Level 2 E-training, week 8

Due to 23:59, Friday, 30 October 2020

Problem 1. Let ABC be a non-degenerate triangle with perimeter $2s$. Let M be the midpoint of BC and L be the foot of altitude from A onto BC. If $BC = a, AL = h$, and $AM = m$, then prove that

$$h \leq \sqrt{s(s-a)} \leq m$$

Problem 2. Let p be an odd prime number and a, b be different integers coprime with p such that $p|a-b$ both Prove that

$$v_p(a^n - b^n) = v_p(a-b) + v_p(n)$$

for every $n \in \mathbb{N}$.

Problem 3. Let $f(x) = x^2 + ax + b$ be a monic quadratic polynomial having 2 different zeros $r < s$ with $s > 0$. Suppose that the polynomial $f(f(x))$ has exactly 3 distinct zeros. Prove that $s \leq 1$.

Problem 4. Can we partition the set $\{1, 2, \ldots, 2020\}$ into 3 subsets A, B, C such that the sets $A+B, B+C$ and $C+A$ are pairwise disjoint? (Remark: for $X, Y \subseteq \mathbb{R}$ we define $X + Y = \{x + y | x \in X, y \in Y\}$)

Problem 5. Let Ω, Γ be two circles on the plane meeting at exactly two different points A, B. The points C, D are outside $\Omega \cup \Gamma$ such that

$$\frac{\mathcal{P}_\Omega(C)}{\mathcal{P}_\Gamma(C)} = \frac{\mathcal{P}_\Omega(D)}{\mathcal{P}_\Gamma(D)}$$

Show that A, B, C and D are either collinear or concyclic.

Problem 6. Find all triples (a, b, c) of positive integers such that the numbers

$$a^2 + b + c + 1, \quad a + b^2 + c + 1, \quad a + b + c^2 + 1$$

are all perfect squares.

Problem 7. Let $n \geq 4$ be an even positive integer and $\mathcal{S}$ be a balanced set of n points. Show that $\mathcal{S}$ contains 4 different points A, B, C and O such that O is the circumcenter of ABC.

$$Note: \ A \ balanced \ set \ is \ defined \ in \ \textbf{Week7, Problem7}$$

Problem 8. Find all real quadruples (x, y, z, t) satisfying

$$x(y+z+t)^2 = y(x+z+t)^2 = z(x+y+t)^2 = t(x+y+z)^2$$