— Combinatorics for L4 —

— January Camp, 2022 — Permutations and Residues —

Warm-up.

- Suppose that p is prime. Find all integers n such that the p numbers
$$1, \quad 1+n, \quad 1+n+n^2, \quad \ldots, \quad 1+n+n^2+\ldots+n^{p-1}$$
are all different modulo p.

1. Find all positive integers n with the following property: there exists a permutation $(a_1,a_2,\ldots,a_n)$ of the set $\{1,2,\ldots,n\}$ such that $a_1+a_2+\ldots+a_k$ is divisible by k for $k=1,2,\ldots,n$.

2. For every positive integer n determine the number of permutations $(a_1,a_2,\ldots,a_n)$ of the set $\{1,2,\ldots,n\}$ with the following property: $2(a_1+a_2+\ldots+a_k)$ is divisible by k for $k=1,2,\ldots,n$.

3. Find all positive integers n with the following property: there exists a permutation $(a_0,a_1,\ldots,a_{n-1})$ of the set $\{0,1,2,\ldots,n-1\}$ such that the residues of
$$a_0, \quad a_0+a_1, \quad a_0+a_1+a_2, \quad \ldots, \quad a_0+a_1+a_2+\ldots+a_{n-1}$$
modulo n form a permutation of this set as well.

4. Let n be a positive integer. Prove that that the residues of
$$0, \quad 0+1, \quad 0+1+2, \quad \ldots, \quad 0+1+2+\ldots+(n-1)$$
modulo n form a permutation of the set $\{0,1,2,\ldots,n-1\}$ if and only if n is a power of two.

5. On a circle of length 2^n $(n\geq 2)$ given are points P_0, P_1, $\ldots$, P_{2^n-1} which are vertices of a regular 2^n-gon, denoted in such a way that the length of the arc $P_{k-1}P_k$ (measured clockwise) is equal to k for $k=1,2,\ldots,2^n-1$. Let
$$\mathcal{E}=\{P_k : 2\,|\,k\} \quad \text{and} \quad \mathcal{F}=\{P_k : k<2^{n-1}\}.$$
Prove that sets $\mathcal{E}$ and $\mathcal{F}$ are congruent (treated as plane figures).