Problems on graph theory

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1. In a simple graph on n vertices, $n-1$ of its vertices have degrees $1, 2, \dots, n-1$. What is the degree of the n-th vertex?

2. In a group of people with at least one pair of acquaintances, every two persons with the same number of acquaintances have no acquaintances in common. Prove that there is a person knowing exactly one other person.

3. In a connected simple graph on n vertices, every vertex has degree at least d. Prove that there is a path of length at least $\min\{2d, n-1\}$.

4. Every vertex of a simple graph has degree 3. Prove that this graph has an even cycle (i.e. of even length).

5. In a graph with 300 vertices, each vertex has degree 3. At most how many cycles of length 4 can this graph contain?

6. At most how many squares on a chessboard 8×8 can we cut along both diagonals so that the chessboard does not fall apart?

7. In a group of $2n+1$ persons, for every group of n persons there is a person (other than themselves) knowing them all. Prove that there is a person knowing all other $2n$ persons.

8. Given a tree with 100 leaves, prove that one can add 50 edges so that the so obtained graph has no bridges. (A bridge is an edge whose removal disconnects the graph.)

9. Find the smallest $n > 4$ for which there is a group of n people in which any two acquaintances have no acquaintances in common, whereas any two non-acquaintances have exactly two acquaintances in common.

10. If a graph on n vertices contains no triangles, prove that it has at most $n^2/4$ edges.

11. If a graph has $2n$ edges and n^2+1 triangles, prove that it contains at least n triangles.

12. Prove that every tournament has a Hamiltonian path.

13. Prove that every planar graph on n vertices has at most $3n-6$ edges, and that every bipartite planar graph has at most $2n-4$ edges.

14. Every edge of a convex polyhedron is assigned $+$ or $-$. Prove that there is a vertex having at most two facial angles with differently signed edges.

15. Prove that the vertices of a planar graph can be colored with 5 colors so that no edge connects two vertices of the same color.