$\boxed{46}$ $a,b,c \in \mathbb{R}_+$: Prove

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq \underbrace{abc}_{}(a^3b^3 + b^3c^3 + c^3a^3) + a^2b^2c^2(a^3+b^3+c^3)$$

a^4b^4c

$\boxed{47}$ $a,b,c \in \mathbb{R}$

$$\sqrt{2(a^2+b^2)} + \sqrt{2(b^2+c^2)} + \sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2 + 3(b+c)^2 + 3(c+a)^2}$$

~ $\boxed{22:45}$

$\boxed{46}$

$x = ab^2$, $y = bc^2$, $z = ca^2$

$x^2 z = a^2b^4ca^2 = a^4b^4c$

$\boxed{49,\ 48,\ 50}$

Solution (of 46)

$x = ab^2$, $y = bc^2$, $z = ca^2$

$$\boxed{x^3 + y^3 + z^3 + 3xyz \geq x^2z + xz^2 + x^2y + xy^2 + yz^2 + y^2z}$$

Schur inequality

$x, y, z \in \mathbb{R}_+$

- $x^3+y^3+z^3+3xyz \geq x^2z+xz^2+x^2y+xy^2+yz^2+y^2z$

- $x^3+y^3+z^3+3xyz-x^2z-xz^2-x^2y-xy^2-yz^2-y^2z \geq 0$

- $\underbrace{\overbrace{x(x-y)(x-z)}^{\geq 0}}_{A} + \underbrace{\overbrace{y(y-x)(y-z)}^{\leq 0}}_{B} + \overbrace{z(z-x)(z-y)}^{\geq 0} \geq 0$

$x^3-x^2z-yx^2+xyz$

Proof?, WLOG $x \geq y \geq z$

$$x(x-y)(x-z) \geq x(x-y)(y-z) \geq$$

$$\geq y(x-y)(y-z) = -y(y-x)(y-z)$$

$$\downarrow$$

$$A+B \geq 0$$

$\square$

- $(x-y+z)(y-z+x)(z-x+y) \leq xyz$

$(xy-xz+x^2-y^2+yz-xy+yz-z^2+zx)(z-x+y) \leq xyz$

$$xyz - xz^2 + x^2z - y^2z + yz^2 - xyz + yz^2 - z^3 + z^2x -$$
$$- x^2y + x^2z - x^3 + xy^2 - xyz + x^2y - xyz + xz^2 - x^2z +$$
$$+ xy^2 - xyz + yx^2 - y^3 + y^2z - xy^2 + y^2z - z^2y + xyz \leq xyz$$

$$3xyz + x^3 + y^3 + z^3 \geq xy^2 + x^2y + \dots +$$

$\boxed{\text{Generalization}}$

$$\alpha \in \mathbb{R}, \quad \beta > 0$$

$$x^\alpha (x^\beta - y^\beta)(x^\beta - z^\beta) + y^\alpha (y^\beta - x^\beta)(y^\beta - z^\beta) +$$
$$+ z^\alpha (z^\beta - x^\beta)(z^\beta - y^\beta) \geq 0$$

$$\boxed{\alpha = \beta = 1}$$

As mule ingosta ca

AM-GM

C-S

Hölder

Allo of you: $\longrightarrow$ 15 most problem: $\boxed{49}$

49

$$\boxed{abc = 1}$$

$a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for some $x, y, z \in \mathbb{R}_+$

Why?

Of course $a = \frac{x}{y}$, $b = \frac{y}{z}$

$$c = \frac{1}{ab} = \frac{1}{\frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z}} = \boxed{\frac{z}{x}}$$

generally $x_1 x_2 \cdots x_n = 1$

$x_1 = \frac{a_1}{a_2}$, $x_2 = \frac{a_2}{a_3}$, $x_3 = \frac{a_3}{a_4}$ . . .

$a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$

IMO 2004

$$\underbrace{\left(\frac{x}{y}-1+\frac{z}{y}\right)}_{\cdot y}\underbrace{\left(\frac{y}{z}-1+\frac{x}{z}\right)}_{\cdot z}\underbrace{\left(\frac{z}{x}-1+\frac{y}{x}\right)}_{\cdot x}\leq 1 \quad \Big|\cdot xyz$$

$$(x-y+z)(y-z+x)(z-x+y)\leq xyz.$$

Schur!

$\boxed{47}$

$$\left(\sqrt{2(a^2+b^2)}+\sqrt{2(b^2+c^2)}+\sqrt{2(c^2+a^2)}\right)^2 \geq \left(\sqrt{3(a+b)^2+3(b+c)^2+3(c+a)^2}\right)^2$$

$$\| $$

$$\cancel{4}a^2+\cancel{4}b^2+\cancel{4}c^2+4\sqrt{(a^2+b^2)(b^2+c^2)}+$$

$$+4\sqrt{(a^2+b^2)(c^2+a^2)}+4\sqrt{(c^2+a^2)(b^2+c^2)} \geq$$

$$\geq \cancel{6}a^2+\cancel{6}b^2+\cancel{6}c^2+6(ab+bc+ca) \quad \Big| :2$$

$$\Downarrow \quad \text{Now?}$$

$2\sqrt{}$

$$2\sqrt{(\vec{a}^2+b^2)(b^2+c^2)} +$$

$$+\ 2\sqrt{(\vec{a}^2+b^2)(c^2+a^2)} + 2\sqrt{(c^2+a^2)(b^2+c^2)} \geq$$

$$\geq \quad a^2+b^2+c^2+3ab+3bc+3ca$$

$\boxed{C-S}$

$$\sqrt{(a^2+b^2)(b^2+c^2)} \geq \sqrt{(b^2+a^2)(b^2+c^2)} \geq$$

$$\geq \sqrt{\left(\sqrt{(b^2)^2}+\sqrt{(ac)^2}\right)^2} = \boxed{b^2+ac}$$

$$2\sqrt{(\vec{a}^2+b^2)(b^2+c^2)} +$$

$$+\ 2\sqrt{(\vec{a}^2+b^2)(c^2+a^2)} + 2\sqrt{(c^2+a^2)(b^2+c^2)} \geq$$

$$\geq \boxed{2(b^2+ac)+2(a^2+bc)+2(c^2+ab) \geq a^2+b^2+c^2+3ab+3bc+3ca}$$

$\square$

$$a^2+b^2+c^2 \geq ab+bc+ca$$

$$\boxed{(a-b)^2+(b-c)^2+(c-a)^2 \geq 0}$$

$\boxed{48}$

$$\sqrt{(a^2+b^2)(c^2+b^2)} \not\geq \sqrt{(ac+b)^2}$$

$P(x) \in \mathbb{R}[x]$

$$P(1) \geq \frac{1}{P(1)} \rightsquigarrow P(1)^2 \geq 1$$

$$\Downarrow$$

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)} \rightsquigarrow P(x)\, P\left(\frac{1}{x}\right) \geq 1.$$

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad a_i \in \mathbb{R}_+$$

$$P(1)^2 \geq 1$$

$$(a_0 + a_1 + \dots + a_n)^2 \geq 1$$

$$\left(a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0\right)\left(a_n \left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \dots + a_1 \cdot \frac{1}{x} + a_0\right)$$

$$\overset{C-S}{\geq} \left(\sqrt{a_n \cdot x^n \cdot a_n \cdot \frac{1}{x^n}} + \dots + \sqrt{a_1 x \cdot \frac{1}{x} a_1 x}\ \sqrt{a_0 \cdot a_0}\right)^2$$

$$\parallel$$

$$(a_0 + a_1 \dots + a_n)^2 \geq 1.$$