— Combinatorics for L3 —

— March 18, 2019 — Chessboard and rooks —

21. A big rectangle is subdivided by segments parallel to its sides into small rectangles. It is known that each small rectangle has at least one dimension of integral length. Prove that the big rectangle has at least one dimension of integral length.

22. Can three faces of a $8 \times 8 \times 8$ cube having a common vertex be covered with 64 strips 1×3? The strips can be folded along the edges of the cube.

23. Over a half of squares of 7×7 board are occupied by rooks. Prove that there exists a rook that is *encircled* i.e. there is a rook attacking it from each of the four directions.

24. On some squares of an $m \times n$ chessboard rooks are placed. It is known that every rook is attacked by at most two different rooks. Determine, in terms of m, $n \geq 2$, the maximum number of rooks for which this situation is possible.

25. On an $n \times n$ chessboard r rooks are placed in such a way that the following property holds: if a cell of coordinates (i, j) is unoccupied, then in the i-th row and in the j-th column there are together at least n rooks. Prove that $r \geq n^2/2$.

26. **(Homework)** Let S be a set of $2n$ cells of $n \times n$ chessboard with the following properties:

- each row and each column contains exactly 2 cells;
- one can walk through all the cells in S with a rook and come back to the initial cell.

Prove that the cells of the board can be painted with n colors in such a way that:

- each row and each column contains exactly one cell of each color;
- the set S contains exactly two cells in each color.

27. **(Homework)** Given is an $n \times n$ board, where n is even, and any Hamiltonian walk of a chess king. Consider a graph formed by the sides and vertices of board squares lying inside the cycle induced by the leap rook's walk. Prove that this graph is a tree.