Email training, N5

Level 3, October 11-17

Problem 5.1. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.

Solution 5.1. We consider the sequence

$$a_1 = 1, a_2 = 2,$$
$$a_{2n+1} = 2a_{2n},$$
$$a_{2n+2} = a_{2n+1} + r_n,$$

where r_n is the smallest natural number that cannot be written in the form $a_i - a_j$, with $i, j =\leq 2n+1$. It satisfies to the conditions of the problem

Problem 5.2. Prove the identity

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \ldots + (-1)^n \frac{\binom{n}{n}}{x+n}.$$

Solution 5.2. By applying the identity

$$\frac{1}{(x+a)(x+b)} = \frac{1}{a-b}\Big(\frac{1}{x+b} - \frac{1}{x+a}\Big)$$

multiple times one may get the following relation

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \sum_{k=0}^{n} \frac{A_k}{x+k}.$$

By multiplying both sides by $x(x+1)(x+2)\ldots(x+n)$ and by putting $n = -k$ one gets

$$n! = A_k \cdot (-k) \cdot (-k+1) \cdot (-k+2) \cdot \ldots \cdot (-1) \cdot 1 \cdot 2 \cdot \ldots \cdot (n-k)$$

so

$$A_k = \frac{(-1)^k A_k}{k!(n-k)!} = (-1)^k \binom{n}{k}.$$

Problem 5.3. Prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Solution 5.3. Let's apply Bernoulli inequality.

$$\Big(1 + \frac{5}{n}\Big)^n > 1 + n \cdot \frac{5}{n} = 6,$$

therefore

$$1 + \frac{5}{n} > 6^{1/n}.$$

Also

$$\Big(1 + \frac{-5}{6n}\Big)^n > 1 + n \cdot \frac{-5}{6n} = \frac{1}{6},$$
$$\Big(\frac{6n-5}{6n}\Big)^n > \frac{1}{6},$$
$$6 > \Big(\frac{6n}{6n-5}\Big)^n,$$
$$6^{1/n} > \frac{6n}{6n-5} = 1 + \frac{5}{6n-5}.$$

Problem 5.4. Let $x, y, z \geq 0$ and $x + y + z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

Solution 5.4. One has

$$3(x + y + z) = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx).$$

Hence it follows that

$$xy + yz + zx = \frac{1}{2}(3x - x^2 + 3y - y^2 + 3z - z^2).$$

Then

$$\sqrt{x} + \sqrt{y} + \sqrt{z} - (xy + yz + zx) =$$

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \frac{1}{2}(x^2 - 3x + y^2 - 3y + z^2 - 3z)$$

$$= \frac{1}{2}\sum_{cyc}(x^2 - 3x + 2\sqrt{x}) = \frac{1}{2}\sum_{cyc}\sqrt{x}(\sqrt{x} - 1)^2(\sqrt{x} + 2) \geq 0.$$

Problem 5.5. Let $a, b, c > 0$. Prove that

$$\frac{a+b}{a^2+b^2} + \frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution 5.5. By applying the AM-GM for the denominator one gets

$$\frac{a+b}{a^2+b^2} \leq \frac{a+b}{2ab} = \frac{1}{2}\Big(\frac{1}{a} + \frac{1}{b}\Big).$$

By applying the same estimation for 2 other expressions of the left side and by taking the sum we get the desired statement.

Problem 5.6. Let $n > 3$, $x_1, x_2, \ldots, x_n > 0$ and $x_1 x_2 \ldots x_n = 1$. Prove that

$$\frac{1}{1 + x_1 + x_1x_2} + \frac{1}{1 + x_2 + x_2x_3} + \ldots + \frac{1}{1 + x_n + x_nx_1} > 1.$$

Solution 5.6.

$$\frac{1}{1 + x_1 + x_1x_2} + \frac{1}{1 + x_2 + x_2x_3} + \ldots + \frac{1}{1 + x_n + x_nx_1} >$$

$$\frac{1}{1 + x_1 + x_1x_2 + x_1x_2x_3 + \ldots + x_1x_2\ldots x_{n-1}} +$$

$$\frac{1}{1 + x_2 + x_2x_3 + x_2x_3x_4 + \ldots + x_2x_3\ldots x_n} + \ldots +$$

$$\frac{1}{1 + x_n + x_nx_1 + x_nx_1x_2 + \ldots + x_nx_1\ldots x_{n-2}}.$$

Denote $S = 1 + x_1 + x_1x_2 + \ldots + x_1x_2\ldots x_{n-1}$. By multiplying the nominator and denominator of second term by x_1, of the third term by x_1x_2 and son on in n-th term by $x_1x_2\ldots x_{n-1}$ and by taking into account that $x_1x_2\ldots x_n = 1$ one gets

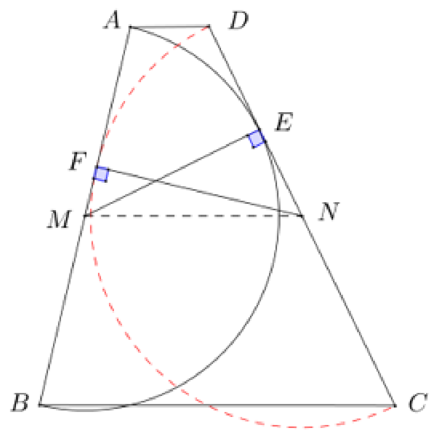
$$\frac{1}{1 + x_1 + x_1x_2} + \frac{1}{1 + x_2 + x_2x_3} + \ldots + \frac{1}{1 + x_n + x_nx_1} >$$

$$\frac{1}{S} + \frac{x_1}{S} + \frac{x_1x_2}{S} + \ldots + \frac{x_1x_2\ldots x_{n-1}}{S} = 1.$$

Problem 5.7. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

Solution 5.7. -

Let M and N be the midpoints of AB and CD, and let E,F be their projections on CD and AB, respectively. We know that the line CD is a tangent to the circle on AB as diameter, so ME is a radius in that circle, and $ME = AM=MB$. Then the line AB is a tangent to the circle on CD as diameter if and only if FN is a radius in that circle, which equivalent to $FN=DN=NC$.



Which equivalent to:

$$\frac{1}{2}AM \times FN = \frac{1}{2}DN \times ME, \quad \frac{1}{2}MB \times FN = \frac{1}{2}NC \times ME$$

Equivalent to:

$$K_{AMN} = K_{DMN}, \quad K_{BMN} = K_{NMC}$$

Equivalent to:

$$AD \parallel MN, \quad MN \parallel BC$$

Equivalent to:

$$AD \parallel MN \parallel BC$$

And we are done.

(we used K_{AMN} as the area of the triangle AMN, and so on.)