Primes and factorization

Lesson by Senya, group L4+

Problem 1. As a first exercise, let's prove Bertrand's Postulate (recall that this is a statement that says that between any n ad $2n$ there is a prime number). Of course you all know how to do it since grade 5, but here is a quick plan on how to aprroach it anyway:

i) Show that no prime $\frac{2n}{3} < p < n$ divides $\binom{2n}{n}$;

ii) Consider $\binom{2n}{n}$. Then for any prime p we have $v_p(\binom{2n}{n}) = \sum_{k=1}^{\infty}\left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2\left\lfloor \frac{n}{p^k} \right\rfloor\right)$, right? Thus show that

(a) For any prime p we have $v_p(\binom{2n}{n}) \leq log_p(2n)$. Thus if p^a divides $\binom{2n}{n}$ then $p^a \leq 2n$;

(b) Moreover, prove that if $p > \sqrt{2n}$ then p^2 does not divide $\binom{2n}{n}$, i.e for such p it is at most p that divides $\binom{2n}{n}$;

iii) Show that $\binom{2t+1}{t} < 4^t$, and then show that product of all the primes between 1 and N is at most 4^N;

iv) Show that $\binom{2n}{n} > \frac{4^n}{2n+1}$;

v) The important ideas are all above, now we are just left with estimating the $\binom{2n}{n}$ in two different ways and get a contradiction if we assume that there are no primes between n and $2n$. So, by assuming that there are no primes between n and $2n$ and by noting that

$$\frac{4^n}{2n+1} < \binom{2n}{n} < \left(\prod_{1<p\leq\sqrt{2n}} p\right) \cdot \left(\prod_{\sqrt{2n}<p\leq 2n/3} p\right) \cdot \left(\prod_{2n/3<p\leq n} p\right)$$

and using the things proved above, get to a contradiction (well, you will only get a contradiction for n big enough, but this is good enough as we can consider n small enough by hand)

Problem 2. For positive integers m and n it is given that $mn \mid m^2 + n^2 + m$. Prove that m must be a perfect square.

Problem 3. Ali claims that he can erase one of the factorials in the product $1! \cdot 2! \cdot \ldots \cdot 60!$ in such a way that the number remaining will be a perfect square. Is he right? Is he lying? What is he up to?

Problem 4. Find all odd integers $n > 1$ such that for any two of its divisors a, b that are co-prime, the numbers $a + b - 1$ is also a divisor of n.

Problem 5. There is a function f defined on positive integers and taking positive integers as values such that the following conditions are satisfied:

i) $f(p) = 1$ for any prime p;

ii) $f(ab) = af(b) + bf(a)$ for any two positive integers a and b

Find all fixed points of this functions (i.e all such k that $f(k) = k$).

Problem 6. There is a primer p and there is a set S with p numbers not divisible by p. Prove that any remainder r modulo p it is possible to pick some numbers from S such that their sum will give remainder r when divided by p.