W1 - P1

$$\frac{\varphi(n)}{n} = \frac{24}{35}$$

Solution

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

$$\Rightarrow \quad \frac{\varphi(n)}{n} = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

$$\frac{(p_1 - 1) \cdots (p_k - 1)}{p_1 \cdots p_k} = \frac{24}{35} \Rightarrow 35 \mid p_1 \cdots p_k$$

$\Rightarrow$ w|o g $p_1 = 5$, $p_2 = 7$

$$\frac{\cancel{4 \times 6}}{\cancel{5 \times 7}} \times \left(1 - \frac{1}{p_3}\right) \times \cdots \left(1 - \frac{1}{p_k}\right)$$

$$= \frac{\cancel{24}}{\cancel{35}} \rightarrow 1$$

if $\exists\ p \mid n$; $p \nmid 35$

$\Rightarrow k \geq 3 \longrightarrow \lightning$

$$\Rightarrow \boxed{n = 5^{\alpha_1} \cdot 7^{\alpha_2} \ ; \ \alpha_1, \alpha_2 \geq 1}$$

$\blacksquare$

W1 - P2

$$2(a^3+b^3+c^3)\sum\frac{1}{ab+bc}\geq 3\sum a$$

$$\sum_{cyc}\frac{1}{ab+bc}\geq\frac{9}{2(ab+bc+ca)}$$

$$\Rightarrow LHS\geq\frac{9(a^3+b^3+c^3)}{ab+bc+ca}$$

ETST:

$$3(a^3+b^3+c^3)\geq(a+b+c)(ab+bc+ca)$$

AM-GM

$$\frac{a^3+a^3+b^3}{3} \geq a^2 b$$

$$\vdots$$

$$\frac{b^3+c^3+c^3}{3} \geq b c^2$$

$$a^3+b^3+c^3 \geq 3abc$$

Σ $\Downarrow$

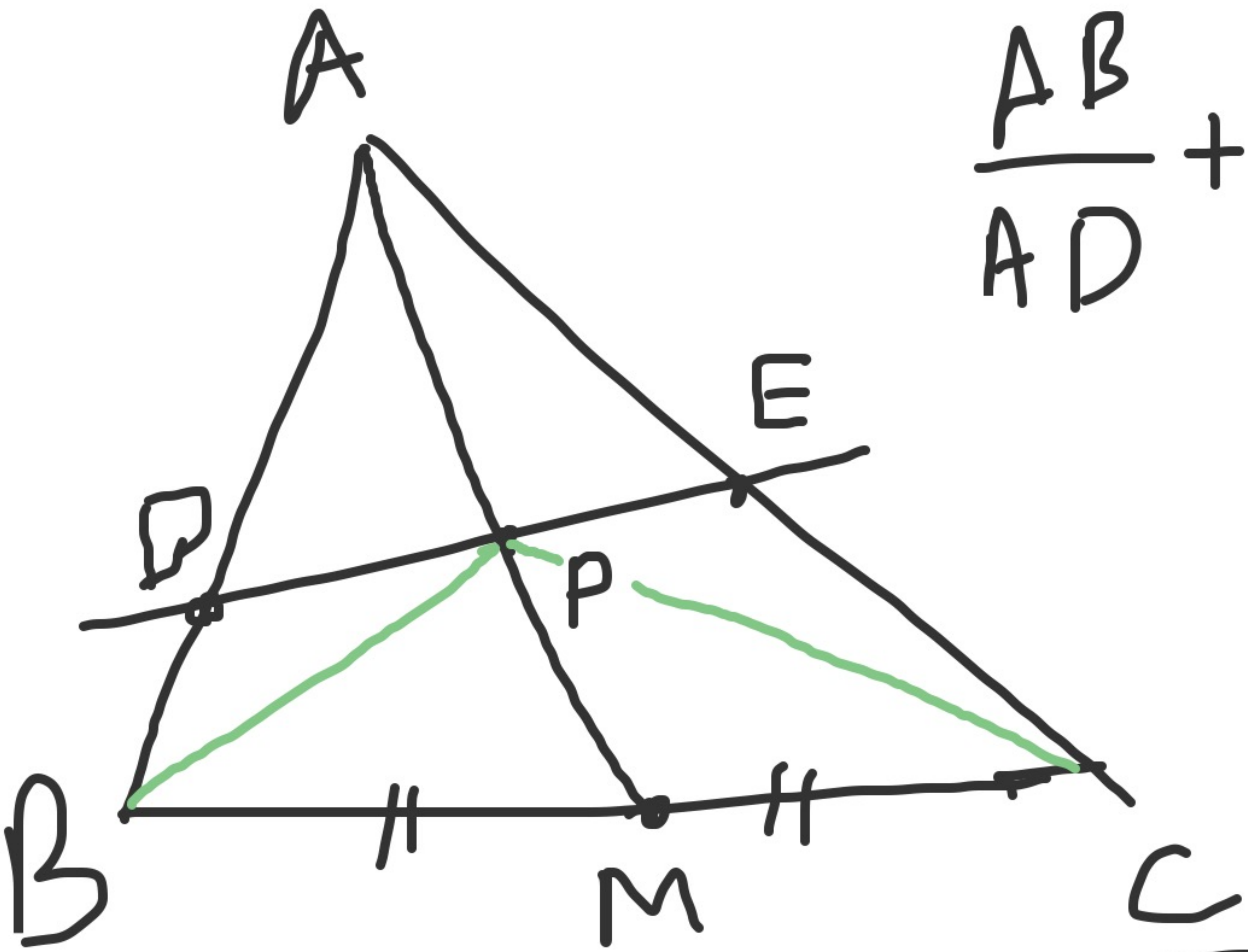
Hölder

$$(1+1+1)(1+1+1)(a^3+b^3+c^3)$$

$$\geq (a+b+c)^2 (a+b+c)$$

$$\geq 3(ab+bc+a)(a+b+c)$$

W1-P3, W2-P3



$$\frac{AB}{AD}+\frac{AC}{AE}$$

$$\frac{AB}{AD}=\frac{[ABP]}{[ADP]} \quad ; \quad \frac{AC}{AE}=\frac{[ACP]}{[AEP]}$$

$$\boxed{[ABP]=[ACP]=\frac{AP}{2AM}\cdot[ABC]}$$

$$\frac{AD\cdot AE}{AB\cdot AC}=\frac{[ADE]}{[ABC]}=\frac{[ADP]+[AEP]}{[ABC]}$$

$$\boxed{\frac{AB}{AD}+\frac{AC}{AE}} = \frac{AB \cdot AC}{AD \cdot AE}\left(\frac{AE}{AC}+\frac{AD}{AB}\right)$$

$$= \frac{[ABC]}{\cancel{[ADP]}+[AEP]}\left(\underbrace{\frac{\cancel{[AEP]}}{[ACP]}+\frac{[ADP]}{[ABP]}}_{=}\right)$$

$$= \frac{[ABC]}{[ACP]} = \frac{[ABC]}{\frac{AP}{2AM}[ABC]}$$

$$= \boxed{\frac{2AM}{AP}} \qquad \blacksquare$$

W1-P4

$$a < b < c \leq d$$

or

$a \mid \gcd(b, c, d)$; $d = a + b + c$

---

sol Let $a < b < c$ be the smallest 3 numbers

if $\nexists\, d \in S$ such that $d = a + b + c \Rightarrow a \mid x \;\; \forall x \in S$ ✓

if $\exists\, d = a + b + c \in S$

$\Rightarrow$ consider (a, b, c, x)

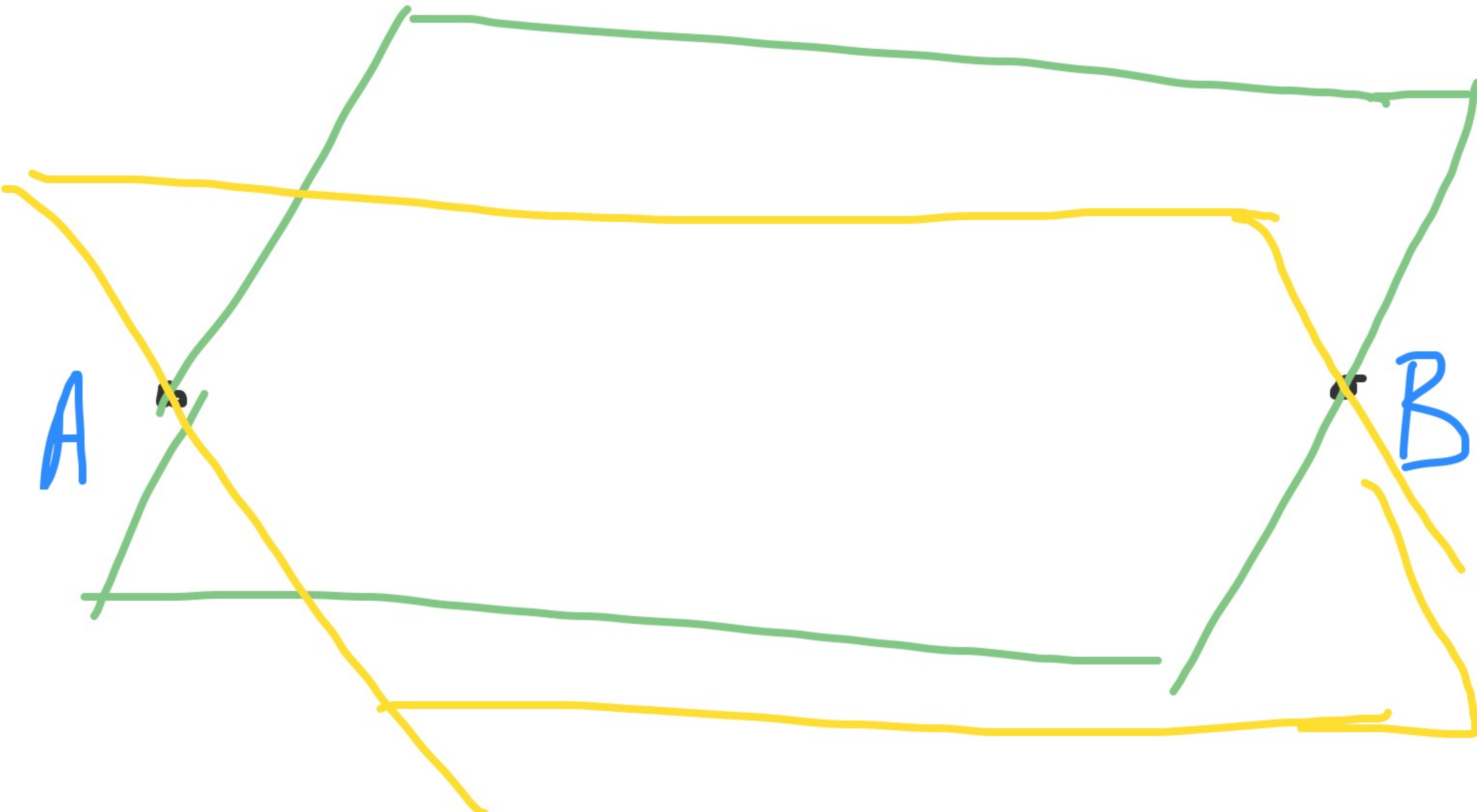
such that $x \neq d, a, b, c$

$\Rightarrow a \mid b, c, x$

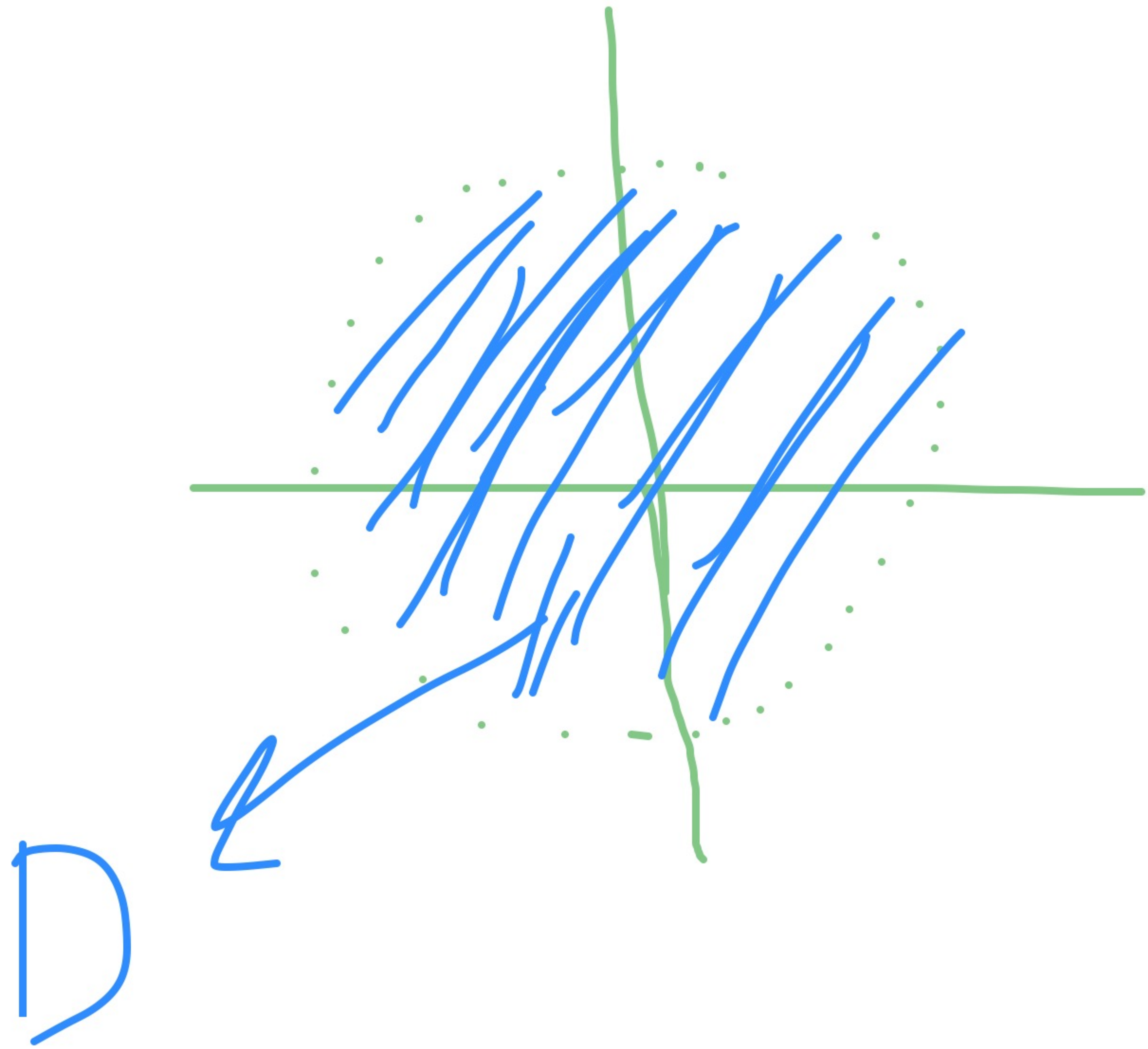
$d = a + b + c \Rightarrow a \mid d$

$a \mid a \quad a \mid b \quad a \mid c$ $\blacksquare$

W1- P 5 | consider the farthest two points in S



then consider all planes passing through $\overline{AB}$ (AB is diameter) together they will form a sphere



- S bounded
- S contains its boundary

W2-P1)

$k > \frac{12}{7\varepsilon}$

$$6 < \frac{4x+1}{3y} + \frac{4y+1}{z} + \frac{3z+1}{2x}$$

is 6 best?

Am-Gm

$$\frac{4x}{3y} + \frac{4y}{z} + \frac{3z}{2x} \geq 6$$

$=$

$$\varepsilon > \underbrace{\frac{1}{3y} + \frac{1}{z} + \frac{1}{2x}} > 0$$

$\sum \Downarrow$ ✓

"=" $(x,y,z) = (3k, 2k, 4k)$; $k \to \infty$

$\boxed{\text{w2-P4}}$

$$8n^6-4n^3+1 \in P$$

$$(a^3+b^3+c^3-3abc)$$

$$8n^6+8n^3+1-3\cdot 2n^2\cdot 2n\cdot 1$$

$$=\underbrace{(2n^2+2n+1)}_{>1}(\underbrace{4n^4-4n^3+2n^2-2n+1}_{=1})$$

$$4n^3(n-1)+2n(n-1)+1=1 \Longrightarrow n=1$$

$p=5 \Longrightarrow$ only solution $\boxed{n=1}$ ■

$\boxed{\text{W2-P5}}$?

$$\tau(n) < 2\sqrt{n}$$

$$n = d_1 \cdot \frac{n}{d_1}$$

$$= \cdots\cdots = d_k \cdot \frac{n}{d_k}$$

$$1 = d_1 < d_2 < \cdots < d_k \leq \sqrt{n} \leq \frac{n}{d_k}$$

$$< \cdots\cdots < \frac{n}{d_1} = n$$

($\because n = ab \Rightarrow$ either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$)

now

Case 1: $\tau(n) = 2k-1$

$$\Rightarrow \quad d_k = \sqrt{n} = \frac{n}{d_k}$$

and $\quad k \leq \sqrt{n}$

$$\Rightarrow \quad \tau(n) = 2k-1 < 2\sqrt{n}$$

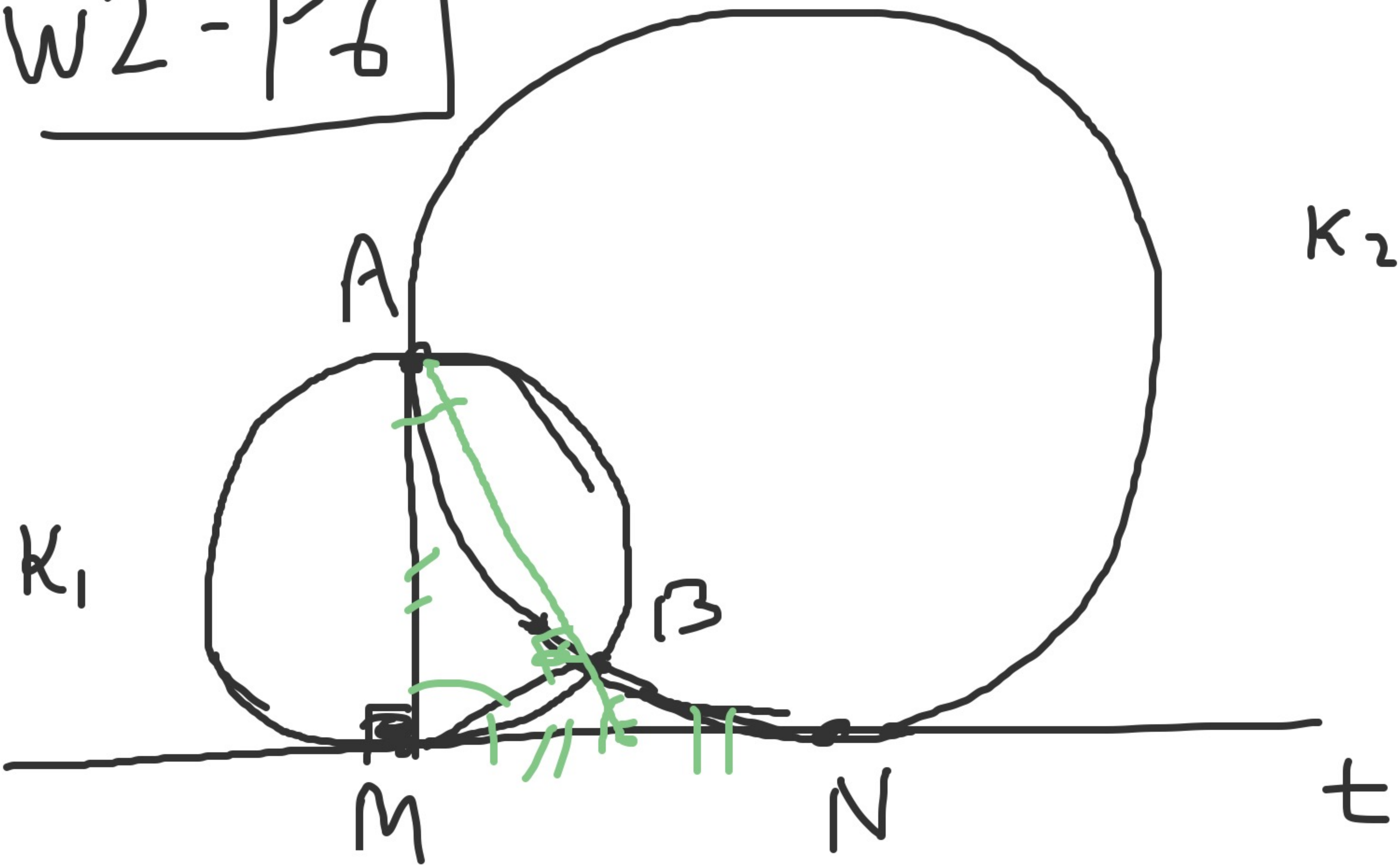
Case 2: $\tau(n) = 2k$;

$$d_k < \sqrt{n} < \frac{n}{d_k} \; ; \sqrt{n} \notin \mathbb{Z}$$

$$\Rightarrow \; k < \sqrt{n}$$

$$\Rightarrow \; \tau(n) = 2k < 2\sqrt{n}$$

W2 - P6



w2 - P7

$d(u^2-v^2) = a$

$c = d(u^2+v^2)$

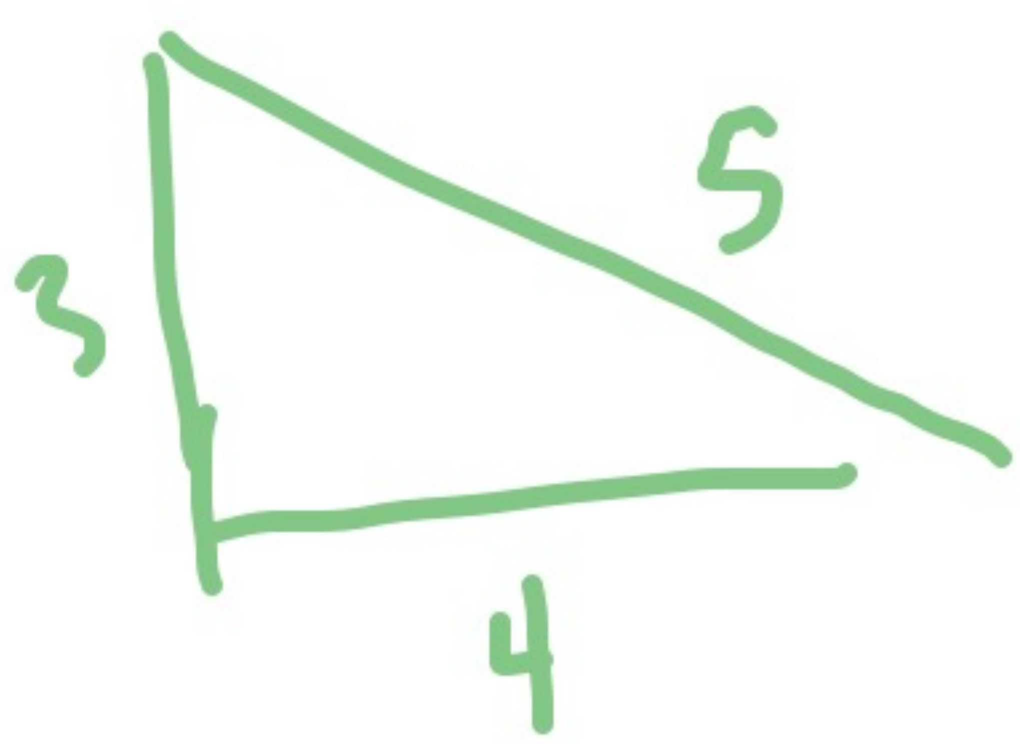
$b = d(2uv)$

$[\,u > v > 0,\ (u,v) = (u+v,2) = 1\,]$

$a^2+b^2=c^2$

$\frac{ab}{2}$

ETST: $12 \mid ab$



$\longrightarrow \frac{ab}{2} = 6$

(mod 3)

$$a^2+b^2 \equiv \begin{matrix} 0+0 \\ 1+0 \\ 0+1 \\ \cancel{1+1} \end{matrix}$$

$$c^2 \equiv 0 \quad \text{or} \quad 1 \Rightarrow 3 \mid ab$$

(mod 4) $$a^2+b^2 \equiv \begin{matrix} 0+0 \\ 1+0 \\ 0+1 \\ \cancel{1+1} \end{matrix}$$

$$c^2 \equiv 0, 1$$

$$\Rightarrow 2 \mid ab$$

(mod 8):

$$a^2+b^2 \equiv \begin{cases} 0+0 \longrightarrow 4 \mid ab \\ 0+1 \longrightarrow 4 \mid ab \\ 0+4 \longrightarrow 4 \mid ab \\ 4+4 \longrightarrow 4 \mid ab \\ \cancel{4+1} \end{cases}$$

$$c^2 \equiv 0, 1, 4$$

$$\Rightarrow \quad 3, 4 \mid ab \Rightarrow 12 \mid ab$$

$$\text{or} \quad 6 \mid \frac{ab}{2}, \quad \boxed{GCD = 6}$$

$\blacksquare$