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L3 — Sequences 2

1. (**Test**) Sequence x_1, x_2, $x_3, \dots$ of real numbers is given by $x_1 = 1$ and $x_{n+1} = x_n^2 + x_n$ for $n \geqslant 1$. Prove that for all $n \geqslant 1$ holds

$$\sum_{i=1}^{n} \frac{1}{1+x_i} + \prod_{i=1}^{n} \frac{1}{1+x_i} = 1.$$

2. The sequence (a_n) is defined by $a_1 = 0$,

$$a_{n+1} = \frac{a_1 + a_2 + \ldots + a_n}{n} + 1.$$

Prove that $a_{2019} > \frac{1}{2} + a_{1000}$.

3. Given $x_1, x_2, ..., x_n$ real numbers, prove that there exists a real number y, such that,

$$\{y - x_1\} + \{y - x_2\} + ... + \{y - x_n\} \leqslant \frac{n-1}{2}.$$

4. Determine all positive reals α for which exists sequence of positive reals (x_i) satisfying

$$x_{n+2} = \sqrt{\alpha x_{n+1} - x_n}, \quad n = 1, 2, 3, \dots.$$

5. Sequence a_0, a_1, a_2, $\dots$ satisfies $a_0 = 2018$ and

$$a_{n+1} = \frac{a_n^2}{a_n + 1} \text{ for } n = 0, 1, 2, \dots.$$

Prove that for $n \in \{0, 1, 2, \dots, 1010\}$ holds $\lfloor a_n \rfloor = 2018 - n$.

6. Let $c \geqslant 1$ be integer. Sequence a_1, $a_2, \dots$ is given by $a_1 = c$ and

$$a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-1)} \text{ for } n \geqslant 1.$$

Prove that each term of the sequence is integer.

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7. (**Bonus**) Given positive real numbers a, b, c, d that satisfy equalities

$$a^2 + d^2 - ad = b^2 + c^2 + bc \quad \text{and} \quad a^2 + b^2 = c^2 + d^2,$$

find all possible values of the expression $\dfrac{ab+cd}{ad+bc}$.