**Topic 1.**

## ADDITIVE FUNCTIONAL EQUATIONS

**Problem 1.** (L) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(2022x^3+y+f(y))=2y+2022x^2f(x) \text{ for all } x,y\in\mathbb{R}.$$

**Problem 2.** (L) Prove that for each positive integer $n$, there exist at most 3 functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$2022\left[(x+y)f(x+y)-f(x^2+y^2)\right]=xf_n(y)+yf_n(x)$$

for all $x,y\in\mathbb{R}$, in which $f_n(x)=f(f(\dots f(x)\dots))$.

**Problem 3.** (O) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(x)+2f(y))=x^2+y+f(y) \text{ for all } x,y\in\mathbb{R}.$$

**Problem 4.** (O) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(3(f(xy))^2+(xy)^2)=(xf(y)+yf(x))^2 \text{ for all } x,y>0.$$

**Problem 5.** (L) Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfy these conditions:

i) $f(x)+f(y)+f(z)+3xyz=0$ for all $x+y+z=0$.

ii) $\max\{f(x)-f(y)\}=2$ for all $x\ge y\ge 0$.

**Problem 6*.** (V) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$2f(x)f(x+y)-f(x^2)=\frac{x}{2}(f(2x)+4f(f(y))) \text{ for all } x,y\in\mathbb{R}.$$

Additional problems.

**Problem 7.** (V) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y^2+z^3)=f(x)+f(y)^2+f(z)^3,\ \forall x,y,z\in\mathbb{R}.$$

**Problem 8.** (V) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f\ \ |x|+y+f(y)\ \ =2y+|f(x)| \text{ for all } x,y\in\mathbb{R}.$$

**Problem 9.** (O) Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ such that for all $x,y>0$ then

i) $f(x)+f(y)\le\dfrac{f(x+y)}{2}$.

ii) $(x+y)\left[yf(x)+xf(y)\right]\ge xyf(x+y)$.

**Problem 10*.** (V) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $0<x<y$ then

i) $xf(y^2)-yf(x^2)>0$.

ii) $f(xf(y^2)-yf(x^2))=(y-x)f(xy)$.



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**Topic 2.**

**ROOTS OF POLYNOMIAL**

**Problem 11.** (L) Find all positive integer $n$ such that there exists some polynomial $P(x)\in\mathbb{R}[x]$ such that $P(x)>-3,\forall x$ and $P(0)=P(2)=P(-2)=0$.

**Problem 12.** (V) Find all odd integer $n$ such that the exist $g(x)\in\mathbb{R}[x]$ such that

$$4x^{2n+1}-3x-1=(x-1)g(x)^2.$$

**Problem 13.** (O) Let $c_1,c_2,\ldots,c_{20}\in\{0,1\}$ and consider the following polynomial

$$P(x)=c_1x^{38}+c_2x^{36}+\cdots+c_{19}x^2+c_{20}.$$

a) Suppose that $P(1)=-10$, prove that there exist at most $6$ coefficients equal to $5$ in the expansion of $P(x)^2$.

b) For $n\ge 20$, denote $c_{n+1}$ as the smallest real root of the polynomial

$$P_n(x)=x^{2n}+c_1x^{2n-2}+c_2x^{2n-4}+\cdots+c_{n-1}x^2+c_n.$$

Prove that $c_{n+1}$ always exist and for all $n\ge 20$, $c_{n+1}<c_n$.

**Problem 14.** (O) Condiser polynomial $P(x)=x^3+ax^2+bx+1,\ (a,b\in\mathbb{R})$ with graph $(C)$ and $(C)$ cuts $x-$axis at $B,C,D$ with $x-$coordinate are $u,v,w$ in that order such that $|u|<|v|<|w|$, cuts $y-$axis at $A$. The circle $(ABD)$ cut $y-$axis again at $E$. Suppose that $\max\{u,v,w\}>2$.

a) Find the smallest length of the segment $CE$. Denote $\Omega$ as the family of polynomial $P(x)$ attaining that minimum value.

b) Consider $P_0(x)\in\Omega$, find the maximum value of the sum of the coefficients of $P_0(P_0(x))$.

**Problem 15.** (L) Given monic $P(x)$ polynomial in $\mathbb{R}[x]$ of degree 15 and has 15 distinct non-integer roots. Suppose that each of $P(2x^2-4x)=0$ and $P(4x-2x^2)=0$ have exactly 20 distinct real roots. Prove that one can find two polynomials $G(x),H(x)$ such that

$$P(x)=G(x)H(x)\text{ and }|G(c)|>|H(c)|,\forall c\in(-1;1).$$

**Problem 16*.** (O) Let $c$ be postive real number, consider the polynomial $P(x)=cx(x-2)$. Suppose that the equation $P_n(x)=0$ has $2^n$ distinct real roots for all positive integer $n$. Prove that $c\ge 1$.

*Note that:* $P_n(x)=P(P(\ldots P(x)\ldots))$ *with* $n$ *times of union of* $P$.

<u>Additional problems.</u>

**Problem 17.** (L) Given the function $f(x)=x^2-2x$. Find the condition of real number $m$ such that $\underbrace{f(f(\ldots f(x)\ldots))}_{\text{2022 times}}=m$ has $2^{2022}$ distinct real roots.



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**Problem 18.** (V) Consider the sequence of polynomial $P_1(x)=x$, $P_{n+1}(x)=\left(P_n(x)-a\right)^2$ for $n\geq 1$. Suppose that $P_n(x)=2$ has $2^{n-1}$ distinct real roots for all $n\in\mathbb{Z}^+$. Prove that $a\geq 2$.

**Problem 19.** (L) Consider polynomial $f(x)=ax^2+bx+c$ with $a\neq 0$.

a) Suppose that $f(f(x))=0$ has unique real root $a$. Prove that $f'(a)=1$.

b) On the Cetersian coordination, the graphs of $y=f(x)$ and $x=f(y)$ meets at four points $A,B,C,D$ form an quadrilateral $ABCD$. Prove that $AC\perp BD$ and $ABCD$ is not a trapezoid.

**Problem 20*.** (O) Consider the monic, integer polynomial $P(x)$ of degree $n$ with $n$ distinct real roots. Suppose that $P(x)$ is irreducible on $\mathbb{Z}[x]$ and there exist some integer polynomial $Q(x)$ of degree smaller than $n$ and such that $P(x)\mid P(Q(x))$.

a) Prove that the number of such $Q(x)$ is at most $n$.

b) Suppose that $\deg P=3$, the sum of its roots is $0$ and $Q(x)$ is monic. Find the maximum value of $Q(2)$.

__Topic 3.__

## MULTI-VARIABLE INEQUALITY

**Problem 21.** (O) Find the maximum value of $c$ such that for any positive integer $n$ and any sequence $(x_n)$ such that $0=x_0<x_1<x_2<\cdots<x_n=1$ then

$$\sum_{i=1}^{n}x_i^3(x_i-x_{i-1})>c.$$

**Problem 22.** (O) For integer $n\geq 3$, given numbers $a_1\leq a_2\leq\ldots\leq a_n$ having the sum equals to $0$. Prove that

$$a_1^2+a_2^2+\cdots+a_n^2+na_1a_n\leq 0.$$

**Problem 23.** (O) Consider positive integer $n$ such that there exist real numbers $x_1,x_2,\ldots,x_n$ in which $i\leq x_i\leq 2i$ for $i=1,2,\ldots,n$ and

$$\frac{x_1^2+x_2^2+\cdots+x_n^2}{(x_1+2x_2+3x_3+\cdots+nx_n)^2}=\frac{27}{4n(n+1)(2n+1)}.$$

a) Prove that $n\equiv 0,4,8 \pmod 9$.

b) Find all $n$ satisfy the given conditions.

**Problem 24.** (L) Consider real numbers $x_1,x_2,\ldots,x_{2022}$ having the sum equals $0$ and

$$|x_1|+|x_2|+\ldots+|x_{2022}|=2022.$$

Find the maximum and minimum value of $T=x_1x_2\ldots x_{2022}$.



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**Problem 25.** (O) Find all tuples $(a_1,a_2,\dots,a_{2n})$ of positive real numbers such that:

i) $a_{2k-1}=\dfrac{1}{a_{2k}}+\dfrac{1}{a_{2k-2}}$ for $k=1,2,\dots,n$ and suppose that $a_0=a_{2n}$.

ii) $a_{2k}=a_{2k-1}+a_{2k+1}$ for $k=1,2,\dots,n$ and suppose that $a_{2n+1}=a_1$.

**Problem 26*.** (V) For positive integer $n$, consider $n$ red balls and $n$ green balls are arrange on the line. For the ball $X$ at positive $i$, denote $x_i$ as the number of pairs of ball the lying on two sides of $X$ and having the color differ from $X$. Find the maximum value of

$$T=x_1+x_2+\dots+x_{2n}.$$

<u>Additional problems.</u>

**Problem 27.** (L) For integer $n\in[2;21]$, consider non-negative real numbers $x_1,x_2,\dots,x_n$ having the sum $\dfrac{7}{4}$. Find the maximum value of $S=5(x_1^2+x_2^2+\dots+x_n^2)-2(x_1^3+x_2^3+\dots+x_n^3)$.

**Problem 28.** (O) For integer $n\ge 3$, consider $4n$ non-negative real numbers $x_1,x_2,\dots,x_{2n}$ and $y_1,y_2,\dots,y_{2n}$ such that

i) $S=x_1+x_2+\dots+x_{2n}=y_1+y_2+\dots+y_{2n}>0$.

ii) $x_ix_{i+2}\ge y_i+y_{i+1}$ for all $i=1,2,\dots,2n-1$ (with $x_{2n+1}=x_1$, $x_{2n+2}=x_2$, $y_{2n+1}=y_1$).

a) Prove that for $n=3$, $S\ge 12$.

b) Suppose that $n\ge 4$, find the minimum value of $S$.

**Problem 29.** (O) For integer $n\ge 3$ and numbers $a_1,a_2,\dots,a_n$ that not all equal to 0. Prove that

$$\frac{12}{n^3-n}(a_1^2+a_2^2+\dots+a_n^2)\ge \min_{1\le i<j\le n}(a_i-a_j)^2.$$

**Problem 30*.** (O) Prove that there exist 100 positive real numbers $a_1,a_2,\dots,a_{100}$ such that

i) $a_ia_{i+1}=2i+1$ for all $i=1,2,\dots,99$.

ii) $a_ia_j\le i+j$ for all $i\ne j$.

<u>Topic 4.</u>

**SOME TECHNIQUES ON FUNCTION EQUATION**

**Problem 31.** (L) For $k$ is the real number, consider $f:\mathbb{R}\to\mathbb{R}$ such that

$$2f(kxy+f(x+y))=xf(y)+yf(x) \text{ for all } x;y\in\mathbb{R}.$$

a) Suppose that $f(0)\ne 0$ then find $k$.

b) Suppose that $k=-\dfrac{1}{2}$, prove that $f(-f(2))=f(2)$.



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**Problem 32.** (V) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xf(y)-f(x))=2f(x)+xy \text{ for all } x,y\in\mathbb{R}.$$

**Problem 33.** (L) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+f(y))=f(x)+\frac{x}{2}f(2y)+f(f(y)) \text{ for all } x,y\in\mathbb{R}.$$

**Problem 34.** (O)

a) Suppose that $f:\mathbb{R}^+\to\mathbb{R}^+$ such that there exist some positive real numbers $a,b,c$ in which $f(x)\ge cx,\ \forall x>0$ and $f(x+a)=f(x+b),\forall x>0$. Prove that $a=b$.

b) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(x+y+f(x))=f(x+y)+f(y) \text{ for all } x,y>0.$$

**Problem 35.** (V)

a) Consider functions $f,g,h:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(g(x)+y)=h(x)+g(y) \text{ for all } x,y>0.$$

Prove that $\dfrac{g(x)}{h(x)}$ is a constant for all $x$.

b) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that $f\left(\dfrac{f(x)}{x}+y\right)=1+f(y)$ for all $x,y>0$.

**Problem 36*.** (V) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

a) $f(x+y+f(x))=f(2x)+f(y)$ for all $x,y>0$.

b) $f(2021x+y+f(x))=f(2022x)+f(y)$ for all $x,y>0$.

Additional problems.

**Problem 37.** (V) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)f(y)=f(xy-1)+xf(y)+yf(x) \text{ for all } x,y.$$

**Problem 38.** (V) Consider functions $f,g:\mathbb{R}\to\mathbb{R}$ such that $g(2022)>0$ and

$$\begin{cases} f(x-g(y))=f(-x+2g(y))+xg(y)-6 \\ g(y)=g(2f(x)-y) \end{cases} \text{ for all } x,y\in\mathbb{R}.$$

Prove that $g$ is constant.

**Problem 39.** (L) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(x+2y+f(x+y))=f(2x)+f(3y) \text{ for all } x,y>0.$$

**Problem 40*.** (O) Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(f(x)f(f(x))+y)=xf(x)+f(y) \text{ for } x,y\in\mathbb{R}^+.$$



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Topic 4.

**SEQUENCE OF THE REAL NUMBERS**

**Problem 41.** (O) Consider sequence $a_1, a_2, \ldots, a_{2022}$ of real numbers with at least one positive. The term $a_k$ is called « good » is one of the following value is positive

$$a_k,\ a_k + a_{k+1},\ \ldots,\ a_k + a_{k+1} + \cdots + a_{2022}$$

Prove that the sum of all good numbers in this sequence is positive.

**Problem 42.** (O) Consider 2022 real numbers $a_1, a_2, \ldots, a_{2022}$. Prove that there exist indices $m, k$ (with $m$ can be $0$ or $2022$) such that

$$\left|\sum_{i=1}^{m} a_i - \sum_{i=m+1}^{2021} a_i\right| \le |a_k|.$$

**Problem 43.** (V) Find the smallest real number $M$ such that for all positive sequence $(u_n)$ in which $\sum_{k=1}^{n} u_k \le \frac{u_{n+1}}{2}$ for all $n$ then

$$\sum_{k=1}^{n} \sqrt{u_k} \le M \sqrt{\sum_{k=1}^{n} u_k},\ \forall n.$$

**Problem 44*.** (O) For $c \in \mathbb{R}^+$, consider the infinite sequence $(x_n)$ such that $0 < x_i < c, \forall i$ and $|x_i - x_j| \ge \frac{1}{j}$ for all $1 \le i < j$. Prove that $c \ge \ln 4$.

**Problem 45*.** (O) For $c \in \mathbb{R}^+$, consider the infinite sequence $(x_n)$ such that $0 < x_i < c, \forall i$ and for any $n \in \mathbb{Z}^+$, the numbers $a_1, a_2, \ldots, a_n$ divide $(0; c)$ into sub-intervals having the length less than $\frac{1}{n}$. Prove that $d \le \ln 2$.

**Problem 46.** (V) For $a \in (1; 2)$, consider the sequence $(x_n)$ of positive real numbers such that

$$u_n^a \ge u_1 + u_2 + \cdots + u_{n-1}, \forall n \ge 2.$$

Prove that there exist constant $c > 0$ such that $u_n \ge cn, \forall n$.

Additional problems.

**Problem 47.** (L) Prove that there does not exist the sequence $(x_n)$ of real numbers such that

$$x_1 = 2 \text{ and } \frac{2x_n^2 + 2}{x_n + 3} < x_{n+1} \le \frac{2x_n + 2}{x_n + 3} + 2022 \text{ for all } n = 1, 2, 3, \ldots$$

**Problem 48.** (L) On the Catersian coordination $Oxy$, consider sequence of points $A_n(x_n, y_n)$ in which $(x_n), (y_n)$ are sequences of positive real numbers such that



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$$x_{n+1} = \sqrt{\frac{x_n^2 + x_{n+2}^2}{2}}, y_{n+1} = \left(\frac{\sqrt{y_n} + \sqrt{y_{n+2}}}{2}\right)^2 \text{ for all } n \geq 1.$$

Suppose that $O, A_1, A_{2022}$ belong to a line$(d)$ and $A_1 \neq A_{2022}$. Prove that all of $A_2, A_3, \dots, A_{2021}$ lying on the same side of the line $(d)$.

**Problem 49.** (O) Let $(a_n)$ be a sequence with $a_1 = a_2 = 1$ and

$$a_n = a_{a_{n-1}} + a_{n-a_{n-1}} \text{ for all } n \geq 3.$$

Prove that $a_{2n} \leq 2a_n$ for all $n$.

**Problem 50*.** (V) Let $a_1, a_2, \dots, a_n$ be some real numbers $(n \geq 3)$. Denote $A, B$ as the number of ordered pairs $(i, j)$ such that $|x_i - x_j| \leq 1$ and $|x_i - x_j| \leq 2$ respectively. Prove that $B \leq 3A$. Can the number $3$ be replaced by another smaller number?



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