Warm-Up Session with the IMO Team

June 2nd, 2022 — Łukasz Bożyk

1 Modulo starter

Problem 1. Source: Polish JMO 2022 finals, P2

Find all $n \geq 1$ such that both

$$1\underbrace{77\ldots7}_{n} \quad \text{and} \quad 3\underbrace{77\ldots7}_{n}$$

are primes.

Solution
For $n \equiv 0, 2 \pmod 3$ one of these numbers is clearly divisible by 3. For $n \equiv 1 \pmod 3$ the second number is divisible by 37 because it's a concatenation of numbers divisible by 37 (37 and several segments $777 = 21 \cdot 37$). So only $n = 1$ works.

Problem 2. Source: Polish MO 2014 finals, P5

Solve $2^x + 17 = y^4$ over positive integers.

Solution
For even x we get $17 = (y^2 - 2^{x/2})(y^2 + 2^{x/2})$, so $(x, y) = (6, 3)$. For odd x we get a contradiction modulo 17 (even by writing down the entire table). Tricky argument:

$$17 = y^4 - 2^x \mid y^{16} - 16^x$$

and $y^{16} \equiv 0, 1 \pmod{17}$ by LFT, $16^x \equiv \pm 1 \pmod{17}$ (depending on parity) — hence x cannot be odd.

Problem 3. Source: Polish MO 2017 finals, P4

Can all positive integers be colored with 5 colors in such a way that in every 5-tuple $(n, 2n, 3n, 4n, 5n)$, $n \in \mathbb{Z}_+$ every color appears exactly once?

Hint 1. Try similar problems for numbers of colors smaller than 5.

Hint 2. The ansewer is yes.

Hint 3. Use the exponents in the prime factorization or some modulo.

Solution
(I) Color each n with $2v_2(n) + v_3(n) + 3v_5(n) \pmod 5$. It's easily checked that for each n the colors of n, $2n$, $3n$, $4n$, $5n$ are all different.
(II) Color each $n = 11^\alpha \cdot m$ $(11 \nmid m)$ with $m \pmod{11}$ and let the colors be $\{\pm 1\}$, $\{\pm 2\}$, $\{\pm 3\}$, $\{\pm 4\}$, $\{\pm 5\}$. Then no two numbers among n, $2n$, $3n$, $4n$, $5n$ will have the same color as it would mean that either their sum, or their difference, is divisible by 11.

2 Combinatorial main course

Problem 4. Source: I asked the guy from whom I know this and he does not remember

Consider an infinite board with square cells, each containing a light bulb, which is initially switched off. In one move one can change state (on/off) of all bulbs within a 3×3 or 4×4 square. Is it possible that after a finite number of moves there are only four light bulbs which are switched on, and they form a 2×2 square?

Hint 1. The answer is No.

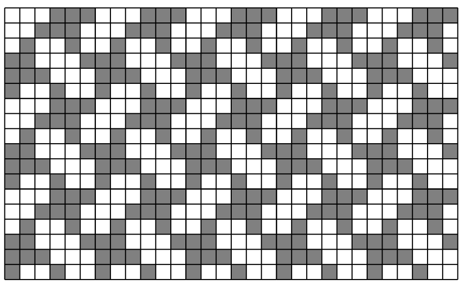
Hint 2. Try to color the plane such that every 3×3 and every 4×4 contains an even number of colored cells and some 2×2 contains an odd number of colored cells.

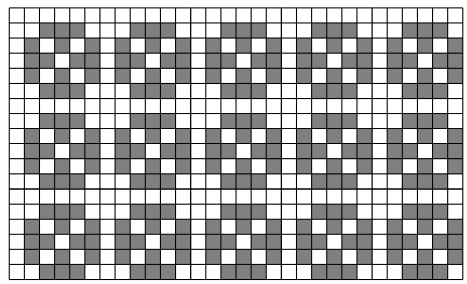
Hint 3. It's good to look for a periodic coloring. What should be the period?

Hint 4. Try to find a 6×6 periodic coloring (so with period 6 in both directions). Moreover, you can assume that the 6×6 base matrix is symmetric.

Solution

Suppose that a sequence of moves leaving some square 2×2 lit is possible and color the plane in either of the following two plane colorings (6×6-periodic) in such a way that the lit square has an odd number of gray cells.





Note that each 3×3 and each 4×4 square (in both colorings) have an even number of gray cells. Therefore every possible move keeps the parity of the number of light bulbs which are switched on and placed on gray cells. But initially this number was 0, so it cannot be changed to 1 or 3.

Problem 5. Source: Polish MO 2022 2nd round, P6

In a badminton tournament n players took part, where n is an odd positive integer. Each pair of participants played exactly *two* games with each other, and no game ended in a draw. It turned out that everyone won and lost equally many times. Prove that you can cancel precisely half of all the games and keep this property satisfied.

Hint 1. Consider the *draw* graph, i.e. the graph where you connect two participants if and only if the score between them is $1 : 1$.

Solution

We will show how to cancel exactly one game between any two players (note that this is a bit stronger than what the problem requires). There are two possible scores between two players: $2 : 0$ or $1 : 1$. In the first case, we simply cancel any of the two equivalent games. For the second case consider the graph whose vertices are all participants and with an edge present if and only if the score was $1 : 1$. The problem's conditions imply that every vertex has even degree. Therefore all edges of this graph can be decomposed into cycles (by applying Euler's theorem to every connected component of this graph). Now it is enough to traverse each of these cycles once and cancel all games in a fixed direction.

Problem 6. Source: Iran TST 2006, P6; Polish MO 2nd round 2017, P5; Sands-Sauer-Woodrow 1982 paper

Given is a tournament with $n \geq 2$ participants. Each two of them played against each other exactly once and there were no tie games. Each game was colored either red, or blue. Prove that there exists a *monochromatic master*, i.e. a participant M with the following property:

> for every player $A \neq M$ there exist players $A_1 = M$, A_2, …, $A_k = A$ such that A_i defeated A_{i+1} for all $i = 1, \ldots, k-1$ and all of these $k-1$ games have the same color.

Hint 1. Suppose for the sake of contradiction that there exists a tournament with n participants for which there is no monochromatic master and consider one with *the smallest* n (the so-called minimal counterexample).

Hint 2. Prove that for every participant A there is a *unique* participant $M(A)$ such that $M(A)$ becomes a monochromatic master after removing A from the graph.

Hint 3. Prove that the permutation M introduced in the previous hint is *cyclic*, i.e. for every participant A the elements

$$A,\ M(A),\ M(M(A)),\ M(M(M(A))),\ \ldots,\ \underbrace{M(M(\ldots M}_{n-1}(A)\ldots))$$

are all different.

Hint 4. Get the final contradiction assuming that the arcs $A \to M(A)$ and $M(A) \to M(M(A))$ have different colors.

Solution

Suppose for the sake of contradiction that there is a tournament T with no monochromatic master and take such tournament with the smallest possible number of participants n. It's easy to check that $n \geq 3$ since in the tournament with 2 players the winner is the master.

We know that every subtournament with $n-1$ players has at least one monochromatic master. Denote by $M(A)$ a monochromatic master of the tournament $T - A$ (all the games not invoving participant A); if there is more than one such master, choose $M(A)$ arbitrarily.

We will prove that if $A \neq B$, then $M(A) \neq M(B)$, i.e. that M is a bijection (because the set of participants is finite). Indeed, if $M(A) = M(B) = M$, then M has a monochromatic path to B in $T - A$ (because $M = M(A)$ and $A \neq B$) and M has a monochromatic path to all players different than B (because $M = M(B)$). But this means that M is a monochromatic master in T — contradiction.

Take an arbitrary participant A and iterate M until reaching A again, i.e. consider the cycle

$$(A, M(A), M(M(A)), \ldots)$$

if the permutation M containing A. If this cycle involves k participants, then they form a tournament in which no one is a monochromatic master (for all B there is no monochromatic path from $M(B)$ to B in T, so there can be no such path in any subtournament of T). This means, by the choice of n, than $k = n$ and M is a cyclic permutation.

Note that from the definition of M the direct game between A and $M(A)$ is won by A, denote this by $A \to M(A)$ (otherwise A would be reachable from $M(A)$ so $M(A)$ would be a master in T). Observe that not all of such games can have the same color, as M is a cyclic permutation (and going along this monochromatic cycle we can reach the preceding element, which gives a contradiction). Therefore there is some participant A such that $A \to M(A)$ is red and $M(A) \to M(M(A))$ is blue.

Note that A is reachable by a monochromatic path P from $M(M(A))$ (who is a master in $T - M(A)$). If P is red, then we can use P and $A \to M(A)$ to monochromatically reach $M(A)$ from $M(M(A))$ — contradiction. If P is blue, then we can use $M(A) \to M(M(A))$ and P to monochromatically reach A from $M(A)$ — contradiction.