INEQUALITIES

HOMEWORK

a, b, c – lengths of sides of triangle

(1)

Prove: $a^2+b^2+c^2 \geq 4\sqrt{3}\, S_{ABC}$

S_{ABC} is area of triangle

$p = \frac{a+b+c}{2}$

$a=b=c \Rightarrow$ equality

$a=x+y,\ b=y+z,\ c=z+x$,

$$S_{ABC} = \sqrt{p\cdot(p-a)(p-b)(p-c)} = \sqrt{\underbrace{\frac{a+b+c}{2}}_{x+y+z}\underbrace{\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)}_{xyz}}$$

$$\frac{a+b+c}{2} - a = x+y+z-x+y = z$$

$$= \sqrt{xyz(x+y+z)}$$

$$(x+y)^2+(y+z)^2+(z+x)^2 \geq 4\sqrt{3}\sqrt{xyz(x+y+z)}$$

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$(A^2+B^2+C)^2 \geq 3(AB+BC+CA)$ so

~~(A~~

(2)

$\{\underbrace{(x+y)}_{A}{}^2+\underbrace{(y+z)}_{B}{}^2+\underbrace{(z+x)}_{C}{}^2 \geq$ ~~$(x+y)(y+$~~

$\cancel{A^2+B^2+C^2 \geq AB+BC+CA = x^2+y^2+z^2+(xz+xy+yz) =}$

$\cancel{= (x+y+z)^2+xy+yz+zx}$

$\cancel{\left((x+y+z)^2+xy+yz+zx\right)^2 \geq 48\, xyz(x+y+z)}$

$(x+y)^2 \geq (2\sqrt{xy})^2 = 4xy$

$(y+z)^2 \geq 4yz$

$(z+x)^2 \geq 4zx$

$\Rightarrow \left((x+y)^2+(y+z)^2+(z+x)^2\right)^2 \geq$

$\geq 16(xy+yz+zx)^2 =$

$= 16(x^2y^2+y^2z^2+z^2x^2)+32xyz(x+y+z) \geq$

$48\,(xyz(x+y+z)$, so we need

$\cancel{16}(x^2y^2+y^2z^2+z^2x^2) \geq \cancel{16}\{xyz(x+y+z)$

?

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$$A_i = \frac{a_i^3}{a_i^2 + a_i \cdot a_{i+1} + a_{i+1}^2}, \quad B_i = \frac{a_{i+1}^3}{a_i^2 + a_i \cdot a_{i+1} + a_{i+1}^2}$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) \quad ?$$

then

$$A_i - B_i = \frac{a_i^3 - a_{i+1}^3}{a_i^2 + a_i \cdot a_{i+1} + a_{i+1}^2} = a_i - a_{i+1}$$

$$\sum_{i=1}^{n} (A_i - B_i) = \cancel{a_1} - \cancel{a_2} + \cancel{a_2} - \cancel{a_3} + \cancel{a_3} - \cancel{a_4} + \dots \cancel{a_n} - \cancel{a_1} = 0.$$

so $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i$.

$$\sum_{i=1}^{n} A_i \geq \frac{a_1 \dots a_n}{3}$$

$$\| $$

$$\sum_{i=1}^{n} B_i$$

$$\sum_{i=1}^{n} A_i \geq \frac{a_1 + a_2 + \dots + a_n}{3} \leftarrow \text{want}$$

$$\sum_{i=1}^{n} (A_i + B_i) \geq \frac{2(a_1 + \dots + a_n)}{3}$$

$$\sum_{i=1}^{n} (A_i + B_i) \geq \frac{2(a_1 + \dots + a_n)}{3} \leftarrow \text{Enough}$$

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So we need $x^2y^2+y^2z^2+z^2x^2 \geq xyz(x+y+z)$

$A = xy, \quad B = yz \quad C = zx$

$A^2+B^2+C^2 \geq AB+BC+CA$

Problem 12

$a_i > 0$

$$\frac{a_1^3}{a_1^2+a_1a_2+a_2^2} + \dots + \frac{a_n^2}{a_1^2+a_1a_n+a_n^2} \geq \frac{a_1+a_2+\dots+a_n}{3}$$

$$\frac{a_1^3}{a_1^2+a_1a_2+a_2^2} = A_1 \qquad \frac{a_n^2}{a_1^2+a_1a_n+a_n^2} = A_n$$

$$A_i := \frac{a_i^3}{a_i^2+a_ia_{i+1}+a_{i+1}^2}$$

$$B_i := \frac{a_{i+1}^3}{a_i^2+a_ia_{i+1}+a_{i+1}^2}$$

$i = 1 \dots n \qquad a_{n+1} = a_1$

$$\boxed{\sum_i A_i = \sum_i B_i}$$

$$\frac{\varepsilon}{n-1x} \leq \frac{\varepsilon^{n+1}+\varepsilon^{n+1}+\varepsilon^{x}}{\varepsilon^{n-1}\varepsilon^{x}}$$

$$(\varepsilon^{n+1}+\varepsilon^{n+1}+\varepsilon^{x})(n-1x) \leq (\varepsilon^{n-1}\varepsilon^{x})\varepsilon$$

$$\varepsilon^{n+1}+\varepsilon^{n+1}+\varepsilon^{x} \leq \varepsilon^{n+1}+\varepsilon^{n+1}+\varepsilon^{x}$$

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$$\boxed{\frac{x^3+y^3}{x^2+xy+y^2} \geq \frac{x+y}{3}}$$

$$3x^3+3y^3 \geq x^3+x^2y+xy^2+x^2y+xy^2+y^3$$

$$2(x^3+y^3) \geq 2x^2y+xy^2$$

$$x^3+y^3 \geq x^2y+xy^2$$

$$\sum_{i=1}^{1}\left(\frac{a_i+a_{i+1}}{3}\right) = \frac{\sum\sum a_i}{3}$$

$$\frac{a_1+a_2 \quad a_2+a_3}{3} \ldots$$

Problem 13 $\quad a, b, c > 0$

(6)

$$a\sqrt{b^2-bc+c^2} + c\sqrt{a^2-ab+b^2} \geq b\sqrt{a^2+ac+c^2}.$$

$$a^2(b^2-bc+c^2) + c^2(a^2-ab+b^2) \geq b^2(a^2+ac+c^2)$$
$$+ 2ac\sqrt{(b^2-bc+c^2)(a^2-ab+b^2)}$$

$$2\cancel{ac}\sqrt{(b^2-bc+c^2)(a^2-ab+b^2)} \geq b^2ac+a^2bc+c^2ab-2a^2c^2 \;\Big|:ac$$

$$2\sqrt{(b^2-bc+c^2)(a^2-ab+b^2)} \geq b^2+ab+cb-2ac \quad \wedge (2)$$

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$$2\sqrt{(b^2-bc+c^2)(a^2-ab+b^2)} \geq b^2+ab+cb-2ac$$

$$4(b^2-bc+c^2)(a^2-ab+b^2) \geq \cancel{b^4}+\cancel{a^2b^2}+\cancel{c^2b^2}+\cancel{4a^2c^2}+\cancel{2b^3a}+\cancel{2b^3c}-\cancel{4abc^2}$$

$$+\cancel{2ab^2c}-\cancel{4a^2bc}-\cancel{4abc^2}.$$

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$$\parallel$$

$$\cancel{4b^2a^2}-\cancel{4ab^3}+\cancel{4b^4}-\cancel{4bca^2}+\cancel{4b^2ac}-\cancel{4b^3c}+$$

$$+\cancel{4a^2c^2}-\cancel{4c^2ab}+\cancel{4c^2b^2}$$

$$3b^4+3a^2b^2+3c^2b^2+6ab^2c-6ab^3-6b^3c \geq 0$$

$$b^4+a^2b^2+c^2b^2+2ab^2c-2ab^3-2b^3c \geq 0 \quad |:b^2$$

$$b^2+a^2+c^2+2ac-2ab-2bc \geq 0$$

$$\underline{(a+c-b)^2 \geq 0}$$

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$a, b, c > 0 \qquad abc = 1$

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

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$a, b, c > 0$ reals

NEXT

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{2\sqrt[3]{abc}} \geq \frac{\left(a+b+c+\sqrt[3]{abc}\right)^2}{(a+b)(b+c)(c+a)}$$

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \frac{a_3^2}{b_3} \geq \frac{(a_1+a_2+a_3)^2}{b_1+b_2+b_3} \qquad \text{C-S.}$$

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$$\frac{1}{a^3(b+c)} = \frac{\frac{1}{a^2}}{a(b+c)} = \frac{\left(\frac{1}{a}\right)^2}{a(b+c)} = \frac{\left(\frac{abc}{a}\right)^2}{ab+ac} = \frac{(bc)^2}{ab+ac}.$$

(8)

$$\sum_{cyc} \frac{1}{a^3(b+c)} \geq \sum_{cyc} \frac{(bc)^2}{ab+bc} \underset{C-S}{\geq} \frac{(ab+ac+bc)^2}{2(ab+bc+ca)} = \frac{ab+bc+ca}{2} \geq$$

Engel form

$\{abc=1\}$.

$$\geq \frac{3\sqrt[3]{ab\cdot bc\cdot ca}}{2} = \frac{3\sqrt[3]{(abc)^2}}{2} = \frac{3}{2}.$$

<u>Next</u>

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{2\sqrt[3]{abc}} \geq \frac{(a+b+c+\sqrt[3]{abc})^2}{(a+b)(b+c)(c+a)}.$$

$$\frac{c^2}{c^2(a+b)} + \frac{a^2}{a^2(b+c)} + \frac{b^2}{b^2(c+a)} + \frac{(\sqrt[3]{abc})^2}{2abc} \geq \frac{(c+a+b+\sqrt[3]{abc})^2}{\underbrace{c^2(a+b)+a^2(b+c)+b^2(c+a)+2abc}_{=(a+b)(b+c)(c+a)}}$$

$\boxtimes$

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$a, b, c \geq 1$ reals

$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leq \sqrt{a(bc+1)}$$

$$bc = ((b-1)+1)((c-1)+1) = \left(\left(\sqrt{b-1}\right)^2 + 1^2\right)\left(1^2 + \left(\sqrt{c-1}\right)^2\right) \overset{C-S}{\geq} \quad (9)$$

$$\overset{*}{\geq} \left(\sqrt{b-1} + \sqrt{c-1}\right)^2$$

$$a(bc+1) = ((a-1)+1)\cdot(1+bc) = \left(\left(\sqrt{a-1}\right)^2 + 1^2\right)\left(1^2 + bc\right) \overset{C-S}{\geq}$$

$$\overset{(*)}{\geq} \left(\left(\sqrt{a-1}\right)^2 + 1^2\right)\left(1^2 + \left(\sqrt{b-1} + \sqrt{c-1}\right)^2\right) \geq$$

$$\left(\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1}\right)^2 \quad (2)$$

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Test Problems

$$\left(\frac{-12000}{821}\right) = \left(\frac{-12\,000 \bmod 821}{821}\right) = \left(\frac{315}{821}\right) \qquad (10)$$

$$\left(\frac{400\cdot(-30)}{811}\right) = \left(\frac{20^2(-30)}{821}\right) = \left(\frac{20^2}{821}\right)\left(\frac{-30}{821}\right)$$

$$\left(\frac{-30}{821}\right) = \left(\frac{-1}{821}\right)\left(\frac{3}{821}\right)\left(\frac{2}{821}\right)\left(\frac{5}{821}\right) = 1$$

$$\left(\frac{-1}{821}\right) = (-1)^{\frac{821-1}{2}} = 1 \qquad \left(\frac{3}{821}\right) = -1 \qquad \left(\frac{2}{821}\right) = -1 \qquad \left(\frac{5}{821}\right) = 1$$

$\pm 1 \bmod 8$

3 is q. r if $\pm 1 \bmod 12$

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$$\left(\frac{5}{821}\right)\left(\frac{821}{5}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{821-1}{2}} = 1$$

(14)

$$\left(\frac{821}{5}\right) = \left(\frac{1}{5}\right) = 1 \rightsquigarrow \left(\frac{5}{821}\right) = 1.$$

(1)

$$\left(\frac{2^{2019}}{953}\right) = \left(\frac{2\cdot\left(2^{1009}\right)^2}{953}\right) = \left(\frac{2}{953}\right) = (-1)^{\frac{953^2-1}{8}} = (-1)^{113526} = 1$$

5 iq. r. iff $p>2$ and $\underline{p \equiv \pm 1 \bmod 10}$

$\Leftarrow$ $p \equiv \pm 1 \bmod 10$

$$\left(\frac{5}{p}\right)\left(\frac{p}{5}\right) = (-1)^{\frac{5-1}{2}\cdot\frac{p-1}{2}} = 1$$

$p=10k+1$ $\quad \left(\frac{p}{5}\right) = \left(\frac{10k+1}{5}\right) = \left(\frac{1}{5}\right) = 1$

$p=10k-1$ $\quad \left(\frac{p}{5}\right) = \left(\frac{10k-1}{5}\right) = \left(\frac{-1}{5}\right) = (-1)^{\frac{5-1}{2}} = 1$

($\Leftarrow$) gato.

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$$\left(\frac{5}{p}\right) = 1 \implies p = \pm 1 \mod 10$$

exclude $p \equiv 3 \vee 7 \mod 10$

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-1

$$\left(\frac{5}{p}\right) \cdot \left(\frac{p}{5}\right) = 1$$

$p = 10k+3$

$$\left(\frac{p}{5}\right) = \left(\frac{10k+3}{5}\right) = \left(\frac{3}{5}\right) = -1$$

$p = 10k+7$

$$\left(\frac{p}{5}\right) = \left(\frac{10k+7}{5}\right) = \left(\frac{7}{5}\right) = \left(\frac{2}{5}\right) = -1$$

$\lightning$

ok?

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$a^2 + b^2 + c^2 = 7d^2$ $\qquad$ $4x^2 + 4x = 4x(x+1)$

$\boxed{\text{mod } 4}$ $\quad a^2 \equiv 0, 1 \pmod 4$ $\qquad$ (13)

1) d is odd the

$7d^2 \equiv 7 \cdot 1 \equiv 3 \bmod 4$ so a, b, c are odd.

$a = 2a_1 + 1$

$b = 2b_1 + 1$

$c = 2c_1 + 1$

$d = 2d_1 + 1$

$$\underbrace{(4a_1^2 + 4a_1 + 1) + (4b_1^2 + 4b_1 + 1) + (4c_1^2 + 4c_1 + 1)}_{3 \pmod 8}$$

$$\underline{LHS \equiv 3 \bmod 8}$$

$$RHS = 7 \cdot (2d_1 + 1)^2 = 7(4d_1^2 + 4d_1 + 1) \equiv$$

$$\equiv 7 \cdot 1 \equiv 7 \bmod 8$$

so $2 \mid d$, $4 \mid a^2 + b^2 + c^2 \Rightarrow 2 \mid a, 2 \mid b, 2 \mid c$

$d = 2d_1$, $a = 2a_1$, $b = 2b_1$, $c = 2c_1$

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$\cancel{4}a_1^2 + \cancel{4}b_1^2 + \cancel{4}c_1^2 = 7 \cdot \cancel{4}d_1^2$ – descent $\rightsquigarrow$

$a = b = c = d = 0$

Coment (Euler)

n is of the form $a^2+b^2+c^2$ iff n is not of the form $4^s(8k+7)$, $k, s \geq 0$

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$n^4 - n^2 + 1$ is of the form $\underline{2k+1}$

$\boxed{p \equiv 1 \bmod 4}$

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$n^4 - n^2 + 1 = \underline{(n^2-1)^2 + n^2} \rightsquigarrow$

$n^4 - n^2 + 1 = (n^2+1)^2 - 3n^2$

$p \mid n^2 - 1 \quad p \mid n$ ↯

$(n^2+1)^2 \equiv 3n^2 \bmod p$

$(n, n^2+1) = 1$ so

$3 \equiv \left((n^2+1) \cdot n^{-1}\right)^2 \bmod p$

so $\left(\frac{3}{p}\right) = 1$

$\Vert$

$\boxed{p \equiv \pm 1 \bmod 12}$

$p \equiv 1 \bmod 12$

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$a, b > 0$

$ab-2 \mid a^2+b^2+ab$ Find all values of $\frac{a^2+b^2+ab}{ab-2}$

$$\frac{a^2+b^2+ab}{ab-2} = k \Rightarrow \frac{a^2+b^2+1}{ab-2} = \cancel{k-2}$$

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a^2+b^2+ab

$$\frac{a^2+b^2+2+ab-2}{ab-2} = k \rightsquigarrow \frac{a^2+b^2+2}{ab-2} = k-1$$

WLOG $\underline{a \geq b}$.

if $a=b$ then

$$k-1 = \frac{2a^2+2}{a^2-2} = \frac{2a^2-4+6}{a^2-2} = 2 + \frac{6}{a^2-2}, \text{ so}$$

$$\boxed{k=-3,\ k=6} \quad a^2-2 = \cancel{1}, -1, \cancel{2}, -2, \cancel{3}, -3, \cancel{6}, -6$$

$a=1$ $a=2$ $a \neq 0$ $\Rightarrow$ $a=1,\ a=2$

$a=-1$ $a=-2$

$$k-1 = \frac{2\cdot 1+2}{1-2} = -4 \qquad k-1 = \frac{2\cdot 4+2}{4-2} = \frac{10}{2} = 5$$

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$a > b$

$$\frac{a^2+b^2+2}{ab-2} = k-1$$

$$a^2 - a(b(k-1)) + b^2 + \cancel{2} + 2k - \cancel{2} = 0$$

$$a^2 - a(b(k-1)) + b^2 + 2k = 0$$

$a-2 =$ -1, 1, 2, $\cancel{-2}$, 3, $\cancel{-3}$, 4, $\cancel{-6}$

$a = 1$

$a = 3$

$a = 1, 3, 4, 5, 8$

$$\frac{a^2+b^2+2}{ab-2} \geq \frac{2a}{b}$$

$$a^2b + b^3 + 2b \geq 2a^2b - 4a$$

$$b^3 + 2b \geq a^2b - 4a$$

$$= a(ab-4)$$

$$k-1 = \frac{a^2+3}{a-2} = \cancel{-4}$$

$$b^3 + 2b \geq (b+1)^2 b - 4(b+1)$$

$$b^3 + 2b^2 + b - 4b - 4$$

$$2b^2 - 5b - 4 \leq 0$$

$\cancel{b=2}$ $\quad 2 \cdot 9 - 15 - 4$

$b = 1$ $\quad 4$

$b = 2$

$b = 3$

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