November camp – KSA training 2022. Level 4 & 4+.

SOME TRICKS ON FUNCTIONAL EQUATION

- Injective at single point.
- Inequality of FE.

Problem 1. Prove that the following non-constant functions $f:\mathbb{R}\to\mathbb{R}$ are injective at 0:

a) $f(x+xy^2f(x))+f(x^2yf(y)-y)=f(x)-f(y)+2x^2y^2$ for all $x,y\in\mathbb{R}$.

b) $2f(x)f(x+y)-f(x^2)=\frac{x}{2}f(2x)+2xf(f(y))$ for all $x,y\in\mathbb{R}$.

c) $f(x+2022f(xy))=f(x)+2022xf(y)$ for all $x,y\in\mathbb{R}$.

Problem 2. Find all function $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for $x,y,z>0$ pairwise distinct then

$$f(x)^2-f(y)f(z)\le f(xy)f(y)f(z)[f(yz)-f(zx)].$$

Problem 3.

a) Consider funnction $f:\mathbb{R}\to\mathbb{R}$ such that

$$x^2f(x-y^2)\ge(x+2y+1)f(x) \text{ for all } x,y\in\mathbb{R}.$$

Calculate $f(2022)$.

b) Prove that there does not exist function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x-f(y))\le x-yf(x) \text{ for all } x,y\in\mathbb{R}.$$

c) Find all function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)-x-f(y)\le f(xy)-yf(x) \text{ for all } x,y\in\mathbb{R}.$$

Problem 4. Find all surjective function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(yf(x+y)+x)=f(y)^2+f((x-1)f(y)) \text{ for all } x,y\in\mathbb{R}.$$

Problem 5. Find all function $f:[0;+\infty)\to[0;+\infty)$ such that

i) $f(f(x))=x^4$ for all $x\ge 0$;

ii) There exist some constant c such that $f(x)\le cx^2$ for all $x\ge 0$.

Problem 6. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfy the following conditions

i) $f(2x)\ge 2f(x)$ for all $x>0$;

ii) f is strictly increasing $(0;+\infty)$.

iii) $f(f(x)f(y)+x)=f(xf(y))+f(x)$ for all $x,y>0$.

Problem 7*. Consider the function $f:\mathbb{R}\to\mathbb{R}$ such that

$$(f(x)+y)(f(x-y)+1)=f(f(xf(x+1))-yf(y-1)) \text{ for all } x,y\in\mathbb{R}.$$

a) Prove that $f(x)$ is injective at 0.

b) Find all functions $f(x)$ satisfy the given condition.