PROBLES $\boxed{\text{}}$

$a, b > 0 \in \mathbb{Z}$, $ab-2 \mid a^2+b^2+ab$.

Find all values of $\dfrac{a^2+b^2+ab}{ab-2}$

(1)

Proof $\dfrac{a^2+b^2+ab}{ab-2} = k$, $\Rightarrow$ $\dfrac{a^2+b^2+\cancel{4}2}{ab-2} = k-1$

NLOG $a \geq b$. if $a=b$ then

$$\frac{2a^2+\cancel{\ }2}{a^2-2} = \frac{2a^2-4}{a^2-2} + \frac{6}{a^2-2} = 2+\frac{6}{a^2-2} \in \mathbb{Z}$$ so

$a^2-2 = +\cancel{1}, -1, 2, -\cancel{2}, \cancel{3}, -\cancel{3}, \cancel{6}, -\cancel{6}$

$\underline{a=1 \quad a=2}$

$k-1 = \dfrac{2a^2+2}{a^2-2} = \dfrac{2\cdot 1^2+2}{1^2-2} = \dfrac{4}{-1} = -4 \Rightarrow \boxed{k=-3}$

$k-1 = \dfrac{2\cdot 2^2+2}{2^2-2} = \dfrac{10}{2} = 5 \Rightarrow \boxed{k=6}$

WLOG : $\boxed{a > b}$

$\frac{a^2+b^2+2}{ab-2} = k-1$ so $a^2 - a((k-1)b) + b^2+2k = 0$

$a^2+b^2+\cancel{2} = abk - ab - 2k + \cancel{2}$

$x_2 = (k-1)b - a = \frac{b^2+2k}{a} > 0$. $\in \mathbb{Z}$ (2)

(a,b) – minimal sum.

$x_2+b \geq a+b \rightsquigarrow x_2 \geq a$.

$(k-1)b - a \geq a \Rightarrow (k-1) \geq \frac{2a}{b}$.

Bt $\frac{a^2+b^2+2}{ab-2} = k-1 \geq \frac{2a}{b} \iff a^2b+b^3+2b \geq 2a^2b-4a$

$\iff b^3+2b \geq a(ab-4)$, but $\underline{a > b+1}$, so

$\cancel{b^3}+2b \geq (b+1)(\cancel{a}(b+1)\cdot b-4) =$

$= (b+1)^2 b - 4(b+1) = \cancel{b^3}+2b^2+b-4b-4$

$\boxed{2b^2-5b-4 \leq 0}$ $\boxed{b=1,2,3,}$

1°) $b=1$

$\mathbb{Z} \ni k-1 = \dfrac{a^2+3}{a-2} = \dfrac{a^2-4}{a-2} + \dfrac{7}{a-2} = a+2+\dfrac{7}{a-2}$ so

$a-2 \in \{-1, 1, 7, -7\} \Rightarrow \underline{a = 1, 3, 9}$

$k-1 = \dfrac{1^2+3}{1-2} = -4 \Rightarrow \underline{k=-3}$

3

$k-1 = \dfrac{3^2+3}{3-2} = 12 \Rightarrow \boxed{k=13}$

$k-1 = \dfrac{9^2+3}{7} = \dfrac{84}{7} = 12 \Rightarrow \underline{k=13}$

2°) $\underline{b=2}$

$k-1 = \dfrac{a^2+6}{2a-2} = \overset{a=2a_1}{\dfrac{4a_1^2+6}{4a_1-2}} = \dfrac{2a_1^2+3}{2a_1-1} = \dfrac{2a_1^2-a_1+a_1+3}{2a_1-1} =$

$= a_1 + \dfrac{a_1+3}{2a_1-1}$

$a_1+3 \geq 2a_1-1$

$a_1 \leq 4$

$\underline{a_1 = 1, 4} \Rightarrow a = 2, 8$

$k-1 = \dfrac{2^2+6}{2\cdot 2-2} = \dfrac{10}{2} = 5 \Rightarrow \underline{k=6}$

$$\frac{8^2+6}{2\cdot 8-2} = \frac{64+6}{14} = \frac{70}{14} = 5 \Rightarrow \underline{k=6}$$

4

$$\frac{1+11}{1} = 12 \Rightarrow \underline{k=13}$$

3) $b=3$

$$\frac{a^2+9+2}{3b-2} = k-1 \qquad \frac{35^2+11}{3\cdot 35-2} = \frac{1236}{103} = 12 \Rightarrow k=13$$

$$\frac{a^2+11}{3a-2} = \qquad 3a-2 \mid 3a^2+33 = (3a-2)\cdot a + 33+2a$$

so $3a-2 \mid 33+2a$

$(3, 3a-2)=1$

$2a+33 \geq 3a-2$

$a \leq 35$

$3a-2 \mid 99+6a =$

$= 2\cdot(3a-2)+103$

$\boxed{3a-2 \mid 103}$ =

$3a-2 = \cancel{-1}, \cancel{1}, 103, \cancel{-103}$

$a=1 \qquad a=\frac{105}{3} = 35$

$$\boxed{k=-3, 6, 13}$$

6 $\underline{x^2+5=y^3}$

if x is od, y is even so $x^2+5 \equiv 2 \mod 4$ ↯

$y^3 \equiv 0 \mod 4$

$\underline{\text{So } x \text{ is even}}$, $\underline{y^3 \equiv 1 \mod 4}$ $\Rightarrow$ $\boxed{y \equiv 1 \mod 4}$

$$x^2+4 = y^3-1 = (y-1)\underbrace{(y^2+y+1)}$$

$$x^2+2^2 \qquad \equiv 1^2+1+1 \mod 4$$

$$\boxed{3 \mod 4}$$ ↯

7 $x^2+3y \qquad y^2+3x$

$\cancel{x \geq y}$ $y \leq x$

$\cancel{y^2 \leq x^2 \leq x^2+3y}$

$$x^2 < x^2+3y \leq x^2+3x < \underline{x^2+4x+4}$$

$$x^2 \qquad\qquad (x+2)^2$$

$$\underline{\cancel{x^2}+3y=(x+1)^2} = \cancel{x^2}+2x+1$$

$$3y=2x+1$$

6

$3y = 2x+1$

$y^2 + 3x$ is square $\rightsquigarrow$ $4y^2 + 12x$ is square

$4y^2 + (3y-1)\cdot 6 = 4y^2 + 18y - 6$

$4y^2 + 18y - 6 = \ell^2$

1, 1	
11, 16	
16, 11	

For $y \geq 3$

$(2y+\cancel{5})^2$

$$\boxed{4y^2 + 12y + 9 < 4y^2 + 18y - 6 < (2y+5)^2}$$

$\cancel{y=0}$

$\boxed{y<3}$

$(2y+3)^2$

$4y^2 + 12y + 9$

$6y > 15$

$4y^2 + 18y - 6 = (2y+4)^2$

$= 4y^2 + 16y + 16$

$2y = 22$

$y = 11 \rightsquigarrow x = 16$

$y=1 \rightsquigarrow 4 + 18 - 6 = 16 = 4^2 \rightsquigarrow x = 1$

$y = 2 \rightsquigarrow 4\cdot 2^2 + 12\cdot 2 + 9 = 16 + 24 + 9 = 49 = 7^2$

$y = 7$

$3\cdot 2 \neq 2x+1$

$\boxed{\text{HM 1}}$

$\underline{a+b+c+d=1}$ Prove

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{1}{2}$$

$$\sum\frac{a^2}{a+b}\geq\frac{(a+b+c+d)^2}{2(a+b+c+d)}=\frac{a+b+c+d}{2}=\frac{1}{2}$$

$\boxed{\text{HM 2}}$

p.e. $p(x)=a_n x^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$, $a_i>0$.

$P(1)\geq\frac{1}{P(1)}$. $P(1)^2\geq1$.

Prove

$$P\left(\frac{1}{x}\right)\geq\frac{1}{P(x)} \quad \text{for any } x.$$

$$P\left(\frac{1}{x}\right)\cdot P(x)=\left(a_n\left(\frac{1}{x}\right)^n+a_{n-1}\left(\frac{1}{x}\right)^{n-1}+\dots a_1\cdot\frac{1}{x}+a_0\right)\left(a_nx^n+\dots+a_1x+a_0\right)$$

$$\geq\left(\sqrt{a_n\left(\frac{1}{x}\right)^n a_nx^n}+\sqrt{a_{n-1}\left(\frac{1}{x}\right)^{n-1}a_{n-1}x^{n-1}}+\dots+\sqrt{a_0^2}\right)^2=$$

$$=(a_0+a_1+\dots+a_n)^2=P(1)^2\geq\underline{1}$$

$$\sqrt{2(a^2+b^2)} + \sqrt{2(b^2+c^2)} + \sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2 + 3(b+c)^2 + 3(c+a)^2}$$

square both sides ...

$$4a^2+4b^2+4c^2 + \sum_{cyc} \cancel{4}\sqrt{(a^2+b^2)(b^2+c^2)} \geq 6(a^2+b^2+c^2) + 6(ab+bc+ca)$$

$$\boxed{\sum_{cyc} 2\sqrt{(a^2+b^2)(b^2+c^2)} \geq a^2+b^2+c^2 + 3(ab+bc+ca)}$$

$$\sqrt{(a^2+b^2)(b^2+c^2)} \geq \boxed{\ldots}$$

$$\boxed{\sqrt{2(a^2+b^2)} \geq a+b}$$

$$\sqrt{2(b^2+c^2)} \geq b+c$$

$$\sum_{cyc} 2\sqrt{(a^2+b^2)(b^2+c^2)} \geq \sum_{cyc} (a+b)(b+c) = \sum_{cyc} ab+ac+b^2+bc =$$

$$= a^2+b^2+c^2 + 3(ab+bc+ca)$$

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2+\left(d+\frac{1}{d}\right)^2 \geq 25$$ (1)

$$a+b+c+d=8$$

$$\left(\sum \left(a+\frac{1}{a}\right)^2\right)\underbrace{\left(1^2+1^2+1^2+1^2\right)}_{4} \overset{C-S}{\geq} \left(\sum_{cyc}\left(a+\frac{1}{a}\right)\right)^2 =$$

$$= \left(\underbrace{a+b+c+d}_{8}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 \geq (8+2)^2 = 100$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geq \frac{(1+1+1+1)^2}{a+b+c+d} = \frac{4^2}{8} = 2$$

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab} \geq \frac{9}{10} \qquad \underline{a+b+c=1}$$

(12)

$$\| $$

$$\underbrace{\frac{a^2}{a+abc}+\frac{b^2}{b+abc}+\frac{c^2}{c+abc}}\geq \frac{9}{10} \qquad \text{WANT}$$

$$\geq \frac{(a+b+c)^2}{(a+b+c)+3abc} = \frac{1}{1+3abc} \geq \frac{9}{10}$$

$$\frac{1}{1+3abc} \geq \frac{9}{10} \iff 10 \geq 9+27abc$$

$$abc \leq \frac{1}{27}$$

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = \left(\frac{1}{3^3}\right) = \frac{1}{27}$$

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq a+b+c$$

(13)

Next $\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1} \geq 1$ if $abc=1$.

$$\frac{(a^2+bc)^2}{(a^2+bc)(b+c)}+\frac{(b^2+ca)^2}{(b^2+ca)(c+a)}+\frac{(c^2+ab)^2}{(a+b)(c^2+ab)} \geq$$

$$\geq \frac{(a^2+b^2+c^2+ab+bc+ca)^2}{\sum_{cyc} a^2b+a^2c+b^2c+bc^2} = \frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2(a^2b+ba^2+ac^2+a^2c+bc^2+b^2c)}$$

$$(a^2+b^2+c^2+ab+bc+ca)^2 \geq 2(a+b+c)(a^2b+ab^2+bc^2+b^2c+ac^2+a^2c) \qquad (14)$$

b^2c^2

$$LHS = a^4+b^4+c^4+\cancel{a^2b^2}+\cancel{b^2c^2}+\cancel{c^2a^2}+2a^2b^2+2a^2c^2+\cancel{2a^3b}+\cancel{2a^2bc}+\cancel{2a^3c}+$$

$$+\cancel{2b^3a}+\cancel{2b^3c}+2b^2ca+2c^2ab+\cancel{2c^3b}+\cancel{2c^3a}+\cancel{2ab^2c}+2abc^2+\cancel{2bc^2a}$$

$$RHS = \cancel{a^3b}+\cancel{a^2b^2}+\cancel{ab^2c}+\cancel{ab^2c}+\cancel{a^2c^2}+\cancel{a^3c}+\cancel{a^2b^2}+\cancel{b^2c^2}+\cancel{ab^3}+\cancel{b^3c}+$$

$$+\cancel{abc^2}+\cancel{ba^2c}+\cancel{ca^2b}+\cancel{cab^2}+\cancel{bc^3}+\cancel{b^2c^2}+\cancel{ac^3}+\cancel{a^2c^2}$$

$$\boxed{a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2} \quad \text{true}$$

$$x=a^2 \quad y=b^2 \quad z=c^2$$

$$x^2+y^2+z^2 \geq xy+yz+zx$$

Rearrangement Inequality

$\boxed{\begin{array}{l} x_1 \le x_2 \le \dots \le x_n \\ y_1 \le y_2 \le \dots \le y_n \end{array}}$ and $\boxed{\text{any}}$ reals

then for any permutation σ of $\{1, 2, \dots, n\}$ we have

$$x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1 \le x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \dots + x_n y_{\sigma(n)} \le x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

opp

$$a^2 \le b^2 \le c^2$$

$s = a+b+c$

$$\frac{1}{b+c} \le \frac{1}{a+c} \le \frac{1}{a+b} \quad \text{so}$$

$$\frac{1}{s-a} \le \frac{1}{s-b}$$

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \ge \frac{b^2}{b+c} + \frac{c^2}{c+a} + \frac{a^2}{a+b}$$

$$\frac{a^2-b^2}{b+c} + \frac{b^2-c^2}{c+a} + \frac{c^2-a^2}{a+b} \ge 0$$

$+b \quad +c \quad +a$

Chebyshev's Inequality

$$a_1 \leq a_2 \leq \dots \leq a_n \qquad \text{any reals}$$
$$b_1 \leq b_2 \leq \dots \leq b_n$$

then: $$\frac{a_1b_1 + a_2b_2 + \dots + a_nb_n}{n} \geq \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)\left(\frac{b_1 + b_2 + \dots + b_n}{n}\right)$$

(trivial application of Chebyshev)

From Chebyshev

$QM \geq AM$:

$$\sqrt{\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \dots + x_n}{n}$$

$$\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} \geq \frac{x_1 + x_2 + \dots + x_n}{n} \cdot \frac{x_1 + x_2 + \dots + x_n}{n}.$$

$$x_1 \leq x_2 \leq \dots \leq x_n$$
$$x_1 \leq x_2 \leq \dots \leq x_n$$

$$\frac{a^2-b^2}{b+c} + b = \frac{a^2-b^2+b^2+bc}{b+c} = \frac{a^2+bc}{b+c}$$

$a, b, c > 0$.

Ex. $$\frac{a^3}{b^2+c^2} + \frac{b^3}{a^2+c^2} + \frac{c^3}{a^2+b^2} \geq \frac{a+b+c}{2}$$ (16)

S — Sym

$$a \leq b \leq c$$

$$\frac{a^2}{S-a^2} = \frac{a^2}{b^2+c^2} \leq \frac{b^2}{c^2+a^2} \leq \frac{c^2}{a^2+b^2}$$

so By Chebyshev

$$\frac{a^3}{b^2+c^2} + \frac{b^3}{a^2+c^2} + \frac{c^3}{a^2+b^2} \geq \frac{a+b+c}{3}\left(\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2}\right) \geq$$

$$\geq \frac{a+b+c}{2}$$ so we need

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \geq \frac{3}{2}$$

Nesbitt inequality

$$\boxed{\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}} \quad (*)$$

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$x=a^2, \quad y=b^2, \quad z=c^2$

$$\frac{a^2+1}{b^2+c^2}+\frac{b^2+1}{c^2+a^2}+\frac{c^2+1}{a^2+b^2}\geq\frac{3}{2}+3$$

$$a^2+b^2+c^2\left(\frac{1}{b^2+c^2}+\frac{1}{c^2+b^2}+\frac{1}{a^2+b^2}\right)\geq\frac{9}{2}$$

$$\geq\frac{(1+1+1)^2}{2(a^2+b^2+c^2)} \quad \boxed{C-S}$$

$$(x+y+z)^2 \geq 3(xy+yz+zx)$$

$$(*) \Leftrightarrow \frac{x^2}{xy+xz}+\frac{y^2}{zy+xy}+\frac{z^2}{xz+zy}\geq\frac{3}{2}$$

$$\overset{C-S}{\geq}\frac{(x+y+z)^2}{2(xy+yz+zx)}\geq\frac{3}{2}$$

bcoz.

Holder Inequality

$a_1, a_2, \dots, a_n \in \mathbb{R}_+$ and take $b_1, b_2, \dots, b_n$

$p, q > 1$ reals such that $\frac{1}{p} + \frac{1}{q} = 1$

Then:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.$$

Equality holds iff $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \dots = \frac{a_n^p}{b_n^q}$.

Corollary: if $p = 2 = q$ then $\frac{1}{2} + \frac{1}{2} = 1$, so we have C-S.

Generalised Holder

$$(a_{11} + a_{12} + \dots + a_{1n})(a_{21} + a_{22} + \dots + a_{2n}) \dots (a_{k1} + \dots + a_{kn}) \geq$$

$$\geq \left(\sqrt[k]{a_{11} a_{21} \dots a_{k1}} + \sqrt[k]{a_{12} a_{22} \dots a_{k2}} + \dots \sqrt[k]{a_{1n} a_{2n} \dots a_{kn}} \right)^k$$

$$\left(a_{11}^{p_1} + a_{12}^{p_1} + \dots + a_{1n}^{p_1}\right)^{\frac{1}{p_1}} \left(a_{21}^{p_2} + a_{22}^{p_2} + \dots + a_{2n}^{p_2}\right)^{\frac{1}{p_2}} \dots \left(a_{k1}^{p_k} + a_{k2}^{p_k} + \dots + a_{kn}^{p_k}\right)^{\frac{1}{p_k}}$$

$$\geq \left(a_{11} a_{21} \dots a_{k1} + a_{12} a_{22} \dots a_{k2} + a_{1n} a_{2n} \dots a_{kn} \right)$$

if $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} = 1$.

Ex 1 $\underset{\substack{\| \\ (1+1+1)}}{3}(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq (ab + bc + ca)^3$.