Test-4, January 7

Level 3

Problem 1. Find all positive integers x, y such that $2^x + 5^y + 2$ is a perfect square.

Problem 2. Let $\triangle ABC$ be a triangle and ω its circumcircle. The exterior angle bisector of $\angle BAC$ intersects ω at point D. Let X be the foot of the altitude from C to AD and let F be the intersection of the internal angle bisector of $\angle BAC$ and BC. Show that BX bisects segment AF.

Problem 3. Two players, A and B, choose a number from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in turns, until no number is left. Then, each player sums all the numbers that he has chosen. We say that a player wins if the sum of his chosen numbers is a prime and the sum of the numbers that his opponent has chosen is composite. In the contrary, the game ends in a draw. The player A starts first. Does there exist a winning strategy for any of the players?

Problem 4. Let a_0 be a fixed positive integer. We define an infinite sequence of positive integers $\{a_n\}_{n\geq 1}$ in an inductive way as follows: if we are given the terms $a_0, a_1, ...a_{n-1}$, then a_n is the smallest positive integer such that $\sqrt[n]{a_0 \cdot a_1 \cdot ... \cdot a_n}$ is a positive integer. Show that the sequence $\{a_n\}_{n\geq 1}$ is eventually constant.