Preparation for Saudi Arabia Team 2021

May/June Session: Level 4

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Lesson 1

The Method of Infinite Descent

Problems:

1. Find all integers x, y and z such that $x^3 + 2y^3 = 4z^3$.
2. Show that there are no natural numbers x, y and z such that $x^4 + y^4 = z^2$.
3. Determine all possible positive interger n such that $(x + y + z)^2 = nxyz$ has positive integer solutions (x, y, z).
4. Let a and b be two positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $\dfrac{a^2 + b^2}{ab + 1}$ is a perfect square.
5. Let a, b be natural numbers such that $a \cdot b$ divides $a^2 + b^2 + 3$. Prove that $\frac{a^2+b^2+3}{ab} \in \{4, 5\}$
6. If m and n are numbers of equal parity such that $m > n$ and $m^2 - n^2 + 1 | m^2$ prove that $m^2 - n^2 + 1$ is a perfect square.
7. Prove that if $P = \frac{x^2+1}{y^2} + 4$ is square number then $P = 9$.
8. Let a, b be two positive integers, such that $ab \neq 1$. Find all the integer values that $f(a, b)$ can take, where
$$f(a, b) = \frac{a^2 + ab + b^2}{ab - 1}.$$
9. Do there exist positive integers m and n such that $\dfrac{n^2 - 1}{m^2 - n^2 - 1}$ is also a positive integer?
10. If a, b, c are positive integers such that
$$0 < a^2 + b^2 - abc \leq c,$$
show that $a^2 + b^2 - abc$ is a perfect square.