Level 2 E-training, week 4

Due to 23:59, Friday, 2 October 2020

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = 3x + 1$$

for every $x \in \mathbb{R}$. Show that $f(-\frac{1}{2}) = -\frac{1}{2}$

Problem 2. Let ABC be a triangle inscribed in the circle ω. Let K be the midpoint of arc BC of ω containing A. Points D, E lie on segment AB, AC, respectively, such that $BD = CE$. Show that the points A, D, E, K are concyclic.

Problem 3. Let n be an odd positive integer. Show that

$$\tau(n) \equiv \sigma(n) \ (mod \ 2)$$

Problem 4. Let $m, n \in \mathbb{N}$. On a line l, there are m ants crawling towrads the same direction and n ants crawling towards the opposite direction. Whenever two ants face each other, they reverse their direction and we call this moment a *switch*. How many switches will occur in the process?

Problem 5. Suppose that $a > b \geq 3$ are integers. Prove that $b^a > a^b$.

Problem 6. Find all $n \in \mathbb{N}$ such that

$$\tau(n) + \phi(n) > n$$

Problem 7. The cells of a 8×8 table are initially white. Alice and Bob play a game. First Alice paints n of the fields in red. Then Bob chooses 4 rows and 4 columns from the table and paints all fields in them in black. Alice wins if there is at least one red field left. Find the least value of n such that Alice can win the game no matter how Bob plays.

Problem 8. Let ABC be an acute-angled triangle. The tangents at B, C to the circumcircle of ABC meet at T. If M is the midpoint of BC, show that $\angle BAT = \angle CAM$.