$\boxed{28}$ For each $n > 0$ find smallest possible value of

$$W_n(x) = x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \dots + (2n-1)x^2 + 2nx$$

$\underline{n=1}$

$$x^2 + 2x \geqslant -1 \qquad \text{"="} \text{ iff } \boxed{x=-1}$$

$$x^2+2x+1 = (x+1)^2 \geqslant 0$$

$$x := -1 \rightsquigarrow W_n(-1) = -n$$

$$\Downarrow$$

$$\underbrace{-1 \text{ is a root of } W_n(-1)+n}$$

Prove that -1 is double root of $W_n(-1)+n$!

$$W_n(x) + n = (x+1)^2 \cdot \text{Something}$$

$$x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \dots + (2n-1)x^2 + 2nx$$

$n=3$

$$x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 3$$

$$x^6 + 2x^5 + x^4$$

$$2x^4 + 4x^3 + 2x^2$$

$$3x^2 + 6x + 3$$

$$x^4(x+1)^2 + 2x^2(x+1)^2 + 3(x+1)^2$$

$$W_3(x) + 3 = \underbrace{(x+1)^2}_{\geq 0}\Big(\underbrace{x^4 + 2x^2 + 3}_{>0}\Big) \geq 0$$

$$W_3(x) \geq -3 \qquad \text{"="} \iff x = -1$$

$W_n(x) + n =$

$$x^{2n} + 2x^{2n-1} + x^{2n-2} = (x+1)^2 x^{2n-2}$$

$$2x^{2n-2} + 4x^{2n-3} + 2x^{2n-4} = (x+1)^2 2x^{2n-4}$$

$$\vdots$$

$$kx^{2n-2k+2} + 2k^{2n-2k+1} + kx^{2n-2k} = (x+1)^2 kx^{2n-2k}$$

$$\vdots$$

$$nx^2 + 2nx + n = (x+1)^2 \cdot n$$

$$= (x+1)^2 \underbrace{\left(x^{2n-2} + 2x^{2n-4} + 3x^{2n-6} + \ldots + (n-1)x^2 + n \right)}_{>0}$$

≥ 0 "=" $\Leftrightarrow x = -1$

Is there any criterion for double root?

Derivatives

$$(x^2+2x+1)' = 2x+2$$

$$(x^n)' = nx^{n-1}$$

$$(f+g)' = f'+g'$$

$$(c)' = 0$$

$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

Criterion:

Suppose that a is a root of P.

$$P(a) = 0$$

then a is k- multiple root iff

$$P(a) = P'(a) = P''(a) = \dots = P^{(k-1)}(a) = 0$$

and $P^{(k)}(a) \neq 0$

$$P(x) = x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 3$$

$$P(-1) = 1 - 2 + 3 - 4 + 5 - 6 + 3 = 0$$

$$P'(x) = 6x^5 + 10x^4 + 12x^3 + 12x^2 + 10x + 6$$

$$P'(-1) = \cancel{-6} + \cancel{10} - \cancel{12} + \cancel{12} - \cancel{10} + \cancel{6} = 0$$

$$=$$

$$\Downarrow$$

(-1) is ≥ 2 multipl root.

$$P''(x) = (P'(x))' = 30x^4 + 40x^3 + 36x^2 + 24x + 10$$

$$P''(-1) = 30 - 40 + 36 - 24 + 10$$

$$= 12 \neq 0$$

$\Rightarrow$ $\boxed{-1 \text{ is doble root}}$!

$$P(x) = (x+1)^2 \cdot (\text{something})$$

a is k-multiple root iff

$$(x-a)^k \mid P(x)$$
$$(x-a)^{k+1} \nmid P(x)$$

$\boxed{2q}$ $\boxed{n \geq 2}$ $n-1$

$\underline{n=2}$

$a_1 + 2a_2 - a_1 \geq 2a_2$ ✓

$\boxed{a_1 \geq a_2 \geq a_3 > 0}$

$\underline{n=3}$

$$a_1 a_2 + (2a_2 - a_1)(2a_3 - a_2) \geq 2a_2 a_3$$

$$a_1 a_2 + 4a_2 a_3 - 2a_2^2 - 2a_1 a_3 + a_1 a_2 \geq 2a_2 a_3$$

$$\cancel{2}a_1 a_2 + \cancel{2}a_2 a_3 \geq \cancel{2}a_2^2 + \cancel{2}a_1 a_3$$

$$a_1 a_2 + a_2 a_3 \geq a_1 a_3 + a_2^2$$

$$a_1 a_2 - a_2^2 + a_3(a_2 - a_1) \geq 0$$

$$a_2(a_1 - a_2) - a_3(a_1 - a_2) \geq 0$$

$$(a_2-a_3)(a_1-a_2) \geqslant 0 \quad \checkmark$$

Inequality

$$a_1 \geqslant a_2 \geqslant \dots \geqslant a_n > 0, \quad n \geqslant 2$$

$$a_1 a_2 \dots a_{n-1} + (2a_2-a_1)(2a_3-a_2)\dots(2a_n-a_{n-1}) \geqslant$$
$$\geqslant 2a_2a_3 \dots a_n .$$

$\boxed{\text{Try induction}}$!

$\boxed{n=2}$

Assume it for $n=k$

$$a_1 a_2 \dots a_k + (2a_2-a_1)(2a_3-a_2)\dots(2a_{k+1}-a_k) \geqslant$$
$$\geqslant 2a_2a_3 \dots a_{k+1}$$

$$P = a_1 a_2 \dots, a_{k-1}$$

$$Q = (2a_2-a_1)(2a_3-a_2)\dots(2a_k - a_{k-1})$$

$$R = 2a_2a_3 \dots, a_k$$

Induction assumption is

$$\boxed{P + Q \geq R}$$

We want

$$a_k \cdot P + (2a_{k+1} - a_k) Q \geq a_{k+1} R$$

$$a_1 \geq a_2 \geq \cdots \geq a_{k+1} > 0 .$$

$$a_i \geq a_{i+1} \rightsquigarrow a_i \geq 2a_{i+1} - a_i$$

$$a_1 \geq |2a_2 - a_1|$$

$$a_2 \geq |2a_3 - a_2|$$

$$\vdots$$

$$a_{k-1} \geq |2a_k - a_{k-1}|$$

$$\Downarrow$$

$$P \geq Q$$

$$x \geq y > 0$$

$$\Downarrow$$

$$x \geq |x - 2y|$$

$$-x \leq -x + 2(x-y)$$

$$= x - 2y < x$$

$$a_k \cdot P + (2a_{k+1} - a_k) Q \geq a_{k+1} R$$

WANT

$$a_k P + 2a_{k+1} Q \geq a_k \cdot Q + a_{k+1} R$$

Since $P \geq Q$

$$a_k P + 2a_{k+1} Q \geq a_k \cdot Q + a_{k+1} P + a_{k+1} Q \geq$$

$$a_k P + a_{k+1} Q \geq a_k \cdot Q + a_{k+1} P$$

$$(a_k - a_{k+1})(P - Q) \geq 0 \quad \text{true}$$

$$\geq a_k Q + a_{k+1} R \qquad \square$$

imp is

$$\boxed{P + Q \geq R}$$

and