Email training, N2

Level 3, September 20-26

Problems with Solutions

Problem 1.1. Prove that for any positive integer n at least on coefficient of the polynomial

$$P(x) = \left(x^4 + x^3 - 3x^2 + x + 2\right)^n$$

is negative.

Solution 1.1. Denote

$$P(x) = a_{4n}x^{4n} + a_{4n-1}x^{4n-1} + a_{4n-2}x^{4n-2} + \ldots + a_2x^2 + a_1x + a_0.$$

Lets note, that $P(0) = 2^n$ and $P(1) = (1 + 1 - 3 + 1 + 2)^n = 2^n$, which means, that $P(1) - P(0) = a_{4n} + a_{4n-1} + \ldots + a_2 + a_1 = 0$. Since $a_{4n} = 1$ one can conclude, that at least one coefficient within $a_1, a_2, \ldots, a_{4n-1}$ is negative.

Problem 1.2. Let polynomial

$$P(x) = \underbrace{((\ldots((}_{k} x - 2)^2 - 2)^2 - \ldots)^2 - 2)^2$$

is given. Find coefficient at x^2.

Solution 1.2. Let

$$P_k(x) = \underbrace{((\ldots((}_{k} x - 2)^2 - 2)^2 - \ldots)^2 - 2)^2 = \ldots + a_kx^2 + b_kx + c_k.$$

One has $a_1 = 1$, $b_1 = 4$ and $c_1 = 2$.

Since $P_k(x) = P_{k-1}(x) - 2)^2$, therefore

i) $c_k = c_{k-1}^2 - 2$,

ii) $b_k = 2b_{k-1}c_{k-1}$,

iii) $a_k = 2a_{k-1}c_{k-1} + b_{k-1}^2$.

Its obvious, that $c_k = 2$, from which immediately follows, that $b_k = 4^k$. By putting those values into *iii)* on gets

$$a_k = 4a_{k-1} + 4^{2k-2}.$$

Lets calculate a_k. Denote $x_k = \frac{a_k}{4^{k-1}}$. Then one has $x_1 = 1$ and

$$x_{k+1} = \frac{a_{k+1}}{4^k} = \frac{4a_k + 4^{2k}}{4^k} = \frac{a_k}{4^{k-1}} + 4^k = x_k + 4^k.$$

From this one immediately obtains, that

$$x_k = x_1 + 4 + 4^2 + \ldots + 4^{k-2} + 4^{k-1} = \frac{4^k - 1}{3}.$$

So $a_k = 4^{k-1}x_k = 4^{k-1}\frac{4^{k-1}(4^k-1}{3}$.

Problem 1.3.

3. Let $f(x) = x^2 - 6x + 5$. Draw on the plane the set of pairs (x, y) that satisfy to the following system on inequalities

$$\begin{cases} f(x) + f(y) \leq 0 \\ f(x) - f(y) \geq 0 \end{cases}.$$

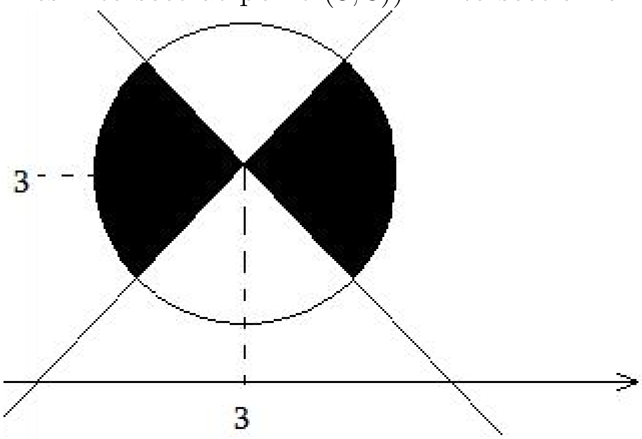
Solution 1.3. The first inequality is equivalent to

$$(x-3)^2+(y-3)^2 \leq 8$$

which solutions is the circle (with interior) with center $(3,3)$ and radius $2\sqrt{2}$.
The second inequality is equivalent to

$$(x-y)(x+y-6) \geq 0$$

which solution the left and right regions obtained by intersection of lines $x-y=0$ and $x+y=6$ (those lines intersect at point $(3,3)$). Intersection of both solutions will be the figure painted black



below.

Problem 1.4. Prove that for any 2 positive integers m and n with $m>n$ holds the following inequality

$$lcm(m,n)+lcm(m+1,n+1) > \frac{2mn}{\sqrt{m-n}} \tag{2.4.1}$$

where $lcm(a,b)$ is the least common multiplier of a and b (for example $lcm(6,8)=24$).

Solution 1.4. Since $lcm(a,b) \cdot gcd(a,b) = ab$ one can rewrite the inequality (2.4.1) in the following form

$$\frac{mn}{gcd(m,n)}+\frac{(m+1)(n+1)}{gcd(m+1,n+1)} > \frac{2mn}{\sqrt{m-n}}$$

$$\frac{mn}{gcd(m-n,n)}+\frac{(m+1)(n+1)}{gcd(m-n,n+1)} > \frac{2mn}{\sqrt{m-n}} \tag{2.4.2}$$

Lets prove the following inequality, which is stronger than (2.4.2)

$$\frac{mn}{gcd(m-n,n)}+\frac{mn}{gcd(m-n,n+1)} > \frac{2mn}{\sqrt{m-n}}$$

$$\frac{1}{gcd(m-n,n)}+\frac{1}{gcd(m-n,n+1)} > \frac{2}{\sqrt{m-n}} \tag{2.4.3}$$

Let $gcd(m-n,n)=x$ and $gcd(m-n,n+1)=y$. Since n and $n+1$ are coprime, then x and y are coprime as well. Also, lets note that x and y are divisors of $m-n$, which means $xy|(m-n)$, therefore $m-n \geq xy$. Now, let back to inequality (2.4.3).

$$\frac{1}{(m-n,n)}+\frac{1}{(m-n,n+1)} = \frac{1}{x}+\frac{1}{y} > 2\sqrt{\frac{1}{x}\cdot\frac{1}{y}} \geq \frac{2}{\sqrt{m-n}}.$$

Problem 1.5. Let convex s-gon is divided to q quadrilaterals such that b of them are not convex. Prove that

$$q \geq b+\frac{s-2}{2}.$$

Solution 1.5. Let p be the number of vertices inside the s-gon. The total sum of angles of all quadrilaterals is $180(s-2)+360p$ which is equal $360q$. So

$$180(s-2)+360p=360q$$

by dividing both side by 180 one gets

$$q=p+\frac{s-2}{2}.$$

To complete the proof one needs to show that $p \geq b$.
Non-convex quadrilateral can't have angle bigger 180^o at the vertex belonging to s-gon, therefor that point should be one of the p points inside the s-gon. Also 2 different non-covex quadrilaterals can have angle bigger than 180^o on the same vertex inside the polygon $(180+180>360)$. It means, that the number of vertices inside the polygon p can't be less than the number of non-convex quadrilaterals b.

Problem 1.6. Let positive numbers are written along the circle, such that all of them are less than 1. Prove that one can split the circle to 3 parts such that for each two arcs the sums of numbers written on them differs by at most 1.

Solution 1.6. By weight of arc lets denote the sum of numbers written on it. So we have three arcs and three numbers. By variance of partition of the circle by 3 arcs we will denote the difference between the highest and lowest weights. Consider the partition having the minimum variance. Lets prove that the variance is at most 1 (it will solve the problem).
Let 3 weights are $a \leq b \leq c$ and $c-a>1$. Take the number r from arc c (on the border with a) and move it to arc a. We will have new particion with weights $a+r, b, c-r$. Then

$$-1 \leq -r \leq b-a-r=b-(a+r)<b-a \leq c-a,$$

$$-1 \leq -r \leq (c-r)-b=c-b-r<c-b \leq c-a,$$

$$-1 \leq (c-a)-2 \leq (c-a)-2r-(c-r)-(a+r)<c-a.$$

So the new variance of new partition is less than $c-a$, which contradicts to the definition of a, b and c. So $c-a \leq 1$.

Other solution. Consider partition having least some $a^2+b^2+c^2$. Again, if $c>a+1$ then $(a+r)^2+b^2+(c-r)^2<a^2+b^2+c^2$.

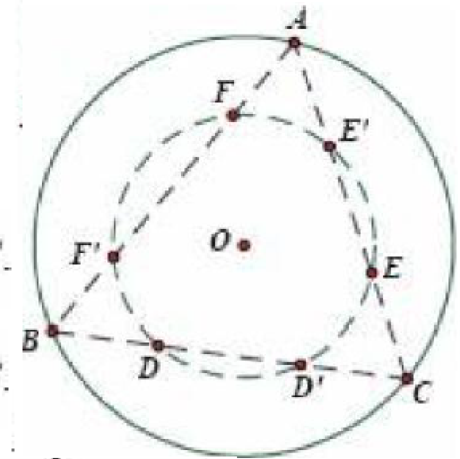
Problem 1.7. Let the triangle ABC is given and D, E, F are on sides BC, AC, AB, respectively, such that

$$\frac{BD}{CD}=\frac{CE}{AE}=\frac{AF}{BF}.$$

Show that if the circumcircle of ABC and DEF coincide, then ABC is equilateral.

Solution 1.7. -

Let the circumcircle of ΔDEF intersect BC at D, D', AC at E, E' and AB at F, F'. Notice that the midpoints of BC and DD' coincide, i.e., D and D' are symmetric about the midpoint of BC.



Let $\dfrac{BD}{CD}=\dfrac{CE}{AE}=\dfrac{AF}{BF}=k.$

We have $BD=\dfrac{k}{k+1}BC$ and $BD'=CD=\dfrac{1}{k+1}BC.$

Similarly, $BF=\dfrac{1}{k+1}AB$ and $BF'=AF=\dfrac{k}{k+1}AB.$

We have $BD\cdot BD'=\dfrac{k}{(k+1)^2}BC^2$ and $BF\cdot BF'=\dfrac{k}{(k+1)^2}AB^2.$

Since $BD \cdot BD' = BF \cdot BF'$ (Tangent Secant Theorem), we must have $AB^2 = BC^2$, i.e., $AB = AC$.

Similarly, $BC = AC$ and the conclusion follows.