Email training, N3

Level 3, September 27-October 3

Problem 3.1. Find all positive integers n such that the products of its digits is equal $n^2 - 10n - 22$.

Solution 3.1. Let's prove that $S(n) \leq n$. Indeed

$$n = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10a_1 + a_0 \geq$$
$$10^k a_k \geq a_k \cdot a_{k-1} \cdot \ldots \cdot a_1 \cdot a_0 = S(n)$$

So, for all n satisfying condition of the problem on has $n \geq n^2 - 10n - 22$, which means $n \leq 12$.

On other side $S(n) = n^2 - 10n - 22 \geq 0$, which means $n > 11$. So, the only candidate for answer is $n = 12$. By checking one concludes that it satisfies to the conditions of the problem. $12^2 - 10 \cdot 12 - 22 = 2 = 1 \cdot 2$.

Answer: 12.

Problem 3.2. Prove there exist infinitely many positive integers divisible by 2021 and each of them containing the same number of digits 0, 1, …, 9.

Solution 3.2. Since $gcd(10, 2021) = 1$, then, according to Euler's theorem $10^{\phi(2021)} - 1$ is divisible by 2021. So

$$10^{\phi(2021)} - 1 = \underbrace{99......99}_{\phi(2021)} = 9 \cdot \underbrace{11\ldots11}_{\phi(2021)}$$

is divisible by 2021. Since numbers 9 and 2021 are coprime, then $\underbrace{11\ldots11}_{\phi(2021)}$ is divisible by 2021.

So

$$N = \underbrace{11\ldots11}_{\phi(2021)}\underbrace{22\ldots22}_{\phi(2021)}\ldots\underbrace{99\ldots99}_{\phi(2021)}\underbrace{00\ldots00}_{\phi(2021)}$$

is multiple of $\underbrace{11\ldots11}_{\phi(2021)}$, which is divisible by 2021. It means, that N is divisible by 2021. By "gluing" numbers N together one gets infinitely many integers satisfying to the conditions of the problem.

Problem 3.3. Find all values of a for which the equation $x^3 + ax^2 + 56x - 4 = 0$ has 3 roots forming consecutive terms of a geometric progression.

Solution 3.3. Denote the roots of the equation by x_1, x_2, x_3 in the order of geometric sequence. Then $x_1 \cdot x_3 = x_2^2$. According to the Viet theorem we have the system

$$\begin{cases} x_1x_2x_3 = 4 \\ x_1x_2 + x_2x_3 + x_1x_3 = 56 \\ x_1 + x_2 + x_3 = -a \end{cases} \Leftrightarrow$$

$$\begin{cases} x_2^3 = 4 \\ x_2(x_1 + x_2 + x_3) = 56 \\ x_1 + x_2 + x_3 = -a \end{cases} \Rightarrow$$

$$\begin{cases} x_2 = \sqrt[3]{4} \\ x_2(-a) = 56 \end{cases} \Rightarrow a = -\frac{56}{\sqrt[3]{4}} = -28\sqrt[3]{2}$$

Answer: $a = -28\sqrt[3]{2}$.

Problem 3.4. Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate the sum
$$\sum_{k=0}^{2021} f\left(\frac{k}{2021}\right).$$

Solution 3.4. Lets prove, that $f(x) + f(1-x) = 1$. Indeed
$$f(x) + f(1-x) = \frac{9^x}{9^x+3} + \frac{9^{1-x}}{9^{1-x}+3} =$$
$$\frac{9^x}{9^x+3} + \frac{9}{9+3\cdot 9^x} = \frac{9^x}{9^x+3} + \frac{3}{3+\cdot 9^x} = 1:$$
So
$$\sum_{k=0}^{2021} f\left(\frac{k}{2021}\right) = \sum_{k=0}^{1010}\left(f\left(\frac{k}{2021}\right) + f\left(\frac{2021-k}{2021}\right)\right) = 1011.$$
Answer: 1011.

Problem 3.5. One cuts a grid of size 8×8 by a straight line. Find the maximal possible number of cells that are cut by the line.

Solution 3.5. There are together 14 horizontal and vertical lines inside the board. Without them the drawn line would cut only ony cell (the original big board). Lets att those 14 lines one by one. When adding one line, it can add the number of cutted cells by at most one. Therefore there could be at most 15 cells cutted by line. It remains to draw such a line. Let draw the diagonal of the board and take a line which is parallel to the diagonal and is a little bit far from it. It will cut the cells $a1$, $a2$, $b2$, $b3$, $c3$, $c4$, $d4$, $d5$, $e5$, $e6$, $f6$, $f7$, $g7$, $g8$, $h8$.
Answer: 15.

Problem 3.6. In the cells of the grid 10×10 are written positive integers, all of them less than 11. It is known that the sum of 2 numbers written in the cells having common vertex is a prime number. Prove that there are 17 cells containing the same number.

Solution 3.6. Let's do contradiction, which means no number is written more than 16 times. Lets split the board to 25 squares of size 2×2.

Let A is the number of squares that don't contain number 1. Then $A \geq 25 - 16 = 9$. Since we have at most 16 numbers 1. then $25 - A \leq 16$, so $A \geq 9$. Let split integers from 2 to 10 to the groups $\{3, 6, 9\}$, $\{5\}$, $\{7\}$, $\{2, 4, 8, 10\}$. From the condition of the problem follows, that in each square can be at most one number from each group. Since the square has 4 cells, it means that in each square that don't contain 4, has numbers from all groups. So in the mentioned above A squares there are both 5 and 7. In other squares can be at most $16 - A$ times number 5 and $16 - A$ times number 7. So, in other squares number 5 and 7 may appear at most $16 - A + 16 - A = 32 - 2A$ times, but in each square should be at least one of them. So there are $32 - 2A \geq 25 - A$ which means $A \leq 7$. But we had $A \geq 9$, which is contradiction.