, respectively. Prove that AP is a radical axis of circumcircles of triangles PDG and PFE.

Problem 16. Let $ABCD$ be a parallelogram. Points K and L lie on the sides AB and AD, respectively. Line segments DK and BL intersect at P. Point Q is chosen such that $AKQL$ is a parallelogram. Prove that P, Q, C are collinear.

Problem 17. Let $ABCD$ be a convex quadrilateral. A line k intersects DA, AB, BC and CD at X, Y, Z and T, respectively. Prove that

$$\frac{DX}{XA} \cdot \frac{AY}{YB} \cdot \frac{BZ}{ZC} \cdot \frac{CT}{TD} = 1.$$

Ceva

Problem 18. Let ABC be a triangle with $\sphericalangle A = 100°$, $\sphericalangle B = 60°$, and let $M \in BC$ and $N \in AC$ be points for which $\sphericalangle BAM = 30°$ and $\sphericalangle ABN = 20°$. Prove that the lines AM, BN and the bisector of $\sphericalangle ACB$ are concurrent.

Problem 19. Let ABC be a right triangle with right angle at C. On sides BC and CA build squares $BEFC$ and $CGHA$, respectively. Let D be the feet of altitude from C to AB. Prove that AE, BH and CD concur.

Problem 20. A circle meets the sides BC, CA, and AB of triangle ABC at points A_1; A_2, B_1; B_2, and C_1; C_2. Prove that the lines AA_1, BB_1, and CC_1 are concurrent if and only if the lines AA_2, BB_2, and CC_2 are concurrent

Problem 21. Let ABC be a triangle. Prove that lines joining midpoints of the sides with midpoints of the corresponding altitudes pass through a single point.

Problem 22. Let $ABCDEF$ be a hexagon inscribed in a circle ω. Show that the diagonals AD, BE, CF are concurrent if and only if

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Problem 23. Prove that in triangle ABC interior bisector of angle A, median of triangle from B and altitude from C concur iff

$$\tan A = \frac{\sin C}{\cos B}.$$

Problem 24. In an acute triangle ABC a semicircle ω centered on the side BC is tangent to the sides AB and AC at points F and E, respectively. If X is the intersection of BE and CF, show that $AX \perp BC$.

Problem 25. Prove that in regular 30-gon diagonals A_1A_{19}, A_3A_{24} and A_8A_{28} concur.

Problem 26. Let P be a point inside equilateral triangle ABC. Let AP, BP, CP meet sides BC, CA, AB are A_1, B_1, C_1. Prove

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$$

When does equality hold?