Intensive Training

Geometry

Day 5

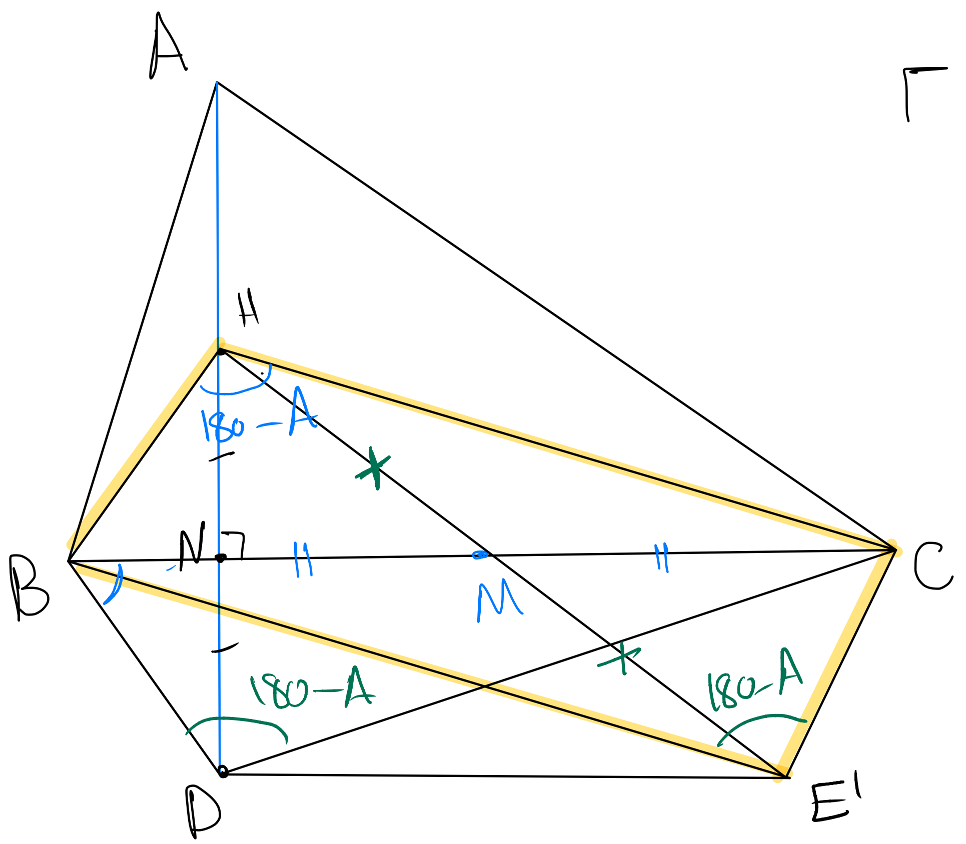
3 April 2021

Includes solutions for:

JBMO 2017 – P3

BMO 2018 – P1

Lemma 1: H, M, and E are collinear



Let E' be the reflection of B over M.

$\Rightarrow BHCE'$ is a paralelogram $\Rightarrow \angle BHC = \angle BE'C$

$E' \in (ABC) \Leftrightarrow ABE'C$ cyclic $\Leftrightarrow \angle BE'C = 180 - A$

$\Leftrightarrow \angle BHC = 180 - A$ ✓

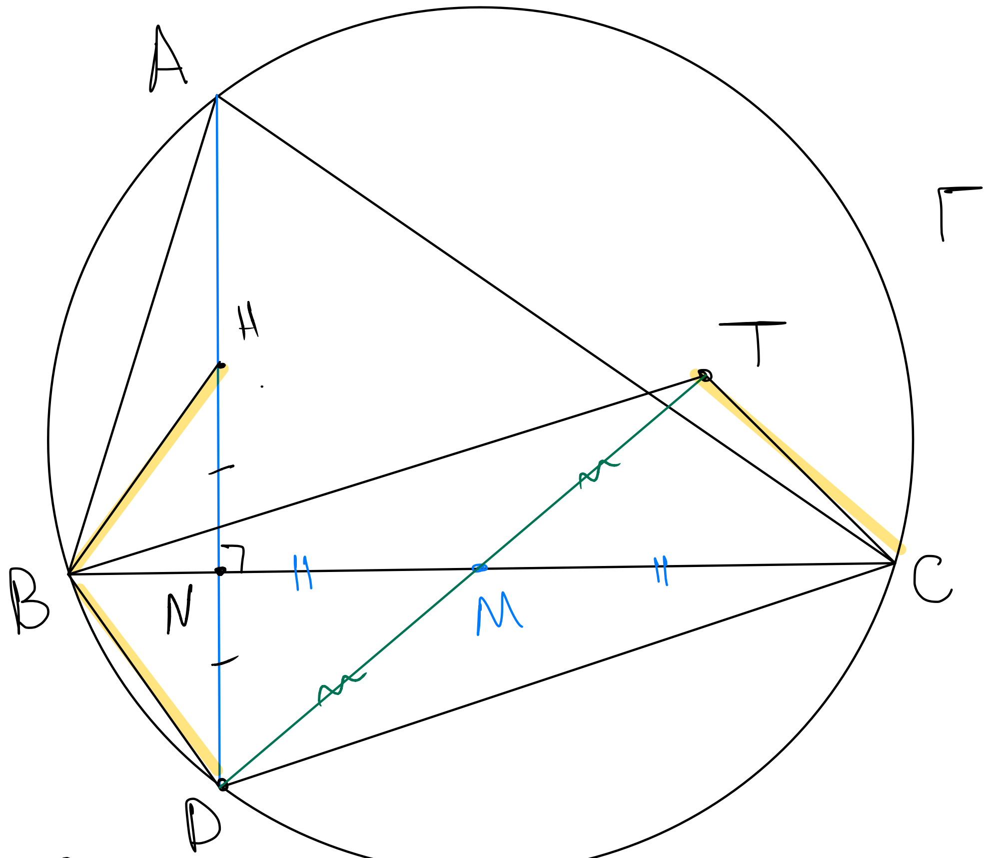
N midpoint HB, M midpoint HE' $\Big\}$ $NM \parallel DE'$ $\Rightarrow DE' \perp AD$

$\Rightarrow \angle ADE' = 90$

$\Rightarrow O \in AE' \Rightarrow E' = E$

□

Lemma 2: H is the reflection of D over BC and $T \in (BHC)$. Also, (BHC) is the reflection of Γ over BC.



Let $N = AD \cap BC$.

* H is the reflection of D over BC $\Rightarrow$ $BH = BD$
* T is the reflection of D over M $\Rightarrow$ $BD = CT$

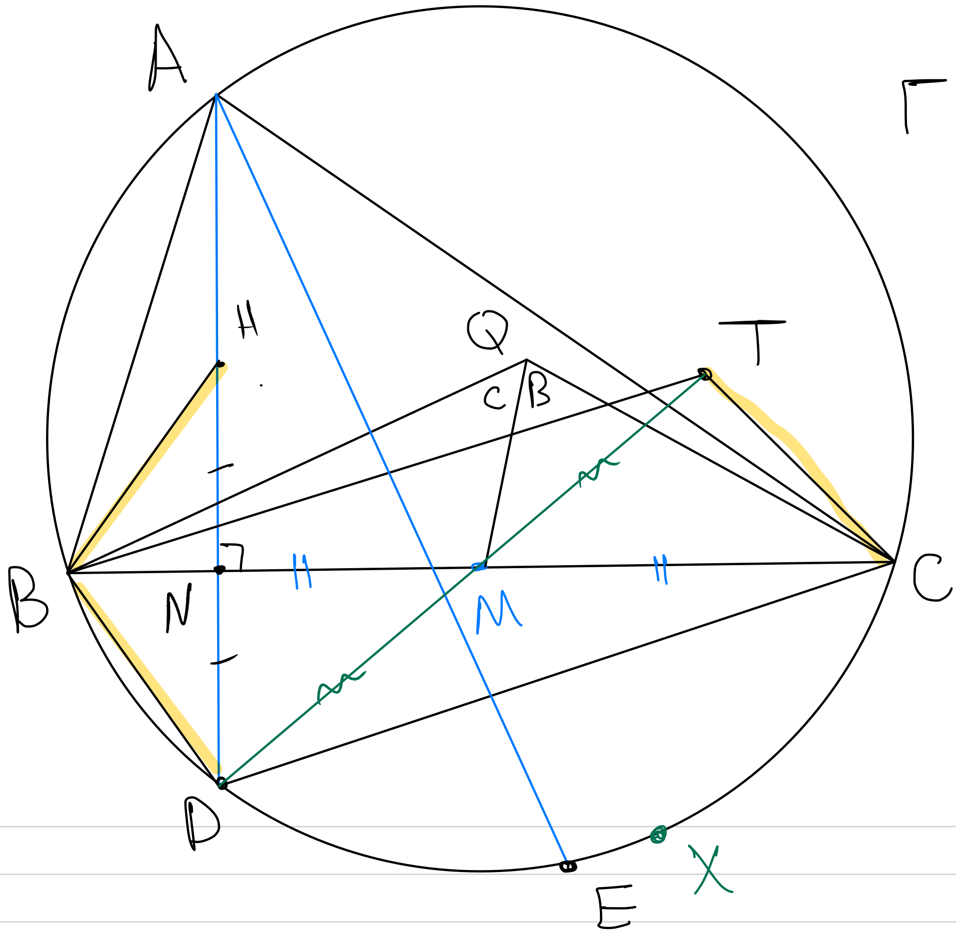
$\Rightarrow MN \parallel HT \Rightarrow HT \parallel BC$ and $BH = TC$.

$\angle BTC = \angle BDC \Rightarrow \angle BTC = 180 - A = \angle BHC$

$\Rightarrow T \in (BHC)$

the reflection of (ABC) over BC is (BHC)

15. Let ABC be an acute triangle such that $AB \neq AC$,with circumcircle Γ and circumcenter O. Let M be the midpoint of BC and D be a point on Γ such that $AD \perp BC$. let T be a point such that $BDCT$ is a parallelogram and Q a point on the same side of BC as A such that $\angle BQM = \angle BCA$ and $\angle CQM = \angle CBA$. Let the line AO intersect Γ at E $(E \neq A)$ and let the circumcircle of $\triangle ETQ$ intersect Γ at point $X \neq E$. Prove that the point A, M and X are collinear. (JBMO 2017 P3)



From lemma 1, H, M, E are collinear and $HM = ME$

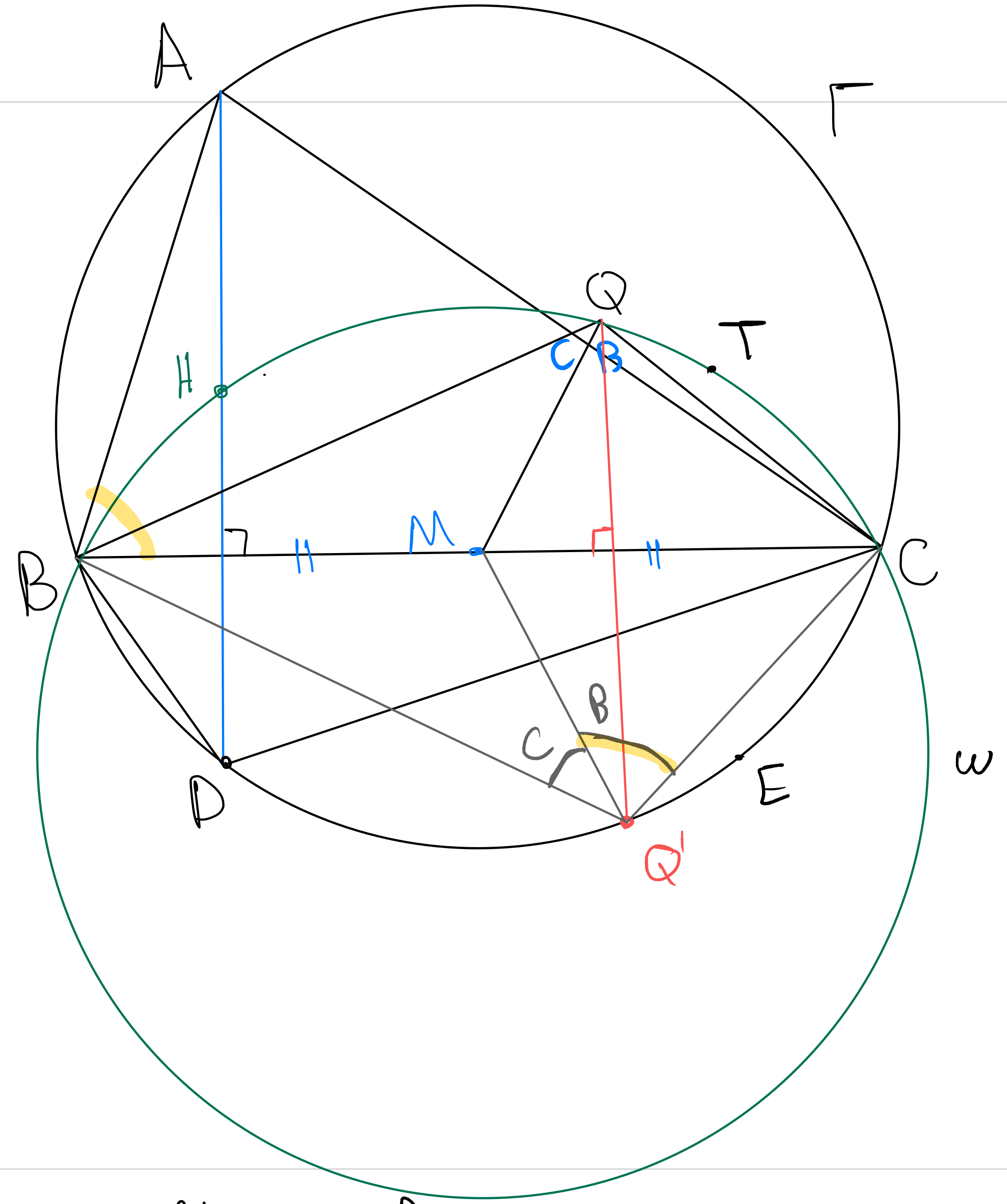
From lemma 2, $BHTC$ cyclic and $HT \parallel BC$, $HT \perp AD$

$\Rightarrow$ $HTED$ parallelogram $\Rightarrow$ a rectangle

$\Rightarrow$ E is the reflection of T over BC.

$\angle BQC = B + C = 180 - A \Rightarrow Q \in (BHC)$

So we will consider the reflection of Q over BC, Q'



Q' is the reflection of Q over BC $\Rightarrow$ $Q' \in (ABC)$

$$\begin{cases} \angle BQ'M = \angle BQM = C \\ \angle CQ'M = \angle CQM = B \end{cases}$$

$\Rightarrow$ $\angle AQ'C = \angle ABC = B$ (cyclic $ABQ'C$)

$\Rightarrow$ $\angle AQ'C = \angle MQ'C$ $\Rightarrow$ A, M, Q' collinear

Now we just need to show that $ETQQ'$ is a cyclic

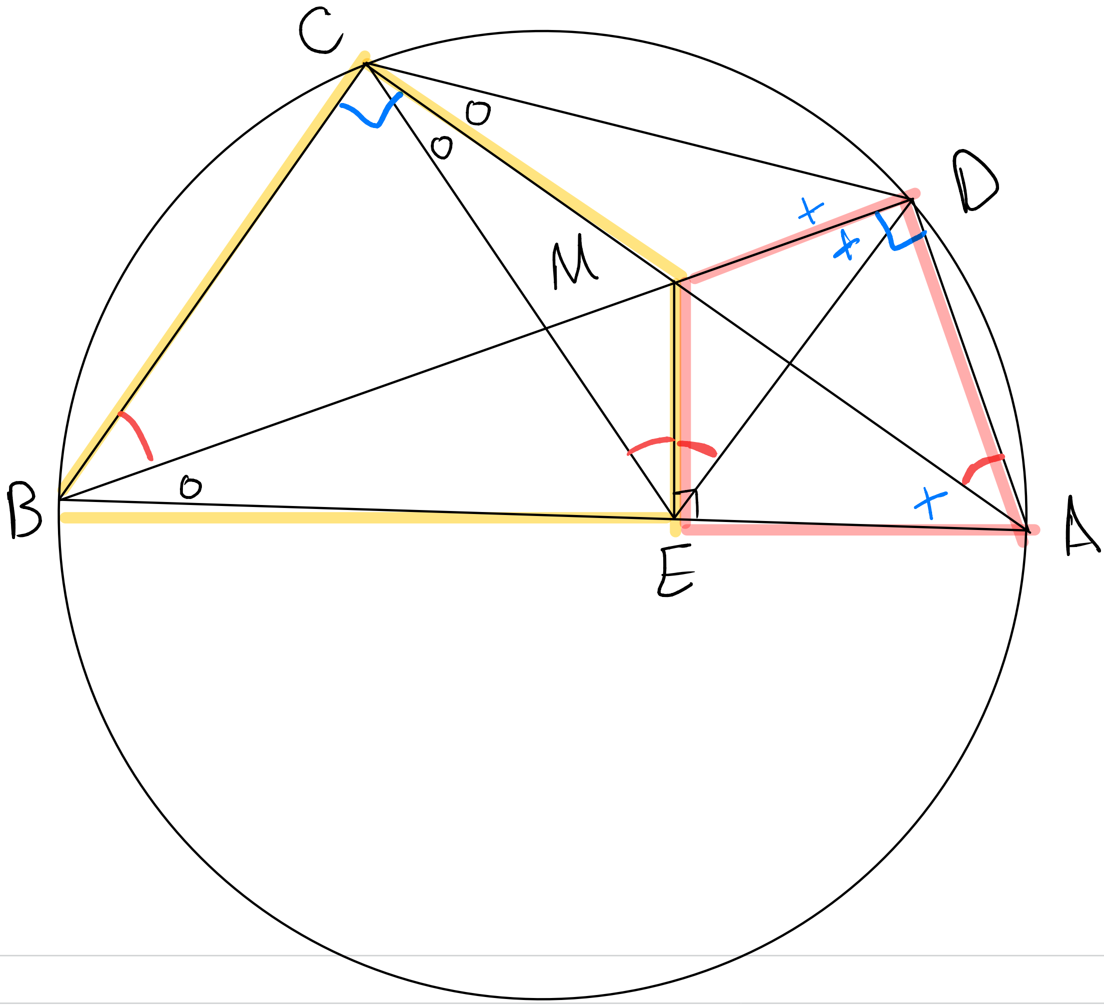
Q' is the reflection of Q over BC $\Rightarrow$ $QQ' \parallel TE$ * $\Rightarrow$ $ETQQ'$ cyclic

E " " " " T over BC $\quad QT = Q'E$

Therefore $Q' = (QTE) \cap \Gamma$ $\Rightarrow$ $Q' = X$

$\Rightarrow$ A, M, X collinear $\square$

16. A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$,and AB is not paralel to CD.Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k. (BMO 2018 P1).



We can show that the second direction holds:

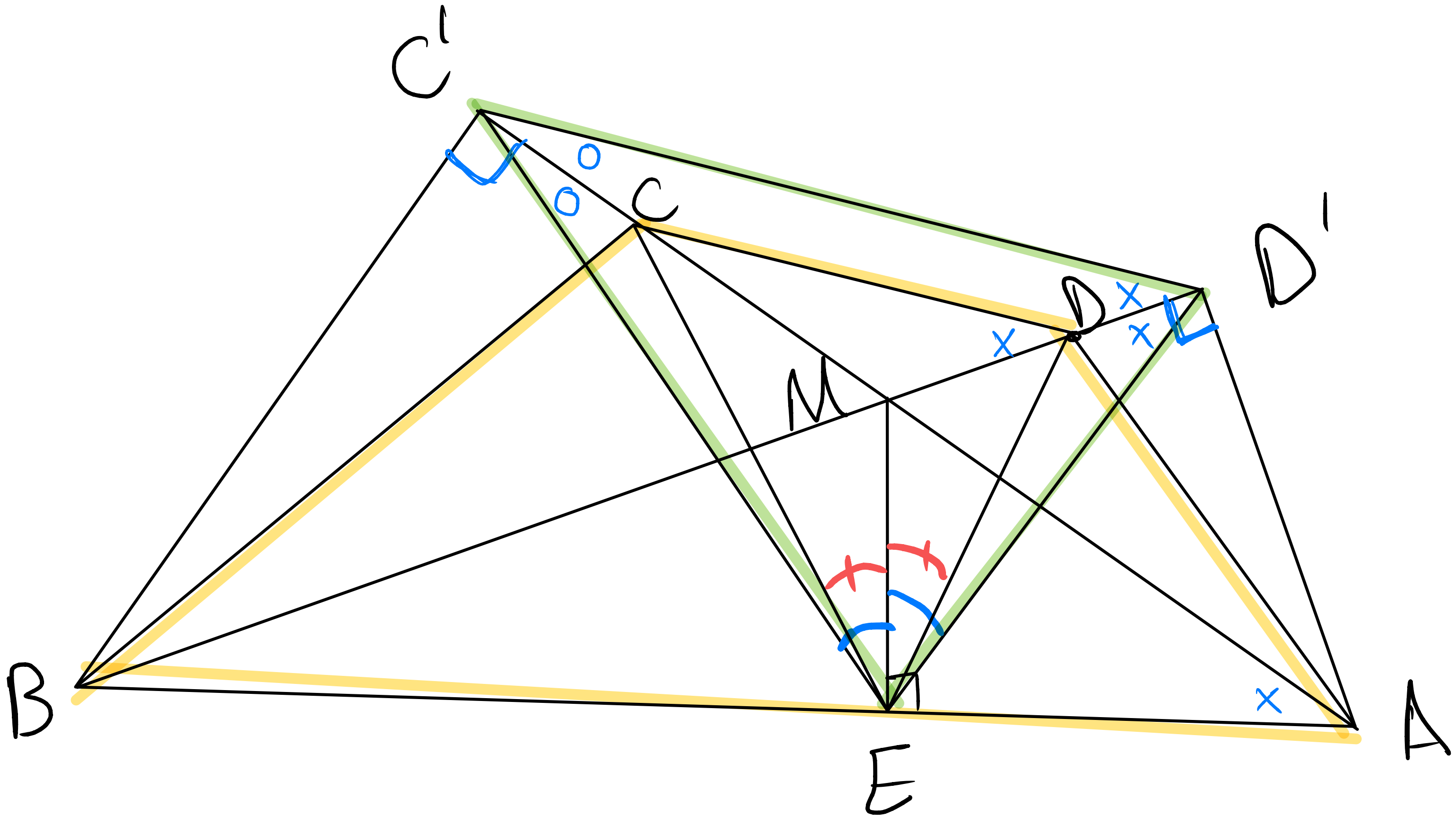
If AB is diameter of k, $AC \cap BD = M$ and $E \in AB$ such that $M \perp AB$ then EM bisects $\angle CDE$.

Proof: $AEMD$ cyclic and $BEMC$ cyclic

$$\underbrace{\angle MED = \angle MAD}_{(AEMD)} = \underbrace{\angle CAD = \angle CBD}_{(ABCD)} = \underbrace{\angle CBM = \angle CEM}_{(CBEM)}$$

$$\Rightarrow \angle MED = \angle CEM \quad \square$$

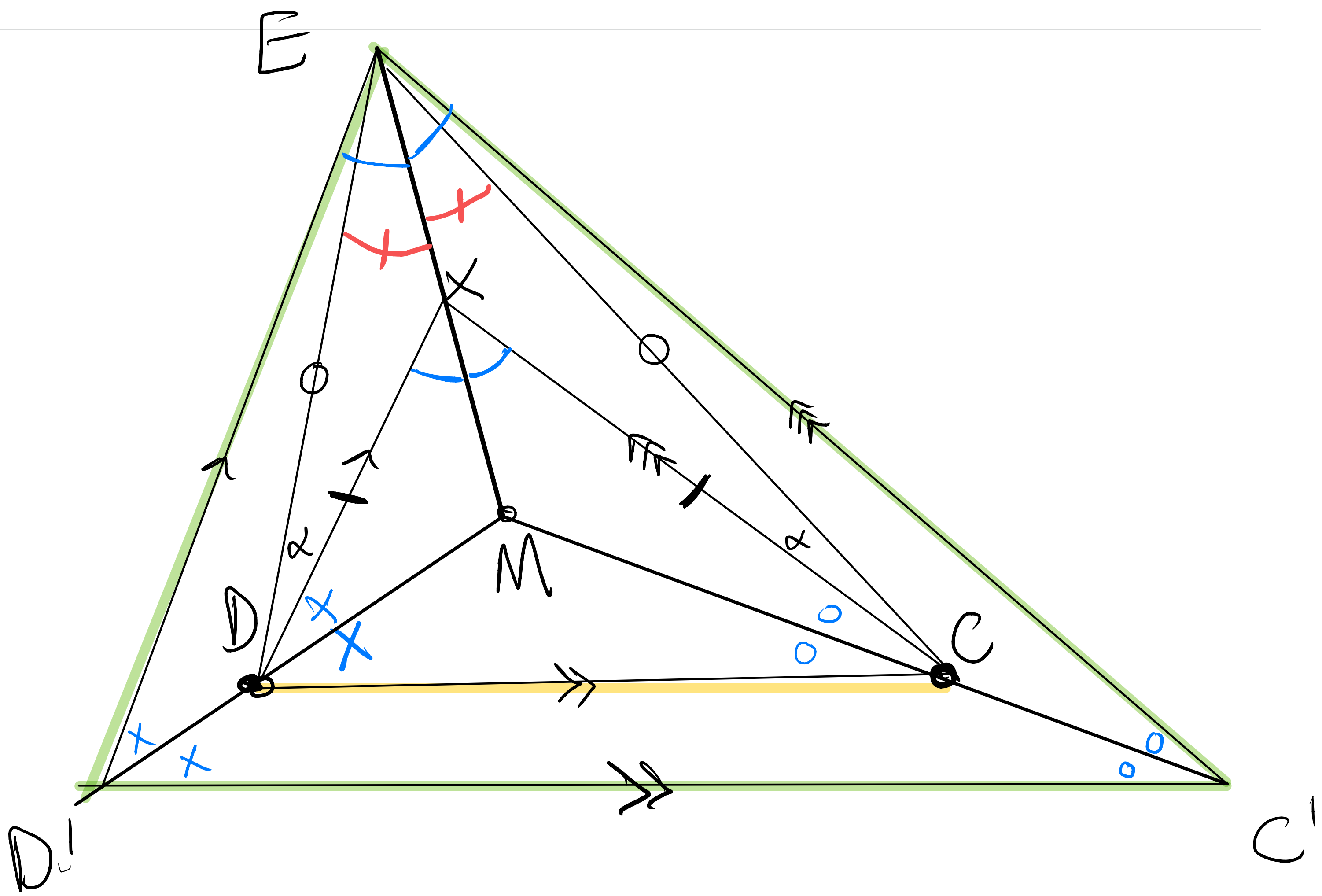
Also, we can show that M is the incenter for ΔCDE.



Let C' and D' be AM and BM such that C', D' are on the circle with diameter AB. From $*$, we know that EM bisects $\angle C'ED'$ and M is the incenter for $\Delta C'ED'$.

$$\underbrace{\angle C'D'B = \angle C'AB}_{(ABC'D')} \overbrace{ = \angle CDB}^{(ABCD)} \Rightarrow CD \parallel C'D'$$

$$\left\{\begin{array}{l} ABC'D' \text{ is a cyclic} \\ ABCD \text{ is a cylic} \end{array}\right. \Longleftrightarrow C'D' \parallel CD$$



M is the incenter of $ED'C'$

Let $X \in EM$ such that $DX \parallel D'E$

$$\underbrace{\frac{MX}{XE} = \frac{MD}{DD'}}_{XD \parallel ED'} \overset{DC \parallel D'C'}{=} \frac{MC}{CC'} \Rightarrow XC \parallel EC'$$

Therefore, M is the incenter for $\triangle XDC$

$\Rightarrow \angle EXC = \angle EXD \Rightarrow \triangle EXC \cong \triangle EXD$

$\Rightarrow ED = EC$

$\Rightarrow EM \perp DC \Rightarrow AB \parallel DC$

which is a contradiction, so $C = C'$, $D = D'$ $\square$