P4 P5 P6

TEST

$f: \mathbb{R} \to \mathbb{R}$

$$f(x) + y f(y) = x + f(y^2 - x) + f(f(x))$$

$$\boxed{f(x^2) = x f(x)}$$

$$f(x+y) = f(x) + f(y)$$

$$\boxed{f(x) = x f(1)}$$

$$\sum_{i=1}^{n} x_i = n \quad , \quad \sum_{i=1}^{n} (x_{i-1} - x_i + x_{i+1})^2 = n$$

$$x_0 = x_n , \qquad x_{n+1} = x_1$$

$$\sqrt{\frac{x_1^2 + \dots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \dots + x_n}{n}$$

$$\geq \frac{|x_1| + \dots + |x_n|}{n} \geq$$

APPLY this in P4.

$$\sqrt{\frac{\sum_{i=1}^{n} (x_{i-1} - x_i + x_{i+1})^2}{n}} \geq \frac{\sum_{i=1}^{n} (x_{i-1} - x_i + x_{i+1})}{n} =$$

$$= \frac{\sum_{i=1}^{n} x_i}{n} = \frac{n}{n} = 1 \; !$$

$$n = \underbrace{\sum_{i=1}^{n} (x_{i-1} - x_i + x_{i+1})^2}_{= n} \geq n$$

$$\forall_{i} \quad (x_{i-1} - x_i + x_{i+1}) = \text{const}$$

$$\sum_{i} (x_{i-1} - x_i + x_{i+1}) = n$$

$$\Downarrow$$

$$\forall_{i=1,\dots,n} \quad \boxed{x_{i-1} - x_i + x_{i+1} = 1}$$

$$(x_1, x_2, \dots, x_n) \quad \text{s.t}$$

$x_0 := x_n$

$x_{n+1} := x_1$

$x_0 = a$

$x_1 = b$

$x_2 = b - a + 1$

$x_3 = 2 - a$

$x_4 = 2 - b$

$x_5 = a - b + 1$

$x_6 = a$

$x_7 = b$

$x_8 = b - a + 1$

$\{a, b, b-a+1, 2-a, 2-b, a-b+1\}$

$(a, b, b-a+1, \ldots - - - - )$

If $6 | n$ then you can take arbitrary $a, b$ and you have solution

What if $6 \nmid n$ ?

$$\begin{cases} x_0 = a \\ x_1 = b \end{cases} \quad - - - -$$

$$x_0 = x_r$$

$$x_1 = x_{r+1}$$

$$\begin{cases} a = b \\ b = b-a+1 \end{cases} \qquad \begin{cases} a = 2-a \\ b = 2-b \end{cases}$$

$$\begin{cases} a = 1-a+b \\ b = 2-a \end{cases} \qquad \begin{cases} a = 2-b \\ a = 1+a-b \end{cases}$$

$$\begin{cases} a = 1+a-b \\ b = a \end{cases} \qquad \Rightarrow \quad a = b = 1$$

If $6 \nmid n \Rightarrow$ $x_0 = 1$
$x_1 = 1$

$(1, 1, 1, - - - - - 1)$

Table $n = 12$

$x_1\ x_2, - - - ,\ x_{12}$

$x_0 = a$
$x_1 = b$

| | | | | | | |
|---|---|---|---|---|---|---|
| $a$, | $b$, | $b-a+1$, | $2-a$, | $2-b$, | $a-b+1$ | |
| $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | |
| $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_{12}$ |
| $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ | $\shortparallel$ |
| $x_6$ | $x_7$ | $x_8$ | $x_9$ | $10$ | $x_{11}$ | $a$ |

$n \equiv 7$

$x_0 = a \quad x_2 = b-a+1 \quad x_4 = 2-b \quad x_6 = a$
$x_1 = b \quad x_3 = 2-a \quad x_5 = a-b+1 \quad x_7 = b$
$x_8 = b-a+1$

$$\boxed{x_{i-1} - x_i + x_{i+1} = 1}$$

$x_{n+1} = x_1$

$b - a + 1 = b$

$a = 1$

$i = 5$

$x_8 = x_1$

$x_4 - x_5 + x_6 = 1$

$x_{i-1} - x_i + x_i$

$x_6 = 1 - x_4 + x_5 = 1 - (2-b) + a - b + 1$

$= a$

$i = 7$

$x_6 - x_7 + x_8 = 1$

$x_8 = 1 - x_6 + x_7 = b - a + 1$

P5'

1° case: $y$ - minimal !

$x \geq y$, $z \geq y$

Look at 1st equation,

$$2x^3 + (z^2+1) \geq 2yx^2 + (z^2+1) \geq$$

AM-GM

$$\geq 2yx^2 + 2z \geq$$

$$\geq 2yx^2 + 2y = 2y(x^2+1)$$

$x = y$, $z = 1$ $z = y$

$x = y = z = 1$

2° $y \geq x$, $z \geq x$.

$$2z^5 + 3(y^2+1) \geq 2z^5 + 3 \cdot 2y \dots$$

$$(z^5 + z^5 + y + y) + 4y \geq$$

$$\geq \underbrace{(z^5 + z^3x^2 + x + x)} + 4x$$

$$\geq 4\sqrt[4]{z^8x^4} + 4x = \underline{4x(z^2+1)}$$

3° At home!

$$\begin{cases} x = \dfrac{2z}{1-z^2} \\ \boxed{y = \dfrac{2x}{1-x^2}} \\ z = \boxed{\dfrac{2y}{1-y^2}} \end{cases} \longrightarrow \begin{array}{l} \boxed{y = \tan(2\alpha)} \\ z = \tan(4\alpha) \end{array}$$

$$\boxed{\tan(\alpha+\beta) = ?}$$

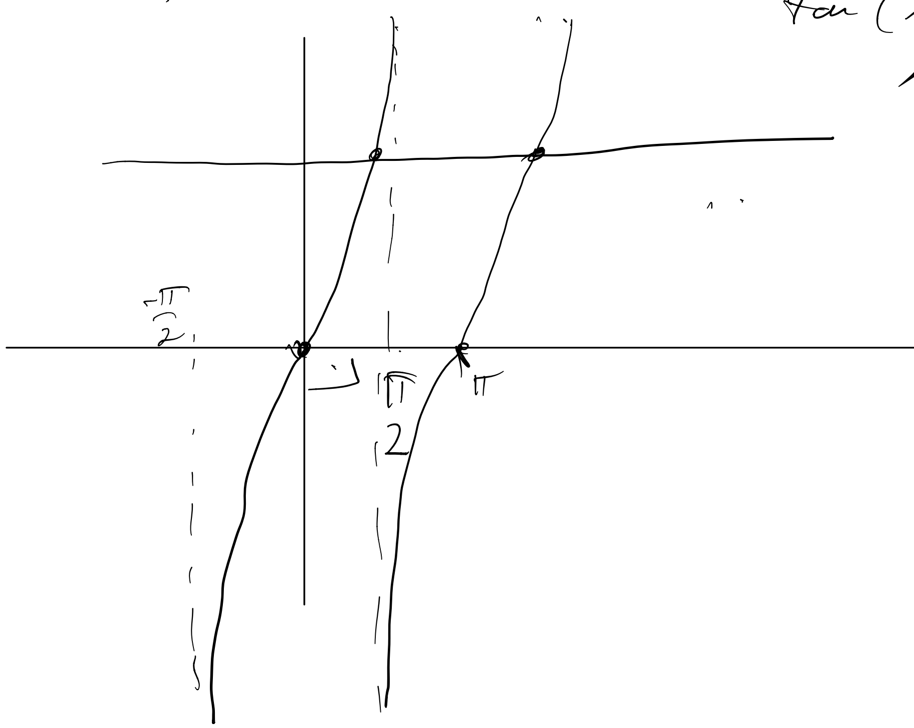
$$\tan(2\alpha) = \frac{2\,tg(\alpha)}{1 - tg^2(\alpha)}$$

$x \in \mathbb{R}$ why I can take

$$-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2} \quad \text{s.t.}$$

$$\boxed{x = \tan(\alpha)}$$

Surjectivity of $\tan(x)$




First equation transforms into

$$\tan(\alpha) = \tan(8\alpha)$$

$$8\alpha - \alpha = k\pi$$

$$7\alpha = k\pi$$

$$\alpha = \frac{k\pi}{7}$$

$$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$$

$$-\frac{\pi}{2} \leq \frac{k\pi}{7} \leq \frac{\pi}{2}$$

$$-3{,}5\cancel{\pi} \leq k\cancel{\pi} \leq 3{,}5\cancel{\pi}$$

$$k = -3, -2, -1, 0, 1, 2, 3$$

7 solutions:

$\tan(\alpha)$, $\tan(2\alpha)$, $\tan(4\alpha)$

$$\alpha = \frac{k\pi}{7}$$

$$k = -3, -2, -1, 0, 1, 2, 3$$

$$\begin{cases} x = \tan \frac{\pi}{7} \\ y = \tan \frac{2\pi}{7} \\ z = \tan \frac{4\pi}{7} \end{cases}$$

. . .

$\boxed{\text{7 solutions}}$

$\boxed{P8}$

Interpret $\boxed{\text{" polynomial } f \text{ has no real root "}}$

$\boxed{\text{always}}$ $f > 0$ or $f < 0$ $\boxed{\text{Const sign}}$

$\boxed{7, \ 8, \ 9, \ 10, \ 11}$