November camp – KSA training 2022. Level 4 & 4+.

ON THE INTEGER POLYNOMIALS

Problem 1.

a) Consider integer polynomial $P(x)$ such that $P(-1)=29, P(-3)=11, P(-2)=2022$. Prove that there does not exist $a \in \mathbb{Z}$ such that $P(P(a))=a^2$.

b) Consider integers $b_1, b_2, \ldots, b_n$ that have the sum is 2022 and take a as some divisor of $\left\lfloor \frac{2022!}{3^{1006}} \right\rfloor$. Prove that the following polynomial does not have rational root

$$P(x)=ax^{2n+2}+b_1x^{2n}+b_2x^{2n-2}+\cdots+b_nx^2+a.$$

Problem 2.

a) Find all monic integer polynomials $P(x)$ of second degree such that there exist some integer polynomial $Q(x)$ satisfying $P(x)Q(x)$ have all coefficients in the set $\{-1,0,1\}$.

b) Given 2023 non-zero integers with non-zero sum. Prove that there exist a permutation of these numbers, denote as $c_0, c_1, \ldots, c_{2022}$ such that $P(x)=c_{2022}x^{2022}+\cdots+c_1x+c_0$ does not have any integer root.

Problem 3.

a) Let $P(x)$ be a integer polynomial of degree 2022 and has 2022 integer roots with the product equals to 0. Find the number of integer solutions of $P(P(x))=0$.

b) For some positive integer k, suppose that there exist infinitely many monic integer polynomials of distinct degree and share the non-zero coefficients of $x^0, x^1, \ldots, x^{k-1}$, each of them have all roots are integers, not neccesary distinct. Find the largest value of k.

Problem 4. Find all non-constant integer polynomial $P(x)$ such that $P(P(n)+n)$ is a prime for infinitely many positive integers n.

Problem 5. Find all integer polynomial $P(x)$ such that there exist infinitely many integer a such that the sequence (u_n) defined by

$$u_0=a,\ u_n=P(u_{n-1}), \forall n \geq 0$$

is periodic from the first term.

Problem 6. Let T be the set of 2023 real numbers such that for all $a \in T$, number $\varphi(T)-a$ is odd integer, in which $\varphi(T)$ is the product of all elements in T. Prove that $T \cap \mathbb{Q}=\varnothing$.

Problem 7*. Let $P(x)$ and $Q(x)$ be two non-constant polynomials with non-negative integer coefficients in which the coefficient of $P(x)$ not exceed 2021 and $Q(x)$ has at least one coefficient bigger than 2021. Suppose that $P(2022)=Q(2022)$ and $P(x), Q(x)$ share some common rational root $\frac{p}{q} \neq 0$, $\gcd(p,q)=1$. Prove that

$$|p|+n|q| \leq Q(n)-P(n) \text{ for all } n=1,2,\ldots,2021.$$