# Preparation for Saudi Arabia Team 2021

## May/June Session: Level 3



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## Lesson 5

# Miscellaneous (Inversion and Double Counting)

**Problems:**

1. Let $\triangle ABC$ be given such that $\measuredangle ACB = 2\measuredangle CAB$. Let circle $s$ with center $S$ be tangent to the circumcircles of $\triangle ADC$, $BDC$ and line $AC$, where point $D$ is the intersection of the angle bisector of $\measuredangle ACB$ witgh $AB$. Prove that $CS \perp AB$.

2. Fix a circle $\Gamma$, a line $\ell$ to tangent $\Gamma$, and another circle $\Omega$ disjoint from $\ell$ such that $\Gamma$ and $\Omega$ lie on opposite sides of $\ell$. The tangents to $\Gamma$ from a variable point $X$ on $\Omega$ meet $\ell$ at $Y$ and $Z$. Prove that, as $X$ varies over $\Omega$, the circumcircle of $XYZ$ is tangent to two fixed circles.

3. A group of $n$ students attends several classes such that if two students attend the same two classes, then the number of students in the two classes is different from each other. Prove that the largest possible number of classes is $(n-1)^2$.

4. There are $n$ people at a party. Prove that there are two people at the party such that at least $\lfloor \frac{n}{2} \rfloor - 1$ people know either both of them or neither of them. (If person $A$ knows person $B$, then $B$ knows $A$.)

5. Let $S$ be a set of $n$ persons such that any person is acquainted to exactly $k$ other persons in $S$, any two persons that are acquainted have exactly $l$ common acquaintances in $S$ and any two persons that are not acquainted have exactly $m$ common acquaintances in $S$. Prove that:
$$m(n-k) - k(k-l) + k - m = 0.$$

6. Show that among $n$ points there are at least $\sqrt{n - \frac{3}{4}} - \frac{1}{2}$ different distances between pairs of points.

7. Let $n > 1$ be a positive integer and let a set $P$ of $n^2$ points be given that form an grid of $(n-1) \times (n-1)$ unit squares. Let $S$ be the set of all squares whose vertices are in $P$ and let $a_k$ be the number of unordered pairs of different points of $P$ such that both of them together are vertices of exactly $k$ squares in $S$. Prove that $a_0 = a_2 + 2a_3$.

8. Let $n$ and $k$ be positive integers and let $S$ be a set of $n$ points in the plane such that no three points of $S$ are collinear, and for every point $P$ of $S$ there are at least $k$ points of $S$ equidistant from $P$. Prove that:
$$k < \frac{1}{2} + \sqrt{2 \cdot n}$$