HW $a+b+c \leq 4 \leq ab+bc+ca$

$\Longrightarrow \geq 2$ inequalities are true among:

$|a-b| \leq 2, \ |b-c| \leq 2, \ |c-a| \leq 2$

Sol The numbers $|a-b|, |b-c|, |c-a|$ can be seen as $x, y, x+y$, where $x, y \geq 0$

Now:

$$x^2+y^2+(x+y)^2 = 2\sum a^2 - 2\sum ab$$

$\Longleftrightarrow$

$$x^2+xy+y^2=(a+b+c)^2-3(ab+bc+ca)$$
$$\leq 16-12=4$$

$\Rightarrow x^2 \leq 4,\ y^2 \leq 4$

$\Rightarrow x \leq 2,\ y \leq 2$ as desired $\blacksquare$

Let $a > b > 1$ such that

$$(ab+1)^2+(a+b)^2 \leq 2(a+b)(a^2-ab+b^2+1)$$

Find the minimum of

$$Q = \frac{\sqrt{a-b}}{b-1}$$

Sol)

$$(a+b)^2+(ab+1)^2 \leq 2(a+b)(a^2-ab+b^2+1)$$

$$\left(-2(a+b)(ab+1)\right)$$

$$\Longleftrightarrow (a-1)^2(b-1)^2 \leq 2(a+b)(a-b)^2$$

BMO 2019 $\quad 0 \leq a \leq b \leq c$

$a+b+c = ab+bc+ca > 0$

Prove:

$$\sqrt{bc}\,(a+1) \geq 2$$

and find all equality cases.

Sol. Let $x = \sqrt{bc}$

$$a+b+c = a(b+c) + x^2$$

- if $a \geq 1 \Rightarrow c, b \geq 1 \Rightarrow a+b+c \leq ab+bc+ca$

$\Rightarrow$ $\boxed{a=b=c=1, \text{ and it is equality case}}$

- Now $a < 1$

$$a + b + c = a(b+c) + x^2$$

$$\Leftrightarrow \quad x^2 - (1-a)(b+c) + a = 0$$

$$\Rightarrow \quad x^2 - 2x(1-a) + a \geq 0$$

$$\Rightarrow \quad x \geq 1 - a + \sqrt{1 - a + a^2}$$

$$\text{or} \quad x \leq 1 - a - \sqrt{1 - a + a^2} \leq 0$$

$$\left(1 - a = \sqrt{1 - 2a + a^2} \leq \sqrt{1 - a + a^2} \right)$$

Now, it is enough to verify

$$(1+a)\left(1 - a + \sqrt{1 - a + a^2}\right) \overset{?}{\geq} 2$$

$$\Leftrightarrow \sqrt{1-a+a^2} \overset{?}{\geqslant} \frac{2}{a+1}+a-1=\frac{1+a^2}{1+a}$$

$$\Leftrightarrow (1-a+a^2)(1+2a+a^2) \overset{?}{\geqslant} 1+2a^2+a^4$$

$$\Leftrightarrow a^3-2a^2+a \overset{?}{\geqslant} 0$$

$$\Leftrightarrow a(1-a)^2 \overset{?}{\geqslant} 0$$ ✓

"="

case 1	case 2
$a=0$	$a=1$
$b=c=2$	$b=c=1$

$a+b+c=ab+bc+ca>0$

Let $a, b, c > 0$. Prove:

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geqslant \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

Sol

$$\left.\begin{aligned}
\frac{a^3}{b^2}+\frac{a^3}{b^2}+b &\geqslant 3\frac{a^2}{b} \\
\frac{b^3}{c^2}+\frac{b^3}{c^2}+c &\geqslant 3\frac{b^2}{c} \\
\frac{c^3}{a^2}+\frac{c^3}{a^2}+a &\geqslant 3\frac{c^2}{a}
\end{aligned}\right\} \text{AM-GM}$$

$$\left.\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geqslant a+b+c\right] \text{Cauchy}$$

Summing up, we get the result $\blacksquare$

HW Let $x \in \mathbb{R}^+$ and $n > 1$ prove that

$$\frac{x^n + 1}{2} \geqslant \frac{x^{n-1} + \cdots + x^2 + x}{n-1}$$