Email training, N2

August 31- September 7

Problem 2.1. For a positive integer n, let $f(n)$ denote the greatest odd divisor of n. For example, $f(11) = 11$ and $f(12) = 3$. Prove that for any positive integer n,

$$f(n+1) + f(n+2) + \ldots + f(2n) = n^2.$$

Solution 2.1. We give solution by induction on n. For $n = 1$ one has

$$f(2) = 1 = 1^2.$$

Assume that the statement is correct for n and let's prove it for $n+1$.

$$f(n+2) + f(n+3) + \ldots + f(2(n+1)) =$$
$$\Big(f(n+1) + f(n+2) + \ldots + f(2n)\Big) + f(2n+1) + \big(f(2n+2) - f(n+1)\big) =$$
$$n^2 + 2n + 1 = (n+1)^2,$$

since $f(2n+1) = 2n+1$ as well $f(2n+2) = f(2(n+1)) = f(n+1)$.

Problem 2.2. Find the smallest positive integer m for which there exists a positive integer n such that

$$\left|\frac{n}{m} - \frac{2}{5}\right| \le \frac{1}{100}.$$

Solution 2.2. Multiplying both sides by $5m$, we get

$$0 < \big|5n - 2m\big| \le \frac{m}{20}.$$

Since $|5n - 2m|$ is a positive integer, then $m \ge 20$.

For $m = 20$, the inequality is satisfied iff we can find n such that $5n - 40 = \pm 1$, which has no solution.

For $m = 21$, the inequality is satisfied iff we can find n such that $5n - 42 = \pm 1$, which has no solution.

For $m = 22$, we observe that the inequality is satisfied iff we can find n such that $5n - 44 = \pm 1$ which is happens when $n = 9$.

Answer: $m = 22$.

Problem 2.3. Find the greatest common divisor of $5^{300} - 1$ and $5^{200} + 6$.

Solution 2.3. Let $n = 5^{100}$. Then $5^{300} - 1 = n^3 - 1$ and $5^{200} + 6 = n^2 + 6$. We start by simplifying $d = gcd(n^3 - 1, n^2 + 6)$. Since

$$n(n^2+6) - 6(n^2+6) - 6n + 1,$$

we see that $d = gcd(6n+1, n^2+6)$. Then we see that

$$n(6n+1) - 6(n^2+6) = n - 36,$$

Since 6 and $6n+1$ are coprime, this means that $d = gcd(6n+1, n-36)$. Finally, we have

$$6n + 1 - 6(n - 36) = 217 = 7 \cdot 31,$$

and since 6 and $6n+1$ are coprime we see that $d = gcd(n-36, 7 \cdot 31)$. To calculate this we reduce $5^{100} - 36$ modulo 7 and module 31. Since $5^6 \equiv 1[7]$, then

$$5^{100} - 36 \equiv 5^4 - 1 \equiv 2 - 1 - 1[7],$$

and since $5^3 \equiv 1[31]$, then

$$5^{100} - 36 \equiv 5 - 5 = 0[31].$$

So $d = 7$.

Answer: 7.

Problem 2.4. Find all pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{lcm(x,y)} + \frac{1}{gcd(x,y)} = \frac{1}{2}.$$

Solution 2.4. We put $x = du$ and $y = dv$ where $d = gcd(x, y)$. So we have $(u, v) = 1$. From the conclusion of the problem we have

$$\frac{1}{du} + \frac{1}{dv} + \frac{1}{d} + \frac{1}{duv} = \frac{1}{2},$$

$$u + v + uv + 1 = \frac{duv}{2},$$

or

$$2(u+1)(v+1) = duv.$$

Since $gcd(v, v+1) = 1$ therefore v divides $2(u+1)$.

Case 1. $u = v$. Then $u = v = 1$ and we get $d = 2(1+1)(1+1) = 8$ which leads $x = y = 8$.

Case 2. $u < v$. Then $u + 1 \leq v$ so $2(u+1) \leq 2v$ so $\frac{2(u+1)}{v}$ is equal either 1 or 2.

If $\frac{2(u+1)}{v} = 1$ then we have $(d-2)u = 3$ which means $(d, u) = (3, 3)$ or $(d, u) = (5, 1)$. So we get $(x, y) = (9, 24)$ or $(x, y) = (5, 20)$.

If $\frac{2(u+1)}{v} = 2$ then we have $(d-2)u = 4$ which means $(d, u) = (3, 4)$ or $(d, u) = (4, 2)$ or $(d, u) = (6, 1)$. So we get $(x, y) = (12, 15)$ or $(x, y) = (8, 12)$ or $(x, y) = (6, 12)$.

Case 3. $u > v$. This is identical to the case 2.

Answer: $(8, 8), (9, 24), (24, 9), (5, 20), (20, 5), (12, 15),$
$(15, 12), (8, 12), (12, 8), (6, 12), (12, 6)$.

Problem 2.5. Ali has choosen 8 cells of the chessboard 8×8 such that no any two lie on the same line or in the same row (we call it general configuration). On each step Baba chooses 8 cells in general configuration and puts coins on them. Then Ali shows all coins that are out of cells chosen by Ali. If Ali shows even number of coins then Baba wins, otherwise Baba removes all coins and makes the next move. Find the minimal number of moves that Baba needs to guarantee the win.

Solution 2.5. Now lets show how Baba can win in 2 moves. On first step he puts 8 coins on the diagonal of the board. If he does not win then on that diagonal there are odd number of cells chosen by Ali. Let on the diagonal cell A is chosen and the diagonal cell B is not chosen. On next step Baba puts coins on 6 diagonal cells (but A and B) as well puts 2 coins on two other vertices of the rectangle with vertices A and B. Note, that these 2 cells are not chosen, since they are on the same row or column with the cell A which is chosen. So the answer will be even. It's obvious that he can't win in 1 move, since we may assume that Ali chooses his cells after Baba makes his first move.

Answer: 2 moves.

Problem 2.6. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the number of functions $f : S \to S$ such that $f(f(x)) = x$ for all $x \in S$.

Solution 2.6. Let A be an arbitrary set and let $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Then the elements of A can be divided into two categories: Elements a such that $f(a) = a$, and elements b such that $f(b) \neq b$. In the latter case, let $c = f(b)$. Then $f(c) = b$. Thus, the set A is partitioned into singletons of the form $\{a\}$ (where $f(a) = a$), and pairs of the form $\{b, c\}$ (where $f(b) = c$ and $f(c) = b$). Conversely, any partition of A into singletons and pairs determines a function f such that $f(f(x)) = x$ for all $x \in A$. Thus, the number of such functions f on a set A is equal to the number of partitions of A into singletons and pairs. So, we count the number of partitions of $S = \{1, 2, 3, 4, 5, 6, 7\}$ into singletons and pairs. For a positive integer n, let t_n denote the number of such partitions of a set with n elements. Let $S_n = \{x_1.x_2, \ldots, x_n\}$, and consider a partition of S_n into singletons and pairs. In any such partition, x_n is either a singleton or a member of a pair.

If x_n is a singleton, then the remaining $n-1$ elements are partitioned into singletons and pairs, so the number of such partitions is simply t_{n-1}.

If x_n is a member of a pair, then the pair is of the form $\{x_k, x_n\}$, where $1 \leq k \leq n-1$, and the remaining $n-2$ elements are partitioned into singletons and pairs. There are $n-1$ choices for k, so the number of such partitions is $(n-1)t_{n-2}$.

Therefore,

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

for all $n \geq 3$. We see that $t_1 = 1$ and $t_2 = 2$, so

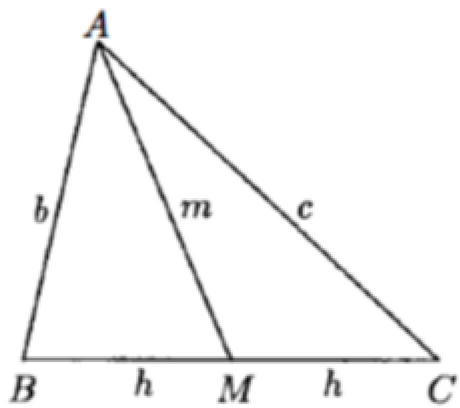
$$\begin{aligned}
t_3 &= t_2 + 2t_1 = 4, \\
t_4 &= t_3 + 3t_2 = 10, \\
t_5 &= t_4 + 4t_3 = 26, \\
t_6 &= t_5 + 5t_4 = 76, \\
t_7 &= t_6 + 6t_5 = 232.
\end{aligned}$$

Answer: 232.

Problem 2.7. In the triangle ABC the median AM is drawn. Is it possible that the radius of the circle inscribed to the triangle ABM be twice bigger than the radius of the circle inscribed to the triangle ACM?

Solution 2.7. -

Let b, c, m and $2h$ be the lengths of AB, AC, AM respectively, and let r_B and r_C be the radii of the inscribed circles for triangles ABM, ACM.



Since the area of a triangle is given by half the circumference times the in-radius, and since triangles ABM, ACM have equal area (equal base and height) we have

$$\frac{1}{2}(b+h+m)r_B = \frac{1}{2}(c+h+m)r_C.$$

So, if $r_B = 2r_C$ then

$$b+h+m = \frac{1}{2}(c+h+m),$$

leading to

$$h+m+2b = c.$$

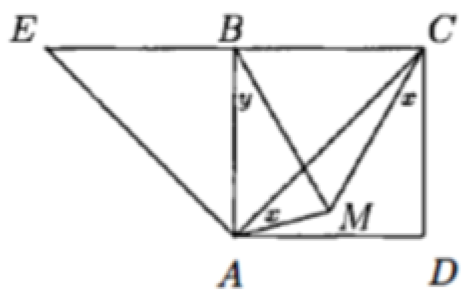
But h, m and c are sides of ΔAMC so $c \leq h+m$. Hence $b = 0$. and ΔABC is degenerate with $A = B$.

So the required solution is impossible unless both radii are zero.

Problem 2.8. Let a point M is chosen inside the square $ABCD$ such that $\angle MAC = \angle MCD$. Find $\angle ABM$.

Solution 2.8. -

Extend the line CB to E, as shown, with $BC = BE$, and construct the line AE.



Since $\angle ACM = (45 - x)^\circ$, and $\angle CAM = x^\circ$,

$$\angle AMC = (180 - x - (45 - x))^\circ = 135^\circ.$$

Furthermore, since $\angle AEB = 45^\circ$, quadrilateral $ECMA$ is cyclic. We now note that $\angle EAC = 90^\circ$, and so EC is a diameter of this exscribed circle. Therefore $BA = BM = BC$ (all radii of the exscribed circle). Thus ΔBAM is isosceles and $y = 180 - 2\angle BAM = 180 - 2(45 + x) = 90 - 2x$.