Homework

1) $\dfrac{a\sqrt{2}+b}{b\sqrt{2}+c} \in \mathbb{Q}$

Prove $a+b+c \mid ab+bc+ca$.

$ab+bc+ca = ab+bc+b^2 = \boxed{b(a+b+c)}$

$$\frac{a\sqrt{2}+b}{b\sqrt{2}+c} = \frac{(a\sqrt{2}+b)(b\sqrt{2}-c)}{(b\sqrt{2}+c)(b\sqrt{2}-c)} = \frac{2ab - ac\sqrt{2} + b^2\sqrt{2} - bc}{2b^2-c^2} =$$

$$= \frac{2ab - bc + \sqrt{2}(b^2-ac)}{2b^2-c^2} = \omega \in \mathbb{Q}$$

$$2ab - bc + \sqrt{2}(b^2-ac) = \omega(2b^2-c^2) \rightsquigarrow$$

$$\rightsquigarrow \sqrt{2}\underbrace{(b^2-ac)}_{\in \mathbb{Q}} = \underbrace{\omega(2b^2-c^2) - 2ab + bc}_{} = \omega_2 \in \mathbb{Q}$$

So $b^2 - ac = 0$

$b^2 = ac$

$$\boxed{\sqrt{2} = \frac{\omega_2}{b^2-ac} \in \mathbb{Q}}$$

Skanowano w CamScanner

2) $a = \overline{a_7 a_6 a_5 a_4 a_3 a_2 a_1 a_0}$ – 8-digit number

(2)

$$b = \overline{a_0 a_7 a_6 a_5 a_4 a_3 a_2 a_1} = a_0 \cdot 10^7 + \overline{a_7 a_6 a_5 a_4 a_3 a_2 a_1} =$$

$$\begin{array}{l} a_0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ + \ \overline{a_7 a_6 a_5 a_4 a_3 a_2 a_1} \end{array}$$

$$\overline{a_7 a_6 a_5 a_4 a_3 a_2 a_1} = \frac{a - a_0}{10} \quad \text{so} \quad a_0 \cdot 10^7 + \frac{a - a_0}{10}$$

$$= \frac{a - a_0}{10}$$

$$10b = a_0 \cdot 10^8 + a - a_0 \rightsquigarrow \boxed{10b = \underbrace{a_0(10^8 - 1)}_{101|} + \underbrace{a}_{|101}}$$

$$101 \mid a \Rightarrow 101 \mid b$$

$$101 \mid 10^8 - 1 = (10^4 - 1)(10^4 + 1) = (10^2 - 1)(\underbrace{10^2 + 1}_{101})(10^4 + 1)$$

$\underline{(10, 101) = 1}$ so $101 \mid b$.

Skanowano w CamScanner

③ Find all positive integers such $n^3 - 7n$ is square

$\boxed{4^3 - 7\cdot 4 = 64 - 28 = 36 = 6^2}$

Suppose $7 \nmid n$ then $\gcd(n, n^2-7) = 1$

$n(n^2-7)$ is square

$\Downarrow$ trick since $(n, n^2-7)=1$

$n = a^2$ and $n^2 - 7 = b^2$ so

$(a^2)^2 - 7 = b^2 \rightsquigarrow a^4 - b^2 = 7 \Rightarrow$

$\Rightarrow \underbrace{(a^2-b)}_{1}\underbrace{(a^2+b)}_{7} = 7$

$\boxed{\begin{cases} a^2 - b = 7 \\ a^2 + b = 1 \end{cases}} \Rightarrow 2a^2 = 8 \rightsquigarrow a = 2 \rightarrow$

$n = 4$

$A \cdot B = X^2$

and $(A,B) = 1$

$\Downarrow$

A, B are square

$AB = X^n$

A, B are coprime

$\Downarrow$

A, B are n-th powers.

$7 \mid n$ $\rightarrow$ $n = 7 \cdot m$ for some m.

$$n^3 - 7n = 49m(7m^2-1) = 7^2 \underbrace{m(7m^2-1)} = X^2$$

so $m(7m^2-1)$ is square.

$\gcd(m, 7m^2-1) = 1$! $\Longrightarrow$ the same trick :)

$m = c^2$, $7m^2 - 1 = d^2$ for some c, d.

$7c^4 = d^2 + 1$ $\rightarrow$ $7 \mid d^2 + 1 = d^2 + 1^2$ $\rightarrow$ $\lightning$ mod 4 odd

$7 \mid A^2 + B^2 \Rightarrow 7 \mid A,\ 7 \mid B$

$-c^4 \equiv d^2 + 1 \mod 4$

$0, 1, 2, 3$

$0, 1, 0, 1$

$d^2 \equiv -1 \mod 7$

$0, 1, 2, 3, 4, 5, 6$

$\boxed{0\ 1\ 4\ 2\ 2\ 4\ 1}$ no 6 here

$a_1, a_2, \dots, a_n$ – arithmetic progression, $i \mid a_i$ for $i = 1, 2, 3, \dots, n-1$

but $n \nmid a_n$. Prove n is a prime power!

$a_1, a_1+d, a_1+2d, \dots$

$a_1 = a_1$, $a_2 = a_1 + d$, $a_3 = a_1 + 2d$, $\dots$, $a_k = a_1 + (k-1)d$

$a_n = a_1 + (n-1)d$.

For $i \in \{1, 2, 3, \dots, n-1\}$ $i \mid a_i = a_1 + (i-1)d$, so

$i \mid a_1 + (i-1)d = id + (a_1 - d) \Rightarrow i \mid a_1 - d$

for $i \in \{1, \dots, n-1\}$.

$n = a \cdot b$ where $(a, b) = 1$, and $1 < a, b < n$.

$n = \boxed{p_1^{\alpha_1} \cdots}\boxed{p_k^{\alpha}}$

$a \quad b$

$a, b \in \{1, \dots, n-1\}$, so $a \mid a_1 - d$, $b \mid a_1 - d \Rightarrow$

$n = ab \mid a_1 - d$, so $n \mid (a_1 - d) + nd = \boxed{a_n}$ $\lightning \Rightarrow n = p^{\alpha}$.

Skanowano w CamScanner

Are there exist. four different numbers. s.t. $ad = bc$

$n^2 \le a, b, c, d < (n+1)^2$ for some positive n ?

$\boxed{d < (n+1)^2}$

$\boxed{a = pq, \quad d = rs, \quad b = pr, \quad c = qs}$

$\boxed{p+q \ge 2\sqrt{pq}}$

WLOG: $n^2 \le a < b < c < d$

$c > a$

$qs > p\cancel{q} \Rightarrow s > p \rightsquigarrow$

$b > a$

$pr > \cancel{p}q \Rightarrow r > q \rightsquigarrow$

$\boxed{\begin{array}{l} s \ge p+1 \\ r \ge q+1 \end{array}}$

AM-GM

$d = rs \ge (q+1)(p+1) = pq + p + q + 1 \ge pq + 2\sqrt{pq} + 1 =$

$= (\sqrt{pq} + 1)^2 = (\sqrt{a} + 1)^2 \ge (\sqrt{n^2} + 1)^2 = (n+1)^2$

so $d \ge (n+1)^2 \rightsquigarrow$ contradiction $\lightning$

Skanowano w CamScanner

Prove that $n^2+1 \mid n!$ for infinitely many number n.

$\boxed{\text{Idea}}$

$\underbrace{n^2+1}_{\text{factor}}$ $\Big|$ $n! = 1\cdot 2\cdot 3 \cdot \ldots \cdot n$

1) $n = 2k^2$ Sophie Germain identity

$$n^2+1 = (2k^2)^2+1 = 4k^4+1 = (2k^2+1)^2 - 4k^2 = (2k^2+1)^2-(2k)^2 =$$

$$= \underbrace{(2k^2+1-2k)}_{A}\underbrace{(2k^2+1+2k)}_{B} = A\cdot 5\cdot s \mid n!$$

$2k^2+1-2k < n = 2k^2$ so $A \mid n!$

take $k = 25m+1$, then $2k^2+2k+1 = 2(25m+1)^2+2(25m+1)+1 =$

$= 5\cdot s$, $5 \nmid s$. $\quad 2\cdot 1^2+2\cdot 1+1$

$\cancel{2\cdot 2^2+2\cdot 2+1}$ $\cancel{8}$

$1 < 5$; $s = \dfrac{2k^2+2k+1}{5} < 2k^2 = n$ $\quad \boxed{25A+5}$

Skanowano w CamScanner

2) $\boxed{n = 4k^2+2k+1}$

1 ↓ Easy 2 ↓ Mean 3 ↓ Hard!

$$n^2+1 = (4k^2+2k+1)^2+1 = \left([4k^2+1]+2k\right)^2+1 =$$

$$= (4k^2+1)^2 + 2\cdot 2k(4k^2+1) + (2k)^2 + 1 = \underline{2(4k^2+1)(2k^2+2k+1)}$$

$$\cancel{2k^2+2k+1 < 4k^2+2k+1} \qquad (4k^2+2k+1)!$$

Suppose: $2 < 2k^2+2k+1 < 4k^2+1$

$k \geq 2$

3) <u>Pell equations</u>

$n^2 - 5m^2 = -1$ ⇝ pseudo pell eq. and here infinitely many <u>solutions</u>.

$\boxed{n^2+1 = 5m^2}$ and $\boxed{n > 2m}$ so

$n^2+4 = 5m^2 \mid n!$ ⇝ ∎

Skanowano w CamScanner

Problem 1 $\quad a, b \in \mathbb{Z}_+$, $\quad a+b \mid ab$

Prove $\quad \gcd(a,b) \geq \sqrt{a+b}$.

Problem 2 $\quad x, y, z \in \mathbb{Z}_+$ such that

$$\frac{x+1}{y} + \frac{y+1}{z} + \frac{z+1}{x} \in \mathbb{Z}.$$

Prove: $\quad \gcd(x,y,z) \leq \sqrt[3]{xy+yz+zx}$.

Problem 1 $\quad d = \gcd(a,b)$, $a = dx$, $b = dy$ when $\gcd(x,y) = 1$.

$dx+dy \mid d^2xy \rightsquigarrow$ • $x+y \mid dxy$ but

• $(x+y; xy) = 1$, so $x+y \mid d \rightsquigarrow x+y \leq d$

$a+b = dx+dy = d(x+y) \leq d \cdot d = d^2 \rightsquigarrow \gcd(a,b) \geq \sqrt{a+b}$.

Skanowano w CamScanner

Problem 3

$a, b, c \in \mathbb{Z}_+$ $\quad \gcd(a,b,c) = 1$ and

$$a^2 + b^2 + c^2 = 2(ab + bc + ca)$$

Prove: a, b, c are squares.

Problem 4

$a, b, c, d \in \mathbb{Z}$ non-zero such that equation.

Hint?

$$ax^2 + by^2 + cz^2 + dt^2 = 0 \implies x = y = z = t = 0.$$

Does it follow that a, b, c, d have the same sign?

a, b, c, d

$x^2 + y^2 + z^2 + t^2 = 0 \longrightarrow x = y = z = t = 0$

Skanowano w CamScanner

② Problem.

$$\frac{x+1}{y} + \frac{y+1}{z} + \frac{z+1}{x} \in \mathbb{Z}$$

$d = \gcd(x,y,z)$, $x = da$, $y = db$, $z = dc$

$\underline{(a,b,c) = 1}$.

$$\frac{x+1}{y} + \frac{y+1}{z} + \frac{z+1}{x} = \frac{x^2z + xz + y^2x + yx + z^2y + zy}{xyz} =$$

$$= \frac{d^3(a^2c + b^2a + c^2b) + xy + yz + zx}{d^3 abc} \in \mathbb{Z}.$$

so $d^3 \mid xy + yz + zx \rightsquigarrow$

$$\gcd(x,y,z) \leq \sqrt[3]{xy + yz + zx} \quad \square$$

Skanowano w CamScanner

Problem 5

$\boxed{12:45}$

Solve:

$$x^2+y^2+z^2-2xyz=0$$ in integers.

$\boxed{\gcd(a,b,c)=1}$

Problem 3 Solution.

$$a^2+b^2+c^2=2(ab+bc+ca)$$

$\downarrow$

$$\begin{pmatrix} \text{sqm} \to (a+b-c)^2=4ab \\ \text{sqm} \to (a-b+c)^2=4ac \\ \text{sqm} \to (-a+b+c)^2=4bc \end{pmatrix}$$

so ab, bc, ca are squares.

$$\boxed{\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=1}$$

let $d=\gcd(a,b)>1$.

$\Downarrow$

$\gcd(a,b,c)>1$ too. $\lightning$

$X\cdot Y=A^2$

$\Downarrow$

X, Y is sq if $(X,Y)=1$

Skanowano w CamScanner

For Ali $a, b, c \in \mathbb{Z}_+$. Prove:

$$\operatorname{lcm}(a,b) \neq \operatorname{lcm}(a+c, b+c).$$

Problem 3

a, b, c, d have the same sign $a; b, c, d > 0$

$$ax^2 + by^2 + cz^2 + dt^2 = 0 \qquad \underline{x = y = z = t = 0}.$$

$$\boxed{a = b = 1, \quad c = d = -3}$$

$$x^2 + y^2 = 3z^2 + 3t^2 \quad \text{(Fermat descent method)}.$$

so $3 \mid x^2 + y^2 \Rightarrow 3 \mid x$ and $3 \mid y$, so

$$x = 3x_1, \quad y = 3y_1$$

$$\cancel{9}^{3} x_1^2 + \cancel{9}^{3} y_1^2 = \cancel{3} z^2 + \cancel{3} t^2$$

$$3x_1^2 + 3y_1^2 = z^2 + t^2 \longrightarrow 3 \mid z^2 + t^2 \text{ i.e. } z = 3z_1, \; t = 3t_1$$

Skanowano w CamScanner

$3x_1^2 + 3y_1^2 = 9z_1^2 + 9t_1^2$ so $\underline{x_1^2 + y_1^2 = 3z_1^2 + 3t_1^2}$

$\Downarrow$

1°) $x^2 + y^2 = 3z^2 + 3t^2$, take (x, y, z, t) such that $|x| + |y| + |z| + |t|$ smallest

$\Downarrow$

$x_1^2 + y_1^2 = 3z_1^2 + 3t_1^2$ with solution (x_1, y_1, z_1, t_1) such that

$$|x_1| + |y_1| + |z_1| + |t_1| = \frac{|x| + |y| + |z| + |t|}{3} < |x| + |y| + |z| + |t| . \lightning$$

so $|x| + |y| + |z| + |t| = 0 \Rightarrow \underline{x, y, z, t = 0}$

2°) $(x, y, z, t) \longrightarrow \left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}, \frac{t}{3}\right) \longrightarrow \left(\frac{x}{3^2}, \frac{y}{3^2}, \frac{z}{3^2}, \frac{t}{3^2}\right) \ldots$

so x, y, z, t are divisible by <u>arbitrary</u> power of 3

$x = y = z = t = 0$. $\blacksquare$

Skanowano w CamScanner

$\sqrt{2} = \frac{m}{n} \rightsquigarrow 2n^2 = m^2 \rightsquigarrow 2 \mid m \Rightarrow m = 2m_1 \Rightarrow$

$2n^2 = 4m_1^2 \Rightarrow n^2 = 2m_1^2 \Rightarrow 2 \mid n^2 \Rightarrow$

$\Rightarrow n = 2n_1 \Rightarrow 4n_1^2 = 2m_1^2 \Rightarrow$

$2n_1^2 = m_1^2$

$|m_1| + |n_1| = \frac{|m| + |n|}{2} < |m| + |n|$ ↯

$|m| + |n|$ – min.

Rodle.

$\boxed{x^2 + y^2 + z^2 = 2xyz}$ $\Rightarrow$ $\underline{x = y = z = 0}$

mod 2/4

$2 \mid x^2 + y^2 + z^2 \rightsquigarrow$ 0 or 2 odd numbers.

so $2 \mid x$, $2 \mid y$, $2 \mid z$ and so

$0 \equiv 2xyz = x^2 + y^2 + z^2 \equiv 2 \bmod 4$ ↯

$\frac{x}{2^n}, \dots \frac{y}{2^n}$

Skanowano w CamScanner

$x = 2x_1, \quad y = 2y_1, \quad z = 2z_1 \Longrightarrow$

$$4x_1^2 + 4y_1^2 + 4z_1^2 - 16x_1y_1z_1 = 0$$

$$x_1^2 + y_1^2 + z_1^2 - 4x_1y_1z_1 = 0$$

$$\downarrow$$

$$\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2}\right) \longrightarrow \longrightarrow \cdots \longrightarrow \left(\frac{x_1}{2^n}, \frac{y_1}{2^n}, \frac{z_1}{2^n}\right)$$

$$\Downarrow$$

$$\boxed{x = y = z = 0}$$

$2 \mid x_1, \; 2 \mid y_1, \; 2 \mid z_1 \rightsquigarrow$

$$x_2^2 + y_2^2 + z_2^2 - 8x_2y_2z_2 = 0$$

Skanowano w CamScanner