# Preparation for Saudi Arabia Team 2021

## May/June Session: Level 4



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## Lesson 3

# Miscellaneous combinatorics

**Problems:**

1. We are given a polyhedron with at least 5 vertices, such that exactly 3 edges meet in each of the vertices. Prove that we can assign a rational number to every vertex of the given polyhedron such that the following conditions are met:

   $(i)$ At least one of the numbers assigned to the vertices is equal to 2020.

   $(ii)$ For every polygonal face, the product of the numbers assigned to the vertices of that face is equal to 1.

2. Let $n$ be a positive integer. There are given $n$ lines such that no two are parallel and no three meet at a single point.

   (a) Prove that there exists a line such that the number of intersection points of these $n$ lines on both of its sides is at least
   $$\left\lfloor \frac{(n-1)(n-2)}{10} \right\rfloor.$$
   The points on the line are not counted.

   (b) Find all $n$ for which there exists a configurations where the equality is achieved.

3. Given is an $n \times n$ board, with an integer written in each square. For each move, one can choose any square, and add 1 to all $2n-1$ numbers in its row and column. Find the largest $N(n)$, such that for any initial choice of integers, one can make a finite number of moves so that there are at least $N(n)$ even numbers on the board.

4. In the nation of Graphia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

   We wish to close half the roads of Graphia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1.

5. Let $k$ and $n$ be integers with $1 \leq k < n$. Alice and Bob play a game with $k$ pegs in a line of $n$ holes. At the beginning of the game, the pegs occupy the $k$ leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the $k$ rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of $n$ and $k$ does Alice have a winning strategy?

6. Each point of a three-dimensional space is colored in one of two colors. Prove that there exists a triangle congruent to a given triangle whose vertices are all of one color.