Test-10, April 25, 2021
Level 3

Problem 1. Anna and Ben are playing with a permutation p of length 2020, initially $p_i = 2021 - i$ for $1 \leq i \leq 2020$. Anna has power A, and Ben has power B. Players are moving in turns, with Anna moving first.

In his turn player with power P can choose any P elements of the permutation and rearrange them in the way he/she wants.

Ben wants to sort the permutation, and Anna wants to not let this happen. Determine if Ben can make sure that the permutation will be sorted (of form $p_i = i$ for $1 \leq i \leq 2020$) in finitely many turns, if

a) $A = 1000, B = 1000$

b) $A = 1000, B = 1001$

c) $A = 1000, B = 1002$

Problem 2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2) + 4y^2 f(y) = (f(x-y) + y^2)(f(x+y) + f(y)).$$

Problem 3. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

Problem 4. Find all positive integers n such that n can be expressed as $n = d_1 + d_2 + \ldots + d_{\varphi(n)+1}$, where $d_1, d_2, \ldots, d_{\varphi(n)+1}$ are positive factors of n, not necessarily distinct, and $\varphi(n)$ is the number of positive integers up to n that are relatively prime to n.