Sol2

$$x^2 = 2xyz - x$$
$$y^2 = 2xyz - y$$
$$z^2 = 2xyz - z$$

$$x^2 + x = y^2 + y = z^2 + z = \underbrace{2xyz}_{P}$$

Define: $f(t) = \underline{t^2 + t} - P$

$f(x) = f(y) = f(z) = 0$ (3 zeros for a quadratic)

$\Longrightarrow$ WLOG $x = y$

$z = 2x^2 - 1$; $x = 2x(2x^2 - 1) - 1$

$\Longleftrightarrow$ $(2x+1)^2(x-1) = 0$

$\longrightarrow$ $(x, y, z) = (1, 1, 1), (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

Prove that:

1) $xy + yz + zx \geq 2x\sqrt{y-1} + 2y\sqrt{z-1} + 2z\sqrt{x-1}$

$\forall \quad x, y, z \geq 1$

---

2) ماهي أكبر مساحة ممكنة لرباعي دائري محاط بدائرة نصف قطرها $\sqrt{2}$ ؟

---

1) $y = (y-1) + 1 \geq 2\sqrt{y-1}$

AM-GM

$$\left.\begin{aligned} \Rightarrow xy &\geq 2x\sqrt{y-1} \\ yz &\geq 2y\sqrt{z-1} \\ zx &\geq 2z\sqrt{x-1} \end{aligned}\right\} \sum \rightarrow LHS \geq RHS$$

③ ما هي أكبر مساحة ممكنة لرباعي (أ - دائري) (ب - ⊗) محيطه 4 ؟

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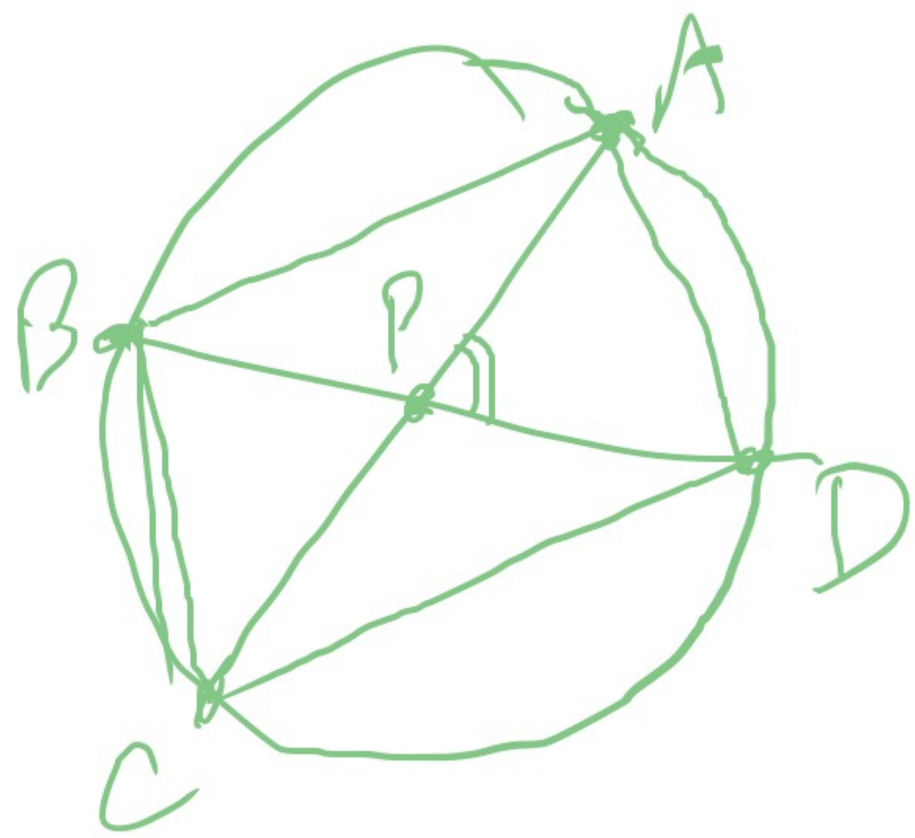
2 : Sol 1



$$[ABCD] = \frac{1}{2} AC \cdot BD \sin\angle P$$

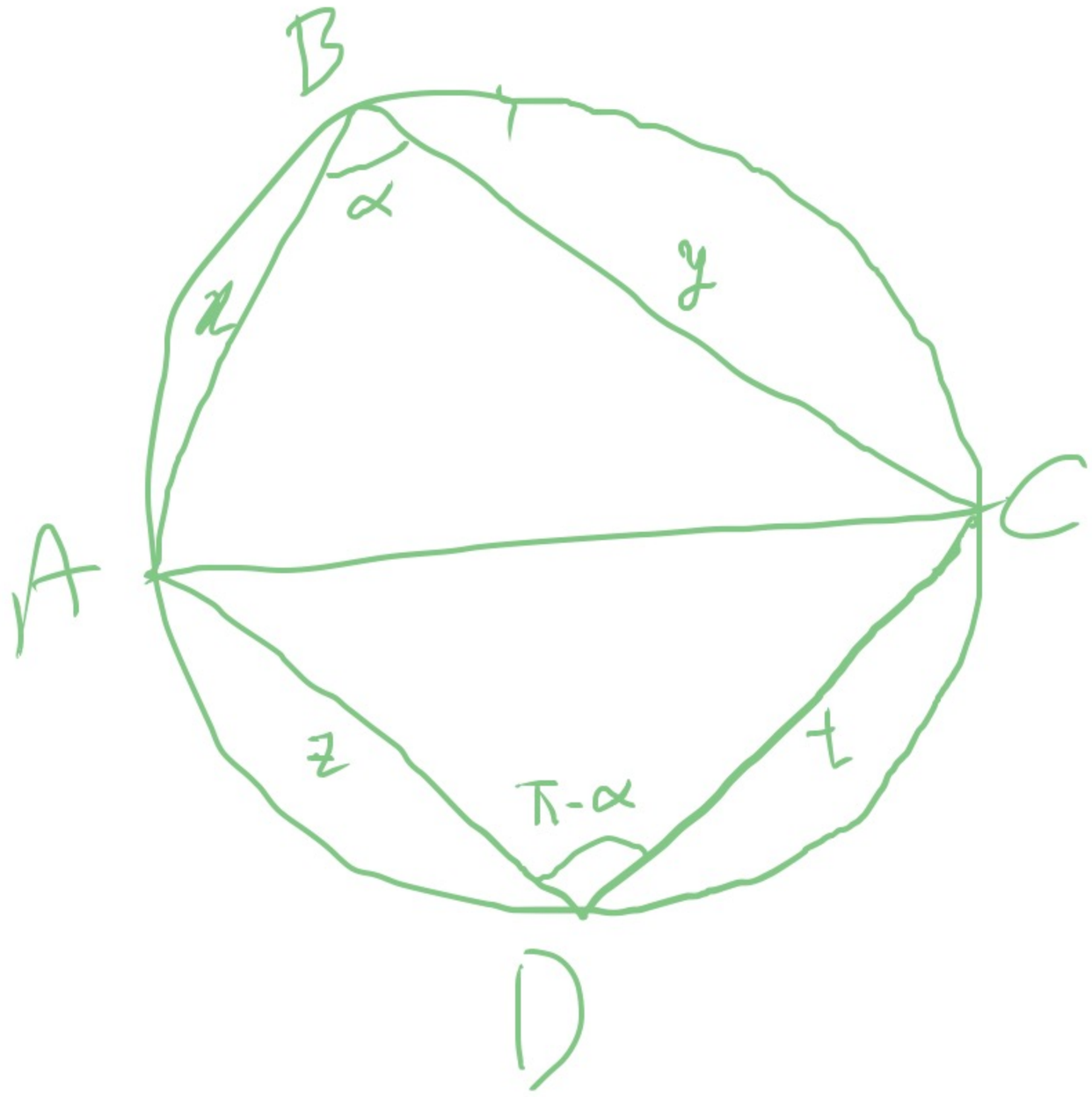
$$AC \leq 2\sqrt{2} \quad ; \quad BD \leq 2\sqrt{2}$$

$$\sin\angle P \leq 1$$

$$\Rightarrow [ABCD] \leq \frac{1}{2}(2\sqrt{2})(2\sqrt{2})(1) = 4$$

Sin

Sol:



$$[ABCD] = \frac{1}{2}xy\sin\alpha + \frac{1}{2}zt\sin(\pi-\alpha)$$

$$= \frac{1}{2}\sin\alpha\,(xy+zt)$$

$$\leq \frac{1}{2}(xy+zt)$$

$$\leq \frac{1}{2}\left(\frac{x^2+y^2}{2} + \frac{z^2+t^2}{2}\right)$$

$$= 4$$

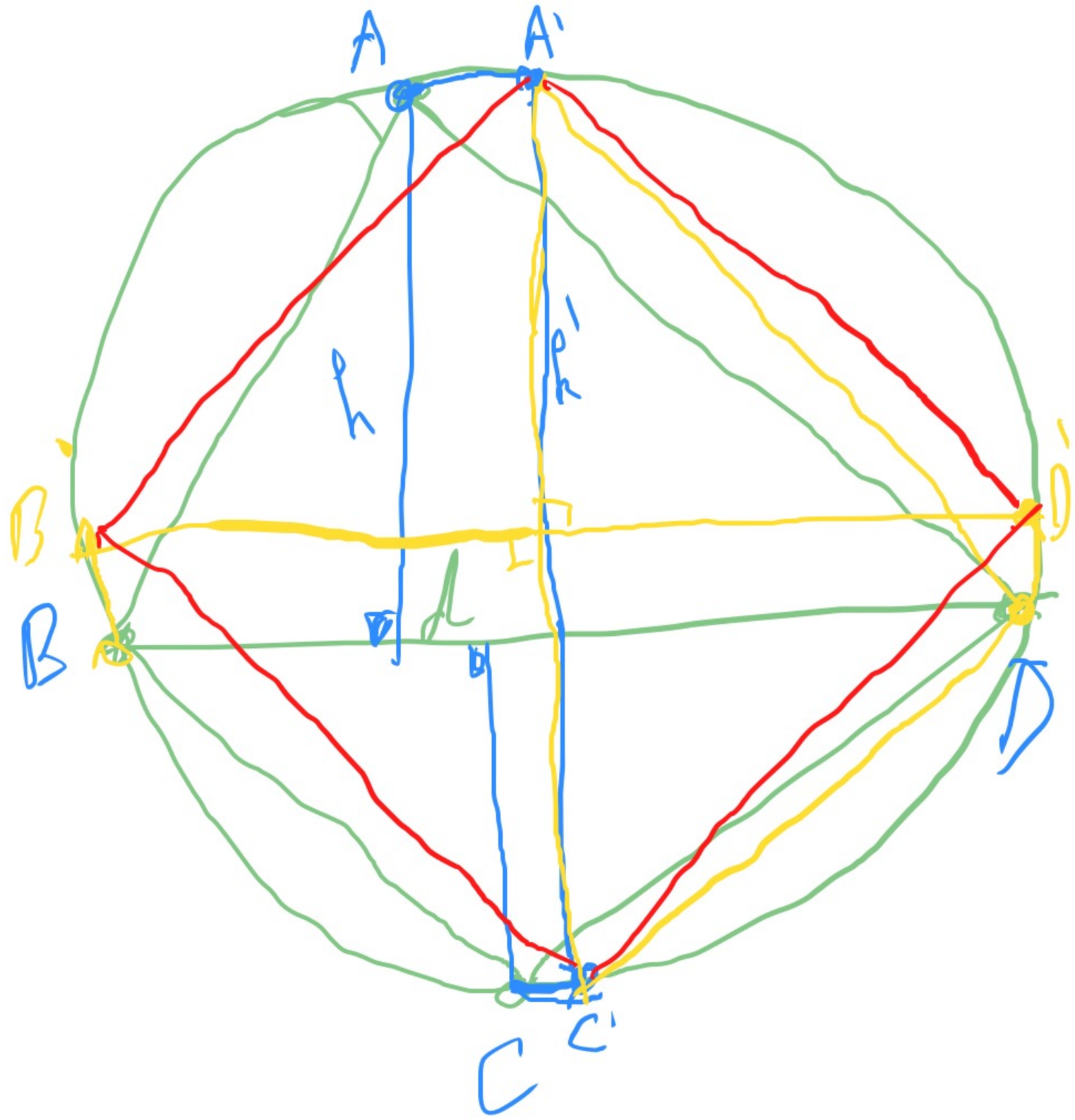
$\frac{\sqrt{3}}{4}(xy+zt)$

$\alpha = 60^\circ$

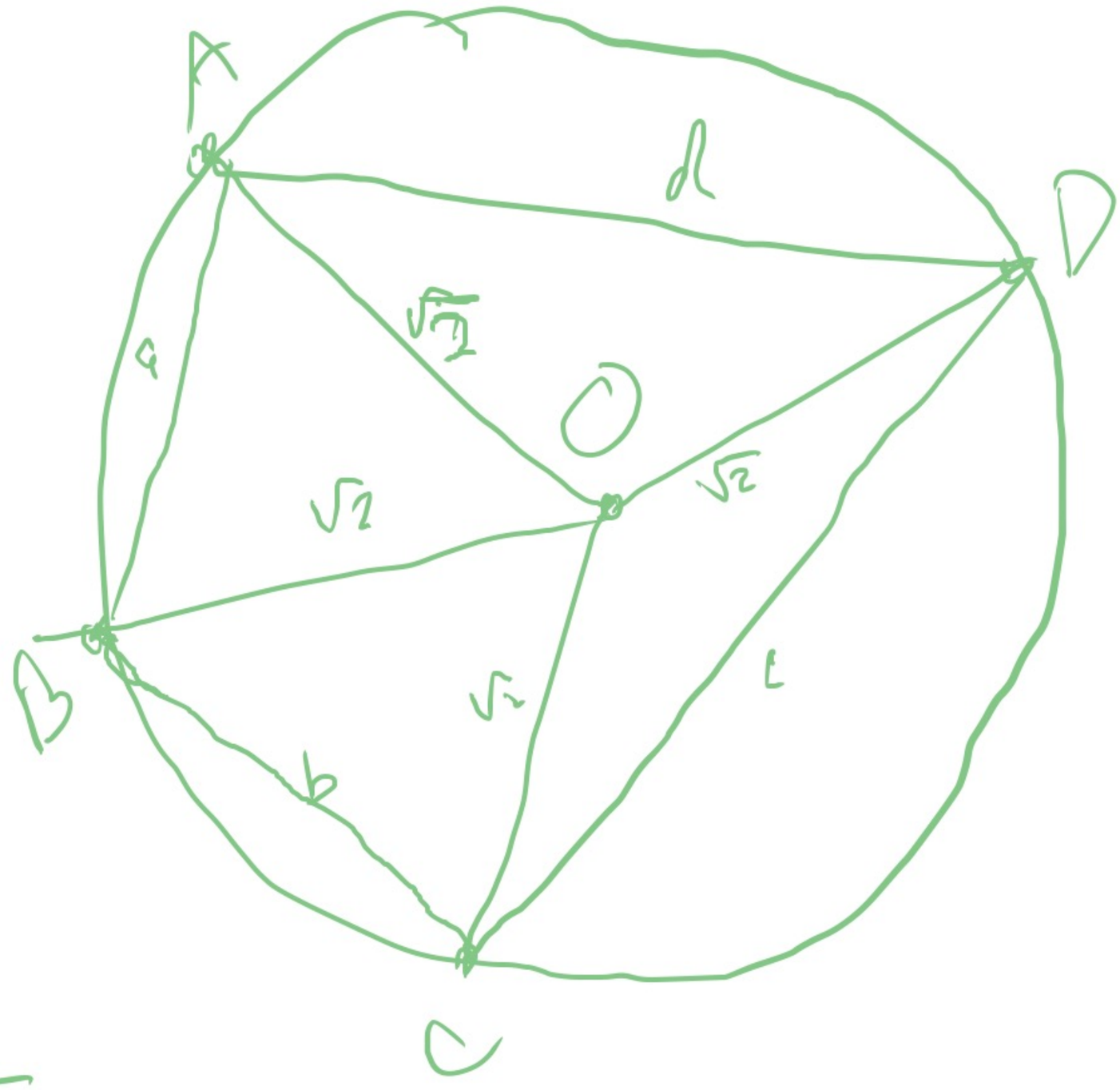
$AC = \sqrt{6}$

$x^2+y^2-xy = 6$ &

$z^2+t^2+zt = 6$ &



$$[ABCD] \leq [A'BCD]$$
$$\leq [A'BC'D]$$
$$\leq [A'B'C'D'] = 4$$



$[ABCD] =$

$$\sum \sqrt{\left(\sqrt{2}+\frac{a}{2}\right)\left(\sqrt{2}-\frac{a}{2}\right)\frac{a}{2}\cdot\frac{a}{2}}$$

$$= \sum \sqrt{\underbrace{\left(2-\frac{a^2}{4}\right)}_{X}\underbrace{\frac{a^2}{4}}_{Y}} \underset{AM-GM}{\leq} \sum \frac{2-\frac{a^2}{4}+\frac{a^2}{4}}{2} = 4$$

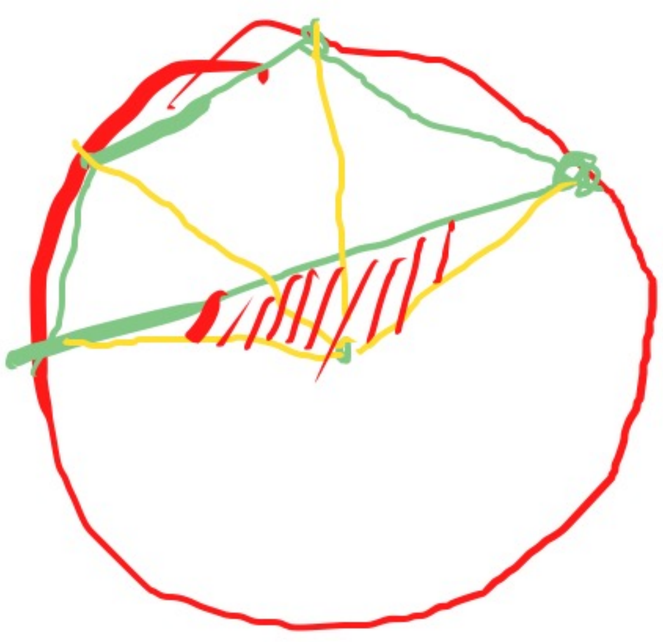
$$\sqrt{XY} \leq \frac{X+Y}{2}$$

$$2-\frac{a^2}{4} = \frac{a^2}{4} \iff a=2$$

$b=2$

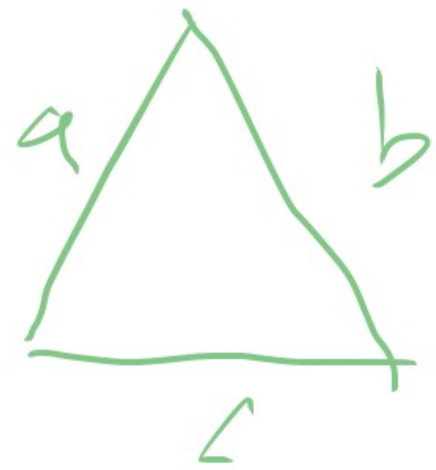
$c=2$

$d=2$



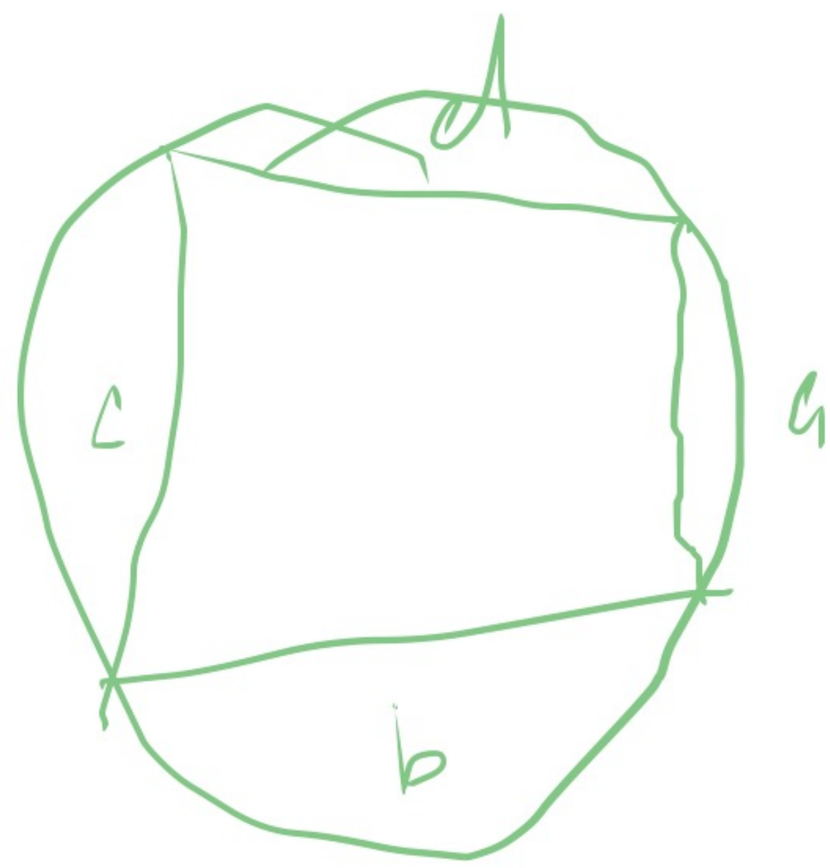
$$[ABCD] = \sqrt{a} + \sqrt{b} + \sqrt{c} - \sqrt{d} \leq 1+1+1+0 = 3$$

Heron:



$$s = \frac{a+b+c+0}{2}$$

$$\frac{\sqrt{s(s-a)(s-b)(s-c)(s-0)}}{}$$



$$s = \frac{a+b+c+d}{2}$$

$$\boxed{A = \sqrt{(s-a)(s-b)(s-c)(s-d)}}$$

$$s = 2 \quad ; \quad A = \sqrt{(2-a)(2-b)(2-c)(2-d)}$$

$$(2-a)+(2-b)+(2-c)+(2-d) \geq 4\sqrt[4]{(2-a)(2-b)(2-c)(2-d)} = 4\sqrt{A}$$

$$= 4 \quad \Longrightarrow \quad A \leq 1 \quad , \quad \text{"="}: \ a=b=c=d=1$$

square

$$\begin{cases} x+y+z=2021 = A \\ x^2+y^2+z^2=6063^2 = 9A^2 \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{2021} = \frac{1}{A} \end{cases}$$

SOL 1)

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 1$$

$$\Leftrightarrow \underline{(x+y+z)(xy+zx+yz)} = xyz$$

$$\Leftrightarrow (x+y)(y+z)(z+x) = 0$$

$(A-x)(A-y)(A-z)$

$$\left[\sum x^2y + \sum xy^2 + 3xyz\right]$$

WLOG $\quad y+z=0 \Rightarrow x=A$

$$A^2+2y^2 = 9A^2 \Leftrightarrow y^2 = 4A^2 \Rightarrow y = \pm 2A$$

$$\boxed{(x,y,z) = (A, 2A, -2A) \text{ \& permutations}}$$

Sol 2

$$\left(\begin{array}{ll} a+b+c & \text{———} \\ ab+bc+ca & \text{———} \\ abc & \text{———} \end{array}\right.$$

$$f(x) = X^3 - (a+b+c)X^2 + (ab+bc+ca)X - abc$$

$$= (X-a)(X-b)(X-c)$$

$$\underbrace{x+y+z=A}_{①} \quad ; \quad x^2+y^2+z^2 = 9A^2$$

$$\Rightarrow xy+yz+zx = -4A^2 \quad ①,②$$

$$③ \Rightarrow xyz = -4A^3$$

$$f(X) = X^3 - AX^2 - 4A^2X + 4A^3 = (X-x)(X-y)(X-z)$$

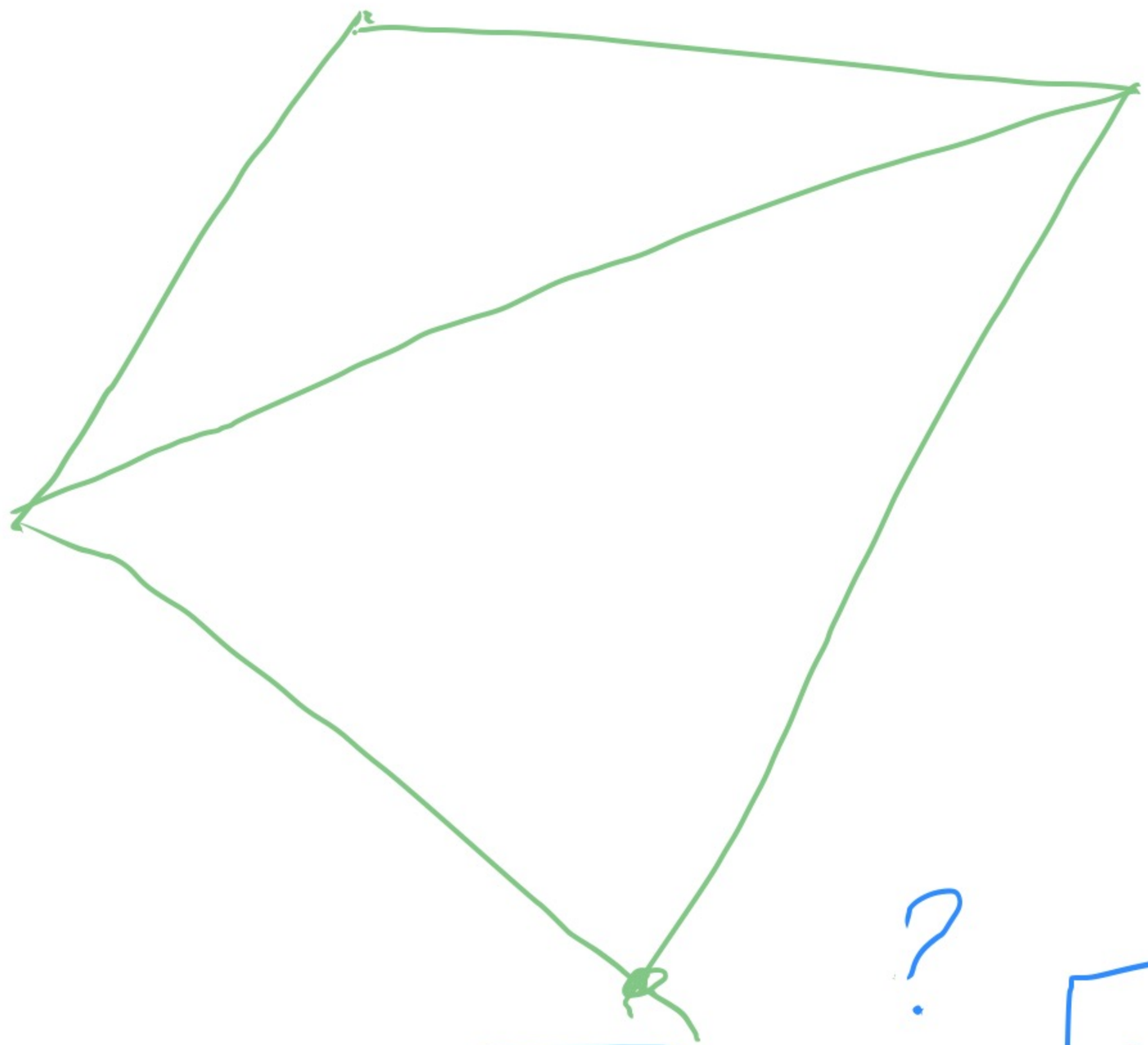
$$= X^2(X-A) - 4A^2(X-A)$$

$$= (X-A)(X^2-4A^2)$$

$$= (X-A)(X-2A)(X+2A)$$

$$(x, y, z) = \sim(A, 2A, -2A)$$

أوجد أكبر مساحة ممكنة لرباعي محيطه 4.



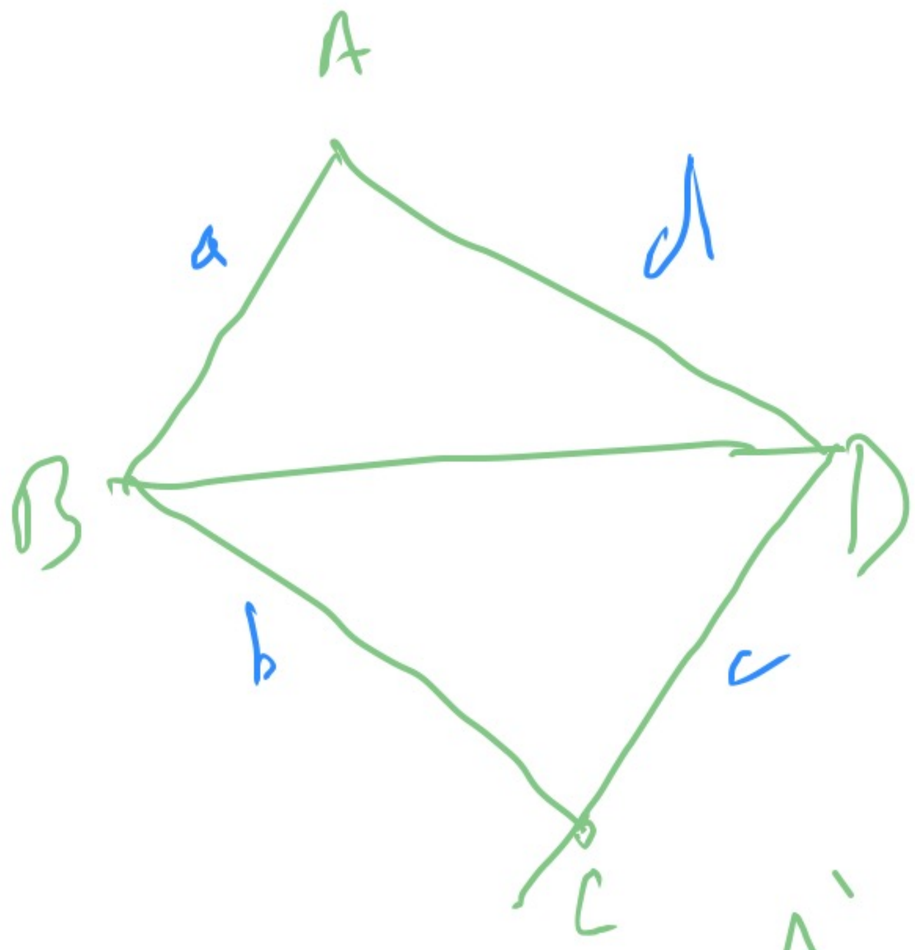
$$\sqrt{s(s-a)(s-b)(s-c)} \overset{?}{\leq} \sqrt{s\left(s-\frac{a+c}{2}\right)(s-b)\left(s-\frac{a+c}{2}\right)}$$

$$\Leftrightarrow (s-a)(s-c) \overset{?}{\leq} \left(s-\frac{a+c}{2}\right)^2 = \left(\frac{s-a+s-c}{2}\right)^2$$

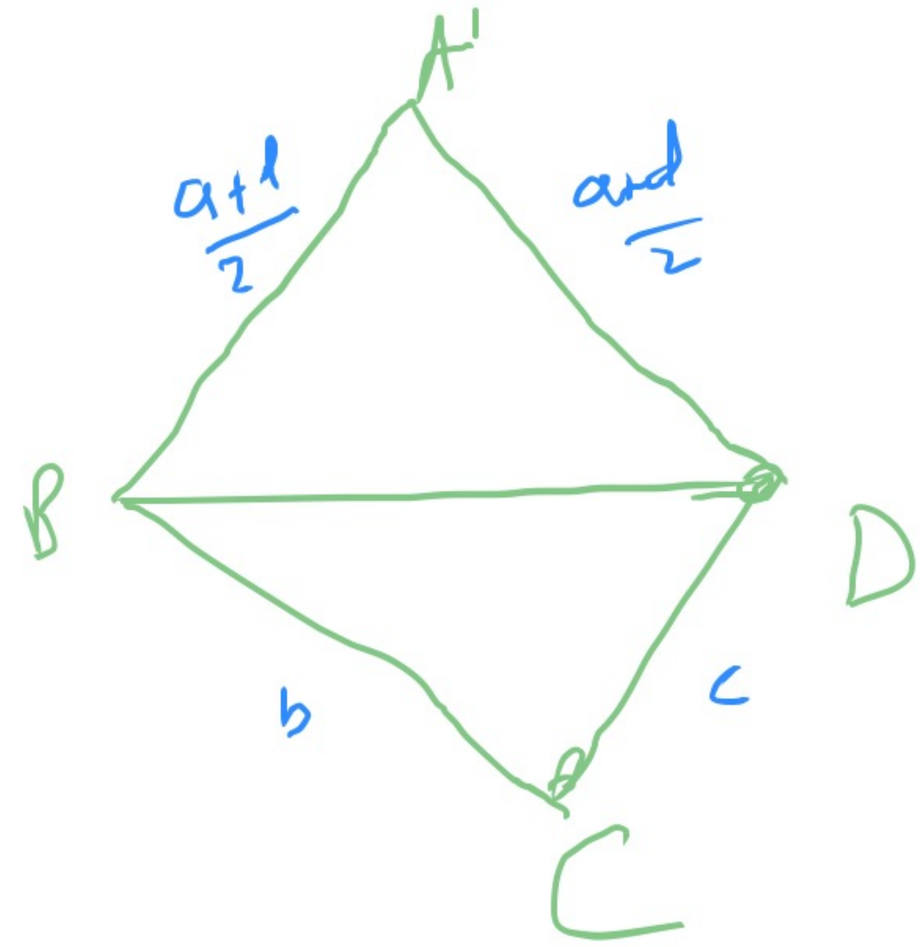
✓

$$s = \frac{a+b+c}{2}$$

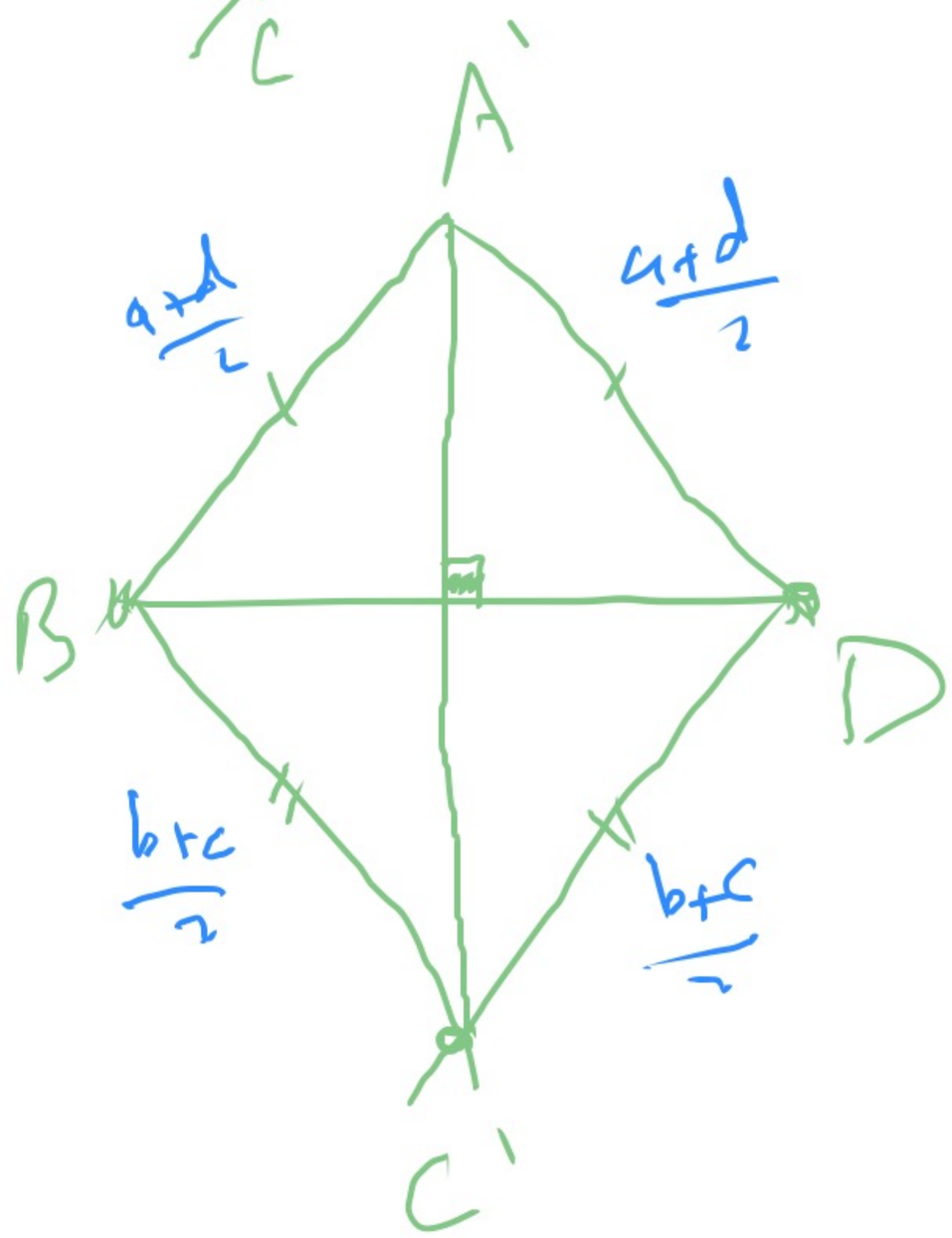
$$s = \frac{a+b+c}{2}$$



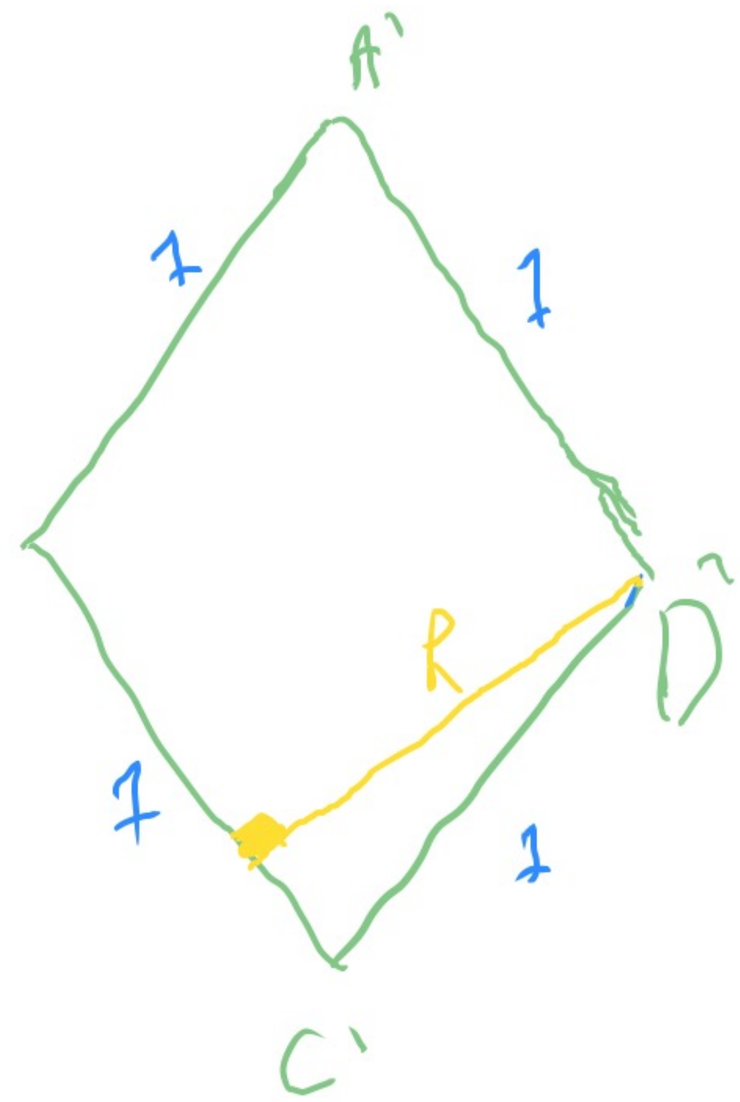
$\leq$



$\leq$

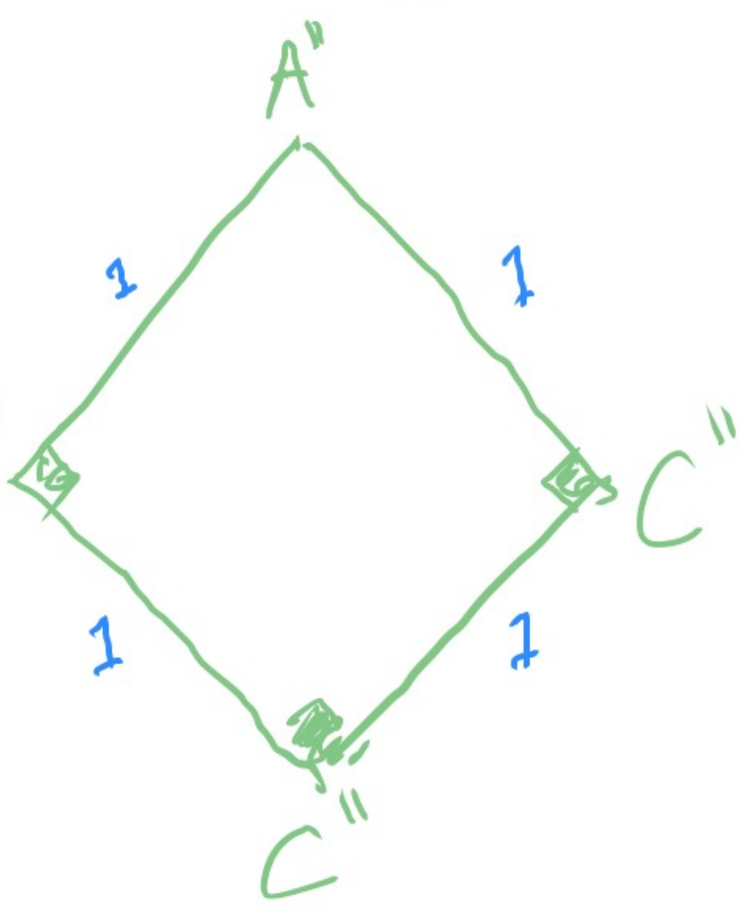


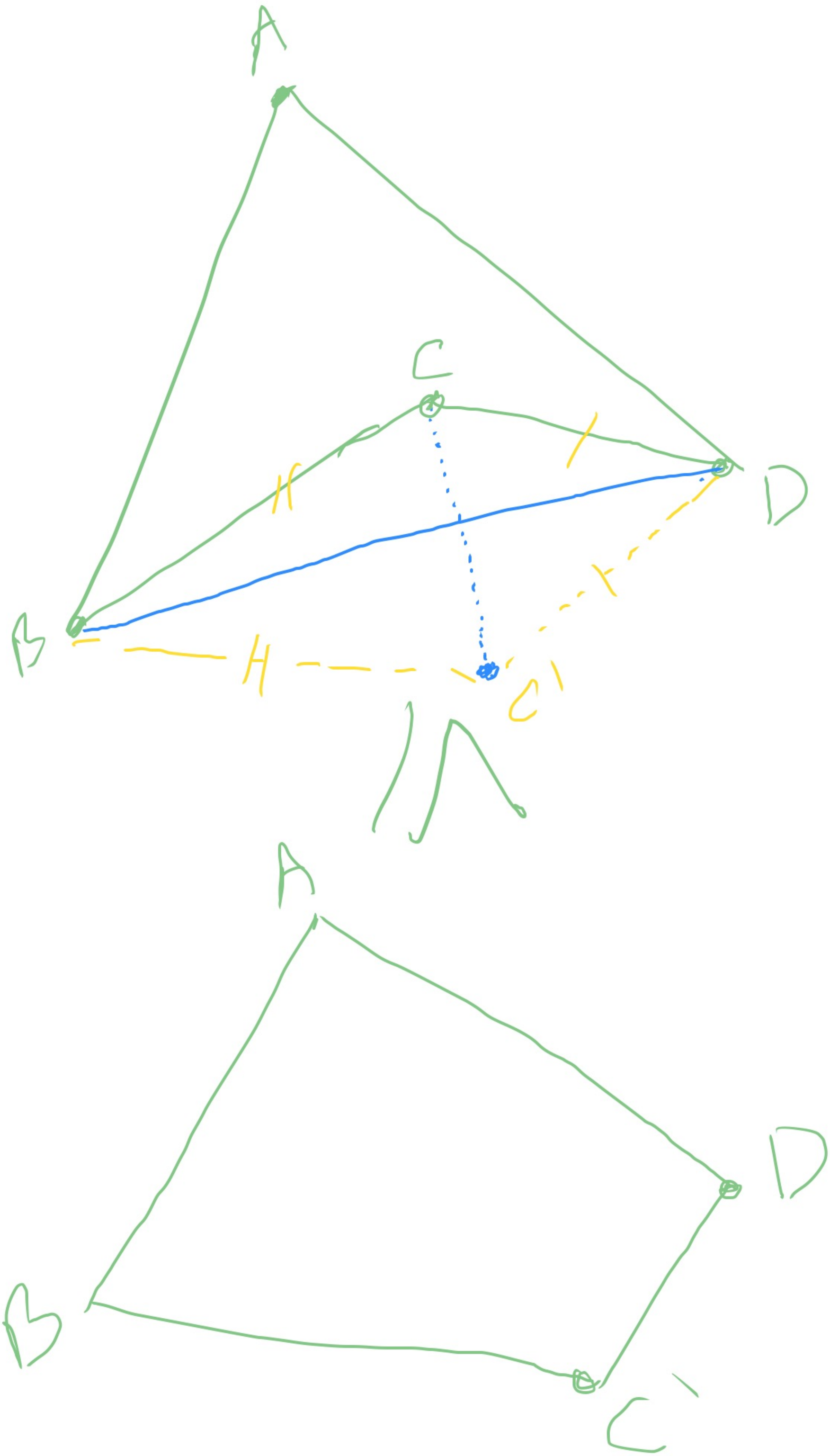
$\leq$ (خطوتين)



$R \leq 1$

$\leq$




A
C
D
B
H
C'
A
D
B
C'

معطى

و

أثبت أن

$$x+y+z \geq x^3+y^3+z^3$$

$$x,y,z>0$$

$$3 \geq x+y+z$$

$$x^3+1+1 \geq 3x$$

$$y^3+1+1 \geq 3y$$

$$z^3+1+1 \geq 3z$$

$$\Rightarrow \quad x^3+y^3+z^3+6 \geq 3(x+y+z)$$

$$\parallel$$

$$x+y+z+6$$

$$\Leftrightarrow \quad \boxed{3 \geq x+y+z}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \overset{?}{\geq} x+y+z.$$

$$\downarrow$$

$$x^3+\frac{1}{x} \geq 2x$$

$$y^3+\frac{1}{y} \geq 2y$$

$$z^3+\frac{1}{z} \geq 2z$$

$$\sum x^3+\sum \frac{1}{x} \geq 2(x+y+z)$$

$$(x+y+z)+\sum \frac{1}{x} \geq$$

أثبت أن

$$a^2b^2(a^2+b^2-2) \geq (a+b)(ab-1)$$

لكل $a,b>0$

$$\Leftrightarrow \quad a^4b^2+a^2b^4+a+b \overset{?}{\geq} 2a^2b^2+a^2b+ab^2$$

$$3\underbrace{(a^4b^2+a^2b^4)}_{\geq\, 2a^3b^3} + \underbrace{2(a+b)}_{\geq\, 2\sqrt{ab}} \geq 10a^2b^2$$

$$\underbrace{m(ab)^3}+n\sqrt{ab} \geq (m+n)\sqrt[m+n]{(ab)^{3m}\cdot(ab)^{n/2}}$$

$$\underbrace{(ab)^3+\cdots+(ab)^3}_{m}+\underbrace{\sqrt{ab}+\cdots+\sqrt{ab}}_{n}$$

$$(ab)^{\frac{3m+\frac{n}{2}}{m+n}}=(ab)^2$$

$$3m+\frac{n}{2}=2m+2n$$

$$2m=3n$$

من AM-GM :

$$\boxed{3(a^4b^2+a^2b^4)+2(a+b) \geq 10a^2b^2}$$

إذن يكفي إثبات أن

$$\boxed{2(a^4b^2+a^2b^4)+3(a+b) \geq 5(a^2b+ab^2)}$$

$a+b=2x, \quad ab=y^2 \quad ; \quad x \geq y$

$$2y^4(4x^2-2y^2)+6x \overset{?}{\geq} 10xy^2$$

$$\Leftrightarrow \quad 8x^2y^4+3x \overset{?}{\geq} 5xy^2+2y^6$$

$$2x^2y^4 \geq 2y^6$$

$$2x^2y^4+3x \overset{?}{\geq} 5xy^2$$

$$\Leftrightarrow \quad 2xy^4+3 \overset{?}{\geq} 5y^2$$

$$LHS \geq y^5+y^5+1+1+1 \geq 5y^2$$

$$a^4b^2 + a^4b^2 + a + a + b \geq 5a^2b$$

$$a^2b^4 + a^2b^4 + a + b + b \geq 5ab^2$$

$$\Downarrow$$

$$2(a^4b^2 + a^2b^4) + 3(a+b) \geq 5a^2b + 5ab^2$$

أثبت أن :

$$1 \leq \frac{(x+y)(x^3+y^3)}{(x^2+y^2)^2} \leq \frac{9}{8}$$

لكل $x, y > 0$

$$\Leftrightarrow (x^2+y^2)^2 \overset{?}{\leq} (x+y)(x^3+y^3)$$

من كوشي $\left((\sqrt{x})^2+(\sqrt{y})^2\right)\left((x\sqrt{x})^2+(y\sqrt{y})^2\right)$ هي صحيحة

$$8(x+y)(x^3+y^3) \overset{?}{\leq} 9(x^2+y^2)^2$$

$$8(x+y)^2(x^2-xy+y^2)$$

$$8(x^2+y^2+2xy)(x^2+y^2-xy) \overset{?}{\leq} 9(x^2+y^2)^2 \Leftrightarrow (x^2+y^2-4xy)^2 \geq 0$$

$$8(a+2b)(a-b) \overset{?}{\leq} 9a^2 \Leftrightarrow 0 \overset{?}{\leq} a^2-8ab+16b^2 = (a-4b)^2$$

$$(x,y) = (x, x(2\pm\sqrt{3}))$$

$a, b \in \mathbb{R}$; $a^3+b^3-6ab=-11$

Prove $-\frac{7}{3} < a+b < -2$

L R

$a^3+b^3+2^3-3\cdot a\cdot b\cdot 2 = -3$

$\Leftrightarrow (a+b+2)(a^2+b^2+4-2a-2b-ab)=-3$

$\downarrow$

$\frac{1}{2}\left((a-b)^2+(2-a)^2+(2-b)^2\right)$

>0

$\Rightarrow a+b+2<0$ (R ✓)

L $\Leftrightarrow -\frac{1}{3} < a+b+2$

$\Leftrightarrow \frac{1}{3} \overset{?}{>} \underbrace{-a-b-2}_{x} > 0$

$x\left((a-b)^2+\underbrace{(2-a)^2+(2-b)^2}\right) = 6$

≥ 0

$\geq \frac{(2-a+2-b)^2}{2} = \frac{(6+x)^2}{2} > 18$

$\Leftrightarrow x < \frac{1}{3}$

$A, B \neq \emptyset$; $\quad A \cap B = \emptyset$

$A \cup B = \{1, 2, \dots, 11\}$

$P_A = \prod_{a \in A} a$; $P_B = \prod_{b \in B} b$

Find min & max of

$$X = P_A + P_B$$

<u>max</u> : $\quad P_A P_B = 11!$

$$X = P_A + P_B \overset{?}{\leq} 1 + 11! = 1 + P_A P_B$$

$$\Leftrightarrow \quad 0 \overset{?}{\leq} (P_A - 1)(P_B - 1) \quad \checkmark$$

<u>Example:</u> $A = \{1\}$, $B = \{2, 3, \dots, 11\}$

$$\Rightarrow \quad \boxed{\max = 1 + 11!} = 39\,916\,801$$

$\underline{\min}$ $\quad P_A + P_B \underset{\text{AM-GM}}{\geq} \left\lceil 2\sqrt{P_A P_B} \right\rceil = \left\lceil 2\sqrt{11!} \right\rceil$

$\downarrow$

$12\,635.9$

$$\Rightarrow X = P_A + P_B \geq \underbrace{12\,636}_{6318}$$

$6303,\ \cancel{6314},\ \cancel{6325},\ 6336$

$6333\quad \cancel{6326}\quad \cancel{6311}\quad 6300$

$$6336 = 2^6 \cdot 3^2 \cdot 11 = 11 \times 9 \times 8 \times 4 \times 2 \times 1$$

$$6300 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 = 3 \times 5 \times 6 \times 7 \times 10$$

for $A = \{1, 2, 4, 8, 9, 11\}$

$B = \{3, 5, 6, 7, 10\}$

we have $X = 12636$

Thus, $\min = 12636$

$a, b \in \mathbb{R}$, $c > 0$, $a \neq b$

$$a^4 - 2021a = b^4 - 2021b = c$$

أثبت أن

$$\sqrt{c} > -ab > 0$$

$$(a+b)(a^2+b^2) = 2021 \quad (\text{so } a+b \neq 0)$$

$$c = a^4 - 2021a = a^4 - a(a+b)(a^2+b^2) = -ab(a^2+ab+b^2)$$

but $a^2+ab+b^2 = \frac{1}{2}(a^2+b^2+(a+b)^2) \geq 0$

$$\Rightarrow \boxed{-ab > 0}$$

Now : $a^2+ab+b^2 \geq -ab \quad (\because (a+b)^2 \geq 0)$

$$\Leftrightarrow c = -ab(a^2+ab+b^2) \geq (-ab)^2$$

$$\Leftrightarrow \sqrt{c} \geq -ab \quad ; \text{ if } \sqrt{c} = -ab \Leftrightarrow a+b=0 \quad ↯ \Rightarrow \boxed{\sqrt{c} > -ab}$$

أوجد أكبر عدد صحيح K يحقق الآتي:

إذا كان $m,n \in \mathbb{N}$ بحيث

$$m^3+n^3 > (m+n)^2$$

فإن

$$m^3+n^3 \geq (m+n)^2+K$$

ماهي أصغر قيمة موجبة للمقدار

$$m^3+n^3-(m+n)^2$$

حيث $m,n \in \mathbb{N}$ ؟

$$A = (m+n)\left(m^2-mn+n^2-(m+n)\right)$$

$$= (m+n)\left(m^2-2mn+n^2+(mn-m-n+1)-1\right)$$

$$= (m+n)\left((m-n)^2+(m-1)(n-1)-1\right)$$

$A \cancel{=1}$, $A=2$, ..., $A=4$, $A=6$, $A=8$, $A=10$

$$A = (m+n)\left((m-n)^2 + (m-1)(n-1) - 1\right)$$

WLOG assume $n \geq m$

<u>Case 1:</u> $m = 1$ $\Rightarrow$ $A = (n+1)(n-2)n$ $(\Rightarrow n \geq 3)$

$$\geq 4 \cdot 1 \cdot 3 = 12$$

$(\text{"}=\text{"}: \quad n=3,\ m=1)$

<u>Case 2:</u> $m > 1$; $n = m$

$\Rightarrow$ $A = 2n^2(n-2)$ $(\Rightarrow n \geq 3)$

$$\geq 2 \cdot 3^2 \cdot 1 = 18$$ $(\text{"}=\text{"}: \ n = m = 3)$

<u>Case 3:</u> $n > m > 1$ $\Rightarrow$ $m \geq 2,\ n \geq 3$
$n - m \geq 1$

$$\Rightarrow A = \underbrace{(m+n)}_{\geq 5}\left(\underbrace{(m-n)^2}_{\geq 1} + \underbrace{(m-1)}_{\geq 1}\underbrace{(n-1)}_{\geq 2} - 1\right)$$

$$\geq 10$$ $(\text{"}=\text{"}: \ n = 3,\ m = 2)$

$\Rightarrow$ $\boxed{\text{Answer is } 10}$

واجب

لتكن $x, y, z \in \mathbb{N}$

بحيث $x \neq y \neq z \neq x$

أثبت أن :

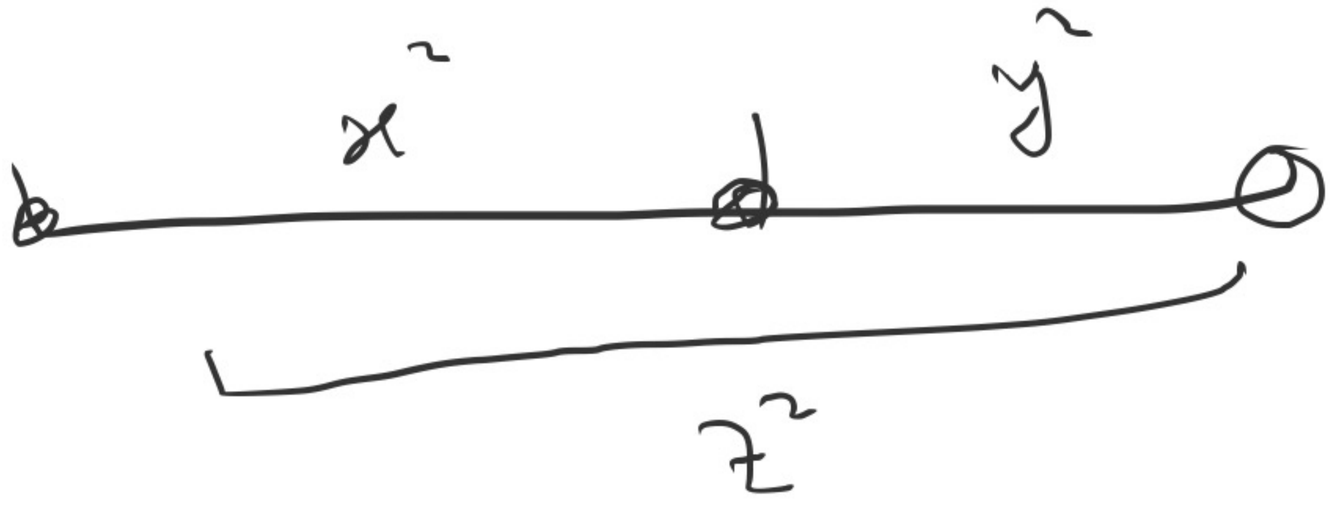
$$(x+y+z)(xy+yz+zx-2) \geq 9xyz$$

وحدد حالة التساوي

Test 5

P3

نفرض العكس



$x^2+y^2=z^2$

0, 9, 16, 25

A B B C

$n-9$, n, $n+7$, $n+16$

A B B C

$\Rightarrow$ n, $n+7$ always have the same color.

0, 7, 14, 21, ..., 49

تناقض

Test 7, P4

$$n = \frac{2^a - 2^b}{2^c - 2^d} = \frac{2^b(2^{a-b} - 1)}{2^d(2^{c-d} - 1)}$$

WLOG $a > b$; $c > d$

$$n = \frac{2^{b-d}(2^{a-b} - 1)}{2^{c-d} - 1} = 2^k \cdot \frac{2^x - 1}{2^y - 1}$$

$k \geq 0$, $x, y > 0$

$$\underbrace{1 = \frac{2-1}{2-1} \quad ; \quad 3 = \frac{2^2-1}{2-1}}_{\downarrow} \quad ; \quad \underbrace{5 = \frac{2^4-1}{2^2-1}}$$

$$2^k - 1 = \frac{2^k - 1}{2 - 1} \qquad 2^k + 1 = \frac{2^{2k} - 1}{2^k - 1}$$

$$7 = \frac{2^3 - 1}{2 - 1} \ , \quad 9 = \frac{2^6 - 1}{2^3 - 1}$$

$$11 = \frac{2^x - 1}{2^y - 1} \Rightarrow x > y > 1$$

$$\left(y > 1 \quad \text{because} \quad 11 \neq 2^x - 1 \right)$$

(mod 4):

$$11(2^y - 1) = 2^x - 1$$

$$\equiv \qquad\qquad \equiv$$

$$11(-1) \equiv -1 \quad [4]$$

Contradiction !!

$$\boxed{\text{Answer}: \quad n = 11}$$

$$n = \frac{2^a - 2^b}{2^c - 2^d} \rightarrow n \mid 2^a - 2^b \qquad a \equiv b \; [\quad]$$

$n > 5$, mod 4
$n = 7$; mod 3 $\quad n = 9$ mod 6 $\quad$ 11 mod 10

Test 8 | P1

$$a + \gcd(a, b+c) = \gcd(a,b) + \gcd(a,c)$$

$$\overset{?}{\Longrightarrow} \quad a \mid b \quad \text{or} \quad a \mid c$$

$\gcd(a,b) \mid a \quad ; \quad \gcd(a,c) \mid a$

if $\gcd(a,b) < a \Rightarrow \gcd(a,b) \leq \frac{a}{2}$

if $\gcd(a,c) < a \Rightarrow \gcd(a,c) \leq \frac{a}{2}$

if $a \nmid b$ and $a \nmid c$

$\Rightarrow \gcd(a,b) \leq \frac{a}{2} \; ; \; \gcd(a,c) \leq \frac{a}{2}$

$\Rightarrow a < LHS = RHS \leq \frac{a}{2} + \frac{a}{2} = a$

$\Rightarrow\Leftarrow$!!

Test8, P3

$$x+y+z=0$$
$$x^2+y^2+z^2=2$$

$$A=|(x-y)(y-z)(z-x)| \qquad \max=?$$

$$(x-y)^2+(y-z)^2+(x-z)^2=2\sum x^2-2\sum xy$$
$$=3\sum x^2-(x+y+z)^2$$
$$=6$$

WLOG $x\geq y\geq z$

$$6\geq 3\sqrt[3]{(x-y)^2(y-z)^2(x-z)^2}=3A^{\frac{2}{3}}$$

$$\Longleftrightarrow \quad A\leq 2\sqrt{2}$$

$$0<\underbrace{x-y}_{a}=\underbrace{y-z}_{b}=\underbrace{x-z}_{a+b} \quad ; \quad x-z=(x-y)+(y-z)$$

$$a^2+b^2+(a+b)^2=6 \quad ; \quad A=ab(a+b)$$

$$6 = \frac{a^2}{1} + \frac{b^2}{1} + (a+b)^2 \geq \frac{(a+b)^2}{2} + (a+b)^2 = \frac{3}{2}(a+b)^2$$

Cauchy

$$\Longleftrightarrow \quad \boxed{a+b \leq 2} \quad (1)$$

$$\Longrightarrow \quad 2 \geq a+b \geq 2\sqrt{ab}$$

AM-GM

$$\Longrightarrow \quad \boxed{ab \leq 1} \quad (2)$$

$$\overset{(1)\times(2)}{\Longrightarrow} \quad A = ab(a+b) \leq 2$$

"=": $a = b = 1$

$\downarrow \qquad \downarrow$

$x-y \qquad y-z$

$$\begin{cases} x+y+z = 0 \\ x-y = 1 \\ y-z = 1 \end{cases}$$

$\downarrow$

$(x, y, z) = (1, 0, -1)$

$a+b = x-z$

$$\begin{cases} x+y+z=0 \\ x^2+y^2+z^2=2 \end{cases}$$

$\rightarrow \quad xy+yz+xz=-1$

$$(x-z)^2 = x^2+z^2-2xz = 2-y^2-2(-1-xy-yz)$$

$$= 4-y^2+2xy+2yz$$

$$= 4-y^2+2y(x+z) = 4-y^2-2y^2$$

$$= 4-3y^2 \leq 4$$

$\Longleftrightarrow \quad x-z \leq 2$

$x, y, z \in \mathbb{N}$; $x \neq y \neq z \neq x$

Prove:

$$(x+y+z)(xy+yz+xz-2) \geq 9xyz$$

$$\Updownarrow$$

WLOG $x < y < z$

$$(x+y+z)(xy+yz+xz)-9xyz \overset{?}{\geq} 2(x+y+z)$$

$$\|$$

$$\sum_{cyc} x(y^2+z^2)+3xyz-9xyz = \sum_{cyc} x(y^2+z^2)-6xyz$$

$$= \sum_{cyc}\left[x(y^2+z^2)-2xyz\right] = \sum_{cyc} x(y-z)^2$$

$$\Leftrightarrow \quad x(z-y)^2+y(z-x)^2+z(y-x)^2 \overset{?}{\geq} 2(x+y+z)$$

Case1: $y-x>1 \Rightarrow$

$$LHS \geq x+9y+4z > 2x+8y+4z > 2(x+y+z)$$

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Case 2: $y - x = 1$, $z \geq x+2$

$$x(z-x-1)^2 + (x+1)(z-x)^2 + z \overset{?}{\geq} 4x+2+2z$$

$$\Leftrightarrow \underbrace{x(z-x-1)^2} + \underbrace{(x+1)(z-x)^2} \overset{?}{\geq} 4x+z+2$$

$$\Leftrightarrow x\left[(z-x-1)^2-1\right] + (x+1)\left[(z-x)^2-4\right] \overset{?}{\geq} z-x-2$$

$$\Leftrightarrow x(z-x-2)(z-x) + (x+1)(z-x-2)(z-x+2) \overset{?}{\geq} z-x-2$$

Case 2.1: $z-x-2=0$

equality case, $(x,y,z) = (x, x+1, x+2)$

Case 2.2: $z-x-2>0$

$$\Leftrightarrow \underset{\geq 2}{x}\,\underset{\geq 3}{(z-x)}\,\underset{\geq 2}{(x+1)}\,\underset{\geq 5}{(z-x+2)} \overset{?}{\geq} 1 \quad \checkmark$$

No equality case

$n \in \mathbb{N}$; $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n > 0$

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 1$$

Prove :

$$|x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| \leq 2 - \min_i \frac{x_i}{y_i} - \min_j \frac{y_j}{x_j}$$

$$\left[\begin{array}{l} \min_i \dfrac{x_i}{y_i} = \min\left(\dfrac{x_1}{y_1}, \dfrac{x_2}{y_2}, \dots, \dfrac{x_n}{y_n}\right) \\ \min_j \dfrac{y_j}{x_j} = \min\left(\dfrac{y_1}{x_1}, \dfrac{y_2}{x_2}, \dots, \dfrac{y_n}{x_n}\right) \end{array} \right]$$

$1 \geq a = \min \dfrac{x_i}{y_i} \implies x_i \geq a y_i \quad \forall\, 1 \leq i \leq n$

$1 \geq b = \min \dfrac{y_j}{x_j} \implies y_j \geq b x_j \quad \forall\, 1 \leq j \leq n$

$(1-b)x_i = x_i - b x_i \geq x_i - y_i \geq a y_i - y_i = -(1-a) y_i$

$\implies |x_i - y_i| \leq \max((1-a)y_i, (1-b)x_i) \leq (1-a)y_i + (1-b)x_i$

$\implies LHS \leq 1 - a + 1 - b = 2 - a - b = RHS$

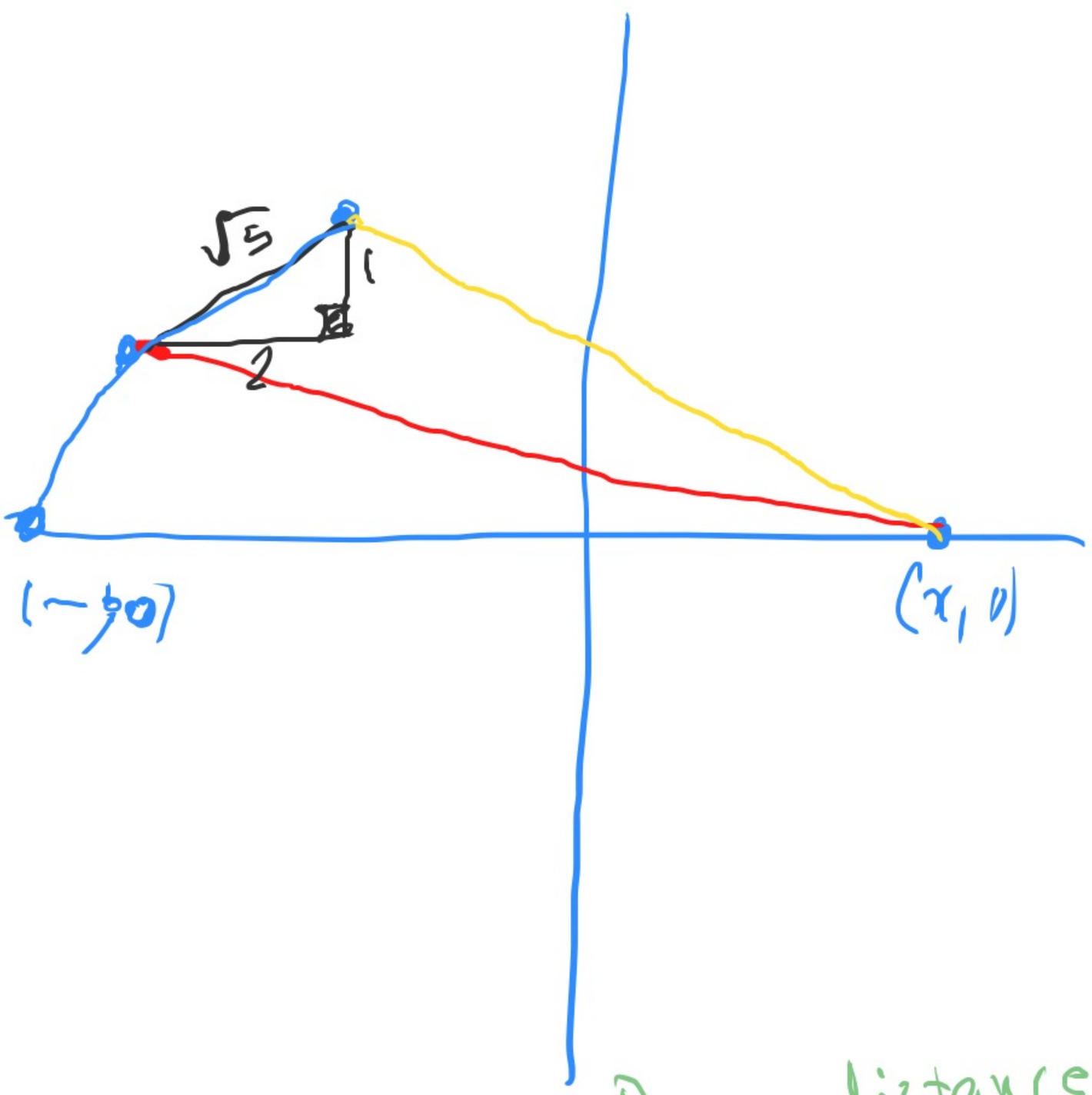
Let $x \in \mathbb{R}$. Find max of

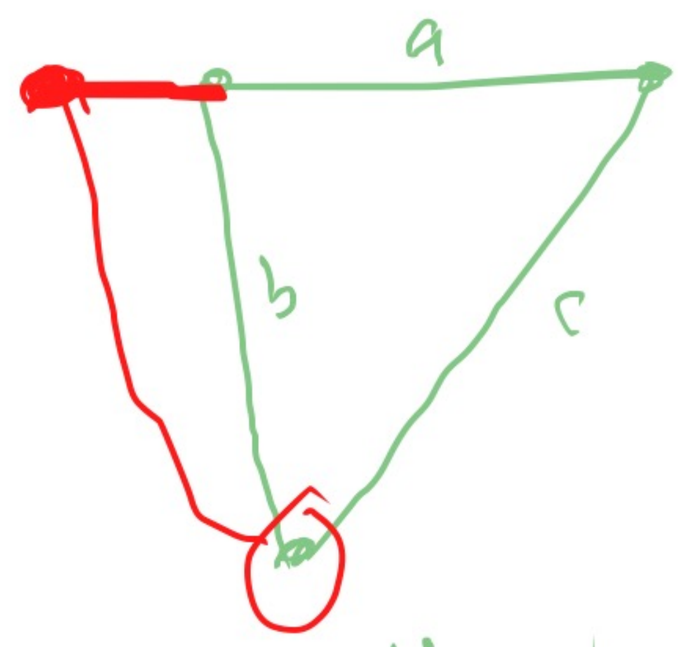
$$D = \left| \sqrt{x^2 + 8x + 17} - \sqrt{x^2 + 4x + 8} \right|$$

$$\sqrt{x^2 + 8x + 17} = \sqrt{(x+4)^2 + 1^2} \qquad (x,0), (-4,1)$$

$$\sqrt{x^2 + 4x + 8} = \sqrt{(x+2)^2 + 2^2} \qquad (x,0), (-2,2)$$

$(x, 0)$





$|b - c| \leq a$

$\Rightarrow$ max D = distance between $(-4,1)$; $(-2,2)$

$= \sqrt{5}$

plug $x = -6 \Rightarrow D = |\sqrt{5} - 2\sqrt{5}| = \sqrt{5}$

(1) $a, b, c > 0$, $abc = 1$ Prove

$$\sum_{cyc} \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

---

(2) $a, b, c, d > 0$ Prove that

$$\sum_{cyc} \frac{a^3}{a^2+ab+b^2} \geq \frac{a+b+c+d}{3}$$

---

(3) $a, b, c, d > 0$; $a+b+c+d = 4$; Prove

$$\sum_{cyc} \frac{a^4}{(a+b)(a^2+b^2)} \geq 1$$

①

$$\sum_{cyc} \frac{1}{c^3(a+b)} = \sum_{cyc} \frac{a^2b^2}{c(a+b)}$$

$$= \sum_{cyc} \frac{(ab)^2}{ca+bc} \underset{\text{Cauchy}}{\geq} \frac{(ab+bc+ca)^2}{2(ab+bc+ca)}$$

$$= \frac{ab+bc+ca}{2}$$

$$\underset{\text{AM-GM}}{\geq} \frac{3\sqrt[3]{a^2b^2c^2}}{2} = \frac{3}{2}$$

(2)

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}=\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a}$$

$$a-b+b-c+c-a=0$$

$$LHS=\sum_{cyc}\frac{a^3}{a^2+ab+b^2}=\sum_{cyc}\frac{b^3}{a^2+ab+b^2}$$

$$\left(\because \Leftrightarrow a-b+b-c+c-a+a-a=0\right)$$

$$2\cdot LHS=\sum_{cyc}\frac{a^3+b^3}{a^2+ab+b^2}\overset{?}{\geq}\sum_{cyc}\frac{a+b}{3}=2RHS$$

$$\frac{a^3+b^3}{a^2+ab+b^2}\overset{?}{\geq}\frac{a+b}{3}\Leftrightarrow 3(a^2-ab+b^2)\overset{?}{\geq}a^2+ab+b^2\Leftrightarrow 2(a-b)^2\geq 0$$

$\blacksquare$

3) $a^4 - b^4 = (a-b)(a+b)(a^2+b^2)$

$$\Rightarrow LHS = \sum_{cyc} \frac{a^4}{(a+b)(a^2+b^2)} = \sum_{cyc} \frac{b^4}{(a+b)(a^2+b^2)}$$

$$\Rightarrow 2LHS = \sum_{cyc} \frac{a^4+b^4}{(a+b)(a^2+b^2)} \overset{?}{\geq} 2RHS = 2$$

$$2 = \frac{a+b+c+d}{2}$$

$$= \sum_{cyc} \frac{a+b}{4}$$

$$\frac{a^4+b^4}{(a+b)(a^2+b^2)} \overset{?}{\geq} \frac{a+b}{4}$$

$$\Leftrightarrow 2(a^4+b^4) \overset{?}{\geq} \frac{(a+b)^2(a^2+b^2)}{2}$$

but $2(a^4+b^4) = (1+1)(a^4+b^4) \underset{\text{Cauchy}}{\geq} (a^2+b^2)^2$

$$= (a^2+b^2)(a^2+b^2) \underset{\text{Cauchy}}{\geq} \frac{(a+b)^2}{2}(a^2+b^2) \checkmark$$

4) $a, b, c > 0$ Prove

$$\sum_{cyc} \frac{a}{(b+c)^2} \geq \frac{9}{4(a+b+c)}$$

$$\Longleftrightarrow$$

$$(a+b+c)\left(\frac{a}{(b+c)^2} + \frac{b}{(c+a)^2} + \frac{c}{(a+b)^2}\right) \overset{?}{\geq} \frac{9}{4}$$

$$\text{LHS} \geq \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^2$$

Cauchy

also ; $$\sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{a^2}{ab+ac} \underset{\text{Cauchy}}{\geq} \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3}{2}$$

$$\Longrightarrow \text{LHS} \geq \left(\sum_{cyc} \frac{a}{b+c}\right)^2 \geq \left(\frac{3}{2}\right)^2 = \frac{9}{4} \quad \blacksquare$$

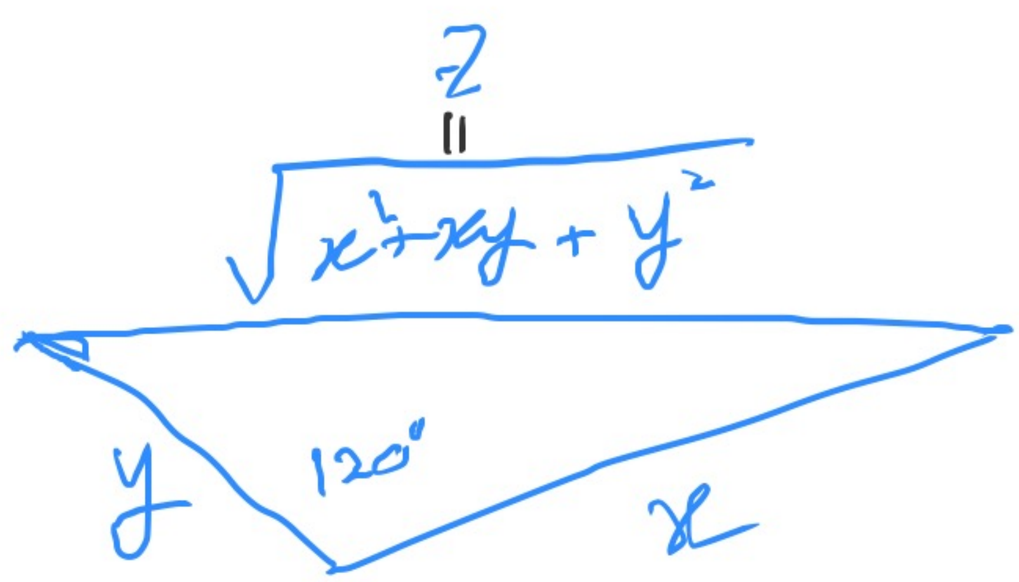
5) (i) $a, b, c > 0$ Prove

$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+c^2} \geq \sqrt{c^2+ca+a^2}$$

(ii) $a, b, c \in \mathbb{R}$ Prove

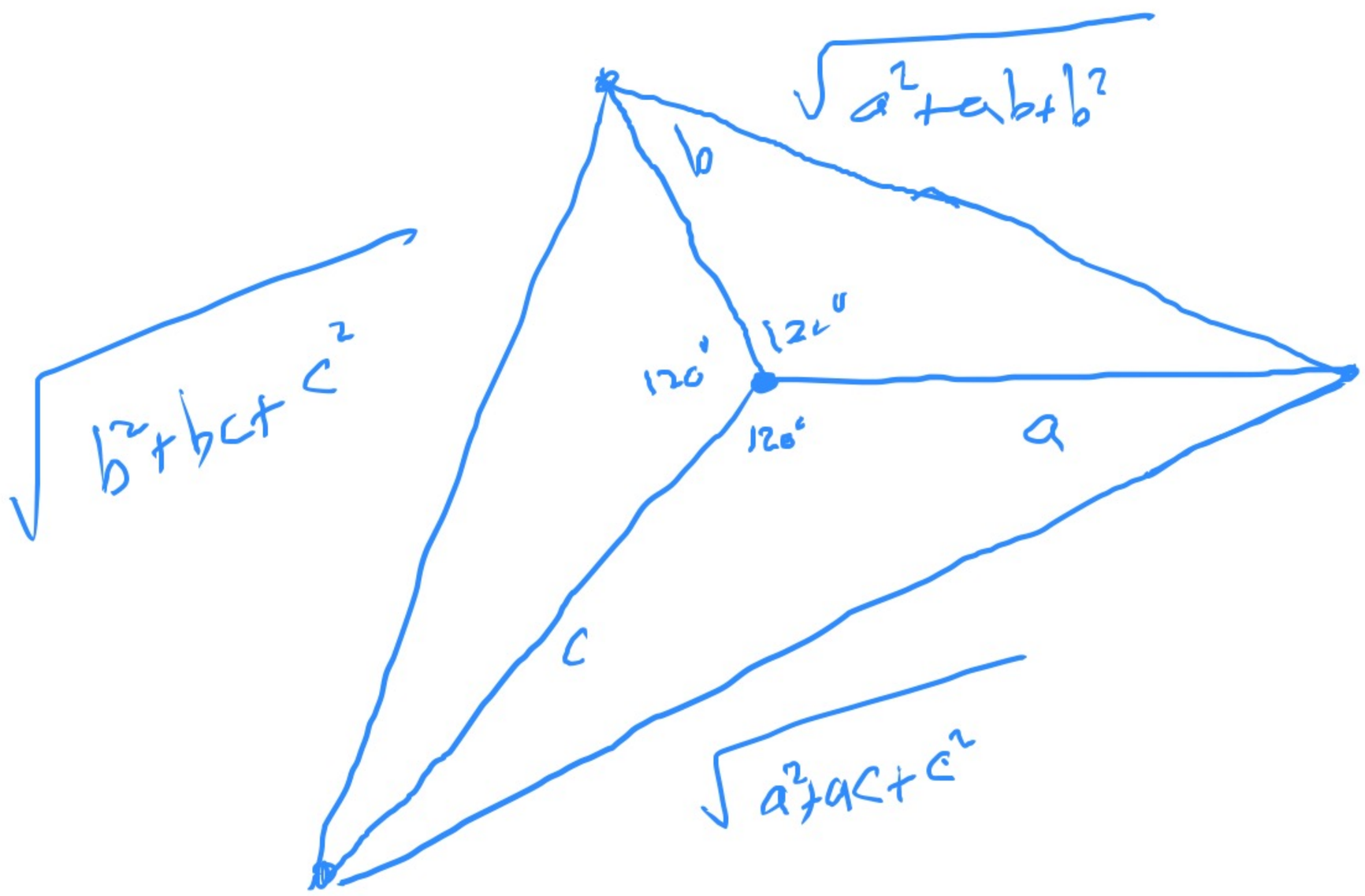
$$\sqrt{a^2+ab+b^2}+\sqrt{b^2+bc+c^2} \geq \sqrt{c^2+ca+a^2}$$

$a, b, c \longrightarrow x=-a,\ y=-b,\ z=-c$



$$z^2 = x^2+y^2-2xy\cos 120° = x^2+y^2+xy$$

5 (i)



(ii) if $a, b, c < 0$

change $(a, b, c) \longrightarrow (-a, -b, -c)$

if $a, b < 0 \leq c$

$b, c < 0 \leq a$

$c, a < 0 \leq b$

change $(a, b, c) \longrightarrow (-a, -b, -c)$

so, assume WLOG ≥ 2 numbers are positive

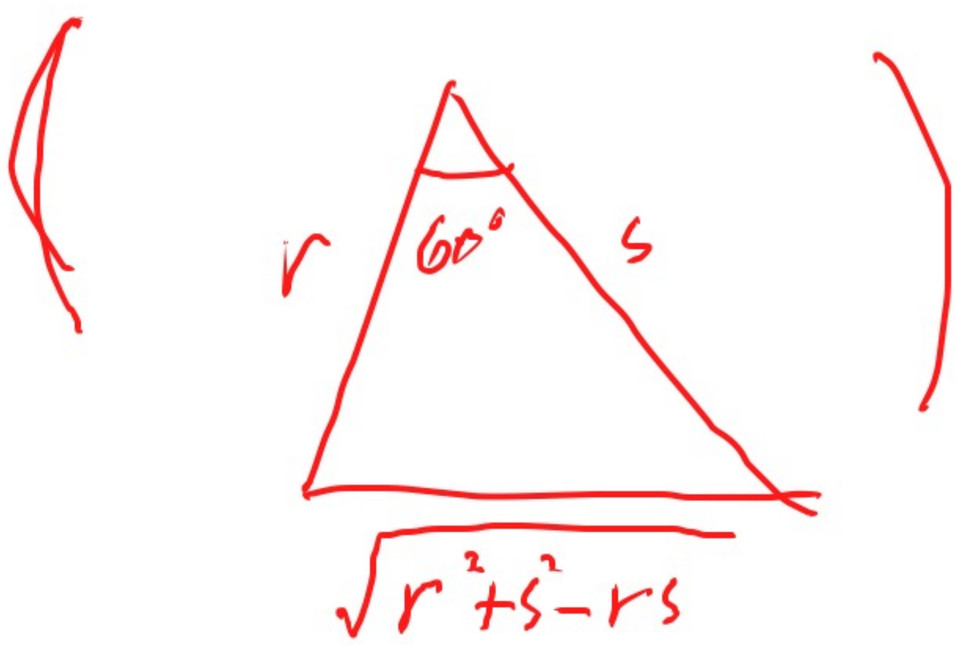
if $a, b, c > 0$; Done from (i)

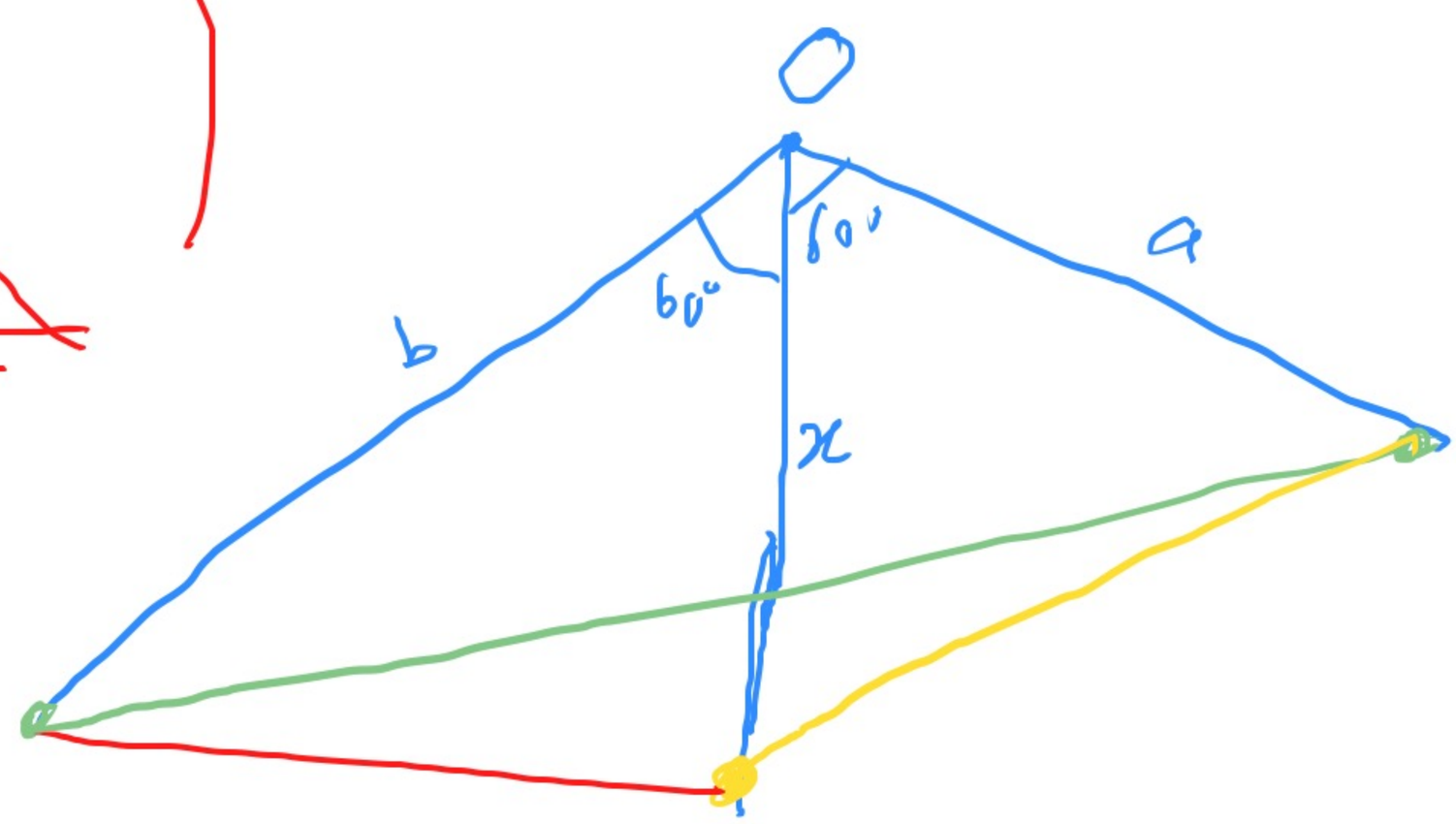
Case 1: $a, b \geq 0 > c$

let $x = -c$

we need:

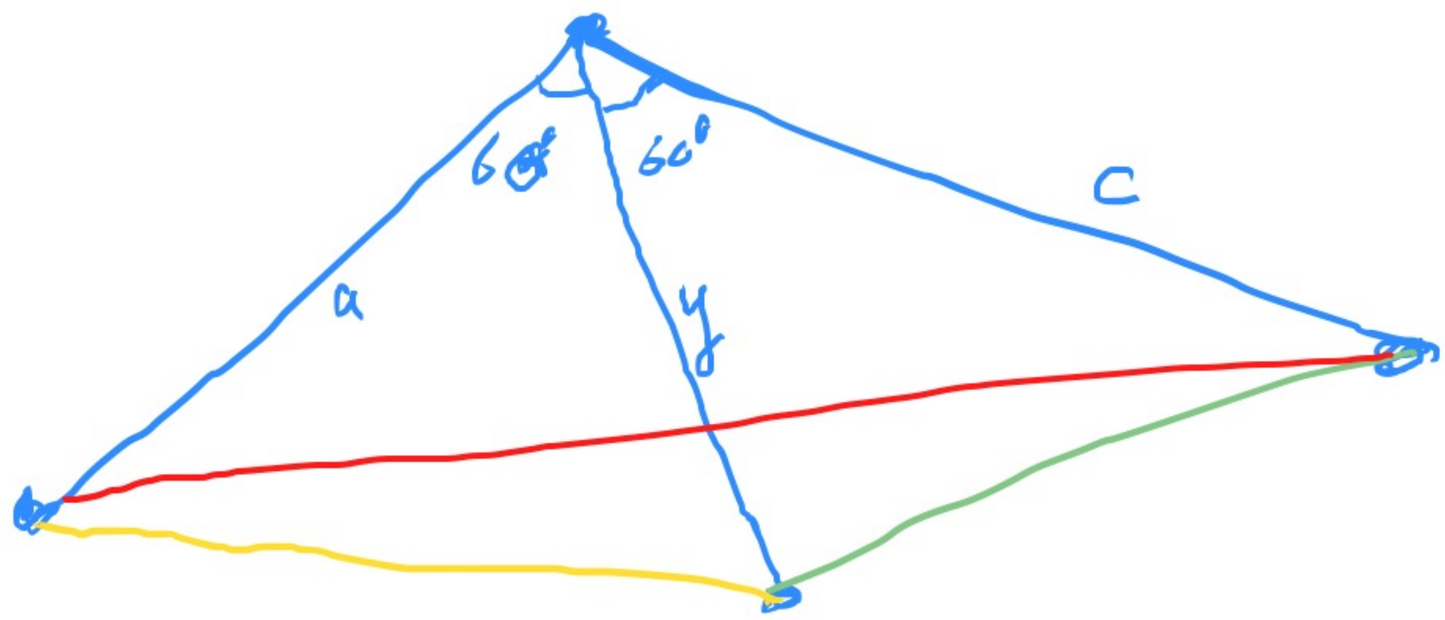
$$\boxed{\sqrt{a^2+ab+b^2}} + \boxed{\sqrt{b^2-bx+x^2}} \overset{?}{\geq} \boxed{\sqrt{a^2-ax+x^2}}$$





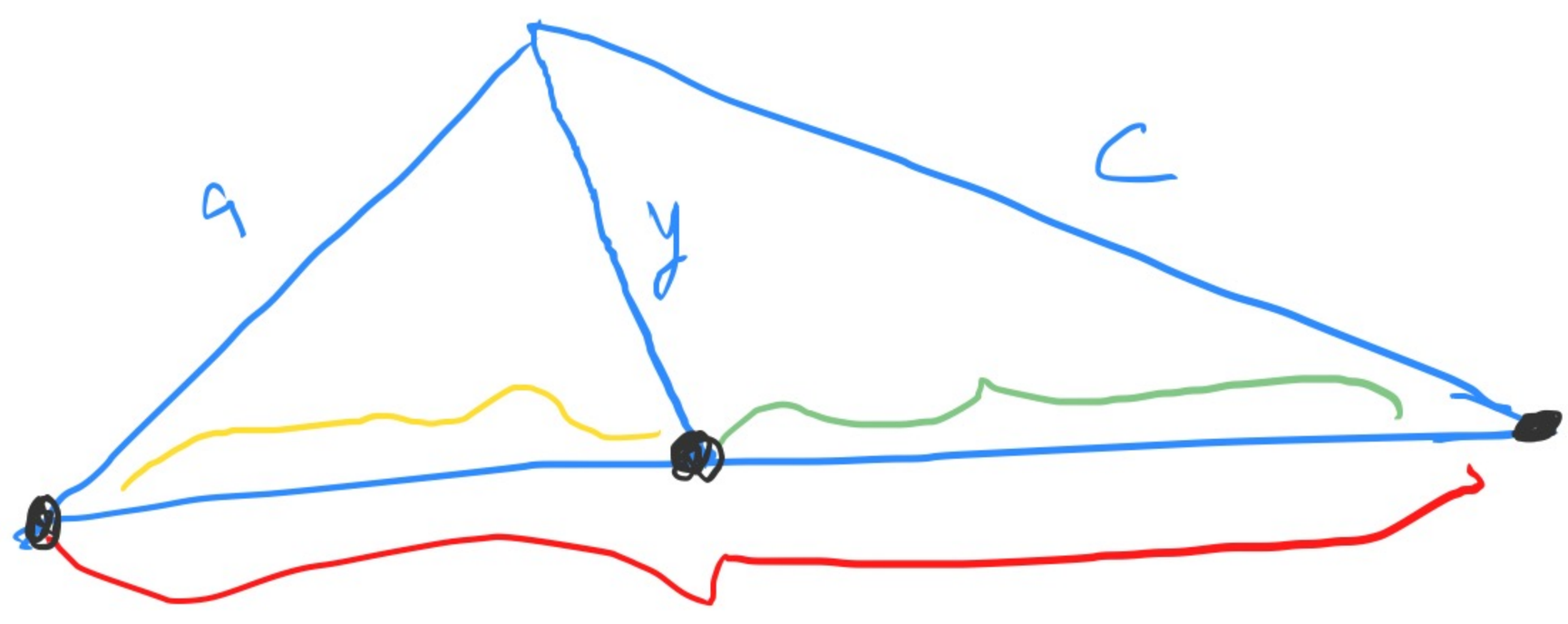
Case 2: $a, c \geq 0 > b$; Let $y = -b$

$$\boxed{\sqrt{a^2 - ay + y^2}} + \boxed{\sqrt{y^2 - cy + c^2}} \overset{?}{\geq} \boxed{\sqrt{a^2 + ac + c^2}}$$



Case 3: $b, c \geq 0 > a$

similar to Case 1



Test 9

P1

$m, n \in \mathbb{Z}$

$$m^5 + mn^4 + n^5 = 1$$

$$(m^2 + mn + n^2)(m^3 - m^2n + n^3) = 1$$

$$\Rightarrow \quad m^2 + mn + n^2 = m^3 - m^2n + n^3 = 1$$

$$\left(\because \quad m^2 + mn + n^2 \geq 0 \right)$$

$$\left.\begin{array}{l} m^2 + mn + n^2 \geq mn \\ m^2 + mn + n^2 \geq -mn \end{array}\right\} \Rightarrow m^2 + mn + n^2 \geq |mn|$$

$$\Rightarrow 1 \geq |mn|$$

$$\Rightarrow (m,n) = (\pm 1, \pm 1), (0,n), (m,0)$$

$$\boxed{(1,-1), (0,1), (1,0)}$$

Test 9, P2

$x, y, z \in \mathbb{R}$

$$(x^2+y^2+z^2)^3 \geq (x^3+y^3+z^3-3xyz)^2$$

$$(x+y+z)^2(x^2+y^2+z^2-xy-yz-xz)^2$$

$$\Big(x^2+y^2+z^2+2(xy+yz+zx)\Big)\Big(x^2+y^2+z^2-xy-yz-zx\Big)\Big(x^2+y^2+z^2-xy-yz-zx\Big)$$

$$\underset{\text{AM-GM}}{\leq} \left(\frac{3(x^2+y^2+z^2)}{3}\right)^3 = (x^2+y^2+z^2)^3 \checkmark$$

$$\left[a, b, c \geq 0 \Rightarrow abc \underset{\text{AM-GM}}{\leq} \left(\frac{a+b+c}{3}\right)^3\right]$$

"=": $xy+yz+zx=0$

P1 | $a, b, c > 0$. $ab + bc + ca = 1$

$$\sum_{cyc} \frac{a}{\sqrt{a^2+1}} \leq \frac{3}{2}$$

$$\sum_{cyc} \frac{a}{\sqrt{a^2+ab+ac+bc}} = \sum_{cyc} \frac{a}{\sqrt{(a+b)(a+c)}}$$

$$= \sum_{cyc} \sqrt{\frac{a}{a+b}} \cdot \sqrt{\frac{a}{a+c}}$$

$$\leq \sum_{cyc} \frac{1}{2}\left(\frac{a}{a+b} + \frac{a}{a+c}\right)$$

$$= \frac{1}{2}\left(\frac{a}{a+b} + \frac{a}{a+c} + \frac{b}{a+b} + \frac{b}{b+c} + \frac{c}{a+c} + \frac{c}{b+c}\right)$$

$$= \frac{3}{2}$$

P2 $a, b, c > 0$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

Let $x = \frac{a}{b}$; $y = \frac{b}{c}$; $z = \frac{c}{a}$

$\Rightarrow xyz = 1$

$$x+y+z \overset{?}{\geq} \frac{1+xy}{1+y} + \frac{1+yz}{1+z} + \frac{1+zx}{1+x}$$

$$\Leftrightarrow (x+y+z)(2+x+y+z+xy+yz+zx) \overset{?}{\geq} \sum_{cyc} (1+xy)(1+z)(1+x)$$

$$2(x+y+z) + (x^2+y^2+z^2) + 2(xy+yz+zx) + \sum x^2y + \sum xy^2 + 3$$

$$(1+xy+z+x+1+x^2y+xz+x)$$

$$(2+2x+z+xy+xz+x^2y)$$

$$RHS = \cancel{6}^{3} + \cancel{3}(x+y+z) + \cancel{2(xy+yz+xz)} + \cancel{(x^2y+y^2z+z^2x)}$$

$$LHS = \cancel{2(x+y+z)} + (x^2+y^2+z^2) + \cancel{2(xy+yz+xz)} + \cancel{(x^2y+y^2z+z^2x)} + (xy^2+yz^2+zx^2) \cancel{+3}$$

$$\Leftrightarrow \quad \overbrace{xy^2+yz^2+zx^2}+\underbrace{x^2+y^2+z^2} \overset{?}{\geq} 3+\underbrace{x+y+z}$$

but $\quad xy^2+yz^2+zx^2 \underset{\text{AM-GM}}{\geq} 3$

$$x^2+y^2+z^2 \underset{\text{Cauchy}}{\geq} \frac{(x+y+z)^2}{3} = (x+y+z)\cdot\frac{x+y+z}{3}$$

$$\underset{\text{AM-GM}}{\geq} x+y+z$$

$\blacksquare$

P3

$a, b, c > 0$

$$(a+b)(a+c) \geq 2\sqrt{abc(a+b+c)}$$

Sol 1

$$a^2+ab+ac+bc = a(a+b+c) + bc$$

$$\underset{\text{AM-GM}}{\geq} 2\sqrt{a(a+b+c)\cdot bc}$$

$$= 2\sqrt{abc(a+b+c)}$$

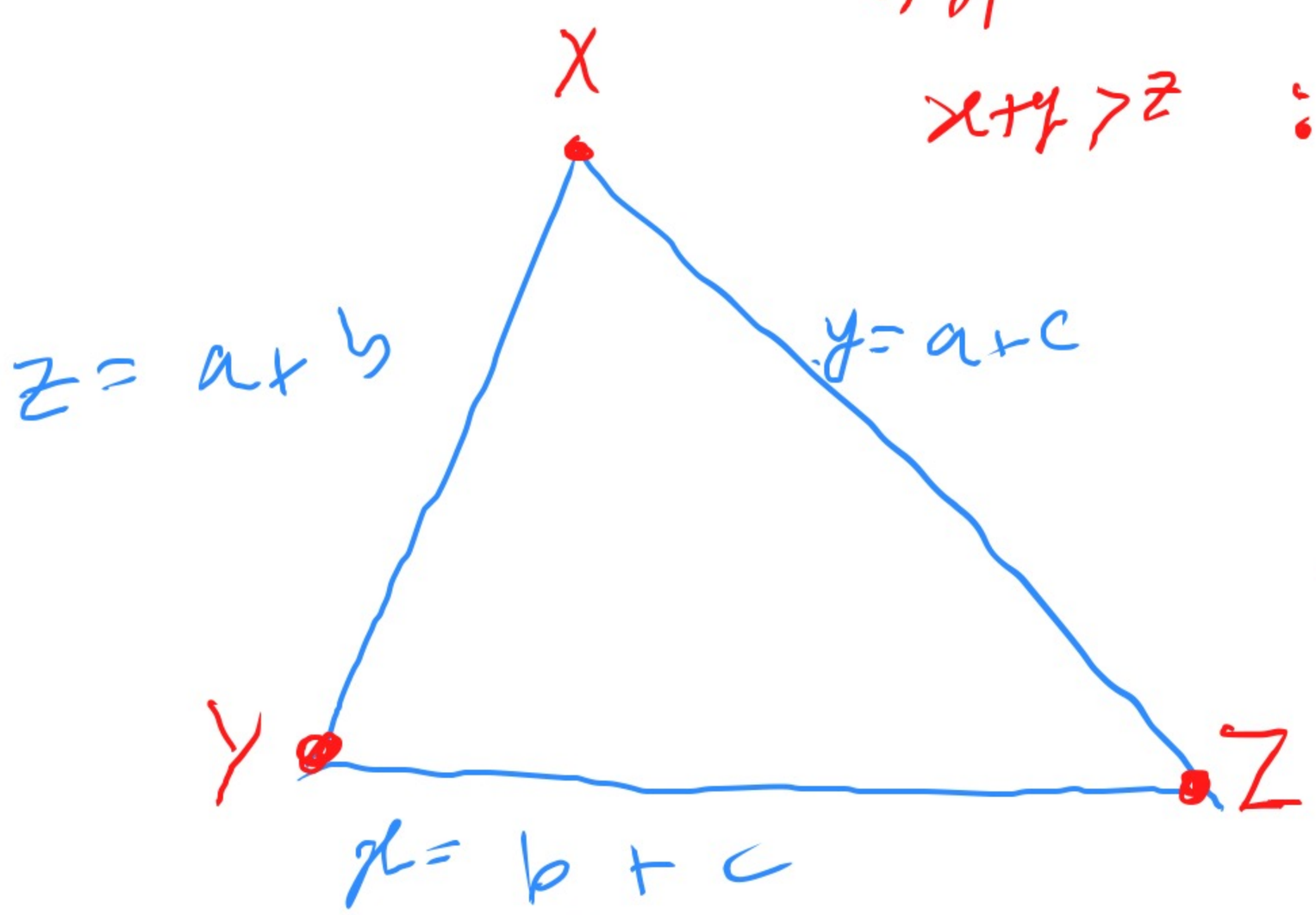
Sol 2

x, y, z

$x+y > z \quad ; \quad c = \frac{x+y-z}{2} > 0$

$b = \frac{x+z-y}{2} > 0$

$a = \frac{y+z-x}{2} > 0$



$$c = \frac{x+y-z}{2} = \frac{x+y+z}{2} - z = s-z \; ; \; b = s-y \; ; \; a = s-x$$

$$a+b+c = s$$

$$(a+b)(a+c) \geq 2\sqrt{abc(a+b+c)}$$

$$\Updownarrow$$

$$yz \overset{?}{\geq} 2\sqrt{s(s-x)(s-y)(s-z)}$$

$$\Leftrightarrow \quad \frac{1}{2}yz \overset{?}{\geq} [XYZ]$$

$$\parallel$$

$$\frac{1}{2}yz \sin\angle X$$

since $\sin\angle x \leq 1$; we're done $\blacksquare$