Saudi Arabia – Math Camp

Geometry – Inversion

Level 4

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Properties of Inversion – Part 2

Now let's talk about lines and circles.

- Let s be a line **passing through** O. We have $I(s) = s$. This does not mean that each point on s goes to itself, but the set of points goes to itself.
- Let t be a line **not passing through** O. The figure $I(t)$ is a circle passing through O. Take P on t such that $OP \perp t$ and let $P' = I(P)$. For each $A \neq P$ on t we have $\angle OA'P' = \angle OPA = 90° \Rightarrow A'$ is on the circle of diameter OP'. It is clear that each point on the circle (except O) is the inverse of a point on t.
- Let k_1 be a circle **passing through** O. The figure $I(k_1)$ is a line not passing through O.
- Let k_1 be a circle **not passing through** O. The figure $I(k_1)$ is a circle that does **not pass through** O. Let O_1 be the center of k_1 and let the line OO_1 cut k_1 at points A and B. Consider the points A' and B' inverses of A and B, respectively. The circle k_2 have diameter $A'B'$ and center O_2. We will prove that $I(k_1) = k_2$. Let $X \in k_1$. We have
$$\angle AXB = 90° \Rightarrow \angle OXB - \angle OXA = 90°$$
$$\Rightarrow \angle OB'X' - \angle OA'X' = 90° \Rightarrow \angle A'X'B' = 90°.$$
- It is important to remember that $I(O_1)$ almost always is not O_2, but we can use that O, O_1 and O_2 are collinear.
- Using the figures as sets of points, we can see that the number of points of intersection is the same before and after the inversion except by point O. This is especially useful with tangency. If two circles are tangent on O, then the inverses do not have intersection and are parallel lines.

Problems

6. Let p be the semiperimeter of a triangle ABC. Points E and F are taken on line AB such that $CE = CF = p$. Prove that the circumcircle of EFC is tangent to the excircle of ABC corresponding to AB.

7. Let C_1, C_2, C_3, C_4 be four different circles such that C_i is externally tangent to C_{i+1} for $i = 1,2,3,4$ $(C_5 = C_1)$. Prove that the four tangent points are collinear or concyclic.

8. (IMO Shortlist/2003) Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that
$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}$$

9. (EGMO/2016) Two circles, ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at a point T_1, and internally tangent to ω_2 at a point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

10. (Russia/2013) Let ω be the incircle of a triangle ABC, and let I be its incenter. Let Γ be the circumcircle of the triangle AIB. Denote by X and Y the two points of intersection of ω and Γ. Denote by Z the point of intersection of the common tangents to ω and Γ. Prove that the circumcircles of the triangles ABC and XYZ are tangent to each other.