Email training, N3

September 8-14, 2019

Problem 3.1. Let (a_n) be sequnce of positive integers such that first k members $a_1, a_2, ..., a_k$ are distinct positive integers, and for each $n > k$, number a_n is the smallest positive integer that can't be represented as a sum of several (possibly one) of the numbers $a_1, a_2, ..., a_{n-1}$. Prove that $a_n = 2a_{n-1}$ for all sufficently large n.

Problem 3.2. Let p be a prime and let $f(x)$ be a polynomial of degree d with integer coefficients. Assume that the numbers $f(1), f(2), \ldots, f(p)$ leave exactly k distinct remainders when divided by p, and $1 < k < p$. Prove that

$$\frac{p-1}{d} \leq k - 1 \leq (p-1)\left(1 - \frac{1}{d}\right).$$

Problem 3.3. Three prime numbers p, q, r and a positive integer n are given such that the numbers

$$\frac{p+n}{qr}, \frac{q+n}{rp}, \frac{r+n}{pq}$$

are integers. Prove that $p = q = r$.

Problem 3.4. $a_1, a_2, ..., a_{100}$ are permutation of $1, 2, ..., 100$. $S_1 = a_1, S_2 = a_1 + a_2, ..., S_{100} = a_1 + a_2 + ... + a_{100}$. Find the maximum number of perfect squares from S_i

Problem 3.5. There are 100 students taking an exam. The professor calls them one by one and asks each student a single person question: "How many of 100 students will have a "passed" mark by the end of this exam?" The students answer must be an integer. Upon receiving the answer, the professor immediately publicly announces the student's mark which is either "passed" or "failed."

After all the students have got their marks, an inspector comes and checks if there is any student who gave the correct answer but got a "failed" mark. If at least one such student exists, then the professor is suspended and all the marks are replaced with "passed." Otherwise no changes are made.

Can the students come up with a strategy that guarantees a "passed" mark to each of them?

Problem 3.6. In a social network with a fixed finite setback of users, each user had a fixed set of followers among the other users. Each user has an initial positive integer rating (not necessarily the same for all users). Every midnight, the rating of every user increases by the sum of the ratings that his followers had just before midnight.

Let m be a positive integer. A hacker, who is not a user of the social network, wants all the users to have ratings divisible by m. Every day, he can either choose a user and increase his rating by 1, or do nothing. Prove that the hacker can achieve his goal after some number of days.

Problem 3.7. A circle with center O is inscribed in an angle. Let A be the reflection of O across one side of the angle. Tangents to the circle from A intersect the other side of the angle at points B and C. Prove that the circumcenter of triangle ABC lies on the bisector of the original angle.

Problem 3.8. The segment AB intersects two equal circles at 4 points and is parallel to the line joining their centres. From the point A tangents to the circle nearest to A are drawn, and from the point B tangents to the circle nearest to B are drawn. The tangent lines form a quadrilateral which contains both circles. Prove that a circle can be drawn that touches all four sides of the quadrilateral.

Solution submission deadline September 14, 2019