Saudi Arabia – Online Math Camp
April 2021. – Level L2

Number Theory

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Problems – April 7

1. Find all triples of positive integers a, b, c such that $a \mid 2b+1$, $b \mid 2c+1$ and $c \mid 2a+1$.
2. Positive integers a, b, c, d satisfy $ab = cd$. Prove that there exist positive integers x, y, u, v such that $a = xu$, $b = yv$, $c = xv$, $d = yu$.
3. If a, b, c and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ are integers, prove that abc is a perfect cube.
4. Suppose x, y, z are rational numbers with $xyz = 1$ such that both $x+y+z$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ are integers. Prove that $|x| = |y| = |z| = 1$.
5. Find all pairs of positive integers (a, b) such that $a^{b^2} = b^a$.
6. Find all positive integers n that cannot be written in the form $n = [a, b] + [b, c] + [c, a]$. (By $[x, y]$ we denote the LCM *(least common multiple)* of x and y.)
7. Suppose that positive integers x, y are such that $\frac{x^2+y^2+x}{xy}$ is an integer. Prove that x is a perfect square.
8. Prove that the largest power of 2 dividing $\frac{(2n)!}{n!}$ is 2^n.
9. Prove that:
   (a) if $2^n - 1$ is prime, then n is prime;
   (b) if $2^n + 1$ is prime, then n is a power of 2.
10. If $p > 3$ is a prime number, prove that $\frac{2^{2p}+1}{5}$ is a composite number.
11. Let $a > 1$ be an integer. Prove that:
    (a) $a^n - 1 \mid a^m - 1$ if and only if $n \mid m$;
    (b) $a^n + 1 \mid a^m + 1$ if and only if $n \mid m$ and $\frac{m}{n}$ is odd.
12. Prove that $5^{2^n} - 1$ is divisible by 2^{n+2}, but not by 2^{n+3}.
13. Prove that there are infinitely many positive integers n for which $n \mid 2^n + 1$.
14. Prove that every multiple of $2^n - 1$ has at least n binary units.
15. Let $a, b \in \mathbb{N}$ be such that $a!b!$ is divisible by $a! + b!$. Prove that $3a \geqslant 2b + 2$.