Problem 1. $p(x) = \underbrace{((\ldots((}_{k} x-2)^2-2)^2 \ldots)^2-2)^2$

coefficient at x^0 is always 2 because $2^2-2=2$

coefficient at x^1 is 4^k because it's 4 times coefficient at x^1 when $(k-1)$

Assume $f(k)$ is coefficient at x^2 for k

$f(1)=1$ $\quad f(2)=20$

$f(k) = \underset{2\cdot 2\cdot f(k-1)x^2}{4f(k-1)} + \underset{(-4^{k-1}x)^2}{4^{2(k-1)}}$

by induction: $f(k) = 4^{2(k-1)} + 4^{2k-3} + \ldots + 4^{k-1} = \frac{4^{k-1}(4^k-1)}{3}$

$f(2) = 4^2+4 = 20$ ✓

$k \to k+1$

$f(k+1) = 4f(k)+4^{2k} = 4(4^{2(k-1)}+\ldots+4^{k-1})+4^{2k} = 4^{2k}+\ldots+4^k$ ✓

problem 3 $\quad lcm(m,n) = \dfrac{mn}{gcd(m,n)}$

Assume $gcd(m,n) = d_1 \quad gcd(m+1, n+1) = d_2$

$gcd(m, m+1) = 1 \Rightarrow gcd(d_1, d_2) = 1$

$d_1 \mid m-n \quad , \quad d_2 \mid (m+1)-(n+1) = m-n \Rightarrow d_1 d_2 \mid m-n$

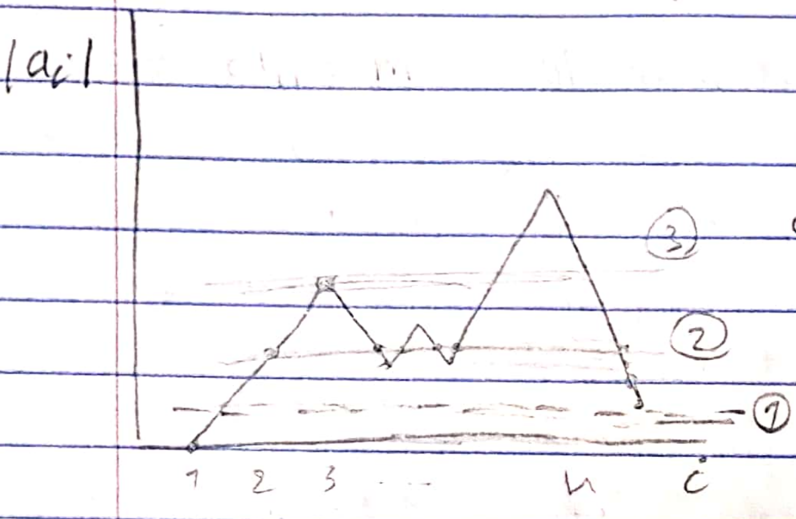
$\Rightarrow m-n \geq d_1 d_2$

$$\frac{mn}{d_1} + \frac{(m+1)(n+1)}{d_2} > \frac{mn}{d_1} + \frac{mn}{d_2} \overset{\text{Am-Gm}}{\geq} \frac{2mn}{\sqrt{d_1 d_2}} \geq \frac{2mn}{\sqrt{m-n}}$$

Problem 2 $S = \Sigma a_i$

if a_k is positive $\Rightarrow |a_{k+1}| = |a_k + 1| = |a_k| + 1$

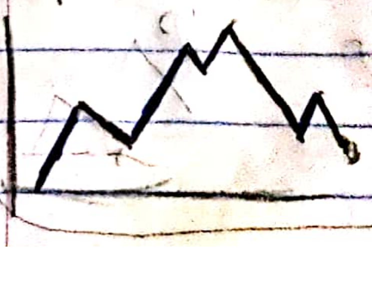
if a_k is negative $\Rightarrow |a_{k+1}| = |a_k + 1| = |a_k| - 1$



take any horizontal line. it will be one of three cases
if it was bellow $|a_n|$ then it will cut odd number of times

if the cut was down to up then $|a_{k+1}| = |a_k| + 1$ so a_k is positive
if up to down then negative
we start down and end up so positive more than negative so sum of this line is positive
case 2 if it wasn't bellow $|a_n|$ than we start and end down so it cuts even time, sum is zero
case 3 now we just need to sum up the changing points
up to down is negative others is positive and it is minus the number of black lines. $\le \frac{n}{2} \Rightarrow S \ge \frac{n}{2}$

but the end is up. and blue is up black is down sum is n $\Rightarrow$ black $\le \frac{n}{2}$



Problem 4

if $p_3 \equiv 1 \mod 3$ $(p_3 > 3)$

$\Rightarrow |p_4 - 2p_3| \equiv 1 \mod 3$

$\Rightarrow p_4 - 2 \equiv 1$ or $-1 \mod 3$

$\Rightarrow p_4 \equiv 1$ or $0 \mod 3 \underset{p_4>3}{\Rightarrow} p_4 \equiv 1 \mod 3$

$\Rightarrow p_4 = 2p_3 - 1$

similarily $p_{i+1} = 2p_i - 1$ $(i \geq 3)$

if $p_3 \equiv -1 \mod 3$

$|p_4 - 2p_3| \equiv 1 \mod 3$

$p_4 + 2 \equiv 1$ or $-1 \mod 3$

$\Rightarrow p_4 \equiv 0$ or $-1 \mod 3$

$\Rightarrow p_4 \equiv -1 \mod 3$

$\Rightarrow p_4 = 2p_3 + 1 \mod 3$

similarily $p_{i+1} = 2p_i + 1$

case 1 : $p_{i+1} = 2p_i - 1$ $(i \geq 3)$

$p_{k+1} \equiv -1 \mod p_k$ for $(k > 3)$

there is infinite $p_i \Rightarrow \exists\ p_{k+t_1} \equiv p_{k+t_2} \mod p_k$ $(t_2 > t_1 > 1)$

$\Rightarrow 2p_{k+t_1-1} - 1 \equiv 2p_{k+t_2-1} + 1 \mod p_k$

$\Rightarrow p_{k+t_1-1} \equiv p_{k+t_2-1} \mod p_k$

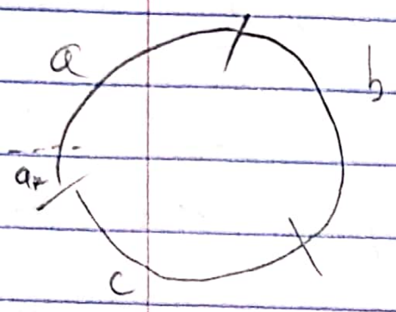
completing like this $\Rightarrow p_k \equiv p_{k+t_2-t_1} \mod p_k$

$\Rightarrow p_k \mid p_{k+t_2-t_1}$ And $p_{k+t_2-t_1}$ is prime bigger than p_k ↯

case 2 : $p_{i+1} = 2p_i + 1$ $(i \geq 3)$

same as case 1

problem 6 divide the circle in three arcs a, b, c



Assume At the beginning $a \geq b \geq c$

then take element a_k from the biggest and put it in the smallest ($a \to c$)

$\Rightarrow$ $b-c$ will decrease by $a_k < 1$

by doing that in some step we will have $1 > b-c \geq 0$

And if $b \geq a \geq c \Rightarrow 1 > b-c > b-a \geq 0$, $1 > b-c \geq a-c \geq 0$

Done.

same as case $b \geq c \geq a$

if $a \geq b \geq c$ take element a_k from a and give it to the smallest (c)

$1 > b-c-a_k \geq -1$ and $a-b$ will decrease less than 2
so $|b-c|<1$ always

So At some step $|a-b|<1$ because period between 1 an -1 is longer than 2

if $b \geq a \geq c$ or $b \geq c \geq a$ or $a \geq c \geq b$ or $c \geq a \geq b$ Done.

if $a \geq b \geq c$ or $c \geq b \geq a$

WLOG $a \geq b \geq c$ now take element a_k from a and put it in c until the order changes

if it $a \geq c \geq b$, $|a-b|<1$ Done. if it $c \geq a \geq b$ $|b-c|<1$ Done.

previous step was $b \geq c$ and c increase by less than 1

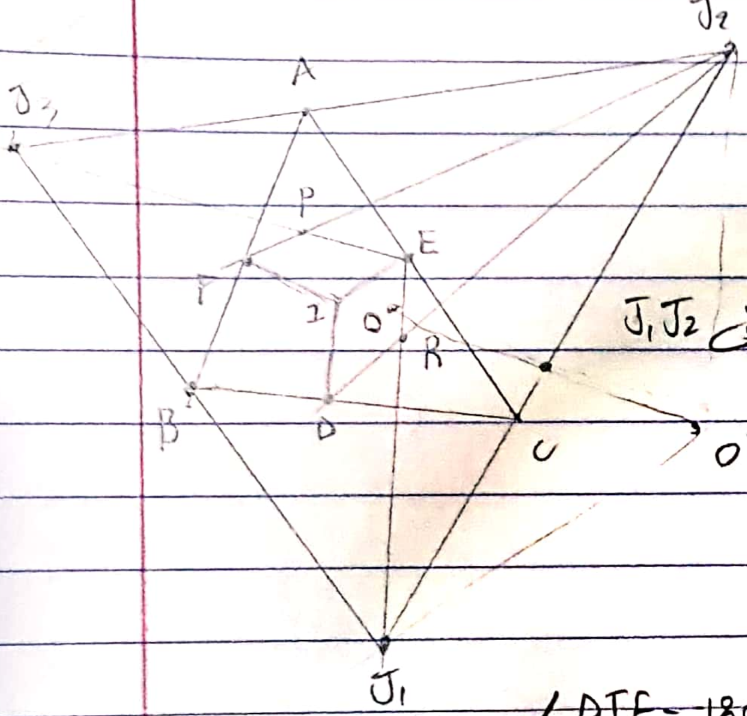
and the same for $b \geq c \geq a$ and $b \geq a \geq c$

if $c \geq b \geq a$ if $c-a<1$ Done if $c-a \geq 1$

then in the previous step $a \geq c$ and $a-c$ decrease by less than 2

it was $\Rightarrow a \geq b \geq c$ and $|a-c|<1$ so we are Done.

Problem 7



افرض O مركز الدائرة $J_1J_2J_3$ ونصف قطرها R

من الواضح أن $\triangle DEF \sim \triangle J_1J_2J_3$

افرض نسبة التشابه $\frac{DE}{J_1J_2} = k$

افرض I' , O' انعكاسات I , O حول منتصف J_1J_2

بما أن I تقاطع $J_1J_2J_3$

إذن I' على الدائرة $J_1J_2J_3$

$\Rightarrow OI' = R \Rightarrow IO' = R$

$OIO'I'$ parallelogram

$\angle DIE = 180 - \angle ACB$

$\angle J_1J_3J_2 = 90 - \frac{\angle ACB}{2} \Rightarrow \angle J_1OJ_2 = 180 - \angle ACB = \angle J_1O'J_2$

إذن $J_1O'J_2$ و DIE مثلثان متحاكيان محاكاة سالبة إذن مركزها

هو تقاطع J_1E و J_2D وهو R

إذن المحاكاة معاملها k ومركزها R ترسل O' إلى I

$\Rightarrow \frac{RI}{RO'} = k \Rightarrow \frac{RI}{IO'} = \frac{k}{1+k} \Rightarrow RI = R\cdot\frac{k}{1+k}$

بالمثل $QI = PI = R\cdot\frac{k}{1+k}$

إذن I مركز الدائرة المحيطة بالمثلث PQR