Yesterday:

$$P(x) \geq 0 \Rightarrow P(x) = A(x)^2 + B(x)^2$$

$A, B \in \mathbb{R}$.

$$P(x) = \underbrace{(x-x_1)^{\alpha_1} \cdots (x-x_k)^{\alpha_k}}_{\text{real root part}} \underbrace{(x^2 + p_1 x + q_1) \cdots (y^2 + p_\ell x + q_\ell)}_{\text{irreducible q. poly.}}$$

$\Delta < 0$:

$p_i^2 - 4q_i < 0$.

$P(x) \geq 0 \leadsto 2 | \alpha_i \rightarrow$ real part is square.

$$P(x) = c^2 \cdot (x^2 + p_1 x + q_1) \cdots (y^2 + p_\ell x + q_\ell)$$

Since $x^2 + ax + b$ with $\Delta < 0$

$\|$ $\quad p = -\frac{b}{2a}$

$(x-p)^2 + q$ where $q = \frac{-\Delta}{4a} > 0$

$$\boxed{(x-p)^2 + (\sqrt{q})^2}$$

<u>Homework</u>

$$P(x) = U_1(x)^2 + U_2(x)^2 + \cdots + U_k(x)^2 \geq 0$$

$$\Downarrow$$

$$P(x)^2 = V_1(x)^4 + V_2(x)^4 + \cdots + \underline{V_\ell}(x)^4 \quad . \quad \square$$

$$P(x) = (x-x_1)^{\alpha_1} \cdots (x-x_k)^{\alpha_k} (x^2+p_1 x+q_1) \cdots (x^2+p_\ell x+q_\ell)$$

$\boxed{1^\circ}$ α_i are even

$$P(x)^2 = \left((x-x_1)^{2\alpha_1} \cdots (x-x_k)^{2\alpha_k} (x^2+p_1 x+q_1) \cdots (x^2+p_\ell x+q_\ell)\right)^2$$

$$= \underbrace{(x-x_1)^{4\alpha_1} \cdots (x-x_k)^{4\alpha_k}}_{A^4} (\quad \cdots \quad)^2$$

$(A^4+B^4)(C^4+D^4) =$ sum of 4 power.

$(x^2+px+q)^2 =$ sum of 4 power.

$f(x) = x^2+px+q$, $\Delta_f < 0$

$\boxed{f(x)^2 = v_1^4 + v_2^4}$ for some $v_1, v_2 \in \mathbb{R}[x]$

$f(\underline{x+u})^2$ for some u,

$$f(x) = \left(x + \frac{p}{2}\right)^2 - \frac{\Delta}{4}$$

$$f\left(x - \frac{p}{2}\right)^2 = \left(x^2 - \frac{\Delta}{4}\right)^2$$

but $-\frac{\Delta}{4} > 0$ $\quad t := \sqrt{-\frac{\Delta}{4}}$

$(x^2+t^2)^2 =$ sum of 4 power.

$$\parallel$$

$$\frac{1}{2}\left(\left(x - \frac{1}{\sqrt{3}}t\right)^4 + \left(x + \frac{1}{\sqrt{3}}t\right)^4\right) =$$

sum of 4 power. $\square$

$$(x^2+px+q)^2 =$$

$$= \frac{1}{2}\left(\left(x + \frac{p}{2} - \frac{1}{\sqrt{3}} \cdot \left(-\frac{\Delta}{4}\right)\right)^4 + \left(x + \frac{p}{2} + \frac{1}{\sqrt{3}} \cdot \left(-\frac{\Delta}{4}\right)\right)^4\right)$$

Do you remember Vieta formulas?

$$f(x) = a_d x^d + \dots + a_0 \quad \text{and}$$

$x_1, x_2, \dots, x_d$ are roots of f.

$$\sum_{i=1}^{d} x_i = -\frac{a_{d-1}}{a_d}$$

$$\sum_{1 \le i < j \le d} x_i \cdot x_j = \frac{a_{d-2}}{a_d}$$

$$\sum_{i_1, \dots, i_k} \prod x_{i_1} x_{i_2} \cdots x_{i_k} = (-1)^k \frac{a_{d-k}}{a_k}$$

$$x_1 x_2 \cdots x_d = (-1)^d \frac{a_0}{a_d}$$

$$x^2 + ax + b$$

$$x_1 + x_2 = -a$$

$$x_1 x_2 = b$$

$\boxed{27}$ W $\deg W = 4$ polynomial $\underline{W \in \mathbb{Z}[x]}$

$x_1\ x_2 \cdot x_3\ x_4$ - roots

$x_3 + x_4 \in \mathbb{Q}$ $\qquad \sqrt{2}-1 \quad \sqrt{2}+1$

$x_3 \cdot x_4 \notin \mathbb{Q}$

$\Rightarrow$ $x_1 + x_2 = x_3 + x_4$

$$W(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$\parallel$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$a_0, a_1, a_2, a_3, a_4 \in \mathbb{Z}.$$

$$-\frac{a_3}{a_4} = \underbrace{x_1 + x_2}_{A} + \underbrace{x_3 + x_4}_{B} = A + B \in \mathbb{Q}$$

$$\frac{a_2}{a_4} = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 =$$

$$= AB + x_1 x_2 + x_3 x_4 \in \mathbb{Q}$$

$$-\frac{a_1}{a_4} = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 =$$

$$= A \cdot x_3 x_4 + B \cdot x_1 x_2 \in \mathbb{Q}$$

$$\frac{a_0}{a_4} = x_1 x_2 x_3 x_4 = x_1 x_2 \cdot x_3 x_4 \in \mathbb{Q}.$$

$B \in \mathbb{Q}$, $x_3 x_4 \notin \mathbb{Q}$

Since $A+B \in \mathbb{Q} \longrightarrow A \in \mathbb{Q}$

$$\underbrace{AB}_{\mathbb{Q}} + x_1 x_2 + \underbrace{x_3 x_4}_{\notin \mathbb{Q}} \in \mathbb{Q} \rightsquigarrow \boxed{x_1 x_2 \notin \mathbb{Q}}$$

$$A \cdot x_3 x_4 + B x_1 x_2 \in \mathbb{Q}$$

$$(A-B) x_1 x_2 = \underset{\mathbb{Q}}{A} \Big(\underbrace{x_1 x_2 + x_3 x_4}_{\mathbb{Q}} \Big) - \underbrace{\Big(A \cdot x_3 x_4 + B x_1 x_2 \Big)}_{\mathbb{Q}}$$

so

$(A-B) x_1 x_2 \in \mathbb{Q}$

but $x_1 x_2 \notin \mathbb{Q} \longrightarrow \boxed{A-B=0}$

$\boxed{A=B}$

$\boxed{30}$

Homework 31, 30 and next