Level 2 E-training, week 3

Due to 23:59, Friday, 25 September 2020

Problem 1. Let x be a real number and n be a positive integer. Prove that

$$\lfloor nx \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{n} \rfloor + \ldots + \lfloor x + \frac{n-1}{n} \rfloor$$

Problem 2. On a board there are 7 nails, each two connected by a rope. Each rope is colored in one of 7 given distinct colors. Is it possible that, for each three distinct colors, there will be three nails connected with ropes of these three colors?

Problem 3. Consider the line t in the plane and draw 3 circles tangent to t and externally tangent to each other. Prove that, for some permutation of their radii (a, b, c), one has

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$$

Problem 4. Let $n > 2$ be an integer. Show that

$$\varphi(\varphi(n)) \leq \frac{\varphi(n)}{2}$$

Problem 5. The positive reals x, y, z satisfy the equation $x^2 + y^2 + z^2 = 2(xy + yz + zx)$. Prove that

$$\frac{x+y+z}{3} \geq \sqrt[3]{2xyz}$$

Problem 6. Let ABC be a non-equilateral acute-angled triangle with circumcenter, incenter, orthocenter O, I, H, respectively. Suppose that the circumcircle of OIH passes through some vertex of ΔABC. Prove that one of the angles of ΔABC is $60°$.

Problem 7. Suppose that $a \in \mathbb{Z}$ and $p \in \mathbb{P}$ such that $p | a^{p^2} - 1$. Prove that $p^3 | a^{p^2} - 1$

Problem 8. Let $n \in \mathbb{N}$. Deemah and Bayan play the following game on a pile of stones: Initially there are n stones in the pile, Deemah and Bayan take turns alternatively. In her turn, a player chooses a prime number p and a nonnegative integer k and removes p^k stones from the pile, the game ends when the pile is empty and the last one removing stones wins. If Deemah plays first, find all n for which Bayan has a winning strategy.