Email training, N2

August 31- September 7

Problem 2.1. Let n be a positive integer. Prove that

$$\frac{n^2-n}{2} \leq \sum_{k=1}^{n^2}\{\sqrt{k}\} \leq \frac{n^2-1}{2}$$

where $\{x\}$ is the fractional part of x.

Solution 2.1. We prove the result by induction on n. It is easy to verify the result for the base case $n=1$. Assume that the result holds for some positive integer $n=m$, so

$$\frac{m^2-m}{2} \leq \sum_{k=1}^{m^2}\{\sqrt{k}\} \leq \frac{m^2-1}{2},$$

For $1 \leq i \leq 2m$ one has $m < \sqrt{m^2+i} < m+1$, so

$$\{\sqrt{m^2+i}\} = \sqrt{m^2+i} - m < \sqrt{m^2+i+\frac{i^2}{4m^2}} - m = \frac{i}{2m}.$$

Hence,

$$\sum_{k=1}^{(m+1)^2}\{\sqrt{k}\} = \sum_{k=1}^{m^2}\{\sqrt{k}\} + \sum_{i=1}^{2m}\{\sqrt{m^2+i}\} <$$

$$\frac{m^2-1}{2} + \frac{1}{2m}\sum_{i=1}^{2m} i = \frac{(m+1)^2-1}{2}.$$

Also

$$\{\sqrt{m^2+i}\} = \sqrt{m^2+i} - m = \frac{i}{\sqrt{m^2+i}+m} > \frac{i}{m+1+m} = \frac{i}{2m+1}.$$

Hence

$$\sum_{k=1}^{(m+1)^2}\{\sqrt{k}\} = \sum_{k=1}^{m^2}\{\sqrt{k}\} + \sum_{i=1}^{2m}\{\sqrt{m^2+i}\} >$$

$$\frac{m^2-m}{2} + \frac{1}{2m+1}\sum_{i=1}^{2m} i = \frac{(m+1)^2-(m+1)}{2}.$$

Problem 2.2. Determine all functions $f : R \to R$ such that

$$f(xf(y) + 2y) = f(xy) + xf(y) + f(f(y)).$$

Solution 2.2. -

Setting $x = 0$ and $y = 0$ in the functional equation yields $f(f(0)) = 0$. So there is at least one zero point of f. Let a be any of them. Setting $y = a$ gives us $f(2a) = f(ax) + f(0)$. If $a \neq 0$, then f is a constant function and we know that $f(a) = 0$, so it is a zero function, which is indeed a solution.

It remains to investigate the case where 0 is the only zero point of f, i.e. $f(a) = 0$ if and only if $a = 0$. Furthermore, taking $x = 0$ in the functional equation we obtain

$$f(2y) = f\big(f(y)\big). \tag{1}$$

If we prove an injectivity of f, the previous identity yields $f(y) = 2y$, what is the second solution, as we can easily check.

Now we prove the injectivity of f. Firstly, let us examine the set of the fixed points of f. This set is non-empty because 0 is one of its points. Assume that p is any of the fixed points, i.e. $f(p) = p$. Setting $x = -1$, $y = p$ in the functional equation gives

$$p = f\big(-f(p) + 2p\big) = f(-p) - f(p) + f\big(f(p)\big) = f(-p).$$

Now we set $x = 1$, $y = -p$ in the functional equation and we obtain using proved $f(-p) = p$

$$p = f\big(f(-p) - 2p\big) = f(-p) + f(-p) + f\big(f(-p)\big) = 3p.$$

This yields that $p = 0$ is the only fixed point of f.

Secondly, we choose x so that $xf(y) + 2y = xy$, which yields $x = 2y/\big(y - f(y)\big)$. This can be done for each $y \neq 0$, since 0 is the only fixed point of f. This substitution gives us

$$f\big(f(y)\big) = \frac{2yf(y)}{f(y) - y} = \frac{2f^2(y)}{f(y) - y} - 2f(y).$$

In order to finish the proof of injectivity let us assume that non-zero real numbers a, b satisfy $f(a) = f(b)$. We have already proved that $f(a) = f(b) \neq 0$. The previous identity yields

$$\frac{2f^2(a)}{f(a) - a} - 2f(a) = f\big(f(a)\big) = f\big(f(b)\big) = \frac{2f^2(b)}{f(b) - b} - 2f(b)$$

and it follows that $a = b$. The proof of the injectivity is thereby finished.

Problem 2.3. Find the greatest common divisor of $5^{300}-1$ and $5^{200}+6$.

Solution 2.3. Let $n = 5^{100}$. Then $5^{300}-1 = n^3-1$ and $5^{200}+6 = n^2+6$. We start by simplifying $d = gcd(n^3-1, n^2+6)$. Since

$$n(n^2+6) - 6(n^2+6) - 6n + 1,$$

we see that $d = gcd(6n+1, n^2+6)$. Then we see that

$$n(6n+1) - 6(n^2+6) = n - 36,$$

Since 6 and $6n+1$ are coprime, this means that $d = gcd(6n+1, n-36)$. Finally, we have

$$6n + 1 - 6(n-36) = 217 = 7 \cdot 31,$$

and since 6 and $6n+1$ are coprime we see that $d = gcd(n-36, 7\cdot 31)$. To calculate this we reduce $5^{100}-36$ modulo 7 and module 31. Since $5^6 \equiv 1[7]$, then

$$5^{100} - 36 \equiv 5^4 - 1 \equiv 2 - 1 - 1[7],$$

and since $5^3 \equiv 1[31]$, then

$$5^{100} - 36 \equiv 5 - 5 = 0[31].$$

So $d = 7$.

Answer: 7.

Problem 2.4. Determine the smallest positive integer n for which the following statement holds true: From any n consecutive integers one can select a non-empty set of consecutive integers such that their sum is divisible by 2019.

Solution 2.4. -

The prime factorization of 2019 is $3 \cdot 673$. Let $p = 673$.

For each integer k, color the three numbers $kp-1, kp, kp+1$ red, and and the six numbers $kp + \frac{p-5}{2}, kp + \frac{p-3}{2}, kp + \frac{p-1}{2}, kp + \frac{p+1}{2}, kp + \frac{p+3}{2}, kp + \frac{p+5}{2}$ blue. Now the integers are colored periodically. In a period of length $p = 673$, there are 3 red integers, then 332 uncolored integers, then 6 blue integers and finally 332 uncolored integers.

The sum of the integers in a red interval is $3kp = 2019 \cdot k$, and the sum of the integers in a blue interval is $6(kp + \frac{p}{2}) = 2019 \cdot (2k+1)$. So if there is a colored interval (we mean a maximal one throughout) in the given n consecutive integers, one can choose it. It is easy to see, that among any $340 = 332 + (6-1) + (3-1) + 1$ consecutive integers, there must be a colored interval. Thus the smallest n (that we look for) satisfies $n \leq 340$.

Now we will show that it is not possible to choose consecutive integers in the desired way from the set $A = \{335, 336, \ldots, 673\}$. ($|A| = 339$ and thus $n \geq 340$.) Assume that there exists $\{a, a+1, \ldots b\} \subseteq A$ such that

$$2019 \mid a + (a+1) + \cdots + b = \frac{(b-a+1)(a+b)}{2}.$$

That means either $673 \mid b-a+1$, or $673 \mid a+b$. Since

$$0 < 1 \leq b-a+1 \leq 339 < 673,$$

673 must divide $a+b$. Taking into account that

$$671 = 335 + 336 \leq a+b \leq 673 + 673 = 2 \cdot 673,$$

we conclude that $a+b$ must be 673 or $2 \cdot 673$. It means either $a = 335$ and $b = 338$, or $a = 336$ and $b = 337$, or $a = b = 673$. But $2019 \nmid 335 + 336 + 337 + 338 = 1346$, $2019 \nmid 336 + 337 = 673$ and $2019 \nmid 673$, a contradiction.

Comment. The same proof works for every odd number $m = p \cdot q$, where p is a 'big' prime divisor of m. We need that $p > \sqrt{3m}$. Then the answer is $n = \frac{p+3q}{2} - 1$.

Problem 2.5. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the number of functions $f : S \to S$ such that $f(f(x)) = x$ for all $x \in S$.

Solution 2.5. Let A be an arbitrary set and let $f : A \to A$ be a function such that $f(f(x)) = x$ for all $x \in A$. Then the elements of A can be divided into two categories: Elements a such that $f(a) = a$, and elements b such that $f(b) \neq b$. In the latter case, let $c = f(b)$. Then $f(c) = b$. Thus, the set A is partitioned into singletons of the form

$\{a\}$ (where $f(a) = a$), and pairs of the form $\{b, c\}$ (where $f(b) = c$ and $f(c) = b$). Conversely, any partition of A into singletons and pairs determines a function f such that $f(f(x)) = x$ for all $x \in A$. Thus, the number of such functions f on a set A is equal to the number of partitions of A into singletons and pairs. So, we count the number of partitions of $S = \{1, 2, 3, 4, 5, 6, 7\}$ into singletons and pairs. For a positive integer n, let t_n denote the number of such partitions of a set with n elements. Let $S_n = \{x_1.x_2, \ldots, x_n\}$, and consider a partition of S_n into singletons and pairs. In any such partition, x_n is either a singleton or a member of a pair.

If x_n is a singleton, then the remaining $n-1$ elements are partitioned into singletons and pairs, so the number of such partitions is simply t_{n-1}.

If x_n is a member of a pair, then the pair is of the form $\{x_k, x_n\}$, where $1 \leq k \leq n-1$, and the remaining $n-2$ elements are partitioned into singletons and pairs. There are $n-1$ choices for k, so the number of such partitions is $(n-1)t_{n-2}$.

Therefore,

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

for all $n \geq 3$. We see that $t_1 = 1$ and $t_2 = 2$, so

$$t_3 = t_2 + 2t_1 = 4,$$

$$t_4 = t_3 + 3t_2 = 10,$$

$$t_5 = t_4 + 4t_3 = 26,$$

$$t_6 = t_5 + 5t_4 = 76,$$

$$t_7 = t_6 + 6t_5 = 232.$$

Answer: 232.

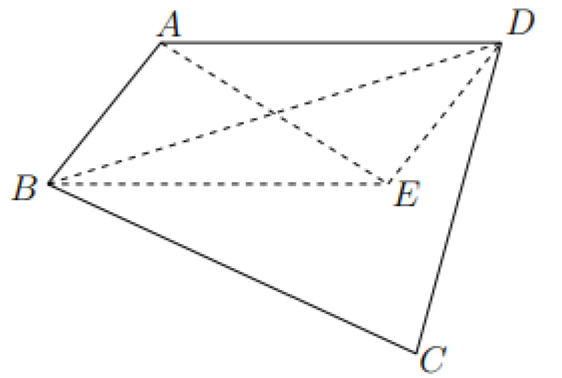
Problem 2.6. Let $n \geq 3$ be an integer. We say that a vertex A_i $(1 \leq i \leq n)$ of a convex polygon $A_1A_2 \ldots A_n$ is Bohemian if its reflection with respect to the midpoint of the segment $A_{i-1}A_{i+1}$ (with $A_0 = A_n$ and $A_1 = A_{n+1}$) lies inside or on the boundary of the polygon $A_1A_2 \ldots A_n$. Determine the smallest possible number of Bohemian vertices a convex n-gon can have (depending on n).

(A convex polygon $A_1A_2 \ldots A_n$ has n vertices with all inner angles smaller than 180^o.)

Solution 2.6. -

Lemma. *If $ABCD$ is a convex quadrilateral with $\angle BAD + \angle CBA \geq \pi$ and $\angle BAD + \angle ADC \geq \pi$ then A is a Bohemian vertex of $ABCD$.*

Proof. Let E be the reflection of A in $ABCD$. It is clearly seen that E belongs to the halfplanes containing C determined by lines AB and AD. Since $\angle BAD + \angle CBA \geq \pi$ and $\angle BAD + \angle EBA = \pi$, point E belongs to the (closed) halfplane containing points A, D determined by the line BC. Analogously, using the assumption $\angle BAD + \angle ADC$ we infer that E belongs to the closed halfplane containing points A, B determined by the line CD.



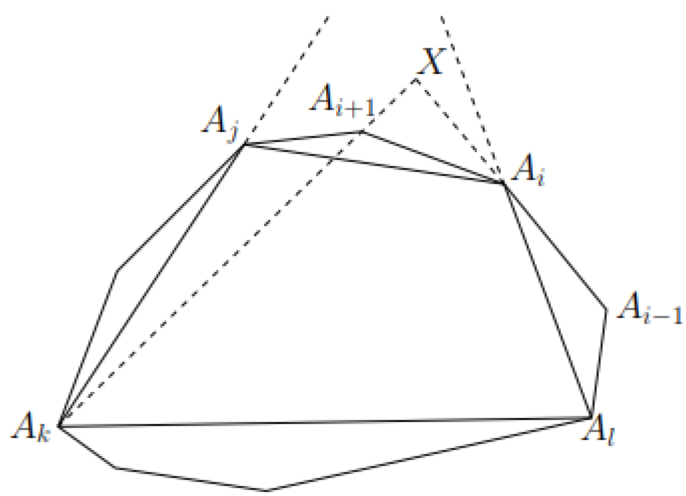
Therefore E lies inside or on the boundary of $ABCD$. Thus A is Bohemian.

Consider a convex n-gon $A_1A_2\dots A_n$. Choose any four vertices A_i, A_j, A_k, A_l with $i<j<k<l$ as in the picture below. Consider quadrilateral $A_iA_jA_kA_l$. It is clear that one of the points A_i, A_j, A_k, A_l satisfies assumption of the lemma, let's say this point is A_i. We claim that A_i satisfies the assumption of the lemma in quadrilateral $A_{i-1}A_iA_{i+1}A_k$. Observe that the point $X := A_kA_{i+1} \cap A_iA_{i-1}$ lies in the triangle bounded by lines A_kA_j, A_jA_i and A_iA_l. So

$$\angle A_kA_{i+1}A_i + \angle A_{i+1}A_iA_{i-1} = \pi + \angle A_kXA_i \geq \pi.$$

(Note: it may happen that X does not exist. It happens iff $j=i+1$, $l=i-1$ and $A_kA_j \parallel A_lA_i$. In that case $\angle A_kA_{i+1}A_i + \angle A_{i+1}A_iA_{i-1} = \pi$.)

Analogously $\angle A_{i+1}A_iA_{i-1} + \angle A_iA_{i-1}A_k \geq \pi$. Using lemma we conclude that A_i is a Bohemian vertex of quadrilateral $A_{i-1}A_iA_{i+1}A_k$. This implies that A_i is a Bohemian vertex of $A_1A_2\dots A_n$ since the quadrilateral $A_{i-1}A_iA_{i+1}A_k$ is a subset of the n-gon and the reflexion point is the same.



Therefore, amongst any four vertices of a convex n-gon there exists a Bohemian vertex. So, every n-gon has at least $n-3$ Bohemian vertices.

An example of a convex n-gon with exactly $n-3$ Bohemian vertices is the following: take any kite $A_1A_2A_3A_4$ with $A_4A_1 = A_1A_2 < A_2A_3 = A_3A_4$ and place points $A_5,\dots,A_n$ very close to A_1. Then A_2, A_3, A_4 are not Bohemian vertices of $A_1A_2\dots A_n$.

Problem 2.7. Let M is the midpoint of AC and let H be the foot point of the altitude from vertex B of triangle ABC. Let P and Q be the orthogonal projections of A and C on the bisector of angle B. Prove that the four points M, H, P and Q lie on the same circle.

Solution 2.7. -

Solution. If $|AB| = |BC|$, the points M, H, P and Q coincide and the circle degenerates to a point. We will assume that $|AB| < |BC|$, so that P lies inside the triangle ABC, and Q lies outside of it.

Let the line AP intersect BC at P_1, and let CQ intersect AB at Q_1. Then $|AP| = |PP_1|$ (since $\triangle APB \cong \triangle P_1PB$), and therefore $MP \parallel BC$. Similarly, $MQ \parallel AB$. Therefore $\angle AMQ = \angle BAC$. We have two cases:

(*i*) $\angle BAC \leq 90°$. Then A, H, P and B lie on a circle in this order. Hence $\angle HPQ = 180° - \angle HPB = \angle BAC = \angle HMQ$. Therefore H, P, M and Q lie on a circle.

(*ii*) $\angle BAC > 90°$. Then A, H, B and P lie on a circle in this order. Hence $\angle HPQ = 180° - \angle HPB = 180° - \angle HAB = \angle BAC = \angle HMQ$, and therefore H, P, M and Q lie on a circle.

Problem 2.8. Let $ABCD$ is a trapezium with $AD \parallel BC$. Let P is the point on the line AB such that $\angle CPD$ is maximal. Let Q is the point on the line CD such that $\angle BQA$ is maximal. Given that P lies on the segment AB, prove that $\angle CPD = \angle BQA$.

Solution 2.8. -

Solution. The property that $\angle CPD$ is maximal is equivalent to the property that the circle CPD touches the line AB (at P). Let O be the intersection point of the lines AB and CD, and let ℓ be the bisector of $\angle AOD$. Let A', B' and Q' be the points symmetrical to A, B and Q, respectively, relative to the line ℓ. Then the circle AQB is symmetrical to the circle $A'Q'B'$ that touches the line AB at Q'. We have

$$\frac{|OD|}{|OA'|} = \frac{|OD|}{|OA|} = \frac{|OC|}{|OB|} = \frac{|OC|}{|OB'|}.$$

Hence the homothety with centre O and coefficient $|OD|/|OA|$ takes A' to D, B' to C, and Q' to a point Q'' such that the circle $CQ''D$ touches the line AB, and thus Q'' coincides with P. Therefore $\angle AQB = \angle A'Q'B' = \angle CQ''D = \angle CPD$ as required.