$$\sum_{cyc} \frac{x}{\sqrt{y^{1/4}+z^{1/4}}} \geq \frac{\left(\sum \sqrt{x}\right)^{7/4}}{\sqrt{2\sqrt{27}}}$$ 7

(*1)

Let $x=a^4$, $y=b^4$, $z=c^4$

$$(*) \Leftarrow \sum \frac{a^4}{\sqrt{b+c}} \overset{?}{\geq} \frac{(a^2+b^2+c^2)^{7/4}}{\sqrt{2\sqrt{27}}}$$

$$\text{LHS} \geq \frac{(a^2+b^2+c^2)^2}{\sum\sqrt{a+b}} \overset{?}{\geq} \frac{(a^2+b^2+c^2)^{7/4}}{\sqrt{2\sqrt{27}}}$$

$$\Leftrightarrow (a^2+b^2+c^2)^{1/4} \geq \frac{\sum\sqrt{a+b}}{\sqrt{2\sqrt{27}}}$$

$\Longleftrightarrow$

$$\sqrt[4]{27}\,\sqrt[4]{4(a^2+b^2+c^2)} \overset{?}{\geq} \sum \sqrt{a+b}$$

$$\sqrt[4]{27}\cdot\sqrt[4]{2(a^2+b^2)+2(b^2+c^2)+2(c^2+a^2)}$$

$$\geq \sqrt[4]{27}\cdot\sqrt[4]{(a+b)^2+(b+c)^2+(c+a)^2}$$

$$= \sqrt[4]{(1+1+1)(1+1+1)(1+1+1)\left(\sum (a+b)^2\right)}$$

$$\geq \sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$$

Hölder $\blacksquare$

$a, b, c > 0$ LG

$$\sum_{cyc} \frac{1}{ab(b+1)(c+1)} \geq \frac{3}{(1+abc)^2}$$

$$p = \frac{a+b+c}{3}, \quad q = \frac{ab+bc+ca}{3}$$

$$r = abc$$

$$\sum_{cyc} \frac{c}{(b+1)(c+1)} \overset{?}{\geq} \frac{3r}{(1+r)^2}$$

$$\iff \frac{p+q}{1+3p+3q+r} \overset{?}{\geq} \frac{r}{(1+r)^2}$$

$$\iff (p+q)(1+r)^2 \overset{?}{\geq} r(1+r+3p+3q)$$

$$\iff (p+q)(1-r+r^2) \geq r(1+r)$$

by AM-GM:

$$p+q = \frac{a+b+c+ab+bc+ca}{3}$$

$$\geq 2\sqrt[6]{a^2b^2c^2} = 2\sqrt{r}$$

ETS : (let $s=\sqrt{r}$)

$$2s(1-s^2+s^4) \overset{?}{\geq} s^2(1+s^2)$$

$$\iff s(s-1)^2(2s^2+3s+2) \overset{?}{\geq} 0 \checkmark$$

$0 \le a,b,c,d \le 1 \le x,y,z,t$ $\boxed{8}$

$$\sum a + \sum x = 8$$

$$\overset{?}{\Longrightarrow} \sum a^2 + \sum x^2 \le 28$$

$$a^2 + b^2 = \_\_\_\_$$

$$a+b \le 2\sqrt{\frac{a^2+b^2}{2}}$$

$$a+b = \_\_\_\_\_\_$$

$$a^2+b^2 \le 0^2 + (a+b)^2$$

Expected equality case:

$(1,0,0,0,1,1,1,5)$

$$x = 8 - \sum a - y - z - t \leq 5$$

similarly; $1 \leq y, z, t \leq 5$

$$\Rightarrow (x-5)(x-1) \leq 0$$

$$\Rightarrow x^2 \leq 6x - 5$$

$$y^2 \leq 6y - 5$$

$$z^2 \leq 6z - 5$$

$$t^2 \leq 6t - 5$$

$0 \leq a, b, c, d \leq 1 \Rightarrow a^2 \leq a, b^2 \leq b$

$c^2 \leq c, d^2 \leq d$

$$\Rightarrow a^2 \leq 6a, b^2 \leq 6b, c^2 \leq 6c, d^2 \leq 6d$$

Now, sum up, we get :

$$\sum a^2 + \sum x^2 \leqslant 6\sum a + \sum (6x-5) = 48-20 = 28 \quad \blacksquare$$

5

$$ab+bc+ca=3$$

$$\sum_{cyc} \frac{a}{\sqrt{a^3+5}} \leqslant \frac{\sqrt{6}}{2}$$

$$a^3 + a^3 + 1 \geq 3a^2 \quad \text{(AM-GM)}$$

$$\Rightarrow a^3 + 5 \geq \frac{3}{2}a^2 + \frac{9}{2}$$

$$= \frac{3}{2}(a^2 + 3)$$

$$= \frac{3}{2}(a^2 + ab + bc + ca)$$

$$= \frac{3}{2}(a+b)(a+c)$$

$$\Rightarrow \sum_{cyc} \frac{a}{\sqrt{a^3+5}} \leq \frac{\sqrt{2}}{\sqrt{3}} \sum_{cyc} \frac{a}{\sqrt{(a+b)(a+c)}}$$

$$\overset{?}{\leq} \frac{\sqrt{6}}{2}$$

ETS:

$$\sum_{cyc} \frac{a}{\sqrt{(a+b)(a+c)}} \overset{?}{\leq} \frac{3}{2}$$

Now let $x = \sqrt{b+c}$, $y = \sqrt{c+a}$, $z = \sqrt{a+b}$

$$\Rightarrow \quad 2a = y^2 + z^2 - x^2$$

$$2b = x^2 + z^2 - y^2$$

$$2c = x^2 + y^2 - z^2$$

$$\sum_{cyc} \frac{y^2 + z^2 - x^2}{yz} \overset{?}{\leq} 3$$

$$\Leftrightarrow \sum_{cyc} x(y^2+z^2-x^2) \overset{?}{\leq} 3xyz$$

$$\Leftrightarrow x^3+y^3+z^3+3xyz \overset{?}{\geq} \sum_{cyc} xy(x+y)$$

✓ (Schur's inequality)

Schur: $x^3+y^3+z^3+3xyz \geq \sum xy(x+y) \quad \forall x,y,z>0$

Proof $(z \geq y \geq x)$

$$\Leftrightarrow xyz \geq \sum_{cyc} xy(x+y) - \sum_{cyc} x^3 - 2xyz$$

$$= (x+y-z)(y+z-x)(x+z-y)$$

Case 1: if $x+y-z \leq 0$

$$\Rightarrow y+z-x \geq x+z-y > 0$$

$$\Rightarrow RHS \leq 0 < LHS$$

Case 2: $z < x+y$, so $\exists u, v, t > 0$

$x = u+v$
$y = v+t$
$z = t+u$

$$\Rightarrow LHS = (u+v)(v+t)(t+u)$$

$$RHS = 8uvt$$

in $\triangle ABC$:

$\boxed{9}$

$$\cos A+\cos B+\cos C \leq \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$$

$$\cos A+\cos B \overset{?}{\leq} 2\sin\frac{C}{2}$$

$$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2}$$

$$\leq 2\sin\frac{C}{2}$$

we sum the similar inequalities

and the the result will follow $\blacksquare$