Hello ! ■

1) $\boxed{x^2+y^2=3z^2}$ (*) $\rightsquigarrow$ $3 \mid x^2+y^2 \Rightarrow 3 \mid x,\ 3 \mid y \Rightarrow x = 3x_1,\ y = 3y_1$

$\Rightarrow \cancel{9}^{3} x_1^2 + \cancel{9}^{3} y_1^2 = \cancel{3}^{1} z^2 \Rightarrow 3x_1^2+3y_1^2 = z^2 \Rightarrow 3 \mid z^2 \Rightarrow z = 3z_1$

$\Rightarrow 3x_1^2+3y_1^2 = 9z_1^2 \Rightarrow \boxed{x_1^2+y_1^2 = 3z_1^2}$

Finally:

- (x,y,z) (*) s.t. $|x|+|y|+|z|$ - smallest possible

  (x_1,y_1,z_1) : $|x_1|+|y_1|+|z_1| = \frac{1}{3}\left(|x|+|y|+|z|\right)$ $\rightsquigarrow$ $x=y=z=0$.

- for any $n \in \mathbb{N}$. $\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)$ is solution of (*) so $x=y=z=0$

■

Skanowano w CamScanner

(2)

$$\boxed{x^3+3y^3+9z^3-3xyz=0} \Rightarrow 3|x \Rightarrow x=3x_1$$

$$27x_1^3+3y^3+9z^3-3\cdot 3x_1yz=0 \quad |:3$$

$$9x_1^3+y^3+3z^3-3x_1yz=0 \Rightarrow 3|y \Rightarrow y=3y_1$$

$$9x_1^3+27y_1^3+3z^3-9x_1y_1z=0 \quad |:3$$

$$3x_1^3+9y_1^3+z^3-3x_1y_1z=0 \Rightarrow 3|z \Rightarrow \underline{z=3z_1}$$

$$3x_1^3+9y_1^3+27z_1^3-9x_1y_1z_1=0 \quad |:3$$

$$\boxed{x_1^3+3y_1^3+9z_1^3-3x_1y_1z_1=0}$$

$\underline{x=y=z=0}$ (Fermat descent)

3

1. 2. 4

$a^2+b^2+c^2=\boxed{7}d^2$

If we consider mod 7 $\longrightarrow$ $7 \nmid a$ $7 \nmid b$ $7 \nmid c$

no cohen.

$a_1^2+a_1 = \underline{a_1(a_1+1)}$

$a^2 \equiv 0, 1, 4, 2$

$\boxed{\text{modulo } 4}$?

$x^2 \equiv 0, 1 \mod 4$

If $2 \nmid d$ then. $a^2+b^2+c^2 \equiv 7 \mod 4$ $\equiv 3$

a^2+

$2 \nmid a$ $2 \nmid b$ $2 \nmid c$

$2a_1+1$ $2b_1+1$ $2c_1+1$

$(2a_1+1)^2+(2b_1+1)^2+(2c_1+1)^2 = \underbrace{4\overbrace{(a_1^2+a_1)}^{\vdots 2} + 4\overbrace{(b_1^2+b_1)}^{\vdots 2} + 4\overbrace{(c_1^2+c_1)}^{\vdots 2}} + 3 \equiv 3 \mod 8$

$-d^2 \equiv 7d^2 = a^2+b^2+c^2 \equiv 3 \mod 8$ $\quad$ $d^2 \equiv 5 \mod 8$ $\vdots 8$ $\nexists$ dcos.

$-3=5$

$2 \mid d$ so $\underset{0,1}{a^2} + \underset{0,1}{b^2} + \underset{0,1}{c_1^2} \equiv 0 \mod 4$

$\Downarrow$

$2 \mid a, b, c$

$a = 2a_1 \quad b = 2b_1, \quad c = 2c_1 \quad d = 2d_1$

$$\boxed{\cancel{4}a_1^2 + \cancel{4}b_1^2 + \cancel{4}c_1^2 = 7 \cdot \cancel{4}d_1^2}$$

Fermat descent $\Rightarrow$ done !

- $x^4 + y^4 + z^4 = 9u^4$ $\quad$ $\cancel{\text{if}}$ $5 \nmid x \Rightarrow \boxed{x^4 \equiv 1 \mod 5}$

Little Fermat theorem. $\quad x^{p-1} \equiv 1 \pmod p$

Suppose. $\quad 5 \nmid u \Rightarrow$

$$\underset{0,1 \bmod 5}{x^4} + \underset{0,1}{y^4} + \underset{0,1}{z^4} \equiv 9 \cdot 1 \pmod 5 \Rightarrow$$

$9 \cdot 1 = 4$

$5 \mid u \quad \underset{0,1}{x^4} + \underset{0,1}{y^4} + \underset{0,1}{z^4} \equiv 0 \mod 5$ so

$5 \mid x, 5 \mid y, 5 \mid z$ so

Fermat descent Finish !

$p>2$ $\quad x, y \in \{1, 2, \dots, \frac{p-1}{2}\}$. We know $x(p-x)y(p-y)$

is square. Prove that $\underline{x=y}$

$p \nmid b-a$ because:

$p^2 > (p-x)(p-y) = kb^2 > b^2 \Rightarrow$

$p > b > b-a$

$$\underbrace{xy}_{ka^2} \cdot \underbrace{(p-x)(p-y)}_{k \cdot b^2} = A^2$$

$p(p-x-y)$ $\quad b>a$

$$\underbrace{(p-x)(p-y) - xy}_{kb^2-ka^2 = k(b-a)(b+a)} = p^2-px-py \longrightarrow \text{divisible by } p.$$

so

$p \mid \boxed{k(b-a)}(b+a)$ $\quad (*)$

- $p \mid k(b-a)$ $\quad$ if $p \mid k \Rightarrow (k \mid xy \Rightarrow p \mid xy \Rightarrow p \mid x$ or $p \mid y)$

$p > b-a$? , $\quad p>b$

$p^2 > kb^2 > b^2$

$\boxed{p^2 > (p-x)(p-y) = kb^2 > b^2}$

so $p \nmid b-a$

$p \mid k(b-a)(b+a) \Rightarrow \underline{p \mid a+b}$ $\quad x \neq y$

$a^2 \leq ka^2 = xy < \left(\frac{x+y}{2}\right)^2 \Rightarrow a < \frac{x+y}{2}$

$b^2 \leq kb^2 = (p-x)(p-y) < \left(\frac{p-x+p-y}{2}\right)^2$ so

$b < \frac{(p-x)+(p-y)}{2}$

so $p \mid a+b \Rightarrow p \leq a+b$

$p \leq a+b < \frac{x+y}{2} + \frac{(p-x)+(p-y)}{2} = p$ $\lightning$

so $\underline{x=y}$.

1) Solve system of equations in integers.

$$\begin{cases} x^2+6y^2=z^2 \\ 6x^2+y^2=t^2 \end{cases}.$$

2) $y^4 = x^3+7$ in integers.

1) $+\begin{cases} x^2+6y^2=z^2 \\ 6x^2+y^2=t^2 \end{cases} \Rightarrow 7x^2+7y^2=z^2+t^2$

$\Downarrow$

$7 \mid z^2+t^2 \Rightarrow 7 \mid z$ and $7 \mid t \rightsquigarrow$

$z=7z_1, \quad t=7t_1$

$7x^2+7y^2=49z_1^2+49t_1^2 \Rightarrow$

$x^2+y^2=7z_1^2+7t_1^2 \rightsquigarrow 7 \mid x^2+y^2 \Rightarrow x=7x_1, y=7y_1$

$7x_1^2+7y_1^2=z_1^2+t_1^2 \Rightarrow$ Fermat descent $\underline{x=y=z=t=0}$.

$$y^4 = x^3 + 7$$

(3) Find all (k, m) s.t. $k^2 + 4m$ and $m^2 + 5k$ are squares ↓ positive integer.

---

(13) mod 13

x	0	1	2	3	4	5	6	7	8	9	10	11	12
x^3 mod 13	0	1	8	1	12	8	8	5	5	1	12	5	12
x^3+7 mod 13	7	8	12	8	6	2	2	12	12	8	6	12	6

2, 6, 8, 7, 12

4

y	0	1	2	3	4	5	6	7	8	9	10	11	12
y^4 mod 13	0	1	3	3	9	~~9~~ 1	9	9	1	9	3	3	1

0, 1, 3, 9

Let $p>2$ prime, k – positive integer. Then $1^k, 2^k, 3^k, \dots, (p-1)^k$ give

$\boxed{\dfrac{p-1}{\gcd(p-1,k)}}$ different residue modulo p.

$y^4 = x^3 + 7$

$0^4, 1^4, 2^4, 3^4, \dots, 12^4$ ~ 10 residues ⟶ ( )

$0^3, 1^3, 2^3, 3^3, \dots, 12^3$ ~ 10 res ⟶ ( )

Idea

$\boxed{\gcd(\text{export}, p-1) \text{ large}}$

$\boxed{\dfrac{p-1}{\gcd(p-1,k)} \text{ is small}}$ $\Longleftrightarrow$ $\gcd(p-1,k)$ is large

optimal $k \mid p-1$

Good to choose p s.t. $3 \mid p-1$, $4 \mid p-1$ ⇝ $p=13$.

(k, m) positive integer.

k^2+4m, m^2+5k are squares.

$\underline{m \geq k}$

$$(m+3)^2 = m^2+6m+9 > m^2+5m \geq m^2+5k > m^2$$

$$(m+3)^2 > m^2+5k > m^2$$

$$m^2+5k = \text{square}$$

$$(m+2)^2 \qquad (m+1)^2 \qquad \ldots$$

hint. :)

For Ali Find all a, b positive integer s.t. $2a^2+1$, $2b^2+1$, $2(ab)^2+1$ are squares.

$m^2+5k=(m+1)^2=m^2+2m+1 \Rightarrow \quad 5k=2m+1$,

$k^2+4m=k^2+2\cdot 2m = k^2+10k-2 \leftarrow$ perfect square.

$k^2+10k-2 < k^2+10k+25=(k+5)^2$

$\uparrow$

square $\quad \Rightarrow \quad k^2+10k-2 \leq (k+4)^2 = k^2+8k+16$, so

$2k \leq 18 \Rightarrow \boxed{k \leq 9}$

$2m=5k-1 \Rightarrow k$ should be odd.

$k^2+10k-2$ for $k=1,3,5,7,9$ are equal $\boxed{9}, 37, 73, 117, \boxed{169}$

$k=1,\ k=9$ and so $m=\frac{5k-1}{2}=2,\ 22$

$\boxed{(1,2),(9,22)}$.

Skanowano w CamScanner

$m^2+5k = (m+2)^2 \Rightarrow$ $\quad 4m = 5k-4$, so $\frac{k^2+4k}{m^2+5k}$ $\quad \boxed{m \geq k}$

$(m+2)^2 = m^2+4m+4$

$$k^2+4m = k^2+5k-4$$

$$\underbrace{k^2+5k-4}_{\text{sqlar}} < k^2+6k+9 = (k+3)^2 \quad \text{so}$$

$$k^2+5k-4 \leq (k+2)^2 = k^2+4k+4 \quad \text{so}$$

$\boxed{k \leq 8}$, $\quad m = \frac{5}{4} \cdot k - 1$ so $4 | k$.

$k = 4, 8 \rightsquigarrow k^2+5k-4 = 32, 100$

$k = 8$, $\quad m = \frac{5}{4}k - 1 = 9$

$\boxed{8, 9}$

We are left with $m < k$

$$(k+2)^2 = k^2+4k+4 > k^2+4k > k^2+4m > k^2$$

sqlue $\quad$ olave

$k^2+4m = (k+1)^2$

olavey

$\cancel{m^2+5k} =$ $\quad k^2+4m=(k+1)^2 = k^2+2m+1$

$\underbrace{4m}_{\text{even}} = \underbrace{2m+1}_{\text{odd}}$ $\quad ↯$

$\boxed{(1,2),\ (9,22),\ (8,9)}$

Square between square

↓ $\qquad$ ↓

n-th power $\qquad$ n-th power

square (n-th power)

<u>Idea</u> $\quad Y^n \underbrace{< A <}\ X^n$

$A \in \{(Y+1)^n, (Y+2)^n, \dots, (X-1)^n\}$

$(a+3)^2 < A < (a+5)^2$

$\boxed{A = (a+4)^2}$

Find all integers x, y s.t

$$x^2+x+1=y^2.$$

Solve

$$2^x+17=y^4$$ in positive integer.

$\boxed{\text{Solution}}$

if $x>0$ $\quad \underset{x^2+2x+1}{\underset{\parallel}{(x+1)^2}} > x^2+x+1 > x^2$ $\lightning$

if $x \leq -2$ $\quad x^2 > x^2+x+1 \geq \underset{x^2+2x+1}{\underset{\parallel}{(x+1)^2}}$ $\lightning$

$\boxed{x=0, -1}$

$x=0$

$y^2=1 \Rightarrow y=1,-1$

$x=-1$

$y^2=1-1+1=1$

$-1, 1$

$(0,1)$ $(0,-1)$ $(-1,1)$ $(-1,-1)$. $\blacksquare$

$\boxed{2^x + 17 = y^4}$ in positive integers.

It will be nice if $2 | x$

$2^{2x_1} + 17 = y^4 \Rightarrow y^4 - 4^{x_1} = 17$

$(y^2 - 2^{x_1})(y^2 + 2^{x_1}) = 17$

$\phantom{(y^2 - 2^{x_1})} \; \| \qquad\quad \|$

$\phantom{(y^2 - 2^{x_1})} \; 1 \qquad\quad 17$

$$+\begin{cases} y^2 - 2^{x_1} = 1 \\ y^2 + 2^{x_1} = 17 \end{cases}$$

$2y^2 = 18 \Rightarrow y = 3$

$2^{x_1} = y^2 - 1 = 9 - 1 = 8$

$x_1 = 3$

$\Downarrow$

$x = 2x_1 = 6$

$2^6 + 17 = \cancel{y}\, 3^4$

$\; \| \qquad\qquad \|$

$64 + 17 \quad\; 81$

$\boxed{(6,3)}$ ~~s[illegible]~~,

$(6,3)$

$2^x + 17 = y^4$

$y^{16}, \quad 16^x$

$y^{16} - 16^x = (\underbrace{y^4}_{A})^4 - (\underbrace{2^x}_{B})^4 = A^4 - B^4 = (\underbrace{A - B}_{\underbrace{y^4 - 2^x}_{17}})(A^3 + A^2B + AB^2 + B^3)$

so $\boxed{17 \mid y^{16} - 16^x}$

Notice that $17 \nmid y$, if yes $17 \mid y^4 - 17 = \underline{2^x}$ ↯

so $\underline{17 \nmid y} \Rightarrow$ LFT: $\boxed{y^{16} \equiv 1 \pmod{17}}$

$1 \equiv y^{16} \equiv 16^x \pmod{17}$

$16^x \equiv (-1)^x$

$16^x \equiv (-1)^x \bmod 17$

$(-1)^x \equiv 1 \pmod{17}$

$\Downarrow$

x is even

Revision

Little Fermat theorem

p – prime: a – arbitrary number $\quad p \mid a^p - a$

$(p,a)=1$ then $a^{p-1} \equiv 1 \pmod p$.

For example $a^2 \equiv 1 \pmod 3$ $\quad a^4 \equiv 1 \mod 5$ $\quad a^{16} \equiv 1 \pmod{17}$

For example $\quad p \equiv 3 \mod 4$ then $p \mid a^2+b^2 \Rightarrow p \mid a \wedge p \mid b$.

$$a^2+b^2 \equiv 0 \pmod p$$

$$a^2 \equiv -b^2 \pmod p \quad \wedge \quad \frac{p-1}{2} = \frac{4k+3-1}{2} = \underbrace{2k+1}_{\text{odd}}$$

$$(a^2)^{\frac{p-1}{2}} \equiv -(b^2)^{\frac{p-1}{2}}$$

$$a^{p-1} \equiv -b^{p-1} \mod p$$

LFT.

Now. Suppose $p \nmid a \Rightarrow p \nmid b$ $(p \mid a^2+b^2)$. so $a^{p-1} \equiv b^{p-1} \equiv 1 \pmod p$

$1 \equiv -1 \mod p \Rightarrow p=2$ $\lightning$

Excus

$p = 3k+2$ — prime.

$1^3, 2^3, 3^3, \dots, (p-1)^3$ are differen (mod p).

1) $i^3 \equiv j^3 \pmod p$ $\qquad \frac{p-\cancel{2}}{3} = k$

$i^{p-2} \equiv j^{p-2} \pmod p$ $\quad | \cdot ij$

LFT $\quad i^{p-1} j \equiv j^{p-1} i \pmod p$

$\quad \overset{|||}{1} \qquad \qquad \overset{|||\ LFT}{1}$

$j \equiv i \pmod p$

$\frac{p-1}{\gcd(p-1, 3)}$ diff remainders.

$\gcd(p-1, 3) = 1$ $\quad p-1$.