Test 1

Levels 3 and 4, November 30

Problem 1.1. Find all positive integers k such that the product of the first k primes increased by 1 is a power of an integer (with an exponent greater than 1).

Problem 1.2. Let the function f is given. It is known that each line on the plane xOy has as many intersections with f as with the parabola $y = x^2$. Prove that $f(x) = x^2$.

Problem 1.3. Given is triangle ABC with $AB > AC$. Circles o_B, o_C are inscribed in angle BAC with o_B tangent to AB at B and o_C tangent to AC at C. Tangent to o_B from C different than AC intersects AB at K, and tangent to o_C from B different than AB intersects AC at L. Line KL and the angle bisector of BAC intersect BC at points P and M, respectively. Prove that $BP = CM$.

Problem 1.4. At a gala banquet, $12n + 6$ chairs, where $n \in N$, are equally arranged around a large round table. A seating will be called a proper seating of rank n if a gathering of $6n + 3$ married couples sit around this table such that each seated person also has exactly one sibling (brother/sister) of the opposite gender present (siblings cannot be married to each other) and each man is seated closer to his wife than his sister. Among all proper seatings of rank n find the maximum possible number of women seated closer to their brother than their husband. (The maximum is taken not only across all possible seating arrangements for a given gathering, but also across all possible gatherings.)