Email training, N6

September 29- October 5, 2019

Problem 6.1. Find the biggest integer n such that $n^3 + 100$ is divisible by $n + 10$.

Problem 6.2. Find all integers x and y for which $(2x + y)(5x + 3y) = 7$.

Problem 6.3. Let a be an odd integer and m is such that $2^m | a + 1$ and $2^{m+1} \not| a + 1$. Prove that for any positive integer k one has

$$2^{k+m+1} | (2a + 1)^{2^k} - 1 \quad \text{and} \quad 2^{k+m+2} \not| (2a + 1)^{2^k} - 1$$

Problem 6.4. Find the number of positive integers n letss than 10000, for which $2^n - n^2$ is divisible by 7.

Problem 6.5. Find all positive integers n such that

$$3^{n-1} + 5^{n-1} | 3^n + 5^n.$$

Problem 6.6. The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

Problem 6.7. In square $ABCD$, M is the midpoint of AD and N is the midpoint of MD. Prove that $\angle NBC = 2\angle ABM$.

Problem 6.8. Let ABC is an isosceles triangle with $AB = AC = 2$. There are 100 points $P_1, P_2, \ldots, P_{100}$ on the side BC. Denote $m_i = AP_i^2 + BP_i \cdot CP_i$. Find the value of $m_1 + m_2 + \ldots + m_{100}$.

Solution submission deadline October 5, 2019