Training Session in Mathematics
November-December 2021
Level 4
Nikola Petrović

Lesson 1
Tilings and partitions

Problems:

1. Is it possible to partition a circle into mutually congruent shapes such that some of the shapes do not contain the center of the circle (neither as an interior point nor as a boundary point)?

2. A regular $4n$-gon is partitioned into parallelograms. Prove that there are at least n rectangles among these parallelograms.

3. A rectangle of dimensions $a \times b$, where $a \geq b$, is partitioned into right-angled triangles such that each two adjacent triangles shares a common side and this side is the hypothenuse for one of the triangles and one of the legs (catheti) for the other triangle. Prove that $a \geq 2b$.

4. An isosceles trapezoid is called *proper* if exactly one pair of its sides is parallel (thus, parallelograms and rectangles are not proper). A partition of a rectangle into n proper isosceles trapezoids is called strict if the union of any $i\,(2 \leq i \leq n)$ trapezoids in the partition does not form a proper isosceles trapezoid. Prove that for any $n, n \geq 9$ there is a strict partition of a 2020×2021 rectangle into n proper isosceles trapezoids.

5. An $n \times n$ board is filled in with integers such that each pair of integers on adjacent fields differ by at most 1. Prove that there is an integer which appears at least n times on the board. Two fields are considered adjacent if they share a common side.

6. An L-tromino is a 2×2 square with one of its units removed. For which values of M, N can an $M \times N$ rectangle be tiled by L-trominos such that there is no smaller rectangle which is tiled by a subset of these L-trominos?

7. Show that any triangle can for any integer $n \geq 6$ be divided into n quadrilaterals that are both cyclic and inscribed (i.e. have both a circumcircle and incircle) such that no two are mutually congruent.

8. Consider now two sepearate tilings of a $2n \times 2n$ board: one with red dominos and the other with blue dominos. We say two squares are red neighbours if they are covered by the same red domino in the red tiling and we similarly define blue neighbours.

   Suppose we can assign a non-zero integer to each of the squares such that the number on any square equals the difference between the numbers on its red and blue neighbours, i.e the number on its red neighbour minus the number on its blue neighbour. Show that n is divisible by 3.

9. Does there exist an n-gon M_0 such that:

   - one can cover the boundary of the n-gon with copies of the n-gon: M_1, M_2, $\ldots M_k$ such that no two n-gons M_i, M_j, $i \neq j$, $0 \leq i, j \leq k$, have a common interior point, each M_i, $i = 1, \ldots, k$, has at least one common boundary with M_0 and M_0 is located strictly in the interior of the union of M_i, $i = 0, 1, \ldots, k$;
   - it is not possible to tile the plane with copies of M_0?

10. An $n \times n$ square is divided into unit squares. Let a unit side be a side of a unit square. For which n can we cover the board with equilateral right-angles triangles of side $\sqrt{2}$ and hypothenuse 2 so that the vertices of all the triangles lie on the vertices of unit squares and all unit sides lie in the interior or on the boundary of exactly one triangle?

11. On an infinite square grid we place finitely many cars, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that:

    (a) the cell immediately in front of each car is empty, and

    (b) no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.).

    In a move, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times. Is an infinite sequence of moves where each car is used infinitely many times still possible if we drop the first condition?

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Lesson 2
Graph theory

Problems:

1. Each edge of a polyhedron is given an orientation such that each vertex of the polyherdron has at least one edge entering and one edge exiting it. Prove that there exists a face of the polyherdon whose edges form an oriented cycle.

2. Let $n \geq 3$ be distinct points be marked on a plane so that no three of them lie on the same line. All points are connected with the segments. All segments are painted one of the four colors so that if in some triangle with the vertices at the marked points two sides have the same color, then all its sides have the same color. In addition, each of the four colors is used to paint at least one edge. What is the largest possible value of n?

3. In the country of Graphia there are 100 towns, each numbered from 1 to 100. Some pairs of towns may be connected by a (direct) road and we call such pairs of towns adjacent. No two roads connect the same pair of towns.

   Peter, a foreign tourist, plans to visit Graphia 100 times. For each i, $i = 1, 2, \ldots, 100$, Peter starts his i-th trip by arriving in the town numbered i and then each following day Peter travels from the town he is currently in to an adjacent town with the lowest assigned number, assuming such that a town exists and that he hasn't visited it already on the i-th trip. Otherwise, Peter deems his i-th trip to be complete and returns home.

   It turns out that after all 100 trips, Peter has visited each town in Graphia the same number of times. Find the largest possible number of roads in Graphia.

4. Let n and k be given natural numbers. In a candy shop one morning there were $2n$ pieces of each flavor of candy. That day a total of m children bought candies at the shop and each of the m children bought 2 pieces candy of (mutually) different flavor. It is known that in each subset of $k+1$ children that bought candy that day there are 2 children who bought at least one pair of candies of the same flavor. Find the largest possible value of m. The store is not replenished with new candy during the day.

5. Turan's theorem: Let G be a graph on n vertices and m is a positive integer with $2 \leq m \leq n$. Suppose G does not contain a complete graph of order m in G. Prove that the number of edges in G is at most $\frac{n^2}{2}\left(1 - \frac{1}{m-1}\right)$.

6. A graph has 2017 vertices. Some pairs of vertices are connected with either blue or red edges. In every subset of 4 vertices there is an even number of edges, at least one edge is red and if there are blue edges then there are more blue edges than red edges. Prove that there are 673 vertices such that each pair of vertices is connected with a red edge.

7. At a university dinner, there are 2017 mathematicians who each order two distinct entrees, with no two mathematicians ordering the same pair of entrées. The cost of each entree is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entree (with ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

8. In a group of m girls and n boys, any two persons either know each other or do not know each other. For any two boys and any two girls, there are at least one boy and one girl among them,who do not know each other. Prove that the number of unordered pairs of (boy, girl) who know each other does not exceed $m + \frac{n(n-1)}{2}$.

9. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size.

   Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

10. In the simple and connected graph G let x_i be the number of vertices with degree i. Let $d > 3$ be the biggest degree in the graph G. Prove that if :

$$x_d \geq x_{d-1} + 2x_{d-2} + ... + (d-1)x_1$$

   Then there exists a vertex with degree d such that after removing that vertex the graph G is still connected.

11. A social club has $2k + 1$ members, each of whom is fluent in the same k languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let A be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of A.

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Lesson 3
Arrays and Tables

Problems:

1. Let the set of integers from 1 to $2n$ be partitioned into two sets of n numbers. Let $a_1 > a_2 > \cdots > a_n$ be the numbers of the first set and $b_1 < b_2 < \cdots < b_n$ be the numbers of the second set. Show that the expression $|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$ does not depend on the choice of partition and find the expression as a function of n.

2. There are n students competing in 6 competitions to enter the National Olympiad team of 6 students. In each competition the students are ordered from the highest to the lowest score and there are no ties. The committee will chose the students by randomly selecting a permutation of competitions and then in order select from each competition the highest rated student that has not yet qualified. Is there a configuration of outcomes such that for each student there is at least one permutation in which the student qualifies for the team for

   (a) $n = 13$,

   (b) $n = 14$?

3. Let $n(\geq 2)$ be an integer. Let $2n^2$ contestants participate in a chess competition, where any each pair of contestants plays exactly once. A game can end in either a win for one player over the other or a draw. It is known that:

   - If A wins B and B wins C, then A wins C.
   - There are at most $\frac{n^3}{16}$ draws.

   Prove that it is possible to choose n^2 contestants and label them $P_{ij}(1 \leq i, j \leq n)$, such that for any $i, j, i', j' \in \{1, 2, ..., n\}$, if $i < i'$, then P_{ij} wins $P_{i'j'}$.

4. Let $n \geq 2$ be a natural number. An $n \times n$ board is filled in with integers from 1 to n^2 such that each number is used exactly once. Two fields are considered adjacent if they have a common edge. It is known that two numbers in adjacent fields differ by at most n. Prove that there is a 2×2 subset of the board such that the pairs of diagonally opposite numbers in the subset have the same sum.

5. Given positive integers n and k, $n > k^2 > 4$. In a $n \times n$ grid, a k-group is a set of k unit squares lying in different rows and different columns. Determine the maximal possible N, such that one can choose N unit squares in the grid and color them, with the following condition holds: in any k-group from the colored N unit squares, there are two squares with the same color, and there are also two squares with different colors.

6. A board $n \times n$ is filled with real numbers such that for each selection of n fields for which no two belong to the same row or column the sum of all elements is the same (as for any other such selection). Let A be the largest among the smallest numbers of each row and let B be the smallest number among the largest numbers of each column. Show that $A = B$.

7. Let n and $k < \frac{n}{2}$ be given numbers. We are given n balls labelled 1 to n of which some are made of silver and the remaining from lead. The balls are identical in appearance, except that a lead ball is slightly heavier than a silver ball. (All silver balls have mutually equal weight and so do all the lead balls). The weight difference is insufficient to be detectable by touch. To identify which balls are made of silver, we use a scale that compares the weight of the balls placed on each side and determines if one side is lighter, heavier or of equal weight in comparison with the other side. We are allowed to select sequences A and B of k consecutive balls each and place on each side the balls belonging to sets $A \backslash B$ and $B \backslash A$. If we know that the number of silver balls is at least 1 and at most d, find the largest value of d for which it will be possible to identify all the silver balls in a finite number of measurements.

8. In a football league, there are $n \geq 6$ teams. Each team has a homecourt jersey and a road jersey with different color. When two teams play, the home team always wear homecourt jersey and the road team wear their homecourt jersey if the color is different from the home team's homecourt jersey, or otherwise the road team shall wear their road jersey. It is required that in any two games with 4 different teams, the 4 teams' jerseys have at least 3 different color. Find the least number of color that the n teams' $2n$ jerseys may use.

9. Around a round table there are an even number of people seated on the same number of evenly distributed seats. After the break, the same people returned and sat down, except perhaps in a different arrangement. Let the distance between two people be one plus the number of chairs between them going along the arc that is equal to or smaller than a semi-circle. Show that there are at least two people such that the distance between them didn't change in the new arrangement with respect to the old arrangement.

10. Prove that for arbitrary positive integer $n \geq 4$, there exists a permutation of the subsets that contain at least two elements of the set $G_n = \{1, 2, 3, \cdots, n\}$: $P_1, P_2, \cdots, P_{2^n-n-1}$ such that $|P_i \cap P_{i+1}| = 2, i = 1, 2, \cdots, 2^n - n - 2$.

11. A snake of length k is an animal which occupies an ordered k-tuple $(s_1, \ldots, s_k)$ of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. If the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead. The snake has turned around if it occupied $(s_1, s_2, \ldots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \ldots, s_1)$ instead.

    Determine whether there exists an integer $n > 1$ such that: one can place some snake of length $0.9n^2$ in an $n \times n$ grid which can turn around.