Topic 1.

REVIEW ON FUNCTIONAL EQUATION

- Substitute variables.
- Injectivity and surjectivity.
- Case works for $(f(x)-x)(f(x)+x)=0, \forall x \in \mathbb{R}$ and new approach.

Problem 1. (L)

a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x-f(x))=x-f(x), \forall x$. Prove that the equation $f(x)=0$ has unique root.

b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x^2)=f(f(f(x)))$ and $f(xy)=xf(y)+yf(x)$ for all x,y. Prove that $f(x)\equiv 0, \forall x$.

Problem 2. (L) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x+y)=f(x)+f(y)$ for all $x\in\mathbb{R}$. Find $f(x)$ in the following cases:

a) $f(x^2)=xf(x)$ and $f(1)=2022$.

b) $2f(x)^2=f(x^2)+xf(x), \forall x$.

Problem 3. (O) Find all function $f:\mathbb{R}\to\mathbb{R}$ such that:

a) $f(x^2+xy+f(y))=f(x)^2+xf(y)+y$ for all x,y.

b) $f(f(x-y))=f(x)-f(y)+f(x)f(y)-xy$ for all x,y.

Problem 4. (O) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

a) $f((x+y)^2)=f(x)f(x+2y)+yf(y)$ for all x,y.

b) $f(x)f(yf(x)-1)=x^2f(y)-f(x)$ for all x,y.

Problem 5. (L) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that f is injective at 0 and

$$f(f(x+y)f(x-y)+21xy+y^2)=20yf(x)+xf(x+y) \text{ for all } x,y.$$

Additional problems.

Problem 6. (V) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

a) $f(y^2)=f(y-x)f(x+y)+x^2$ for all x,y.

b) $f(f(x+y))=f(x+y)+f(x)f(y)-xy$ for all x,y.

Problem 7. (O) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

a) $f(f(x)+y)=f(x^2-y)+4yf(x)$ for all x,y.

b) $f(x^2+xy)=f(x)f(y)+yf(x)+xf(x+y)$ for all x,y.

Problem 8. (O) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

a) $f(f(x)+y)=f(x+y)+xf(y)-xy-x+1$ for all $x,y\in\mathbb{R}$.

b) $xf(x+xy)=xf(x)+f(x^2)f(y)$ for all $x,y\in\mathbb{R}$.

Topic 2.

FUNCTION EQUATION WITH POLYNOMIAL

- Focus on the degree, the leading coefficient.
- Some situations: $P(P(x)),P(x)^m,P(x^n),\ldots$
- Working on $P(x+a)-P(x)$.

Problem 9. (L) Find all polynomials $P(x)\in\mathbb{R}[x]$ such that:

a) $P(2x+3)=P(2x)+x$ for all x.

b) $P(P(x))=(x^2+x+1)P(x)$ for all x.

Problem 10. (L) Find all polynomials $P(x)\in\mathbb{R}[x]$ such that:

a) $(x+10)P(2x)=(8x-32)P(x+6),\forall x$ and $P(1)=210$.

b) $(x^2+2x)P(x+1)=(x^2+4x+3)P(x)+2x^2+2x,\forall x\in\mathbb{R}$.

Problem 11. (L)

a) Suppose that $a,b\in\mathbb{R}\setminus\{0\}$ and $\frac{b}{a}\notin\mathbb{Z}$, prove that there does not exist $P(x)$ non-constant such that $xP(x-a)=(x-b)P(x)$ for all x.

b) For $c\in\mathbb{R}$, suppose that there exists polynomial $P(x)$ such that $P(x)^2-P(x^2)=cx^{2022},\forall x$. Prove that $c\geq-\frac{1}{4}$.

Problem 12. (O) Find all polynomials $P(x)\in\mathbb{R}[x]$ such that:

a) $(x-2)Q(3x+2)=3^{22}xQ(x),\forall x\in\mathbb{R}$.

b) $P((x+1)^{2022})=(P(x)+3x+1)^{2022}-(x+1)^{2022}$ for all x.

Problem 13. (O) Find all polynomials $P(x)\in\mathbb{R}[x]$ such that

a) $P(x^3+x^2+1)=P(x+2)P(x^2+1),\forall x$.

b) $P(x^2)=P(x)P(x-1),\forall x\in\mathbb{R}$.

c) $P(2x^3+x)=P(x)P(2x^2)$ for all x.

Problem 14. (V) Find all polynomials $P(x)\in\mathbb{R}[x]$ satisfy

a) $P(x-y)+P(y-z)+P(z-x)=3(P(x)+P(y)+P(z))$ for all x,y,z such that $x+y+z=0$.

b) $(x-y)P(z)+(y-z)P(x)+(z-x)P(y)=0$ for all x,y,z such that $x+y+z=0$.

c) $P(x)^2+P(y)^2+P(z)^2 = P(x+y+z)^2+2$ for all x, y, z such that $xy+yz+zx=-1$.

Additional problems.

Problem 15. Find all polynomials $P(x)\in\mathbb{R}[x]$ satisfy

a) $(x+4y)P(z)+(y+4z)P(x)+(z+4x)P(y)=xP(2y-z)+yP(2z-x)+zP(2x-y)$ for all x, y, z such that $3(x+y+z)=xyz$.

b) $P(x)P(x^2)...P(x^n) = P(x^{\frac{n(n+1)}{2}})$ for all x, given that $n\in\mathbb{Z}^+$ and $n>1$.

Problem 16. Let $P(x), Q(x), R(x)$ be non-constant real coefficient polynomials such that

$$P(x^2-x)+xQ(x^2-x)=(x^2-4)R(x) \text{ for all } x.$$

a) Prove that equation $Q(x)=R(x-3)$ has at least two distinct real roots.

b) Suppose that the sum of degree of $P(x), Q(x), R(x)$ is 5 and $R(x)$ monic. Find the minimum value of $M=P^2(0)+8Q^2(3)$.

Topic 3.

POWER OF POINT TO CIRCLE. RADICAL AXIS

- Some situations for radical axis of 2 circles, radical center of 3 circles.
- Point-circle.
- Coaxial circles.

Problem 17. (L) Let ABC be a triangle with H, O are orthocenter and circumcenter. Denote M, N as midpoints of AB, AC and BD, CE are altitudes of ABC. MN cuts DE at K.

a) Prove that $AK\perp HO$.

b) Line DE cuts BC at T and the tangent line from A of (O) cuts MN at S. Prove that $ASTK$ is a parallelogram.

Problem 18. (V) Let ABC be a triangle with incircle (I) tangent to BC, CA, AB at D, E, F. Denote K as the projection of D onto EF.

a) Prove that the power from K to the circles $(BE), (CF)$ are equal.

b) Denote H as orthocenter of ABC, prove that KD is angle bisector of HKI.

Problem 19. (O) Let ABC be a triangle inscribed in a circle (O) with BC is fixed and A moving around (O). Denote H as its orthocenter and BE, CF are its altitudes. Let M, N, I be the midpoints of BC, AH, EF. Line AM cuts (O) agains at T. Prove that $\mathcal{P}_{I/(ANT)}=const.$

Problem 20. (O) Let A be a point lying outside circle (O) and AB, AC are tangent line of (O). Take D, E, M on (O) such that $MD=ME$. Suppose that MB, MC cut DE at R, S respectively and take $X\in OB$, $Y\in OC$ such that $RX\perp DE$, $SY\perp DE$. Prove that $XY\perp AM$.

Problem 21. (L) Let AB be a chord of circle (O) and M be midpoint of the arc AB. A circle (I) lying on the different side with M, respect to AB that tangent to AB and internally tangent to (O), given that. The lines pass through M and perpendicular to AI, BI intersect AB at C, D respectively. Prove that $AB = 2CD$.

Problem 22. (L) Let ABC be a triangle with circumcenter O, orthocenter H and M is midpoint of AH. Denote $(O_1),(O_2)$ as circles that both pass through H and tangent to BC at B, C respectively.

a) Prove that O_1, O_2, M are collinear.

b) Line O_1O_2 cuts BC at T. The circle (MOT) cuts line HO again at K. Prove that the center of (AOK) belongs to a midline of triangle ABC.

Additional problems.

Problem 23. (O) Let ABC be a triangle and M, N, P are midpoints of BC, CA, AB. Denote I as incenter of MNP. Prove that I is the radical center of 3 excircles of ABC.

Problem 24. (O) Prove that in a triangle, three Appolonius circles are coaxial.

Topic 4.

VIETA & BEZOUT THEOREM

- Vieta formula.
- Bezout theorem.
- About divisibility of polynomial.

Problem 25. (V)

a) Given $P(x)$ is a polynomial of degree 6 and $P(1)=P(-1), P(2)=P(-2), P(3)=P(-3)$. Prove that $P(2022)=P(-2022)$.

b) Let $P(x)$ is a monic polynomial of degree 4 and $P(1)=2, P(2)=5, P(3)=10$. Prove that $P(0)+P(4)$ is constant.

Problem 26. (O)

a) Find all monic polynomials of degree 3 such that

$$|P(1)|=|P(2)|=|P(3)|=|P(5)|=|P(6)|=|P(7)|.$$

b) Find all quadratic polynomials $P(x)$ such that $P(13)=2017$ and

$$x^2-2x+2 \le P(x) \le 15x^2-30x+16, \forall x.$$

Problem 27. (O) Let $P(x)$ be monic polynomial of degree 3 with 3 distinct roots. Consider polynomial $Q(x)=x^2+2x+2022$ such that $P(Q(x))$ has no real root. Prove that $P(2022)>1$.

Problem 28. (L) Let $P(x)=x(x^2-1)(x^2-a)-1$ with $a \ge 5$.

a) Prove that $P(x)$ has 5 distinct real roots x_1, x_2, x_3, x_4, x_5.

b) Calculate $f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)$ in two cases: $f(x) = x^2 - 2022$ and $f(x) = x^2 + 1$.

Additional problems.

Problem 29. (O) Let a, b, c, d be real numbers such that $\begin{cases} a+b+c+d > 0 \\ ab+bc+cd+da+ac+bd > 0 \\ abc+bcd+cda+dab > 0 \\ abcd > 0 \end{cases}$.

Prove that a, b, c, d are all positive.

Problem 30. (O)

a) Let $P(x)$ be monic polynomial of degree 2021 and has roots $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2022}$. Find the sum of coefficient of all odd-degree terms of $P(x)$.

b) Find all positive integers n such that in the monic polynomial $f_n(x)$ of degree n and has real roots $-n, -\frac{n}{2}, \ldots, -1$, the sum of coefficient of all odd-degree terms of $f_n(x)$ is also odd.

Problem 31. (L) Let $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ for $a, b, c \in \mathbb{R}$ such that it has four roots equal to the side length of some parallelogram. Find the minimum value of $T = a + b + c$.

Problem 32. (O) Find all polynomials $P(x)$ of degree 3 such that

$$(x^3 - 2x + 1 - P(x))(2x^3 - 5x^2 + 4 - P(x)) \le 0, \forall x \in \mathbb{R}.$$

Topic 4.

ISOGONAL CONJUGATE

- Conditions for isogonal conjugate lines, the anti-parallel figure.
- About symmedian and its properties.
- About Humpty-Dumpty point and the properties.
- About the completed quadrilateral and Gauss, Steiner line, Miquel point.
- Isogonal conjugate pair of points and the properties.

Problem 33. (L) Let ABC be a triangle and point D is on BC. Denote DE, DF, AK as the symmedians of triangle ADB, ADC, ABC. Prove that:

a) If D is the midpoint of BC then $DFEK$ is isosceles trapezoid.

b) If AD is the angle bisector of ABC then AK, BF, CE are concurrent.

Problem 34. (O) Let $ABCD$ be a cyclic quadrilateral. The lines AB, CD meet at E, and lines AD, BC meet at F. Prove that the angle bisector of $\angle E, \angle F$ and the Gauss line of the complete quadrilateral $ABCD.EF$ are concurrent.

Problem 35. (L) Let ABC be a triangle with M, N are midpoints of AB, AC, centroid G. Circles $(BMG), (CNG)$ meet again at K.

AoMP

a) Prove that AK is the symmedian of triangle ABC.

b) The line AK cuts $BC,(O)$ at T,D respectively. DB,DC cut $(ABT),(ACT)$ again at R,S. Prove that $OD \perp RS$ and T is the midpoint of RS.

Problem 36. (V) Let ABC be a triangle with $AB = AC$. Take P inside triangle ABC such that $\angle BPC = 180° - \angle A$. Line PB,PC cut AC,AB at D,E respectively. Denote I,J as the ex-centers of triangle ABD and ACE. The circles (ADE) cuts IJ again at T.

a) Prove that TP passes through a fixed point when P is moving.

b) Denote O as center of (ADE), prove that $OI = OJ$.

Problem 37. (O) Let ABC be a triangle with circumcircle (O) and the tangent line at B,C of (O) meet at T. Take $A' \in (O)$ such that $AA' \parallel BC$. Denote H,K as projections of T onto AB,AC and M as mipoint of BC. Circle (HKM) cuts BC again at L. Prove that $\angle AA'L = 90°$.

Problem 38. (O) Let $ABCD$ be a parallelogram with $\angle CAD = 90°$. Denote H as the projection of A onto CD and the tangent line of (ABD) at D cuts AC at K. Prove that $\angle KBA = \angle HBD$.

Problem 39. (O) Let ABC be a triangle with circumcircle (O). A circle (O') tangent to BC and internally tangent to (O). The angle bisector of $\angle BAC$ cuts (O') at D,E. Prove that D,E are isogonal conjugate.

Additional problems.

Problem 40. (O) Let ABC be a triangle with $\angle B = 2\angle C$. The perpendicular bisector of BC cuts AC at D. Denote E as reflection of D over A. Prove that BE is parallel to the symmedian of vertex A in triangle ABC.

Problem 41. (L) Let ABC be a triangle with circumcircle (O). Denote AD as its angle bisector and I as the incenter. The circle (BID) cuts AB again at M and the circle (CID) cuts AC again at N. Denote $IB \cap DM = X$, $IC \cap DN = Y$. Prove that the altitude, the median of triangle IXY pass through the midpoint of the arc BC of (O).

Problem 42. (L) Let ABC be a triangle with a circle passes through B,C cuts AB,AC at F,E respectively. Denote $BE \cap CF = T$ and $AT \cap BC = D$. Suppose that T is the Lemoine point of triangle DEF, prove that ABC must be isoceles triangle and (DEF) is its incircle.

Topic 5.

INEQUALITY ON SEQUENCE

- The increasing sequence with upper bound will have the limit.
- The sequence $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ is divergent.
- In the strictly increasing sequence of integers, $u_{n+m} \ge u_n + m$ for $m,n \in \mathbb{Z}^+$.

Problem 43. (O)

a) Let (a_n) be a sequence with $a_0 > 0$ and $a_n = \dfrac{a_{n-1}}{\sqrt{1+2021a_{n-1}^2}}, \forall n$. Prove that $a_{2021} < \dfrac{1}{2021}$.

b) Let (a_n) be a sequence with $a_1 = 1$ and $a_{n+1}^2 + a_{n+1} = a_n$, $\forall n \geq 1$. Prove that $a_{2021} > \dfrac{1}{2021}$.

Problem 44. (L)

a) Prove that there does not exist the sequence (x_n) of real numbers such that

$$x_1 = 2 \text{ and } \frac{2x_n^2+2}{x_n+3} < x_{n+1} \leq \frac{2x_n+2}{x_n+3} + 2022 \text{ for all } n = 1,2,3,\ldots$$

b) Similar question with $x_{n+2} = \ln\left(\dfrac{x_{n+1}^2+2022}{x_n^2+2022}\right)$ for all $n = 1,2,3,\ldots$

Problem 45. (L)

a) Consider the strictly increasing sequence (u_n) of positive integers such that for any 2022 consecutive positive integers, there are at least one number belongs to u_n. Prove that the following sequence is unbounded

$$v_n = \frac{1}{u_2^2-u_1^2} + \frac{1}{u_3^2-u_2^2} + \cdots + \frac{1}{u_{n+1}^2-u_n^2}.$$

b) Consider the strictly increasing sequence (u_n) of positive integers such that:

i) $u_1 > 1$ and all of terms of u_n are pairwise coprime.

ii) The sum $\dfrac{1}{\sqrt{u_1u_2}} + \dfrac{1}{\sqrt{u_3u_4}} + \cdots + \dfrac{1}{\sqrt{u_{2n-1}u_{2n}}}$ is unbound when $n \to +\infty$.

Prove that this sequence contains infinitely many prime numbers.

Problem 46. (O) Given (u_n) be a sequence of positive integer such that there exist constant c such that

$$u_{2n} + u_{2n-1} = c \cdot u_n \text{ for all } n \in \mathbb{Z}^+.$$

a) Prove that c is an integer.

b) Suppose that (u_n) is strictly increasing, find the minimum value of c.

Problem 47. (O) Does there exist infinite sequence (x_n) such that $0 < x_i < 1 + \dfrac{2021}{2022}, \forall i$ and

$$|x_i - x_j| \geq \frac{2}{i+j} \text{ for all } 1 \leq i < j?$$

Additional problems.

Problem 48. (O) For $k \in (0;1/2)$ and $a,b \in (0;1)$, consider sequences $(a_n),(b_n)$ such that

$$a_0 = a,\ a_{n+1} = \frac{1+a_n}{2} \text{ and } b_0 = b,\ b_{n+1} = (b_n)^k.$$

Prove that there exists n such that $a_n < b_n$.

Problem 49. (L) Consider sequence (a_n) with $a_1, a_2 > 0$ and $a_{n+2} = |a_{n+1} - a_n|, \forall n \geq 1$. Find the conditions of a_1, a_2 such that this sequence contains 0.

Problem 50. (O) Suppose that the equation $x^3 + 2x = 1$ has the unique real root r. Find all strictly increasing sequences (a_n) such that

$$\frac{1}{2} = r^{a_1} + r^{a_2} + r^{a_3} + \cdots$$