Cell Combinatorics

Lesson by Senya, group L4+

Problem 1. Some rectangle R with odd sides has been divided into smaller rectangles. Prove that among those smaller rectangles there is at least one such that the distances from its sides to the sides of R are of the same parity.

Problem 2. There is a cube $3 \times 3 \times 3$. Prove that it is impossible to cover the three faces having a vertex a common using 1×3 rectangles (no overlaps are allowed).

Problem 3. Define a distance between two cells on a board as the minimal amount of steps a king needs to get from one of them to another. One coloured several squares on the board 100×100 so that there are no two cells such that the distance between them is exactly 15. What is the biggest amount of cells could be coloured?

Problem 4. There is a board 12×15. What is the largest possible number of 1×3 can one place on the board in such a way that no two rectangles touch each other, even via a corner?

Problem 5. Let's call a *block* a staircase with three steps consisting of 12 unit squares. Find all n such that the cube $n \times n \times n$ can be divided in such staircases.

Problem 6. Chessboard has been divided into dominos. Call two dominos *knight-connected* if it is possible to jump from one of them to another with a knight. What is the minimal amount of colours needed so that no matter how the board is divided one can always colour the dominos so that no two dominos of the same colour are knight-connected?

Problem 7. There is a board 100×100 and there is either plus or minus in each of its cells. In one move you can select any *cross* (i.e. union of a row and a column) and change the signs in it to the opposite ones.

(a) With the help of the moves described, how can one get a board with only pluses in its cells?

(b) What is the smallest amount of moves needed to achieve that no matter what the initial board is?

Problem 8. Given board 100×100, each of its cells is either black or white. It turned out that in any two columns there is the same amount of black cell, whereas in any two rows there is different amount of black cells. What is the biggest possible amount of the pair of neighbouring cells that are of different colours?

Problem 9. Fix an integer $n \geq 2$. An $n \times n$ sieve is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A stick is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Problem 10. Let $n \geq 2018$ be an integer. A board of size $3n \times 3n$ has been tiled with rectangular tiles of size 3×1 (which may be rotated) so that every cell of the board is covered by exactly one tile. Prove that one may color the cells of the board in three colors so that:

(i) there is the same number of cells of each color;

(ii) no two cells of the same color have a common side; and

(iii) no tile covers cells of all three colors.

Problem 11. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contain two marked cells.