# Competition Preparation for Saudi Arabia Team: Level 4

Nikola Petrović

## Lesson 8

## Miscellaneous

**Problems:**

1. A unit square is partitioned into rectangles. From each rectangle we select the smaller of the two sides or any side if the rectangle is a square. Prove that the sum of selected sides is at least 1.

2. A rectangle is partitioned into smaller rectangles such that each smaller rectangle has at least one side of integer length. Prove that the original rectangle also has a side of integer length.

3. Given a set $A$ with $n^2$ elements and $n \geq 2$, let $F$ be a collection of subsets of $A$, each with $n$ elements, such that each two subsets of the collection share at most one common element. Prove that $|F| \leq n^2 + n$.

4. Positive integer points on a line are colored with one of two colors. Show that there exists an infinite sequence of natural numbers $a_1$, $a_2$, $a_3$, $\ldots$, such that points $a_1$, $\frac{a_1+a_2}{2}$, $a_2$, $\frac{a_2+a_3}{2}$, $a_3$, $\frac{a_3+a_4}{2}$, $\ldots$ are all of the same color.

5. A car races along a circular track along which there are several gas stations, each with a fixed (non-replenishable) amount of gas. The total amount of gas on the stations is just large enough to complete one round trip along the track. Prove that there exists a gas station from which the car can start with an empty (and large enough) tank and complete a round trip (i.e. return to the starting station without running out of gas). Assume that the car consumes a fixed amount of gas per unit length.

6. A partition of $n$ is the division of the set $\{1, 2, \ldots, n\}$ into several nonempty disjoint sets. Prove that the number of partitions of $n$ equals the number of partitions of $n + 1$ so that no two consecutive numbers are in the same set. (For example the number of partitions of $\{1, 2, 3\}$ is 5: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$ and $\{\{1, 2, 3\}\}$, and the number of partitions of $\{1, 2, 3, 4\}$ such that no two consecutive numbers are in the same set is also 5: $\{\{1\}, \{2\}, \{3\}, \{4\}\}$, $\{\{1, 3\}, \{2\}, \{4\}\}$, $\{\{1, 4\}, \{2\}, \{3\}\}$, $\{\{2, 4\}, \{1\}, \{3\}\}$ $\{\{1, 3\}, \{2, 4\}\}$.)

7. On a spherical planet $X$ there are $2n$ gas stations. Each station is paired with a gas station located on the diametrically opposite point on the planet. Each station has a fixed (non-replenishable) amount of gas. The arrangements of stations and the amounts of gas at each station are such that it is possible for a car with an empty (and large enough) reservoir to start from any station and reach its paired station (filling up its reservoir with any gas it encounters along the way). For which $n$ does it follow that it is possible for the car to start from some gas station and visit all the other gas stations on the planet? Assume that the car consumes a fixed amount of gas per unit length.

8. In every square of a $2n \times 2n$ board there is a lightbulb that can be on or off. In each step one can chose a row or column in which at least $n$ lightbulbs are on and change the state of all the lightbulbs in this row or column.

   (a) Prove that there is a starting arrangement of bulbs such that no matter how we play there will always be at least $2n(n + 1)$ lightbulbs that are on.

   (b) Prove that we can always reach a state with at most $2n(n + 1)$ lightbulbs on.

9. On a $2021 \times 2021$ board we place on each square a coin. In each move we remove a coin which is adjacent to a positive even number of coins as long as at least one such coin exists. Find $k$, the lowest possible number of coins we can obtain and prove that if we end up with $k$ coins then no two coins are adjacent. A square is adjacent to another square if it has at least one common vertex.

10. A snake is a collection of $k$ unit squares $A_1A_2\dots A_k$ such that squares $A_i$ and $A_j$ are adjacent if and only if $|i-j|=1$, with $A_1$ and $A_k$ being the ends of the snake. Two squares are adjacent if they share a common edge. Let a $(2n+1)\times(2n+1)$ board be partitioned into two snakes. Prove that the end of one snake is in the center of the board.