Email training, N1

Level 3, September 13-19, 2021

1. Problems with Solutions

Problem 1.1. Does there exist a polynomial $P(x)$ of second degree with integer coefficients such that the leading coefficient is not divisible by 2018 and all numbers $P(1)$, $P(2)$, ..., $P(2018)$ give different residues mode 2018.

Solution 1.1. Yes, there exisits. For example $1009x(x+1)+x$. First term is always divisible by 2018 since $x(x+1)$ is even for all integer numbers x. It remains to consider term x which gives different residues mod 2018.

Problem 1.2. Prove that for any 2 positive integers m and n with $(m,n) \neq (1,1)$ the value of expression

$$S = \frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{m+n-1}$$

is not an integer.

Solution 1.2. If $n \leq m$ then it's obvious that

$$S = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n-1} < n \cdot \frac{1}{m} \leq 1$$

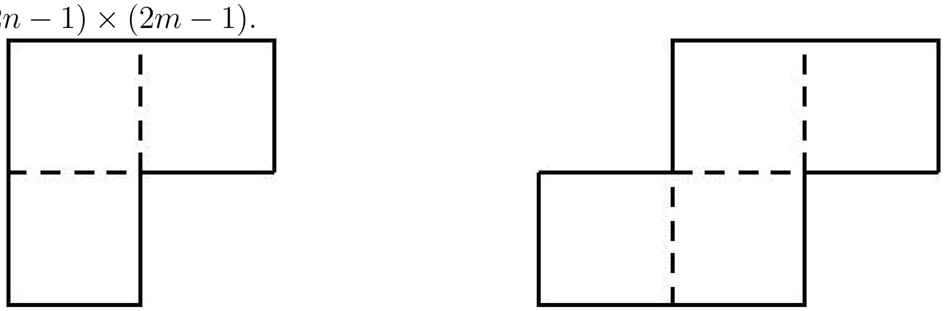
which means that S isn't integer number.

Now let us consider the case when $n > m$: Then between integers$m$ and $m+n-1$ there is at least one power of 2. Let k be the biggest integer for which one has $m \leq 2^k \leq m+n-1$: Note, that in the expression

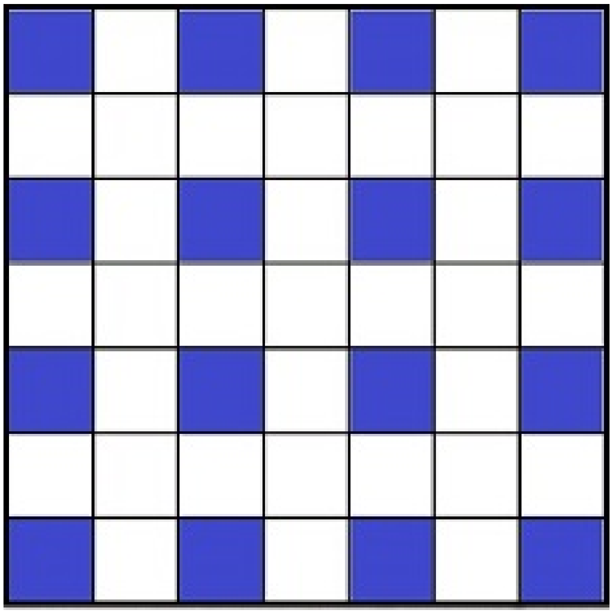
$$2^k S = \frac{2^k}{m} + \frac{2^k}{m+1} + \cdots + \frac{2^k}{2^k} + \ldots + \frac{2^k}{m+n-1}$$

(after substracting nominators and denominators) all denominators are odd numbers and all nominators are even numbers but $\frac{1}{1}$: By calculating the some one can see that the common denominator is odd number and all terms in nominators are even number but one term. So one has that the expression has the form $\frac{a}{b}$ where both a and b are odd. It means that $S = \frac{a}{2^k b}$ is not integer, even if $b = 1$.

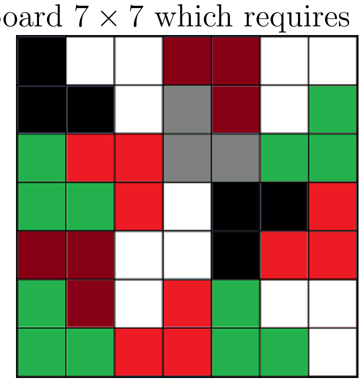
Problem 1.3. Let m and n are positive integers greater than 3. Determine the least number of figures (see picture below) needed to cover the rectangle of size $(2n-1) \times (2m-1)$.



Solution 1.3. Let's consider the following tiling.



There are mn blue cells. Each Z figure covers exactly one blue cell and each L figure covers at most one blue cell, so one needs at leasthas mn figures. Let's prove that the figure is possible to cover with exactly mn figures. The starting point is the board 7×7 which requires 16 figures and below is the example



When adding 2 rows or 2 columns, then it can be covered by the sequence $L - Z - Z - \ldots - Z - Z - L$ like below