Email training, N4

October 2-8

Problem 4.1. Let a, b, c are solutions of equation $x^3+x^2-3x-1=0$. Construct an equation which roots are $a+1$, $b+1$ and $c+1$.

Solution 4.1. Since a, b, c are solutions of the equation $x^3+x^2-3x-1=0$ then according to Viet's theorem one has

$$\begin{cases} a+b+c=-1 \\ ab+ac+bc=-3 \\ abc=1 \end{cases}.$$

If $a+1$, $b+1$ and $c+1$ are solutions of the equation $x^3+px^2+qx+r=0$ then, according to Viet's theorem one has

$$\begin{cases} p= & -(a+1+b+1+c+1)=-(-1+3)=-2 \\ q= & (a+1)(b+1)+(a+1)(c+1)+(b+1)(c+1) \\ & =ab+bc+ac+2(a+b+c)+3=-2 \\ r= & -(a+1)(b+1)(c+1)=-(abc+ab+bc+ac+a+b+c+1)=2 \end{cases}$$

So one gets the equation $x^3-2x^2-2x+2=0$.

Solution 2. Note that if the roots of the equation $P(x)=0$ are a, b, c, then the roots of the equation $P(x-1)=0$ will be $a+1$, $b+1$ and $c+1$. It remains to conclude that

$$P(x-1)=(x-1)^3+(x-1)^2-3(x-1)-1=$$

$$x^3-3x^2+3x-1+x^2-2x+1-3x+3-1=$$

$$x^3-2x^2-2x+2.$$

Answer: $x^3-2x^2-2x+2=0$.

Problem 4.2. Let a, b and c are pairwise different numbers. Solve the system of equations

$$\begin{cases} z+ay+a^2x+a^3=0 \\ z+by+b^2x+b^3=0 \\ z+cy+c^2x+c^3=0. \end{cases}$$

Solution 4.2. Consider the polynomial t^3+xt^2+yt+z. The numbers a, b, c are the roots of the polynomial. According to Viet's theorem one has

$$\begin{cases} a+b+c=-x \\ ab+bc+ca=y \\ abc=-z \end{cases}$$

Answer: $x=-a-b-c,\ y=ab+bc+ca,\ z=-abc.$

Problem 4.3. Solve equation in integers

$$x!+13=y^2.$$

Solution 4.3. Note that for the numbers bigger than 4 their factorial is divisible by 10, so their last digit is 0. Therefore for $x \geq 5$ the last digit of the left side of the equation is 3 so it can't be perfect square. So $x \leq$. By verifying each case $x=1,2,3,4$ we conclude that the equation has no solution in integers.

Problem 4.4. Let numbers $x_1, x_2, \ldots, x_n$ are given and each of them is equal either $+1$ or -1. Prove that if

$$x_1x_2+x_2x_3+\ldots+x_nx_1=0$$

then n is divisible by 4.

Solution 4.4. There are n terms in the sum, and half of them are $+1$ and half of them are equal -1, so n is even. Now lets prove that there are even number of -1's. From this we will conclude the statement of the problem. To obtian -1 we need consecutive written numbers $+1$ and -1. Lets write numbers $x_1, x_2, \ldots, x_n$ around the circle and by starting from $x_!$ go around the circle. Note that we move from $+1$ to -1 as many times as we move from -1 to $+1$. So there are even number of terms -1 in the sum $x_1x_2+x_2x_3+\ldots+x_nx_1$.

Problem 4.5. Chess king has started from some cell and by passing over each cell exactly ones came back to original position. Prove that the king has done even number of diagonal moves.

Solution 4.5. The king has made 64 moves. Note that when king makes diagonal move then it doesn't change it's cell color, otherwise he changes it. Since he came back to the same cell, it means that he made even number of non-diagonal moves. Since he made in total 64 moves, therefore the number of diagonal moves is even as well.

Problem 4.6. Let k is given and numbers from 1 to 100 are written on the board. Ali erases from the board arbitrary k numbers. Is it true that Bob may choose k numbers written on the board, which sum is equal to 100. Consider cases when a) $k=8$, b) $k=9$.

Solution 4.6. a) Let's group numbers from 1 to 24 to the pairs such that their sum is equal 25.

$$(1;24),\quad (2;23),\quad (3;22),\quad \ldots,\quad (11;14),\quad (12;13):$$

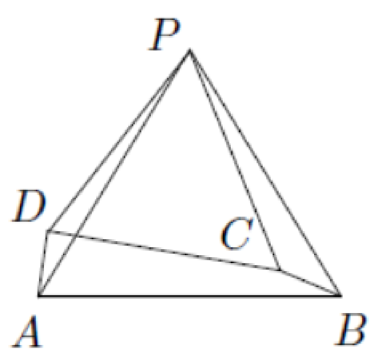
In total there are 12 pairs. If we remove any 8 numbers then there will be at lease $12-8=4$ pairs where no any number is removed. If we take all numbers from these 4 groups then we get 8 numbers which sum is equal $4\cdot 25=100$.

b) Lets remove numbers from 1 to 9. If we choose any 9 numbers from the remaining numbers then their sum will be at least $10+11+12+13+14+15+16+17+18=126>100$. So we can't choose 9 numbers which sum is equal 100.

Answer: a) possible, b) not possible.

Problem 4.7. . Let ABCD be a quadrangle, $|AD|=|BC|$, $\angle A+\angle B=120^o$ and let P be a point exterior to the quadrangle such that P and A lie at opposite sides of the line DC and the triangle DPC is equilateral. Prove that the triangle APB is also equilateral.

Solution 4.7. -

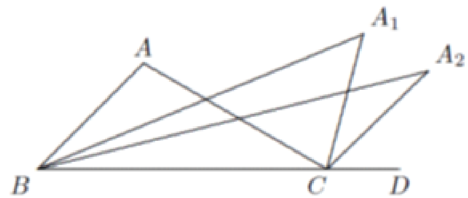


Note that ∠ADC + ∠CDP + ∠BCD + ∠DCP = 360°. Thus ∠ADP = 360° − ∠BCD − ∠DCP = ∠BCP. As we have |DP| = |CP| and |AD| = |BC|, the triangles ADP and BCP are congruent and |AP| = |BP|. Moreover, ∠APB = 60° since ∠DPC = 60° and ∠DPA = ∠CPB.

Problem 4.8. In triangle ABC, $\angle A=96^o$. Extend BC to an arbitrary point D. The angle bisectors of angle ABC and ACD intersect at A_1, and the angle bisectors of A_1BC and A_1CD intersect at A_2, and so on. The angle bisectors of A_4BC and A_4CD intersect at A_5 . Find the size of $\angle A_5$ in degrees.

Solution 4.8. -

Since A_1B and A_1C bisect $\angle ABC$ and $\angle ACD$ respectively, $\angle A = \angle ACD - \angle ABC = 2(\angle A_1CD - \angle A_1BC) = 2\angle A_1$, therefore $\angle A_1 = \frac{1}{2}\angle A$.



Similarly, we have $A_{k+1} = \frac{1}{2}A_k$ for $k = 1, 2, 3, 4$. Hence

$$A_5 = \frac{1}{2}A_4 = \frac{1}{4}A_3 = \frac{1}{2^3}A_2 = \frac{1}{2^4}A_1 = \frac{1}{2^5}A = \frac{96^\circ}{32} = 3^\circ.$$