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Topic 2.

POLYNOMIAL 1

Note that. *(Bezout's theorem) If polynomial* $f(x)$ *has root* $x = x_0$ *then* $f(x) = (x - x_0)g(x)$ *for some polynomial* $g(x)$.

Corollary. Suppose $\deg f = n$ and f has n root $x_1, x_2, \dots, x_n$ then

$$f(x) = c(x - x_1)(x - x_2)\dots(x - x_n).$$

Problem 1.

Let $P(x)$ be a integer coefficient polynomial with at least 9 distinct integer roots. Suppose that there exist $x_0 \in \mathbb{Z}$ such that $|P(x_0)| < 2019$. Prove that $P(x_0) = 0$.

Problem 2.

a) Let $P(x)$ be a quadratic polynomial with $x^2 - 2x + 3 \le P(x) \le 2x^2 - 4x + 4$ for all x. Given that $P(3) = 7$, find $P(7)$.

b) Let $P(x)$ is a cubic polynomial and $Q(x) = (x^3 - 2x + 1 - P(x))(2x^3 - 5x^2 + 4 - P(x))$. Suppose that $Q(x) \le 0, \forall x$ and $P(0) = 3$, find $Q(-1)$.

Problem 3.

Consider $P(x) = 5x^3 - 40x^2 + 100x - 79$. Prove that $P(x)$ has three distinct real roots a, b, c and they are the sidelength of some triangle ABC. Prove that the length of the medians of ABC also form another triangle.

Problem 4.

Consider $P(x), Q(x), R(x) \in \mathbb{R}[x]$ as three non constant polynomials and satisfy

$$P(x^2 - x) + xQ(x^2 - x) = (x^2 - 4)R(x) \text{ for all } x.$$

a) Prove that the equation $Q(x) = R(x - 3)$ has at least two distinct real roots.

b) Suppose that the sum of degree of $P(x), Q(x), R(x)$ is 5 and $R(x)$ is monic. Find the minimum value of $M = P^2(0) + 8Q^2(3)$.

Problem 5.

Find all integer k such that the polynomial $P(x) = x^{n+1} + kx^n + 13x^2 + 4x + 2019$ has integer root for infinite many positive integers $n \ge 3$.

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Topic 3.

POLYNOMIAL 2

Problem 1. Consider two polynomials

$$P(x)=x^5+5x^4+5x^3+5x^2+1 \text{ and } Q(x)=x^5+5x^4+3x^3-5x^2-1.$$

Find all prime p such that there exist some nonnegative integer x_0 less than p and $P(x_0), Q(x_0)$ are all divisible by p.

Problem 2. Suppose that polynomials P,Q of real coefficient such that

$$[P(x)]=[Q(x)] \text{ for all } x.$$

Prove that $P(x)=Q(x)$.

Problem 3. Find all polynomial $f\in\mathbb{Z}[x]$ such that for all integer $n>2019$, then

$$f(n)>1 \text{ and } f(n+g(n))=n+g(f(n)),$$

in which $g(n)$ is the greatest prime divisor of integers $n>1$.

Problem 3. Let f be a integer polynomial such that for primes $p>2019$ then

$$f(p)>0 \text{ and } 2(f(p)!)+1 \text{ is divisible by } p.$$

a) Prove that $f(x)=x-c$ for some $c\in\mathbb{Z}$.

b) Find c.

Problem 4. Let f be a polynomial with integer coefficients such that: for each positive number a, there is some positive integer b such that $f^{(b)}(a)=a$.

a) Is there any a such that $b\geq 3$?

b) Find all f satisfy the given condition.

Problem 5. H Lz

a) Prove that for all $n\in\mathbb{Z}^+$ and $x>0$ then $\dfrac{x^{2n}+x^{2n-2}+\cdots+x^2+1}{x^{2n-1}+x^{2n-3}+\cdots+x}\geq\dfrac{n+1}{n}$.

b) Find the smallest $n\in\mathbb{Z}^+$ such that there exist some polynomial $P(x)$ of degree $2n$ such that : all of its coefficients are in the range $[7;8]$ and it has some real root.

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Topic 4.

POLYNOMIAL 3

Problem 1.

a) Suppose that polynomial $x^{2018} \pm x^{2017} \pm \cdots \pm x \pm 1$ has no real root. At most how many signs $-$ we can have?

b) Consider equation $(x-1)(x-2)\cdots(x-16) = (x-1)(x-2)\cdots(x-16)$. At least how many factors we need to delete to make the remain equation has no real root?

Problem 2.

a) Prove that if $a, b, c \in \mathbb{Z}$ such that $a \cdot \sqrt[3]{4} + b \cdot \sqrt[3]{2} + c = 0$ then $a = b = c = 0$.

b) Let $P(x)$ be a polynomial with nonnegative integer coefficients and $P(\sqrt[3]{2}) = 2019$. Find the minimum value of the sum of all coefficients of P.

Problem 3.

Prove that the polynomial $P(x) = (x^2 - 11x + 10)^4 + 23$ cannot be expressed as the product of three integer polynomials.

Problem 4.

Denote $P(x) \in \mathbb{Z}[x]$ be a monic polynomial with 3 distinct irrational roots a, b, c such that their sum is equal to 0. Suppose that there are some $m, n \in \mathbb{Z}$ for $a = b^2 + mb + n$. Prove that

$$(m-a)(m-b)(m-c) = 1.$$

Problem 5.

a) Suppose that $P(x)$ is polynomial such that $P(x)P(x+1) = P(x^2+1)$. Prove that the degree of $P(x)$ must be even.

b) Find all polynomial $P(x)$ such that

$$P(-x)P(3x) + P^2(2x) = P(x)P(5x) \text{ for all } x \in \mathbb{R}.$$

Problem 6.

a) Find all nonconstant polynomial $P(x) \in \mathbb{Z}[x]$ such that for any $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 \neq c^2$ then $a^2 + b^2 - c^2 \mid P(a) + P(b) - P(c)$.

b) Find condition of two real numbers a, b and $ab \neq 0$ such that there exist a polynomial of degree 2019 such that

$$xP(x-a) = (x-b)P(x) \text{ for all } x \in \mathbb{R}.$$

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Topic 5.

POLYNOMIAL 5

Problem 1.

Let $f(x)\in\mathbb{Z}[x]$ be a polynomial satisfies $f\left(\frac{a}{b}\right)=0$ for some integers a,b and $\gcd(a,b)=1$. Prove that there is some $g(x)\in\mathbb{Z}[x]$ such that $f(x)=(bx-a)g(x)$.

Problem 2.

a) Prove that $P(x)=x^{2^n}+1$ is irreducible for all $n\geq 1$.

b) For prime p, prove that the polynomial $\dfrac{x^p-1}{x-1}$ is irreducible.

Problem 3.

Denote $f(x)=x^{16}+a_{15}x^{15}+\cdots+a_1x+a_0$ as a polynomial with $a_k\in\{4,8,12\}$ for $0\leq k\leq 15$.

a) Prove that $f(x)+1$ is irreducible.

b) Is there any $f(x)$ reducible?

Problem 4.

For odd prime p, how many pairs (k,l) such that $1\leq k<l\leq p$ and

$$x^p-p(x^k+x^l)+1$$

is irreducible?

Problem 5.

For $k\geq 2$, denote $65^k=\overline{a_na_{n-1}\ldots a_1a_0}$ and consider polynomial

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.$$

Prove that $f(x)$ has no rational root.

Problem 6.

A student partitions the set $\mathbb{Z}^+$ into 2019 non empty subsets $S_1,S_2,\ldots,S_{2019}$. Prove that there is a set among them such that from this, we can make infinitely many irreducible cubic polynomial.

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Topic 7.

POLYNOMIAL 6

Problem 1.

a) Prove that if $P(x) \in \mathbb{R}[x]$ has more than 2 distinct mononials, then $P^2(x) - P(x^2)$ has at least 2 distinct mononials.

b) Suppose that $P(x) \in \mathbb{R}[x]$ and $P^2(x) - P(x^2) = cx^{2018}$ for some real number c. Find the smallest value of c. And for that number, find all ~~value~~ polynomial satisfied.

Problem 2.

Consider the polynomial $P(x) = x^3 - 3x$.

a) Prove that there exsit a, b, c are distinct real numbers such that

$$P(a) = b, P(b) = c, P(c) = a.$$

b) Suppose that there are 9 numbers (a_i, b_i, c_i) with $i = 1, 2, 3$ such that

$$P(a_i) = b_i, P(b_i) = c_i, P(c_i) = a_i.$$

Prove that three sums $s_i = a_i + b_i + c_i$ cannot be all equal.

Problem 3.

Consider the positive integer n and not divisible by 5, prove that $\left(x^8 + x^3 + \frac{1}{x^2} + \frac{1}{x^7}\right)^n$ has no constant.

Problem 4.

Denote f as a polynomial such that: for each rational number a, there exist rational number b such that $f(b) = a$.

a) Prove that $f \in \mathbb{Q}[x]$.

b) Prove that f is linear.

Problem 5.

Consider an integer polynomial $P(x)$. Denote m as the sum of all the coeficients of x^n with $(n, 3) = 1$ in the expansion of $(P(x))^3$. Prove that $6 \mid m$.

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Topic 8.

POLYNOMIAL 7

Problem 1.

Let $f(x)=x^2+ax+b$ be a quadratic function with $a,b\in\mathbb{R}$ such that $f(f(x))=0$ has four distinct real roots and the sum of 2 roots among them is equal to -1. Prove that $b\le -\frac{1}{4}$.

Problem 2.

Given polynomial $P(x)=x^4+ax^3+bx^2+cx$ and each of equation $P(x)=1, P(x)=2$ has 4 real roots (not necessary distinct). Prove that if four roots of $P(x)=1$ are x_1,x_2,x_3,x_4 and satisfy the equation $x_1+x_2=x_3+x_4$, then the same holds for $P(x)=2$.

Problem 3.

Suppose that $P_i(x)=x^2+b_ix+c_i$ with $1\le i\le n$ are pairwise distinct polynomials of degree 2 with real coefficients such that for any $1\le i<j\le n$, the polynomial $P_i(x)+P_j(x)$ has only one real root. Find the greatest possible value of n.

Problem 4.

Does there exist polynomials $P(x), Q(x)$ of integer coefficients and $\deg P, Q>1$ such that

$$P(Q(x))=(x-1)(x-2)\ldots(x-9)?$$

Problem 5.

Polynomial $P(x)$ is called "nice" if it is monic and its cofficients are in $\{-1,0,1\}$.

a) Suppose that $\deg P(x)=100$ and it is divisible by x^7-1, at least how many non-zero coefficients $P(x)$ can have?

b) Consider $a\in(1,2)$, prove that there exist nice polynomial $P(x)$ such that $|P(a)|<\frac{1}{2019}$.

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Topic 9.

POLYNOMIAL 8

Problem 1.

Denote α is the positive real root of $x^2+x=5$. Consider positive integer $n\geq 3$ and nonnegative integer $c_0,c_1,...,c_n$ satisfy

$$c_0+c_1\alpha+c_2\alpha^2+...+c_n\alpha^n=2019.$$

What is the remainder of $c_0+c_1+...+c_n$ when divide by 3?

Problem 2.

Let $P(x)$ be a polynomial of degree n and has only roots $x=0,x=2,x=3$ (no complex root). Suppose that $P'(x)$ is divisible by $8x^2-24x+7$. Find the minimum value of n.

Problem 3.

Given $P(x)=x^3+2x^2-7x-16$, $Q(x)=x^3-10x^2-800$. Prove that each of them has exactly one positive real root, namely u,v and $\sqrt{\frac{v}{2}}=\sqrt{u}+1$.

Problem 4.

On the board, there are two polynomial x^2-2x and $2x^3-3x^2-4$. At each step, if there aresome polynomial $f(x),g(x)$ we can write more the following

$$f(x)\pm g(x), f(x)g(x), cf(x), cg(x) \text{ for any } c\in\mathbb{R}.$$

Can we obtain the following after finite step?

a) x^n-1 for $n\in\mathbb{Z}^+$?

b) $(x-2)^n$ for $n\in\mathbb{Z}^+$?

Bài 5.

Let $P(x),Q(x),R(x)$ be polynomials with degree 3, 2, 3 and satisfy the equation

$$P^2(x)+Q^2(x)=R^2(x),\forall x\in\mathbb{R}$$

Prove that $T(x)=P(x)Q(x)R(x)$ has at least 6 real root (not necessary distinct).