— Geometry for L4 —

— November 28, 2021 — Isogonal Conjugates —

12. Given is triangle ABC with $AC = BC$. Points P, Q lie inside this triangle with

$$\angle PAC = \angle ABQ \quad \text{and} \quad \angle PBC = \angle BAQ.$$

Prove that C, P, Q are collinear.

13. Let $ABCD$ be a convex quadrilateral whose diagonals intersect at E. Points P, Q lie outside $ABCD$ and satisfy

$$\angle PAB = \angle CAD,\ \angle PBA = \angle CBD,\ \angle QCD = \angle ACB,\ \angle QDC = \angle ADB.$$

Prove that P, E, Q are collinear.

14. Given is triangle ABC with $AC = BC$ and M — midpoint of side AB. Point P lies inside ABC and satisfies $\angle PAB = \angle PBC$. Prove that $\angle APC + \angle BPM = 180°$.

15. The incircle of triangle ABC is tangent to the side BC at point D. The line perpendicular to AD passing through D intersects angle bisectors of ABC, ACB at points P, Q, respectively. Prove that $PD = DQ$.

16. Let P be a point inside an acute triangle ABC. Lines AP, BP, CP intersect the circumcircle of triangle ABC again at points D, E, F, respectively. Points K, L, M are symmetric to D, E, F with respect to lines BC, CA, AB, respectively. Prove that the circumcircle of KLM passes through the orthocenter of ABC.

17. Given is an acute triangle ABC. Points X, Y, Z lie on BC, CA, AB, respectively, such that XYZ is an equilateral triangle of smallest possible area inscribed into ABC. Prove that perpendiculars to YZ, ZX, XY passing through A, B, C, respectively, are concurrent.

18. Triangle ABC is inscribed in circle ω. Point P is the midpoint of the arc BC of ω containing A. Circle with diameter CP intersects the bisector of angle BAC at points K and L (points A, K, L lie in that order on this line). Moreover, M is symmetric to L with respect to line BC. Prove that the circumcircle of BKM passes through the midpoiont of segment BC.

19. Let ABC be an acute triangle with $AB < AC$, and let D and E be points on side BC such that $BD = CE$ and D lies between B and E. Suppose there exists a point P inside ABC such that $PD \parallel AE$ and $\angle PAB = \angle EAC$. Prove that $\angle PBA = \angle PCA$.

20. **(P5/IMO2004)** In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. Point P lies inside $ABCD$ with $\angle PCB = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.