Saudi Arabia 2022 – Math Camp

Level 4

Geometry – Projective Geometry

Instructor: Regis Barbosa

Harmonic Division

Definition (Cross-Ratio line): Let A, B, C, D be four points on a line. The cross-ratio $(A, B; C, D)$ is defined as

$$(A, B; C, D) = \frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}}$$

where $\overrightarrow{XY}$ is directed length.

Definition (Cross-Ratio circle): Let A, B, C, D be four points lying on a circle. The cross-ratio $(A, B; C, D)$ is defined as

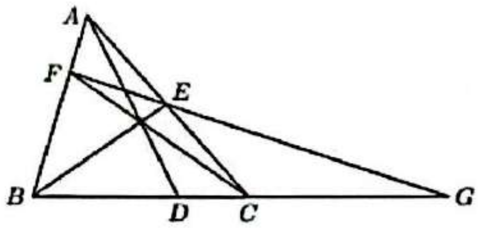
$$(A, B; C, D) = \pm \frac{CA}{CB} : \frac{DA}{DB}$$

where we take $+$ if the segments AB and CD do not intersect and take $-$ otherwise.

Definition (Harmonic Bundle): Let A, B, C, D be four points on a line. If $(A, B; C, D) = -1$, then $(A, B; C, D)$ is called a *harmonic bundle* or a harmonic division or simply described as harmonic.

Definition (Harmonic Quadrilateral): Let A, B, C, D be four points lying on a circle in this order. If $(A, C; B, D) = -1$, then the quadrilateral $ABCD$ is called a *harmonic quadrilateral.* In other words a cyclic quadrilateral $ABCD$ is harmonic iff $AB \cdot CD = DA \cdot BC$.

Theorem 1: Let D, E, F be points on the interior of sides BC, CA, AB of the triangle ABC. If G is the point of intersection of line FE with the extend side BC (suppose C is between B and G). Then $(B, C; D, G) = -1$ iff AD, BE and CF are concurrent.



Theorem 2 (Projection using P): Let A, B, C, D be four points lying in this order on line d and let P be a point not lying on this line. Take another line d' and consider the four points of intersection A', B', C', D' of the lines PA, PB, PC, PD with d'. Then $(A, C; B, D) = (A', C'; B', D')$.

Important: what does it happen when $PD' || d'$?

This configuration can be denoted as $P(A, C; B, D)$ and is called a pencil. This can be written as $(A, C; B, D) \stackrel{P}{=} (A', C'; B', D')$.

Theorem 3 (Projection using P): Let A, B, C, D be four points lying on a circle in this order. Let P also on this circle. The lines PA, PB, PC, PD intersect a line d at points A', B', C', D'. Then $(A, C; B, D) = (A', C'; B', D')$.

Examples

1. (IMO Shortlist/1995) The incircle of ABC touches BC, CA, and AB at D, E, and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

2. (Cono Sur/2022) Given is a triangle ABC with incircle ω, tangent to BC, CA, AB at D, E, F. The perpendicular from B to BC meets EF at M, and the perpendicular from C to BC meets EF at N. Let DM and DN meet ω at P and Q. Prove that $DP = DQ$.

Problems

1. (AIME II/2016) Triangle ABC is inscribed in circle ω. Points P and Q are on side AB with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4$, $PQ = 3$, $QB = 6$, $BT = 5$ and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. Let A, B, C, D be four points on a line and M be the midpoint of AC. Prove that $(A, C; B, D)$ is a harmonic bundle iff $MB \cdot MD = MA^2$.

3. (Brazil/2013) Let Γ be a circle and A a point outside Γ. The tangent lines to Γ through A touch Γ at B and C. Let M be the midpoint of AB. The segment MC meets Γ again at D and the line AD meets Γ again at E. Given that $AB = a$, $BC = b$, compute CE in terms of a and b.

4. (Brazil/2011) Let ABC be a triangle and H its orthocenter. The lines BH and CH intersect AC and AB at points D and E, respectively. The circumcircle of ADE intersects the circumcircle of ABC at $F \neq A$. Prove that the bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on BC.