January Camp - 2020

Number theory L4+ Factorials

Problems

1. Let $n > 1$ be an odd integer. Find the remainder of $((n-2)!!)^2$ when it is divided by n.

2. Let p be a prime number and $N = 1+2+\ldots+(p-1) = (p-1)p/2$. Find the remainder when $(p-1)!$ is divided by N.

3. Suppose there exists a permutation $a_1, a_2, \ldots, a_n$ of $0, 1, 2, \ldots, n-1$ such that when divided by n, the remainders of
$$a_1,\ a_1a_2,\ a_1a_2a_3,\ \ldots,\ a_1a_2\ldots a_n$$
are distinct numbers. What may the values of n be?

4. For integers n and q satisfying inequalities $n \geq 5$ and $n \geq q \geq 2$, prove that $\lfloor (n-1)!/q \rfloor$ is divisible by $q-1$.

5. Let p and $p+2$ be twin primes. Find the remainder of $4(p-1)!$ when it is divided by $p(p+2)$.

6. Let $p > 3$ be a prime. Positive integers m and n are relatively prime and
$$\frac{m}{n} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(p-1)^2}.$$
Prove that p divides m and p^2 divides
$$(p-1)!\left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{p-1}\right).$$

7. Prove that the difference between any two terms of the sequence
$$\binom{2}{1}, \binom{4}{2}, \binom{8}{4}, \ldots, \binom{2^{n+1}}{2^n}, \ldots$$
is always divisible by 2^{2020}, apart from a finite number of cases.

8. Let n be a positive integer and $p \geq 5$ a prime number. Prove that
$$\binom{np}{0} + \binom{np}{p} + \binom{np}{2p} + \cdots + \binom{np}{np} \equiv 2^n \pmod{p^3}.$$

Homework

1. Let $a_n = 2^n\binom{3^n}{2^n} + 3^n$. Prove that $\nu_3(a_n) > \frac{4n}{3}$.