Practice Problems- 5

30 June, 2020

Level 2

Homework Problems

31. The product of a few primes is ten times as much as the sum of the primes. What are these (not necessarily distinct) primes?

32. A 10-digit number is said to be *interesting* if its digits are all distinct and it is a multiple of 11111. How many interesting integers are there?

1. [MOSP 1998]

   (a) Prove that the sum of the squares of 3, 4, 5, or 6 consecutive integers is not a perfect square.

   (b) Give an example of 11 consecutive positive integers the sum of whose squares is a perfect square.

2. [MOSP 1998] Let $S(x)$ be the sum of the digits of the positive integer x in its decimal representation.

   (a) Prove that for every positive integer x, $\frac{S(x)}{S(2x)} \leq 5$. Can this bound be improved?

   (b) Prove that $\frac{S(x)}{S(3x)}$ is not bounded.

More Problems ☺

33. [Russia 1999] Do there exist 19 distinct positive integers that add up to 1999 and have the same sum of digits?

40. Fractions in modular arithmetic.

(1) [ARML 2002] Let a be the integer such that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{23}=\frac{a}{23!}.$$

Compute the remainder when a is divided by 13.

(2) Let $p > 3$ be a prime, and let m and n be relatively prime integers such that

$$\frac{m}{n}=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}.$$

Prove that m is divisible by p.

(3) [Wolstenholme's Theorem] Let $p > 3$ be a prime. Prove that

$$p^2 \mid (p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right).$$

3. Most positive integers can be expressed as a sum of two or more consecutive positive integers. For example, $24 = 7 + 8 + 9$ and $51 = 25 + 26$. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore *interesting*. What are all the interesting integers?

4. Set $S = \{105, 106, \ldots, 210\}$. Determine the minimum value of n such that any n-element subset T of S contains at least two non-relatively prime elements.