Lecture 5

Training Camp

23 November 2020

Level 2

Let $\mathcal{F}$ be a set of subsets of the set $\{1, 2, \ldots, n\}$ such that

(1) if A is an element of $\mathcal{F}$, then A contains exactly three elements;

(2) if A and B are two distinct elements in $\mathcal{F}$, A and B share at most one common element.

Let $f(n)$ denote the maximum number of elements in $\mathcal{F}$. Prove that

$$\frac{(n-1)(n-2)}{6} \leq f(n) \leq \frac{(n-1)n}{6}.$$

مثال $F = \{ \{1,2,3\}, \{4,5,6\}, \ldots, \{n-2, n-1, n\} \}$

عناصر F لا يمكن أن تتشارك في أكثر من عنصر (مثال $\{1,2,3\}, \{3,2,5\}$ غير ممكنة)

$$f(n) = \max_{F \text{ on } \{1,2,\ldots,n\}} |F|$$

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We will prove the upper bound first for any set F that satisfies (1) and (2). We can count the number of pairs in each element of F. for example

$$F = \{ \underbrace{\{1,2,3\}}_{\downarrow 3}, \underbrace{\{3,7,9\}}_{\downarrow 3}, \underbrace{\{9,1,5\}}_{\downarrow 3} \}$$

pairs are:

(1,2)	(3,7)	(9,1)
(2,3)	(3,9)	(1,5)
(1,3)	(7,9)	(9,5)

and we have 9 pairs

Let $\mathcal{F}$ be a set of subsets of the set $\{1, 2, \ldots, n\}$ such that

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Let $f(n)$ denote the maximum number of elements in $\mathcal{F}$. Prove that

$$\frac{(n-1)(n-2)}{6} \leq f(n) \leq \frac{(n-1)n}{6}.$$

Every element in F results 3 pairs, and there is no two elements in F that result the same pair since they share at most one element in common.

$$\# \text{ pairs} = 3f(n) \leq \binom{n}{2}$$

$$\Longrightarrow \quad \boxed{f(n) \leq \frac{n(n-1)}{6}}$$

For the lowerbound, it's enough to show an example for F that has more than $\frac{(n-1)(n-2)}{6}$ elements.

Define T_i to be a set of 3 element subsets

$$T_i = \{ \{a,b,c\} \mid a+b+c \equiv i \pmod{n} \}$$
$$a,b,c \in \{1,2,\ldots,n\}$$

* T_i satisfies (1) $\longrightarrow$ direct

* T_i satisfies (2).

Assume that there are two elements in T_i that have two elements in common

$\{x,y,z\}$ and $\{x,y,z'\}$

$x+y+z \equiv i \pmod{n}$, $x+y+z' \equiv i \pmod{n}$

$\Rightarrow$ $z \equiv z' \pmod{n}$, $1 \le z, z' \le n$ $\Rightarrow$ $z = z'$ $\Rightarrow\Leftarrow$!!

Now we have $T_1, T_2, \ldots, T_n$

Since every subset of size 3 of $\{1, 2, \ldots, n\}$ is in one of the T_i's, $|T_1 \cup T_2 \cup \cdots \cup T_n| = \binom{n}{3}$

There is no intersection between T_i, T_j

if $\{a, b, c\} \in T_i \cap T_j$ then $a+b+c \equiv i \pmod{n}$, $a+b+c \equiv j \pmod{n}$

$n \mid i-j$ but $1 \leq |i-j| < n$ $\Rightarrow\Leftarrow$!!

Therefore,

$$|T_1| + |T_2| + |T_3| + \cdots + |T_n| = \binom{n}{3}$$

$\exists\, T_i$ such that $|T_i| \geq \underbrace{\frac{1}{n}\binom{n}{3}}_{\text{average}} = \frac{(n-1)(n-2)}{6}$

$$\Rightarrow \boxed{f(n) \geq \frac{(n-1)(n-2)}{6}}$$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

① z is even

$$5^z \equiv 7^t \pmod 3$$

$$\Rightarrow (-1)^z \equiv 1^t \pmod 3 \Rightarrow z \text{ is even}$$

$$z = 2z_1$$

② y is odd

$$2^x 3^y + 5^{2z_1} \equiv 0 \pmod 7$$

$$2^x 3^y + 2^{2z_1} \equiv 0 \pmod 7$$

$$2^x (-4)^y \equiv -2^{2z_1} \pmod 7$$

$$(-1)^y 2^x 2^{2y} \equiv -2^{2z_1} \pmod 7$$

$$(-1)^{y+1} \equiv 2^{x+2y-2z_1} \pmod 7$$

but $2^m \equiv -1 \pmod 7$ has no solutions $\Rightarrow$ y is odd

and $3 \mid x + 2y - 2z_1$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

3 $x > 1$

Assume that $x = 1$

$$2 \cdot 3^y + 5^z = 7^t$$

taking it (mod 4):

$$2(-1)^y + 1 \equiv (-1)^t \pmod 4$$

$$-2 + 1 \equiv (-1)^t \pmod 4 \Rightarrow t \text{ is odd}$$

taking it (mod 5)

$$2(-2)^y \equiv 2^t \pmod 5$$

$$-2^{y+1} \equiv 2^t \pmod 5$$

$$2^{y+1-t} \equiv -1 \pmod 5 \Rightarrow y+1-t \text{ is even}$$

but y and t are both odd $\Rightarrow\Leftarrow$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

(4) t is even

Proof: $2^x 3^y + 5^{2z_1} = 7^t$

$5^{2z_1} \equiv 7^t \pmod 4$

$1 \equiv (-1)^t \pmod 4 \Rightarrow$ t is even $t = 2t_1$

$2^x 3^y + 5^{2z_1} = 7^{2t_1}$

$2^x 3^y = (7^{2t_1} - 5^{2z_1}) = (7^{t_1} + 5^{z_1})(7^{t_1} - 5^{z_1})$

3 can't divide both of $7^{t_1} + 5^{z_1}$ and $7^{t_1} - 5^{z_1}$

$$\begin{cases} 7^{t_1} + 5^{z_1} = 2^u 3^y \\ 7^{t_1} - 5^{z_1} = 2^v \end{cases} \quad \text{or} \quad \begin{cases} 7^{t_1} + 5^{z_1} = 2^u \\ 7^{t_1} - 5^{z_1} = 2^v 3^y \end{cases}$$

4 can't divide both of $7^{t_1} + 5^{z_1}$ and $7^{t_1} - 5^{z_1}$, so $\min(u, v) = 1$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

Consider:

$$\begin{cases} 7^{t_1} + 5^{z_1} = 2^u \\ 7^{t_1} - 5^{z_1} = 2^v \, 3^y \end{cases}$$

but $7^t + 5^{z_1} > 2$, so $u \neq 1 \Rightarrow v = 1$

$$\Rightarrow \begin{cases} 7^{t_1} + 5^{z_1} = 2^u \\ 7^{t_1} - 5^{z_1} = 2 \cdot 3^y \end{cases}$$

by adding them

$$\Rightarrow 2 \cdot 7^{t_1} = 2^u + 2 \cdot 3^y, \Rightarrow 7^{t_1} = 2^{u-1} + 3^y$$

taking it (mod 7)

$$2^{u-1} + (-2)^y \equiv 0 \pmod 7$$

$$2^{u-1} + 2^y \equiv 0 \pmod 7 \Rightarrow \underline{2^{y-u+1}} \equiv -1 \pmod 7$$

but $2^m \equiv -1 \pmod 7$ has no solutions $\Rightarrow\Leftarrow$!!

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

Consider:

$$\begin{cases} 7^{t_1} + 5^{z_1} = 2^u \cdot 3^y \\ 7^{t_1} - 5^{z_1} = 2^v \end{cases}$$

$\min(u, v) = 1$

<u>Case 1</u>: $u = 1$

by adding the two equations

$$2 \cdot 7^{t_1} = 2 \cdot 3^y + 2^v$$

$$\Rightarrow \quad 7^{t_1} = 3^y + 2^{v-1}$$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

$$7^{t_1} = 3^y + 2^{x-1}$$

(mod 3):

$$(+1)^{t_1} \equiv (-1)^{x-1} \pmod 3 \Rightarrow x-1 \text{ is even}$$

$$5^{z_1} = 3^y - 2^{x-1}$$

(mod 5)

$$(-2)^y - (2)^{x-1} \equiv 0 \pmod 5$$

$$-2^y - 2^{x-1} \equiv 0 \pmod 5$$

$$2^y + 2^{x-1} \equiv 0 \pmod 5 \Rightarrow 2^{y-(x-1)} \equiv -1 \pmod 5$$

$$\Rightarrow 2 \mid y-(x-1)$$

but y is odd and $x-1$ even

$\Rightarrow\Leftarrow$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

<u>Case 2</u>: $v = 1$

$$\begin{cases} 7^{t_1} + 5^{z_1} = 2^u \cdot 3^y & (1) \\ 7^{t_1} - 5^{z_1} = 2^1 & (2) \end{cases}$$

(1) − (2):

$$5^{z_1} = 2^{u-1} 3^y - 1$$

(mod 4)

$$1 \equiv 2^{(u-1)} (-1)^y - 1 \pmod 4$$

$$2 \equiv -2^{u-1} \pmod 4 \Rightarrow 2^{u-1} \equiv 2 \pmod 4$$

$$\Rightarrow u - 1 = 1$$

$$\Rightarrow u = 2$$

$$5^{z_1} = 2 \cdot 3^y - 1$$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

$$5^{z_1} = 2 \cdot 3^y - 1$$

$$5^{z_1} + 1 = 2 \cdot 3^y$$

taking mod 3

$$(-1)^{z_1} + 1 \equiv 0 \pmod 3 \Rightarrow z_1 \text{ is odd}$$

By using LTE $\quad 3 \mid 5+1, \quad z_1$ is odd

$$v_3(5^{z_1}+1) = v_3(5+1) + v_3(z_1)$$

$$y = 1 + v_3(z_1)$$

$$\Rightarrow \quad v_3(z_1) = y-1 \Rightarrow \boxed{3^{y-1} \mid z_1}$$

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.

$$5^{z_1} + 1 = 2 \cdot 3^y \Rightarrow \underline{2 \cdot 3^y} > \underline{5^{z_1}} \quad (3)$$

$$3^{y-1} \mid z_1 \Rightarrow z_1 \geq 3^{y-1} \Rightarrow 5^{z_1} \geq 5^{3^{y-1}} \quad (4)$$

(3), (4):

$$2 \cdot 3^y > 5^{3^{y-1}}$$

$$6 \cdot 3^{y-1} > 5^{3^{y-1}}$$

$x = 3^{y-1}$

$$6x > 5^x$$

if $x = 1 \Rightarrow y = 1 \Rightarrow (x, y, z, t) = (3, 1, 2, 2)$

if $x > 1 \Rightarrow 5^x > 6x$ by induction

Determine all positive integers k such that

$$\frac{\tau(n^2)}{\tau(n)} = k,$$

for some n.

Let n be a positive integer greater than two. Prove that the Fermat number f_n has a prime divisor greater than $2^{n+2}(n+1)$.

Solve in non-negative integers the equation $2^a 3^b + 9 = c^2$

Prove that if p is a prime number, then $7p + 3^p - 4$ is not a perfect square.