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Geometry – Level 3

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Power of the point – problems

Problem 1. Let ABC be a non-right triangle with orthocenter H and let M, N be points on its sides AB and AC. Prove that the common chord of circles with diameters CM and BN passes through H.

Problem 2. Let the incircle ω of triangle ABC touches BC, CA, and AB at D, E, and F, respectively. Let Y_1, Y_2, Z_1, Z_2, and M be the midpoints of BF, BD, GE, CD, and BC, respectively. Let $Y_1Y_2 \cap Z_1Z_2 = X$. Prove that $MX \perp BC$.

Problem 3. Let ω_1, ω_2 be two circles. One of their common external tangents is tangent to ω_1 at A, the second one is tangent to ω_2 at D. Line AD intersects the circles ω_1, ω_2 for the second time at B, C, respectively. Prove that $AB = CD$.

Problem 4. Let $ABCD$ be a circumscribed quadrilateral with $BC = 2AB$. Suppose that perpendicular bisector of BC and bisector of angle DCB intersect at X. Prove that AX and BD are perpendicular.

Problem 5. Let $ABCDEF$ be a convex hexagon in which $AB = AF$, $BC = CD$, $DE = EF$ and $\sphericalangle ABC = \sphericalangle EFA = 90°$. Prove that $AD \perp CE$.

Problem 6. Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.

Problem 7. Given is triangle ABC. Points P and Q were chosen such that

$$\sphericalangle PBC = \sphericalangle QCB = 90° \quad \text{and} \quad AP = PB, \;\; AQ = CQ.$$

Tangent to circumcircle of ABC passing through point A intersects BC at R. Prove that P, Q and R are collinear.

Problem 8. Convex hexagon $ABCDEF$ satisfies $AB = BC$, $CD = DE$, $EF = FA$. Show that lines containing altitudes of triangles BCD, DEF and FAB from vertices C, E, A, respectively, are concurrent.

Problem 9. An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle circumscribed on ABC crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.

Problem 10. Let $ABCD$ be a cyclic quadrilateral ($AB \neq CD$). Quadrilaterals $AKDL$ and $CMBN$ are rhombi with equal sides. Prove that points K, L, M, N lie on a single circle.

Problem 11. Let circles ω_1 and ω_2, with centres in O_1, O_2, respectively, intersect at two distinct points P and Q. Their common tangent, closer to point P, touches the circles at A, B respectively. Let the perpendicular from A to the line BP meet O_1O_2 at C. Prove that $\sphericalangle APC = 90°$.

Problem 12. A circle Ω, its chord AB and the midpoint W of the minor arc AB are given. Take an arbitrary point C on the major arc AB. The tangent to the circle at C meets the tangents at A and B at points X and Y, respectively. Lines WX and WY meet AB at points N and M respectively. Prove that the length of segment NM does not depend on point C.

Problem 13. Lines b and c passing through vertices B and C of triangle ABC are perpendicular to sideline BC. The perpendicular bisectors of AC and AB meet b and c at points P and Q, respectively. Prove that the line PQ is perpendicular to median AM of triangle ABC.

Problem 14. Let ABC be a triangle with $\sphericalangle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Problem 15. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. Let P be a point on the line XY such that $P \notin BC$. The line GP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Inversion – Problems

Problem 16. Circles k_1, k_2, k_3, k_4 are such that k_2 and k_4 each touch k_1 and k_3. Show that the tangency points are collinear or concyclic.

Problem 17. Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\measuredangle PAB$.

Problem 18. Let M, N, P be the point where the incircle of scalene triangle ABC touches sides BC, CA, AB, respectively. Prove that the orthocenter of triangle MNP, the incenter I of the triangle ABC and the circumcenter O of the triangle ABC are collinear.

Problem 19. Let p be the semiperimeter of triangle ABC. Points E and F are on line AB such that $|CE| = |CF| = p$. Prove that the circumcircle of triangle CEF is tangent to the excircle of triangle ABC with respect to the side AB.

Problem 20. Points A, B, C are given on a line in this order. Semicircles ω, ω_1, ω_2 are drawn on AC, AB, BC respectively as diameters on the same side of the line. A sequence of circles (k_n) is constructed as follows: k_0 is the circle determined by ω_2 and k_n is tangent to ω, ω_1, k_{n-1} for $n \geq 1$. Prove that the distance from the center of k_n to AB is $2n$ times the radius of k_n.

Problem 21. Let ω be the circumcircle of triangle ABC. Circle ω_1 is inscribed in angle BAC and touches ω internally at T. Let D be the point of tangency of BC and the A-excircle. Prove that $\measuredangle BAT = \measuredangle DAC$.

Problem 22. Let ω be a circle tangent internally to circle Ω. Let A be a point on Ω, and let AX and AY be tangents to ω. Consider circles ω_1, ω_2 tangent internally to Ω and tangent to ω at X and Y, respectively. Prove that there exists common exterior tangent of circles ω, ω_1, ω_2.

Power of the point – solutions

Problem 1. Let ABC be a non-right triangle with orthocenter H and let M, N be points on its sides AB and AC. Prove that the common chord of circles with diameters CM and BN passes through H.

Sketch. Let D and E be feet of B and C on AC and AB, respectively. Then apply **three axis theorem** to circle $(BCED)$ and circles with diameters CM and BN. □

Problem 2. Let the incircle ω of triangle ABC touches BC, CA, and AB at D, E, and F, respectively. Let Y_1, Y_2, Z_1, Z_2, and M be the midpoints of BF, BD, GE, CD, and BC, respectively. Let $Y_1Y_2 \cap Z_1Z_2 = X$. Prove that $MX \perp BC$.

Sketch. **Three axis theorem** for circles ω, B, C. □

Problem 3. Let ω_1, ω_2 be two circles. One of their common external tangents is tangent to ω_1 at A, the second one is tangent to ω_2 at D. Line AD intersects the circles ω_1, ω_2 for the second time at B, C, respectively. Prove that $AB = CD$.

Sketch. Compare power of E wrt ω_1 and power of R wrt ω_2. □

Problem 4. (Polish MO 2020) Let $ABCD$ be a circumscribed quadrilateral with $BC = 2AB$. Suppose that perpendicular bisector of BC and bisector of angle DCB intersect at X. Prove that AX and BD are perpendicular.

Proof. Let M be a midpoint of BC and take N on segment CD such that $CN = CM$. Then $AB = BM = MC = CN$. Moreover $AB + CD = BC + AD$, by *described quadrilateral theorem*, so

$$2 \cdot AB + DN = AB + CN + DN = AB + CD = BC + AD = 2 \cdot AB + AD,$$

hence $AD = DN$.

Take circles $\odot(B, AB)$, $\odot(C, AB)$, $\odot(D, AD)$. From the above, circles in pairs $(\odot(B, AB), \odot(C, AB))$ and $(\odot(C, AB), \odot(D, AD))$ are externally tangent.

From the **radical axis theorem** if follows that X is a radical centre of $\odot(B, AB)$, $\odot(C, AB)$ and $\odot(D, AD)$. In particular AX is radical axis of $\odot(B, AB)$ and $\odot(D, AD)$, hence we are done. □

Discussion. https://om.mimuw.edu.pl/static/app_main/problems/om71_1r.pdf

Problem 5. (Baltic Way 2019) Let $ABCDEF$ be a convex hexagon in which $AB = AF$, $BC = CD$, $DE = EF$ and $\sphericalangle ABC = \sphericalangle EFA = 90°$. Prove that $AD \perp CE$.

Proof. Draw circles $\odot(C, CD)$ and $\odot(E, ED)$. Clearly AB, AF are tangents, so AD is the **radical axis** of these circles. Hence we are done. □

Discussion. https://artofproblemsolving.com/community/c6h1954642p13500917

Problem 6. (Baltic Way 2016) Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.

Proof. Let $Q = AB \cap CD$. Note that $QA \cdot QB = QC \cdot QD$, so the power of Q to the circle centreed at P with radius $PA = PB$ is equal to the power of Q to the circle centreed at M with radius $MC = MD$. Since these circles are congruent and Q lies on their **radical axis**, Q lies on the perpendicular bisector of their centres, as desired. □

Discussion. https://artofproblemsolving.com/community/c6h1334580p7212380

Problem 7. Given is triangle ABC. Points P and Q were chosen such that

$$\sphericalangle PBC = \sphericalangle QCB = 90° \quad \text{and} \quad AP = PB, \;\; AQ = CQ.$$

Tangent to circumcircle of ABC passing through point A intersects BC at R. Prove that P, Q and R are collinear.

Proof. Consider radical axis of point A, circumcircle ω of ABC and circle with diameter BC. Then radical axis of

- A and ω is line AA;
- A and circle with diameter BC is line PQ;
- ω and circle with diameter BC is line BC.

Therefore lines AA, PQ and BC intersect at one point - namely R, thus P, Q, R are collinear. □

Problem 8. Convex hexagon $ABCDEF$ satisfies $AB = BC$, $CD = DE$, $EF = FA$. Show that lines containing altitudes of triangles BCD, DEF and FAB from vertices C, E, A, respectively, are concurrent.

Proof. Denote by

1) ω_1 - circle with centre in point B and radius $AB = BC$;
2) ω_2 - circle with centre in point D and radius $CD = DE$;
3) ω_3 - circle with centre in point F and radius $EF = FA$.

Let $A' = \omega_1 \cap \omega_2$. Notice that line AA' is radical axis of circles ω_1 and ω_2. Therefore it's perpendicular to line connecting centres of this circles - $AA' \perp BF$. Hence line AA' contains altitude of triangle FAB from vertex A.

Analogously we show that lines CC' and EE' contain altitudes of triangles BCD and DEF from vertices C and E, respectively. But lines AA', BB', CC' are also radical axes of circles ω_1, ω_2 and ω_3, thus they are concurrent and we are done. $\square$

Problem 9. An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle circumscribed on ABC crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.

Proof. Notice that quadrilateral $EFBC$ is cyclic, because $\sphericalangle BEC = \sphericalangle BFC = 90°$. Denote circumcirle of $EFBC$ by ω_1. Moreover

$$\sphericalangle EFA = 180° - \sphericalangle BFE = \sphericalangle ACB = \sphericalangle EAQ,$$

Last equality comes from fact that $AQ \parallel BC$.

It's not hard to notice that circumcircle ω_2 of triangle AFE is tangent to line AQ. It follows from radical axis concurrence theorem applied to point A and circles ω_1 and ω_2 that point Q lies on radical axis of point A and ω_1.

Once again applying this theorem to point A, circumcirle of ABC and ω_1 we obtain that P lies on radical axis of A and ω_1.

Thus line PQ is radical axis of point A and circle ω_1. In particular PQ is perpendicular to line connecting centers of this circles - namely A and M. Hence $PQ \perp AM$ and we are done. $\square$

Problem 10. Let $ABCD$ be a cyclic quadrilateral ($AB \neq CD$). Quadrilaterals $AKDL$ and $CMBN$ are rhombi with equal sides. Prove that points K, L, M, N lie on a single circle.

Proof. Let a be the length of sides of rhombi. Consider circles (K, a), (M, a) and $(ABCD)$. Then from three axis theorem it follows that perpendicular bisector of KM passes through $O := AB \cap CD$. Similarly we get that perpendicular bisectors of LN and KL pass thorugh O, so O is circumcenter of quadraliteral $KLMN$. $\square$

Problem 11. Let circles ω_1 and ω_2, with centres in O_1, O_2, respectively, intersect at two distinct points P and Q. Their common tangent, closer to point P, touches the circles at A, B respectively. Let the perpendicular from A to the line BP meet O_1O_2 at C. Prove that $\sphericalangle APC = 90°$.

Proof. Let $N = AC \cap BP$. Powers of point O_1 with respect to point P and circumcircle of ANB are equal (notice that O_1A is tangent to circumcircle of ANB). Similarly it holds for O_2. Thus we can obtain that O_1O_2 is the radical axis of point P and circumcircle of ANB. Since C lies on O_1O_2 we get

$$CN \cdot CA = CP^2.$$

From now on it is easy to get that $\sphericalangle APC = 90°$. □

Problem 12. A circle Ω, its chord AB and the midpoint W of the minor arc AB are given. Take an arbitrary point C on the major arc AB. The tangent to the circle at C meets the tangents at A and B at points X and Y, respectively. Lines WX and WY meet AB at points N and M respectively. Prove that the length of segment NM does not depend on point C.

Proof. Consider circle ω, touching XY at C and touching AB (at point T). It is easy to see that WX is the radical axis of point A and ω (one just need to notice that W T and C are collinear by homothety and apply the definition of homothety), i.e. it passes through the midpoint N of segment AT. Similarly WY passes through the midpoint M of segment TB. Thus $MN = \frac{AB}{2}$. □

Problem 13. Lines b and c passing through vertices B and C of triangle ABC are perpendicular to sideline BC. The perpendicular bisectors of AC and AB meet b and c at points P and Q, respectively. Prove that the line PQ is perpendicular to median AM of triangle ABC.

Proof. Construct a circle centred at P and passing through A. It is tangent to BC at B, because $PA = PB$. Similarly, the circle centred at Q and passing through A is tangent to BC at C. The powers of M with respect to these circles are equal, so the radical axis of these circles is AM. It is perpendicular to the line connecting centres of these circles, namely line PQ. Hence we are done. □

Problem 14. Let ABC be a triangle with $\sphericalangle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Proof.  Consider the circles $\omega_1(B, BC)$, $\omega_2(A, AC)$ and $\omega(F, FK)$, where the circle ω has its center on BK and it is internally tangential to the other two circles ω_1, ω_2 at K and L_1, respectively. The radical axes of the three circles will be intersected at point T of altitude CD. We have that $X = CD \cap AK$.

Let the circle (T, TK), where $TK = TL_1$, intersects ω_2 at point I. The point B belongs to the radical axis L_1I, since the triangle is a right one at the vertex C and $BC = BK$.

Similarly, the straight line AK is the radical axis of the circles ω_1, (T, TK). Thus, because of uniqueness of the points , we deduce that $L_1 = L$ and hence $F = M$. This ends the proof. □

Problem 15. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. Let P be a point on the line XY such that $P \notin BC$. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Proof. First assume that P lies outside the circles. Line XY is the radical axis of the two circles, so the concurrence of BN, CM and XY by the Radical Lemma implies th at points B, C, M, N are concyclic. Using the radical axis again a in the second direction, we realize it suffices to prove that points A, D, M, N are also concyclic.

But this is easy! We just recall that AC and BD are diameters, write

$$\sphericalangle DNM + 90 = \sphericalangle BNM = 180^\circ - \sphericalangle MCB = \sphericalangle DAM + 90^\circ,$$

and conclude that $ADMN$ is cyclic.

The case when P belongs to segment XY is treated similarly. Another option is to regard angles as directed mod 180°. □

Inversion – solutions

Problem 16. Circles k_1, k_2, k_3, k_4 are such that k_2 and k_4 each touch k_1 and k_3. Show that the tangency points are collinear or concyclic.

Proof. Let k_1 and k_2, k_2 and k_3, k_3 and k_4, k_4 and k_1 touch at A, B, C, D, respectively. An inversion with center A maps k_1 and k_2 to parallel lines ${k_1}^*$ and ${k_2}^*$, and k_3 and k_4 to circles ${k_3}^*$ and ${k_4}^*$ tangent to each other at C' and tangent to ${k_2}^*$ at B' and to ${k_4}^*$ at D'. It is easy to see that A', B', C' and D' are collinear. Therefore B, C, D lie on a circle through A. □

Problem 17. Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\sphericalangle PAB$.

Proof. Invert through C. Semicircle ω maps to the semicircle ω' with diameter $P'Q'$, circle k to the tangent to ω' parallel to $P'Q'$, and line AB to a circle l centered on $P'Q'$ which touches k (so it is congruent to the circle determined by ω'). Circle l intersects ω' and $P'Q'$ in A' and B' respectively. Hence $P'A'B'$ is an isosceles triangle with

$$\sphericalangle PAC = \sphericalangle A'P'C = \sphericalangle A'B'C = \sphericalangle BAC.$$

□

Problem 18. Let M, N, P be the point where the incircle of scalene triangle ABC touches sides BC, CA, AB, respectively. Prove that the orthocenter of triangle MNP, the incenter I of the triangle ABC and the circumcenter O of the triangle ABC are collinear.

Proof. The incenter of triangle ABC and the orthocenter of triangle MNP lie on the Euler line of the triangle ABC. The inversion with respect to the incircle of ABC maps points A, B, C to the midpoints of NP, PM, MN, so the circumcircle of ABC maps to the nine-point circle of triangle MNP which is also centered on the Euler line of MNP. It follows that the center of circle ABC lies on the same line. □

Problem 19. Let p be the semiperimeter of triangle ABC. Points E and F are on line AB such that $|CE| = |CF| = p$. Prove that the circumcircle of triangle CEF is tangent to the excircle of triangle ABC with respect to the side AB.

Proof. The inversion with center C and radius p maps points E and F and the excircle to themselves, and the circumcircle of triangle CEF to line AB which is tangent to the excircle. The statement follows from the fact that inversion preserves tangency. □

Problem 20. Points A, B, C are given on a line in this order. Semicircles ω, ω_1, ω_2 are drawn on AC, AB, BC respectively as diameters on the same side of the line. A sequence of circles (k_n) is constructed as follows: k_0 is the circle determined by ω_2 and k_n is tangent to ω, ω_1, k_{n-1} for $n \geq 1$. Prove that the distance from the center of k_n to AB is $2n$ times the radius of k_n.

Proof. Under the inversion with center A and squared radius $AB \cdot AC$ points B and C exchange positions, ω and ω_1 are transformed to the lines perpendicular to BC at C and B, and the sequence (k_n) to the sequence of circles $({k_n}^*)$ inscribed in the region between the two lines. Obviously, the distance from the center of $({k_n}^*)$ to AB is $2n$ times its radius. Since circle k_n is homothetic to $({k_n}^*)$ with respect to A, the statement immediately follows. □

Problem 21. Let ω be the circumcircle of triangle ABC. Circle ω_1 is inscribed in angle BAC and touches ω internally at T. Let D be the point of tangency of BC and the A-excircle. Prove that $\measuredangle BAT = \measuredangle DAC$.

Proof. We apply inversion with centre A and radius $\sqrt{AB \cdot AC}$ and observe that ${\omega_1}^*$ is still inscribed in $\measuredangle BAC$ and as ω_1 touched ω internally, ${\omega_1}^*$ touches BC externally, hence ω^* is the A-excircle of triangle ABC. Thus T and D correspond in the $\sqrt{bc}$-inversion and the conclusion follows. □

Problem 22. Let ω be a circle tangent internally to circle Ω. Let A be a point on Ω, and let AX and AY be tangents to ω. Consider circles ω_1, ω_2 tangent internally to Ω and tangent to ω at X and Y, respectively. Prove that there exists common exterior tangent of circles ω, ω_1, ω_2.

Proof. Consider inversion with respect to a circle Γ with centre A and radius AX. Then ω, ω_1, ω_2 are orthogonal to Γ, therefore after inversion they are fixed. On the other hand, the image of Ω under this inversion is a line tangent to ω, ω_1 and ω_2! □