— Algebra (+ some NT) for L3 —

— June Camp, 2022 — Łukasz Bożyk —

Problem 1.

We know that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

But what about coefficients of polynomials of the form

$$P_n(x) := x(x+1)(x+2)\dots(x+n-1)?$$

Denote by $\left[{n \atop k}\right]$ the coefficient by x^k in $P_n(x)$, e.g.

$$P_4(x) = x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x,$$

so $\left[{4 \atop 2}\right] = 11$. Prove that if $1 \le k \le n$, then

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}.$$

Solution: We have $P_{n+1}(x) = (x+n)P_n(x)$ by definition, so the coefficient of x^k is the same on both sides.

Problem 2.

Express as products:

(A) $x^4 + x^2 + 1$

(B) $x^5 + x^4 + 1$

(C) $x^3 + y^3 + z^3 - 3xyz$

(D) $(x+y+z)^3 - x^3 - y^3 - z^3$

(E) $(x+1)^4 + x^4 + 1$

(F) $(x+y+z)^5 - x^5 - y^5 - z^5$

Solution:

(A) $x^4 + 2x^2 + 1 - x^2 = (x^2+1)^2 - x^2 = (x^2-x+1)(x^2+x+1)$

(B) $x^5 - x^2 + x^4 + x^2 + 1 = x^2(x-1)(x^2+x+1) + (x^2-x+1)(x^2+x+1) = (x^3-x+1)(x^2+x+1)$

(C) $x^3 + y^3 + z^3 - 3xyz = (x+y)^3 + z^3 - 3xy(x+y+z) = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

(D) $(x+y+z)^3 - x^3 - y^3 - z^3 = 3(x+y)(y+z)(z+x)$

(E) $(x+1)^4 + x^4 + 1 = 2(x^4 + 2x^3 + 3x^2 + 2x + 1) = 2(x^2+x+1)^2$

(F) $(x+y+z)^5 - x^5 - y^5 - z^5 = 5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$

Problem 3.

Initially, number 1 is written on a board. If there is a on the board, then you can write on the board any $b \geq 1$ such that $a+b+1 \mid a^2+b^2+1$. For example in your first move you can write 4 on the board because $1+4+1 \mid 1^2+4^2+1$, and in your second move you can write 37 on the board because $4+37+1 \mid 4^2+37^2+1$ (and 4 is already written). Can every positive integer eventually appear on the board?

Solution: If n^2 is on the board, we can write n, because

$$n^4+n^2+1=(n^2+n+1)(n^2-n+1).$$

If k^2 is on the board, we can write $(k+1)^2$, because

$$k^4+(k+1)^4+1=2(k^2+k+1)^2=((k+1)^2+k^2+1)(k^2+k+1).$$

For every $n \geq 2$ using the second observation $n-1$ times (for $k=1,2,3,\ldots,n-1$), and then the first observation, we see that n can appear on the board.

Bonus problem: fix $c \in \mathbb{Z}_+$ and change the divisibility condition to $a+b+c \mid a^2+b^2+c$.

Solution: It is enough to prove that for an appropriately defined function $f_c\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ we have edges $\{n, f_c(n)\}$ and $\{f_c(n), f_c(n+1)\}$ for every n. The most natural candidate (following directly from the divisibility assumption) is

$$f_c(n)=n^2+n(c-1)+\frac{1}{2}c(c-1),$$

and it turns out to work: $f_c(n)+f_c(n+1)+c=n^2+(n+c)^2+c$ and

$$f_c(n)^2-n^2+f_c(n+1)^2-(n+c)^2=(f_c(n)+n)(n^2+(n+c)^2+c).$$

Problem 4.

(A) Do there exist positive real numbers a, b, c, x such that

$$a^2+b^2=c^2 \quad \text{and} \quad (a+x)^2+(b+x)^2=(c+x)^2?$$

(B) Suppose that positive integers a, b, c satisfy $a^2+b^2=c^2$. Prove that $\frac{1}{2}(c-a)(c-b)$ is a square of an integer.

Solution: (A) We have $x^2=2x(a+b-c)$, so $x=2(c-a-b)$. But $a+b>c$ because a, b, c are side lengths of a (right) triangle. Therefore $x<0$, contradiction.

(B) Note that

$$\left(\frac{a+b-c}{2}\right)^2=\frac{c^2-ac-bc+ab}{2}=\frac{1}{2}(c-a)(c-b)$$

and the number $\frac{a+b-c}{2}$ is an integer because $a+b$ is of the same parity as c.

Problem 5.

Everywhere below if something appears in denominator, we assume it is nonzero. Note that we don't assume that the numbers are positive.

(A) Suppose that $a^3+b^3+1=3ab$. Find all possible values of $a+b$.

(B) Suppose that $(a+\sqrt{a^2+1})(b+\sqrt{b^2+1})=1$. Find all possible values of $a+b$.

(C) Suppose that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}$, where $a \neq b \neq c \neq a$. Find all possible values of abc.

(D) Suppose that $x+y+z=0$ and $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{x}{z}+\frac{z}{y}+\frac{y}{x}+1$. Find all possible values of both sides of this equality.

(E) Suppose that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1$. Find all possible values of $\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}$.

(F) Suppose that $x^2 - yz = y^2 - zx = z^2 - xy = 2$. Find all possible values of $xy + yz + zx$.

(G) Suppose that $x^2 - y = y^2 - z = z^2 - x$ and $x \neq y \neq z \neq x$. Find all possible values of $(x+y)(y+z)(z+x)$.

(H) Suppose that $a^2 + a = b^2$, $b^2 + b = c^2$, $c^2 + c = a^2$. Find all possible values of $(a-b)(b-c)(c-a)$.

Solution: (A) We have

$$a^3 + b^3 + 1 - 3ab = (a + b + 1)(a^2 + b^2 + 1 - a - b - ab),$$

so either $a + b = -1$, or $\frac{1}{2}((a-1)^2 + (b-1)^2 + (a-b)^2) = 0$, which means that $a = b = 1$ and $a + b = 2$.

(B) Multiplying both sides by $\sqrt{a^2+1} - a$, we get

$$b + \sqrt{b^2+1} = \sqrt{a^2+1} - a, \quad \text{so} \quad a + b = \sqrt{a^2+1} - \sqrt{b^2+1}.$$

Analogously multiplying the initial equation by $\sqrt{b^2+1} - b$, we get

$$a + b = \sqrt{b^2+1} - \sqrt{a^2+1}.$$

Therefore $a + b = 0$ (any pair $(a, -a)$ satisfies the given equation).

(C) We have $a - b = \frac{1}{c} - \frac{1}{b} = \frac{b-c}{bc}$ and two symmetric equalities. Multiplying them together, we get

$$(a-b)(b-c)(c-a) = \frac{(a-b)(b-c)(c-a)}{(abc)^2} \quad \text{so} \quad abc = \pm 1.$$

Now it is enough to see that both values are possible, take $(a, b, c) = (\pm 1, \mp\frac{1}{2}, \mp 2)$.

(D) Denote the common value of the two sides of the given equality as S. Then

$$2S = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{z}{y} + \frac{y}{x} + 1 = \frac{-x}{x} + \frac{-y}{y} + \frac{-z}{z} + 1 = -2,$$

so $S = -1$. But how to prove that a triple x, y, z exists? Let us try to argue that such a triple with $x = 1$ exists. This is equivalent to proving that there is a nonzero y such that

$$\frac{1}{y} + \frac{y}{-1-y} + \frac{-1-y}{1} = \frac{1}{-1-y} + \frac{-1-y}{y} + \frac{y}{1} + 1.$$

But this reduces to a cubic equation which is readily seen to have a (nonzero) real root.

(E) Multiply the given equation by $a + b + c$. Then we'll get that the result is 0. But how to prove that a triple a, b, c exists? Let us argue that such a triple with $c = -1$ exists. It reduces to proving that one can choose some real a, b such that $a + b \neq 0$, $a \neq 1 \neq b$ and

$$\frac{a}{b-1} + \frac{b}{a-1} - \frac{1}{a+b} = 1.$$

But this reduces to $a^3 + b^3 = ab + 1$, so it is enough to take e.g. $b = 2$ and a to be the solution of $x^3 + 7 = 2x$ (it is easily seen that neither 1, nor -2 is a solution; and this is a cubic, so a real solution exists).

(F) By $x^2 - yz = y^2 - zx$, we get $(x-y)(x+y+z) = 0$, so $x = y$ or $x + y + z = 0$ and similarly for other equalities. So if $x + y + z \neq 0$, then $x = y = z$, which is a contradiction: $x^2 - yz = 0 \neq 2$. Therefore $x + y + z = 0$ and

$$0 = (x+y+z)^2 = (x^2 - yz) + (y^2 - zx) + (z^2 - xy) + 3(xy + yz + zx) = 6 + 3(xy + yz + zx),$$

which means that $xy + yz + zx = -2$. This is in fact achievable, e.g. $(x, y, z) = (0, \sqrt{2}, -\sqrt{2})$.

(G) Write $(x-y)(x+y) = y - z$ and two similar equalities and multiply them together to get

$$(x-y)(y-z)(z-x)(x+y)(y+z)(z+x) = (x-y)(y-z)(z-x).$$

Because $x \neq y \neq z \neq x$, we obtain $(x+y)(y+z)(z+x) = 1$. But how to prove that a triple x, y, z exists? Let us try to prove there exist one with $x = 1$, i.e. there are y, z such $1 - y = y^2 - z = z^2 - 1$ and $y \neq 1 \neq z \neq y$. Plugging $y = 2 - z^2$ to $y^2 = z^2 + z - 1$, we get

$$z^4 - 5z^2 - z + 5 = 0 \iff (z-1)(z^3 - z^2 - 4z - 5) = 0.$$

The cubic factor is readily seen to have its (at least one) real root different than ± 1 and -2, so in particular $y = 2 - z^2 \neq z$ and $y \neq 1$.

(H) If one number is zero, then all of them are and the desired expression is 0 as well. Suppose that $abc \neq 0$. Adding up the three equations, we get $a + b + c = 0$. Therefore

$$a = b^2 - a^2 = (b-a)(b+a) = -c(b-a),$$

and similarly $c = -b(a-c)$, $b = -a(c-b)$. Multiplying up the three obtained equations and using $abc \neq 0$, we get that $(a-b)(b-c)(c-a) = 1$. But how to prove that a triple a, b, c exists? Plugging $c = -a - b$ in the initial conditions, we get the system $a^2 + a = b^2$, $b = 2ab + a^2$, $2ab + b^2 = a + b$ (any two of them imply the third one). In particular $b(1-2a) = a^2$ and so

$$(1-2a)^2(a^2+a) = b^2(1-2a)^2 = a^4 \implies a^3 = (a+1)(1-2a)^2.$$

This is a cubic (again), so it has at least one real root and the solution can be produced based on it. Fun fact: the (only, up to cyclic permutation of variables) solution of the system has its geometric interpretation. If in a regular 9-gon $A_1A_2\ldots A_9$ we have $A_1A_4 = 1$, $A_1A_2 = a$, $A_1A_3 = b$, and $A_1A_5 = -c$, then a, b, c satisfy given system of equations (which can be directly seen by applying Ptolemy's theorem to three different trapezoids within this polygon).

Problem 6.

Positive integers a, b satisfy $\dfrac{a}{b} > \sqrt{2}$. Prove that $\dfrac{a}{b} - \dfrac{1}{2ab} > \sqrt{2}$.

Solution: [I] We have $a^2 > 2b^2$, so $a^2 \geq 2b^2 + 1$ because a^2, b^2 are integers. Therefore $(a - b\sqrt{2})(a + b\sqrt{2}) \geq 1$, so

$$\frac{a}{b} - \sqrt{2} = \frac{a - b\sqrt{2}}{b} \geq \frac{1}{b(a + b\sqrt{2})} > \frac{1}{2ab},$$

where the last inequality follows from $b\sqrt{2} < a$.

[II] The desired inequality is equivalent to (by squaring both sides)

$$\frac{a^2-1}{b} + \frac{1}{4a^2b^2} > 2.$$

This follows from $a^2 - 1 \geq 2b^2$ and $\frac{1}{4a^2b^2} > 0$.

[III] Suppose for the sake of contradiction that $a - \frac{1}{2a} < b\sqrt{2}$ (the equality cannot hold since the left-hand side is rational whereas the right-hand side is not). After joining this with $b\sqrt{2} < a$ and squaring all three expressions, we get

$$a^2 - 1 + \frac{1}{4a^2} < 2b^2 < a^2,$$

which is a contradiction since an integer $2b^2$ is sandwiched between two consecutive integers $a^2 - 1$ and a^2.

Warm-up. Given numbers s and p, do there exist real numbers with sum s and product p? If so, how to find them?

We consider a quadratic equation $x^2 - sx + p = 0$. If it has two (possibly equal) solutions x_1, x_2, then they satisfy $x_1 + x_2 = s$ and $x_1x_2 = p$. On the other hand, any pair x_1, x_2 of sum s and product p has to comprise of two roots of this quadratic equation. So we know that such (real) numbers exist if and only if $s^2 \geq 4p$ (that is: if the quadratic above has real solutions).

Problem 7.

Let $x, y \in \mathbb{R}$ be such that $x = y(3-y)^2$ and $y = x(3-x)^2$. Find all possible values of $x + y$.

Solution: If $x = 0$, then $y = 0$ and vice versa (and $x + y = 0$). Further assume that $xy \neq 0$. If $x = y \neq 0$, then $(3-x)^2 = 1$, so $(x, y) = (2, 2)$ or $(x, y) = (4, 4)$. This gives sums $x + y$: 4 or 8. Further assume that $x - y \neq 0$.

Subtracting the two given equations and dividing by $x - y$ gives $10 = 6(x + y) - (x^2 + xy + y^2)$, so

$$10 = 6s - s^2 + p,$$

where $s := x + y$, $p := xy$. Multiplying the two given equations and dividing by xy gives $|(3-x)(3-y)| = 1$, so

$$p = 3s - 8 \quad \text{or} \quad p = 3s - 10.$$

Plugging the first possibility above, we get $s^2 - 9s + 18 = 0$, so $(s, p) = (3, 1)$ (and $\{x, y\} = \{\frac{1}{2}(3+\sqrt{5}), \frac{1}{2}(3-\sqrt{5})\}$) or $(s, p) = (6, 10)$ (contradiction, since we should have $s^2 \geq 4p$). In the second case, we get $s^2 - 9s + 20 = 0$, so $(s, p) = (4, 2)$ (and $\{x, y\} = \{2 + \sqrt{2}, 2 - \sqrt{2}\}$) or $(s, p) = (5, 5)$ (and $\{x, y\} = \{\frac{1}{2}(5 - \sqrt{5}), \frac{1}{2}(5 + \sqrt{5})\}$).

Finally, $s \in \{0, 3, 4, 5, 8\}$.

Problem 8.

[JBMO 2020] Find all triples (a, b, c) of real numbers such that the following system holds:

$$\begin{cases} a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ a^2 + b^2 + c^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \end{cases}$$

Solution: Denote $p := a + b + c$, $q := ab + bc + ca$, $r := abc$. The first equation (upon multiplying both sides by abc) can be restated as $pr = q$. The first equation squared minus the second equation gives $qr = p$. This means that either (case 1) $p = q$ and $r = 1$, or (case 2) $p = -q$ and $r = -1$.

In the first case we know that a, b, c are roots of

$$x^3 - px^2 + px - 1 = 0,$$

which clearly has a root $x = 1$. We therefore get that if a, b, c is a solution, then it is of the form 1, b, $\frac{1}{b}$ for some $b \neq 0$. It is easily verified that every such triple satisfies the given system.

Similarly in the second case we get that a, b, c are roots of

$$x^3 - px^2 - px + 1 = 0,$$

and so one of them has to be -1, and the two others: b and $\frac{1}{b}$ for some $b \neq 0$. Again, it's easily verified that such triples satisfy problem's conditions.

Problem 9.

Does there exist an integer $n > 1$ such that $9n + 16$ and $16n + 9$ are both perfect squares?

Solution: Say $9n + 16 = k^2$ and $16n + 9 = l^2$, where k, l are positive integers. Then

$$(4k - 3l)(4k + 3l) = 16k^2 - 9l^2 = 16^2 - 9^2 = 175 = 5 \cdot 5 \cdot 7.$$

Therefore (as $4k + 3l > 0$, also $4k - 3l > 0$) we have $(4k - 3l, 4k + 3l)$ equal to one of $(1, 175)$, $(5, 35)$, $(7, 25)$. This gives k: 22, 5, 4 respectively and n: 52, 1, 0. Therefore $n = 52$ is the only such integer greater than 1.

Problem 10.

The Fibonacci numbers are defined as follows: we start with two ones and every next term is the sum of the previous two. So the sequence starts like this: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... Calculate

$$0.1 + 0.01 + 0.002 + 0.0003 + 0.00005 + 0.000008 + 0.0000013 + 0.00000021 + \ldots,$$

where the i-th summand has i decimal places ending with the i-th Fibonacci number.

SOLUTION: Let s be the sought sum. We know that $s + 10s = 100s - 10 \implies s = \frac{10}{89}$.

PROBLEM 11.

Prove that $2^{10} + 5^{12}$ is composite.

SOLUTION: $2^{10} + 5^{12} = \left(2^5 + 5^6\right)^2 - 1000^2 = 14657 \cdot 16657$

PROBLEM 12.

Point P lies inside an equilateral triangle ABC. Lines PA, PB, PC intersect sides BC, CA, AB at points D, E, F, respectively. Is it possible that exactly three among six triangles AFP, BFP, BDP, CDP, CEP, AEP have equal areas?

SOLUTION: Where could the 3 equal areas be? If $[AFP] = [BFP]$, then F is the midpoint of AB and so CP is the axis of symmetry, making $[AEP] = [BDP]$ and $[CEP] = [CDP]$ — it is not possible to have exactly three equal areas in this setting (there will always be an even number of triangles with equal areas). Similarly $[BDP] \neq [CDP]$ and $[CEP] \neq [AEP]$.

If $[AEP] = [BDP]$, then $[ABE] = [ABD]$, so $DE \parallel AB$ and the trapezoid $ABDE$ is isosceles. Therefore CP is again an axis of symmetry, so this cannot be the case. Similarly $[AFP] \neq [CDP]$ and $[BFP] \neq [CEP]$.

The only configuration not excluded so far is a *windmill*: $[AFP] = [BDP] = [CEP]$ (or symmetrically $[BFP] = [CDP] = [AEP]$). We will prove that this is not possible, and therefore show that the answer in the entire problem is No.

Denote $s = [AFP] = [BDP] = [CEP]$ and moreover $a = [AEP]$, $b = [BFP]$, $c = [CDP]$. Then

$$\frac{s}{b} = \frac{[AFP]}{[BFP]} = \frac{AF}{BF} = \frac{[APC]}{[BPC]} = \frac{a+s}{b+s}, \quad \text{so} \quad s(b+s) = b(a+s)$$

Similarly we get

$$s(c+s) = c(b+s) \qquad s(a+s) = a(c+s).$$

If we add up the three equations, we will get $s^2 = \frac{1}{3}(ab + bc + ca)$. If we multiply them up instead, we will get $s = \sqrt[3]{abc}$, so $s^2 = \sqrt[3]{ab \cdot bc \cdot ca}$. Therefore s^2 is both AM and GM of the triple ab, bc, ca, which means that $ab = bc = ca = s^2$ and so $a = b = c = s$, contradiction.

PROBLEM 13.

Positive numbers a, b, c, d satisfy

$$abcd = 4 \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 = 10.$$

Find the greatest possible value of $ab + bc + cd + da$.

SOLUTION: Suppose without loss of generality that $p := ac \geq 2$ (the roles of pairs $\{a, c\}$ and $\{b, d\}$ are symmetric and $ac \cdot bd = 4$, so we can do that). Denote $s = a^2 + c^2$. From the problem's conditions we get $bd = 4/p$ and $b^2 + d^2 = 10 - s$. Also, clearly, $s \geq 2p$. Now

$$\begin{aligned}
(ab + bc + cd + da)^2 &= (a+c)^2(b+d)^2 \\
&= (s + 2p)(10 - s + 8/p) \\
&= -s^2 + 10s + 16 + 20p - 2s(p - 4/p) \\
&= -(s-5)^2 + 41 + 20p - 2s(p - 4/p) \\
&\leq -(s-5)^2 + 41 + 20p - 4p(p - 4/p) \\
&= -(s-5)^2 - (2p-5)^2 + 82 \\
&\leq 82,
\end{aligned}$$

so $ab + bc + cd + da \leq \sqrt{82}$. This value is in fact obtainable: to have equalities in the above inequalities, we need $s = 2p = 5$, i.e. $a^2 = c^2 = \frac{5}{2}$, which means that $a = c = \sqrt{\frac{5}{2}}$. The numbers b, d should satisfy $bd = \frac{8}{5}$ and $b^2 + d^2 = 5$, so $b + d = \sqrt{\frac{41}{5}}$ and since $\frac{41}{5} \geq \frac{32}{5}$, such numbers exist. One can even directly calculate them: $\{b, d\} = \{\frac{\sqrt{41}-3}{2\sqrt{5}}, \frac{\sqrt{41}+3}{2\sqrt{5}}\}$.

Problem 14.

Positive integers a, b, c satisfy $a^2 + ab - 1 = c^2$. Prove that $b \geq \sqrt{4c-3}$.

Solution: Using the assumption $a^2 + ab = c^2 + 1$ we see that the desired inequality is equivalent to

$$\begin{aligned} b^2 &\geq 4c - 3 \\ b^2 + 4(ab + a^2) &\geq 4c - 3 + 4(c^2 + 1) \\ (b + 2a)^2 &\geq (2c + 1)^2 \\ b + 2a &\geq 2c + 1. \end{aligned}$$

Now by AM-GM

$$b + 2a = a + (a + b) \geq 2\sqrt{a(a+b)} = 2\sqrt{c^2+1} > 2\sqrt{c^2} = 2c,$$

and since $b + 2a$ and $2c$ are integers, $b + 2a \geq 2c + 1$ follows.

Remark. In the equality case we have $b = 2c - 2a + 1$, which plugged into the initial condition gives $c = a + \sqrt{a-1}$ (and in consequence $b = 2\sqrt{a-1} + 1$). These numbers are integers only if $a - 1 = n^2$ for $n \geq 0$, i.e. $(a, b, c) = (n^2 + 1, 2n + 1, n^2 + n + 1)$, where n is a nonnegative integer.

Bonus problem 1.

A big swimming pool has the shape of a circular ring divided into $2n$ sectors, where $n \geq 2$ is a positive integer. Over the border of every pair of neighboring sectors there is a circus hoop in the air. Initially in every sector there is one dolphin. During a show the dolphins perform some acrobatics, and a single acrobatics is a joint jump of two dolphins (from neighboring sectors) through the hoop between their sectors resulting in exchanging their positions. Suppose that during the show each pair of dolphins performed together exactly one jump. Prove that some hoop has never been used.

Solution: Note that each dolphin performed exactly $2n - 1$ jumps, so none of them could go around the entire pool. Therefore every dolphin moved by some arc of the pool, either clockwise, or anticlockwise (from the initial sector to the terminal one). Choose the dolphin c with the *longest clockwise arc*, or in other words — the dolphin which performed the greatest number of clockwise jumps. We will prove that this dolphin performed actually *only* clockwise jumps. Indeed, if there was some anticlockwise jump of c, then this jump was performed together with some dolphin d who jumped clockwise at that moment. But it was their only common jump, so before the jump d was earlier (clockwise) than c, and after the jump d was later (clockwise) than c. But this would mean that d has a longer arc — contradiction.

As c performed $2n - 1$ clockwise jumps, it used all the hoops apart from exactly one (the one between the initial sector and the terminal sector). Call it H. We will prove that actually H could never be used by any pair of dolphins. Suppose the contrary and consider *the earliest acrobatics using* H. Say it was performed by dolphin x jumping anticlockwise and dolphin y jumping clockwise. Because before the jump x was in c's initial sector (and could have gotten to this sector only anticlockwise, as H was not used yet), x and c must have jumped together already before. Moreover y and c have not jumped together (as y is still clockwise later than c on its way around the pool). But after the jump of x and y through H, dolphin y loses the possibility to jump together with c forever: x is on their way (and neither c, x, nor y, x can perform another jump) and c is jumping only clockwise. Contradiction.

Bonus problem 2.

A variable line ℓ passes through the center of an equilateral triangle ABC and intersects the segments AC and BC at points K and L, respectively. Point P satisfies $AL = LP$ and $BK = KP$. Prove that the distance of the point P from the line ℓ is independent of the choice of ℓ.

Solution: Consider point D in space such that $ABCD$ is a regular tetrahedron (i.e. whose all faces are equilateral triangles). Triangles ALB and DLB are congruent (two equal sides and 60° degrees in between), so are triangles AKB and AKD. So $AL = LD$ and $BK = KD$, i.e. triangles KLD and KLP are congruent (they have all sides equal). In particular — they have equal altitudes onto the side KL, i.e. points P and D have equal distances from line ℓ. But the distance from D to ℓ is just the altitude of tetrahedron $ABCD$, so it does not depend on the choice of ℓ.

By the way, we proved that the projection of P onto ℓ is the center of ABC and that the desired constant distance is precisely $AB \cdot \sqrt{2/3}$ (that's the altitude of a regular tetrahedron).

Problem 15.

Let n be a positive integer. Prove that $\sqrt{100^n + 2}$ has at least n consecutive zeros after the decimal point.

Solution: It is enough to prove that

$$\sqrt{100^n + 2} < 10^n + 10^{-n},$$

which is readily seen after squaring.

Problem 16.

Prove that for every $x > 0$:

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}} = 1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \ldots}}}}.$$

Solution: Denote the left-hand side by L and the right-hand side by R (and suppose that these numbers are actually well-defined, so the infinite calculations can be reasonably understood). Then we have

$$L = \sqrt{x + L} \iff x = L^2 - L \iff 4x + 1 = (2L - 1)^2 \iff L = \frac{1 \pm \sqrt{4x + 1}}{2}.$$

But we know that $L > 0$, so there can be no minus sign above (as $4x + 1 > 1$ and therefore $1 - \sqrt{4x + 1} < 0$), i.e. $L = \frac{1}{2}\left(1 + \sqrt{4x + 1}\right)$.

On the other hand, we have

$$R = 1 + \frac{x}{R} \iff x = R^2 - R,$$

which leads to the same equation, the same set of solutions $R = \frac{1 \pm \sqrt{4x+1}}{2}$ and $R > 0$ again makes the minus sign not possible, so $R = \frac{1}{2}\left(1 + \sqrt{4x + 1}\right) = L$.

Problem 17.

On the board there are four positive integers. It is known that they are equal to

$$a + b + c, \quad a + bc, \quad b + ca, \quad c + ab$$

for some positive integers $2 \le a \le b \le c$, but it is not given which value corresponds to which expression. Can you uniquely determine the values of a, b, c given only the numbers from the board?

Solution: Note that if x, y are integers greater than 1, then $(x - 1)(y - 1) > 0 \iff xy > x + y$. In particular it follows that the smallest number on the board is $a + b + c$, denote this number by s. Let the remaining three expressions be x, y, z, and name them in such a way that $x \ge y \ge z$. As $(a+bc)-(b+ca) = (b-a)(c-1) \ge 0$ and similarly $(b + ca) - (c + ab) = (c - b)(a - 1) \ge 0$, we know that $a + bc \ge b + ca \ge c + ab$, so $x = a + bc$, $y = b + ca$, $z = c + ab$. Now note that

$$x - s + 1 = (b - 1)(c - 1), \quad y - s + 1 = (c - 1)(a - 1), \quad z - s + 1 = (a - 1)(b - 1),$$

so

$$a = 1 + \sqrt{\frac{(y-s+1)(z-s+1)}{x-s+1}}, \quad b = 1 + \sqrt{\frac{(z-s+1)(x-s+1)}{y-s+1}}, \quad c = 1 + \sqrt{\frac{(x-s+1)(y-s+1)}{z-s+1}},$$

which means that we can get the values of a, b, c back given the information on the board.

Problem 18.

Positive real numbers a, b, c, d that satisfy

$$a^2 + d^2 - ad = b^2 + c^2 + bc \quad \text{and} \quad a^2 + b^2 = c^2 + d^2.$$

Find all possible values of the expression $\dfrac{ab+cd}{ad+bc}$.

Solution: Consider a quadrilateral whose consecutive side lengths are a, b, c, d, the angle between a and d is $60°$, and the angle between b and c is $120°$ (such a quadrilateral exists by the first condition and the law of cosines). Now if the angles between a and b, and between c and d were not right, then one would be greater than $180°$ (and one would be smaller), so from the law of cosines again we would get a contradiction with the second assumption. Therefore these angles are right (and the second condition is just a double Pythagoras). Having constructed the quadrilateral, it is now just enough to see that in the desired ratio the numerator is twice the area of this quadrilateral, and the denominator is $4/\sqrt{3}$ times the area, so the ratio has to be equal to $\sqrt{3}/2$ (achievable for any quadruple corresponding to a polygon with described properties, e.g. $(a,b,c,d) = (\sqrt{3}, 1, 1, \sqrt{3})$).

Problem 19.

Prove that every integer n can be expressed in the form

$$\pm 1^2 \pm 2^2 \pm 3^2 \pm \ldots \pm k^2$$

for some positive integer k and some choice of signs.

Solution: First of all note that it is enough to prove the statement for nonnegative integer (as we will then obtain any negative integer by just negating all signs). Moreover, as for every k:

$$k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2 = 4,$$

having the desired expression for n, we can extend it to an expression for $n+4$. This means that it is enough to cope with 0, 1, 2, 3, and here are the base examples:

$$0 = -1 - 4 + 9 - 16 + 25 + 36 - 49, \quad 1 = 1, \quad 2 = -1 - 4 - 9 + 16, \quad 3 = -1 + 4.$$

Problem 20.

Given is a finite sequence of real numbers, whose sum is zero but not all of them are zeroes. Prove that there exists a permutation $(a_1, a_2, \ldots, a_n)$ of this sequence such that

$$a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n + a_na_1 < 0.$$

Solution: Consider the sum of such sums for all possible permutations. We will prove that it's negative (therefore proving that at least one of such sums has to be negative). For any two distinct numbers x_i, x_j, the term x_ix_j will appear the same number of times in the sum of sums, say C (we can actually compute that $C = 2n(n-2)!$). Therefore the sum of sums is just

$$\frac{C}{2}\left((x_1 + x_2 + \ldots + x_n)^2 - (x_1^2 + x_2^2 + \ldots + x_n^2)\right) = -\frac{C}{2}(x_1^2 + x_2^2 + \ldots + x_n^2) < 0$$

(the inequality is strict because not all the squares are zeroes).

Problem 21.

Each positive integer was given some color. Suppose that for every pair of integers $a, b > 1$ the numbers $a + b$ and ab have the same color. Prove that all integers greater than 4 have the same color.

Solution: For every $n > 1$ we have

$$n + 3 = (n + 1) + 2 \sim (n + 1) \cdot 2 = 2n + 2 \sim 2n \cdot 2 = 4n \sim 4 + n,$$

where $\sim$ denotes having the same color. This means that all integers greater than 4 have the same color.

Bonus problem: Each positive integer was given some color. Suppose that for every pair of integers $a, b > k$ the numbers $a + b$ and ab have the same color, where $k \geq 1$ is fixed. Prove that for some K all integers greater than K have the same color.

Solution: [I] We will prove the desired statement with $K = 5k + 1$. Take arbitrary n and m such that $n - m > k$ and $m > k$. We have

$$n + 2 = (n - m + 1) + (m + 1) \sim (n - m + 1)(m + 1) = nm + n - m^2 + 1 = (n + 1) + m(n - m) \sim$$
$$\sim (n + 1)m(n - m) = (nm + m)(n - m) \sim nm + m + n - m = n(m + 1) \sim n + m + 1$$

Now let $n = 4k$ and take $m = k+1, \ldots, 3k-1$ to see that $4k+2$ has the same color as all the numbers in the interval $[5k+2, 7k]$, in particular — all numbers in this interval have the same color C. For $n = 5k$ and $m = k+1, \ldots, 4k-1$ we get that $5k + 2$ (which is of color C) has the same color as every integer in the interval $[6k + 2, 8k]$, which makes the entire interval $[5k+2, 8k]$ of color C (as $6k+2 \leq 7k+1$). Repeating similar reasoning for $n = rk$ and consecutive $r = 6, 7, \ldots$, we eventually get that all integers $\geq 5k + 2$ are of the same color.

[II] We can prove it also a bit simpler for a larger $K = k + 2^{k+1} - 1$ as seen below

$$n + 2^{k+1} = (n + 2^k) + 2^k \sim n2^k + 2^{k+1} = (n2^k + 2^k) + 2^k \sim (n + 1)2^{k+1} \sim n + 2^{k+1} + 1.$$

These $\sim$'s can be performed since $2^k > k$ and only when $n + 1 > k$. In particular this means that every number $\geq k + 2^{k+1}$ will have the same color.

Problem 22.

Prove that there are infinitely many 5-tuples of positive integers (a, b, c, d, e) such that $a < b < c < d < e$ and $a^2 + b^2 + c^2 = d^2 + e^2$.

Solution: Note that if we find one such 5-tuple, then we can multiply each of its terms by any $n \geq 2$ to get infinitely many 5-tuples. And the base one can be for instance $(10, 11, 12, 13, 14)$.

In order to find it, one may for example restrict attention to consecutive numbers, i.e. plug

$$(a, b, c, d, e) = (c - 2, c - 1, c, c + 1, c + 2)$$

into the expression in the statement to get $c = 12$ (or $c = 0$, clearly impossible).

Problem 23.

Prove that there are infinitely many triples of positive integers (a, b, c) such that $a^3 + b^3 + 2 = c^3$.

Solution: Let us try to find a family of solutions with $b = n - 1$ and $c = n + 1$ for some $n \geq 2$ (the reason being trying to get rid of the 2 on the left-hand side). Then we get $a^3 = 6n^2$ which can be achieved by putting $n = 6k^2$ (then $a = 6k$). We get solutions $(6k, 6k^2 - 1, 6k^2 + 1)$, where $k \geq 1$ is a positive integer.

By the way, it is not known if there are any other solutions that those of the form mentioned above (up to swapping a with b).

Problem 24.

Find all triples (a, b, c) of positive integers, such that

$$\frac{a^2 + b}{b^2 + a} = c \qquad \text{and} \qquad \frac{a^2 + c}{c^2 + a} = b.$$

SOLUTION: One can easily get $(b-c)(1-bc+a)=0$ by getting rid of fractions and subtracting the two equations. This means that $b=c$ or $a=bc-1$. We will handle these cases separately.

1° If $b=c$, then both of the given equations take the form $a^2+b=b^3+ab$, i.e. $a^2=b(b^2+a-1)$. If $b=1$, then $a^2=a$, meaning $a=1$.

Suppose that $b\neq 1$ and let p be any prime divisor of b. From $a^2=b(b^2+a-1)$ follows that p is a divisor of a, so it is not a divisor of b^2+a-1. This means (as p was chosen arbitrarily) that b and b^2+a-1 are coprime and since their product is a perfect square, we have

$$b=n^2 \quad \text{and} \quad b^2+a-1=k^2$$

for some positive integers n, k. In particular $a=nk$.

Now $b^2+a-1=k^2$ can be rewritten as

$$k^2=n^4+nk-1 \iff 4n^4+n^2-4=(2k-n)^2.$$

But since $b\neq 1$, we have $n\geq 2$ and $4n^4+n^2-4\geq (2n^2)^2$, with the only equality case $n=2$. On the other hand,

$$4n^4+n^2-4<4n^4+4n^2+1=(2n^2+1)^2.$$

Because $4n^4+n^2-4=(2k-n)^2$, it is equivalent to $(2n^2)^2\leq(2k-n)^2<(2n^2+1)^2$, so the only possibility is having an equality in the left inequality. This means that $n=2$, $2n^2=2k-n$, $k=5$, and we obtain $(a,b,c)=(10,4,4)$.

2° If $a=bc-1$, then

$$a^2+b=b^2c+ac \iff a(a-c)=b(bc-1), \iff a-c=b,$$

which means that $a=b+c$, implying

$$bc-1=b+c \iff (b-1)(c-1)=2.$$

Therefore$(b,c)=(2,3)$ or $(b,c)=(3,2)$ and in both cases $a=5$.

It is easy to check that all four triples (a,b,c): $(\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{10},\mathbf{4},\mathbf{4})$, $(\mathbf{5},\mathbf{2},\mathbf{3})$, $(\mathbf{5},\mathbf{3},\mathbf{2})$ satisfy the initial system of equations.

PROBLEM 25.

Find all triples (a,b,c) of integers satisfying the system of equations

$$a+b=c, \qquad a^2+b^3=c^2.$$

SOLUTION: We have

$$b^3=c^2-a^2=(c-a)(c+a)=b(c+a),$$

so either $b=0$ (and $a=c$, easily verified to be a solution), or $a+c=b^2$, which yields $a=\frac{1}{2}b(b-1)$, $c=\frac{1}{2}b(b+1)$ (both are integers for every b as either b is even, or $b+1$ and $b-1$ are both even).

PROBLEM 26.

Do there exist prime numbers p, q, r such that

$$(p^2+p)(q^2+q)(r^2+r)$$

is a perfect square?

SOLUTION: If any two of the three primes were equal, say $p=q$, then r^2+r would be a square, which is not possible (e.g. because $r^2<r^2+r<(r+1)^2$). Suppose then w.l.o.g. $p>q>r\geq 2$. Then p is a divisor of $qr(p+1)(q+1)(r+1)$, but since p is coprime with q, r, $p+1$, $q+1$ (q is an odd prime), $r+1$ ($p>r+1$), this is not possible.

PROBLEM 27.

Positive integers a and b are such that both numbers

$$ab \qquad \text{and} \qquad (a+1)(b+1)$$

are perfect squares. Prove that there exists an integer $n \geq 2$ such that $(a+n)(b+n)$ is also a perfect square.

Solution: If $a = b = 1$, then every n works. If $ab \geq 2$, then $n = ab$ works, since

$$(a+ab)(b+ab) = ab \cdot (a+1)(b+1).$$

Problem 28.

Suppose that $1 + 3^n + 5^n$ is a prime number. Prove that n is divisible by 12.

Solution: If n is odd, then $5^n \equiv 2 \pmod 3$, so $1 + 3^n + 5^n \equiv 0 \pmod 3$. If $n = 2m$ and m is odd, then $9^m \equiv -1 \pmod 5$, so $1 + 9^m + 25^m \equiv 0 \pmod 5$. Finally if $n = 4k$, then $1 + 81^k + 625^k \equiv 1 + 4^k + 2^k \pmod 7$ and this is 0 unless k is divisible by 3. By the way, for $n = 12, 36, 48, 62$, our number is prime!

Problem 29.

Prove that if for an integer $n \geq 2$ the number $n2^n + 1$ is prime, then the numbers $n+1$, $n+2$ are both composite.

Solution: Suppose $n+1$ is an odd prime. Then by Fermat's little theorem we have that $2^n - 1$ is divisible by $n+1$, and so is

$$n(2^n - 1) + n + 1 = n2^n + 1.$$

Suppose now $n+2$ is an odd prime. Then, again by Fermat's little theorem, $2^{n+1} - 1$ is divisible by $n+2$, and so is the number $(n+2)2^n - (2^{n+1} - 1) = n2^n + 1$.

Problem 30.

Do there exist integers a, b, c so that each of the quadratics

$$ax^2 + bx + c \qquad \text{and} \qquad (a+1)x^2 + (b+1)x + (c+1)$$

has two integer roots?

Solution: If (a, b, c) is a solution, then so is $(-a-1, -b-1, -c-1)$, so we can without loss of generality assume that a is even. But then since the solutions x_1, x_2 of the first quadratic satisfy $x_1 + x_2 = -\frac{b}{a}$ and $x_1 x_2 = \frac{c}{a}$, both b and c are even as well. But the discriminant of the second quadratic, equal to

$$(b+1)^2 - 4(a+1)(c+1)$$

is $5 \pmod 8$, so cannot be a square. This means that such three numbers do not exist.

Problem 31.

Determine all possible values of $\gcd(n^2 + 2, n^3 + 3)$ where n is an integer.

Solution: Note that if d is a divisor of both $n^2 + 2$ and $n^3 + 3$, then d is also a divisor of $n^6 + 8$ and $n^6 - 9$, so a divisor of 17. Both 1 and 17 can actually be the desired gcd's, which is seen by the examples: $n = 0$ and $n = -7$, respectively.

Problem 32.

Find all pairs (n, p) of positive integers such that $p > n$, p is a prime number and $n^2 + np + p^2$ is a perfect square.

Solution: Suppose that $n^2 + np + p^2 = m^2$, i.e.

$$p(n+p) = (m-n)(m+n),$$

in particular $p \mid m-n$ or $p \mid m+n$. But since

$$m^2 = n^2 + np + p^2 < (n+p)^2,$$

we have $m < n+p$, so $m-n < p$. Therefore $p \mid m+n$, but since $n < p < m$, we have

$$p < p+n < m+n < n+p+n < 3p,$$

so $m+n = 2p$, i.e. $m = 2p-n$. Plugging this into the initial equation, we get $5n = 3p$, so $(n,p) = (3,5)$.