Chapter 3

Formulas

3.1 Identities

An equation is just a mathematical statement. It may be always true, always false, or true only for certain values of the variables. If an equation is true only for some values of the variables, we call it a **conditional equation**, and the values which make it true the **solutions** to the equation. Of course such equations are very important, and you've been trained to solve them since you were small. But there are the other kinds of equations.

An equation that is always false is called a **contradiction**.

An equation that is always true is called an **identity**.

$2x - 1 = 5$ is a conditional equation since it is a true statement only if $x = 3$.

$x + 4 = x$ is a contradiction since it is not true for any value of x.

$x - \frac{1}{x} = \frac{(x+1)(x-1)}{x}$ **appears** to be an identity since it is true for various values of x.

$$x = 2 \;\Rightarrow\; 2 - \tfrac{1}{2} = \tfrac{3}{2} = \tfrac{(2+1)(2-1)}{2}$$

$$x = 1 \;\Rightarrow\; 1 - \tfrac{1}{1} = 0 = \tfrac{(1+1)(1-1)}{1}$$

$$x = 10 \;\Rightarrow\; 10 - \tfrac{1}{10} = 9.9 = \tfrac{(10+1)(10-1)}{10}$$

But this just shows it's true for those three x's. How do we show it's true for **all** x's?

You **cannot** just treat it like a conditional equation. To begin, choose one side of the equation: the Left Hand Side of the equation (LHS) or the Right Hand Side (RHS). Then you do algebra to that expression until it looks like the other side.

Example 3.1: Prove the following is an identity.

$$x - \frac{1}{x} = \frac{(x+1)(x-1)}{x}$$

Solution: We'll start with the Left Hand Side.

$$\begin{aligned}
\text{LHS} &= x - \tfrac{1}{x} \\
&= \tfrac{x^2}{x} - \tfrac{1}{x} \\
&= \tfrac{x^2-1}{x} \\
&= \tfrac{(x+1)(x-1)}{x} = \text{RHS}
\end{aligned}$$

Notice we did not do anything like "solve" the equation. We did not "multiply both sides by x" as there are no "both sides". There's one side (the starting point) and the other side (the destination).

Remember, too, that we already have several identities, and that they can be used to prove new identities. The most important identity we have is $\sin^2 x + \cos^2 x = 1$.

Example 3.2: Prove the following is an identity.

$$\sec(x) = \tan(x) + \frac{\cos(x)}{1+\sin(x)}$$

Solution: It usually easier to start with the more complicated side and simplify. Let's start with the Right Hand Side.

$$\begin{aligned}
\text{RHS} &= \tan(x) + \tfrac{\cos(x)}{1+\sin(x)} \\
&= \tfrac{\sin(x)}{\cos(x)} + \tfrac{\cos(x)}{1+\sin(x)} \\
&= \tfrac{\sin(x)}{\cos(x)}\left(\tfrac{1+\sin(x)}{1+\sin(x)}\right) + \tfrac{\cos(x)}{1+\sin(x)}\left(\tfrac{\cos(x)}{\cos(x)}\right) \\
&= \tfrac{\sin(x)+\sin^2(x)+\cos^2(x)}{\cos(x)(1+\sin(x))} \\
&= \tfrac{\sin(x)+1}{\cos(x)(1+\sin(x))} \\
&= \tfrac{1}{\cos(x)} = \sec(x) = \text{LHS}
\end{aligned}$$

Example 3.3: Prove the following is an identity.

$$\frac{\sec^2(x) - 1}{\sec^2(x)} = \sin^2(x)$$

Example 3.4: Prove the following is an identity.

$$\frac{\cos(-x) + \sin(-x)}{\cos(x)} = 1 - \tan(x)$$

One trick that often works when you have a $1 \pm \sin(x)$ or $1 \pm \cos(x)$ is to multiply by the "conjugate" $1 \mp \sin(x)$ or $1 \mp \cos(x)$.

Example 3.5: Prove the following is an identity.

$$\frac{\cos(x)}{1-\sin(x)} = \sec(x) + \tan(x)$$

Solution:

$$\begin{aligned} \text{LHS} &= \frac{\cos(x)}{1-\sin(x)} \\ &= \frac{\cos(x)}{1-\sin(x)}\left(\frac{1+\sin(x)}{1+\sin(x)}\right) \end{aligned}$$

3.1.1 Practice

Homework 3.1

Prove the identities.

1. $(\cos(x) + \sin(x))^2 = 1 + 2\cos(x)\sin(x)$
2. $\cos^2(\theta)(1 + \tan^2(\theta)) = 1$
3. $\csc(x) - \sin(x) = \cos(x)\cot(x)$
4. $\cos(t) + \tan(t)\sin(t) = \sec(t)$
5. $\cot(-x)\cos(-x) + \sin(-x) = -\csc(x)$
6. $\frac{1-\cos(\alpha)}{\sin(\alpha)} = \frac{\sin(\alpha)}{1+\cos(\alpha)}$
7. $\tan^2(t) - \sin^2(t) = \tan^2(t)\sin^2(t)$
8. $\frac{1}{\sec(t)+\tan(t)} + \frac{1}{\sec(t)-\tan(t)} = 2\sec(t)$

3.2 Sum and Difference Formulas

Most functions do not "distribute" over a sum. That is, for most functions f, and most numbers, x and y,

$$f(x+y) \neq f(x) + f(y)$$

Certainly,

$$(x+y)^2 \neq x^2 + y^2$$

and

$$\ln(x+y) \neq \ln(x) + \ln(y)$$

Likewise, for most angles α and β,

$$\cos(\alpha+\beta) \neq \cos(\alpha) + \cos(\beta) \text{ and } \sin(\alpha+\beta) \neq \sin(\alpha) + \sin(\beta)$$

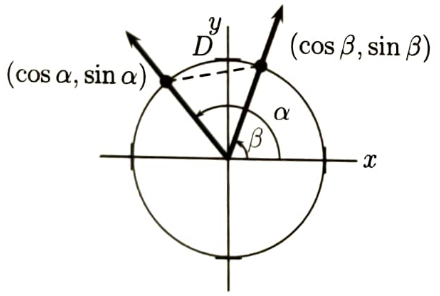
For instance,

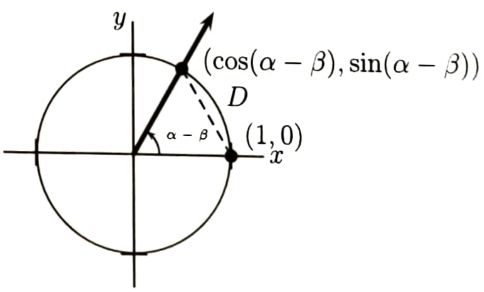
$$\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

but

$$\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \neq 0$$

There is, however, a way to write $\cos(\alpha+\beta)$ in terms of the sine and cosine of α and β. It's more complicated than just "distributing", but it has the advantage of being true.





The **Distance Formula** for the distance between two points, (x_1, y_1) and (x_2, y_2), is:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The square of the distance in the left picture is:

$$\begin{aligned}
D^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\
&= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \\
&= (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)
\end{aligned}$$

The square of the distance in the right picture is:

$$\begin{aligned} D^2 &= (\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2 \\ &= \cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta) \\ &= 2-2\cos(\alpha-\beta) \end{aligned}$$

Setting the two expressions for D^2 equal:

$$\begin{aligned} 2-2\cos(\alpha-\beta) &= 2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta) \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \end{aligned}$$

This expression can be used to derive other expressions for the sine or cosine of the sum or difference of angles.

$$\boxed{\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta}$$

$$\boxed{\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta}$$

Example 3.6: Find the exact value of $\cos\left(\frac{\pi}{12}\right)$.

Solution: First notice that $\frac{\pi}{12}=\frac{\pi}{3}-\frac{\pi}{4}$.
Then,

$$\begin{aligned} \cos\left(\tfrac{\pi}{12}\right) &= \cos\left(\tfrac{\pi}{3}-\tfrac{\pi}{4}\right) \\ &= \cos\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{\pi}{4}\right)+\sin\left(\tfrac{\pi}{3}\right)\sin\left(\tfrac{\pi}{4}\right) \\ &= \tfrac{1}{2}\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{2}}{2} \\ &= \tfrac{\sqrt{2}+\sqrt{6}}{4} \end{aligned}$$

Example 3.7: Find the exact value of $\sin\left(\frac{17\pi}{12}\right)$.

Hint: $\frac{17\pi}{12}=\frac{8\pi}{12}+\frac{9\pi}{12}$

Example 3.8: Find the exact value of: $\cos 20^\circ \cos 50^\circ + \sin 20^\circ \sin 50^\circ$.

Example 3.9: Simplify the expression: $\cos\left(x + \frac{\pi}{2}\right)$.

Solution:

$$\begin{aligned} \cos\left(x + \tfrac{\pi}{2}\right) &= \cos(x)\cos\tfrac{\pi}{2} - \sin(x)\sin\tfrac{\pi}{2} \\ &= \cos(x)\cdot 0 - \sin(x)\cdot 1 \\ &= -\sin(x) \end{aligned}$$

Example 3.10: Simplify the expression: $\sin(x - \pi)$.

We can use the sum and difference rules for sine and cosine to construct similar rules for tangent.

$$\begin{aligned} \tan(\alpha \pm \beta) &= \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} \\ &= \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta} \\ &= \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}\left(\frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}\right) \end{aligned}$$

$$\boxed{\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}}$$

Example 3.11: Find the exact value of $\tan\left(\frac{5\pi}{12}\right)$.

3.2.1 Writing a Sum as a Single Function

One very useful application of the sum and difference rules is to write a sum of a sine and a cosine as a single trig function, say a sine.

Example 3.12: Write $-\sin(x) + \sqrt{3}\cos(x)$ in the form: $k\sin(x+\phi)$.

Solution: We want to find k and ϕ so that

$$k\sin(x+\phi) = -\sin(x) + \sqrt{3}\cos(x)$$

We begin by letting $k = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$.

Dividing both sides of the above equation by k gives,

$$\sin(x+\phi) = -\frac{1}{2}\sin(x) + \frac{\sqrt{3}}{2}\cos(x)$$

Now applying the addition rule for sine,

$$\cos\phi\sin(x) + \sin\phi\cos(x) = -\frac{1}{2}\sin(x) + \frac{\sqrt{3}}{2}\cos(x)$$

We need to chose a ϕ so that,

$$\cos\phi = -\frac{1}{2} \text{ and } \sin\phi = \frac{\sqrt{3}}{2}$$

Cosine is negative while sine is positive, so ϕ is in quadrant II. The reference angle for ϕ is clearly $\pi/3$, hence $\phi = 2\pi/3$.

$$-\sin(x) + \sqrt{3}\cos(x) = 2\sin\left(x + \frac{2\pi}{3}\right)$$

In general, to write $A\sin(x) + B\cos(x)$ as $k\sin(x + \phi)$,

1. Let $k = \sqrt{A^2 + B^2}$.
2. Find ϕ so that:

   $$\cos\phi = \frac{A}{k} \text{ and } \sin\phi = \frac{B}{k}$$

   by finding the quadrant and reference angle for ϕ.

Example 3.13: Write $\sin(x) - \cos(x)$ in the form: $k\sin(x + \phi)$.

Example 3.14: Find k and approximate ϕ so that:

$$3\sin(x) + 4\cos(x) = k\sin(x + \phi)$$

3.2.2 Practice

Homework 3.2

1. Use the addition and subtraction formulas to find the exact value of the expression.

   (a) $\cos(15°)$

   (b) $\sin\left(\frac{19\pi}{12}\right)$

   (c) $\tan\left(\frac{17\pi}{12}\right)$

2. Use the addition and subtraction formulas to find the exact value of the expression.

   (a) $\sin(18°)\cos(27°)+\cos(18°)\sin(27°)$

   (b) $\cos(10°)\cos(80°)-\sin(10°)\sin(80°)$

   (c)
   $$\frac{\tan\left(\frac{\pi}{18}\right)+\tan\left(\frac{\pi}{9}\right)}{1-\tan\left(\frac{\pi}{18}\right)\tan\left(\frac{\pi}{9}\right)}$$

3. Use the addition and subtraction formulas to simplify the expression.

   (a) $\sin\left(x-\frac{\pi}{2}\right)$

   (b) $\cos\left(x+\frac{3\pi}{2}\right)$

   (c) $\tan\left(\frac{\pi}{2}-x\right)$

   (d) $\sin\left(\cos^{-1}\left(\frac{1}{3}\right)+\cos^{-1}\left(\frac{3}{5}\right)\right)$

4. Prove the identities.

   (a)
   $$\cos\left(x+\frac{\pi}{6}\right)+\sin\left(x-\frac{\pi}{3}\right)=0$$

   (b)
   $$\cos(x+y)+\cos(x-y)=2\cos(x)\cos(y)$$

5. Write the expression exactly as $k\sin(x+\phi)$.

   (a) $\sin(x)+\cos(x)$

   (b) $-5\sin(\pi x)-5\cos(\pi x)$

   (c) $3\sin(x)-3\sqrt{3}\cos(x)$

6. Approximate the expression as $k\sin(x+\phi)$.

   (a) $5\sin(x)+12\cos(x)$

   (b) $-4\sin(x)+3\cos(x)$

   (c) $-8\sin(x)-7\cos(x)$

3.3 More Trig Formulas

3.3.1 Product-to-sum and Sum-to-product Formulas

We can also use the sum and difference formulas to write the product of two trig functions as a sum.

$$\begin{array}{rrl} & \sin(\alpha-\beta) & = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ + & \sin(\alpha+\beta) & = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \hline \multicolumn{2}{r}{\sin(\alpha-\beta)+\sin(\alpha+\beta)} & = 2\sin\alpha\cos\beta \end{array}$$

Dividing both sides by 2 we have a formula for $\sin\alpha\cos\beta$.
Similarly we may derive formulas for the other products:

$$\boxed{\sin\alpha\cos\beta = \tfrac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta))}$$

$$\boxed{\sin\alpha\sin\beta = \tfrac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))}$$

$$\boxed{\cos\alpha\cos\beta = \tfrac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))}$$

Example 3.15: Write $\sin(2x)\cos(3x)$ as a sum of two trig functions.

Solution: According to our first product-to-sum formula:

$$\begin{aligned} \sin(2x)\cos(3x) &= \tfrac{1}{2}(\sin(2x-3x)+\sin(2x+3x)) \\ &= \tfrac{1}{2}(\sin(-x)+\sin(5x)) \\ &= \tfrac{1}{2}(-\sin(x)+\sin(5x)) \\ &= \tfrac{1}{2}(\sin(5x)-\sin(x)) \end{aligned}$$

Example 3.16: Write $\sin(7x)\sin(3x)$ as a sum of two trig functions.

Using a simple substitution we can go from a product back to a sum.

$$\text{Let} \quad \begin{array}{l} u = \alpha - \beta \\ v = \alpha + \beta \end{array} \quad \Rightarrow \quad \alpha = \frac{u+v}{2}, \quad \beta = \frac{-u+v}{2} = -\frac{u-v}{2}$$

Hence our formulas become:

$$\boxed{\sin(u) + \sin(v) = 2\sin\left(\tfrac{u+v}{2}\right)\cos\left(\tfrac{u-v}{2}\right)}$$

$$\boxed{\sin(u) - \sin(v) = 2\cos\left(\tfrac{u+v}{2}\right)\sin\left(\tfrac{u-v}{2}\right)}$$

$$\boxed{\cos(u) - \cos(v) = -2\sin\left(\tfrac{u+v}{2}\right)\sin\left(\tfrac{u-v}{2}\right)}$$

$$\boxed{\cos(u) + \cos(v) = 2\cos\left(\tfrac{u+v}{2}\right)\cos\left(\tfrac{u-v}{2}\right)}$$

Example 3.17: Write $\cos(4a) - \cos(6a)$ as a product of trig functions.

Example 3.18: Find the exact value of: $\sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)$.

3.3.2 Double Angle Formulas

Just as functions don't, in general, "distribute" over sums; you also cannot, in general, "factor out" a number from inside a function.

$$e^{2x} \neq 2e^{x} \quad \text{and} \quad \sqrt{2x} \neq 2\sqrt{x}$$

Likewise for the trig functions. As we saw in section 2.3,

$$\sin(2x) \neq 2\sin(x) \quad \text{and} \quad \cos(2x) \neq 2\cos(x)$$

We can easily use the sum and difference formulas to give us formulas for the sine or cosine of twice an angle.

$$\sin(x+x) = \sin(x)\cos(x) + \cos(x)\sin(x)$$

Thus,

$$\boxed{\sin(2x) = 2\sin(x)\cos(x)}$$

Likewise for cosine,

$$\cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x)$$

Using the formula, $\cos^2(x) + \sin^2(x) = 1$ we can form three different expressions for the cosine of double an angle.

$$\boxed{\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 2\cos^2(x) - 1 \\ &= 1 - 2\sin^2(x) \end{aligned}}$$

Example 3.19: Use a double angle formula to evaluate: $\sin\left(\frac{2\pi}{3}\right)$.

Solution:

$$\begin{aligned} \sin\left(2\tfrac{\pi}{3}\right) &= 2\sin\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{\pi}{3}\right) \\ &= 2 \cdot \tfrac{\sqrt{3}}{2} \cdot \tfrac{1}{2} \\ &= \tfrac{\sqrt{3}}{2} \end{aligned}$$

Which we know is correct by the methods of section 2.2.

Example 3.20: Use a double angle formula to evaluate: $\cos\left(\frac{2\pi}{3}\right)$.

Example 3.21: Prove the identity:

$$\frac{\cos(2\theta)}{\sin(2\theta)} = \frac{1}{2}(\cot\theta - \tan\theta)$$

Example 3.22: Simplify the expression: $\sin\left(2\sin^{-1}\left(\frac{2}{3}\right)\right)$

Solution:

$$\begin{aligned} \sin\left(2\sin^{-1}\left(\tfrac{2}{3}\right)\right) &= 2\sin\left(\sin^{-1}\left(\tfrac{2}{3}\right)\right)\cos\left(\sin^{-1}\left(\tfrac{2}{3}\right)\right) \\ &= 2\left(\tfrac{2}{3}\right)\left(\tfrac{\sqrt{5}}{3}\right) \\ &= \tfrac{4\sqrt{5}}{9} \end{aligned}$$

(Using example 2.67.)

Example 3.23: Say θ is an angle in quadrant IV, and $\cos\theta = \frac{1}{4}$. Find $\sin(2\theta)$.

Solution: To find $\sin(2\theta)$ we need both the cosine and sine of θ.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \Rightarrow \sin\theta &= \pm\sqrt{1-\cos^2\theta} \\ \sin\theta &= -\sqrt{1-\left(\tfrac{1}{4}\right)^2} \\ &= -\tfrac{\sqrt{15}}{4} \end{aligned}$$

(Negative since θ is in quadrant IV where sine is negative.)
Thus,

$$\begin{aligned} \sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\ &= 2 \cdot -\tfrac{\sqrt{15}}{4} \cdot \tfrac{1}{4} \\ &= -\tfrac{\sqrt{15}}{8} \end{aligned}$$

Example 3.24: Say θ is an angle in quadrant IV, and $\cos\theta = \frac{1}{4}$.
Find $\cos(2\theta)$. In what quadrant is the angle 2θ?

We may also use the double angle formulas to help us solve an equation.

Example 3.25: Solve the equation.

$$\sin(2x) = \cos(x)$$

Solution:
Using the double angle formula for sine we have

$$\begin{aligned} 2\sin(x)\cos(x) &= \cos(x) \\ 2\sin(x)\cos(x) - \cos(x) &= 0 \\ \cos(x)(2\sin(x) - 1) &= 0 \\ \Rightarrow \cos(x) = 0 &\text{ or } \sin(x) = \tfrac{1}{2} \\ \Rightarrow x = \tfrac{\pi}{2} + \pi n, &\quad x = \tfrac{\pi}{6} + 2\pi n, \text{ or } x = \tfrac{5\pi}{6} + 2\pi n \end{aligned}$$

3.3.3 Half Angle Formulas

To reverse this process, going from the sine or cosine of an angle to the sine or cosine of **half** the angle, we make a simple substitution: $u = 2x$.

$$\begin{aligned} \cos(2x) &= 2\cos^2(x) - 1 \\ \cos(u) &= 2\cos^2\left(\tfrac{u}{2}\right) - 1 \end{aligned}$$

Solving for the cosine of half u gives:

$$\boxed{\cos\left(\tfrac{u}{2}\right) = \pm\sqrt{\tfrac{1+\cos(u)}{2}}}$$

Likewise,

$$\begin{aligned} \cos(2x) &= 1 - 2\sin^2(x) \\ \cos(u) &= 1 - 2\sin^2\left(\tfrac{u}{2}\right) \end{aligned}$$

$$\boxed{\sin\left(\tfrac{u}{2}\right) = \pm\sqrt{\tfrac{1-\cos(u)}{2}}}$$

In both cases the $\pm$ is determined by the **quadrant of** $u/2$ (not u).

Example 3.26: Use a half angle formula to find the exact value of $\sin\frac{\pi}{8}$.

Solution:

$$\sin\tfrac{\pi}{8} = \sin\left(\tfrac{\pi/4}{2}\right) = +\sqrt{\tfrac{1-\cos\frac{\pi}{4}}{2}}$$

$$= \sqrt{\tfrac{1-\frac{\sqrt{2}}{2}}{2}\cdot\left(\tfrac{2}{2}\right)}$$

$$= \sqrt{\tfrac{2-\sqrt{2}}{4}} \qquad = \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

The + comes from the fact that $\pi/8$ is in quadrant I (where sine is positive).

Example 3.27: Use a half angle formula to find the exact value of $\sin\frac{7\pi}{12}$. (*Hint:* Watch out for the signs!)

Example 3.28: Say θ satisfies $\pi \le \theta \le 2\pi$ and $\cos\theta = \frac{2}{3}$.
Find $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$.

Solution: First find the quadrant that $\theta/2$ is in. Dividing through by 2,

$$\pi \le \theta \le 2\pi \quad \Rightarrow \quad \frac{\pi}{2} \le \frac{\theta}{2} \le \pi$$

Thus $\theta/2$ is in Quadrant II. Cosine will be negative, and sine positive.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{2}{3}}{2}\left(\frac{3}{3}\right)} = -\sqrt{\frac{5}{6}}$$

$$\sin\frac{\theta}{2} =$$

Notice we didn't need $\sin\theta$; the half-angle formulas only use cosine.

Also there was **no** "$\cos\frac{2}{3}$"! **2/3 is not an angle!** (At least not in this problem.)

Example 3.29: Say instead θ satisfies $-\pi \le \theta \le 0$ and $\cos\theta = \frac{2}{3}$.
Now what are $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$?

There are also double and half angle formulas for tangent.

$$\tan(x+x) = \frac{\tan(x)+\tan(x)}{1-\tan(x)\tan(x)}$$

Gives,

$$\boxed{\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}}$$

Using the half angle formulas for sine and cosine we have,

$$\tan\left(\frac{u}{2}\right) = \frac{\sin\frac{u}{2}}{\cos\frac{u}{2}} = \frac{\pm\sqrt{\frac{1-\cos(u)}{2}}}{\pm\sqrt{\frac{1+\cos(u)}{2}}} = \pm\sqrt{\frac{1-\cos(u)}{1+\cos(u)}}$$

Multiplying by the conjugate of the bottom gives

$$\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos(u)}{1+\cos(u)}\left(\frac{1-\cos(u)}{1-\cos(u)}\right)} = \pm\sqrt{\frac{(1-\cos(u))^2}{1-\cos^2(u)}} = \pm\left|\frac{1-\cos(u)}{\sin(u)}\right|$$

Miraculously, the absolute values and the $\pm$ cancel with each other.
You get a similar formula if you multiply by the conjugate of the top.

$$\boxed{\tan\left(\frac{u}{2}\right) = \frac{1-\cos(u)}{\sin(u)} = \frac{\sin(u)}{1+\cos(u)}}$$

Example 3.30: Use the half angle formula to find the exact value of $\tan\frac{\pi}{8}$.

Solution:

$$\begin{aligned}\tan\tfrac{\pi}{8} &= \tan\left(\tfrac{1}{2}\cdot\tfrac{\pi}{4}\right) = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= \left(1-\tfrac{\sqrt{2}}{2}\right)\tfrac{2}{\sqrt{2}} = \tfrac{2}{\sqrt{2}} - 1 = \sqrt{2}-1\end{aligned}$$

3.3.4 Practice

Homework 3.3

1. Use a half-angle formula to find the exact value of the expression.

   (a) $\cos(15°)$
   (b) $\sin\left(\frac{3\pi}{8}\right)$
   (c) $\tan(75°)$

2. Use a double angle formula to simplify the expression.

   (a) $2\sin(3\theta)\cos(3\theta)$
   (b) $1 - 2\sin^2(5x)$
   (c) $\frac{2\tan(7t)}{1-\tan^2(7t)}$

3. Prove the identities.

   (a)
   $$\sin(8x) = 2\sin(4x)\cos(4x)$$
   (b)
   $$\frac{1+\sin(2x)}{\sin(2x)} = 1+\frac{1}{2}\csc(x)\sec(x)$$
   (c)
   $$\frac{\sin(4x)}{\sin(x)} = 4\cos(x)\cos(2x)$$

4. Simplify the expressions.

   (a) $\cos\left(2\cos^{-1}\left(\frac{3}{5}\right)\right)$
   (b) $\tan\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

5. Find $\sin(x/2)$ and $\cos(x/2)$ from the given information.

   (a) $\cos(x) = \frac{3}{5}, \quad 0° < x < 90°$
   (b) $\sin(x) = -\frac{5}{13}, \quad 180° < x < 270°$
   (c) $\csc(x) = 3, \quad 90° < x < 180°$

6. Write the sum as a product.

   (a) $\sin(5x) + \sin(3x)$
   (b) $\cos(4x) - \cos(6x)$
   (c) $\sin(x) - \sin(4x)$

7. Write the product as a sum.

   (a) $\sin(2x)\cos(3x)$
   (b) $\cos(5x)\cos(3x)$
   (c) $8\sin(x)\sin(5x)$

8. Solve the equations.

   (a)
   $$\cos(2x) = \sin(x)$$
   (b)
   $$\tan\left(\frac{x}{2}\right) = \sin(x)$$