Email training, N6

September 29-October 5

Problem 6.1. Find the biggest integer n such that $n^3 + 100$ is divisible by $n + 10$.

Solution 6.1.

$$\frac{n^3+100}{n+10} = \frac{n^3+10^3-900}{n+10} = n^2 - 10n + 100 - \frac{900}{n+10}.$$

Since $n^2 - 10n + 100$ is integer, then 900 must be divisible by $n + 10$. The biggest n satisfying to this condition is $n = 900 - 10 = 890$.

Answer: $n = 890$.

Problem 6.2. Find all integers x and y for which $(2x + y)(5x + 3y) = 7$.

Solution 6.2. First multiplier may have values $-1, 1, 7, -7$, and the second one $-7, 7, 1, -1$ respectively. By solving each of the systems we get 4 solutions.

Answer: $(4, -9), (-4, 9), (20, -33), (-20, 33)$.

Problem 6.3. Let a be an odd integer and m is such that $2^m | a + 1$ and $2^{m+1} \not| a + 1$. Prove that for any positive integer k one has

$$2^{k+m+1} | (2a+1)^{2^k} - 1 \quad \text{and} \quad 2^{k+m+2} \not| (2a+1)^{2^k} - 1$$

Solution 6.3. Let's prove the statement by induction. For $k = 1$ we have $(2a + 1)^2 - 1 = 4a(a + 1) \vdots 2^{m+2}$ and $4a(a + 1) \not\vdots 2^{m+3}$. Assume that the statement is proved for k and let's prove it for $k + 1$.

$$(2a+1)^{2^{k+1}} - 1 = \Big((2a+1)^{2^k} - 1\Big)\Big((2a+1)^{2^k} + 1\Big).$$

Note that that the expression in the right brackets is divisible by 2 and isn't divisible by 4. From this follows that

$$(2a+1)^{2^{k+1}} - 1 \vdots 2^{k+m+1} \cdot 2^1 = 2^{(k+1)+m+1}$$

and

$$(2a+1)^{2^{k+1}} - 1 \not\vdots 2^{k+m+1} \cdot 2^1 \cdot 2 = 2^{(k+1)+m+2}$$

Problem 6.4. Find the number of positive integers n letss than 10000, for which $2^n - n^2$ is divisible by 7.

Solution 6.4. Residues of division 2^n by 7 has period 3 $(2, 4, 1)$. Residues of division n^2 by 7 has period 7 $(1, 4, 2, 2, 4, 1, 0)$. Therefore the divisibility of $2^n - n^2$ depends only on the residue that we obtain when divide n by 7. By considering all possible cases we conclude that $2^n - n^2$ is divisible by 7 only when $n \equiv 2, 4, 5, 6, 10, 15[21]$.

We have $10000 = 476 \cdot 21 + 4$. Therefore we have $476 \cdot 6 + 2 = 2858$ numbers satisfying to the conditions of the problem.

Answer: 2858.

Problem 6.5. Find all positive integers n such that

$$3^{n-1} + 5^{n-1} | 3^n + 5^n.$$

Solution 6.5. Notice that $3^{n-1}+5^{n-1}$ also divides 5 times itself:

$$3^{n-1}+5^{n-1}|5\left(3^{n-1}+5^{n-1}\right)=3^n+2\cdot 3^{n-1}+5n,$$

Subtracting the given equation from the one given in the problem we get

$$3^{n-1}+5^{n-1}|3^n+2\cdot 3^{n-1}+5^n-(3^n+5^n)$$

which means

$$3^{n-1}+5^{n-1}|2\cdot 3^{n-1}.$$

However, for $n>1$, we have $3^{n-1}+5^{n-1}>2\cdot 3^{n-1}$, leading to the above divisibility being impossible. We then check that $n=1$ is the only possible solution.

Answer: $n=1$.

Problem 6.6. The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n=100+n^2$, where $n=1,2,3,\ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

Solution 6.6. Since $d_n=gcd(100+n^2, 100+(n+1)^2)$, then d_n must divide the difference between these two, or $d_n=(100+n^2, 2n+1)$. Since $2n+1$ will always be odd, 2 will never be a common factor, hence we can multiply n^2+100 by 4 without affecting the greatest common divisor:

$$d_n=gcd(4n^2+400, 2n+1)=gcd(401, 2n+1).$$

Therefore, in order to maximize the value of d_n, we set $n=200$ to give a greatest common divisor of 401.

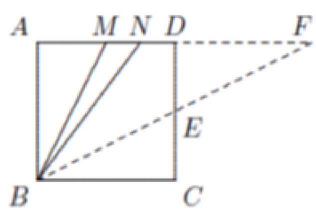
Answer: 401.

Problem 6.7. In square $ABCD$, M is the midpoint of AD and N is the midpoint of MD. Prove that $\angle NBC=2\angle ABM$.

Solution 6.7. -

Solution Let $AB=BC=CD=DA=a$. Let E be the midpoint of CD. Let the lines AD and BE intersect at F.

By symmetry, we have $DF=CB=a$. Since right triangles ABM and CBE are symmetric in the line BD, $\angle ABM=\angle CBE$. It suffices to show $\angle NBE=\angle EBC$, and for this we only need to show $\angle NBF=\angle BFN$ since $\angle DFE=\angle EBC$.



By assumption we have

$$AN=\frac{3}{4}a,\quad \therefore NB=\sqrt{(\frac{3}{4}a)^2+a^2}=\frac{5}{4}a.$$

On the other hand,

$$NF=\frac{1}{4}a+a=\frac{5}{4}a,$$

so $NF=BN$, hence $\angle NBF=\angle BFN$.

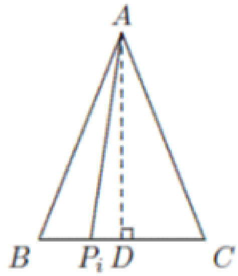
Problem 6.8. Let ABC is an isosceles triangle with $AB = AC = 2$. There are 100 points $P_1, P_2, \ldots, P_{100}$ on the side BC. Denote $m_i = AP_i^2 + BP_i \cdot CP_i$. Find the value of $m_1 + m_2 + \ldots + m_{100}$.

Solution 6.8. -

From A introduce $AD \perp BC$ at D. Then $BD = DC$. Let $BD = DC = x$ and $DP_i = x_i$.

By Pythagoras' Theorem, for $1 \leq i \leq 100$,

$$\begin{aligned} m_i &= AP_i^2 + BP_i \cdot P_iC \\ &= AP_i^2 + (x - x_i)(x + x_i) \\ &= AP_i^2 - x_i^2 + x^2 \\ &= AD^2 + x^2 = AB^2 = 4. \end{aligned}$$



Thus,

$$m_1 + m_2 + \cdots + m_{100} = 400.$$