Test 2

Level 4, December 1

Problem 2.1. Let a_1, a_2, ..., a_n are non-zero integers such that

$$a_1a_2\dots a_n\Big(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\dots+\frac{1}{a_n^2}\Big)$$

is an integer. Prove that $a_1a_2\dots a_n \vdots a_k^2$ for all $k=1,2,\dots,n$.

Problem 2.2. For any positive integer a,b,c prove that there is a non-negative integer k such that

$$gcd(a^k+bc,b^k+ac,c^k+ab)>1$$

Problem 2.3. Consider an acute-angled triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of the triangle CEF and Γ meet at X and C, with $X \neq C$. The line BX and the tangent to Γ through A meet at Y. Let P be the point on segment AB so that $YP = YA$, with $P \neq A$, and let Q be the point where AB and the parallel to BC through Y meet each other. Show that F is the midpoint of PQ.

Problem 2.4. There are $n > 2022$ cities in the country. Some pairs of cities are connected with straight two-ways airlines. Call the set of the cities *unlucky*, if it is impossible to color the airlines between them in two colors without monochromatic triangle (i.e. three cities A, B, C with the airlines AB, AC and BC of the same color).

The set containing all the cities is unlucky. Is there always an unlucky set containing exactly 2022 cities?