KSA Training – May camp

GEOMETRY FOR LEVEL 2

Session 1. **Trainer: Le Phuc Lu (Vietnam)**

Problem 1. Let $ABCD$ be a quadrilteral with $\angle A = \angle B = 90^\circ$, $AB = AD$. Denote E as the midpoint of AD, suppose that $CD = BC + AD, AD > BC$. Prove that

$$\angle ADC = 2\angle ABE.$$

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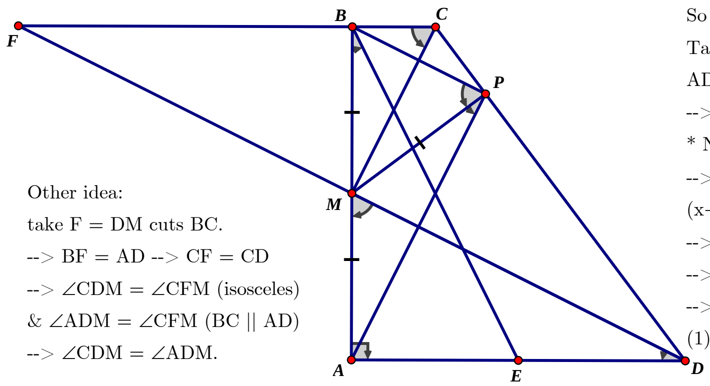

Other idea:
take F = DM cuts BC.
--> BF = AD --> CF = CD
--> ∠CDM = ∠CFM (isosceles)
& ∠ADM = ∠CFM (BC || AD)
--> ∠CDM = ∠ADM.

Take M is the midpoint of AB.
--> △ABE ≅ △ADM since AM = AE, AB=AD
--> ∠ABE = ∠ADM.
So we need to prove: DM is the angle bisector of ∠ADC (*).
Take P as the projection of M onto CD --> two quadrilaterals ADPM and BCPM are cyclic.
--> ∠BPM = --> ∠BCM, ∠APM = ∠ADM (1)
* Note: if denote AB = AD = x, BC = y
--> Pythagores: CD = x+y = $\sqrt{\text{x^2+(x-y)^2}}$
(x+y)^2 = x^2+(x-y)^2 --> x^2+2xy+y^2 = 2x^2-2xy+y^2
--> 4xy = x^2 --> x = 4y --> AD = 4BC.
--> △ABE ≅ △ADM ~ △BMC
--> ∠BCM + ∠ADM = 90 (2)
(1), (2) --> ∠APB = 90 --> MP = MA --> (*) --> done.

Problem 2. Let ABC be a an acute, non-isosceles triangle with altitudes BD, CE. The perpendicular bisector of CE cuts the line DE at T and cuts AC at S. Prove that BT, BS are symmetric with respect to the angle bisector of $\angle ABC$.

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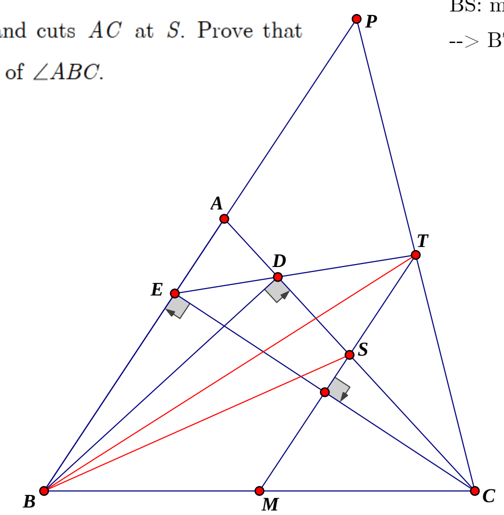
<u>1st way:</u> AB cuts TC at P --> T is the midpoint of PC.
We have: ∠SBC = ∠PBT since △ABC ~ △CBP by angle chasing: ∠P = ∠TEP (ET is the median of the right triangle EPC) = ∠AED = ∠ACB (by cyclic quadrilateral BCDE).

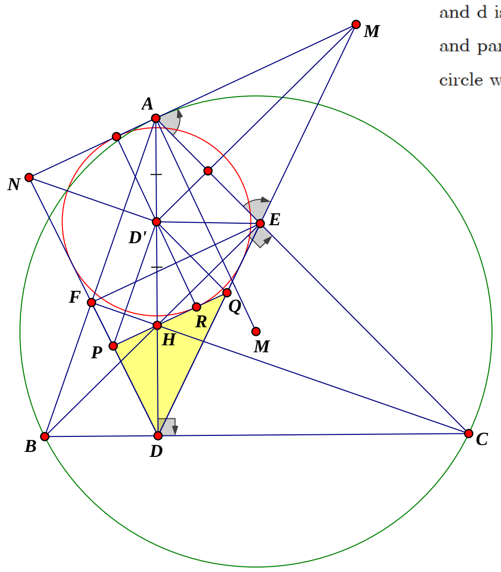
<u>2nd way</u>: Extend ST cuts BC at M --> S, M are midpoints of AC, BC. Note that TEC is isosceles triangle --> ∠MTC = ∠STC = ∠STE = ∠AED = ∠ACB = ∠MCS --> MC is tangent to (CST) --> MS.MT = MC^2, but MC = MB --> MS.MT = MB^2 --> MB is tangent to (BST) --> ∠MBS = ∠STB = ∠ABT.

BS: median
--> BT: symmedian.



Problem 3. Let ABC be an acute, non-isosceles triangle with AD, BE, CF are altitudes and d is the tangent line of the circumcircle of triangle ABC at A. The line throught H and parallel to EF cuts DE, DF at Q, P respectively. Prove that d is tangent to the ex-circle with respect to vertex D of triangle DPQ.

red circle = incircle of triangle DMN.



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We know that: DA is angle bisector of ∠PDQ since ∠HDF = ∠HBF = ∠HCE = ∠HDE. Extend DE, DF cut d at M, N respectively.
Take D' on AH s.t MD' || BE. We need to prove D' is the incenter of triangle DMN.
By angle chasing, we have: ∠NAB = ∠ACB = ∠AFE --> MN || EF || PQ --> D'M is the angle bisector of ∠M in triangle DMN --> D' is the incenter. Angle chasing: ∠MAE = ∠MAC = ∠B = ∠CED = ∠AEM --> AEM is isosceles triangle --> D'M is the perpendicular bisector of AE --> D'A = D'E --> D'A = D'H --> D' is the circumcenter of AEHF. Angle chasing: ∠PHF = ∠HFE = ∠HFP --> PH = PF. --> PD' is the perpendicular bisector of FH --> PD' is angle bisector of P --> D' is the excenter of DPQ, done!

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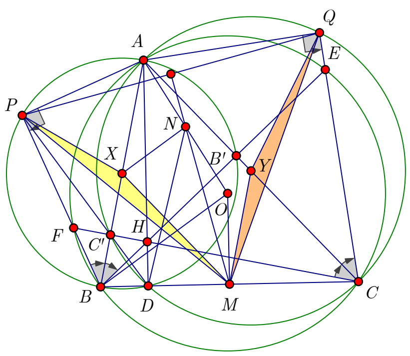
Session 2. **Trainer: Le Phuc Lu (Vietnam)**

Problem 1. Let ABC be an acute, non-isosceles triangle inscribed in (O) and BB', CC' are altitudes. Denote E, F as the intersections of BB', CC' with (O) and D, P, Q are projections of A on BC, CE, BF. Prove that the perpendicular bisectors of PQ bisects two segments AO, BC.



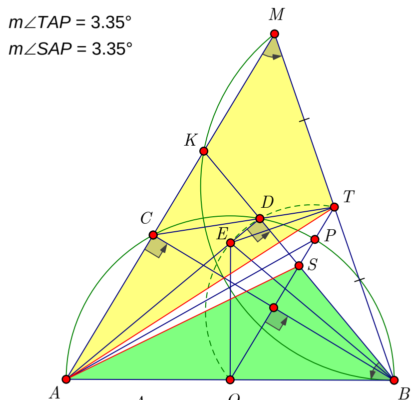
Problem 1. Let ABC be an acute, non-isosceles triangle inscribed in (O) and BB', CC' are altitudes. Denote E, F as the intersections of BB', CC' with (O) and D, P, Q are projections of A on BC, CE, BF. Prove that the perpendicular bisectors of PQ bisects two segments AO, BC.

We know that if H is the orthocenter of ABC then: (H,E) and (H,F) are symmetric w.r.t AB, AC. Denote M, N as the midpoints of BC, AO; we need to prove: MP = MQ and NP=NQ.
Take X, Y are midpoints of AB, AC --> MX = AC/2 = QY and
MY = AB/2 = PX and
∠MXP = ∠MXB+∠BXP = ∠MYC+∠CYQ = ∠MYQ (note that MXAY is a parallelogram and two triangles ABP, AQC are similar) --> △MXP ≅ △MYQ (s.a.s) --> MP = MQ
* How to continue with NP = NQ?
We have: X is the center of the circumcircle of APBD and **XN** || **BO** (midline) and we also have: **BO ⊥ DC'**.
Angle chasing: ∠BDP = ∠BAP=∠BAC=∠BDC' --> **D, C', P are collinear**. So XN ⊥ DP.
But XP = XD --> XN is the perpendicular bisector of DP --> NP = ND.
Similar, NQ = ND --> NP = NQ.
We also can prove that: N is the circumcenter of PQMD.

Problem 2. Given a semicircle (ω) of diameter AB and center O, let C, D are two distinct points on that (ω) such that ray AC meets ray BD at K lying outside (ω). The line passes through O, parallel to AC cuts CD at T, cuts KB at S and cuts (ω) at P. Take M on BT such that $TB = TM$, take E on AD such that $EA = EB$.

1) Prove that AB is tangent to (KBM) and D, E, O, T are concyclic.

2) Prove that AP is the angle bisector of angle SAT.



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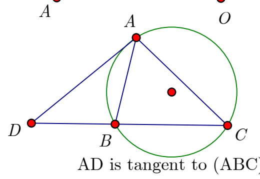
1) OT is the midline of triangle ABM --> OT || AM, but OT ⊥ BC --> AM ⊥ BC, and AC ⊥ BC --> A, C, M are collinear. Triangle CBM is right so CT=TM --> ∠M = ∠TCM = ∠KBA (since A,C,D,B are concyclic) --> AB is tangent to (KBM). ∠EOT = ∠AOT - ∠AOE = 180 - ∠CAB - 90 = 90 - ∠CAB = ∠CBA = ∠CDA = ∠CDE --> O, E, D, T are concyclic.
Here we also have O, B, D, E are concyclic --> 5 points O, B, D, E, T are concyclic --> ET ⊥ BM so E lies on perpendicular bisector of BM --> E is the circumcenter of ABM.
(technique to create this problem: isogonal conjugate + antiparallel)
2) We have OT is the perpendicular bisector of BC (since OC=OB, TC=TB) --> P is the midpoint of the arc BC --> AP is the angle bisector of ∠BAC, so we need to prove: ∠SAB = ∠TAM (*).
△ABK ~ △AMB (a.a) --> **AB/AM** = BK / MB = **SB / MT,** combining with ∠AMT = ∠ABS --> △ABS ~ △AMT (s.a.s) --> (*), done!

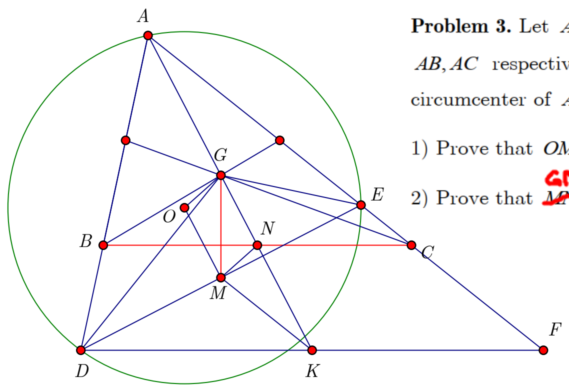


AD is tangent to (ABC) iff ∠DAB = ∠C

Problem 3. Let ABC be an acute, non-isosceles triangle with centroid G. Take D, E on AB, AC respectively such that G is the orthocenter of triangle ADE. Denote O as circumcenter of ADE and M, N as the midpoints of DE, BC.

1) Prove that $OMNG$ is parallelogram.

2) Prove that $GM \perp BC$.

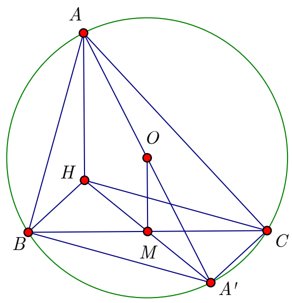


Problem 3. Let ABC be an acute, non-isosceles triangle with centroid G. Take D, E on AB, AC respectively such that G is the orthocenter of triangle ADE. Denote O as circumcenter of ADE and M, N as the midpoints of DE, BC.

1) Prove that $OMNG$ is parallelogram. (JBMO TST 2019's suggestion)

2) Prove that GM $\cancel{MN} \perp BC$.

Lemma: AH = 2OM.



Construct F on AC such that DF || BC
AN cuts DF at K --> K is the midpoint of DF --> MK is the midline of triangle DEF --> MK || EF.
But EF ⊥ DG (since G is orthocenter) --> MK ⊥ DG. And AG ⊥ DE so M is the orthocenter of DGK.
--> GM ⊥ DK --> GM ⊥ BC.

Consider triangle ADE with orthocenter G, circumcenter O --> by applying the lemma, AG = 2 OM and AG || OM.
But AG = 2GN since G is the centroid --> OM = GN, OM || GN --> OMNG is parallelogram.

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Session 3. **Trainer: Le Phuc Lu (Vietnam)**

Problem 1. Let ABC be an acute, non-isosceles triangle with altitude AD and orthocenter H. Denote O_1, O_2 as the centers of circle pass through B, C respectively and tangent to BC, external tangent to the circle of diameter AD. Prove that O_1O_2 bisects HD.

Take E as the tangent point of (O1) & (I) --> E lies on IO1. Then we have: ∠O1EB = ∠O1BE, ∠IEA = ∠IAE (isosceles) ∠O1BE = ∠IAE (O1B || AI) --> ∠O1EB = ∠IEA --> A, E, B are collinear. Same with A, F, C.

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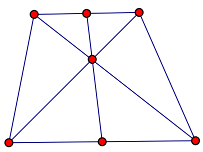
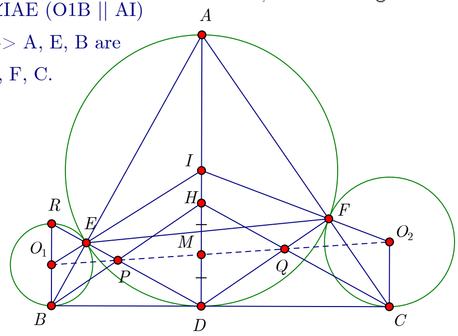


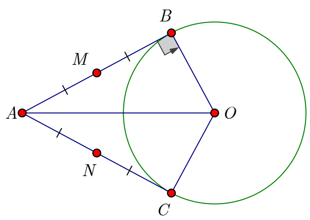


P = BH cuts DE, Q = CH cuts DF. ∠HCB = 90 - ∠B = ∠DAB = ∠BDE --> DE || HC. Similarly: DF || HB. --> HPDQ parallelogram. --> PQ bisects HD (M:midpoint HD). Extend DE cuts (O1) at R, since IE = IA = ID --> ∠AED = 90 --> ∠BER = 90 --> BR is the diameter of (O1) --> O1 is the midpoint of BR. Trapezoid BRHD with P = intersection of BH, DR --> O1, P, M are collinear. Similarly: O2, Q, M are collinear --> 5 points O1, P, M, Q, O2 are all collinear.

well-known fact

Problem 2. Let ABCD be a parallelogram with T as the intersection of two diagonals. A circle (ω) of center O, passes through D and is tangent to BD. Suppose that (ω) cuts the segment CD at E, cuts ray AD at F such that points B, E, F are collinear. Prove that $\angle ATD = \angle BOD$.

point-circle
~ degenerate circle ?
--> circle of radius zero.
radical axis of normal circle and <u>point-circle</u>.



power of M to point-circle A is: MA^2 - 0^2 = MA^2
power of M to (O): MB^2 --> two powers are equal
Similar with N --> MN is radical axis of (O),(A;0).

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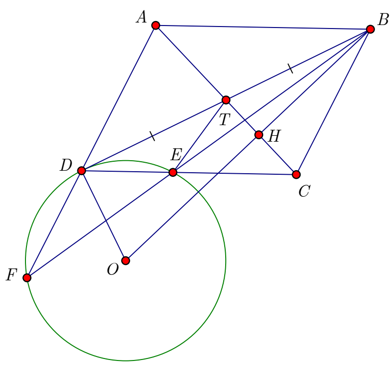
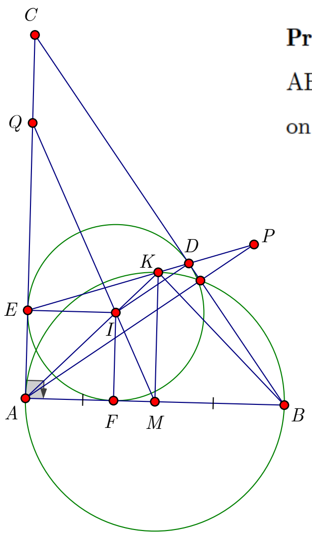


TC is the radical axis of (O) and (B;0).
P_T/(O) = TD^2
P_T/(B,0) = TB^2
--> T lies on the radical axis
△CEB ~ △CBD (a.a) since
∠EBC=∠EFD (BC || AD)
=∠BDE (tangent)
--> CE.CD = CB^2
--> P_C/(O) = P_C/(B;0)
--> TC is the radical axis
--> TC ⊥ BO at H --> DTHO
is cyclic --> done!

Problem 3. Let ABC be a triangle right at A with incenter I. Denote M as midpoint of AB and IM cuts AC at Q. Circle (I) is tangent to BC, AC at D, E respectively. Take P on DE such that $AP \perp BC$. Prove that $AP = AQ$.



Problem 3. Let ABC be a triangle right at A with incenter I. Denote M as midpoint of AB and IM cuts AC at Q. Circle (I) is tangent to BC, AC at D, E respectively. Take P on DE such that $AP \perp BC$. Prove that $AP = AQ$.

AI cuts DE at K --> (lemma) ∠AKB=90 (angle chasing) --> AKB is a isosceles right triangle since ∠KAB = 45 --> AKM is also isosceles right --> AK = AM.$\sqrt{2}$.

Let use Thales theorem.

ID || AP (since both perpendicular to BC)

--> **ID/AP = KI/KA**; KI = AK - IA = AM.$\sqrt{2}$ - EI.$\sqrt{2}$

--> AP = r.KA/KI = r.AM.$\sqrt{2}$ / (AM.$\sqrt{2}$ - EI.$\sqrt{2}$)

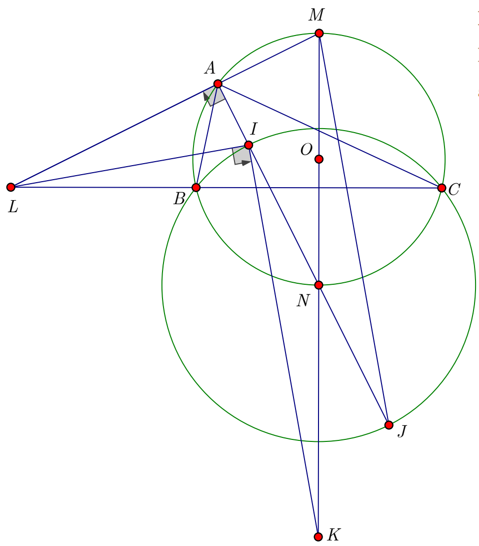
= r.AM / (AM-r) (1)

2*[AQM]=AQ.AM=EI.AQ+IF.AM=r(AQ+AM)

--> AQ(AM-r) = r.AM --> AQ = r.AM/(AM-r) (2).

(1), (2) --> AP = AQ.

Problem 4. *(about the IMO TST)* Let ABC be a non-isosceles triangle with incenter I and let R be the circumradius of this triangle. Denote AL as the external angle bisector of angle BAC with L on BC. Let K be a point on perpendicular bisector of BC such that $IL \perp IK$. Prove that $OK = 3R$.



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(this TST for IMO, May 2nd)

Denote M, N as the intersections of AL, AI with (O) --> M, N are midpoints of the arcs BC. --> M, O, N are collinear.

We need to prove: NM = NK. Take J as the ex-center of ABC --> we know that N is the midpoint of IJ. So we need: IKJM is parallelogram.

But IL ⊥ IK, so we need: IL ⊥ MJ.

By similar triangles: △ABI ~ △AJC (a.a) --> AB/AJ = AI/AC

--> AI.AJ = AB.AC.

By another similar triangles: △ABL ~ △AMC (a.a) --> AB/AM = AL/AC --> **AL.AM** = AB.AC = **AI.AJ** --> △ALI ~ △AJM (s.a.s)

--> ∠ALI = ∠AJM --> LI ⊥ MJ --> done!