Number Theory – group L2

Instructor: Dušan Djukić *Date: 27.11.2021.*

1. Denote by $d_k(n)$ the number of divisors of n that are not less than k. Evaluate $d_1(2021) + d_2(2022) + \cdots + d_{2020}(4040)$.
2. Find all positive integers n for which $2^n + 5n$ is a perfect square.
3. Find all primes p for which $p^2 + 11$ has less than 12 divisors.
4. Find all positive integers x for which $3x^4 + 10x^2 + 3$ is a square.
5. Find all integer solutions (a, b, c, d) of the equation $6(6a^2 + 3b^2 + c^2) = 5d^2$.
6. Find all triples of positive integers (x, y, z) such that each of the numbers $x^2 - 1$, $y^2 - 2$, $z^2 - 4$ is divisible by $x + y + z$.
7. Given n positive integers, denote by d_k the greatest common divisor of all product of k of these integers. Prove that $d_k^2 \mid d_{k-1}d_{k+1}$ for $2 \leqslant k \leqslant n - 1$. (HW)
8. If x, y, z are rational numbers such that $xyz = 1$ and $x + y + z$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ are both integers, prove that $|x| = |y| = |z| = 1$. (HW)
9. For a positive integer n, define $f(n) = [\frac{n}{1}] + [\frac{n}{2}] + \cdots + [\frac{n}{n}]$. Prove that there are infinitely many n for which $\frac{f(n+1)}{n+1} < \frac{f(n)}{n}$. (HW)
10. By $\tau(x)$ we denote the number of divisors of a positive integer x. Prove that there are infinitely many positive integers k for which the equation $\frac{x}{\tau(x)} = k$ has no solutions $x \in \mathbb{N}$. (HW)

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7. Given n positive integers, denote by d_k the greatest common divisor of all product of k of these integers. Prove that $d_k^2 \mid d_{k-1}d_{k+1}$ for $2 \leqslant k \leqslant n-1$.

8. If x, y, z are rational numbers such that $xyz = 1$ and $x+y+z$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ are both integers, prove that $|x| = |y| = |z| = 1$.

9. For a positive integer n, define $f(n) = [\frac{n}{1}] + [\frac{n}{2}] + \cdots + [\frac{n}{n}]$. Prove that there are infinitely many n for which $\frac{f(n+1)}{n+1} < \frac{f(n)}{n}$.

10. By $\tau(x)$ we denote the number of divisors of $x \in \mathbb{N}$. Prove that there are infinitely many positive integers k for which the equation $\frac{x}{\tau(x)} = k$ has no solutions $x \in \mathbb{N}$.

11. Solve the equation $x! + 76 = y^2$ in positive integers.

12. Let p be a prime. If the equation $x^3 + px^2 = y^3$ has a solution in integers, prove that $3 \mid p-1$.

13. Find all positive integers k for which the equation $x(x+k) = y(y+1)$ has a solution in positive integers.

14. Given a positive integer n, does there always exist a positive integer divisible by n that has exactly n divisors?

15. Let $1 = d_1 < d_2 < d_3 < \cdots < d_k = 4n$ be all divisors of $4n$, where $n \in \mathbb{N}$. Prove that there is an index i for which $d_{i+1} - d_i = 2$.

16. Can $n(n+1)(n+2)(n+3)$ be a perfect cube for any $n \in \mathbb{N}$?

17. Positive integers x, y greater than 1 are such that x^2+xy-y is a perfect square. Prove that $x+y+1$ is a composite number.

18. Suppose that all divisors of n have been divided into pairs so that the sum in each pair is a prime. Prove that all these sums are distinct. (HW)

19. Determine all prime numbers $p > 2$ such that both $\frac{p+1}{2}$ and $\frac{p^2+1}{2}$ are perfect squares. (HW)

20. Can all integers greater than 10^{100} be written as a sum of a prime and a perfect square? (HW)

21. If $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n}[\frac{n}{i}]^2 = \sum_{i=1}^{n}(2i-1)[\frac{n}{i}]$. (HW)

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18. Suppose that all divisors of n have been divided into pairs so that the sum in each pair is a prime. Prove that all these sums are distinct.

19. Determine all prime numbers $p > 2$ such that both $\frac{p+1}{2}$ and $\frac{p^2+1}{2}$ are perfect squares.

20. Can all integers greater than 10^{100} be written as a sum of a prime and a perfect square?

21. If $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n}[\frac{n}{i}]^2 = \sum_{i=1}^{n}(2i-1)[\frac{n}{i}]$.

22. Prove that $2^{58}+1$ has at least three distinct prime divisors.

Chinese Remainder Theorem. Let $a_1, a_2, \dots, a_k$ be integers and let $n_1, n_2, \dots, n_k$ be pairwise coprime positive integers. Then the system of congruences

$$x \equiv a_i \pmod{n_i} \quad \text{for } i = 1, 2, \dots, k$$

has a unique solution x modulo $n_1 n_2 \cdots n_k$. $\square$

23. Prove that there exist 201 consecutive positive integers, each of which has a prime divisor not exceeding 103.

24. Suppose a and b are positive integers such that $\gcd(an+2, bn+3) > 1$ for every positive integer n. Prove that $b = \frac{3}{2}a$.

25. Solve the equation $x^2 + x = y^4 + y^3 + y^2 + y$ in positive integers,

The sum of divisors of a positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ equals

$$\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1} \cdots \frac{p_k^{\alpha_k+1}-1}{p_k-1}. \ \square$$

26. By $\sigma(n)$ we denote the sum of divisors of $n \in \mathbb{N}$. Find all n that satisfy $\sigma(n)+\sigma(2n) = \sigma(3n)$.

27. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant a(b-1)+b(c-1)+c(a-1)+1$. Show that equality occurs for infinitely many triples (a, b, c).

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22. Prove that $2^{58}+1$ has at least three distinct prime divisors.

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26. By $\sigma(n)$ we denote the sum of divisors of $n \in \mathbb{N}$. Find all n that satisfy $\sigma(n)+\sigma(2n) = \sigma(3n)$.

27. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant a(b-1)+b(c-1)+c(a-1)+1$. Show that equality occurs for infinitely many triples (a, b, c). (HW)

28. If $a, b > 2$ are integers, prove that 2^a+1 is never divisible by 2^b-1. (HW)

29. Find all positive integers n for which the sum of digits of $n!$ equals 9. (HW)

30. Let a, b, c, d be positive integers such that $b < c$ and $a+b+c+d = ab-cd$. Prove that $a+c$ is a composite number. (HW)

31. Find all pairs (k, n) of positive integers k, n such that $(2^n-1)(2^n-2)(2^n-2^2)\cdots(2^n-2^{n-1}) = k!$. (HW)

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27. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant a(b-1) + b(c-1) + c(a-1) + 1$. Show that equality occurs for infinitely many triples (a, b, c).

28. If $a, b > 2$ are integers, prove that $2^a + 1$ is never divisible by $2^b - 1$.

29. Find all positive integers n for which the sum of digits of $n!$ equals 9.

30. Let a, b, c, d be positive integers such that $b < c$ and $a + b + c + d = ab - cd$. Prove that $a + c$ is a composite number.

31. Find all pairs (k, n) of positive integers k, n such that $(2^n - 1)(2^n - 2)(2^n - 2^2) \cdots (2^n - 2^{n-1}) = k!$.

32. We are given $n \geqslant 3$ consecutive odd three-digit numbers. Prove that these n numbers can be ordered in a sequence $b_1, b_2, \ldots, b_n$ so that the number $\overline{b_1 b_2 \ldots b_n}$, obtained by writing these numbers one after another in the decimal system, be composite.

33. Find all pairs of positive integers (a, b) for which a is odd, b is a power of 2, and $a^2 - ab + b^2$ is a perfect square.

34. Denote by b_n the number of binary unit digits of a positive integer n. We call n *lively* if $b_n \mid n$. Prove that (a) there are no 5 consecutive positive integers that are all lively; (b) there are infinitely many instances of 4 consecutive positive integers that are all lively. (HW)

35. Prove that for every positive integer n there exists a positive integer x such that $\varphi(x) = n!$. (HW)

36. Call an integer an *almost-square* if it is a product of two consecutive integers. Prove that every almost-square can be written as a quotient of two other almost-squares. (HW)

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34. Denote by b_n the number of binary unit digits of a positive integer n. We call n *lively* if $b_n \mid n$. Prove that (a) there are no 5 consecutive positive integers that are all lively; (b) there are infinitely many instances of 4 consecutive positive integers that are all lively.

35. Prove that for every positive integer n there exists a positive integer x such that $\varphi(x) = n!$.

36. Call an integer an *almost-square* if it is a product of two consecutive integers. Prove that every almost-square can be written as a quotient of two other almost-squares.

37. Find all values of n for which there are positive integers a, b, c, d for which $a+b+c+d = n$ and $abc + abd + acd + bcd$ is divisible by n.

38. Find all triples of positive integers a, b, c such that $a^2+b^2 = c^2$ and $a^3+b^3 = (c-1)^3-1$.

39. Denote by $\omega(x)$ the number of distinct prime divisors of a positive integer x. Let a, b and c be arbitrary positive integers. Prove that there is a positive integer n such that $\omega(an + c) \geqslant \omega(bn + c)$.

40. Suppose that positive integers $a_1, a_2, \ldots, a_n$ have the property that each of the quotients $k_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ for $i = 1, 2, \ldots, n$ is an integer (here $a_0 = a_n$ and $a_{n+1} = a_1$). Prove that $2n \leqslant k_1 + k_2 + \cdots + k_n \leqslant 3n$.

41. Is there a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(xf(y)) = \frac{f(x)}{y}$ for all rational $x, y > 0$?

Solutions – group L2

Instructor: Dušan Djukić *Nov.26–Dec.4, 2021*

1. The summand $d_i(2020+i)$ counts the divisors not less than i, so a possible divisor $k \leqslant 2020$ would be counted only through the summands $d_1(2021), \dots, d_k(2020+k)$. Since k divides exactly one of the numbers $2021, 2022, \dots, 2020+k$, it follows that it has been counted exactly once. Thus the divisors from 1 to 2020 have been counted 2020 times in total.

   Additionally, the divisors from 2021 to 4040 have been counted once each. This gives the sum of 4040.

2. Note that n cannot be odd, as then $2^n + 5n \equiv \pm 2 \pmod 5$ cannot be a square. So $n = 2k$, but then $(2^k)^2 < 2^n + 5n < (2^k+1)^2$ for $n \geqslant 10$. It remains to check the small cases $n = 2, 4, 6, 8$. Only $n = 4$ is a solution.

3. If $p \geqslant 5$, then $p^2 + 11$ is divisible by 12. Thus for $p \geqslant 13$ it already has 12 divisors: $1, 2, 3, 4, 6, 12$ and $\frac{p^2+11}{1}, \frac{p^2+11}{2}, \frac{p^2+11}{3}, \frac{p^2+11}{4}, \frac{p^2+11}{6}, \frac{p^2+11}{12}$. For $5 \leqslant p \leqslant 11$ some of these divisors may overlap, so we check them manually. The solutions are $p \in \{2, 3, 5\}$.

4. We have $3x^4 + 10x^2 + 3 = (3x^2+1)(x^2+3)$. The GCD of $3x^2+1$ and x^2+3 divides $3(x^2+3) - (3x^2+1) = 8$, so it is 1, 2, 4 or 8. Therefore $3x^2+1$ and x^2+3 are either squares, or squares multiplied by 2. However, $x^2 + 3 = 2a^2$ is impossible modulo 3, so both factors must be squares. Then $x^2 + 3 = a^2$, which is only possible for $x = 1$. This is a solution indeed.

5. Hint: check the equation modulo 2 (or 4 or 8) and divide by two whatever is even. Be persistent. The only solution will be $(0, 0, 0, 0)$.

6. Denote $s = a + b + c$. Modulo s we have $x^2 \equiv 1$, $y^2 \equiv 2$ and $(x+y)^2 \equiv z^2 \equiv 4$. It follows that $2xy = (x+y)^2 - x^2 - y^2 \equiv 1 \pmod s$, so $4 \cdot 1 \cdot 2 \equiv 4x^2y^2 = (2xy)^2 \equiv 1 \pmod s$. Therefore $s \mid 7$, and for $s = 7$ we have a unique solution $(x, y, z) = (1, 4, 2)$.

7. We will prove that $v_p(d_{k-1}d_{k+1}) \geqslant v_p(d_k^2)$ for every prime p. Let the exponents at p in the given numbers be $r_1 \leqslant r_2 \leqslant \dots \leqslant r_n$. Then $v_p(d_i) = r_1 + \dots + r_i$, so $v_p(d_{k-1}d_{k+1}) = 2(r_1 + \dots + r_{k-1}) + r_k + r_{k+1}$ and $v_p(d_k^2) = 2(r_1 + \dots + r_k) \leqslant v_p(d_{k-1}d_{k+1})$.

8. Suppose that e.g. $v_p(x) = -k < 0$. Since $v_p(x+y+z) \geqslant 0$, one of $v_p(y), v_p(z)$ is $-k$, so the other one (since $xyz = 1$) is $2k$. Thus two of $v_p(x), v_p(y), v_p(z)$ are negative and one is positive. But then among $v_p(\frac{1}{x}), v_p(\frac{1}{y}), v_p(\frac{1}{z})$ only one is negative, so $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ cannot be an integer.

9. Hint: Prove (e.g. by induction) that $f(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Thus $f(n)$ is the average number of divisors of the numbers $1, 2, \ldots, n$. Now prove that every prime n works.

10. We will prove that if $k = p^{p-1}$, where $p \geqslant 5$ is a prime, the given equation has no solutions. Suppose that $x = p^{p-1}\tau(x)$ for some $x \in \mathbb{N}$. Clearly, $r = v_p(x) \geqslant p-1$. Moreover, $p^{p-1} = \frac{x}{\tau(x)} = \frac{p^r}{\tau(p^r)} \cdot \frac{x/p^r}{\tau(x/p^r)}$, which implies $\frac{x/p^r}{\tau(x/p^r)} = \frac{r+1}{p^{r+1-p}}$. If $r = p-1$, then $\frac{x/p^r}{\tau(x/p^r)} = p$, which is impossible because $p \nmid x/p^r$. On the other hand, if $r \geqslant p+1$, then $\frac{x/p^r}{\tau(x/p^r)} \leqslant \frac{p+2}{p^2} < 1$, which is also impossible. Finally, we have a contradiction for $r = p$ as well: $\frac{x/p^r}{\tau(x/p^r)} = \frac{p+1}{p}$ - indeed, note that $\tau(y) \leqslant \frac{y}{2} + 1 < \frac{p}{p+1}y$ for $y \geqslant p+1$.

11. If $x \geqslant 7$, then $x! + 76$ gives remainder 6 modulo 7 and cannot be a square. For smaller n we find the solutions $x = 4$ and $x = 5$.

12. If $p \mid x = pz$, then $x^3 + px^2 = p^3(z^3 + z^2)$, so $z^3 + z^2$ must be a cube as well, but this cannot happen because $z^3 < z^3 + z^2 < (z+1)^3$.

    If $p \nmid x$, then $x^3 + px^2 = x^2(x+p)$ and $\gcd(x^2, x+p) = 1$, so both $x+p$ and x^2 (and also x) must be perfect cubes. But if $x = a^3$ and $x + p = b^3$, then $b - a$ divides $b^3 - a^3 = p$, so $b - a = 1$ (clearly, it cannot be p). Thus $p = (a+1)^3 - a^3 = 3(a^2 + a) + 1$.

13. Multiply by 4 and complete squares to get $(2x + k)^2 - (2y + 1)^2 = k^2 - 1$. Now find suitable representations of $k^2 - 1$ as a difference of squares. You will get that all k except 2 and 3 work.

14. If $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, then take the number $p_1^{p_1^{r_1}-1} \cdots p_k^{p_k^{r_k}-1}$. Why is it divisible by n?

15. Suppose the statement is false. Consider the largest pair of even divisors $(a, a+2)$ of $4n$ (there is one such pair: $(2, 4)$). Then $a + 1$ must also be a divisor and it is odd, so $2a + 2$ is a divisor. Moreover, either $2a$ or $2a + 4$ is not divisible by 8 and hence is a divisor of $4n$. Thus we find a larger pair, namely $(2a, 2a + 2)$ or $(2a + 2, 2a + 4)$.

16. One of the factors $n + 1$ or $n + 2$ is coprime to all other factors and hence be a cube. Elminiating it, we find that $n(n + 1)(n + 3)$ or $n(n + 2)(n + 3)$ is a cube, but it lies between $(n + 1)^3$ and $(n + 3)^3$ if $n > 2$, which is impossible. The cases $n = 1, 2$ do not give a cube either.

17. If $x + y + 1 = p$ is a prime, then $y = p - 1 - x$ and $a^2 = x^2 + xy - y = p(x - 1) + 1$, so $p \mid a^2 - 1 = (a + 1)(a - 1)$, but $1 < a < p - 1$, which is impossible.

18. Hint: prove that if (a, b) is one such pair, then ab cannot exceed n, Deduce that in fact $ab = n$ in all such pairs. Now all pars look like $(a, \frac{n}{a})$, but these sums are distinct for all pairs.

19. Let $p + 1 = 2a^2$ and $p^2 + 1 = 2b^2$. Then $p \mid p^2 - p = 2(b^2 - a^2) = 2(b - a)(b + a)$, but $0 < a, b < p$, so we must have $a + b = p$; hence $b - a = \frac{p-1}{2}$, so $a = \frac{p+1}{4}$. Solving the equation in p yields $p = 7$.

20. Can n^2 with $n > 10^{50}$ always be written in that form? If $n^2 = a^2 + p$, then $p = (n+a)(n-a)$, so $a = n-1$ and $p = 2n-1$, so it is possible only when $2n-1$ is a prime, which is not always the case.

21. Use induction on n (base $n=1$). For the inductive step, when $n-1$ increases to n, only the summands corresponding to $i \mid n$ change, as then $[\frac{n}{i}] = [\frac{n-1}{i}] + 1$. Verify that both sides of the equality get exactly the same increment.

22. $2^{58} + 1 = 4a^4 + 1 = (2a^2 - 2a + 1)(2a^2 + 2a + 1)$, where $a = 2^{14}$. The two factors are coprime and greater than 5. Also, $2a^2 + 2a + 1$ is divisible by 5 but not by 25. Thus $2a^2 + 2a + 1$ gives two distinct prime factors and $2a^2 - 2a + 1$ gives a third one.

23. Take n to be divisible by $100!$. Then each of the numbers $n-100, n-99, \ldots, n+100$ has a prime divisor not exceeding 100, except for $n \pm 1$. We cover these two by setting n so that $101 \mid n+1$ and $103 \mid n-1$ (and $100! \mid n$). Such an n exists by the Chinese Remainder Theorem.

    By the way, $100! + 1$ is divisible by 101.

24. Note that $\gcd(an+2, bn+3)$ divides $a(bn+3) - b(an+2) = 3a - 2b$. Now set $n = |3a - 2b|$ if it is nonzero. Then $\gcd(an+2, bn+3)$ divides 2 and 3, so it is 1, contrary to the assumption. Therefore $|3a - 2b| = 0$.

25. Complete the square: $4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2x+1)^2$ is a square, but it lies between $(2y^2 + y)^2$ and $(2y^2 + y + 1)^2$ for $y \geqslant 2$. The small cases give the unique solution $(x, y) = (5, 2)$.

26. Write $n = 2^a 3^b m$, where $\gcd(m, 6) = 1$. Then $\sigma(2n) = \frac{2^{a+2}-1}{2^{a+1}-1}\sigma(n)$ and $\sigma(3n) = \frac{3^{b+2}-1}{3^{b+1}-1}\sigma(n)$, so the given equation becomes $\frac{2 \cdot 3^{b+1}}{3^{b+1}-1} = \frac{2^{a+2}-1}{2^{a+1}-1}$. This simplifies to $3^{b+1} = 2^{a+2} - 1$. Now prove this is only possible when $a = b = 0$, so the answer is all n not divisible by 2 or 3.

27. The given product of GCD's divides both abc and $(b-1)(c-1)(a-1)$, so it divides the difference, which is $a(b-1) + b(c-1) + c(a-1) + 1$.

    We find an equality case by setting $(a, b, c) = (n, n = 1, n+2)$, with $3 \mid n-1$.

28. Suppose $a = br + q$, where $q \geqslant 0$ and $0 \leqslant r < b$ are integers. If $2^b - 1 \mid 2^a + 1 = 2^r + 1 + 2^r(2^{qb} - 1)$, then also $2^b - 1 \mid 2^r + 1 \leqslant 2^{b-1} + 1 < 2^b - 1$, which is impossible.

29. Hint: Can a number whose sum of digits is 9 be divisible by 11?

30. We have $(a-1)(b-1) = (c+1)(d+1) = n$ for some n, so $a-1$ and $c+1$ are divisors of n whose product is greater than n; hence $\gcd(a-1, c+1) > 1$, so $a + c = (a-1) + (c+1)$ cannot be prime.

31. Take v_2 of both sides: $v_2(LHS) = \frac{n(n-1)}{2}$ and $v_2(RHS) < k$, so $k > \frac{n(n-1)}{2}$, but then $k!$ will be too big: prove that $\frac{n(n-1)}{2}! > 2^{n^2} > LHS$ if $n \geqslant 5$. Check the small cases manually.

32. If $n = 5$, one of $b_1, \dots, b_n$ is divisible by 5, so put it on the last place. If $n = 4$, order them as b_1, b_2, b_4, b_3 to get a number divisible by 11. And if $n = 3$, they will always yield a multiple of 3.

33. Let $b = 2^n$ and $a^2 - ab + b^2 = c^2$. We can rewrite this as $3 \cdot 2^{2n-2} = \frac{3}{4}b^2 = c^2 - (a - \frac{b}{2})^2 = (c + a - 2^{n-1})(c - a + 2^{n-1})$.

    We easily check the cases $n \leqslant 2$ and find no solutions. Assume that $n \geqslant 3$. Then the factors $c + a - 2^{n-1}$ and $c - a + 2^{n-1}$ are even and not both multiples of 4, so one of them equals ± 2 or ± 6. Assuming c is positive, both factors are positive as well. Checking all four possibilities we find only two possibilities for $n \geqslant 3$: $a = 2^{2n-4} + 2^{n-1} - 3$ or $a = 3 \cdot 2^{2n-4} + 2^{n-1} - 1$, and in addition, $a = 3$ for $n = 3$.

34. (a) The numbers $4k + 1$ and $4k + 2$ have the same number of binary unit digits, so they cannot both be lively. However, in every five consecutive numbers one can find $4k + 1$ and $4k + 2$.

    For (b), set n so that $b_n = 6$, $b_{n+1} = 7$, $b_{n+2} = 4$ and $b_{n+3} = 5$. We can take e.g. $n = 2^a + 2^b + 2^c + 14$ with $a > b > c > 4$. Then see how to make $n, n+1, n+2, n+3$ all lively.

35. Take $x = n \prod_p \frac{p}{p-1}$, where the product is over the primes $p \leqslant n + 1$. Prove that it works.

36. Note that $(x - 1)x = \frac{(x^2-1)x^2}{x(x+1)}$.