Winter Camp 2021

Algebra Level 2

Day 9 (5)

8 January 2021

Test 2 problem Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\boxed{f(xy-1) + f(x)f(y) = 2xy - 1} \quad \forall x, y \in \mathbb{R}$$

الحلول $f(x) = x \quad \forall x \in \mathbb{R}$, $f(x) = -x^2 \quad \forall x \in \mathbb{R}$

$f(x) = -x^2$:

<u>check</u>: $-(xy-1)^2 + x^2y^2 = 2xy - 1$ ✓

$f(x) = x$: $xy - 1 + xy = 2xy - 1$ ✓

$y = 0$: $f(-1) + f(x)f(0) = -1$

$\Leftarrow$ لو $f(0) \neq 0$ $\quad f(x) = c$ بالتعويض $\Rightarrow\Leftarrow$

$\Rightarrow f(0) = 0, \quad f(-1) = -1$

$x = y = 1$: $f(1)^2 = 1 \quad \to f(1) = 1$ or $f(1) = -1$

$P(xy, 1)$: $f(xy-1) + f(xy)f(1) = 2xy - 1$

$$\Rightarrow \boxed{f(xy)\,f(1) = f(x)f(y)} \quad (*)$$

$y=\frac{1}{x}$: $f(1)^2 = 1 = f(x)f(\frac{1}{x})$

$y=1$: $\boxed{f(x-1) + f(x)f(1) = 2x-1}$ (**)

الطريقة الأولى

إذا كان $f(1)=1$:

$f(xy) = f(x)f(y)$, $f(x-1) + f(x) = 2x-1$

$x+1$

$y=x$: $f(x^2-1) + f(x)^2 = 2x^2-1$

$= f(x-1)f(x+1) + f(x)^2$

$= (2x-1-f(x))(2x+1-f(x)) + f(x)^2$

$= 4x^2-1 - 4xf(x) + 2f(x)^2$

$\Rightarrow 2x^2 - \cancel{1} = 4x^2 - \cancel{1} - 4xf(x) + 2f(x)^2$

$\Leftrightarrow 0 = x^2 - 2xf(x) + f(x)^2 = (x-f(x))^2$

$\Rightarrow f(x) = x$

إذا كان $f(1)=-1$:

$f(xy) = -f(x)f(y)$, $f(x-1) - f(x) = 2x-1$

$x+1$

$y=x$: $f(x^2-1) + f(x)^2 = 2x^2-1$

$= -f(x-1)f(x+1) + f(x)^2$

$= -(f(x)+2x-1)(f(x)-2x-1) + f(x)^2$

$= \cancel{-f(x)^2} - 2f(x) + 4x^2 - 1 \cancel{+f(x)^2}$

$\Rightarrow 2x^2-1 = -2f(x) + 4x^2 - 1 \Rightarrow f(x) = -x^2.$

الطريقة الثانية

$$f(x)f\left(\frac{1}{x}\right)=1$$

$x \to x+1$: $f(x+1)f\left(\frac{1}{x+1}\right)=1$

- $f(x+1)=\dfrac{2x+1-f(x)}{f(1)}$

- $f\left(\frac{1}{x+1}\right)=f\left(1-\frac{x}{x+1}\right)=\dfrac{-\frac{2x}{x+1}+1-f\left(\frac{-x}{x+1}\right)}{f(1)}$

- $f\left(\frac{-x}{x+1}\right)\underset{(*)}{=}f(1)f(-x)f\left(\frac{1}{x+1}\right)=f(1)f(1)f(-1)f(x)f\left(\frac{1}{x+1}\right)$
$=-f(x)f\left(\frac{1}{x+1}\right)$

$\Rightarrow f\left(\frac{1}{x+1}\right)=\dfrac{\frac{1-x}{x+1}+f(x)f\left(\frac{1}{x+1}\right)}{f(1)}$

$f\left(\frac{1}{x+1}\right)\left(f(1)-f(x)\right)=\frac{1-x}{x+1}$

$\Rightarrow 1=\dfrac{2x+1-f(x)}{f(1)}\cdot\left(\frac{1-x}{x+1}\right)\cdot\dfrac{1}{f(1)-f(x)}$

$\Leftrightarrow f(1)\left(f(1)-f(x)\right)(x+1)=\left(2x+1-f(x)\right)(1-x)$

if $f(1)=1$: $\cancel{x}+\cancel{1}-xf(x)-\cancel{f(x)}=\cancel{2x}-2x^2+\cancel{1}-\cancel{x}-\cancel{f(x)}+xf(x)$

$\Leftrightarrow xf(x)=2x^2 \Rightarrow f(x)=x$

if $f(1)=-1$:

$$-(-1-f(x))(x+1) = 2x-2x^2+1-x-f(x)+xf(x)$$

$$\Leftrightarrow \cancel{x}+\cancel{1}+\cancel{xf(x)}+f(x) = \cancel{x}-2x^2+\cancel{1}-f(x)+\cancel{xf(x)}$$

$$\Leftrightarrow 2f(x) = -2x^2 \Leftrightarrow f(x) = -x^2.$$

الطريقة الثالثة

$$f(x)\,f\left(\frac{1}{x}\right) = 1$$

$P\left(x+1, \frac{1}{x}\right)$: $f\left(\frac{1}{x}\right) + f(x+1)f\left(\frac{1}{x}\right) = 2\left(1+\frac{1}{x}\right)-1$

$$= \frac{2}{x}+1 = \frac{x+2}{x}$$

$$f\left(\frac{1}{x}\right)\left[f(x+1)+1\right] = \frac{x+2}{x}$$

$$\Rightarrow f(x+1)+1 = \frac{x+2}{x}f(x)$$

$P(x+1, 1)$: $f(x) + f(x+1)f(1) = 2x+1$

$$\Leftrightarrow f(x+1) = \frac{2x+1-f(x)}{f(1)}$$

$$\Rightarrow \frac{x+2}{x}f(x) - 1 = \frac{2x+1-f(x)}{f(1)}$$

if $f(1)=1$: $(x+2)f(x) - x = 2x^2+x-xf(x)$

$$(2x+2)f(x) = 2x^2+2x$$

$$\Rightarrow \quad f(x) = x$$

if $f(1) = -1$: $(x+2)f(x) - x = -2x^2 - x + xf(x)$

$$\Leftrightarrow \quad 2f(x) = -2x^2$$

$$\Leftrightarrow \quad f(x) = -x^2.$$

الطريقة الرابعة

حل رغدة

$$f(1) = 1$$

$$f(xy) = f(x)f(y) \Rightarrow f \text{ odd}$$

$$\left.\begin{array}{l} f(xy-1) + f(xy) = 2xy - 1 \\ x \to -x: \ f(-xy-1) + f(-xy) = -2xy - 1 \end{array}\right\}$$

$xy = z$

$$\Rightarrow \quad f(z-1) - f(z+1) = -2$$

$$\Rightarrow \quad f(x+2) - f(x) = 2$$

$P(2,1)$: $f(2) = 2 \rightarrow f(\frac{1}{2}) = \frac{1}{2}$

$$f(\tfrac{1}{2})f(x+2) - f(\tfrac{1}{2})f(x) = 1$$

$$\Leftrightarrow \quad f(\tfrac{x}{2}+1) - f(\tfrac{x}{2}) = 1$$

$$\Rightarrow \quad f(z+1) - f(z) = 1 \quad \forall z \in \mathbb{R}$$

$y=1$: $f(x-1) + f(x) = 2x - 1 \rightarrow f(x) + f(x+1) = 2x+1$

$$\Rightarrow 2f(x+1) = 2x+2 \quad \Rightarrow f(x+1) = x+1$$

$$\Rightarrow f(x) = x \quad .$$

إذا كان $f(1) = -1$:

$$f(xy) = -f(x)f(y)$$

$$f(xy-1) - f(xy) = 2xy - 1$$

$$\Leftrightarrow f(x-1) - f(x) = 2x - 1$$

Problem 1. (Saudi TST 2016) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x[f(x+y) - f(x-y)] = 4yf(x) \qquad \forall x, y \in \mathbb{R}$$

Problem 2. (APMO 2019) Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a, b.

$$f(a)+b \mid a^2+f(a)f(b) \quad \Rightarrow f(a)+b \mid a^2-bf(b)$$

$$b=1: \ f(a)+1 \mid a^2-1 \quad (*)$$

$$f(b) \geq b$$

$$a=b=1: \quad f(1)+1 \mid 1-f(1) \qquad f(1)+1 > |1-f(1)|$$

$$\Rightarrow f(1)=1$$

$$a=1: \ b+1 \mid 1+f(b) \quad \Rightarrow b \leq f(b)$$

$$a=b: \ f(a)+a \mid a^2+f(a)^2 \quad \Rightarrow f(a)+a \mid a^2+(-a)^2$$

$$\Rightarrow f(a)+a \mid 2a^2$$

$a=p$ أولي : $f(p)+p \mid 2p^2 \Rightarrow f(p) \in \{2p^2-p, p^2-p, p\}$

$$\times \quad \times$$

$(*) \Rightarrow f(a) \le a^2-2$

$f(p) \le p^2-2$

$f(p)+1 \mid p^2-1$

$f(p)=2p^2-p$: $2p^2-p+1 \mid 2p^2-2$

$2p^2-p+1 \mid p-3$

$\rightarrow 2p^2-p+1 \le p-3$

$2p(p-1) \le -4$ $\Rightarrow\Leftarrow$

if $f(p)=p^2-p$: $p^2-p+1 \mid p^2-1 \rightarrow p^2-p+1 \mid p-2$

$p^2-p+1 \le p-2 \iff p(p-2) \le -3$ $\Rightarrow\Leftarrow$

$p \ge 3$

$\Rightarrow f(p)=p \quad \forall p \ge 5$

$a=p$: $p+b \mid p^2+pf(b)$

$b=p$: $f(a)+p \mid a^2+pf(a) \Rightarrow \boxed{f(a)+p \mid a^2-f(a)^2}$

$f(a)+p \le |a^2-f(a)^2| \Leftarrow a^2-f(a)^2 \ne 0$ إذا كان

p هو أي عدد أولي ≥ 5

$\Leftarrow$ يوجد p بحيث

$|a^2-f(a)^2| < f(a)+p$

$\Rightarrow f(a)^2-a^2=0 \Rightarrow f(a)=a \quad \forall a \in \mathbb{Z}$

Problem 3. (BMO 2017) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$n + f(m) | f(n) + nf(m) \qquad \forall m, n \in \mathbb{N}$$

الحلول

$f(n) = n^2$

$f(n) = 1$

$$\boxed{n + f(m) \mid f(n) - n^2}$$

$m = n$: $\quad n + f(n) \mid f(n) - n^2 \Rightarrow n + f(n) \mid -n - n^2$

$\Rightarrow n + f(n) \mid n^2 + n \quad \Rightarrow \quad n + f(n) \leq n^2 + n$

$\rightarrow \boxed{f(n) \leq n^2} \quad \forall n \in \mathbb{Z}^+$

$\Rightarrow f(1) = 1 \quad , \quad f(2) \leq 4$

- if $f(2) = 1$:

$n = 2$: $\quad 2 + f(m) \mid 1 - 4 = -3 \Rightarrow 2 + f(m) \mid 3$

$\Rightarrow 2 + f(m) = 3 \rightarrow \boxed{f(m) = 1} \quad \forall m \in \mathbb{Z}^+$

- $f(2) = 2$:

$n = 2$: $\quad 2 + f(m) \mid 2 - 4 = -2 \Rightarrow 2 + f(m) \mid 2$

$\Rightarrow\Leftarrow$

- $f(2) = 3$:

$n = 2$: $\quad 2 + f(m) \mid 3 - 4 = -1 \Rightarrow 2 + f(m) \mid 1$

$\Rightarrow\Leftarrow$

$\cdot f(2)=4$:

$m=2$: $n+4 \mid n^2-f(n)$

$m=1$: $n+1 \mid n^2-f(n)$

$$n+4 \mid (n+3)^2-f(n+3)$$

$\gcd(n+1,n+4)=\gcd(n+1,3)$

فإذن $n \not\equiv -1 [3]$ $\Leftarrow$ $\gcd(n+1,n+4)=1$

$\boxed{\text{افترض } f(n)\neq n^2}$

$\Rightarrow (n+1)(n+4) \mid n^2-f(n)$

$\Rightarrow n^2+5n+4 \leq n^2-f(n)$

$\Leftrightarrow \underbrace{f(n)+5n+4}_{>0} \leq 0$ $\Rightarrow\Leftarrow$

$\Rightarrow f(n)=n^2 \quad \forall\, n \equiv 0,1 [3]$

$n=k \equiv -1 [3]$, $m=k+1$

$$k+(k+1)^2 \mid k^2-f(k)$$

افترض $k^2 \neq f(k)$

$\Rightarrow k+\cancel{k^2}+2k+1 \leq \cancel{k^2}-f(k)$

$\underbrace{f(k)+3k+1}_{>0} \leq 0$ $\Rightarrow\Leftarrow$

$\Rightarrow f(k)=k^2 \quad \Rightarrow f(n)=n^2 \quad \forall\, n\in\mathbb{Z}^+$

الحل الثاني

① $f(n)$ لها قيم لا نهائية لـ $n \in \mathbb{Z}^+$

$$\underbrace{n+f(m)}_{\text{له قيم لا نهائية}} \mid n^2 - f(n) \Rightarrow n^2 = f(n)$$

② $f(n)$ لها قيم محدودة $\Leftarrow$ يوجد k بحيث $f(n) \le k$ لكل n

ويوجد t بحيث $f(n) = t$ لقيم لا نهائية لـ n

- $f(m) = f(n) = t$

$$n+t \mid t+nt \Rightarrow \underbrace{n}_{\text{له قيم لا نهائية}}+t \mid t-t^2$$

$$\Rightarrow t - t^2 = 0 \Rightarrow t = 1$$

- $f(n) = 1$: $n + f(m) \mid 1 + nf(m)$

$$n + f(m) \mid 1 - f(m)^2$$

ثبت m . يوجد عدد لانهائي من الأعداد n .

$$\Rightarrow 1 - f(m)^2 = 0 \Rightarrow f(m) = 1 \quad \forall m \in \mathbb{Z}^+$$

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(y + f(x)) = f(x)^2 + 2yf(x) + f(y) \qquad \forall x, y \in \mathbb{R}$$

Problem 5. Find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(n+1) > f(f(n)) \qquad \forall n \in \mathbb{Z}^+$$

Problem 6. Find all $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(x + \frac{y}{x}) = f(x) + f(\frac{y}{x}) + 2y$$

Problem 7. Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(1) = 1$ and

$$f(m^2 + n^2) = f^2(m) + f^2(n)$$

Problem 8. Find all $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$mf(n) + nf(m) = (m+n)f(m^2 + n^2)$$

Problem 9. (BMO 2009) Find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(f^2(m) + 2f^2(n)) = m^2 + 2n^2$$