January Camp - 2020

Number theory L4 Vieta Jumping

Problems

1. Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Prove that $\frac{a^2+b^2}{ab+1}$ is a perfect square.

2. Let a and b be positive integers such that ab divides a^2+b^2+1. Prove that
$$3ab = a^2+b^2+1.$$

3. Prove that for every real number N, equation
$$x_1^2+x_2^2+x_3^2+x_4^2 = x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4$$
has a solution for which x_1, x_2, x_3, x_4 are all integers larger than N.

4. Let a and b be positive integers such that $4ab-1 \mid (4a^2-1)^2$. Show that $a=b$.

5. Let a and b be odd positive integers such that $2ab+1 \mid a^2+b^2+1$. Show that $a=b$.

6. Find all pairs of positive integers (m,n) such that $mn-1$ divides $(n^2 - n+1)^2$.

Homework

1. Suppose a, b, and c are positive integers such that $0 < a^2+b^2-abc < c$. Show that a^2+b^2-abc is a perfect square.

2. If a and b are positive integers such that $\frac{a^2+b^2}{ab-1} = k$ is an integer, then $k=5$.

3. Let a and b be distinct positive integers such that $2ab+1 \mid a^2+b^2+1$. Show that $2ab+1$ is a perfect square.