Email training, N9

October 20 - 29, 2019

Problem 9.1. Let 4 positive integers are given. It is known that the square of the sum of any 2 of them is divisible to the product of 2 other numbers (like this: $(a+b)^2$ is divisible by cd). Prove that at least three numbers are equal.

Problem 9.2. Prove that for any prime number $p > 2$ there exists a unique pair of positive integers (x, y) with $x > y$ such that

$$\frac{2}{p} = \frac{1}{x} + \frac{1}{y}.$$

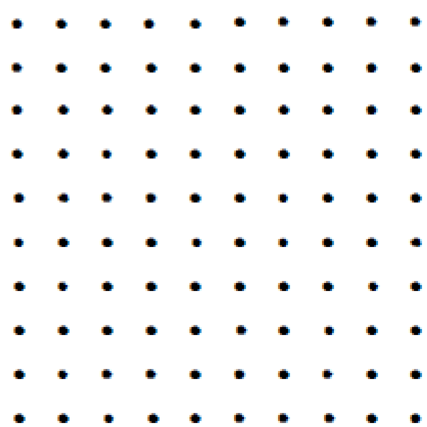
Problem 9.3. Let $P(x)$ and $Q(x)$ be monic polynomials of degree 10 with real coefficients such that equation $P(x) = Q(x)$ does not have a solution in real numbers. Prove that equation $P(x+1) = Q(x-1)$ has a solution in real numbers.

Problem 9.4. Find all positive integers n for which $n^5 + n^4 + 1$ is a prime.

Problem 9.5. Find all integers n for which $x^2 - x + n$ divides $x^{13} + x + 90$.

Problem 9.6. Prove that the sequence $[2^n\sqrt{2}]$ contains infinitely many composite numbers (here $[a]$ is the integer part of a, ie. the maximal integer not exceeding a).

Problem 9.7. Determine the number of squares with all their vertices belonging to the $n \times n$ array of points defined in the figure.



Problem 9.8. Find all positive integers k such that the set

$$\{1990, 1990+1, 1990+2, \ldots, 1990+k\}$$

can be split into 2 sets, such that the sums of numbers in both sets are equal.

Problem 9.9. Let the triangle ABC is given and $\angle A : \angle B : \angle C = 1 : 2 : 4$. Prove that

$$\frac{1}{AB} + \frac{1}{AC} = \frac{1}{BC}.$$

Problem 9.10. It is known that a circle can be inscribed in a trapezium $ABCD$. Prove that the two circles, constructed on its oblique sides (nonparallel sides) as diameters, touch each other

Solution submission deadline October 29, 2019