$$x_1 + \frac{K}{x_2} = \cdots = x_n + \frac{K}{x_1} \qquad (1$$

---

n odd > 2020, $K \in \mathbb{N}$, $x_i \in \mathbb{Q}^*$

(a) $x_1 x_2 \cdots x_n = ?$

$$x_1 - x_2 = \frac{K}{x_2 x_3}(x_2 - x_3) \qquad \left(x_1 + \frac{K}{x_2} = x_2 + \frac{K}{x_3}\right)$$

$$\vdots$$

$$x_n - x_1 = \frac{K}{x_1 x_2}(x_1 - x_2)$$

Clearly, non of $x_i - x_{i+1}$ is 0

$$\Rightarrow \prod_{i=1}^{n}(x_i - x_{i+1}) = \frac{k^n}{\left(\prod_{i=1}^{n} x_i\right)^2} \prod_{i=1}^{n}(x_i - x_{i+1})$$

$$(x_{n+1} = x_1) \iff \boxed{\prod_{i=1}^{n} x_i = \pm k^{\frac{n}{2}}}$$

(b) $(x_1, x_2, \dots, x_n) \neq (a, b, a, b, \dots, a, b)$

we try: $(x_1, x_2, \dots, x_n) = (a, b, c, a, b, c, \dots, a, b, c)$

$K^{\frac{n}{2}} \in \mathbb{Q}$; $K \in \mathbb{N} \implies K \geq 4$

$$\boxed{a + \frac{4}{b} = b + \frac{4}{c} = c + \frac{4}{a} \quad (1)}$$

we try to find a solution

$$a^2b^2c^2 = 64 \to \boxed{abc = 8 \quad (2)}$$

$$(1) \to a + \frac{4}{b} = b + \frac{ab}{2} = \frac{8}{ab} + \frac{4}{a}$$

$$a\left(1 - \frac{b}{2}\right) = (b^2 - 4)/b \quad , \text{ let } b = 2$$

$$\Rightarrow a+2 = \frac{8}{a_h} + \frac{4}{a} = \frac{8}{a}, \quad a \neq 2$$

$$a = -4 \rightarrow c = -1$$

Now, for $\boxed{k=4}$, $n = 2025$,

$x_{3i-2} = -4$, $x_{3i-1} = 2$, $x_{3i} = -1$

$\forall \; 1 \leq i \leq 675$, we have

$$x_j + \frac{4}{x_{j+1}} = -2 \quad \forall \; 1 \leq j \leq 2025 \quad \blacksquare$$

(2

$x_1, y_1, \ldots, x_n, y_n \in \mathbb{R}^+$

$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 1$

Prive:

$$|x_1 - y_1| + \cdots + |x_n - y_n| \leq 2 - \min \frac{x_i}{y_i} - \min \frac{y_i}{x_i}$$

Let $p = \min\limits_{1 \leq i \leq n} \dfrac{x_i}{y_i}$, $(\Rightarrow x_i \geq p y_i)$.

$q = \min\limits_{1 \leq i \leq n} \dfrac{y_i}{x_i}$ $(y_i \geq q x_i)$ $\Rightarrow p, q \leq 1$

$$(1-q)x_i \geq x_i - y_i \geq (p-1) y_i$$

$$\Rightarrow |x_i - y_i| \leq \max\left((1-p)y_i, (1-q)x_i\right)$$

$$\leq (1-p)y_i + (1-q)x_i$$

$\forall\ 1 \leq i \leq n$

Now sum up, we get

$$\sum_{i=1}^{n} |x_i - y_i| \leq 1-p+1-q$$

$$= 2-p-q$$

as desired $\blacksquare$

$$m^3+n^3>(m+n)^2$$ [3

$$\Rightarrow m^3+n^3\geq(m+n)^2+k$$

(Find the least positive value of

$\boxed{m^3+n^3-(m+n)^2}$, $m,n\in\mathbb{N}$)

wlog $m\geqslant n$

Sol 1: Case 1: $m=n$

$$m^3+n^3-(m+n)^2=2m^2(m-2)$$
$$\geqslant 18 \quad (3,3)$$

Case 2: $m>n=1$

$\Rightarrow$

$$m^3+1-(m+1)^2$$

$$=(m+1)m(m-2) \geq 12 \quad (3,1)$$

<u>Case 3:</u> $m > n > 1$

$$m^3+n^3-(m+n)^2$$

$$=(m+n)(m^2-mn+n^2-m-n)$$

$$\left((m-n)^2 \geq 1 \Leftrightarrow m^2+n^2-mn \geq mn+1\right)$$

$$\geq (m+n)(mn+1-m-n)$$

$$=(m+n)(m-1)(n-1) \geq 10 \quad (3,2)$$

Thus, the answer is $K=10$

$\boxed{\text{Sol 2}}$ Let $m=1$, $n=2$

$$\rightarrow K \leq 10$$

<u>Case 1</u>: $m, n \geq 3$

$$m^3+n^3-(m+n)^2$$

$$\geq 3m^2+3n^2-(m+n)^2$$

$$=2(m^2-mn+n^2)$$

$$\geq 2mn \geq 18 > 10$$

case 2 $\quad n=1$

$$m^3+1-(m+1)^2$$

$$=(m+1)m(m-2) \geqslant 12 > 10$$

Cas 3 : $\quad n=2$

$$m^3+8-(m+2)^2$$

$$=(m+2)(m^2-3m+2)$$

$$=(m+2)(m-2)(m-1) \geqslant 10$$

✓

$$A \subseteq \{1, 2, \cdots, 2020\}$$ $\boxed{4}$

$\forall \{a, b, c\} \subseteq A$:

$|a-b| \geq \sqrt{a} + \sqrt{b}$ or

$|b-c| \geq \sqrt{b} + \sqrt{c}$ or

$|c-a| \geq \sqrt{c} + \sqrt{a}$

Find max $|A|$.

$\Longleftrightarrow$ $a < b < c$

$\sqrt{c} - \sqrt{a} \geq 1$

Let $B_k = \{k^2, k^2+1, \cdots, k^2+2k\}$

$\forall 1 \leq k \leq 43$, $B_{44} = \{44^2, \cdots, 2020\}$

Notice that

$B_1 \cup B_2 \cdots \cup B_{44}$ form a partition of $\{1, 2, \ldots, 2020\}$

; (b) gives that

$$|A \cap B_i| \leq 2 \quad \forall\, 1 \leq i \leq 44$$

$$\Rightarrow \quad |A| \leq 88$$

$A = \{1, 2^2, 3^2, \ldots, 44^2$

$1+1, 2^2+2, 3^2+3, \ldots, 44^2+44\}$

$1 < 1+1 < 2^2 < 2^2+2 < \cdots < 44^2 < 44^2+44$

we can easily prove: $\sqrt{n^2+n} - \sqrt{n^2-n} \geq 1$