— COMBINATORICS FOR L3 —

— JANUARY CAMP, 2022 — GRAPH THEORY (2): PLAYING WITH DEGREES —

WARM-UP.

- Make sure you are familiar with the following notions: ISOMORPHIC GRAPHS, REGULAR GRAPH, BIPARTITE GRAPH, COMPLEMENTARY GRAPH
- Prove that sum of all degrees equals twice the number of edges.
- Deduce that in every graph the number of vertices of odd degree is even.
- Prove that in every simple graph there are two vertices of equal degrees.

7. Find all n for which there exists a $(2n+1)$-vertex regular graph of degree n.

8. Given is a bipartite graph with parts A and B of sizes $2n$ and $2n+1$, respectively. Prove that if all vertices from A have equal degrees, then some two vertices from B have equal degrees.

9.

(a) Prove that if an n-vertex graph is isomorphic with its complement, then n is 0 or 1 mod 4.

(b) For every $k \geq 1$ construct a $4k$-vertex graph isomorphic with its complement.

(c) Prove that if a $(4k+1)$-vertex graph is isomorphic with its complement, then it has a vertex of degree $2k$.

10. In an n-vertex triangle-free graph each pair of non-neighbors has exactly two common neighbors. Prove that the graph is regular.

11. Given is a convex polyhedron whose all faces are triangles, and whose vertices are 3-colored. Prove that the number of *rainbow triangles* (faces with one vertex of each color) is even.

12. Given is a non-empty graph with an even number of vertices. Prove that some two vertices of this graph have an even number of common neighbors.