Lecture 1

Winter Camp

Number Theory

2 Jan 2021

Level 2

Check if there exists positive integers a, b and prime number p such that $a^3 - b^3 = 4p^2$

$a^3 - b^3 \equiv 0 \pmod 2 \Rightarrow a \equiv b \pmod 2$

<u>Case 1</u> $a \equiv b \equiv 0 \pmod 2$

$\Rightarrow 8 \mid a^3 - b^3 \Rightarrow 8 \mid 4p^2 \Rightarrow 2 \mid p^2 \Rightarrow p = 2$

$\Rightarrow a^3 - b^3 = 16$

$a = 2x, \; b = 2y$

$\Rightarrow x^3 - y^3 = 2$

$\Rightarrow (x-y)(x^2 + xy + y^2) = 2$

$x^2 + xy + y^2 \geq 3$ so L.H.S $\geq 3 \Rightarrow$ no solutions

Check if there exists positive integers a, b and prime number p such that $a^3 - b^3 = 4p^2$

Case 2 $\quad a \equiv b \equiv 1 \pmod 2$

$$(a-b)(a^2+ab+b^2) = 4p^2$$

since a^2+ab+b^2 is odd, $4 \mid a-b$

it's clear that $a-b \neq 4p^2$ since $a^2+ab+b^2 \geq 3$

- if $a-b = 4p$:

  but $a^2+ab+b^2 > a > a-b$

  $\Rightarrow \quad p > 4p \Rightarrow \Leftarrow$!!

- if $a-b = 4$:

  $a^2+ab+b^2 = p^2$, $\quad a \equiv b \pmod 4$

  $a^2 \equiv b^2 \equiv p^2 \equiv 1 \pmod 4$

  $3a^2 \equiv p^2 \pmod 4$,

  $\Rightarrow \quad 3 \equiv 1 \pmod 4 \Rightarrow \Leftarrow$!!

Find all positive integers n such that there exists a prime number p, such that $p^n - (p-1)^n$ is a power of 3.

$*$ n is even

proof:

$$p^n - (p-1)^n \equiv 0 \pmod 3$$

but $3 \nmid p, (p-1)$ at the same time $\Rightarrow p \not\equiv 0, 1 \pmod 3$

$\Longrightarrow$ $p \equiv 2 \pmod 3 \Rightarrow 2^n \equiv 1 \pmod 3$

$\Rightarrow n$ is even

$n = 2m$

$$p^{2m} - (p-1)^{2m} = 3^a$$

$$(p^m - (p-1)^m)(p^m + (p-1)^m) = 3^a$$

What's $\gcd(a-b, a+b)$ if $\gcd(a,b) = 1$?

it's 1 or 2

Find all positive integers n such that there exists a prime number p, such that $p^n - (p-1)^n$ is a power of 3.

$\Rightarrow \quad \gcd\left(p^m - (p-1)^m,\ p^m + (p-1)^m\right) = 1$

but both terms are powers of 3

$$\begin{cases} p^m - (p-1)^m = 1 \\ p^m + (p-1)^m = 3^a \end{cases}$$

$\longleftarrow$ can't be much bigger ?

$$p^m - (p-1)^m = (p-(p-1))\,\underbrace{\left(p^{m-1} + p^{m-2}(p-1) + \cdots + (p-1)^{m-1}\right)}_{\geq m}$$

$\Rightarrow \quad m = 1 \quad \Rightarrow \quad n = 2$

$p^2 - (p-1)^2 = 2p - 1$ which is a power of 3 when $p = 5$

$\Rightarrow n = 2$ is the only solution

Find all positive integers n such that there exists a prime number p, such that $p^n - (p-1)^n$ is a power of 3.

$p^m - (p-1)^m$ is increasing function of m

$$p^{m+1} - (p-1)^{m+1} > p^m - (p-1)^m$$

Prove that any positive integer less than $n!$ can be represented as a sum of no more than n positive integer divisors of $n!$.

We will use induction

Base Case: $n = 3$. trivial

Induction Hypothesis: same as the problem's statement

and assume that it holds for $n-1$

let k any number less than $n!$

$$k = mn + q \quad , 0 \leq q \leq n-1$$

$* \; m \leq (n-1)!$ because $mn \leq k \leq n! \Rightarrow m \leq (n-1)!$

since $m \leq (n-1)!$ we can use the induction hypothesis:

Prove that any positive integer less than $n!$ can be represented as a sum of no more than n positive integer divisors of $n!$.

$m = d_1 + d_2 + \cdots + d_u$ such that $d_i \mid (n-1)!$

and $u \leq n-1$

$\Rightarrow \quad k = n(d_1 + d_2 + \cdots + d_u) + q$

$k = nd_1 + nd_2 + \cdots + nd_u + q$

① if $q = 0$, $d_i' = n d_i \Rightarrow d_i' \mid n!$

$d_i \mid (n-1)!$

② if $q \geq 1$, $d_i' = n d_i$, $d_i \mid (n-1)! \Rightarrow d_i' \mid n!$

$k = \underbrace{d_1' + d_2' + \cdots + d_u' + q}_{u+1 \text{ divisors of } n!}$ but $q < n \Rightarrow q \mid n!$

but $u \leq n-1 \Rightarrow u+1 \leq n$

Let m and n be integers greater than 1 such that $\gcd(m, n-1) = \gcd(m, n) = 1$. Prove that the first $m-1$ terms of the sequence $n_1, n_2, \ldots,$ where $n_1 = mn + 1$ and $n_{k+1} = n \cdot n_k + 1$, $k \geq 1$, cannot all be primes.

Let n be a given integer with $n \geq 4$. For a positive integer m, let S_m denote the set $\{m, m+1, \ldots, m+n-1\}$. Determine the minimum value of $f(n)$ such that every $f(n)$-element subset of S_m (for every m) contains at least three pairwise relatively prime elements.