Saudi Arabia – Math Camp

Geometry – Inversion

Level 4

Instructor: Regis Barbosa

Definition 1. Let $k(O,r)$ be a circle of center O and radius r. Consider the function on the plane $I: \mathbb{R}^2 \to \mathbb{R}^2$ sending $X \neq O$ to the point $X' \in \overrightarrow{OX}$ such that

$$OX \cdot OX' = r^2$$

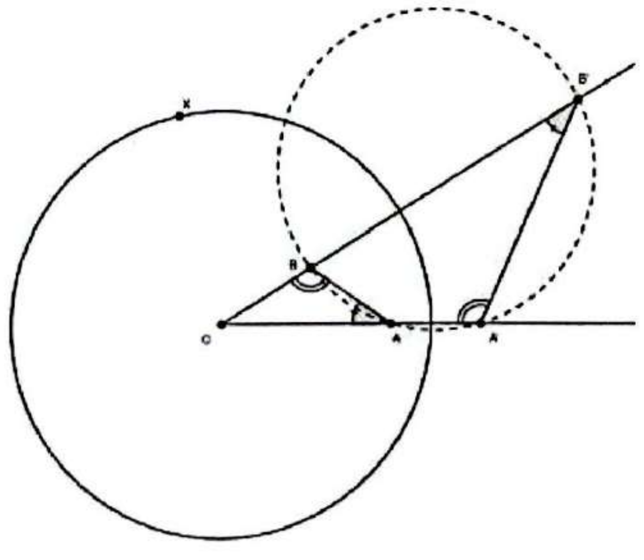
Such a function is called an inversion of the plane with center O and radius r or an inversion with respect to the circle $k(O,r)$.

Properties of Inversion – Part 1

- The inverse points of the points on k are themselves. $X \in k \Leftrightarrow X' = X$.
- I^2 is the identity on the plane. It means that $(X')' = X$ or $I(X) = Y \Rightarrow I(Y) = X$.
- If $O \notin \overleftrightarrow{AB}$, then $OA \cdot OA' = OB \cdot OB' \Leftrightarrow A, A', B$ and B' are concyclic.
- If $O \notin \overleftrightarrow{AB}$, then $OA \cdot OA' = OB \cdot OB' \Leftrightarrow \frac{OA}{OB'} = \frac{OB}{OA'}$ and $\angle AOB = \angle B'OA'$ implying that $\Delta OAB \sim \Delta OB'A'$ (SAS) and

$$\angle OAB = \angle OB'A'$$

$$\frac{A'B'}{BA} = \frac{OA'}{OB} = \frac{r^2}{OA \cdot OB} \Rightarrow A'B' = AB \cdot \frac{r^2}{OA \cdot OB}$$



Problems

1. (IberoAmerican/2015) A line r contains the points A, B, C, D in that order. Let P be a point not in r such that $\angle APB = \angle CPD$. Prove that the angle bisector of $\angle APD$ intersects the line r at a point G such that:

$$\frac{1}{GA} + \frac{1}{GC} = \frac{1}{GB} + \frac{1}{GD}$$

2. (Ptolemy Theorem) Let $ABCD$ be a convex quadrilateral. Prove that

$$AC \cdot BD \leq AB \cdot CD + AD \cdot BC$$

with equality iff the quadrilateral is cyclic.

3. Let P be a point inside the acute angle $\angle ABC$. Show how to draw a line through P that cuts the lines AB at M and BC at N such that $\frac{1}{MP} + \frac{1}{NP}$ is a maximum.

4. (IMO Shortlist/2008) Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

5. (IMO/1996) Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.