Number Theory – group L3

Instructor: Dušan Djukić *Date: 23.2.2022.*

1. Find all primes p, q such that $p^2 - pq - q^3 = 1$.
   What if we do not require q to be prime?

2. Given a positive integer n, define a sequence (a_k) by $a_0 = n$ and $a_{k+1} = \tau(a_k)$, where $\tau(x)$ denotes the number of (positive) divisors of a positive integer x. Find all n for which no term a_k is a perfect square.

3. A function $f : \mathbb{N} \to \mathbb{N}$ is such that $f(f(n)) = \tau(n)$. Prove that if p is prime, then $f(p)$ is prime.

4. Prove that there are infinitely many positive integers n such that $\lfloor \tau(n)\sqrt{3} \rfloor$ divides n.

5. Suppose that $1 \leqslant a_1, a_2, \ldots, a_n \leqslant 2n$ are integers such that $\operatorname{lcm}(a_i, a_j) > 2n$ whenever $i < j$. Prove that $a_1 a_2 \cdots a_n$ divides $(n+1)(n+2)\cdots(2n)$.

6. A triple of positive integers (a, b, c) is *lame* if $c^2 + 1$ divides $(a^2 + 1)(b^2 + 1)$, but not $a^2 + 1$ and $b^2 + 1$. Given c, if there is a lame triple (a, b, c), prove that there is a lame triple in which $ab < c^3$.

7. The sequence (a_n) is defined by $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = a_n(a_{n+1} + 1)$ for all $n \geqslant 1$. Prove that a_{a_n} is divisible by a_n^n for every $n \geqslant 100$.

8. If a positive integer $n > 20$ is not squarefree, prove that there exist positive integers a, b, c such that $ab + bc + ca = n$.

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9. Prove that every integer can be uniquely written in the form $a_0 + a_1(-\frac{3}{2}) + a_2(-\frac{3}{2})^2 + \cdots + a_k(-\frac{3}{2})^k$ for some integers $k \geqslant 0$ and $a_0, a_1, \ldots, a_k \in \{0, 1, 2\}$.

10. Denote $\phi = \frac{-1-\sqrt{5}}{2}$. Can every integer be written as a sum of powers of ϕ, where each power occurs at most 1000 times?

11. If a, b are positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$, prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by a cube greater than 1.

12. Suppose that a and b are integers such that $2^n a + b$ is a perfect square for every $n \in \mathbb{N}$. Prove that $a = 0$.

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<u>*Linear recurrences*</u>
Sequences (x_n) defined by a recurrence relation of the form

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}, \qquad (\spadesuit)$$

with the first k terms given, can be solved in closed form. Here is how.

We first check if there are exponential sequences of the form $x_n = \alpha^n$ that satisfy $(\spadesuit)$. It turns out that the constant α must satisfy $P(x) = x^k - c_1 x^{k-1} - \cdots - c_{k-1} x - c_k = 0$. The polynomial $P(x)$ is called the *characteristic polynomial.*

So, let the zeros of $P(x)$ be $\alpha_1, \ldots, \alpha_\ell$. We allow multiple roots, so let r_i be the multiplicity of the zero α_i. Then the sequence $x_n = \alpha_i^n$ satisfies $(\spadesuit)$. Moreover, even the sequence $x_n = n^k \alpha_i^n$ satisfies $(\spadesuit)$, if $0 \leqslant k \leqslant r_i - 1$ is an integer. In general, every linear combination of the described sequences, and no others, satisfies $(\spadesuit)$.

To sum up, a formula for x_n will have the form

$$x_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \cdots + P_\ell(n)\alpha_\ell^n,$$

where $P_i(x)$ are some polynomials of degree strictly less than r_i.

13. Find a_n in closed form if:
    (a) $a_0 = 0$, $a_1 = 1$ and $a_n = 2a_{n-1} + a_{n-2}$ for $n \geqslant 2$;
    (b) $a_0 = a_1 = 0$, $a_2 = 1$, $a_n = 3a_{n-2} - 2a_{n-3}$.

14. Find all positive integers n for which $(1+\sqrt{2})^n + (1-\sqrt{2})^n$ is divisible by 5.

15. Prove that there is a positive integer n, not divisible by any of the numbers from 2 to 1000, such that the numbers $n^2 - 1$, $n^2 - 2, \ldots, n^2 - 1000$ are all composite.

16. Denote $m = 2^{100}$ and $n = 3^{100}$. Prove that there exist positive integers a, b, c, d such that $am - bn = cm - dn = ad - bc = 1$.

17. Prove that there exist infinitely many positive integers n for which $n!$ is divisible by $n^2 + 1$.

18. What is the largest n for which there exist $2n$ positive integers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ that satisfy $a_i b_j - a_j b_i = 1$ whenever $i < j$?

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19. Sequence (a_n) is defined by $a_1 = 1$ and $a_{n+1} = 3a_n + \sqrt{8a_n^2 + 1}$. Prove that each a_n is an integer.

20. Find all sequences of positive integers (a_n) that satisfy $a_n + a_{n+1} = a_{n+2}a_{n+3} - 1000$ for all n.

21. Let $a_0 = 1$ and $a_{n+1} = \frac{1+a_n}{3+a_n}$. Find a_n in closed form.

22. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. Prove that for each n one of the numbers $5F_n^2 \pm 4$ is a perfect square.

23. Prove that for every $a \in \mathbb{N}$ there is a Fibonacci number that is divisible by a.

24. Prove that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.

25. Find all Fibonacci numbers that are powers of 2 or powers of 3.

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26. Define $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 2a_n + a_{n-1}$. Prove that $2^k \mid a_n$ if and only if $2^k \mid n$.

27. A sequence (a_n) satisfies $a_{n+1} = a_n^3 + 103$ for all $n \in \mathbb{N}$. Prove that this sequence contains at most one perfect square.

28. If a, b are positive integers, can $(a + b)^{15}$ be divisible by $4ab - 1$?

29. (a) If a prime p divides $x^2 + xy + y^2$ for some integer x, y, but $p \nmid xy$, prove that $p \equiv 1 \pmod 3$ or $p = 3$.

    (b) If a prime $p > 3$ divides $x^2 + 3$ for some integer x, prove that $p \equiv 1 \pmod 3$.

30. Let a and b be positive integers. If $\gcd(an + 2, bn + 3) > 1$ for every $n \in \mathbb{N}$, what is b/a?

31. If a, b, c are three distinct nonnegative integers, prove that $\gcd(ab+1, bc+1, ca+1) \leqslant \frac{a+b+c}{3}$.

32. Does there exist an integer x such that $x^2 + 2$ is divisible by 3^{2022}?

33. Is there a positive integer n such that $n! + 1$ is divisible by $n + 100$?

34. Find all triples of positive integers a, b, c such that $a \mid bc + 1$, $b \mid ac + 1$ and $c \mid ab + 1$.

Solutions – group L3

Instructor: Dušan Djukić *Feb.21–Mar.3, 2021*

1. The discriminant of the given quadratic $p^2 - q \cdot p - (q^3+1)$ must be a square, so $d^2 = q^2 + 4(q^3+1)$. This leads to $(d+2)(d-2) = q^2(4q+1)$, but since only one of the factors $d \pm 2$ can be divisible by q, that one is a multiple of q^2, while the other factor (which is less by at most 4) divides $4q+1$. It follows that $q^2 - 4 \leqslant 4q+1$ and hence $q \leqslant 5$. Testing these values of q yields two solutions: $(7,3)$ and $(14,5)$.

2. The sequence decreases until it eventually drops down to 2. Let $a_{k-1} > a_k = 2$. Since $\tau(a_{k-1}) = 2$, a_{k-1} must be an odd prime, and since $\tau(a_{k-2})$ is odd (assuming that $k > 1$), a_{k-2} must be a perfect square. So, if there is no square in the sequence, we must have $k \leqslant 1$, so n is a prime.

3. Iterating one more f we obtain $f(\tau(n)) = f(f(f(n))) - \tau(f(n))$. Now setting $n = p$ to be prime yields $\tau(f(p)) = f(\tau(p)) = f(2)$, so it is enough to prove that $f(2) = 2$. However, we have $\tau(f(2)) = f(\tau(2)) = f(2)$, so $f(2)$ equals 1 or 2.

   It remains to show that $f(2) = 1$ is impossible. Otherwise we would have $f(p) = 1$ for all p, so $f(1) = f(f(p)) = \tau(p) = 2$. On the other hand, then $1 = f(3) = f(\tau(25)) = \tau(f(25))$, so also $f(25) = 1$, contradicting $f(f(25)) = 3$.

4. Setting e.g. $\tau(n) = 8$ we find that n must be divisible by $\lfloor 8\sqrt{3} \rfloor = 13$, which is achievable by taking $n = 13p^3$ for any prime $p \neq 13$ (then indeed $\tau(n) = 8$).

5. Each of the numbers a_i has a multiple in the set $\{n+1, \dots, 2n\}$, but no two share this multiple (because lcm $> 2n$), so each number from $n+1$ to $2n$ has a unique divisor among the a_i. The statement immediately follows.

6. If $c^2 + 1 = 2m$ is even, then m is odd, so m divides $(c+m)^2 + 1$, but $2m$ does not (c is also odd). This enables us to take $a = 1$ and $b = c + m$; clearly, $ab = \frac{(c+1)^2}{2} < c^3$.

   Now let $c^2 + 1$ be odd. It must be composite, so let $c^2 + 1 = mn$, where $m < c < n$. We first choose a so that $a^2 + 1$ is divisible by m, but not by mn - this can be done by simply taking a to be the remainder of c when divided by m, as then $a^2 + 1 < m^2 < mn$.

   It remains to choose b. The numbers c^2+1, $(n-c)^2+1$ and $(n+c)^2+1$ are all divisible by n, but not all are divisible by mn (else $mn \mid (n+c)^2 - (n-c)^2 = 4cn$ and hence $mn \mid n$), so we can take b so that $b < 2n < c^2$. Then $ab < c^3$.

7. An easy induction yields $a_k = (a_{k-1}+1)(a_{k-3}+1)\cdots(a_{k-2i+1}+1)a_{k-2i}$. We will show that $v_p(a_{a_n}) \geqslant n v_p(a_n)$ for every prime p.

   It follows from the recurrence relation that the sequence $v_p(a_i), v_p(a_{i+2}), v_p(a_{i+4}), \dots$ is nondecreasing. Moreover, if $p^k \mid a_i + 1$, then $p^k \mid a_{i+1}$ and $a_{i+2} = a_i(a_{i+1}+1) \equiv -1$

(mod p^k), which implies that the sequence $v_p(a_i+1), v_p(a_{i+2}+1), v_p(a_{i+4}+1), \dots$ is nondecreasing as well.

Now consider the largest ℓ for which $p \mid a_{n-2\ell}$. From $a_{n-2\ell} = a_{n-2\ell-2}(a_{n-2\ell-1}+1)$ it follows that $p \mid a_{n-2\ell-1}+1$, so $v_p(a_{n-2\ell-1}+1) \leqslant v_p(a_{n-2\ell+1}+1) \leqslant \dots \leqslant v_p(a_{n-1}+1) \leqslant v_p(a_{n+1}+1) \leqslant \dots$. So if $k = v_p(a_{n+1}+1)$, it follows that $v_p(a_n) \leqslant \ell k < \frac{1}{2}nk$.

On the other hand, a_n is inductively shown to have the same parity as n, so $a_{a_n} = a_n(a_{n+1}+1)(a_{n+3}+1)\cdots(a_{a_n-1}+1)$ and hence $v_p(a_{a_n}/a_n) \geqslant \frac{1}{2}(a_n-n)k$. It remains to show that $\frac{1}{2}(a_n-n)k \geqslant (n-1)\cdot\frac{1}{2}nk$, which reduces to $a_n \geqslant n^2$, and this holds by induction for $n \geqslant 6$.

8. Since $n = ab+bc+ca$ is equivalent to $n+a^2 = (a+b)(a+c)$, the problem reduces to finding a such that $n+a^2$ is a product of two integers greater than a.

   Let $n = p^2m$, where p is a prime. We first try taking $a = p$, so that $n+p^2 = (m+1)p^2$. This clearly works if $m+1 > p$, or if $m+1$ is composite ($m+1 = uv \Rightarrow n+p^2 = up\cdot vp$).

   It remains to deal with the case when $m+1 = q \leqslant p$ is a prime. Then $n = p^2q - p^2$, so we can take a to be the remainder of p modulo q: then $n+a^2 = p^2q-(p^2-a^2)$ is divisible by q and greater than $n+a^2 > n > pq$. This works unless $q = p$.

   We are left with the case $n = p^2(p-1)$. Then $a = 6$ works if $p > 3$ (which corresponds to $n > 20$), because $n+a^2 = (p+3)(p^2-4p+12)$.

9. Since all summands except a_0 are divisible by 3, a_0 is uniquely determined modulo 3. Subtract a_0, multiply by $-\frac{2}{3}$ (this should decrease its absolute value, except in very small cases) and continue. The basis of induction will be from -6 to 6.

10. Switching from ϕ to $\bar\phi = \frac{-1+\sqrt5}{2}$ only changes the sign at $\sqrt5$, so if $a_0+a_1\phi+\dots+a_k\phi^k = n$ is an integer, then also $a_0+a_1\bar\phi+\dots+a_k\bar\phi^k = n$. But if each a_i is at most 1000, then $n \leqslant 1000(1+\bar\phi+\dots+\bar\phi^k) < \frac{1}{1-\bar\phi} = 500(3+\sqrt5)$. Thus the answer is *no*.

11. Observe that $N = a^2+3ab+3b^2-1$ divides $(a+b^3)+aN = (a+b)^3$. Assume to the contrary that N is not divisible by any cube other than 1. Then for any prime divisor p of $a+b$ we have $v_p(N) \leqslant 2$, but $v_p((a+b)^3) \geqslant 3$, which implies that $N \leqslant (a+b)^2$. But this is false, thus finishing the proof.

12. Note that $3b = 4(2^na+b)-(2^{n+2}a+b)$. But $b = 0$ does not work, and if $a \neq 0$, then we get infinitely many ways to write $3b$ as a difference of two squares, which is impossible. Thus we must have $a = 0$.

13. (a) The characteristic polynomial is $P(t) = t^2-2t-1$ and its zeros are $1+\sqrt2$ and $1-\sqrt2$. It follows that $a_n = A(1+\sqrt2)^n + B(1-\sqrt2)^n$ for some constants A, B. Moreover, $a_0 = 0 = A+B$ and $a_1 = 1 = A(1+\sqrt2)+B(1-\sqrt2)$ give $A = -B = \frac{1}{2\sqrt2}$, so $a_n = \frac{1}{2\sqrt2}[(1+\sqrt2)^n-(1-\sqrt2)^n]$.

    (b) The characteristic polynomial is t^3-3t+2, with a double root $t = 1$ and a root $t = -2$. It follows that $a_n = (A+Bn)\cdot 1^n + C(-2)^n$. Plugging in $n = 0,1,2$ gives us a_0, a_1, a_2 yields $A = -\frac{1}{9}$, $B = \frac{1}{3}$, $C = \frac{1}{9}$, so $a_n = \frac{(-2)^n+3n-1}{9}$.

14. Since $1+\sqrt{2}$ and $1-\sqrt{2}$ are the roots of the polynomial t^2-2t-1, this is the characteristic polynomial of the sequence $a_n=(1+\sqrt{2})^n+(1-\sqrt{2})^n$, so the sequence satisfies the recurrence relation $a_n=2a_{n-1}+a_{n-2}$. Now since $a_0=a_1=2$, we can compute the sequence modulo 5. We get the sequence $2,2,1,4,4,2,3,3,4,1,1,3,2,2,\dots$ which has a period 12 and contains no zero. Hence no a_n is divisible by 5.

15. Take $n=1000!m+1$. Then $n^2-k=m^2\cdot 1000!-2m\cdot 1000!-(k-1)$ is obviously composite for $k=1,3,4,5,\dots,1000$. As for n^2-2, we can set m to make it divisible by e.g. $33^2-2=1087$ which is a prime, and it is enough to take $m\equiv 33 \pmod{1087}$.

    (Alternatively, instead of 1087, that is prime by chance, we could have taken any odd prime divisor of $1000!^2-2$.)

16. Taking a to be a multiplicative inverse of m modulo n we find a,b with $am-bn=1$.

    Take $c=a+n$ and $d=b+m$. Then also $cm-dn=1$. Moreover, $ad-bc=a(b+m)-b(a+n)=1$.

17. We need n such that all prime factors of n^2+1 are less than n. Take an arbitrary k. Then k^2+1 divides n^2+1 whenever $n\equiv \pm k \pmod{k^2+1}$, so let us choose $n=k^2+k+1$. Then $n^2+1=(k^2+1)(k^2+2k+2)$. Finally, taking k even we get $n^2+1=2\cdot(k^2+1)\cdot\frac{k^2+2k+2}{2}$, where the three factors are distinct if $k\geqslant 3$ and less than n, so each factor occurs in the product $n!=1\cdot 2\cdots n$. This secures that $n^2+1\mid n!$.

18. We can have three pairs (a_i,b_i), e.g. $(1,1),(1,2),(2,3)$. We cannot have four. Indeed, first of all, we cannot have a_i,b_i both even, so if $n>3$, there are two fractions with $a_i\equiv a_j$ and $b_i\equiv b_j \pmod 2$, but then $a_jb_i-a_ib_j$ is even.

19. We have $(a_{n+1}-3a_n)^2=8a_n^2+1$, i.e. $a_{n+1}^2-6a_na_{n+1}+a_n^2=1$. Subtracting the analogous equation for $n-1$, which is $a_{n-1}^2-6a_na_{n-1}+a_n^2=1$, we obtain $a_{n+1}^2-a_{n-1}^2=6a_n(a_{n+1}-a_{n-1})$. Canceling $a_{n+1}-a_{n-1}$ (which is obviously positive) yields $a_{n+1}=6a_n-a_{n-1}$. All terms are integers by induction.

20. Subtracting the original relation from the analogous shifted relation $a_{n+1}+a_{n+2}=a_{n+3}a_{n+4}-1000$ yields $a_{n+2}-a_n=a_{n+3}(a_{n+4}-a_{n+2})$. There are two cases.

    (i) $a_{n+3}=1$ for some n. Then $1-a_{n+1}=a_{n+4}(a_{n+5}-1)$, but since all terms are positive, this is only possible if $a_{n+1}=a_{n+5}=1$. By induction, $a_k=1$ for all $k\equiv n+1 \pmod 2$. Then $a_{k+1}=a_{k-1}+1001$ for all such k, so the sequence (a_n) has the form $(1,)\, a,1,a{+}1001,1,a{+}2002,1,\dots$.

    (ii) $a_{n+3}>1$ for all n. Then either $|a_{n+4}-a_{n+2}|<|a_{n+2}-a_n|$ for all n, which is impossible (infinitely decaying nonnegative integers!), or $a_{n+2}=a_n$ for all n. Thus the sequence has the form $a,b,a,b,a,b,\dots$, where a and b must satisfy $a+b=ab-1000$, i.e. $(a-1)(b-1)=1001$, giving 8 possibilities.

21. Write $a_n=\frac{x_n}{y_n}$, assuming the initial values $x_0=y_0=1$. Then $\frac{x_{n+1}}{y_{n+1}}=a_{n+1}=\frac{y_n+x_n}{3y_n+x_n}$, so we can define $x_{n+1}=x_n+y_n$ and $y_{n+1}=x_n+3y_n$.

    We will eliminate y_n. The first relation gives $y_n=x_{n+1}-x_n$ and consequently $y_{n+1}=x_{n+2}-x_{n+1}$, so the second relation becomes $x_{n+2}-x_{n+1}=x_n+3(x_{n+1}-x_n)$, i.e. $x_{n+2}-4x_{n+1}+2x_n=0$. From here we obtain $x_n=A(2+\sqrt{2})^n+B(2-\sqrt{2})^n$, and from the initial values $x_0 = 1$ and $x_1 = 2$ we find $A = B = \frac{1}{2}$, so $x_n = \frac{1}{2}[(2+\sqrt{2})^n + (2-\sqrt{2})^n]$. For y_n we get $y_n = x_{n+1} - x_n = \frac{1}{2}[(1+\sqrt{2})(2+\sqrt{2})^n + (1-\sqrt{2})(2-\sqrt{2})^{n+1}]$.

22. The characteristic polynomial of F_n is $t^2 - t - 1$ with the zeros $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$, so $F_n = A\phi^n + B\bar{\phi}^n$. From $F_0 = 0$ and $F_1 = 1$ we find $A = -B = \frac{1}{\sqrt{5}}$, so $F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$.

    Using $\phi\bar{\phi} = -1$ we obtain $5F_n^2 = \phi^{2n} + \bar{\phi}^{2n} - 2(-1)^n$, so $5F_n^2 + 4(-1)^n = (\phi^n + \bar{\phi}^n)^2$. Finally, note that $L_n = \phi^n + \bar{\phi}^n$ is an integer, because $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$.

23. Since there are only finitely many possible pairs (F_n, F_{n+1}) modulo a, some will repeat: e.g. $F_n \equiv F_m$ and $F_{n+1} \equiv F_{m+1} \pmod{a}$ for some $m < n$. Then by induction $F_{n-k} \equiv F_{m-k} \pmod{a}$, and in particular $F_{n-m} \equiv F_0 = 0 \pmod{a}$, so $a \mid F_{n-m}$.

24. First note that if F_k is the smallest Fibonacci number divisible by a, then $a \mid F_n$ if and only if $k \mid n$.

    It follows from above that $F_{\gcd(m,n)}$ divides both F_m and F_n. On the other hand, If any d divides F_m and F_n, then it must divide F_k for some k that divides both m and n, which completes the proof.

25. Obvious solutions are $F_1 = F_2 = 1$ and $F_3 = 2$, $F_6 = 8$ for powers of 2, or $F_4 = 3$ for powers of 3. We claim that there are no others.

    The Fibonacci sequence modulo 9 is $0, 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, \dots$, so $9 \mid F_n$ if and only if $12 \mid n$, but then F_n is also divisible by $F_{12} = 144$ and cannot be a power of 3.

    Similarly, modulo 16 we get $0, 1, 1, 2, 3, 5, 8, 13, 5, 2, 7, 9, 0, \dots$, so $16 \mid F_n$ if and only if $12 \mid n$. Again, then F_n cannot be a power of 2.

26. The sequence modulo 2 has period 2, so we have the basis of induction: $2 \mid a_n$ if and only if $2 \mid n$. To do the inductive step, it suffices to show that $\frac{a_{2n}}{a_n}$ is an integer divisible by 2 but not by 4.

    The explicit formula for a_n is $a_n = \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n]$, which gives us $\frac{a_{2n}}{a_n} = (1+\sqrt{2})^n + (1-\sqrt{2})^n$. When expanded by the binomial formula, this will give $\frac{a_{2n}}{a_n} = 2 + 2^2\binom{n}{2} + 2^3\binom{n}{4} + \cdots$, which is indeed $2 \pmod{4}$, as desired.

27. We can assume w.l.o.g. that $a_0 = x^2$ is a square. Then $a_0 \equiv 0, 1 \pmod{4}$, which implies $a_1 \equiv 3, 0 \pmod{4}$ and $a_2 \equiv 2, 3 \pmod{4}$. By induction it follows that $a_n \equiv 2, 3 \pmod{4}$ for every $n \geqslant 2$, so a_n is not a square if $n \geqslant 2$.

    It remains to verify that $a_1 = x^6 + 103$ is not a square. If to the contrary $a_1 = y^2$, then $(y - x^3)(y + x^3) = 103$ is a prime, implying that $x^3 = 51$, a contradiction.

28. Assume that $4ab - 1 \mid (a+b)^{15}$ and consider any prime divisor p of $4ab - 1$. We have $p \mid a + b$, so $b \equiv -a$ and $4ab - 1 \equiv -(4a^2 + 1) \pmod{p}$, so $p \mid (2a)^2 + 1$, and this is possible only if $p \equiv 1 \pmod{4}$. But not all prime divisors of $4ab - 1$ can be $1 \pmod{4}$, a contradiction.

29. (a) Suppose that $p = 3k + 2$. Since $p \mid x^3 - y^3$, we have $y^{p-2} = y^{3k} \equiv x^{3k} = x^{p-2}$, and since also $y^{p-1} \equiv x^{p-1}$ by Fermat's theorem, we deduce that $y \equiv x$. But then $p \mid x^2 + xy + y^2 \equiv 3x^2 \pmod p$, so $p \mid 3$, which is a contradiction.

(b) Taking $x + p$ instead of x if needed, we can assume that $x = 2y + 1$ is odd. Then $p \mid x^2 + 3 = 4(y^2 + y + 1)$, which by part (a) for $p > 3$ implies that $p \equiv 1 \pmod 3$.

30. Observe that $d = \gcd(an + 2, bn + 3)$ divides $a(bn + 3) - b(an + 2) = 3a - 2b$. If $n = |3a - 2b| \neq 0$, then $\gcd(an + 2, bn + 3)$ divides n, so it also divides 2 and 3 and therefore equals 1, a contradiction. Thus we must have $|3a - 2b| = 0$, i.e. $\frac{b}{a} = \frac{3}{2}$.

31. Denote $d = \gcd(ab + 1, bc + 1, ca + 1)$. Clearly, d is coprime to a, b, c, but $d \mid (ab + 1) - (ac + 1) = a(b - c)$, so $d \mid b - c$; similarly, $d \mid c - a$ and $d \mid a - b$. Assuming that $a < b < c$, this means that $b \geqslant a + d$ and $c \geqslant a + 2d$, so $a + b + c \geqslant a + 3d \geqslant 3d$, as desired.

32. We prove by induction on n that there is x such that $3^n \mid x^2 + 2$.

Base of induction is $n = 1$: then e.g. $x = 1$.

Inductive step: Assuming there is x with $3^n \mid x^2 + 2$, we will find y with $3^{n+1} \mid y^2 + 2$. We set $y = x + 3^n k$. Then $y^2 + 2 = x^2 + 2 + 2x \cdot 3^n k + 3^{2n} k^2$ is divisible by 3^{n+1} if $2x \cdot k \equiv -\frac{x^2+2}{3^n} \pmod 3$, and such a k obviously exists.

33. We will find n such that $p = n + 100$ is a prime. Since $p \mid (p - 1)! + 1$, we infer $p \mid (p - 1)! - (p - 100)! = (p - 100)! \cdot [(p-1)(p-2) \cdots (p-99) - 1] \equiv -99! - 1$, so it is enough to take for p any prime divisor of $99! + 1$ (then clearly $p > 100$, so $n > 0$).

34. Each of the numbers a, b, c divides $ab+bc+ca+1$. The numbers a, b, c must be pairwise coprime, so $abc \mid ab+bc+ca+1$, i.e. $F = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc}$ is an integer. Let $a \leqslant b \leqslant c$.

For $a \geqslant 3$ we have $F \leqslant \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{60} < 1$, so no solutions.

If $a = 2$, then $b \geqslant 5$ leads to $F < 1$, and $b = 4$ is impossible, so $b = 3$ and $c \mid 2 \cdot 3 + 1$, i.e. $(a, b, c) = (2, 3, 7)$.

If $a = 1$, then $b \mid c + 1$ and $c \mid b + 1$ and we get three more solutions: $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$.