May Online Camp 2021

Number Theory

Level L3

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Contents

(01-06-2021, 14:24)

Problems

Problem 1. A positive integer is called *nice* if it can be represented as a sum of two squares of non-negative integers. Prove that any positive integer is the difference of two nice numbers.

Problem 2. Let p_i for $i = 1, 2, \ldots, k$ be a sequence of consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 3 \ldots$). Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_k$. Prove that in a set $\{1, 2, \ldots, N\}$ there are exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

Problem 3. Find all sets of positive integers $\{x_1, x_2, \ldots, x_{20}\}$ such that

$$x_{i+2}^2 = \operatorname{lcm}(x_{i+1}, x_i) + \operatorname{lcm}(x_i, x_{i-1})$$

for $i = 1, 2, \ldots, 20$ where $x_0 = x_{20}$, $x_{21} = x_1$, $x_{22} = x_2$.

Problem 4. Let $n > 1$ be odd integer. Consider numbers n, $n + 1$, $n + 2$, $\ldots$, $2n - 1$ written on the blackboard. Prove that we can erase one number, such that the sum of all numbers will be not divided any number on the blackboard.

Problem 5. Let $n > 20$ and $k > 1$ be integers such that k^2 divides n. Prove that there exist positive integers a, b, c, such that

$$n = ab + bc + ca.$$

Problem 6. Let a, $b > 1$ be integers such that $a^2 + b$, and $a + b^2$ are primes. Prove $\gcd(ab + 1, a + b) = 1$.

Problem 7. Let p, q be primes such that $p < q < 2p$. Prove that there are two consecutive positive integers, such that largest prime divisor of first number is p, and the largest prime divisor of second number is q.

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that

$$a = \frac{x+y}{z} \quad \text{and} \quad b = \frac{xy}{z}.$$

Problem 9. We say that a positive integer is an almost square, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.

Problem 10. It is known that a cells square can be cut into n equal figures of k cells. Prove that it is possible to cut it into k equal figures of n cells.

Problem 11. Prove that any rational number may be written as

$$\frac{a^2+b^3}{c^5+d^7},$$

where a, b, c, d are positive integers.

Problem 12. Let n be a positive integer. Prove that there exists positive integers a and b, such that

$$a^2+a+1 = (n^2+n+1)(b^2+b+1).$$

Problem 13. Let a, b, c be positive integers. Prove that there is a positive integer n such that

$$(a^2+n)(b^2+n)(c^2+n)$$

is a perfect square.

Problem 14. Let a, b, z be positive integers such that $ab = z^2+1$. Prove that there are positive integers such x, y such that

$$\frac{a}{b} = \frac{x^2+1}{y^2+1}.$$

Problem 15. Prove that there are infinitely many pairwise distinct positive integers a, b, c and d such that $a^2+2cd+b^2$ and $c^2+2ab+d^2$ are squares.

Solutions

Problem 1. A positive integer is called *nice* if it can be represented as a sum of two squares of non-negative integers. Prove that any positive integer is the difference of two nice numbers.

Solution. Note that

$$2a - 1 = a^2 - (a-1)^2 \quad \text{and} \quad 2a = (a^2 + 1^2) - (a-1)^2.$$

We just need to make sure that all nice numbers must be positive which is fine as soon as $a \geq 2$.

But for $a = 1$ we can certainly write $1 = (1^2 + 1^2) - 1^2$ and $2 = 2^2 - (1^2 + 1^2)$. □

Discussion.

Problem 2. Let p_i for $i = 1, 2, \dots, k$ be a sequence of consecutive prime numbers ($p_1 = 2$, $p_2 = 3$, $p_3 = 3 \dots$). Let $N = p_1 \cdot p_2 \cdot \dots \cdot p_k$. Prove that in a set $\{1, 2, \dots, N\}$ there are exactly $\frac{N}{2}$ numbers which are divisible by odd number of primes p_i.

Solution. Let's call the numbers which are in $\{1, 2, \dots, N\}$ and divisible by odd number of p_i's nice. We claim that: If $1 \leq n \leq \frac{N}{2}$, then exactly one of the numbers $\{n, n + \frac{N}{2}\}$ is lucky.

Indeed: let $n = p_{i_1}^{r_1} \cdot p_{i_2}^{r_2} \dots p_{i_m}^{r_m}$. Then

$$n + \frac{N}{2} = p_{i_1}^{r_1} \cdot p_{i_2}^{r_2} \dots p_{i_m}^{r_m} + p_2 \cdot p_3 \dots p_k.$$

Note that n and $n + \frac{N}{2}$ have the same set of prime divisors among $\{p_2, p_3, \dots, p_k\}$. Notice also that the parity of n and $n + \frac{N}{2}$ are different. So one of them is lucky and other is not, as desired.

□

Discussion.

Problem 3. Find all sets of positive integers $\{x_1, x_2, \dots, x_{20}\}$ such that

$$x_{i+2}^2 = \operatorname{lcm}(x_{i+1}, x_i) + \operatorname{lcm}(x_i, x_{i-1})$$

for $i = 1, 2, \dots, 20$ where $x_0 = x_{20}$, $x_{21} = x_1$, $x_{22} = x_2$.

Solution. Firstly, notice that for any i, $\gcd(x_{i+1}, x_i) \geq 2$. Indeed, there exists i so that x_i is divisible by prime p, since otherwise $x_i = 1$ for all i, which does not satisfy the given. Since

$$x_{i+3}^2 = \operatorname{lcm}(x_{i+2}, x_{i+1}) + \operatorname{lcm}(x_{i+1}, x_i) \quad \text{and} \quad x_{i+2}^2 = \operatorname{lcm}(x_{i+1}, x_i) + \operatorname{lcm}(x_i, x_{i-1}),$$

thus

$$(x_{i+3} - x_{i+2})(x_{i+3} + x_{i+2}) = \operatorname{lcm}(x_{i+2}, x_{i+1}) - \operatorname{lcm}(x_i, x_{i-1}).$$

As $p \mid x_i$, we get $p \mid x_{i+2}$, and therefore from the above equality $p \mid x_{i+3}$, inducing all up, we have every x_i divisible by p. Therefore, for every i, $\gcd(x_{i+1}, x_i) \geq 2$.

We sum all the equations up and obtain that

$$2\sum_{i=1}^{20} \operatorname{lcm}(x_{i+1}, x_i) = \sum_{i=1}^{20} x_i^2 \geq \sum_{i=1}^{20} x_i x_{i+1},$$

where equality holds if and only if $x_i = x_{i+1}$ for all i. Now we rewrite

$$\operatorname{lcm}(x_{i+1}, x_i) = \frac{x_i x_{i+1}}{\gcd(x_{i+1}, x_i)}$$

and as $\gcd(x_{i+1}, x_i) \geq 2$, we conclude that the inequality must hold, therefore all integers are equal. Now, we must have $x_i^2 = 2x_i \implies x_i = 2$. $\square$

Discussion.

Problem 4. Let $n > 1$ be odd integer. Consider numbers n, $n+1$, $n+2$, $\ldots$, $2n-1$ written on the blackboard. Prove that we can erase one number, such that the sum of all numbers will be not divided any number on the blackboard.

Solution. Let S be the sum of all numbers on the blackboard ,

$$S = n^2 + \frac{n(n-1)}{2} = \frac{n(3n-1)}{2}.$$

If we erase any number x the sum will be $S - x$. If two of $S - x$'s are divided by the same number then

$$S - x \equiv S - y \pmod{n+i} \Longleftrightarrow x \equiv y \pmod{n+i}$$

which is absurd, so all of $n-1$, $S-x$'s have different divisors from the set of written numbers (on blackboard).

Assume that all $S - x$'s are divided by a written number, then all of them must be divided by exactly one of the written numbers (because there are $n-1$ numbers and $n-1$ sums). But taking $x = n$ the sum is $\frac{n(3n-3)}{2}$ which is divided by both n and $\frac{3n-3}{2}$ (because n is odd and $n < \frac{3n-3}{2} < 2n-1$ is a written integer – contradiction. $\square$

Discussion.

Problem 5. Let $n > 20$ and $k > 1$ be integers such that k^2 divides n. Prove that there exist positive integers a, b, c, such that
$$n = ab + bc + ca.$$

Solution. So note that if $n = ab + bc + ca$, then $n + a^2 = (a + b)(a + c)$, so we have to construct a for each non-squarefree n, such that $n + a^2$ is representable as the product of two numbers, bigger than a.

Consider prime p, such that $n = p^2 l$. So firstly consider whether we can take $a = p$. We want $(l + 1)p^2$ to represented as a product in the above way. If $l + 1 > p$, we have It. If $l + 1$ is composite, then it is st, and take ps and pt. So we are left with the case when it's prime q, so $n = (q - 1)p^2$. Now take $p = mq + r$, where r is the remainder (now we look in the case where r is positive integer) , and choose $a = r$ and the rest is to choose the one number to be $q > r$, the other is bashed to be bigger than r. We are only left to consider $q = p$, then choose $a = 6$ and factor out $p^3 - p^2 + 36$ to see that it works. $\square$

Discussion.

Problem 6. Let a, $b > 1$ be integers such that $a^2 + b$, and $a + b^2$ are primes. Prove $\gcd(ab + 1, a + b) = 1$.

Solution. Assume $p \mid ab + 1, a + b$ for some prime p. Then, we have $(a + 1)(b + 1) = ab + 1 + a + b \equiv 0 \pmod p$. Thus, we have $a + 1 \equiv 0 \pmod p$ or $b + 1 \equiv 0 \pmod p$.

WLOG, we assume $a \equiv -1 \pmod p$. Since, $a + b \equiv 0 \pmod p$, we must have $b \equiv 1 \pmod p$. Then, $b^2 + a \equiv 0 \pmod p$. Therefore, $b^2 + a = p$ must be satisfied. Then, $p > b^2 \geq b \equiv 1$. Therefore, $b = 1$, a contradiction. $\square$

Discussion.

Problem 7. Let p, q be primes such that $p < q < 2p$. Prove that there are two consecutive positive integers, such that largest prime divisor of first number is p, and the largest prime divisor of second number is q.

Solution. We know $qb - pa = 1$ for some positive integers a, b with $1 \leq b \leq p$, $1 \leq a \leq q$; this is straightforward Bézout and noticing that if (a, b) is solution, then so is $(a - q, b - p)$.

If $a \leq \frac{q}{2}$ then $b \leq \frac{p}{2}$, and the largest prime divisor of qb is q, and that of pa is p since $p < \frac{q}{2}$. If $a > \frac{q}{2}$ then $(q - a, p - b)$ satisfies $px - qy = 1$; and repeat same argument, since $q - a < \frac{q}{2}$, $p - b < \frac{p}{2}$. $\square$

Discussion.

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that

$$a = \frac{x+y}{z} \quad \text{and} \quad b = \frac{xy}{z}.$$

Solution. Take

$$x = \frac{b+1}{a}, \quad y = \frac{b(b+1)}{a}, \quad z = \frac{(b+1)^2}{a^2}.$$

□

Discussion.

Problem 9. We say that a positive integer is an almost square, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.

Solution. Note that

$$a(a-1) = \frac{(a^2-1)a^2}{(a-1)a}.$$

□

Discussion.

Problem 10. It is known that a cells square can be cut into n equal figures of k cells. Prove that it is possible to cut it into k equal figures of n cells.

Solution. Note that $nk = s^2$ for some s. By Factor Lemma, pick $n = ab$, $k = cd$, and $s = ac = bd$. Now we can tile with $a \times b$ rectangles! □

Discussion.

Problem 11. Prove that any rational number may be written as

$$\frac{a^2+b^3}{c^5+d^7},$$

where a, b, c, d are positive integers.

Solution. For any positive integers p, q the following holds

$$\frac{p}{q} = \frac{p}{q} \cdot \frac{p^5q^4 + p^{14}q^6}{p^5q^4 + p^{14}q^6} = \frac{p^6q^4 + p^{15}q^6}{p^5q^5 + p^{14}q^7} = \frac{(p^3q^2)^2 + (p^5q^2)^3}{(pq)^5 + (p^2q)^7}.$$

□

Discussion.

Problem 12. Let n be a positive integer. Prove that there exists positive integers a and b, such that
$$a^2 + a + 1 = (n^2 + n + 1)(b^2 + b + 1).$$

Solution. Take $a = n^2$ and $b = n - 1$, then
$$\frac{a^2 + a + 1}{b^2 + b + 1} = \frac{n^4 + n^2 + 1}{(n-1)^2 + (n-1) + 1} = \frac{n^4 + n^2 + 1}{n^2 - n + 1} = n^2 + n + 1.$$
□

Discussion.

Problem 13. Let a, b, c be positive integers. Prove that there is a positive integer n such that
$$(a^2 + n)(b^2 + n)(c^2 + n)$$
is a perfect square.

Solution. Let $n = ab + bc + ca$, then
$$(a^2 + n)(b^2 + n)(c^2 + n) = (a^2 + ab + bc + c)(b^2 + ab + bc + ca)(c^2 + ab + bc + ca) =$$
$$= (a+b)^2(b+c)^2(c+a)^2.$$
□

Discussion.

Problem 14. Let a, b, z be positive integers such that $ab = z^2 + 1$. Prove that there are positive integers such x, y such that
$$\frac{a}{b} = \frac{x^2 + 1}{y^2 + 1}.$$

Solution. Let $x = z + a$ and $y = z + b$. Then
$$\frac{x^2 + 1}{y^2 + 1} = \frac{(z+a)^2 + 1}{(z+b)^2 + 1} = \frac{z^2 + 1 + 2za + a^2}{z^2 + 1 + 2zb + b^2} =$$
$$= \frac{ab + 2za + a^2}{ab + 2zb + b^2} = \frac{a(a + b + 2z)}{b(a + b + 2z)} = \frac{a}{b}.$$
□

Discussion.

Problem 15. Prove that there are infinitely many pairwise distinct positive integers a, b, c and d such that $a^2 + 2cd + b^2$ and $c^2 + 2ab + d^2$ are squares.

Solution. It is enough to take distinct a, b, c and d for which $ab = cd$. For example, $b := 6a$, $c := 2a$ and $d := 3a$. $\square$

Discussion.

References

- Art of Problem Solving - https://artofproblemsolving.com
- Polish Mathematical Olympiad - https://om.mimuw.edu.pl
- Homepage of Dominik Burek - http://dominik-burek.u.matinf.uj.edu.pl