Winter Camp 2021

Algebra Level 2

Day 6 (4)
5 January 2021

Problem 1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y) \qquad \forall x, y \in \mathbb{R}$$

$x=0$: $f(f(y)f(-y)) = -yf(y)$ (1)

$y \to -y$: $f(f(-y)f(y)) = yf(-y)$

$\to f$ odd

$y=0$: $f(f(x)^2) = x^2$

$\forall x \geq 0 \ \exists z : f(z) = x$

(1) $\Rightarrow$ $f(f(y)^2) = yf(y)$

$\to x^2 = xf(x)$

$\to \boxed{f(x) = x \quad \forall x \in \mathbb{R}}$

$y=0$: $f(f(x)^2) = x^2$

$x=0$: $f(f(0)^2) = 0$ $\qquad c = f(0)^2 \qquad f(c) = 0$

$x=c, y=0$: $f(0) = c^2$

$x=2c, y=c$: $f(0) = 4c^2$

$\to c = 0 \to f(0) = 0$

$x=y$: $0 = x^2 - xf(x) \to f(x) = x \quad \forall x \in \mathbb{R}$

$x^2+y = f(x)-y$

$y = \frac{f(x)-x^2}{2}$

$0 = f(x)[f(x)-x^2]$

$\rightarrow f(x)=0$ or $f(x)=x^2$

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2+y) = f(f(x)-y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}$$

$\cancel{x^4+y^2+2x^2y = x^4}$

$f(x^2+y) = f(f(x)-y) + 4f(x)y$, $\boxed{y=0} \rightarrow f(x^2) = f(f(x))$

$y=f(x) \rightarrow f(x^2+f(x)) = f(0)+4f(x)^2$

$y=-x^2 \rightarrow f(0) = f(f(x)+x^2) + 4 \cdot f(x)(-x^2)$

$\rightarrow f(0) = f(x^2+f(x)) - 4f(x)^2 = f(f(x)+x^2) - 4f(x)x^2$

$\rightarrow -4f(x)^2 = -4f(x)x^2 \rightarrow f(x)^2 = f(x)x^2$

$\rightarrow f(x)=0$ or $f(x)=x^2$

$f(a)=0,\ f(b)=b^2 \qquad a\neq b\neq 0$

$x=a,\ y=b$: $f(a^2+b)=f(-b)+4f(a)b=f(-b)=f(b)=b^2$

$x=0$: $f(y)=f(-y) \rightarrow f$ even

$\Rightarrow (a^2+b)^2=b^2 \rightarrow \boxed{a^2=-2b}$ (*) $\quad b<0$

- $f(c)=c^2 \quad c\neq 0,\ c\neq b \quad \Rightarrow a^2=-2c \rightarrow b=c \ \Rightarrow\Leftarrow$
- $f(c)=0,\ c\neq a,\ c\neq 0 \quad \Rightarrow c^2=-2b \Rightarrow c^2=a^2 \Rightarrow c=-a$

$A=\{x \mid f(x)=0,\ x\neq 0\}$ $\qquad A\cup B=\mathbb{R}$

$B=\{x \mid f(x)=x^2,\ x\neq 0\}$ $\qquad A\cap B=\emptyset$

$|A|,|B|\geq 1$: if $|B|\geq 2 \ \Rightarrow\Leftarrow$

if $|A|\geq 2$: $\Rightarrow |A|=2$

(*) $\Rightarrow |B|=1 \ \Rightarrow\Leftarrow$

$\Rightarrow f(x)=0\ \forall x\in\mathbb{R}$ or $f(x)=x^2\ \forall x\in\mathbb{R}$

Problem 3. (BMO 2019) Let $\mathbb{P}$ be the set of all prime numbers. Find all the functions $f : \mathbb{P} \to \mathbb{P}$ such that:

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q \qquad \forall p, q \in \mathbb{P}.$$

- $f(2) = 2$
- f injective
- f increasing

$A(x) = 2^{x+1} - (x+1)^2 - (2^x + x^2) > 0$

$A(3) > 0$

$A(x) = 2^x - 2x - 1$

$A(x) - A(x-1) = 2^{x-1} > 0$

$A(x) > A(3) > 0$

$\to 2^{x+1} - (x+1)^2 > 2^x - x^2$

$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$

$p = 2,\ q = 3$

$f(2)^{f(3)} + 1 = f(3)^{f(2)}$

if $f(2), f(3)$ are odd primes $\Rightarrow$ L.H.S $\equiv 0 [2]$, R.H.S $\equiv 1 [2]$ $\Rightarrow$ c.c

$\Rightarrow f(2) = 2$ or $f(3) = 2$

$\Rightarrow f(2)^2 + 1 = 2^{f(2)}$, but $2^x > x^2 + 1$ for $x \geq 5$

$32 > 25 + 1 = 26$, $2^{x+1} > (x+1)^2 + 1 \iff 2(x^2+1) > x^2 + 2x + 2 \iff$

$x^2 > 2x \iff x > 2$ ✓

$\Rightarrow f(2) = 2$ or 3 but $2^2 + 1 > 2^2$, $3^2 + 1 > 2^3 \Rightarrow$ c.c

$\Rightarrow f(2) = 2 \Rightarrow f(3) = 3$

$p = 2,\ q = x$

$g(x) = 2^x - x^2$

$\Rightarrow 2^{f(q)} + q^2 = f(q)^2 + 2^q \Rightarrow 2^{f(q)} - f(q)^2 = 2^q - q^2$

$g(f(q)) = g(q)$

$2^{x+1} - (x+1)^2 > 2^x - x^2$, $x \geq 3$

$\iff 2^4 - 4^2 > 2^3 - 3^2 \iff 0 > -1$

$\iff 2^x \overset{?}{>} 2x + 1$ which is clearly true for $x \geq 3$.

$\Rightarrow f(q) = q$.

Problem 4. (Japan 2013) Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that

$$f(m) + f(n) = f(mn) + f(m + n + mn) \qquad \forall m, n \in \mathbb{Z}$$

$f(2a) = f(2),\ f(2b+1) = f(1)$

$n=1$: $f(m) + f(1) = f(m) + f(2m+1) \longrightarrow f(2m+1) = f(1)$

$m = 2a$, $n = 2b+1$: $f(2a) + f(1) = f(2a(2b+1)) + f(1)$

$f(2a) = f(2a(2b+1)) \longrightarrow f(2^\alpha b) = f(2^\alpha)$, $2 \nmid b$

$k \geq 2$, $m = 2^k$, $n = 2$: $f(2^k) + f(2) = f(2^{k+1}) + \underbrace{f(2^k + 2 + 2^{k+1})}_{= f(2)}$

$\longrightarrow f(2^k) = f(2^{k+1}) \quad \forall k \geq 2$

$f(2) \overset{?}{=} f(4) = f(8)$

$m=n=2$: $2f(2) = f(4)+f(8) = 2f(4)$

$\rightarrow f(2) = f(4)$

$$\Rightarrow \begin{cases} f(2a) = f(2) & \forall a \in \mathbb{Z} \\ f(2a+1) = f(1) \end{cases}$$

$f: \mathbb{R} \to \mathbb{R}$

- $f(x) \equiv 0$
- $f(x) = x^2 + c$

Problem 5. (BMO 2007) Find all real functions f defined on $\mathbb{R}$ such that

$$f(f(x)+y) = f(f(x)-y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}$$

$y = f(x)$: $f(2f(x)) = f(0) + 4f(x)^2$

$y \to -y$: $f(f(x)-y) = f(f(x)+y) - 4f(x)y$

if $\exists c : f(c) = 0$: $x = c$: $f(y) = f(-y)$

$\exists \ell : f(\ell) \neq 0$

$x = \ell$: $f(f(\ell)+y) = f(f(\ell)-y) + 4f(\ell)y$

$y = \frac{t}{4f(\ell)}$: $f\Big(\underbrace{f(\ell) + \frac{t}{4f(\ell)}}_{a}\Big) = f\Big(\underbrace{f(\ell) - \frac{t}{4f(\ell)}}_{b}\Big) + t$

t يأخذ أي عدد حقيقي

$f(a) - f(b) = t$

$$\boxed{\forall t \ \exists a, b : t = f(a) - f(b)}$$

$$y \to f(y): \boxed{f(f(x)+f(y)) = f(\underbrace{f(x)-f(y)}_{t \in \mathbb{R}}) + 4f(x)f(y)}$$

$$y \leftrightarrow x \Rightarrow f(\underbrace{f(x)-f(y)}_{t}) = f(\underbrace{f(y)-f(x)}_{-t})$$

$$f(t) = f(-t) \to f \text{ even}$$

$$\boxed{y \to f(y)+t}: f(f(x)+f(y)+t) = f(f(x)-f(y)-t) + 4f(x)[f(y)+t]$$

$$x \leftrightarrow y: f(f(y)+f(x)+t) = f(f(y)-f(x)-t) + 4f(y)[f(x)+t]$$

$$\Rightarrow -f(f(x)-f(y)-t) + f(f(y)-f(x)-t) = 4t\,(f(x)-f(y))$$

$$f(x)-f(y) = d \in \mathbb{R}$$

لأن كل الأعداد الحقيقية

$$\forall d, t \in \mathbb{R}$$

$$-f(d-t) + f(d+t) = 4td$$

$$d=t: -f(0) + f(2d) = 4d^2$$

$$\to f(2d) = 4d^2 + f(0) = (2d)^2 + f(0)$$

$$\Rightarrow \boxed{f(x) = x^2 + f(0)}$$

How many ways to represent $21 = a_1 + a_2 + \ldots + a_k$ such that $a_1 = 1, a_{i+1} \in 1,2,3,4$ and $a_{i+1} = a_i \pm 1 \pmod 4$ for $i = 1,2,..,k-1$?
(Note that k varies on each representation).
Alternative statement: We put four numbers 1, 2, 3, 4 around a circle in order. One starts at the number 1 and every step, he moves to an adjacent number on either side. How many ways he can move such that sum of the numbers he visits in his path (including the starting number) is equal to 21?

كم عدد طرق لتمثيل $21 = a_1 + a_2 + \ldots + a_k$ بحيث $a_1 = 1, a_{i+1} \in 1,2,3,4$ ،
$a_{i+1} = a_i \pm 1 (\text{mod} 4)$ لكل $i = 1,2,..,k-1$ ؟ (لاحظ k تتغير في كل تمثيل).
بعبارة أخرى: وضعنا الأعداد الأربعة $1,2,3,4$ حول دائرة بهذا الترتيب. يبدأ أحدهم من العدد 1 وفي كل خطوة ، يتحرك لعدد مجاور في أحد الإتجاهين. كم عدد الطرق التي يمكنه أن يتحرك بها بحيث مجموع الأعداد التي يمر عليها في مساره إبتداءاً من العدد الأول يساوي 21 ؟

a_{2x+1} فردي
a_{2x} زوجي
$\Big\} \to k \equiv 1,2 \,[4]$

بأصغر قيم

$1,2,1,2,\ldots$

$k \le 14 \Leftarrow$

بأكبر قيم

مجموع = 22 $\quad 1,4,3,4,3,4,3$

$7 \le k \Leftarrow$

$\Rightarrow k \in \{9,10,13,14\}$

if $k=9$: الحالة 1

$$1 \;\overset{4}{\underset{ز}{2}}\; \underset{ف}{3} \;\overset{4}{\underset{ز}{2}}\; \underset{ف}{3} \;\overset{4}{\underset{ز}{2}}\; \underset{ف}{3} \;\overset{4}{\underset{ز}{2}}\; \underset{ف}{3} = 21$$

$4 \cdot 3 + 1 = 13 \quad + 2 \cdot 4 = 21$

$\boxed{1}$

② $1 _ 3 _ 3 _ 3 _ 1 = 21$

$3 \cdot 3 + 2 = 11$ $\quad [2,2,2,4]$

$\binom{4}{1} \cdot \binom{4}{1} = 16$

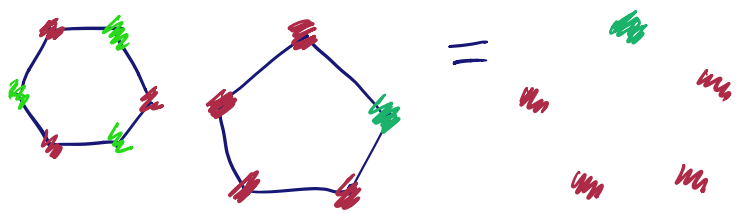
③ $1 _ 3 _ 3 _ 1 _ 1 = 21$

$[2,2,4,4]$

$\binom{4}{2} \cdot \binom{4}{2} = 36$

$\underline{144}$

Problem 4. How many ways are there to color the vertices of a regular pentagon with 4 colors $1, 2, 3, 4$. (The colorings must be different so that we can't get one from the other by a rotation.)



Prime

مضلع p منتظم

n لون

$\boxed{\frac{n^p - n}{p}} + n \in \mathbb{Z}$

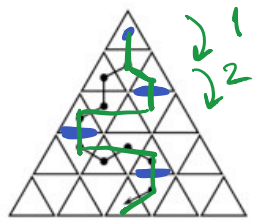
4^5 الحالات مع التكرار

$\frac{4^5 - 4}{5} + 4 = 208$

$\Rightarrow n^p \equiv n [p]$

فيرما

4. (Canada 2005) Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$. $= 2004!$



$(n-1)!$

4	Let n be an odd integer greater than 1 and let $c_1, c_2, \ldots, c_n$ be integers. For each permutation $a = (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, define $S(a) = \sum_{i=1}^{n} c_i a_i$. Prove that there exist permutations $a \neq b$ of $\{1, 2, \ldots, n\}$ such that $n!$ is a divisor of $S(a) - S(b)$.