Email training, N1

August 25 - September 1

Problem 2.1. Let $a \neq 0$ and let x_1 and x_2 are the roots of the equation

$$x^2 + ax - \frac{1}{2a^2} = 0.$$

Prove that

$$x_1^4 + x_2^4 \geq 2 + \sqrt{2}.$$

Solution 2.1. According to Viet theorem one has $x_1 + x_2 = -a$ and $x_1 x_2 = -\frac{1}{2a^2}$. Therefore one may write

$$x_1^4 + x_2^4 = (x_1^2 + x_2^2)^2 - 2x_1^2 x_2^2 =$$

$$((x_1 + x_2)^2 - 2x_1 x_2)^2 - \frac{1}{2a^4} =$$

$$\left(a^2 + \frac{1}{a^2}\right)^2 - \frac{1}{2a^4} = 2 + a^4 + \frac{1}{2a^4}.$$

To get the desired result it's enough to use the inequality $x + y \geq 2\sqrt{xy}$ and conclude that

$$a^4 + \frac{1}{2a^4} \geq 2\sqrt{a^4 \cdot \frac{1}{2a^4}} = \sqrt{2}$$

Problem 2.2. Prove that at least one coefficient of the polynomial

$$P(x) = (x^4 + x^3 - 3x^2 + x + 2)^n$$

is negative.

Solution 2.2. Note that $P(0)$ is the free term coefficient of the polynomial P and $P(1)$ is the total sum of coefficients of P. Since $P(0) = P(1) = 2^n$, then $P(1) - P(0) = 0$. It means that the sum of all coefficients but free term is equal 0. Since the leading coefficient is 1 it means some other coefficient must be negative.

Problem 2.3. Prove that $lcm(1, 2, 3, \ldots, 2n) = lcm(n+1, n+2, \ldots, 2n)$, where lcm is the least common multiplier.

Solution 2.3. At first it's obvious that

$$lcm(n+1, \ldots, 2n) \mid lcm(1, \ldots, 2n).$$

On the other side $d \mid lcm(n+1, \ldots, 2n)$ for all $1 \leq d \leq 2n$. Therefore one has

$$lcm(1, 2, \ldots, 2n) \mid lcm(n+1, \ldots, 2n).$$

From $a|b$ and $b|a$ follows that $a = b$.

Problem 2.4. Four positive integers are given. It is known that the sum of squares of any two of them is divisible by product of other two numbers ($cd|a^2 + b^2$). Prove that at least three numbers are equal.

Solution 2.4. Without loss of generality one may assume that $gcd(a, b, c, d) = 1$. Assume that one of a, b, c, d has prime divisor $p > 2$. Let $p|a$. Since

$$a|b^2 + c^2, a|b^2 + d^2, a|d^2 + c^2,$$

it follows that

$$a|(b^2 + c^2) + (b^2 + d^2) + (d^2 + c^2) = 2(b^2 + c^2 + d^2).$$

Since $p \neq 2$ it follows $p|b^2 + c^2 + d^2$. So one has

$$p|b^2 + c^2 + d^2, p|b^2 + c^2.$$

From this follows that $p|d^2$, ie $p|d$. In the same way one gets $p|b$, $p|c$, ie

$$p|gcd(a, b, c, d).$$

Contradiction. So $p = 2$ and $a = 1, b = 2^x, c = 2^y, d = 2^z$ with $0 \leq x \leq y \leq z$. Since

$$\frac{1 + 2^{2x}}{2^{y+z}} \in N,$$

it follows that either $y = z = 0$ or $x = 0$. In the first case one gets $a = b = c = d = 1$ in the second case one has $\frac{2}{2^{y+z}} \in N$ which means $y + z \leq 1$ ie $y = 0$. So $a = b = c = 1$.

Problem 2.5. The endpoints of N arcs split the circle into $2N$ equal arcs of length 1. It is known that each arc splits the circle into 2 parts of even length. Prove that N is even.

Solution 2.5. Let's paint the endpoints in clockwise order by switching black and white. Since arc lengths are even, then endpoints of arcs will have the same color. It means that N white endpoints are split into pairs. So N is even.

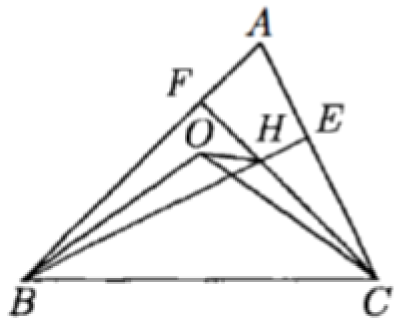
Problem 2.6. The robber's car speed is 90% of policeman's car speed. Robber and policeman are along the line and policeman doesn't know in which direction goes the robber. Prove that the policeman may catch the robber.

Solution 2.6. Imagine that the policeman has two assistants and they move with the speed equal 95% of the policeman car. At the beginning they stay with policeman and one of them goes left and one of them goes right. Then, let policeman goes left until meets the assistant. After that he turns back and goes right until meets another assitant and so on. It's obvious that the policeman will meet assistants infinitely many times and that one of the assistants will pass the robber.

Problem 2.7. Angle A of the acute-angled triangle ABC equals 60^o. Prove that the bisector of one of the angles formed by the altitudes drawn from B and C, passes through the circumcircle 's centre.

Solution 2.7. -

Let BE and CF be altitudes, intersecting at the orthocentre H. Let O be the circumcentre.

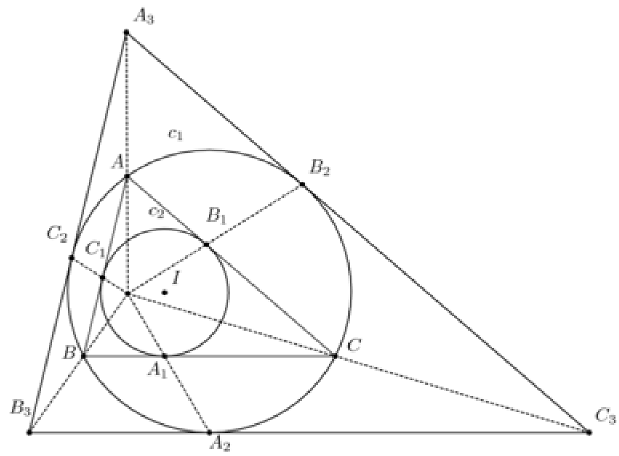


Then $\angle BOC = 2\angle BAC = 120°$. Also $\angle BHC = \angle FHE = 180° - \angle BAC = 180° - 60° = 120°$. Hence $BCHO$ is cyclic and $\angle OHB = \angle OCB = 30°$ (since ΔOBC is isoceles with $\angle BOC = 120°$). But

$\angle BHF = 180° - \angle BHC = 180° - 120° = 60°$. It follows that OH bisects $\angle BHF$.

Problem 2.8. The bisectors of the angles A, B, C of a triangle $\triangle ABC$ intersect with the circumcircle c_1 of triangle ABC at A_2, B_2, C_2 respectively. The tangents of c_1 at A_2, B_2, C_2 intersect each other at A_3, B_3, C_3 (the points A_3 and A lie on the same side of BC,the points B_3 and B on the same side of CA, also C_3 and C on the same side of AB).The incircle triangle ABC is tangent to BC, CA, AB at A_1, B_1, C_1 respectively. Prove that A_1A_2, B_1B_2, C_1C_2, AA_3, BB_3 and CC_3 are concurrent.

Solution 2.8. -



Since A_2 is the midpoint of arc BC, we see that $A_3B_3C_3$ and ABC have corresponding parallel sides, thus they are homothetic from some center, say P, such that AA_3, BB_3, and CC_3 concur at P. Note that A_1 and A_2 are the incircle contact points of the corresponding sides $BC$and B_3C_3 on $\triangle ABC$ and $\triangle A_3B_3C_3$, respectively, so A_1 gets taken to A_2 under this homothety; thus P, A_1, A_2 are collinear. Similarly, we get that A_1A_2, B_1B_2, C_1C_2 all concur at P as well. This completes the proof. $\square$