Email training, N1

August 25- 31, 2019

Problem 1.1. Let $a \neq 0$ and let x_1 and x_2 are the roots of the equation

$$x^2 + ax - \frac{1}{2a^2} = 0.$$

Prove that

$$x_1^4 + x_2^4 \geq 2 + \sqrt{2}.$$

Problem 1.2. Prove that at least one coefficient of the polynomial

$$P(x) = (x^4 + x^3 - 3x^2 + x + 2)^n$$

is negative.

Problem 1.3. Prove that $lcm(1, 2, 3, \ldots, 2n) = lcm(n + 1, n + 2, \ldots, 2n)$, where lcm is the least common multiplier.

Problem 1.4. Four positive integers are given. It is known that the sum of squares of any two of them is divisible by product of other two numbers ($cd|a^2 + b^2$). Prove that at least three numbers are equal.

Problem 1.5. The endpoints of N arcs split the circle into $2N$ equal arcs of length 1. It is known that each arc splits the circle into 2 parts of even length. Prove that N is even.

Problem 1.6. The robber's car speed is 90% of policeman's car speed. Robber and policeman are along the line and policeman doesn't know in which direction goes the robber. Prove that the policeman may catch the robber.

Problem 1.7. Angle A of the acute-angled triangle ABC equals 60^o. Prove that the bisector of one of the angles formed by the altitudes drawn from B and C, passes through the circumcircle 's centre.

Problem 1.8. The bisectors of the angles A, B, C of a triangle $\triangle ABC$ intersect with the circumcircle c_1 of triangle ABC at A_2, B_2, C_2 respectively. The tangents of c_1 at A_2, B_2, C_2 intersect each other at A_3, B_3, C_3 (the points A_3 and A lie on the same side of BC,the points B_3 and B on the same side of CA, also C_3 and C on the same side of AB).The incircle triangle ABC is tangent to BC, CA, AB at A_1, B_1, C_1 respectively. Prove that A_1A_2, B_1B_2, C_1C_2, AA_3, BB_3 and CC_3 are concurrent.

Solution submission deadline August 31, 2019