Email training, N7-8

October 8-19, 2019

Problem 7.1. Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Problem 7.2. Let x, y, z be nonnegative real numbers with $x + y + z = 1$. Show that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.

Problem 7.3. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Problem 7.4. Given a set M of 2019 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4-th power of an integer.

Problem 7.5. Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

Problem 7.6. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 2019$ for every n.

Problem 7.7. Let all vertices of a polygon are lattice points (point with integer coefficients) in a coordinate plane. Prove that the area of the polygon is equal to

$$I + \frac{1}{2}B - 1,$$

where I is the number of lattice points in the interior and B being the number of lattice points on the boundary.

Problem 7.8. Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

Problem 7.9. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Problem 7.10. Let AB and CD are segments lying on the two sides of an angle whose vertex is O, such that A is between O and B, as well as C is between O and D. The line connecting the midpoints of the segments AD and BC intersects AB at M and CD at N. Prove that $\frac{OM}{ON} = \frac{AB}{CD}$.

Problem 7.11. Let all vertices of convex quadrilateral $ABCD$ lie on the circumference of a circle with center O. Let F be the second point of intersection of the circumcircles of the triangles ABO and CDO. Prove that the circle passing through the points A, F and D also passes through the point of intersection of the segments AC and BD.

Solution submission deadline October 19, 2019