Today's plan: $\boxed{36}$

1) $P(x)$, $Q(x)$ are monic poly's

$\deg P = \deg Q = 10$

$\Downarrow$

leading coeff is $\boxed{1}$

$$P(x) = x^{10} + a_9 x^9 + \dots + a_1 x + a_0$$

$$Q(x) = x^{10} + b_9 x^9 + \dots + b_1 x + b_0$$

$P(x) = Q(x)$ has no solution

We will discuss it at $\boxed{15:45}$

$\Downarrow$

$P(x+1) = Q(x-1)$ has real solution.

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$P(x)$ – poly.

$$P(8) + P(11) < 19 < P(12) + P(7)$$

$\Downarrow$

$\exists \ a, b \in \mathbb{R}$:

$$a+b = P(a) + P(b) = 19.$$

$\boxed{42 - 44}$ Inequalities for those who solved.

$\boxed{138}$ (Easy)

$\boxed{36}$

$\deg P(x) = 10$ monic

$P(x) = x^{10} + a_9 x^9 + \dots + a_0$ for some $a_0, \dots, a_9 \in \mathbb{R}$.

$Q(x) = x^{10} + b_9 x^9 + \dots + b_0$ $b_0 \dots b_9$

" $P(x) = Q(x)$ has no real soln. "

$\Downarrow$

$P(x) - Q(x)$ has no real root

$\parallel$

$a_9 x^9 + \dots + a_0 - (b_9 x^9 + \dots + b_0) =$

$= \underbrace{(a_9 - b_9) x^9 + (a_8 - b_8) x^8 + \dots + (a_0 - b_0)}_{\text{polynomial}}$

what if $a_9 - b_9 \neq 0$ $\deg 9$ - odd

" all poly's of odd degree has real root "

$a_9 = b_9$

$$P(x) = x^{10} + a_9 x^9 + \dots + a_0$$

$$Q(x) = x^{10} + a_9 x^9 + \dots + b_0$$

$$P(x+1) = (x+1)^{10} + a_9 (x+1)^9 + a_8 (x+1)^8 + \dots + a_0$$

$$Q(x-1) = (x-1)^{10} + a_9 (x-1)^9 + b_8 (x-1)^8 + \dots + b_0$$

$$P(x+1) - Q(x-1) = (x+1)^{10} - (x-1)^{10} + a_9\left[(x+1)^9 - (x-1)^9\right] + a_8\left[(x+1)^8 - (x-1)^8\right) + \dots\right]$$

$\boxed{\text{deg}}$

$$(x+1)^{10} - (x-1)^{10} = \left(x^{10} + 10x^9 + \underline{\quad}\right) - \left(x^{10} - 10x^9 + \underline{\quad}\right) =$$

$= 20x^9 +$ tail of deg < 9

$$a_9\left((x+1)^9-(x-1)^9\right)=\boxed{20x^8}+\text{tail}$$

$$\deg\left(P(x+1)-Q(x-1)\right)=9=\text{odd}$$

$$\Downarrow$$

has a root.

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polynomial $P(x)$

$$P(8)+P(11)<19<P(12)+P(7)$$

$$\Downarrow$$

$\exists\ a,b\in\mathbb{R}$:

$a+b=P(a)+P(b)=19$.

HINT: $\boxed{P(x)+P(19-x)-19}$

16:15

15 min

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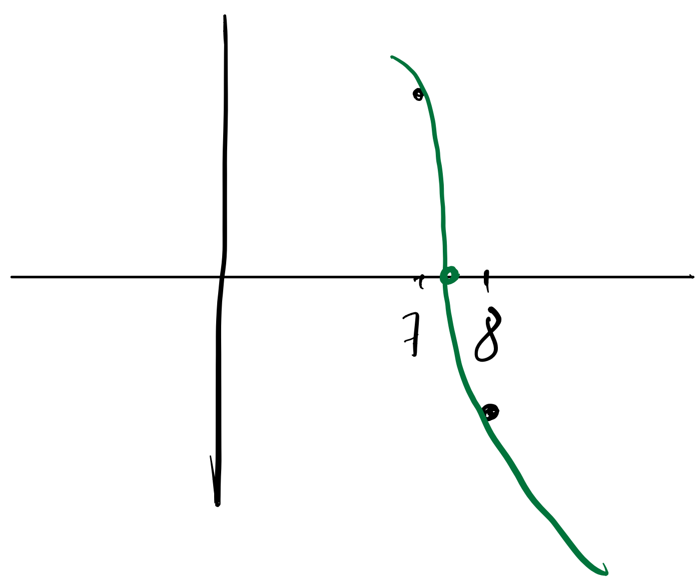
$$Q(x) = P(x) + P(19-x) - 19$$

$$Q(8) = P(8) + P(11) - 19 < 0$$

$$Q(7) = P(7) + P(12) - 19 > 0$$

$$Q(8) < 0$$

$$Q(7) > 0$$



$$\Rightarrow y \in (7,8) \text{ s.t } \boxed{Q(y) = y}$$

$$= P(x) + P(19-x) - 19$$

$$\Downarrow$$

$$P(y) + P(19-y) - 19 = 0$$

$$\overset{?}{\Downarrow}$$

$$P(y) + P(\underbrace{19-y}_{x}) = 19$$

$$\boxed{\begin{array}{l} x+y = 19 \\ P(x)+P(y) = 19 \end{array}}$$

Clear?

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$$P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$$

with roots: $x_1, x_2, \dots, x_n$.

$$Q(x) = x^{n+1} + b_n x^n + \dots + b_0$$

with roots: $y_1 \dots y_{n+1}$

Prove: $P(y_1) \dots P(y_{n+1}) = Q(x_1) \dots Q(x_n)$.

16:45

$$P(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$$

$$Q(x) = (x-y_1)(x-y_2)\cdots(x-y_{n+1})$$

$$\left.\begin{aligned} P(y_1)\cdots P(y_{n+1}) = {} & (y_1-x_1)(y_1-x_2)\cdots(y_1-x_n) \\ & \cdot (y_2-x_1)(y_2-x_2)\cdots(y_2-x_n)\cdots \\ & \vdots \\ & (y_{n+1}-x_1)(y_{n+1}-x_2)\cdots(y_{n+1}-x_n) \end{aligned}\right\} n(n+1)$$

$$\|$$

$$\begin{aligned} & (x_1-y_1)(x_1-y_2)\cdots(x_1-y_{n+1}) \\ & (x_2-y_1)(x_2-y_2)\cdots(x_2-y_{n+1}) \\ & \vdots \qquad\qquad\qquad\qquad\qquad\qquad n(n+1) \\ & (x_n-y_1)\cdots\cdots(x_n-y_{n+1}) \end{aligned}$$

$$LHS = (-1)^{n(n+1)} \cdot RHS$$

$n(n+1)$ → even number !

$$LHS = RHS \qquad \square$$

42, 43, 44

$\boxed{\text{Stat } 44}$ · uh · $\boxed{17-15}$

C-Schwartz Inequality

$a_1, a_2, \ldots, a_n, \quad b_1, \ldots, b_n \in \mathbb{R}$

$$\boxed{(a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2) \geq (a_1b_1+\ldots+a_nb_n)^2}$$

Remember we took Hölder inequality

it is Hölder for $p=2=q$

$$\boxed{\frac{1}{p}+\frac{1}{q}=1}$$

$\boxed{a, b, c \geq 1}$

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \leq \sqrt{a(bc+1)}$$

C-S

$$bc = \Big(\underbrace{(\overset{\geq 0}{b-1})+1}_{b}\Big)\Big(\underbrace{1+(\overset{\geq 0}{c-1})}_{c}\Big) \geq \left(\sqrt{b-1}+\sqrt{c-1}\right)^2$$

$$\left(\left(\sqrt{b-1}\right)^2+1^2\right)\left(1^2+\left(\sqrt{c-1}\right)^2\right) \geq \left(\sqrt{b-1}+\sqrt{c-1}\right)^2$$

$$\left(\sqrt{b-1}+\sqrt{c-1}\right) \le \sqrt{bc}$$

$$a(bc+1) = \left((a-1)+1\right)(1+bc) = \quad \text{C-S}$$

$$= \left(\left(\sqrt{a-1}\right)^2+1^2\right)\left(1^2+\left(\sqrt{bc}\right)^2\right) \geq$$

$$\geq \left(\left(\sqrt{a-1}\cdot 1\right)+\sqrt{bc}\right)^2 =$$

$$= \left(\sqrt{a-1}+\sqrt{bc}\right)^2 \geq \left(\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}\right)^2$$

$\square$

For every: $\boxed{42}$ (try Hölder)

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inequality