TWO-VARIABLE POLYNOMIALS

Ex 1. *(Bezout) Let be given polynomial $P(x,y)$ such that $P(a,y)=0$ for some fixed value of a and for inifinitely many values of y. Prove that $P(x,y)$ is divisible by $x-a$.*

Ex 2. *Does there exist $P(x,y)$ such that $P(x,y)>0, \forall x,y \in \mathbb{R}$ and $P(x,y)$ surjective on $\mathbb{R}^+$?*

Problem 1.

a) Find all polynomials $P(x,y)$ such that

$$xP(z,y)+yP(x,z)=2zP(x,y) \text{ for all } x,y,z \in \mathbb{R}.$$

b) Find all polynomials $P(x,y)$ such that

$$P(x+y,x-y)=2P(x,y) \text{ for all } x,y \in \mathbb{R}.$$

Problem 2. Let $P(x)$ be a polynomial such that $P(0,0)=0$ and

$$P(x+2y,x+y)=P(x,y) \text{ for all } x,y \in \mathbb{R}.$$

Prove that $x^2-2y^2 \mid P(x,y)$ and there exists $Q(x)$ such that $P(x,y)=Q((x^2-2y^2)^2)$.

Problem 3.

a) Find all polynomials $P(x,y)$ such that $P(x^2+y^2,2xy) \mid P^2(x,y)$.

b) Does there exist $P(x,y)$ such that $P^2(x,y)+2021$ is divisible by x^2+y^2+2022?

Problem 4.

a) Let be given polynomial $P(x,y)$ such that

$$P(xy,z^2+1)+P(yz,x^2+1)+P(zx,y^2+1)=0 \text{ for all } x,y,z.$$

Prove that $P(x)\equiv 0$.

b) Let $P(x,y)$ is a non-constant polynomials such that

$$P(x,y)\cdot P(z,t)=P(xz+yt,xt+yz) \text{ for all } x,y,z,t \in \mathbb{R}.$$

Prove that $P(x,y)$ is divisible by $x+y$ or $x-y$.

Problem 5. Suppose that with some $m \in \mathbb{Z}^+$, there exists polynomials $P(x), Q(x)$ and $R(x,y)$ of real coefficients such that for all $a,b \in \mathbb{R}$ with $a^m=b^2$ then

$$P(R(a,b))=a \text{ and } Q(R(a,b))=b.$$

Prove that $m=1$.

Problem 6. Given that set $S=\{(a,b) \mid a\neq b; a,b\in\mathbb{Z}, 1\le a\le 4, 1\le b\le 4\}$ and $f_0(x,y)$ is a non-constant integer polynomials with minimum degree and $f(a,b)=0, \forall (a,b)\in S$. Prove that

$$f_0\left(\frac{3}{2};\frac{5+\sqrt{6}}{2}\right)=0.$$

Problem 7*. Prove that for positive integer n, the polynomial x^n+xy+y^n cannot be written as the product of two 2-variable polynomials with degree less than n.