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$$f(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1}x + 1$$

with nonnegative coeffs and with n real roots

- $f(2) \geq 3^n$
- $\boxed{f(x) \geq (x+1)^n \quad \forall_x \geq 0}$
- $a_k \geq \binom{n}{k} \quad k = 1, 2, \dots, n-1,$

$a_k = ?$

$$f(x) = (x+x_1)(x+x_2) \cdot \dots \cdot (x+x_n)$$

for some $-x_1, -x_2, \dots, -x_n \geq 0$ $\boxed{\text{roots}}$

All roots are negative Why?

$$f(2) = \underbrace{(1+1+x_1)}(1+1+x_2) \cdot \dots \cdot (1+1+x_n)$$

$$\geq 3\sqrt[3]{x_1} \cdot 3\sqrt[3]{x_2} \cdot \dots \cdot 3\sqrt[3]{x_n}$$

$$= 3^n \sqrt[3]{x_1 x_2 \dots x_n} \geq 3^n$$

$3^n$

∎

Hölder inequality

$$\left.\begin{matrix} a_{11}, a_{12}, \dots, a_{1n} \\ a_{21}, a_{22}, \dots, a_{2n} \\ \vdots \\ a_{k1}, a_{k2}, \dots, a_{kn} \end{matrix}\right. \geq 0$$

Then

$$(a_{11} + a_{12} + \dots + a_{1n})(a_{21} + a_{22} + \dots + a_{2n}) \cdot \dots \cdot$$
$$(a_{k1} + \dots a_{kn}) \geq \left( \sqrt[k]{a_{11} a_{21} a_{31} \dots a_{k1}} + \right.$$
$$\left. + \sqrt[k]{a_{12} a_{22} a_{32} \dots a_{k2}} + \dots + \sqrt[k]{a_{1n} a_{2n} \dots a_{kn}} \right)^{k}$$

$$(x + x_1)(x + x_2) \dots (x + x_n) \geq$$
$$\geq \left( \sqrt[n]{x \cdot x \cdot \dots \cdot x} + \sqrt[n]{x_1 x_2 \dots x_n} \right)^n =$$
$$= (x + 1)^n .$$

# Full generalization of Hölder:

Hölder inequality

$a_{11}, a_{12}, \dots, a_{1n}$

$a_{21}, a_{22}, \dots, a_{2n}$

$\vdots$

$a_{k1}, a_{k2}, \dots, a_{kn}$

$\geq 0$

$p_1, p_2 \dots p_k > 0$

$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} = 1$

Then

$$\left(a_{11}^{p_1} + a_{12}^{p_1} + \dots + a_{1n}^{p_1}\right)^{\frac{1}{p_1}} \left(a_{21}^{p_2} + a_{22}^{p_2} + \dots + a_{2n}^{p_2}\right)^{\frac{1}{p_2}} \cdot \dots \cdot$$

$$\left(a_{k1}^{p_k} + \dots a_{kn}^{p_k}\right)^{\frac{1}{p_k}} \geq a_{11}a_{21}a_{31}\dots a_{k1} +$$

$$+ a_{12}a_{22}a_{32}\dots a_{k2} + \dots + a_{1n}a_{2n}\dots a_{kn}$$

$p_i = n$

$a, b, c > 0$

Ex

$$3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq (ab + bc + ca)^3$$

if $k=2$ $\quad p_1, p_2 = 2$

$$\left(a_{11}^2 + a_{12}^2 + \dots + a_{1n}^2\right)\left(a_{21}^2 + a_{22}^2 + \dots + a_{2n}^2\right) \geq$$

$$\left(a_{11}a_{21} + a_{12}a_{22} + \dots + a_{1n}a_{2n}\right)^2$$

Cauchy-Schwarz inequality.

$$\left(a^2b + b^2c + c^2d + d^2a\right) \leq \sqrt[3]{2\left(a^3+b^3+c^3+d^3\right) \cdot \sqrt{a^2b^2+b^2c^2+c^2d^2+d^2e^2}}$$

---

Hölder inequality

$a_{11}, a_{12}, \dots, a_{1n}$

$a_{21}, a_{22}, \dots, a_{2n}$ $\qquad \geq 0$

$\vdots$

$a_{k1}, a_{k2}, \dots, a_{kn}$

$p_1, p_2 \dots p_k \geq 0$

$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_k} = 1$

Then

$$\left(a_{11}^{p_1} + a_{12}^{p_1} + \dots + a_{1n}^{p_1}\right)^{\frac{1}{p_1}}\left(a_{21}^{p_2} + a_{22}^{p_2} + \dots + a_{2n}^{p_2}\right)^{\frac{1}{p_2}} \cdot \dots \cdot$$

$$\left(a_{k1}^{p_k} + \dots a_{kn}^{p_k}\right)^{\frac{1}{p_k}} \geq a_{11}a_{21}a_{31}\dots a_{k1} +$$

$$+ a_{12}a_{22}a_{32}\dots a_{k2} + \dots + a_{1n}a_{2n}\dots a_{kn}$$

$a, b, c \in \mathbb{R}_+ \qquad ab + bc + ca \geq 3$

<u>Prove:</u>

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}\ .$$

coeff $a_k$ = sum of all possible products of numbers among $x_1\, x_2 \dots x_n$ taken $k$-times there are

$\binom{n}{k}$ such products  By AM-GM

$$a_k \geq \binom{n}{k} \sqrt[\binom{n}{k}]{\left( \prod_{i_1 \dots i_k} a_{i_1} \dots a_{i_k} \right)}$$

$$= \binom{n}{k} \sqrt[\binom{n}{k}]{\left( \underset{\overset{\shortparallel}{1}}{x_1\, x_2 \dots x_n} \right)^{\binom{n-1}{k-1}}} = \binom{n}{k}$$

$a_k =$

$$a_k = s_1 + \dots + s_{\binom{n}{k}} \geq \binom{n}{k} \sqrt[\binom{n}{k}]{s_1 \dots s_{\binom{n}{k}}}$$

$$x^3 + a_1 x^2 + a_2 x + 1$$

$$a_2 = x_1 x_2 + x_2 x_3 + x_3 \cdot 1$$

$$a_2 \geq \binom{3}{2} \sqrt[\binom{3}{2}]{(x_1 \dots x_3)^{2 = \binom{3-1}{2-1}}} = 1 \cdot \binom{3}{2}$$

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x + 1$$

$$\|$$

$$(x + x_1)(x + x_2) \cdot (x + x_3)(x + x_4)$$

$$a_2 = \underbrace{x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4}_{\binom{4}{2}^{n}_{k} = \frac{4!}{2! \cdot 2!} = \frac{1 \cdot 2 \cdot 3 \cdot \cancel{4}}{\cancel{2} \cdot \cancel{2}} = 6} \geq$$

$$\geq 6 \sqrt[6]{x_1 x_2 \, x_1 x_3 \, x_1 x_4 \, x_2 x_3 \, x_2 x_4 \, x_3 x_4}$$

$$= 6 \sqrt[6]{x_1^3 x_2^3 x_3^3 x_4^3} = \boxed{6}$$

$$\square$$

$$3 = \binom{4-1}{2-1}$$