HELLO!

$$\boxed{a^n + b^n = c^n}$$

3 most active participants (homework, class work)

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1°) Prove that $7 \mid a^2+b^2 \Rightarrow 7 \mid a,\ 7 \mid b$.

Little Fermat theorem?

Euler theorem?

Wilson theorem?

$d \mid m^2n+1 \mid n$

$d \mid mn^2+1 \cdot m$

$\rightsquigarrow$

$$\left.\begin{array}{l} d \mid m^2n^2+n \\ d \mid m^2n^2+m \end{array}\right\} \Rightarrow d \mid m^2n^2+n-(m^2n^2+m) = n-m$$

so $d \mid n-m$

$$m^3+1-(m^2n+1) = m^2(m-n)$$

Warm up

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- $a+b \mid a^2 \overset{\text{Prove}}{\Longrightarrow} a+b \mid b^2$
- $m+n \mid m+n^2 \overset{\text{Prove}}{\Longrightarrow} m+n \mid m+n^3$.
- m^2n+1, mn^2+1 are divisible by d.

  Prove $d \mid m^3+1$, $d \mid n^3+1$.

- $\boxed{133 \mid 11^{n+2} + 12^{2n+1}}$

1) $a^2 \equiv 0 \pmod{a+b}$

$a+b \mid a^2-b^2 = (a-b)(a+b)$ so

$0 \equiv a^2 \equiv b^2 \pmod{a+b}$

$m+n \mid m+n^2 \rightsquigarrow$

$m+n \mid m+n^3 - (m+n) =$

$= n(n-1)$.

$m+n^3 - \underbrace{(m+n)}_{\mid m+n} = \underbrace{n(n-1)(n+1)}_{m+n}$

so

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5) Prove that

$$\gcd(m,n)\cdot \operatorname{lcm}(m,n) = mn. \qquad m\cdot n.$$

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Prove that

6) $$\frac{\operatorname{lcm}(a,b,c)^2}{\operatorname{lcm}(a,b)\cdot\operatorname{lcm}(b,c)\cdot\operatorname{lcm}(c,a)} = \frac{\gcd(a,b,c)^2}{\gcd(a,b)\cdot\gcd(a,c)\cdot\gcd(b,c)}.$$

$$133 \mid 11^{n+2} + 12^{2n+1} = 11^2\cdot 11^n + 12\cdot(12^2)^n =$$

$$= \underbrace{121}_{-12 \bmod 133}\cdot 11^n + 12\cdot \underbrace{144}_{11 \bmod 133}{}^n \equiv$$

$$\equiv -12\cdot 11^n + 12\cdot 11^n = 0 \pmod{133}.$$

$n=0$

$133 \mid 11^2 + 12 = 133$ ✓

n, $\boxed{n+1}$ $\quad 11^{n+3} + 12^{2(n+1)+1} = 11^{n+3} + 12^{2n+3} = 11\left(11^{n+2} + 12^{2n+1}\right) + 133\cdot 12^{2n+1}$

$11\cdot 12^{2n+1}$

$144\cdot 12^{2n+1}$

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$\gcd(m,n) \cdot \operatorname{lcm}(m,n) = mn$

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$$n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

the same set of primes.

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$\alpha_i \geq 0, \quad \beta_i \geq 0$

$121 = 11^2 \cdot 3^0$

$33 = 11^1 \cdot 3^1$

$v_{11}(121) = 2$

$v_3(33) = 1$

$v_{11}(33) = 1$

$$\boxed{v_p(n) = k \quad \text{s.t.} \quad p^k \mid n, \quad p^{k+1} \nmid n}$$

$$\gcd(m,n) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \cdots p_k^{\min(\alpha_k,\beta_k)}$$

$$v_{p_i}(\gcd(m,n)) = \min(\alpha_i, \beta_i)$$

$$v_{p_i}(\operatorname{lcm}(m,n)) = \max(\alpha_i, \beta_i)$$

RULE

$$A = B$$

$$v_p(A) = v_p(B)$$

for any prime p.

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Problem 7

Let a, b positive integers, p, q distinct primes s.t. $p \mid aq-1$, $q \mid bp-1$.

Prove: $\frac{a}{p} + \frac{b}{q} > 1$.

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$$\frac{\operatorname{lcm}(m,n) \cdot \gcd(m,n)}{L} = \frac{mn}{R}$$

$$v_{p_i}\left(\operatorname{lcm}(m,n) \cdot \gcd(m,n)\right) = v_{p_i}(\operatorname{lcm}(m,n)) + v_{p_i}(\gcd(m,n)) =$$

$$= \max(\alpha_i, \beta_i) + \min(\alpha_i, \beta_i)$$

$$v_{p_i}(mn) = v_{p_i}(m) + v_{p_i}(n) = \beta_i + \alpha_i$$

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$$v_{p_i}\left(\frac{\mathrm{lcm}(a,b,c)^2}{\mathrm{lcm}(a,b)\cdot\mathrm{lcm}(b,c)\cdot\mathrm{lcm}(c,a)}\right) = \frac{(\mathrm{gcd}(a,b,c))^2}{\mathrm{gcd}(a,b)\,\mathrm{gcd}(b,c)\,\mathrm{gcd}(a,c)}$$

L R

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$$v_{p_i}(L) = v_{p_i}(R)$$

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$$b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

$$c = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}$$

$$v_p\left(\frac{p^k}{p^l}\right) = \underline{k-l}$$

...

$$v_{p_i}(\mathrm{lcm}(a,b,c)^2) = 2\max(\alpha_i, \beta_i, \gamma_i)$$

$$v_{p_i}(\mathrm{lcm}(a,b)) = \max(\alpha_i, \beta_i)$$

$$v_{p_i}(\mathrm{lcm}(b,c)) = \max(\beta_i, \gamma_i)$$

$$v_{p_i}(\mathrm{lcm}(c,a)) = \max(\gamma_i, \alpha_i)$$

$$v_{p_i}(\mathrm{gcd}(a,b,c)) = \min(\alpha_i, \beta_i, \gamma_i)$$

$$v_{p_i}(\mathrm{gcd}(a,b)) = \min(\alpha_i, \beta_i)$$

etc...

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$$\alpha_i + \beta_i = \underline{\max(\alpha_i, \beta_i)} + \min(\alpha_i, \beta_i)$$

$\alpha_i \geq \beta_i$

$$\alpha_i + \beta_i \overset{?}{=} \alpha_i + \beta_i$$

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We need:

$$\left(2\overbrace{\max(\alpha_i, \beta_i, \gamma_i)}^{\alpha_i} - \overbrace{\max(\alpha_i, \beta_i)}^{\alpha_i} - \overbrace{\max(\beta_i, \gamma_i)}^{\beta_i} - \overbrace{\max(\gamma_i, \alpha_i)}^{\alpha_i} = \right.$$

$$\left.2\underbrace{\min(\alpha_i, \beta_i, \gamma_i)}_{\gamma_i} - \underbrace{\min(\alpha_i, \beta_i)}_{\beta_i} - \underbrace{\min(\beta_i, \gamma_i)}_{\gamma_i} - \underbrace{\min(\gamma_i, \alpha_i)}_{\gamma_i}\right)$$

$\alpha_i, \beta_i, \gamma_i$

$\alpha_i \geq \beta_i \geq \gamma_i$ ← WLOG

$$2\cancel{\alpha_i} - \cancel{\alpha_i} - \beta_i - \cancel{\alpha_i} \overset{?}{=} 2\cancel{\gamma_i} - \beta_i - \cancel{\gamma_i} - \cancel{\gamma_i}$$

$$-\beta_i = -\beta_i$$

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7) $a, b \in \mathbb{Z}_+$ $\quad p, q$ - distinct primes

$p \mid aq-1, \quad q \mid bp-1$

Prove: $\frac{a}{p} + \frac{b}{q} > 1.$

8) a, b - odd positive integers such that

$a^b \cdot b^a$ is perfect square.

Prove ab is perfect square.

9) Let a, b, c, d positive integers s.t $ab = cd$

Prove $a^{2020} + b^{2020} + c^{2020} + d^{2020}$ is composite number.

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$$\alpha_i + \beta_i = \underline{\max(\alpha_i, \beta_i)} + \min(\alpha_i, \beta_i)$$

7

$\alpha_i \geq \beta_i$

$$\alpha_i + \beta_i \overset{?}{=} \alpha_i + \beta_i$$

We need:

$$\left(\begin{array}{l} 2\overbrace{\max(\alpha_i, \beta_i, \gamma_i)}^{\alpha_i} - \overbrace{\max(\alpha_i, \beta_i)}^{\alpha_i} - \overbrace{\max(\beta_i, \gamma_i)}^{\beta_i} - \overbrace{\max(\gamma_i, \alpha_i)}^{\alpha_i} = \\ 2\underbrace{\min(\alpha_i, \beta_i, \gamma_i)}_{\gamma_i} - \underbrace{\min(\alpha_i, \beta_i)}_{\beta_i} - \underbrace{\min(\beta_i, \gamma_i)}_{\gamma_i} - \underbrace{\min(\gamma_i, \alpha_i)}_{\gamma_i} \end{array} \right)$$

$\alpha_i, \beta_i, \gamma_i$

$\alpha_i \geq \beta_i \geq \gamma_i$ ← WLOG

$$2\cancel{\alpha_i} - \cancel{\alpha_i} - \beta_i - \cancel{\alpha_i} \overset{?}{=} 2\cancel{\gamma_i} - \beta_i - \cancel{\gamma_i} - \cancel{\gamma_i}$$

$$-\beta_i = -\beta_i \quad \blacksquare$$

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7) $\times$ $a, b \in \mathbb{Z}_+$ $\quad p, q$ - distinct primes

(8)

$p \mid aq-1, \quad q \mid bp-1$

Ali

Prove: $\frac{a}{p} + \frac{b}{q} > 1.$

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8) $\times$ a, b - odd positive integers such that

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$a^b \cdot b^a$ is perfect square.

Prove ab is perfect square.

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9) Let a, b, c, d positive integers s.t $ab = cd$

Prove $a^{2020} + b^{2020} + c^{2020} + d^{2020}$ is composite number.

10) ~~ab~~: $a > b > 1$ integers

$a+b \mid ab+1$ and $a-b \mid ab-1$.

Prove $a < \sqrt{3} \cdot b$

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(8) a, b - odd,

$$a^b \cdot b^a = x^2 \quad | \cdot ab$$

$$Y^2 = a^{b+1} \cdot b^{a+1} = x^2 \cdot (ab)$$

$\longrightarrow$ ab is square

$2 \mid b+1, \quad 2 \mid a+1$

<u>$a \geq b$</u>

$$a^b \cdot b^a = (ab)^b \, b^{a-b} = x^2$$

$b^{a-b} = Y^2$ because $2 \mid a-b$

$(ab)^b = Z^2$ but b is odd

$(ab)^{b+1} = Z^2 (ab)$ $\quad 2 \mid b+1$

$(ab)^{b+1} = T^2$

$\Longrightarrow$ ab is square.

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7) $p \mid aq-1 \qquad q \mid bp-1$

$$p \mid aq + bp - 1$$

$$q \mid aq + bp - 1$$

so $pq \mid aq+bp-1$, so $pq \mid aq+bp-1$

In particular $pq \leq aq+bp-1$ so

$$\frac{a}{p} + \frac{b}{q} = \frac{aq+bp}{pq} \geq \frac{pq+1}{pq} = 1 + \frac{1}{pq} > 1 \blacksquare$$

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For Ali:

Prove that there exist infinitely many positive integers n such that $\underline{n^2+1 \mid n!}$

<u>Lemma</u> (Four numbers theorem)

a, b, c, d – positive integers s.t. $ab = cd$

then $a = pq$, $b = rs$, $c = ps$, $d = qr$

for some integers p, q, r, s.

<u>Proof</u>

$$\frac{a}{c} = \frac{d}{b}\frac{b}{d} = \frac{q}{s} \quad \text{where } (q,s) = 1.$$

$$\frac{a}{c} = \frac{q}{s} \rightsquigarrow$$

$$\frac{b}{d} = \frac{q}{s} \rightsquigarrow$$

$$\boxed{\begin{aligned} a &= p \cdot q \\ c &= p \cdot s \\ d &= r \cdot q \\ b &= r \cdot s \end{aligned}}$$

and so $\boxed{ab = cd}$

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$ab = cd$

$a = pq, \quad b = rs, \quad c = pr, \quad d = qs$

p, q, r, s

$$a^{2020} + b^{2020} + c^{2020} + d^{2020} = (pq)^{2020} + (rs)^{2020} + (pr)^{2020} + (qs)^{2020}$$

$$= p^{2020} \cdot q^{2020} + r^{2020} s^{2020} + p^{2020} r^{2020} + q^{2020} s^{2020} =$$

$$= \boxed{\underbrace{\left(p^{2020} + s^{2020}\right)}_{>1} \underbrace{\left(q^{2020} + r^{2020}\right)}_{>1}}$$

composite

$n := 2020$

abcd is composite always.

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$a > b > 1$ integers

$p, q \mid aq + bp - 1$

$pq \mid aq + bp - 1$

$a+b \mid ab+1$ and $a-b \mid ab-1$ (*)

Prove $a < \sqrt{3} \cdot b$.

$$b^2 - 1 = \underbrace{b(a+b)}_{\mid a+b} - \underbrace{(ab+1)}_{\mid a+b}, \text{ so}$$

$$\boxed{a+b \mid b^2-1}$$

$$b^2 - 1 = \underbrace{(ab-1)}_{\mid a-b} - \underbrace{b(a-b)}_{a-b}, \text{ so}$$

$$\boxed{a-b} \mid b^2 - 1$$

$a+b \mid$

$a-b \mid b^2-1$

$d = \gcd(a+b, a-b)$

$d \mid a+b$
$d \mid a-b$ $\Rightarrow$ $d \mid 2a$

$d \mid 2a$

$\gcd(a, d) = 1$ because $d \mid ab+1$

$\boxed{d \mid 2}$

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$\gcd(a+b, a-b) \mid 2$ and $a+b, a-b \mid b^2-1$

$\Downarrow$

$(a+b)(a-b) \mid 2(b^2-1)$

$\Downarrow$

$a^2-b^2 \mid 2b^2-2$ so

$a^2-b^2 \leq 2b^2-2 \Longrightarrow$

$\Longrightarrow 3b^2 \geq a^2+2 > a^2 \rightsquigarrow \underline{a < \sqrt{3} \cdot b.}$ ■

$d \mid a+b \rightsquigarrow d \mid ab+1$ if $\gcd(a,d) > 1$

$d \mid a-b$

$p \mid a, p \mid d$

$p \mid ab+1$

$\Downarrow$

$p \mid 1$ ↯ ■

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