Number Theory

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Problems – April 15

1. Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n for which $d_1^2 + d_2^2 + d_3^2 + d_4^2 = n$.

- The number of divisors of a positive integer $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is $\tau(n) = \prod_{i=1}^{k}(r_i+1)$.

- The sum of divisors of a positive integer $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is $\sigma(n) = \prod_{i=1}^{k}(1 + p_i + p_i^2 + \cdots + p_i^{r_i})$.

2. Given a positive integer n, define a sequence (a_k) by $a_0 = n$ and $a_{k+1} = \tau(a_k)$. Find all n for which no term a_k is a perfect square.

3. If $a \mid b$ and $a < b$, prove that $\frac{\sigma(a)}{a} < \frac{\sigma(b)}{b}$.

4. Prove that there are infinitely many pairs of different positive integers m and n such that $\sigma(m^2) = \sigma(n^2)$?

5. A positive integer n is *perfect* if its sum of divisors $\sigma(n)$ (including itself) equals $2n$. If a perfect number is divisible by 7, prove that it is divisible by 49.

6. Prove that every even perfect number is of the form $n = 2^{k-1}(2^k - 1)$, where k is a positive integer.

7. For $n \in \mathbb{N}$, denote by $f(n)$ the smallest positive integer having exactly n divisors. Thus e.g. $f(5) = 16$ and $f(6) = 12$. Prove that, for any $k \in \mathbb{N}$, $f(2^k)$ divides $f(2^{k+1})$.

8. Is there a positive integer n such that both $n - 2015$ and $\frac{n}{2015}$ are positive integers having exactly 2015 divisors?

9. Find all pairs of positive integers a and b such that $\mathrm{lcm}[a+1, b+1] = a^2 - b^2$.

10. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant ab+bc+ca-a-b-c+1$. Show that equality occurs for infinitely many triples (a, b, c).

11. If a and b are positive integers such that $\mathrm{lcm}[a, b] + \mathrm{lcm}[a+2, b+2] = 2\mathrm{lcm}[a+1, b+1]$, prove that $a \mid b$ or $b \mid a$.

12. Suppose that n is odd and both $\varphi(n)$ and $\varphi(n+1)$ are powers of 2. Prove that either $n = 5$, or $n + 1$ is itself a power of two.