Day 6 (4)

Problem 1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y) \qquad \forall x, y \in \mathbb{R}$$

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}$$

Problem 3. (BMO 2019) Let $\mathbb{P}$ be the set of all prime numbers. Find all the functions $f : \mathbb{P} \to \mathbb{P}$ such that:

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q \qquad \forall p, q \in \mathbb{P}.$$

Problem 4. (Japan 2013) Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that

$$f(m) + f(n) = f(mn) + f(m + n + mn) \qquad \forall m, n \in \mathbb{Z}$$

Problem 5. (BMO 2007) Find all real functions f defined on $\mathbb{R}$ such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}$$

Problem 6. (Saudi TST 2016) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x[f(x+y) - f(x-y)] = 4yf(x) \qquad \forall x, y \in \mathbb{R}$$

Problem 7. (APMO 2019) Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

Problem 8. (BMO 2017) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$n + f(m) | f(n) + nf(m) \qquad \forall m, n \in \mathbb{N}$$

Problem 9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(y + f(x)) = f(x)^2 + 2yf(x) + f(y) \qquad \forall x, y \in \mathbb{R}$$

Problem 10. Find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(n+1) > f(f(n)) \qquad \forall n \in \mathbb{Z}^+$$

Problem 11. Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(x + \frac{y}{x}) = f(x) + f(\frac{y}{x}) + 2y$$

Problem 12. Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(1) = 1$ and

$$f(m^2 + n^2) = f^2(m) + f^2(n)$$

Problem 13. Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$mf(n) + nf(m) = (m + n)f(m^2 + n^2)$$

Problem 14. (BMO 2009) Find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(f^2(m) + 2f^2(n)) = m^2 + 2n^2$$