November camp – KSA training 2022. Level 3.

INJECTIVE AND SURJECTIVE

- Injective: if we have $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ then $a = b$.
- Surjective: for all $y \in \mathbb{R}$, there exists some $x \in \mathbb{R}$ such that $f(x) = y$.
- If f is both injective and surjective, then f is bijective.

Example: let $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x$ for all x. Prove that f is bijective on $\mathbb{R}$.

Problem 1.

a) Prove that if $f(xf(x)) = x^2$ and f is injective, then f is bijective.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x \in \mathbb{R}$,

$$f(x^3 + x)^2 \le f(2x) + 2 \text{ and } f(-2x)^3 \ge 3f(-x^3 - x) + 2.$$

Prove that f is not injective on $\mathbb{R}$.

Problem 2.

a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(xf(y) - f(x)) = 2f(x) + xy \text{ for all } x, y.$$

Prove that f is injective and $f(0) = 1, f(1) = 0$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(|x| + y + f(y)) = 2y + |f(x)| \text{ for all } x, y.$$

Prove that f is surjective and $f(0) = 0$.

Problem 3.

a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(2022x^3 + y + f(y)) = 2y + 2022x^2 f(x) \text{ for all } x, y.$$

Prove that f is bijective.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(xf(x) + 2f(y)) = x^2 + y + f(y) \text{ for all } x, y.$$

Prove that f is bijective.

Problem 4. Find all $c \in \mathbb{R}$ such that there exist some function $f : \mathbb{R} \to \mathbb{R}$ such that:

i) $f(1) = 2022$;

ii) $f(x + y + f(y)) = f(x) + cy$ for all $x, y \in \mathbb{R}$.

Problem 5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \ne 0$ and

$$f(2xy + f(x + y)) = xf(y) + yf(x) \text{ for all } x, y \in \mathbb{R}.$$

Problem 6. Find all injective function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x + y) + x) = f(y)^2 + f((x - 1)f(y)) \text{ for all } x, y \in \mathbb{R}.$$

Problem 7*. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(2xf(x) - 2f(y)) = 2x^2 - y - f(y) \text{ for all } x, y \in \mathbb{R}.$$