Email training, N1

Level 2, September 13-19

Problem 1.1. Let integers x and y are such that $5x + 7y = 111$. Prove that $x + y$ is even.

Solution 1.1. Note that

$$5x + 7y \equiv x + y \equiv 111 \equiv 1[2],$$

which means that $x + y$ is odd.

Alternative solution. $x + y = 111 - 4x - 6y = 2(55 - 2x - 3y) + 1$ which is odd.

Problem 1.2. Is it possible to put signs $+$ and $-$ instead of $*$'s to get correct expression

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 * 10 = 0.$$

Solution 1.2. Since $x + y \equiv x - y[2]$, therefore

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 * 10 \equiv$$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \equiv 55 \equiv 1[2],$$

so it's value can't be 0.

Problem 1.3. Find the number of 3 digit positive integers, such that all digits are even.

Solution 1.3. Note, that the first digit may take any value from the set $\{2, 4, 6, 8\}$, however second and third digit may be any digit from the set $\{0, 2, 4, 6, 8\}$. Since there is no any forbidden combinations, then the total number will be $4 \cdot 5 \cdot 5 = 100$.

Problem 1.4. During the contest 10 students all together have solved 35 problems. It's known that some student solved exactly 1 problem, there is a students that solved exactly 2 problems and there is a student that solved exactly 3 problems. Prove that there is a student that solved at least 5 problems.

Solution 1.4. Note, that the rest 7 students have solved $35 - 1 - 2 - 3 = 29$ problems together. Therefore, according to the Pigeonhole priciple at least one of them solved at least $\left\lceil \frac{29}{7} \right\rceil = 5$ problems.

Problem 1.5. Recover missing digits

$$1 * \cdot * 1 = 1 * *1.$$

Solution 1.5. Since the last digit of the product is 1 and the last digit of the second multiplier is one, so the last digit if the first multiplier is 1. We get $11 \cdot *1 = 1 * *1$. Since $11 \cdot 81 = 891 < 1000$, so we conclude that the only option is $11 \cdot 91 = 1001$.

Problem 1.6. Which number is bigger $(n - 1)! \cdot (n + 1)$ or $n! \cdot n$.

Solution 1.6.

$$\frac{(n-1)! \cdot (n+1)}{n! \cdot n} = \frac{n+1}{n^2},$$

which is less than 1 for $n > 1$ and bigger than 1 for $n = 1$.

Problem 1.7. Let AA', BB' and CC' are the altitudes of the triangle ABC. Let A_1 and A_2 are the projections of A' on AB and AC, respectively, B_1 and B_2 are the projections of B' on BC and BA, as well as C_1 and C_2 are the projections of C' on CA and CB. Prove that:

- $B_2C_1 \parallel BC$,

- The hexagon $A_1B_2C_1A_2B_1C_2$ is cyclic.

Solution submission deadline September 19, 2021
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submission email **imo20etraining@gmail.com**