— Geometry for L2 —

— November 26, 2021 — Computing with Pythagorean theorem —

1. Point D on side AB of an equilateral triangle AB is such that $AD=3$ and $BD=5$. Compute CD.

2. Point P inside a rectangle $ABCD$ is such that $AP=4$, $BP=1$, $CP=7$. Compute DP.

3. Given is a triangle ABC with $AB=n+1$, $BC=n+2$ and $CA=n$, where $n>3$. Circle centered at C and radius n intersects the segment AB at points A and D. Compute AD.

4. Circle o_1 of radius 4 and circle o_2 of radius 9 are externally tangent and line ℓ is one of their common external tangents. Circle ω is tangent to ℓ, o_1 and o_2. Compute all possible radii of ω.

5. Segment AB of length 2 is a diameter of circle γ and a radius of circle Γ. Find the radius of a circle externally tangent to γ, internally tangent to Γ, and tangent to line AB.

6. Points A and D are distinct and on the same side of line BC, with $AB=BC=CD$ and $AD\perp BC$. Lines AD and BC meet at E. Prove that

$$|BE-CE|<AD\cdot\sqrt{3}.$$

— Geometry for L2 —

— November 27, 2021 — Shifting Areas —

Let $[\mathcal{P}]$ denote the area of polygon $\mathcal{P}$.

7. Let $ABCDE$ be a convex pentagon in which $AB \parallel CE$ and $BC \parallel DA$. Prove that $[ABE] = [BCD]$.

8. Let $ABCD$ be a convex quadrilateral and let K and L be the midpoints of sides AB and CD, respectively. Prove that $[AKCL] = \frac{1}{2}[ABCD]$.

9. Let $ABCD$ be a convex quadrilateral and let K, L, M, N be the midpoints of sides AB, BC, CD, DA, respectively. Segment AL intersects DK and BM at P and Q, respectively, and segment CN itnersects DK and BM at S and R, respectively. Prove that $[APK] + [BQL] + [CRM] + [DSN] = [PQRS]$.

10. On sides AB and AD of a convex quadrilateral $ABCD$ lie points K and L, respectively, in such a way, that $AKCL$ is a parallelogram. Diagonal BD intersects segments CK and CL at points P and Q, respectively. Prove that $[APQ] = [BCD]$.

11. Diagonals of a convex quadrilateral $ABCD$ intersect at P. Prove that

$$\sqrt{[ABCD]} \geq \sqrt{[ABP]} + \sqrt{[CDP]}$$

and describe when does the equality hold.

12. Given is a convex hexagon $ABCDEF$ with $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$. Prove that $[ACE] = [BDF]$.

13. Let M be the midpoint of side AB of an acute triangle ABC. Circle centered at M passing through C intersects AC and BC at points P and Q (different than C), respectively. Point R of the segment AB satisfies $[APR] = [BQR]$. Prove that $PQ \perp CR$.

— Geometry for L2 —

— November 29, 2021 — Ratios of Areas —

14. Points K and L lie on sides AC and BC of triangle ABC, respectively, so that

$$\frac{AK}{KC} = \frac{CL}{LB}.$$

(a) Segments AL and BK intersect at P. Prove that $[ABP] = [CKPL]$.

(b) Let M be the midpoint of side AB. Prove that $[CKML] = \frac{1}{2}[ABC]$.

15. Points K and L lie on sides AC and BC of triangle, respectively. Segments AL and BK intersect at P. It is known that among regions ABP, APE, BPD, $CDPE$ three have unit areas. What are the possible areas of the fourth region?

16. (Angle bisector theorem) Let D be a point on side AB of triangle ABC. Prove that $\angle ACD = \angle DCB$ if and only if

$$\frac{AC}{CB} = \frac{AD}{DB}.$$

17. Given is a convex quadrilateral $ABCD$ which is not a parallelogram. Suppose that the line joining midpoints of its diagonals intersets segment BC at P. Prove that $[ABP] + [CDP] = [ADP]$.

18. Diagonals of a convex quadrilateral $ABCD$ intersect at P. The line joining midpoint of side AB with point P intersects segment CD at Q. Prove that

$$\frac{CQ}{DQ} = \frac{[BCP]}{[ADP]}.$$

19. (Ceva's theorem) Points D, E, F lie on sides BC, CA, AB of triangle ABC, respectively. Prove that segments AD, BE, CF have a common point (are *concurrent*) if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

— Geometry for L2 —

— December 1, 2021 — Tangents —

20. In triangle ABC the incircle and the excircle opposite to A touch segment BC at points D and T, respectively. Prove that $BD = CT$.

21. In triangle ABC the excircles opposite to B and C touch segments CA, AB at points E and F, respectively.

(a) Prove that $BF = CE$.

(b) Prove that $AE + AF = BC$.

22. In a quadrilateral $ABCD$ internal angle at C is greater than $180°$. Suppose that there exists a circle tangent to sides BC, CD and to extensions of sides AB, DA. Prove that $AB + BC = CD + DA$.

23. In triangle ABC point I is the incenter and M is the midpoint of side BC. Prove that if $AI = IM$, then one of the sides of ABC is two times longer than some other side of this triangle.

24. Let $ABCD$ be a unit square. Points E and F lie on sides BC and CD, respectively, so that triangle CEF has perimeter 2. Determine the distance from A to EF.

25. Point X is chosen on side BC of triangle ABC. The common tangent of incircles of triangles ABX and ACX which is different than BC intersects segment AX at Y. Prove that the length of CY does not depend on the choice of X.

— GEOMETRY FOR L2 —

— DECEMBER 3, 2021 — ROTATIONS AND EQUILATERAL TRIANGLES —

26. Let P be a point inside an equilateral triangle ABC, and let D, E, F be projections of P onto sides BC, CA, AB, respectively.

(a) Prove that $PD+PE+PF$ does not depend on the choice of P.

(b) Prove that $[AFP]+[BDP]+[CEP]=[AEP]+[BFP]+[CDP]$.

27. On sides BC and CD of parallelogram $ABCD$, equilateral triangles BCP and CDQ are constructed outwards. Prove that triangle APQ is equilateral.

28. Let ABC be an equilateral triangle.

(a) Let P be an arbitrary point inside ABC. Prove that there exists a triangle of side lengths AP, BP, CP.

(b) Let P lie inside ABC so that $\angle APB=150°$. Prove that $AP^2+BP^2=CP^2$.

(c) Let P lie outside ABC so that $\angle APB=120°$. Prove that $AP+BP=CP$.

29. Equilateral triangles ABC and CDE lie on the same side of side AE. Prove that C and the midpoints of AD, BE are vertices of an equilateral triangle.

30. On sides AB and BC of triangle ABC equilateral triangles ABP and ACQ are constructed outside. Prove that the midpoints of segments BC, AP, AQ are vertices of an equilateral triangle.

31. Given is a rhombus $ABCD$ with $\angle BAD=60°$. Point P lies inside this rhombus and satisfies $BP=1$, $DP=2$, $CP=3$. Find AP.

32. Suppose that inside an equilateral triangle ABC there exists a point P with $AP=3$, $BP=4$, $CP=5$. Find AB.