Hello !

1)

$$\begin{cases} \boxed{\begin{matrix}(x+y)^3 = 8z \\ (y+z)^3 = 8x\end{matrix}} \\ (z+x)^3 = 8y \end{cases}$$

$$(2x)^3 = 8x \Rightarrow 8x^3 = 8x \Rightarrow x^3 = x \Rightarrow$$

$$\boxed{x = 0, -1, 1}$$

Manipulations.

$$\boxed{(0,0,0),\ (1,1,1),\ (-1,-1,-1)}.$$

$$(x+y)^3 - (y+z)^3 = 8(z-x)$$

$$\parallel$$

$$\underbrace{(x+y-(y+z))}_{x-z}\left[(x+y)^2 + (x+y)(y+z) + (y+z)^2\right] = 8(z-x)$$

$$(x-z)\left\{\left[\underbrace{(x+y)}_{A}{}^2 + \underbrace{(x+y)}_{A}\underbrace{(y+z)}_{B} + \underbrace{(y+z)}_{B}{}^2 + 8\right]\right. = 0$$

I $\underline{x=z}$

II $A^2 + AB + B^2 + 8 = \left(A + \frac{1}{2}B\right)^2 + \frac{3}{4}b^2 + 8 \geq 8 \neq 0$,

So $\underline{x=z}$ $\underline{y=z}$ $\underline{z=x}$ $\Rightarrow$ $\boxed{x=y=z}$

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2) Which sing squares can be removed from 7x7 so that rest can by tiled with

I coloring

0	1	2	0	1	2	0
1	2	0	1	2	0	1
2	0	1	2	0	1	2
0	1	2	0	1	2	0
1	2	0	1	2	0	1
2	0	1	2	0	1	2
0	1	2	0	1	2	0

II coloring

0	1	2	0	1	2	0
2	0	1	2	0	1	2
1	2	0	1	2	0	1
0	1	2	0	1	2	0
2	0	1	2	0	1	2
1	2	0	1	2	0	1
0	1	2	0	1	2	0

Observation

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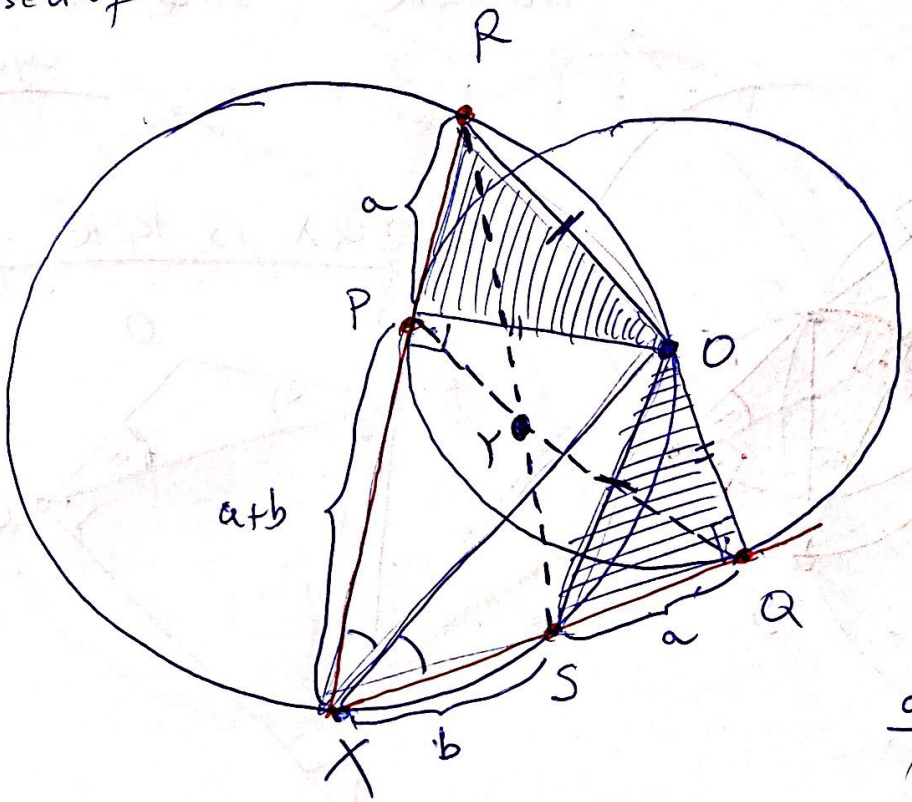
3) YO bisect of SXR

Prove PQ bisects RS



POQX is kite

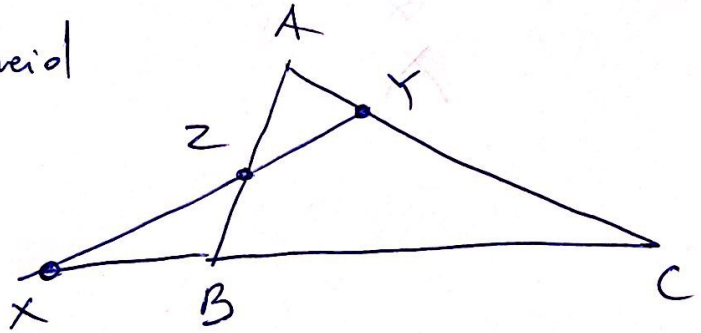
We need $RY = YS$

By Menelaus:

$$\frac{a+b}{a} \cdot \frac{RY}{YS} \cdot \frac{a}{a+b} = 1$$ so

$RY = YS$ ■

1) Reneiol



X, Y, Z are collinear $\iff$ $\boxed{\frac{CY}{YA} \cdot \frac{AZ}{ZB} \cdot \frac{XB}{XC} = 1}$

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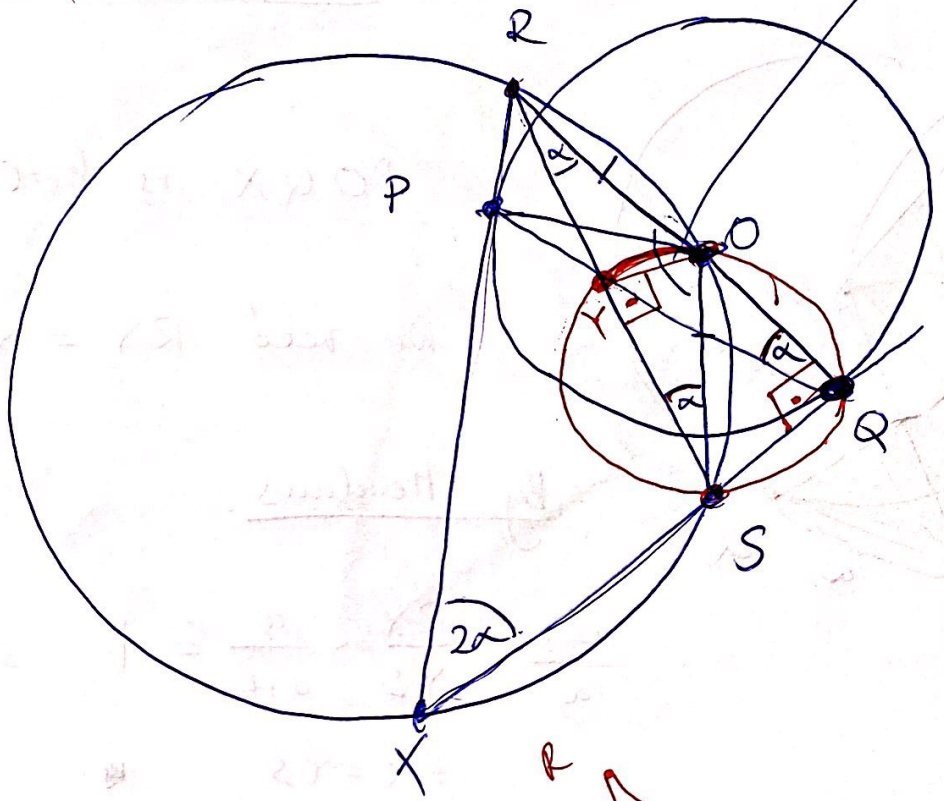
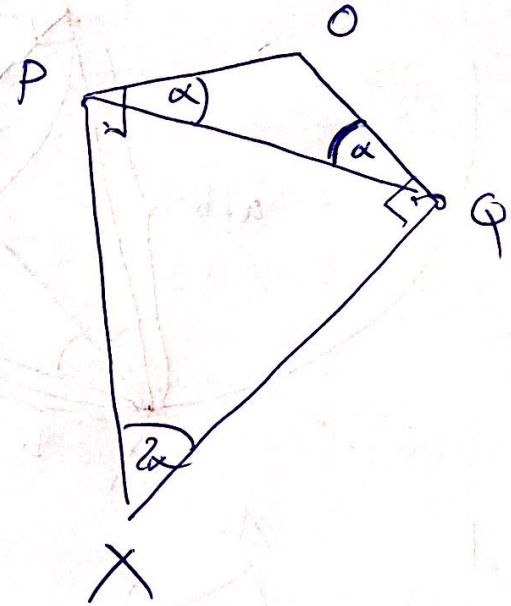
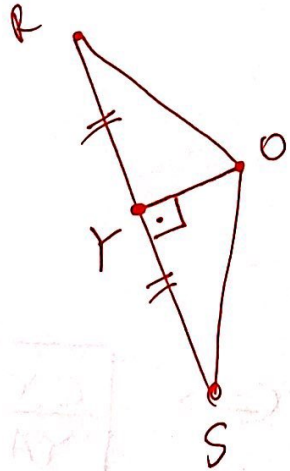
11

$180 - 2\alpha$ since ROSX is cyclic



POQX is kite





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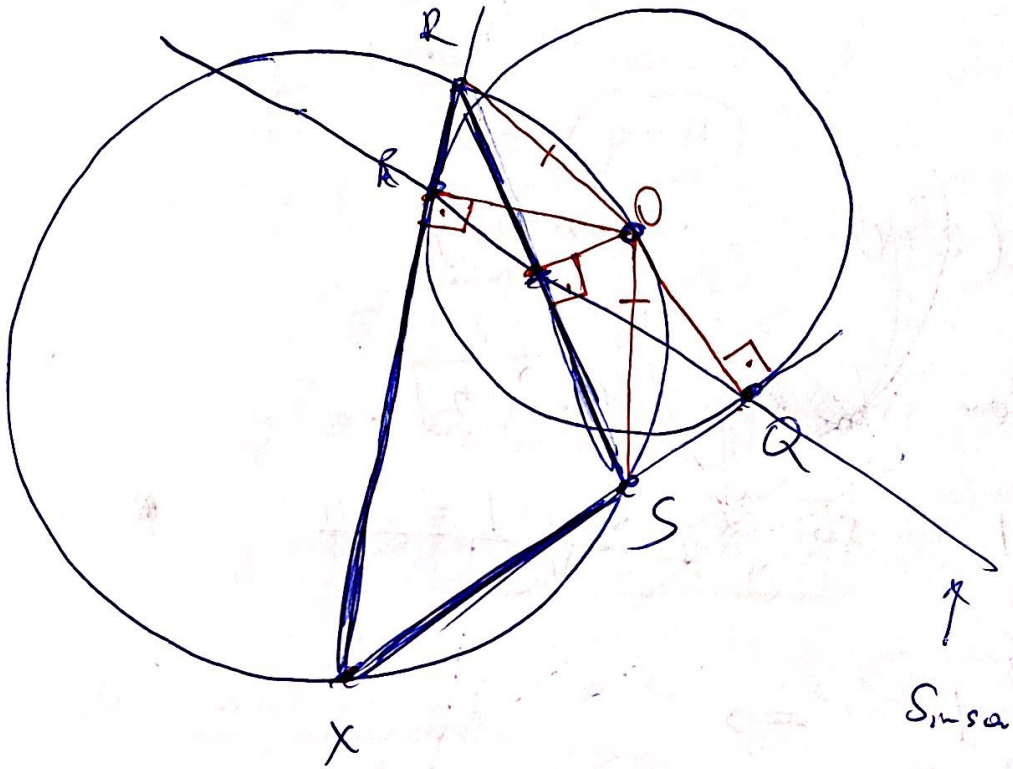
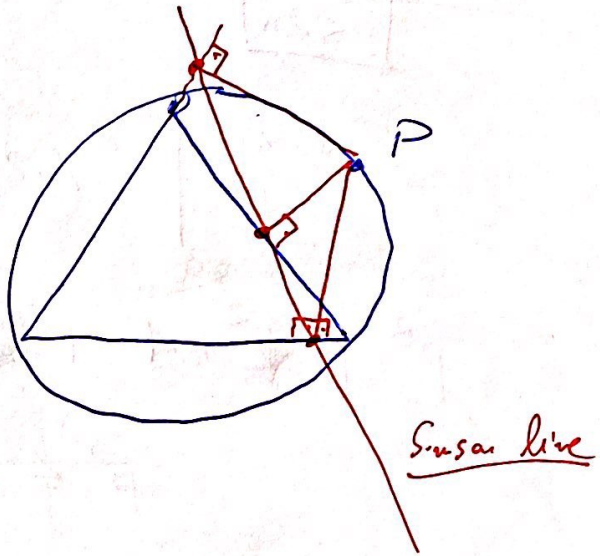


Simson line of O wrt XRS

Simson line



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4) p – prime $\quad n > 0$ integer $\quad p^2 \mid \prod_{k=1}^{n} (k^2+1)$

$\boxed{p < 2n}$

$$p^2 \mid (1^2+1)(2^2+1)\cdots(n^2+1)$$

I) $p^2 \mid k^2+1$ for some $k \in \{1, \dots, n\}$.

$p^2 \leq k^2+1$ so $\underline{p < k+1 \leq 2n}$ ∎

II $p \mid k^2+1, \quad p \mid l^2+1 \implies$

$p \mid (k^2+1) - (l^2+1) = (k-l)(k+l)$ so

$p \mid k-l$ or $p \mid k+l$

$p \leq \max\{k-l, k+l\} < 2n$ ∎

$n + n-1$? $< 2n$.

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1) Solve $\boxed{x^5 = y^2 + 4}$

$2 \mid p-1$

$5 \mid p-1$ $\rightsquigarrow$ $p = 11$

$x^5 \equiv \boxed{-1, 0, 1} \mod 11$

$y^2 \equiv 0, 1, 3, 4, 5, 9 \mod 11$

$y^2 + 4 \equiv \boxed{2, 4, 5, 7, 8, 9}$

No solutions ↯

2) Find all rationals a, b s.t. $a^2 + ab + b^2 = 2$.

$\boxed{y \neq 0}$

$a = \frac{x}{y}$, $b = \frac{z}{y}$

$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6}$

x, y, z — integer.

$\left(\frac{x}{y}\right)^2 + \left(\frac{x}{y} \cdot \frac{z}{y}\right) + \left(\frac{z}{y}\right)^2 = 2 \Rightarrow \boxed{x^2 + xz + z^2 = 2y^2}$

Finish now!

$x = 2x_1$, $z = 2z_1$, $y = 2y_1$

$\cancel{4}x_1^2 + \cancel{4}x_1z_1 + \cancel{4}z_1^2 = \cancel{8}y_1^2 \mid :4 \Rightarrow x = y = z = 0$ ↯

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3) $\underline{p \geq 3}$

$$p \mid \underbrace{11\ldots1}_{p}\ \underbrace{22\ldots2}_{p}\ \ldots\ldots\ \underbrace{99\ldots9}_{p} - 123456789$$

$$\underbrace{11\ldots1}_{p}\ \underbrace{\underbrace{00\ldots0}_{p}\ \ldots\ldots\ \underbrace{00\ldots0}_{p}}_{8p \times 0'}$$

$$\underbrace{22\ldots2}_{p}\ \underbrace{00\ldots0}_{7p \times 0'}$$

$+$

$$\underbrace{111\ldots1}_{p} = \frac{10^p-1}{9} \qquad \underbrace{33\ldots3}_{p}\ \underbrace{0\ldots0}_{6p \times 0'}$$

$$\frac{22\ldots2}{p} = \frac{2}{9}(10^p-1)$$

$$\vdots$$

$$\underbrace{11\ldots1}_{p} \cdot 10^{8p} + \underbrace{22\ldots2}_{p} \cdot 10^{7p} + \underbrace{33\ldots3}_{p} \cdot 10^{6p} \ldots$$

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$$\underbrace{111\ldots1}_{n} = \frac{10^n - 1}{9}$$

$$9\left(\underbrace{1111\ldots1}_{p}\ \underbrace{22\ldots2}_{p}\ \ldots\ \underbrace{99\ldots9}_{p} - 123\ldots456789\right) =$$

$$= 9\left(\underbrace{111\ldots1}_{9p} + \underbrace{111\ldots1}_{8p} + \ldots + \underbrace{111\ldots1}_{p} - \underbrace{11\ldots1}_{9} - \underbrace{11\ldots1}_{8} - \ldots - 1\right) =$$

$$= 10^{9p} + 10^{8p} + \ldots + 10^{p} - 10^{9} - 10^{8} - \ldots - 10$$

$$\frac{10^{9p}-1}{9} \qquad 10^{kp} \equiv 10^{k} \pmod{p} \qquad k = 1 \ldots 9$$

LFT.

$$10^{9p} - 1 + 10^{8p} - 1 \ldots 10^{p} - 1 - (10^{9} - 1) - (10^{8} - 1) \ldots (10 - 1)$$

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$$10^{8p} \cdot \underbrace{11..1}_{p} = 10^{8p} \cdot \frac{10^{p}-1}{9} = \frac{10^{9p}-10^{8p}}{9}$$

$$10^{7p} \cdot \underbrace{22..2}_{p} = 10^{7p} \cdot \frac{2}{9}\left(10^{p}-1\right) = \frac{2}{9}\left(10^{8p}-10^{7p}\right)$$

$$\frac{3}{9}\left(10^{7p}-10^{6p}\right)$$

$$\frac{1}{9}\left(10^{9p}-10^{8p}+2\left(10^{8p}-10^{7p}\right)+3\left(10^{7p}-10^{6p}\right)+\dots\right)$$

$$\frac{1}{9}\left(10^{9p}+10^{8p}+10^{7p}+\dots+10^{2p}+10^{p}\right)-10^{0}\cdot\dots$$

$$123\dots9 = 10^{8}+2\cdot10^{7}+3\cdot10^{6}+\dots+9\cdot10^{0}$$

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$$x^4+y = x^3+y^2$$

~ Square between squares ~

$$x^4-x^3 = y^2-y \rightarrow 4x^4-4x^3 = 4y^2-4y \quad /+1$$

$$\boxed{4x^4-4x^3+1 = (2y-1)^2}$$

If $x \geq 2$ or $x \leq -2$ then square

$|x| \geq 2$

$$(2x^2-x-1)^2 < 4x^4-4x^3+1 < (2x^2-x)^2$$

$$\cancel{4x^4}+x^2+1-\cancel{4x^3}-4x^2+2x \overset{?}{<} \cancel{4x^4}-\cancel{4x^3}+1 \qquad \cancel{4x^4}-\cancel{4x^3}+1 < \cancel{4x^4}+x^2-\cancel{4x^3}$$

$$-3x^2+2x < 0 \qquad \boxed{x^2 > 1}$$

$$3x^2 > 2x$$

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so $\boxed{\text{no}}$ solutions $x \geq 2$ or $x \leq -2$

so $x \in \{-1, 0, 1\}$

$$\boxed{(x,y) \in \{(0,0), (0,1), (1,0), (1,1), (-1,2), (-1,-1)\}}$$

$x = -1$

$$(-1)^4 + y = (-1)^3 + y^2$$

$$1 + y = -1 + y^2 \Rightarrow \boxed{y^2 - y - 2 = 0}$$

$$\Delta = 1 + 8 = 3^2$$

$$\boxed{\begin{aligned} y_1 &= \frac{-1+3}{2} = 1 \\ y_2 &= \frac{-1-3}{2} = -2 \end{aligned}}$$

$$\boxed{(y-1)(y+2) = 0}$$

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Find all positive integers (a,b) s.t. $a^3+6ab+1$ and $b^3+6ab+1$ are cubes.

~ cubes between cubes ~

<u>WLOG</u> $\boxed{a \le b}$

$$b^3 < \underbrace{b^3+6ab+1}_{\text{cube}} \le b^3+6b^2+1 < b^3+6b^2+12b+8 = (b+2)^3$$

$$\cancel{b^3}+6ab+\cancel{1} = (b+1)^3 = \cancel{b^3}+3b^2+3b+\cancel{1}, \text{ so}$$

$$6ab = 3b^2+3b \qquad b \neq 0$$

$$\boxed{b = 2a-1}$$

$$a^3+6ab+1 = a^3+6a(2a-1)+1 = a^3+12a^2-6a+1 \leftarrow \text{cube}$$

$$a^3 \le a^3+6a^2-6a < \underset{\text{cube}}{a^3+12a^2-6a+1} < a^3+12a^2+48a+64 = (a+4)^3$$

$$a^3+12a^2-6a+1 \in \{(a+1)^3, (a+2)^3, (a+3)^3\}$$

1) $a^3+12a^2-6a+1 = (a+1)^3 \rightsquigarrow 9a^2-9a=0 \rightsquigarrow a=0 \vee 1$

2) $a^3+12a^2-6a+1 = (a+2)^3 \rightsquigarrow 6a^2-18a-\boxed{7}=0 \rightarrow$ no sol

3) $a^3+12a^2-6a+1 = (a+3)^3 \rightsquigarrow 3a^2-33a-26=0 \rightsquigarrow$ no sol

So since $a>0$ $\boxed{a=1}$

$b=2a-1=1$ $\boxed{(1,1)}$

Solve in integers $\boxed{2x^6+y^7=11}$

$6 \mid p-1$ $\rightsquigarrow$ $\boxed{43}$ but no such prime works.

$7 \mid$

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$$2x^6 + y^7 = 11$$

$x^6 \bmod 43$ $\quad 1, 4, 11, 16, 21, 35, 41$

$y^7 \bmod 43$ $\quad 1, 6, 7, 36, 37, 42$

$$2 \cdot x^6 + y^7 \not\equiv 11 \mod 43$$

Next $\quad p > 3 \quad \boxed{3k+2} \quad a_k = k^2 + k + 1$ for $k = 1, 2, 3, \ldots, p-1$

$$a_1 a_2 \cdots a_{p-1} \equiv 3 \pmod p$$

$p = 3k+2$ then

$1^3, 2^3, 3^3, \ldots, (p-1)^3$ are different <u>mod p</u>.

given $\{1, 2, \ldots, p-1\}$.

$2^3 - 1, 3^3 - 1, \ldots, (p-1)^3 - 1 \rightsquigarrow \{1, 2, 3, \ldots, p-2\}$.

$$\prod_{k=2}^{p-1} (k^3 - 1) \equiv (p-2)! \pmod p$$

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$\boxed{k^3-1=(k-1)\cdot a_k}$. for $k=2,3,4,\dots,p-1$

$$\prod_{k=2}^{p-1}(k^3-1) \equiv = \prod_{k=2}^{p-1}(k-1)\,a_k = \cancel{(p-2)!}\; a_2 a_3 \cdots a_{p-1} \pmod p$$

$$\equiv \cancel{(p-2)!}$$

$$p \nmid (p-2)! = 1\cdot 2 \cdots (p-2).$$

$$a_2 a_3 a_4 \cdots a_{p-1} \equiv 1 \pmod p$$

$$\underbrace{a_1}_{1^2+1+1} a_2 \cdots a_{p-1} \equiv 3 \pmod p.$$

<u>Prove</u> There are no positive integer a, b s.t. $2a^2+1$, $2b^2+1$, $2(ab)^2+1$ are squares. $2a^2+1$, $2b^2+1$, $2(ab)^2+1$

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Suppose $2a^2+1$, $2b^2+1$, $2(ab)^2+1$ are squares.

WLOG $a \geq b$

$$\underset{\text{square}}{4}\,\underset{\text{square}}{(2a^2+1)}\,\underset{\text{square}}{(2(ab)^2+1)} \text{ is also square}$$

$$\|$$

$$(4a^2b+b)^2+8a^2-b^2+4 \text{ — square}$$

$$(4a^2b+b)^2 < \underset{\text{square}}{(4a^2b+b)^2}+8a^2-b^2+4 < (4a^2b+b+1)^2 =$$

$$= (4a^2b+b)^2+8a^2b+2b+1$$

$\lightning$

1) $p > 3$ – prime. Prove that there exists x, y, k integers s.t.

$0 < 2k < p$ and $kp + 3 = x^2 + y^2$. $kp = x^2 + y^2 - 3$

$0^2, 1^2, \dots, \left(\frac{p-1}{2}\right)^2$ $\quad 3-0^2, 3-1^2, \dots, 3-\left(\frac{p-1}{2}\right)^2$

diff. mod p?

$i^2 \equiv j^2 \mod p$

$p \mid (i-j)(i+j)$

$p \mid (i-j)$ or $(i+j) < p$.

$\neq$ $\leq p$.

Pigeonhole principle $i^2 = j^2$ ....

$3 - i^2 \equiv 3 - j^2$

In total we have $p+1$ numbers.

$$i^2 \equiv 3 - j^2 \pmod p \Rightarrow$$

$$p \mid i^2 + j^2 - 3$$

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$p \mid i^2+j^2-3$

$i^2+j^2-3 \leq \left(\frac{p-1}{2}\right)^2+\left(\frac{p-1}{2}\right)^2-3 < \frac{p^2}{2}$ so

$k := \frac{i^2+j^2-3}{p} < \frac{\frac{p^2}{2}}{p} = \frac{p}{2}$

$0 < \boxed{k < 2p}$

$x := i$

$y := j$

$i^2+j^2 \leq 3$

$0 \leq$

$0 < 2k < p$

$x := i \quad y := j \quad k = \frac{x^2+y^2-3}{p}$

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Lemma (Thue)

Let n is a positive integer, and a – number coprime to n, $(a,n)=1$.

There exist integers $0 < x, y \le \sqrt{n}$ s.t.

$$ax \equiv \pm y \pmod{n}$$

$\boxed{12:45}$. Consider all values

$0, 1, \dots, k$

$(k+1)$

$x \cdot a - y$ where $0 \le x, y \le \lfloor \sqrt{n} \rfloor$

How many numbers we have?

$0, 1, 2, \dots, \lfloor \sqrt{n} \rfloor \rightsquigarrow \lfloor \sqrt{n} \rfloor + 1$ possibilities for x

— " — $\rightsquigarrow \lfloor \sqrt{n} \rfloor + 1$ possibilities for y

$$\boxed{(\lfloor \sqrt{n} \rfloor + 1)^2 > n} \rightsquigarrow \begin{matrix} ax_1 - y_1 \\ ax_2 - y_2 \end{matrix} \text{ give the same rem. modulo } n.$$

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$$ax_1 - y_1 \equiv ax_2 - y_2 \pmod{n}$$

- $x_1 = x_2 \rightsquigarrow$ $\cancel{ax_1} - y_1 \equiv \cancel{ax_1} - y_2 \mod n$

  $y_1 \equiv y_2 \mod n$ ↯

  $0 \le y_1, y_2 \le \lfloor \sqrt{n} \rfloor$

- $y_1 = y_2 \rightsquigarrow$ $ax_1 - \cancel{y_1} \equiv ax_2 - \cancel{y_1} \pmod{n}$

  $\cancel{a}x_1 \equiv \cancel{a}x_2 \pmod{n}$

  $x_1 \equiv x_2 \pmod{n}$ $\quad 0 \le x_1, x_2 \le \lfloor \sqrt{n} \rfloor$ ↯

WLOG $x_1 > x_2$,

$$a(\underbrace{x_1 - x_2}_{X}) \equiv y_1 - y_2 \pmod{n} \qquad 0 < x, y \le \lfloor \sqrt{n} \rfloor$$

$$y := |y_1 - y_2| \qquad \blacksquare$$

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Wilson theorem ?

$$(p-1)! \equiv -1 \pmod{p}$$

$a \in \{1, 2, \dots, p-1\}$ you can find b s.t.

$b := a^{-1}$ $\quad ab \equiv 1 \pmod{p}$

Euclid Lem: $(a, p) = 1 \Rightarrow ax + py = 1$ for some x, y

$\gcd(a,p) = 1$

$1 \quad 2, 3, \dots, p-2 \quad p-1$

$(a, a^{-1}) \equiv 1 \bmod p$

$ax \equiv -1 \bmod p$, $x = b$.

$1 \cdot p-1 (\prod 1) \equiv -1 \pmod{p}$

All $1, 2, \dots, p-1$ has inverse mod p.

$a^{-1} \equiv a \bmod p$

$a^2 \equiv 1 \bmod p \rightsquigarrow p \mid a-1$, $p \mid a+1$ $\rightsquigarrow$ $a = 1$, $a = p-1$

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$5 = 2^2 + 1^2$, $\quad 7 \neq a^2 + b^2$

$17 = 4^2 + 1^2$

<u>Fermat theorem</u> $\boxed{p = 4k+1 \text{ is sum of 2 squares.}}$

<u>Proof</u>

$\boxed{1^\circ}$ $p \equiv 1 \pmod 4 \Longrightarrow \exists\, n : p \mid n^2+1$

$p = 4k+1$ take $n := \left(\frac{p-1}{2}\right)! = (2k)!$

Wilson

$$-1 \equiv (p-1)! \equiv 1 \cdot 2 \cdots \left(\frac{p-1}{2}\right) \cdot \left(\frac{p+1}{2}\right) + \cdots p-1 \equiv$$

$$\equiv 1 \cdot 2 \cdots \left(\frac{p-1}{2}\right) \underbrace{\left(p - \frac{p-1}{2}\right)}_{-\frac{p-1}{2}} \left(p - \frac{p-3}{2}\right) \cdots \underbrace{(p-1)}_{-1} \equiv$$

$$\equiv (-1)^{\frac{p-1}{2}} \left(1 \cdot 2 \cdots \frac{p-1}{2}\right)^2 \equiv (-1)^{2k} \cdot (2k!)^2 \equiv$$

$$\equiv ((2k)!)^2 \pmod p$$

$$\boxed{p \mid \underbrace{(2k!)^2}_{= n} + 1}$$

2°) $\underline{p = 4k+1}$, from 1° $p \mid n^2+1$ for some n.

$\Downarrow$

$\Rightarrow (p, n) = 1 \overset{\text{by Thue Lemma}}{=\!=\!=} \exists \; n^2 \equiv -1 \pmod{p}$

$0 < x, y < \sqrt{p}$ $\notin \mathbb{Z}$

$nx \equiv \pm y \pmod{p}$ ^2 } Thue

$-x^2 \equiv n^2 \cdot x^2 \equiv y^2 \pmod{p}$, so

$$\boxed{p \mid x^2+y^2}$$

$$0 < \underset{\overset{\shortparallel}{p}}{x^2+y^2} < (\sqrt{p})^2 + (\sqrt{p})^2 = 2p$$

$$\boxed{p = x^2+y^2}$$

$\blacksquare$

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Conclusions $p = 4k+1$ – sum of 2 squares.

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

p^k – $p \equiv 1 \pmod 4$ is sum of 2 squares

$$n = \underbrace{\boxed{2^{\alpha}}}_{A^2+B^2} \cdot \underbrace{\boxed{\underbrace{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}_{p_i \equiv 1 \pmod 4}}}_{C^2+D^2} \cdot \underbrace{\underbrace{q_1^{\beta_1} \cdots q_l^{\beta_l}}_{q_i \equiv 3 \pmod 4}}_{E^2+F^2}$$

$2 = 1^2+1^2$

$2^{\alpha} = A^2+B^2$

$2^{2k} = (2^k)^2 + 0^2$

$2^{2k+1} = (2^k)^2 + (2^k)^2$

Supp $2 \mid \beta_i$ $\quad \beta_i = 2 \cdot \gamma_1$

$q_1^{\beta_1} = (q_1^{\gamma_1})^2$

$$(E^2+F^2) \cdot R^2 = (ER)^2 + (FR)^2$$

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$n = 2k + 1$

$$n = (A^2 + B^2) \cdot \underbrace{q_1 \cdot \ldots \cdot q_k}_{4k+3}$$

<u>Thm</u> n is sum of two squares $\boxed{\text{iff}}$ ~~the is~~ exponent v_{q_i} is even for any $q_i \equiv 3 \pmod 4$

$$a^2 + b_1^2 = n = (A^2 + B^2) \cdot q \qquad q \equiv 3 \pmod 4$$

$q \mid a^2 + b^2 \Rightarrow q \mid a \wedge q \mid b$, so $a = q a_1$, $b = q b_1$

$$q^2 a_1^2 + q^2 b_1^2 = (A^2 + B^2)\, q$$

$$q(a_1^2 + b_1^2) = (A^2 + B^2)$$

↯

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Final

- $p = 4k+1 \implies p = a^2+b^2$
- $p = 4k+3 \implies p \mid a^2+b^2 \implies p \mid a \wedge p \mid b$, in particular $p \nmid a^2+1^2$
- n is sum of 2 squares iff exponent of $4k+3$ prime divisor is even.
- Any number of the form $4k+3$ has a ~~div~~ prime divisor $p \equiv 3 \pmod 4$ with odd exponent.

Why?

$(4w+1)(4w_1+1) = 4A+1$

$$4k+3 = \underbrace{\underbrace{p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}}_{4w+1}}_{4B+1} \; \underbrace{\underbrace{q_1^{\beta_1} \cdots q_k^{\beta_k}}_{4w_1+3}}_{4B+1}$$

$(4w+3)(4w_1+3) =$

$= 16ww_1 + 12w + 12w_1 + 9$

$= 4B+1$

Is 2020 sum of 2 squares?

$$2020 = \underset{\substack{\parallel \\ \text{no } p}}{4} \cdot \underset{\parallel}{5} \cdot \underset{\parallel}{101} = A^2 + B^2$$

Skanowano w CamScanner