Hello! :) TOMOPROW TEST

1)

$$n = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \implies n \text{ is sum of 2 squares}$$

$$c^2 n = a^2 + b^2, \implies v_p(c^2 n) = \underbrace{v_p(a^2+b^2)}_{\text{even}}$$

$$\| $$

$$\forall_{p \equiv 3 \pmod 4} \quad v_p(n) - \text{even} \qquad \underbrace{2 \cdot v_p(c)}_{\text{even}} + v_p(n) = \text{even}$$

take any $p \equiv 3 \pmod 4$ $\quad p \mid n$ $\qquad \Downarrow$

$v_p(n)$ $\qquad v_p(n) = \underline{\text{even}}$

by Thm (About sum of 2 squares)

n is $A^2 + B^2$ for some A, B.

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2) Prove that

$$\frac{x^2+1}{y^2-5} \notin \mathbb{Z} \quad \text{for } x, y > 2 \text{ integers}$$

$y^2-5 \nmid x^2+1$, suppose $y^2-5 \mid x^2+1$

- y is even $\quad y^2-5 \equiv 3 \pmod 4 \implies y^2-5 \nmid x^2+1$
- y is odd $\quad y^2-5 \equiv 0 \pmod 4 \implies 4 \nmid x^2+1$

3) $p = 4k+1$ then $p = a^2+b^2 = c^2+d^2 \implies \{|a|,|b|\} = \{|c|,|d|\}$

$a, b, c, d \geq 0$ integers

Proof

$$p = a^2+b^2 = c^2+d^2$$

$$a^2+b^2 = c^2+d^2$$

$$a^2-c^2 = d^2-b^2$$

$$\underbrace{(a-c)}_{A}\underbrace{(a+c)}_{B} = \underbrace{(d-b)}_{C}\underbrace{(d+b)}_{D}$$

$\boxed{AB = CD}$

hint :)

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$$(a-c)(a+c) = (d-b)(d+b)$$

$$\begin{array}{cccc} \| & \| & \| & \| \\ xy & wz & xw & yz \end{array}$$

$$a = \frac{xy+wz}{2}$$

$$b = \frac{yz-xw}{2}$$

so

$$p = a^2+b^2 = \left(\frac{xy+wz}{2}\right)^2 + \left(\frac{yz-xw}{2}\right)^2 =$$

$$= \frac{(x^2+z^2)(y^2+w^2)}{4}, \text{ so}$$

WLOG

so $4p = (x^2+z^2)(y^2+w^2)$, so $p \mid x^2+z^2$ but they

$y^2+w^2 \mid 4$ only if $y=w=1$. then $a=d$, $b=c$.

$4 = A \cdot (y^2+w^2)$

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$s_1, s_2, \dots, s_n$ – pairwise coprime

$1, 2, 3, \dots, n$

not is to use it +

$a \cdot 1, \; a \cdot 2, \; a \cdot 3 \; \dots \; a \cdot n$

$p_1, p_2, \dots, p_k$

$$i = p_1^{\alpha_{1,i}} \cdots p_k^{\alpha_{k,i}}$$

for any $i \in \{1 \dots n\}$

$$a = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

a_i is s_i perfect power. $\quad a_i = \square^{s_i}$

$$a_i = p_1^{x_1 + \alpha_{1,i}} p_2^{x_2 + \alpha_{2,i}} \cdots p_k^{x_k + \alpha_{k,i}}$$

$$x_1 + \alpha_{1,i} \equiv 0 \pmod{s_i}$$
$$x_2 + \alpha_{2,i} \equiv 0 \pmod{s_i}$$
$$\vdots$$
$$x_k + \alpha_{k,i} \equiv 0 \pmod{s_i}$$

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$$\begin{cases} x_1 + \alpha_{1,i} \equiv 0 \pmod{s_1^i} & \text{for } i = 1 \dots n \\ x_1 \equiv -\alpha_{1,i} \pmod{s_1^i} & \text{for } i = 1 \dots n, \end{cases}$$ — has solut by CRT.

$$\begin{cases} x_2 \equiv -\alpha_{2,i} \pmod{s_2} & \text{for } i = 1 \dots n \\ \text{has solut by CRT.} \end{cases}$$

$\vdots$

k systems of equations (congruences) each of them has solu. by CRT.

$a = p_1^{x_1} \cdots p_k^{x_k}$ such the $a \cdot i = \square^{s_i^r}$ — perfect power,

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$1, 2, \dots, n$ by a s.t. a_i is perfect square

$\underline{n=1}$ $\quad a=4. \quad 4\cdot 1 = 2^2$.

Suppose

$$k\cdot a = x_k^{y_k} \text{ for } k = 1, 2, \dots, n.$$

$1, 2, \dots, n+1$. $\leftarrow$ we want find b s.t.

$$kb = \text{power for } k = 1, \dots, n+1.$$

Let m - multiplay (common) of $y_1, y_2, \dots, y_n$

$y_i \mid m$ for ~~$i=1 \dots n$~~.

$$b = (n+1)^m \cdot a^{m+1}$$

$k = 1 \dots n$

$$k\cdot b = k\cdot a\cdot((n+1)\cdot a)^m = x_k^{y_k}\cdot((n+1)\cdot a)^m = \square^{y_k}$$

$$(n+1)\cdot b = (n+1)^{m+1}\cdot a^{m+1} = (a\cdot(n+1))^{m+1} \rightarrow \text{perfect power},$$

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$A\ a_1, A\ a_2, \ldots, A\ a_n$

Prove that for all $n \geq 1$ there is a set A of n positive integers such that for all $1 \leq k \leq n$ and $x_1, x_2, \ldots, x_k \in A$

$$\frac{x_1 + x_2 + \ldots + x_k}{k} \text{ is perfect power.}$$

$\{ x_1 \ldots x_2 \ldots x_n \}$

$\frac{x_1 + x_2 + \ldots + x_k}{k}$ — perfect power.

$\{ 1, 2, 3, 4, 5, 7, 9 \}$

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$\{x_1, x_2, \dots, x_n\}$

$\frac{x_1 + x_2 + \dots + x_k}{k}$ $\qquad$ $\frac{x_1 + x_4}{2}$

<u>Proof</u>

$$\underset{\cdot a}{1}, \quad \underset{\cdot a}{2}, \quad \underset{\cdot a}{3}, \quad \dots, \quad \underset{\cdot a}{n \cdot n!}$$

a_i = perfect power for all $i \in \{1, 2, \dots, n \cdot n!\}$.

$$A := \{ n! \cdot a, 2n! \cdot a, \dots, n \cdot n! \, a \}$$

Take any $x_1, x_2, \dots, x_k \in A$

$b_1 \cdot n! \, a, \; b_2 \cdot n! \, a, \dots, b_k \cdot n! \, a$ where $b_1, \dots, b_k \in \{1, 2, \dots, n\}$

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$$\frac{b_1 \cdot n! \cdot a + b_2 \cdot n! \cdot a + \dots + b_k \cdot n!\, a}{k} = \frac{n!}{k} \cdot a \cdot (b_1 + b_2 + \dots + b_k)$$

$$\Downarrow$$

$k \in \{1, 2, \dots, n\}$ $\qquad$ $b_i \in \{1, \dots, n\}$.

$$a \le a \cdot (b_1 + b_2 + \dots + b_k) \le a \cdot nk$$

$$\frac{n!}{k} \cdot a \cdot m \quad \text{where} \quad 1 \le m \le n \cdot k$$

$$= a \cdot \boxed{\frac{n!}{k} \cdot m}$$ — perfect power.

$$\frac{n!}{k} \cdot m \le \frac{n!}{k} \cdot nk = n! \cdot n$$

$$\in \{1, 2, \dots, n! \cdot n\}$$

$a \cdot i$ = power for any i

$\in \{1, 2, \dots, n! \cdot n\}$

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n is "nice" iff for some a, b, c.

$$n = \gcd(b,c)\cdot\gcd(a,bc) + \gcd(c,a)\cdot\gcd(b,ca) + \gcd(a,b)\gcd(c,ab)$$

Prove 2020 consecutive nice numbers.

$$a = mx^2, \quad b = mx \quad c = x$$

$$n = x\cdot mx^2 + x\cdot mx + mx\,x = mx^3 + 2mx^2 = m\cdot x^2(x+2)$$

$$x^2(x+2) \mid n \implies n = m\cdot x^2(x+2) = \text{nice number}$$

$$a = mx^2, \quad b = mx, \quad c = x$$

Conclusion: any number divisible by $x^2(x+2)$ is nice!

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$$\begin{cases} n \equiv 0 \mod (x_1^2(x_1+2)) \\ n \equiv -1 \mod (x_2^2(x_2+2)) \\ \vdots \\ n \equiv -2019 \mod (x_{2020}^2(x_{2020}+2)) \end{cases}$$

p1+2

BY CRT.

$y_1 = x_1^2(x_1+2)$, $y_2 = x_2^2(x_2+2)$, ..., $y_n = x_n^2(x_n+2)$

$y_1, y_2, \dots, y_n$ are pairwise coprime.

$x_1 = 1$

$x_2 = y_1 - 1$ $\quad y_2 = (y_1-1)^2(y_1+1)$

$x_3 = y_1 y_2 - 1$ $\quad y_3 = (y_1 y_2 - 1)^2(y_1 y_2 + 1)$

$\vdots$

$x_i = y_1 y_2 \cdots y_{i-1} - 1$ ....

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Schur lemma $\quad P \in \mathbb{Z}[X] \leftarrow$ P has integer coefficient

$\exists$ ∞ primes q such that $q \mid P(n)$ for some n.

- $a-b \mid P(a)-P(b)$ for any integer a, b

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0$$

Prove, $a, b \in \mathbb{Z}$ then $\underline{a-b \mid P(a)-P(b)}$

$$P(a)-P(b) = \left(a_n \cdot a^n + a_{n-1} \cdot a^{n-1} + \dots + a_1 \cdot a + a_0\right) - \left(a_n \cdot b^n + a_{n-1} \cdot b^{n-1} + \dots + a_1 \cdot b + a_0\right) =$$

$$= a_n\left(a^n - b^n\right) + a_{n-1}\left(a^{n-1} - b^{n-1}\right) + \dots + a_1(a-b)$$

$\underline{a-b \mid a^n - b^n}$.

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Let $f \in \mathbb{Z}[x]$ non constant, fix n, k positive integers.

Prove there is positive integer a s.t

$$f(a),\ f(a+1),\ f(a+2),\ \ldots,\ f(a+n-1)$$

has at least k distinct prime divisors.

<u>Proof</u> Take distinct primes $\{p_{ij}\}$ $1 \le i, j \le k$ k^2 primes s.t

~~By Schur~~ we can take $\{x_{ij}\}$ integers $1 \le i, j \le k$ s.t

$$\boxed{f(x_{ij}) \equiv 0 \mod (p_{ij})} \quad \text{for all } 1 \le i, j \le k$$

By CRT: I can find $a \ge 1$ s.t

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$$a \equiv x_{ij'} - i + 1 \pmod{p_{ij'}} \text{ for any } i, j'$$

$$a + i - 1 \equiv x_{ij'} \pmod{p_{ij'}}$$

$$\downarrow$$

$$f(a+i-1) \equiv f(x_{ij'}) \equiv 0 \pmod{p_{ij'}}$$

$f(a) \equiv f(b) \pmod{a-b}$

$n \mid a-b$

$n \mid f(a) - f(b)$

$f(a+i-1)$ is divisible by p_{ij} for all $1 \le j' \le k$

$$\begin{array}{c} i \\ p_{i1} \\ p_{i2} \\ \vdots \\ p_{ik} \end{array}$$

$f(a), f(a+1), \ldots, f(a+i-1)$

$\blacksquare$

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Revision

- $P \in \mathbb{Z}[x]$ then there exists infinitely many primes p s.t $p \mid P(n)$. $\{n^2+n+1\}$
- $a-b \mid P(a)-P(b)$ for $a, b \in \mathbb{Z}$

$\boxed{\text{Kobayashi theorem}}$

$\{a_n\}$ — unbounded for any M

$\{a_n\}$ unbounded sequen of integers. s.t $a_k > M$.

has finitely many prime divisors.

For example $a_n = 2^n$. $\leftarrow$ unbounded, 2 divids a_k for any k.

Kobyashi : $a_n + t$, $t \neq 0$. has infinitely many prime divisors.

$2^n + 2020$ IRAN $2^{2^n}+1$, $2^{2^n} \leftarrow$ unbounded

Kobayashi uses

$$Ax^3 - By^3 = t \quad , \quad A, B, t \text{ fixed}$$

has only finitely many <u>solutions</u>.

<u>CRT</u>

$$\begin{cases} x \equiv a_1 & (\text{mod } n_1) \\ x \equiv a_2 & (\text{mod } n_2) \\ \vdots & \vdots \\ x \equiv a_k & (\text{mod } n_k) \end{cases}$$

if $n_1, n_2, \ldots, n_k$ are pairwise coprime !.

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