Email training, N6

Level 2, October 18-24

Problem 6.1. Let $a^2 + b^2 > a + b$ with $a > 0$ and $b > 0$. Prove that

$$a^3 + b^3 > a^2 + b^2.$$

Solution 6.1. If we prove the inequality $(a + b)(a^3 + b^3) > (a + b)^2(a + b)^2$, then from that will follow that $a^3 + b^3 > a^2 + b^2$. Lets verify that. By expanding the brackets we get

$$a^4 + b^4 + a^3b + ab^3 > a^4 + b^4 + 2a^2b^2,$$

which is equivalent to

$$ab(a^2 + b^2) > 2a^2b^2.$$

After subtracting both parts by ab and taking into account that $ab > 0$ we get $a^2 + b^2 > 2ab$, which is obviously correct.

Problem 6.2. Let the sequence x_n is given such that $0 < x_1 < 1$ and $x_{k+1} = x_k - x_k^2$ for all $k \geq 1$. Prove that for all n one has

$$x_1^2 + x_2^2 + \ldots + x_n^2 < 1.$$

Solution 6.2. Since for $0 < a < 1$ one has $a^2 < a$, therefore from $0 < x_1 < 1$ and the recurrence relation follows that $0 < x_2 < 1$, $0 < x_3 < 1$ and so on. Further by using $x_k^2 = x_k - x_{k+1}$ we may write

$$x_1^2 + x_2^2 + \ldots + x_n^2 = (x_1 - x_2) + (x_2 - x_3) + \ldots + (x_n - x_{n+1}) = x_1 - x_{n+1} < 1.$$

Problem 6.3. Find the maximum value of expression $\sqrt{x^2 + y^2}$ if it's known that

$$\{-4 \leq y - 2x \leq 2, \quad 1 \leq y - x \leq 2\}.$$

Solution 6.3. Let's notice, that the given region is a quadrilateral with it's internal region. In fact we need to find the most far point of the quadrilateral from the point $(0, 0)$. It's obvious, that we are looking for the one of the vertices of the quadrilateral. By simple calculation we get the following vertices $A(6, 8)$, $(0, 2)$, $C(-1, 0)$ and $D(5, 6)$. From these points the point A is the most far and has the distance 10.

Answer: 10.

Problem 6.4. Prove that for any numbers a, b, $c > 0$ the following inequality holds

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \frac{2}{a} + \frac{2}{b} - \frac{2}{c}.$$

Solution 6.4. After bringing to the common denominator and eliminating abc we get the following equivalent inequality

$$a^2 + b^2 + c^2 \geq 2bc + 2ac - 2ab$$

which is equivalent to

$$(a + b)^2 + c^2 \geq 2c(a + b).$$

The last one is the known inequality $x^2 + y^2 \geq 2xy$.

Problem 6.5. Prove the inequality

$$\sqrt{a + 1} + \sqrt{2a - 3} + \sqrt{50 - 3a} \leq 12.$$

Solution 6.5. By using Cauchy inequality we get

$$1 \cdot \sqrt{a + 1} + 1 \cdot \sqrt{2a - 3} + 1 \cdot \sqrt{50 - 3a} \leq$$

$$\sqrt{1^2 + 1^2 + 1^2} \cdot \sqrt{\sqrt{a + 1}^2 + \sqrt{2a - 3}^2 + \sqrt{50 - 3a}^2} = 12.$$

Problem 6.6. Let the parabola $y = x^2 + px + q$ is given, which intersects coordinate axes in 3 different points. Consider the circumcircle of the triangle having vertices these 3 points. Prove that there is a point that belongs to that circle, regardless of values p and q. Find that point.

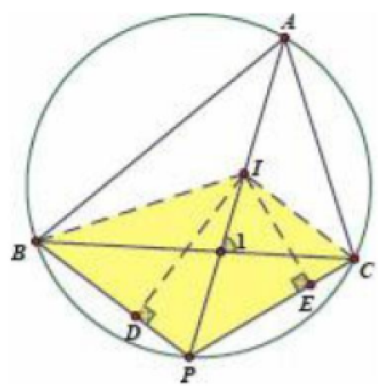
Solution 6.6. For some p and q draw the requested circle and take the intersection point of the circle and y-axis, namely $(0, a)$ (different from $(0, q)$). The circle has 2 chords (axes) that intersect at point $(0, 0)$. By applying the chord rule we get $x_1x_2 = qa$, where x_1 and x_2 are intersection point coordinates of the circle and x-axis. According to the Viet theorem we have $x_1x_2 = q$, so $a = 1$. We conclude that the circle passes the point $(0, 1)$, independently from the values of p and q.

Problem 6.7. Let I be the incenter of ΔABC. Let AI is extended and intersects the circumcircle of ΔABC at P. Draw $ID \perp BP$ at D and $IE \perp CP$ at E. Show that: $ID + IE = AP sin\angle BAC$.

Solution 6.7. -

We have $\angle PIC = \dfrac{\angle BAC}{2} + \dfrac{\angle ACB}{2} = \angle ICP$

So $PI = PC$, similarly we find $PI = PB$, and then $PI = PB = PC$.
However, one may find it difficult to construct a line segment equal to $ID + IE$. Since ID, IE are heights, perhaps we could use the area method. Notice that:



$$[BPCI] = [\Delta BPI] + [\Delta CPI] = \frac{1}{2}BP \cdot ID + \frac{1}{2}CP \cdot IE = \frac{1}{2}BP \cdot (ID + IE).$$

On the other hand, $[BPCI] = \frac{1}{2}BC \cdot PI \sin\angle 1 = \frac{1}{2}BC \cdot BP \sin\angle 1$. (*)

It follows that $ID + IE = BC \sin\angle 1$. Now it suffices to show that $BC \sin\angle 1 = AP \sin\angle BAC$.

Is it reminiscent of Sine Rule? Shall we show that $\dfrac{BC}{\sin\angle BAC} = \dfrac{AP}{\sin\angle 1}$?

Indeed, applying Sine Rule repeatedly gives $\dfrac{BC}{\sin\angle BAC} = \dfrac{AB}{\sin\angle ACB}$ $= \dfrac{AB}{\sin\angle APB} = \dfrac{AP}{\sin\angle ABP}$.

One sees the conclusion by showing $\angle ABP = \angle 1$. But this is true, because both of them equals $\dfrac{\angle BAC}{2} + \angle ABC$