Test 2
Level 3, December 1

Problem 2.1. Consider function $f : R \to R$ such that

$$2(y+1)f(x)f(y-1) = 2yf(xy) - f(2x)$$

for all $x, y \in R$. Find all possible values for $f(2022)$.

Problem 2.2. Let $n \leq 100$ be a positive integer. There are 101 numbers written in a row:

$$0 \cdot n[101], 1 \cdot n[101], \ldots, 100 \cdot n[101].$$

How many pairs of neighbouring numbers are there in this row such that the one on the left is bigger than the one on the right? ($a[b]$ stands for the remainder of a when divided by b, and is a number $0, 1, ..., b-2$ or $b-1$)

Problem 2.3. Given is an equilateral triangle ABC with circumcenter O. Let D be a point on to minor arc BC of its circumcircle such that $DB > DC$. The perpendicular bisector of OD meets the circumcircle at E, F, with E lying on the minor arc BC. The lines BE and CF meet at P. Prove that $PD \perp BC$.

Problem 2.4. There are 100 doors labeled with numbers $1, 2, ..., 100$. You have 100 keys labeled with numbers. Each key corresponds to exactly one door. If the key i corresponds to the door j, then $\mid i - j \mid \leq 1$. At each turn, you may pick doors with numbers i and j and check whether the key i corresponds to the door j. Can you find which key corresponds to which door in

a) 99 turns?
b) 75 turns?
c) 74 turns?