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Number Theory

Level L3

Dominik Burek

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Problems

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$$2a^2+a=3b^2+b.$$

Prove that $a-b$ and $2a+2b+1$ are perfect squares.

Problem 3. Let m, n be a positive integers such that set $\{1, 2, \ldots, n\}$ contains exactly m different prime numbers. Prove that if we choose any $m+1$ different numbers from $\{1, 2, \ldots, n\}$ then we can find number from $m+1$ choosen numbers, which divide product of other m numbers.

Pro☠lem 4. Positive rational number a and b satisfy the equality

$$a^3+4a^2b=4a^2+b^4.$$

Prove that the number $\sqrt{a}-1$ is a square of a rational number.

Pro☠lem 5. Let $a \geqslant 3$ be an integer. Prove that there exists infinitely many positive integers n such that $n^2 \mid a^n-1$.

Problem 6. Consider a sequence $a_n=|n(n+1)-19|$ for integer $n \geqslant 0$. Prove that for any $n \neq 4$ the following holds: If for all integers $k<n$ numbers a_k and a_n are coprime, then a_n is a prime.

Problem 7. Let p a prime number and r an integer such that $p \mid r^7-1$. Prove that if there exist integers a, b such that $p \mid r+1-a^2$ and $p \mid r^2+1-b^2$, then there exist an integer c such that $p \mid r^3+1-c^2$.

Pro☠lem 8. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle, moving in a clockwise direction; that is, the numbers a, b, c, d are replaced by $a-b$, $b-c$, $c-d$, $d-a$. Is it possible that after 2019 of such moves to have numbers a, b, c, d such the numbers $|bc-ad|$, $|ac-bd|$, $|ab-cd|$ are primes?

Problem 9. Let $k>1$ be the integer. Sum of a divisor of k and a divisor of $k-1$ is equal to ℓ and $\ell > k+1$. Prove that at least one number: $\ell-1$ or $\ell+1$ is composite.

Pro☠lem 10. Let k, n be a positive integers such that $k > n!$. Prove that there exist distinct prime numbers $p_1, p_2, \ldots, p_n$ such that $p_i \mid k+i$ for all $i = 1, 2, \ldots, n$.

Problem 11. Let a, b, c, d be a positive integers such that

$$cn + d \mid an + b$$

for any positive integers n. Prove that $a = kc$ and $b = kd$ for some integer k.

Problem 12. Find all positive integers n for which there exist positive integers $x_1, x_2, \ldots, x_n$ such that

$$\frac{1}{x_1^2} + \frac{2}{x_2^2} + \frac{2^2}{x_3^2} + \cdots + \frac{2^{n-1}}{x_n^2} = 1.$$

Problem 13. Let $d(k)$ denote the number of positive divisors of a positive integer k. Prove that there exist infinitely many positive integers M that cannot be written as

$$M = \left(\frac{2\sqrt{n}}{d(n)}\right)^2$$

for any positive integer n.

Problem 14. For any integer $N \geq 2$, let $f(N)$ denotes sum of N and the greatest divisor of N (other than N). Prove that for any integer $A \geq 2$, by iterating f on A we can get a number divisible by 3^{2021}.

Problem 15. Call a positive integer n a *good number*, if there exists prime number p such that $p \mid n$ and $p^2 \nmid n$. Prove that 99% numbers among $1, 2, 3, \ldots, 10^{12}$ are good.

Pro☠lem 16. Integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \ldots a_n$, then prove that

$$(a_1 a_2 \ldots a_n)^{m-1} \geqslant (n!)^m.$$

Pro☠lem 17. Find all positive integers n for which there exist non-negative integers a_1, a_2, $\ldots$, a_n such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Solutions

Problem 1. Let a, b be a positive integers such that $a+b+1$ is a prime divisor of $4ab-1$. Prove that $a=b$.

Solution. We see that

$$a+b+1 \mid 4ab-1+2(a+b+1)=(2a+1)(2b+1),$$

so $a+b+1$ divides $2a+1$ or $2b+1$. WLOG assume, that $a+b+1 \mid 2a+1$, then

$$\frac{2a+1}{a+b+1}=1,$$

because otherwise inequality

$$\frac{2a+1}{a+b+1} \geqslant 2$$

implies that $2b+1 \leqslant 0$ – contradiction. Therefore $2a+1=a+b+1$ i.e. $a=b$. □

Discussion. AoPS

Problem 2. Let a, b be integers such that

$$2a^2+a=3b^2+b.$$

Prove that $a-b$ and $2a+2b+1$ are perfect squares.

Solution. Notice that

$$b^2=2a^2+a-2b^2-b=(a-b)(2a+2b+1). \tag{1}$$

If prime p divides $a-b$ and $2a+2b+1$, then $p \mid b$ and so

$$p \mid (2a+2b+1-2(a-b))-4b=4b+1-4b=1.$$

Therefore $\gcd(a-b, 2a+2b+1)=1$.

If $a-b<0$, then $a-b=-k^2$ and $2a+2b+1=-\ell^2$ for some integers k, l. The number $3b^2+b=b(3b+1)$ is even, so $2a^2+a$ too and so a is even.

Since $a=\frac{1}{4}(-2k^2-\ell^2-1)$, we see that ℓ is odd, i.e. $\ell=2n+1$ for some integer n. Then $a=\dfrac{-k^2-2n(n+1)-1}{2}$. Therefore k is odd, so $k=2m+1$ for some integer m. Thus

$$a=\frac{-2(4m^2+4m+1)-4n(n+1)-2}{4}=-2m^2-2m-n(n+1)-1,$$

hence a is odd – contradiction.

Therefore $a-b>0$ and so from 1 the problem follows. □

Discussion.

Problem 3. Let m, n be a positive integers such that set $\{1, 2, \ldots, n\}$ contains exactly m different prime numbers. Prove that if we choose any $m+1$ different numbers from $\{1, 2, \ldots, n\}$ then we can find number from $m+1$ choosen numbers, which divide product of other m numbers.

Solution. Suppose that problem statement doesn't hold. Then there exists $(m+1)$-elements set $A \subset \{1, 2, \ldots, n\}$, such that no $x \in A$ which divide product of remaining elements in A. Therefore any $x \in A$ has a prime divisor p, whose exponent is greater then exponent of p in a product of numbers in $A \setminus \{x\}$.

Thus to any $x \in A$ we associate a prime number from $\{1, 2, \ldots, n\}$. Since A consists of $m+1$ elements, then by the Pigeonhole Principle some prime p is associated for two different elements $x, y \in A$. Denote by w the product $m-1$ elements of the set $A \setminus \{x, y\}$. There exists non-negative integers k and l such that $p^k \mid x$, $p^k \nmid wy$, $p^l \mid y$ and $p^l \nmid wx$. Then exponent of p in $wy \cdot wx$ is smaller then $k+l$, and simultaneously $p^{k+l} \mid xy \mid wy \cdot wx$ – contradiction. $\square$

Discussion.

Pro☠lem 4. Positive rational number a and b satisfy the equality

$$a^3 + 4a^2b = 4a^2 + b^4.$$

Prove that the number $\sqrt{a} - 1$ is a square of a rational number.

Solution. Note that

$$a(a+2b)^2 = a^3 + 4a^2b + 4ab^2 = 4a^2 + b^4 + 4ab^2 = (2a+b^2)^2,$$

thus

$$a = \frac{(2a+b^2)^2}{(a+2b)^2} \quad \text{and} \quad \sqrt{a} = \frac{2a+b^2}{a+2b}.$$

Therefore $\sqrt{a} \in \mathbb{Q}$. Moreover $x = b$ is a root of quadratic equation

$$x^2 - 2\sqrt{a}x + 2a - a\sqrt{a} = 0.$$

Simultaneously coefficients of these equation are rational, hence its discriminant too. Thus

$$\Delta = (2\sqrt{a})^2 - 4(2a - a\sqrt{a}) = 4a(\sqrt{a} - 1)$$

is a perfect square, in particular

$$\frac{\Delta}{(2\sqrt{a})^2} = \sqrt{a} - 1$$

is a perfect square, too. $\square$

Solution. As in the above solution we have that $\sqrt{a} \in \mathbb{Q}$. Let $c := \sqrt{a}$, then our equality becomes $c^6 + 4c^4b = 4c^4 + b^4$. Hence

$$c^2 + 4b = 4 + \left(\frac{b}{c}\right)^4 = \left(\left(\frac{b}{c}\right)^2 + 2\right)^2 - 4 \cdot \frac{b^2}{c^2},$$

so

$$\left(\frac{2b}{c}+c\right)^2 = c^2 + 4\cdot\frac{b^2}{c^2} + 4b = \left(\left(\frac{b}{c}\right)^2+2\right)^2,$$

thus

$$\left(\frac{b}{c}\right)^2 + 2 = \frac{2b}{c} + c$$

i.e.

$$\sqrt{a} - 1 = c - 1 = \left(\frac{b}{c}\right)^2 - 2\cdot\frac{b}{c} + 1 = \left(\frac{b}{c}-1\right)^2.$$

□

Solution. As in the previous solutions we get that

$$\sqrt{a} = \frac{(2a+b^2)^2}{(a+2b)},$$

so we are left with proving that

$$\sqrt{a} - 1 = \frac{(b^2-2b+a)(a+2b)}{(a+2b)^2}$$

is a square of a rational number. Since a is a perfect square of some rational, it is enough to prove that $a(b^2-2b+a)(a+2b)$ a square of a rational number. But

$$\begin{aligned} a(b^2-2b+a)(a+2b) &= a^2b^2 + a^3 + 2b^3a - 4b^2a = \\ &= a^2b^2 + 4a^2 + b^4 + 2b^3a - 4a^2b - 4b^2a = (2a-b^2-ab)^2. \end{aligned}$$

□

Discussion. AoPS

Pro☠lem 5. Let $a \geqslant 3$ be an integer. Prove that there exists infinitely many positive integers n such that $n^2 \mid a^n - 1$.

Solution. Suppose that $n^2 \mid a^n - 1$ for some n. Let $m := \dfrac{a^n-1}{n}$. We show that $m^2 \mid a^m - 1$.

We see that $n \mid m$. Moreover

$$\frac{a^m-1}{a^n-1} = 1 + a^n + a^{2n} + \ldots + a^{m-n},$$

thus $\dfrac{a^m-1}{a^n-1}$ is sum of $\dfrac{m}{n}$ numbers, which are 1 modulo a^n-1, so modulo $\dfrac{a^n-1}{n^2}$ too. Thus

$$\frac{a^n-1}{n^2} \mid \frac{a^m-1}{a^n-1}$$

i.e.

$$m^2 = (a^n-1)\frac{a^n-1}{n^2} \mid (a^n-1)\frac{a^m-1}{a^n-1} = a^m - 1.$$

□

Discussion. AoPS

Problem 6. Consider a sequence $a_n = |n(n+1)-19|$ for integer $n \geqslant 0$. Prove that for any $n \neq 4$ the following holds: If for all integers $k < n$ numbers a_k and a_n are coprime, then a_n is a prime.

Solution. Let $c_n = n(n+1) - 19$, then $a_n = \pm c_n$. We check that a_0, a_1, a_2, a_3 are primes and $a_4 = 1$. Take $a_n = c_n$, which is a composite number for $n > 4$. It is enough to prove that c_n has a common divisor $d > 1$ with at least one integer from $c_0, c_1, \ldots, c_{n-1}$.

Let $d > 1$ be the smallest divisor of c_n. Then $\frac{c_n}{d}$ also divides c_n and $d \leqslant \frac{c_n}{d}$, thus

$$d^2 \leqslant n(n+1) - 19 < (n+1)^2,$$

so $d \leqslant n$ i.e. $k = n - d \in \{0, 1, \ldots, n-2\}$. Since $c_n - c_k = d(2n - d + 1)$ is divisible by d, we get that d is a common divisor of c_n and c_k. □

Discussion. AoPS

Problem 7. Let p a prime number and r an integer such that $p \mid r^7 - 1$. Prove that if there exist integers a, b such that $p \mid r+1-a^2$ and $p \mid r^2+1-b^2$, then there exist an integer c such that $p \mid r^3 + 1 - c^2$.

Solution. It is easy to exclude cases $r \equiv \pm 1, 0 \pmod{p}$. Therefore

$$\left(\frac{r-1}{p}\right)\left(\frac{r^4+1}{p}\right) = \left(\frac{r-1}{p}\right)\left(\frac{r+1}{p}\right)\left(\frac{r^2+1}{p}\right)\left(\frac{r^4+1}{p}\right) = \left(\frac{r^8-1}{p}\right) = $$
$$= \left(\frac{r^8-r^7}{p}\right) = \left(\frac{r}{p}\right)\left(\frac{r-1}{p}\right).$$

Hence,

$$\left(\frac{r^3+1}{p}\right) = \left(\frac{r^3+r^7}{p}\right) = \left(\frac{r}{p}\right)\left(\frac{r^4+1}{p}\right) = 1. \quad \blacksquare$$

□

Discussion. AoPS

Pro☠lem 8. Four integers are marked on a circle. On each step we simultaneously replace each number by the difference between this number and next number on the circle, moving in a clockwise direction; that is, the numbers a, b, c, d are replaced by $a-b$, $b-c$, $c-d$, $d-a$. Is it possible that after 2019 of such moves to have numbers a, b, c, d such the numbers $|bc-ad|$, $|ac-bd|$, $|ab-cd|$ are primes?

Solution. Obviously, after the first step the sum of a, b, c and d is zero. So after 2019 steps we will get new numbers a', b', c', d', such that $a' + b' + c' + d' = 0$.

Notice that

$$\begin{aligned} a'b' - c'd' &= a'b' + c'(a' + b' + c') = (c' + a')(c' + b'), \\ a'c' - b'd' &= a'c' + b'(a' + b' + c') = (b' + a')(b' + c'), \\ b'c' - a'd' &= b'c' + a'(a' + b' + c') = -(a' + b')(a' + c'). \end{aligned}$$

So we get that:

$$|a'b' - c'd'| \cdot |a'c' - b'd'| \cdot |a'd' - b'c'| = (a' + b')^2(b' + c')^2(c' + a')^2.$$

But the product of three primes can't be a perfect square, so the answer is no. □

Discussion.

Problem 9. Let $k > 1$ be the integer. Sum of a divisor of k and a divisor of $k - 1$ is equal to ℓ and $\ell > k + 1$. Prove that at least one number: $\ell - 1$ or $\ell + 1$ is composite.

Solution. Any divisor of N is equal to N or is at most $N/2$. Therefore if sum of divisors of k and $k - 1$ is greater then $k + 1$ then one of them is equal to k or $k - 1$.

If one is equal to k, then $\ell - 1 = k - 1 + d$, where $d > 1$ and $d \mid k - 1$. Thus $d \mid \ell - 1$, so $\ell - 1$ is composite.

If one of them is $k - 1$, then $\ell + 1 = k + e$, for some $e > 1$ and $e \mid k$. Then $e \mid \ell + 1$, so $\ell + 1$ is composite.

□

Discussion. AoPS

Pro☠lem 10. Let k, n be a positive integers such that $k > n!$. Prove that there exist distinct prime numbers $p_1, p_2, \ldots, p_n$ such that $p_i \mid k + i$ for all $i = 1, 2, \ldots, n$.

Solution. For $i = 1, 2, \ldots, n$ let

$$a_i = \operatorname{lcm}(\text{divisors of } k + i \text{ which not exceed } n).$$

Then $a_i \leqslant n! < k$. Moreover $a_i \mid k + i$, thus

$$\frac{k+1}{a_1}, \quad \frac{k+2}{a_2}, \ldots, \frac{k+n}{a_n}$$

are integers greater than 1.

No we prove that these numbers are coprime. Take any $1 \leqslant i, j \leqslant n$. Since $(k + i) - (k + j) < n$, than $d := \gcd(k + i, k + j) \leqslant n$, so $d \mid a_i$ and $d \mid a_j$. It means that $\frac{k+i}{a_i}$ and $\frac{k+j}{a_j}$ are divisors of $\frac{k+i}{d}$ and $\frac{k+j}{d}$, respectively. But the letter numbers are coprime, so $\frac{k+i}{a_i}$ and $\frac{k+j}{a_j}$ are coprime too.

Finally easy to observe that these numbers satisfy problem statement. □

Discussion.

Problem 11. Let a, b, c, d be a positive integers such that

$$cn + d \mid an + b$$

for any positive integers n. Prove that $a = kc$ and $b = kd$ for some integer k.

Solution. Given condition implies that

$$cn + d \mid a(cn + d) - c(an + b) = ad - bc$$

for any integer $n \geqslant 0$. Therefore $ad = bc$, so

$$\frac{a}{c} = \frac{b}{d} = k,$$

for some $k \in \mathbb{Q}$. But $a + b = k(c + d)$ is divisible by $c + d$ (we put $n = 1$ in problem condition). Therefore $k \in \mathbb{Z}$ and we are done. □

Discussion. AoPS

Problem 12. Find all positive integers n for which there exist positive integers $x_1, x_2, \ldots, x_n$ such that

$$\frac{1}{x_1^2} + \frac{2}{x_2^2} + \frac{2^2}{x_3^2} + \cdots + \frac{2^{n-1}}{x_n^2} = 1.$$

Solution. We will prove by induction that all n except 2 satisfies given condition.

Claim: If n has a solution, then $n + 2$ has a solution.

Proof. Let $(x_1, x_2, \ldots, x_n)$ be a solution for n. Then just check that

$$(x_1, x_2, \ldots, x_{n-1}, 2x_n, 2x_n, 4x_n)$$

is a solution for $n + 2$. Now note that 1 is a solution for $n = 1$, $(2, 2, 4)$ for $n = 3$ and $(3, 3, 3, 6)$ is a solution for $n = 4$ so that all $n \neq 2$ work. □

Finally for $n = 2$ the equation is equivalent to $(x_1^2 - 1)(x_2^2 - 2) = 2$ which is easily seen to not have solution. □

Discussion. AoPS

Problem 13. Let $d(k)$ denote the number of positive divisors of a positive integer k. Prove that there exist infinitely many positive integers M that cannot be written as

$$M = \left(\frac{2\sqrt{n}}{d(n)}\right)^2$$

for any positive integer n.

Solution. Let M be an odd perfect square, and suppose

$$M = \left(\frac{2\sqrt{n}}{d(n)}\right)^2.$$

In particular $\sqrt{n}$ is an integer and hence n is a perfect square. But then $d(n)$ is odd, which implies that M is even – contradiction.

Since there are infinitely many odd perfect squares, our proof is complete. $\square$

Discussion.

Problem 14. For any integer $N \geq 2$, let $f(N)$ denotes sum of N and the greatest divisor of N (other than N). Prove that for any integer $A \geq 2$, by iterating f on A we can get a number divisible by 3^{2021}.

Solution. Note that f takes even values for odd arguments. Moreover taking even number of the form $2^k a$, where $k \geq 1$ and $2 \nmid a$, we see that

$$2^k a \xrightarrow{f} 2^{k-1} \cdot 3a \xrightarrow{f} 2^{k-2} \cdot 3^2 a \xrightarrow{f} \ldots \xrightarrow{f} 3^k a.$$

We will prove inductively, that for any natural n by iterating f, from any integer (≥ 2) we can made odd number divisible by 3^n.

Base case of an induction was at the beginning, since we made from any number, the odd number divisible by 3. Suppose that by iterating f we obtained number of the form $3^n a$, where a is odd number. Then

$$3^n a \xrightarrow{f} 2^2 \cdot 3^{n-1} a \xrightarrow{f} 2 \cdot 3^n a \xrightarrow{f} 3^{n+1} a,$$

which ends inductive step. $\square$

Discussion.

Problem 15. Call a positive integer n a *good number*, if there exists prime number p such that $p \mid n$ and $p^2 \nmid n$. Prove that 99% numbers among $1, 2, 3, \ldots, 10^{12}$ are good.

Solution. Let us observed that the number which is not good has the form a^3b^2 for some integers a, b. Therefore the number of not good numbers (*bad* one's) does not exceed the number of numbers in $\{1, \ldots, 10^{12}\}$ with the form a^3b^2.

Since $a^3b^2 \leqslant 10^{12}$ we see that $a \leqslant 10^4$ and $b \leqslant 10^6$, so the number of bad numbers is not greater that $10^4 \cdot 10^6 = 10^{10}$. Thus there is at least

$$\frac{10^{12} - 10^{10}}{10^{12}} = 99\% \cdot 10^{12}$$

good numbers in the set $\{1, \ldots, 10^{12}\}$. $\square$

Discussion.

Pro☠lem 16. Integers $a_1, a_2, \dots, a_n$ satisfy
$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$
If m is the number of distinct prime factors of $a_1 a_2 \dots a_n$, then prove that
$$(a_1 a_2 \dots a_n)^{m-1} \geqslant (n!)^m.$$

Solution. Let us write $a_i = p^{k_i} \cdot b_i$, where $p \nmid b_i$ for a prime divisor p of $a_1 a_2 \dots a_n$. Then, due to $a_1 < a_2 < \ldots < a_n < 2a_1$ we get that b_i are pairwise distinct. Indeed, if $b_i = b_j$ for some $i < j$ then
$$\frac{a_j}{a_i} = \frac{p^{k_j} \cdot b_i}{p^{k_i} \cdot b_i} = p^{k_j - k_i} \geqslant 2.$$
Thus
$$b_1 b_2 \dots b_n \geqslant n!.$$
Multiplying such inequalities for each $p \mid a_1 a_2 \dots a_n$ we get the result. $\square$

Discussion.

Pro☠lem 17. Find all positive integers n for which there exist non-negative integers a_1, a_2, $\dots$, a_n such that
$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Solution. The answer is $n \equiv 1, 2 \pmod 4$. These are obviously the only n that work, since if $n \equiv 3, 4 \pmod 4$, then
$$1 = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} \equiv 1 + 2 + \cdots + n \equiv 0 \pmod 2,$$
contradiction.

We will show all $n \equiv 1, 2 \pmod 4$ by strong induction, with the following base cases:

- For $n = 1$, take $(a_1) = (0)$.
- For $n = 5$, take $(a_1, \dots, a_5) = (2, 2, 2, 3, 3)$.
- For $n = 9$, take $(a_1, \dots, a_9) = (2, 3, 3, 3, 3, 4, 4, 4, 4)$.

Claim: If $4k + 1$ works, then so does $4k + 2$.

Proof. Note the identities
$$\frac{1}{2^{a_{2k+1}}} = \frac{1}{2^{a_{2k+1}+1}} + \frac{1}{2^{a_{2k+1}+1}} \quad \text{and} \quad \frac{2k+1}{3^{a_{2k+1}}} = \frac{2k+1}{3^{a_{2k+1}+1}} + \frac{4k+2}{3^{a_{2k+1}+1}}.$$
Now suppose $(a_1, \dots, a_{4k+1})$ is a valid solution for $n = 4k + 1$. Setting $b_{2k+1} = b_{4k+2} = a_{2k+1} + 1$ and $b_i = a_i$ for all other i, we obtain a valid solution $(b_1, \dots, b_{4k+2})$ for $n = 4k + 2$. $\square$

Claim: If $4k + 1$ works, then so does $4k + 13$.

Proof. Let $(x_1, \ldots, x_{13}) = (2, 2, 3, 3, 6, 6, 6, 5, 6, 6, 6, 4, 4)$ satisfy
$\dfrac{1}{2^{x_1}} + \dfrac{1}{2^{x_2}} + \cdots + \dfrac{1}{2^{x_{13}}} = \dfrac{1}{3^{x_1}} + \dfrac{2}{3^{x_2}} + \cdots + \dfrac{13}{3^{x_{13}}} = 1$ and $\dfrac{1}{3^{x_1}} + \dfrac{1}{3^{x_2}} + \cdots + \dfrac{1}{3^{x_{13}}} = \dfrac{1}{3}$.
It follows that the following identities hold:

$$\frac{1}{2^{a_{k+1}}} = \frac{1}{2^{a_{k+1}+2}} + \sum_{t=2}^{13} \frac{1}{2^{a_{k+1}+x_t}} \quad \text{and} \quad \frac{k+1}{3^{a_{k+1}}} = \frac{k+1}{3^{a_{k+1}+2}} + \sum_{t=2}^{13} \frac{4k+t}{3^{a_{k+1}+x_t}}.$$

Now suppose $(a_1, \ldots, a_{4k+1})$ is a valid solution for $n = 4k+1$. Setting $b_{k+1} = a_{k+1} + 2$, $b_{4k+t} = a_{k+1} + x_t$, and $b_i = a_i$ for all other i, we obtain a valid solution $(b_1, \ldots, b_{4k+13})$ for $n = 4k+13$. □

□

Discussion.

References

- Art of Problem Solving - `https://artofproblemsolving.com`
- Polish Mathematical Olympiad - `https://om.mimuw.edu.pl`
- Homepage of Dominik Burek - `http://dominik-burek.u.matinf.uj.edu.pl`