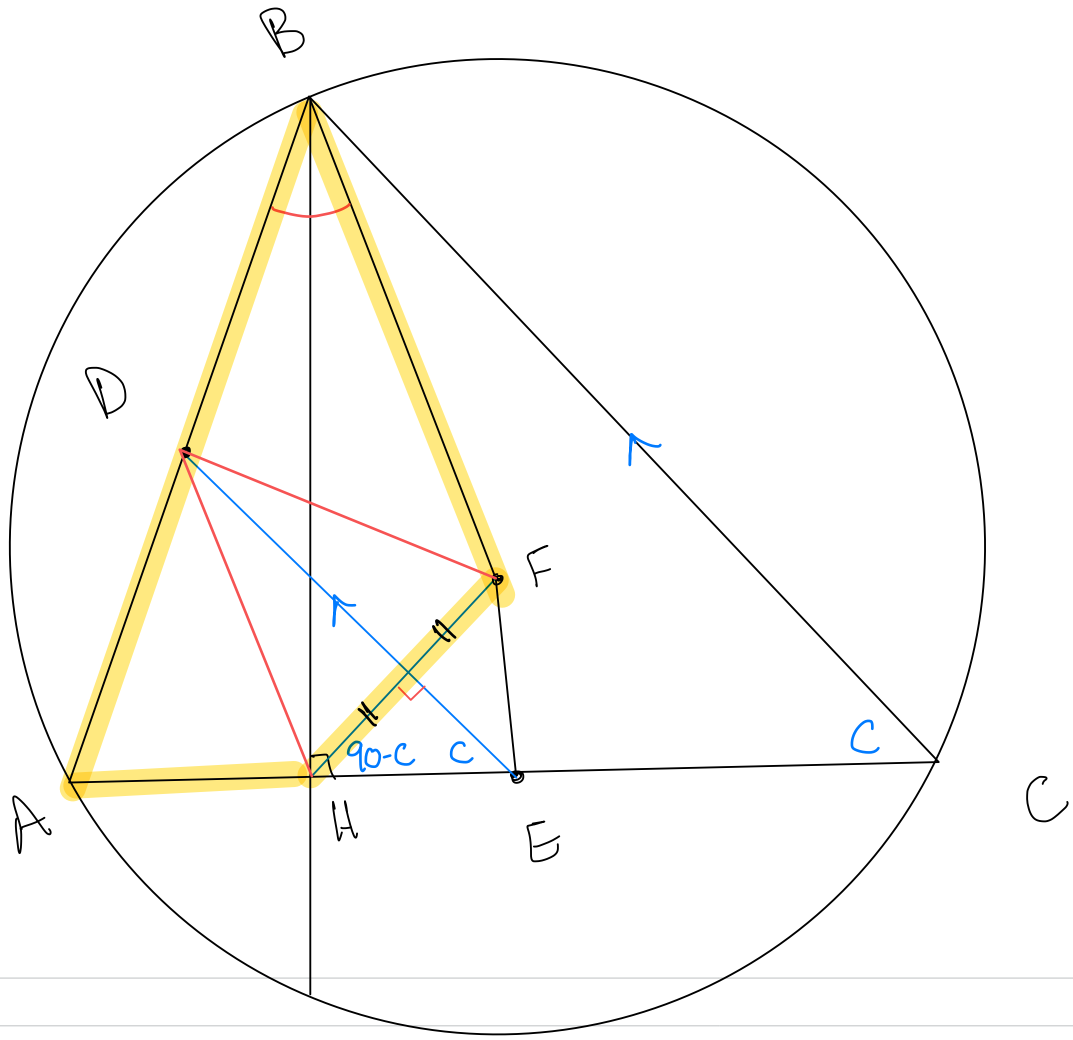
# Intensive Training
## Geometry

Day 4
1 April 2021

Includes solutions for:

Exam 6 – P2
EGMO 2016 – P2
JBMO 2010 – P3
EGMO 2014 – P2

**Problem 2.** In acute-angled triangle $ABC$, $BH$ is the altitude of the vertex $B$. The points $D$ and $E$ are midpoints of $AB$ and $AC$ respectively. Suppose that $F$ be the reflection of $H$ with respect to $ED$. Prove that the line $BF$ passes through circumcenter of $ABC$.




$O \in FB \iff \angle FBA = \angle OBA = 90 - C$

Note that $DE \parallel CB \Rightarrow \angle DEH = C$

$\Rightarrow \angle EHF = 90 - C$

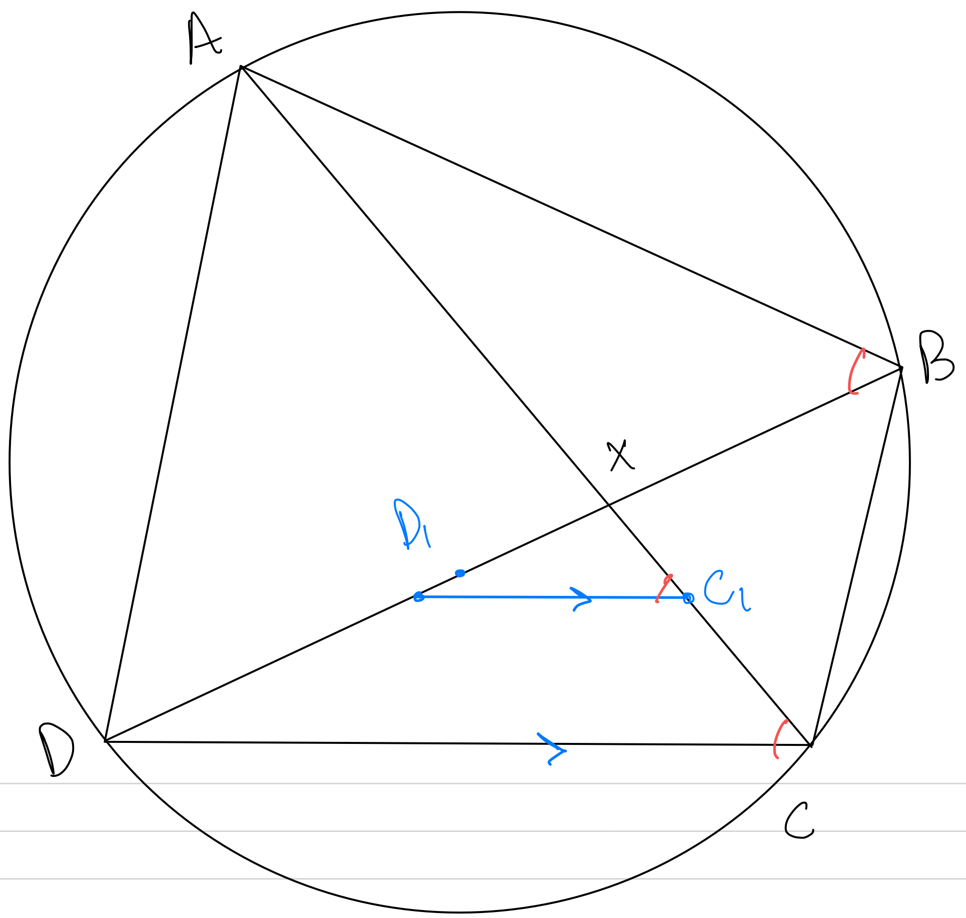
Therefore $O \in FB \iff BFHA$ cyclic

but $BHA$ is a right angle triangle, so

$F \in (BHA) \iff DF = DH$ which's true by reflection around $DE$.

**Lemma:** $ABCD$ is a cyclic and $X = AC \cap BD$.

Then for any two points $C_1, D_1$ on $CX$ and $DX$ resp. such that $\frac{DD_1}{DX} = \frac{CC_1}{CX}$, $ABC_1D_1$ is a cyclic




**Proof.**

$\triangle AXB \sim \triangle DXC$

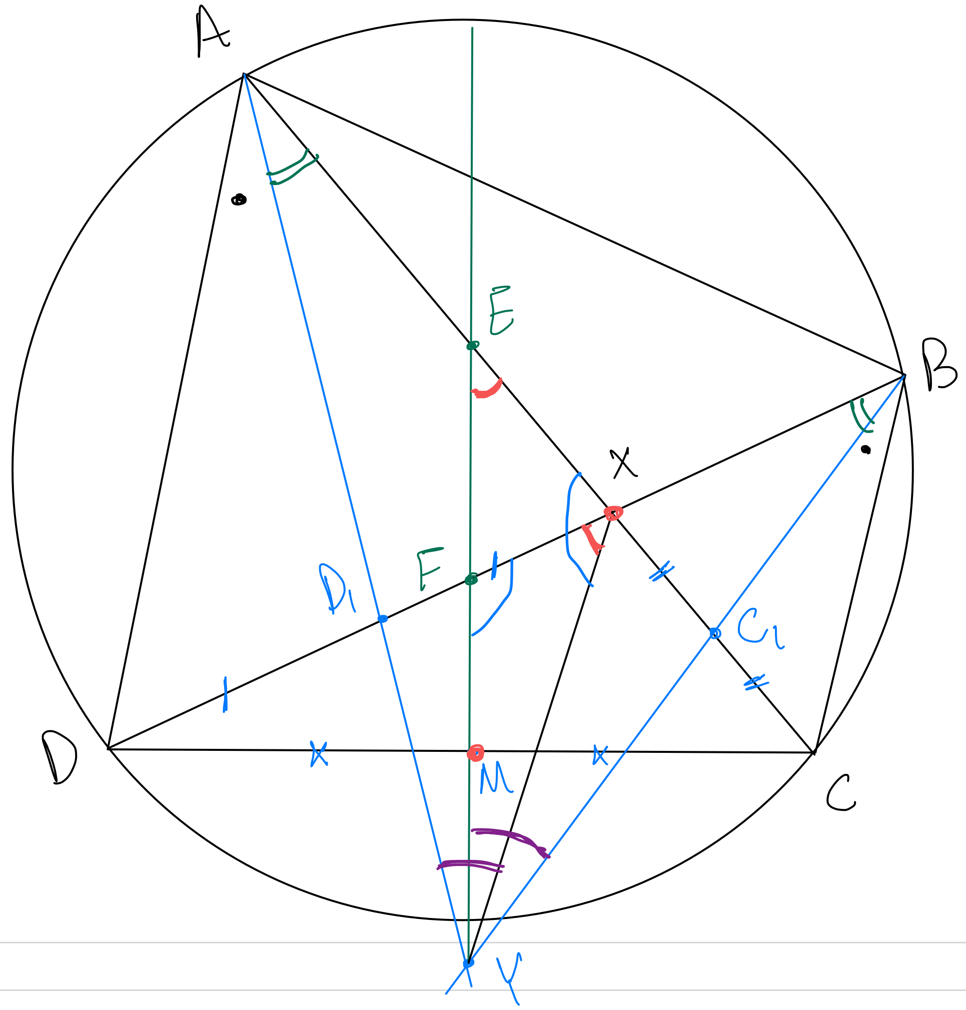
$\triangle DXC \sim \triangle D_1XC_1$ $\quad (C_1D_1 \parallel CD)$

Another way: by the power of the point $X$. $\square$

$AX \cdot XC = BX \cdot XD$

$\Rightarrow AX \cdot XC_1 = BX \cdot XD_1 \Rightarrow ABC_1D_1$ cyclic $\square$

12. Let $ABCD$ be a cyclic quadrilateral, and let diagonals $AC$ and $BD$ intersect at $X$.Let $C_1, D_1$ and $M$ be the midpoints of segments $CX, DX$ and $CD$, respecctively. Lines $AD_1$ and $BC_1$ intersect at $Y$, and line $MY$ intersects diagonals $AC$ and $BD$ at different points $E$ and $F$, respectively. Prove that line $XY$ is tangent to the circle through $E, F$ and $X$. (EGMO 2016 P2)




Proof

By the lemma, $ABC_1D_1$ is a cyclic, so

$\angle D_1AX = \angle C_1BX$. (1)

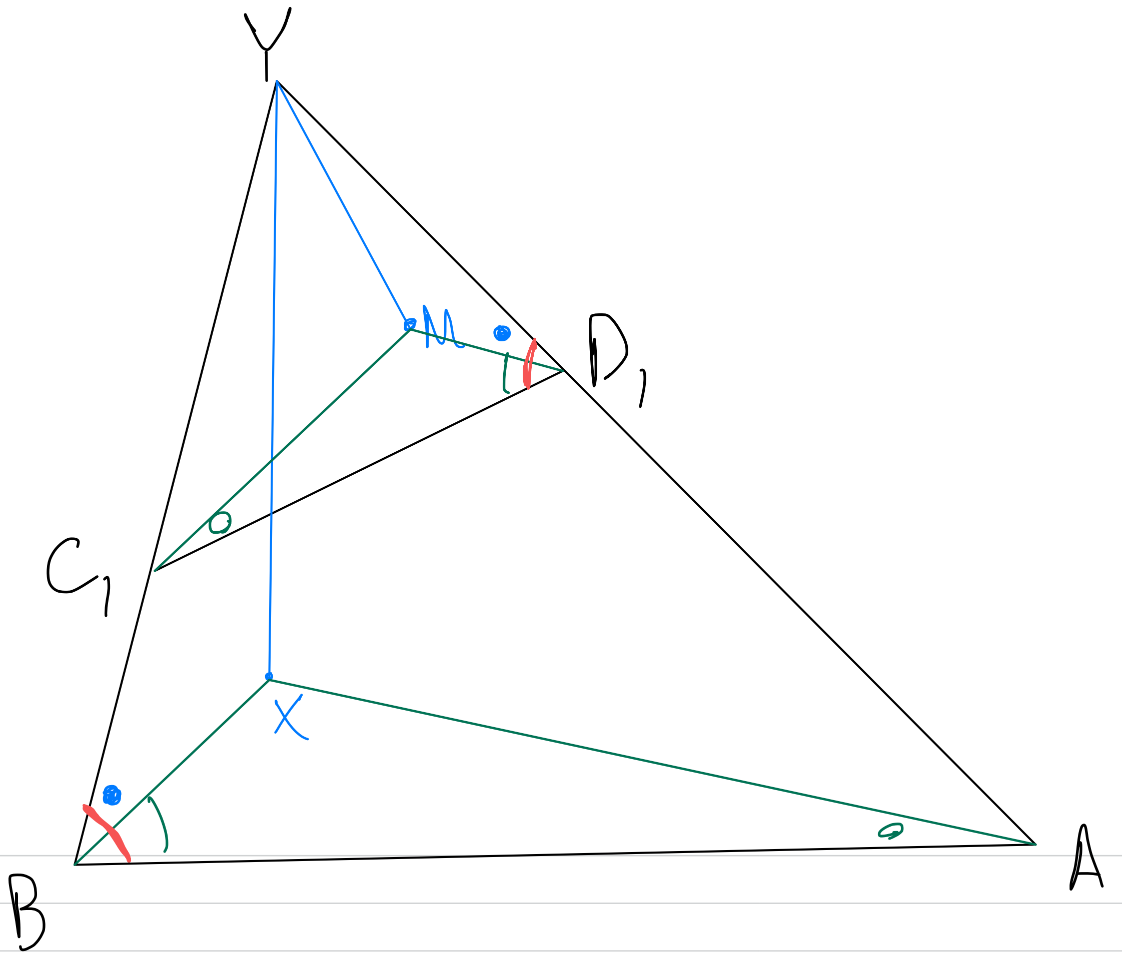
Note that $\angle YEX = \angle FXY \Leftrightarrow \angle EXY = \angle XFY$ (2)

From (1), and (2), we get that

$\boxed{\angle YEX = \angle FXY} \Leftrightarrow \triangle AXY \sim \triangle BFY$

**Lemma 2:** In $\Delta YAB$, $ABC_1D_1$ is a cyclic and $\angle MD_1C_1 = \angle XBA$. Show that $\angle MAD_1 = \angle XYB$

$$\angle MC_1D_1 = \angle XAB$$




**Solution 1:** $\Delta YD_1C_1 \sim \Delta YBA$ and $M$ for $\Delta YD_1C_1$ is the same as $X$ for $\Delta YBA$.

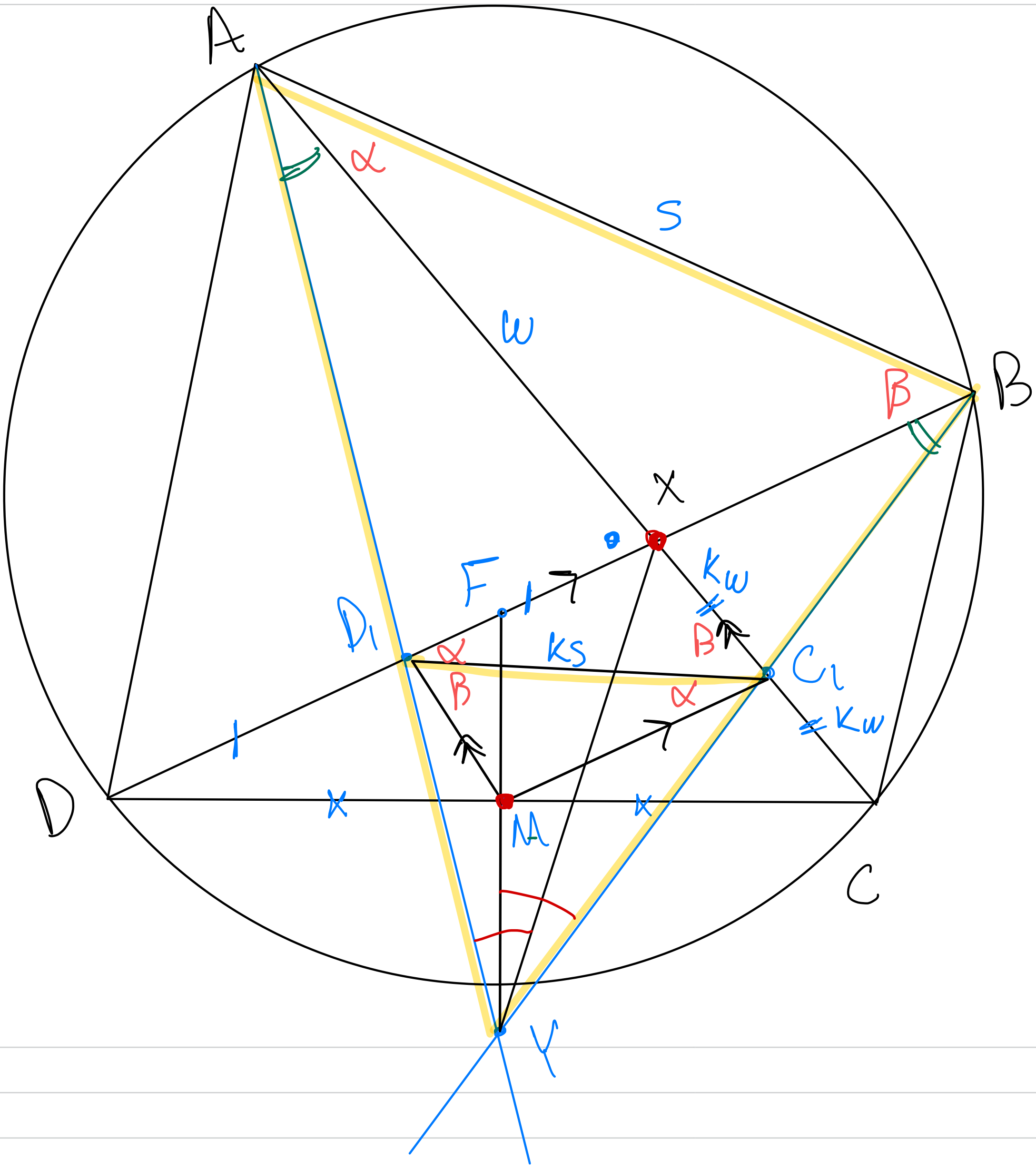
$$\Rightarrow \angle MAD_1 = \angle XYB$$

**Solution 2:**

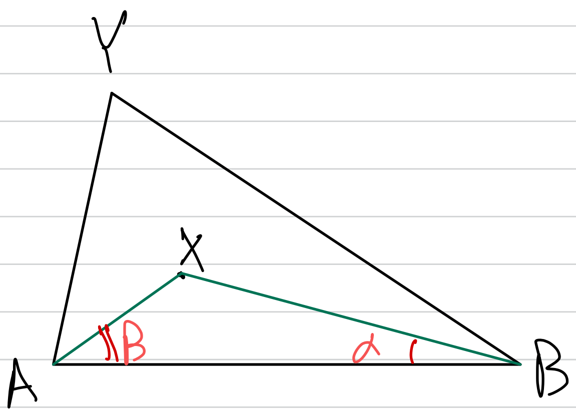
$$\Delta MYD_1 \sim \Delta XAB$$

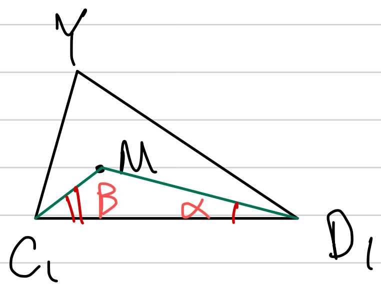
$$\Leftrightarrow \begin{cases} \angle YD_1M = \angle YBX \longrightarrow \checkmark \\ \dfrac{YD_1}{YB} = \dfrac{MD_1}{XB} \end{cases}$$

$$\underbrace{\frac{MD_1}{XB} = \frac{C_1D_1}{AB}}_{\Delta MD_1C_1 \sim \Delta XBA} = \frac{YD_1}{YB} \qquad \Delta YD_1C_1 \sim \Delta YBA$$




$\Delta AXY \sim \Delta BFY \iff \angle XYA = \angle MYB$



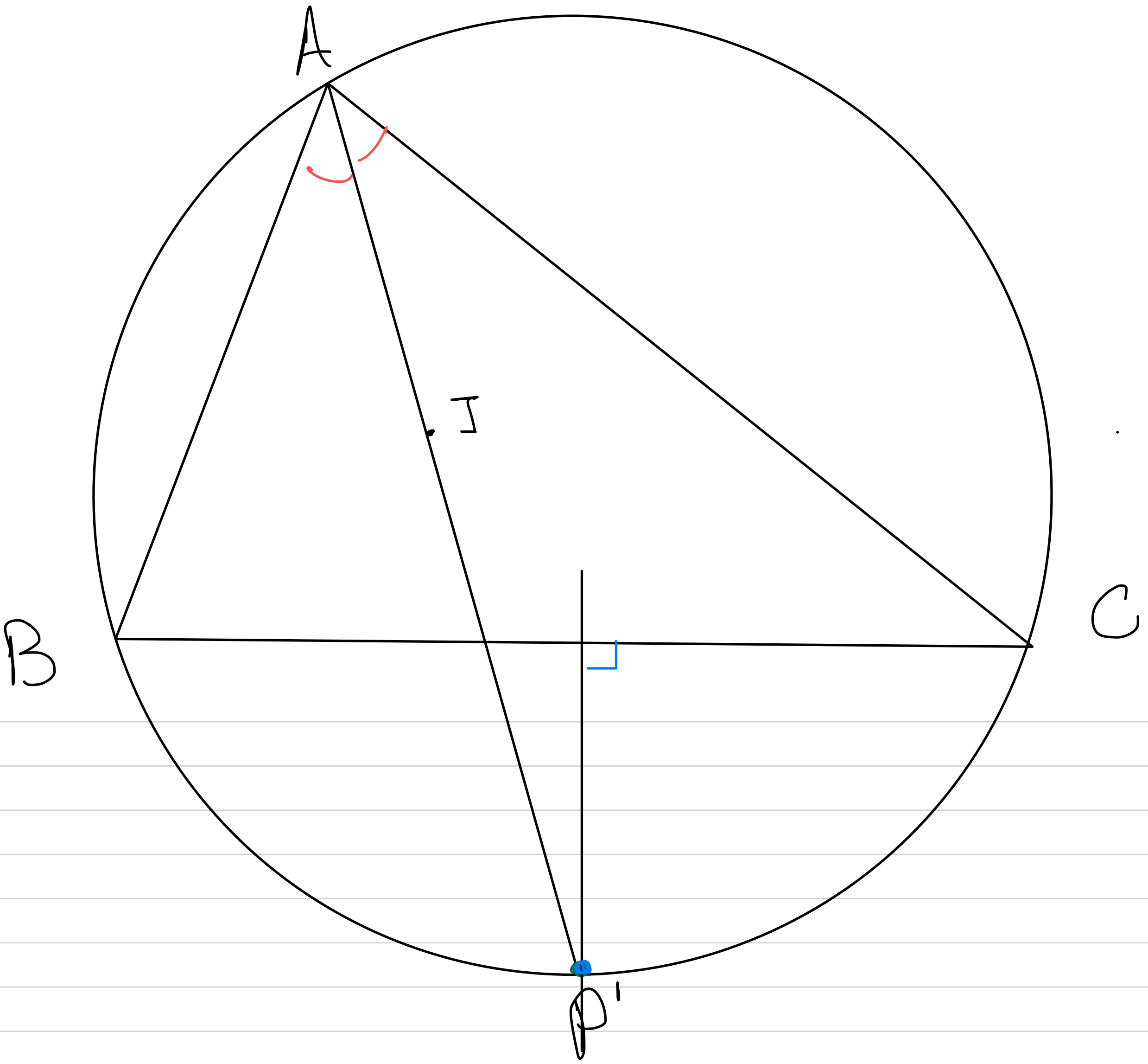




from the lemma 2: $\angle MYB = \angle XYA$

$\Rightarrow \quad \Delta AXY \sim \Delta BFY$

$\Rightarrow \angle YEX = \angle FXY$ $\square$

Lemma 3: If $P$ is on the angle bisector of $A$ and $PB = PC$, then $P \in (ABC)$



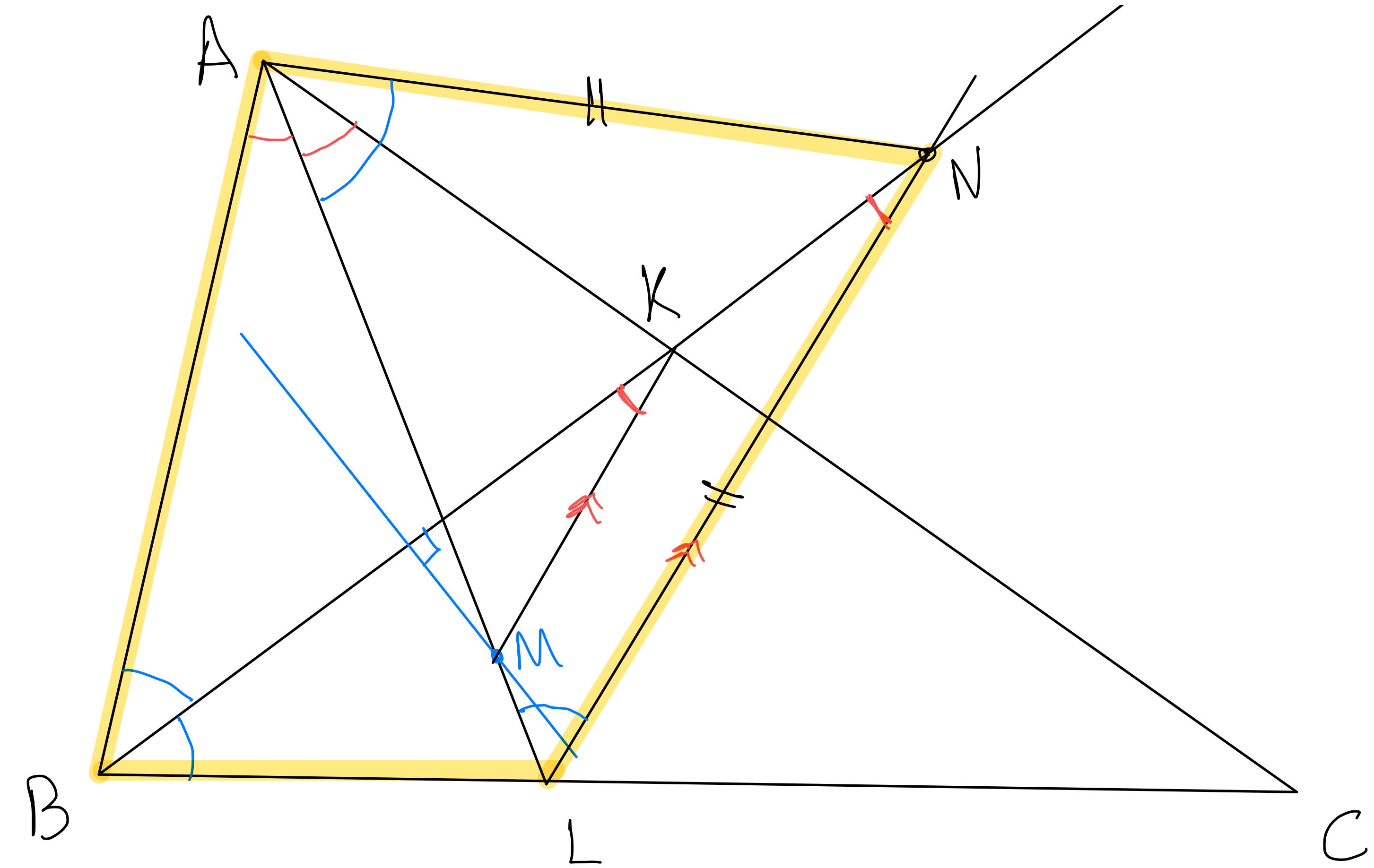

$P' = AI \cap (ABC)$

$$\left.\begin{array}{l}\angle P'AB = \angle P'AC \Rightarrow P'B = P'C \\ \text{and} \quad P' \in AI\end{array}\right\} \Rightarrow P' = P$$

Therefore, $P \in (ABC)$

$\square$

13. Let $AL$ and $BK$ be angle bisectors in the non-isosceles triangle $ABC$ ($L$ lies on the side $BC$, $K$ lies on the side $AC$). The perpendicular bisector of $BK$ intersects the line $AL$ at point $M$. Point $N$ lies on the line $BK$ such that $LN$ is parallel to $MK$. Prove that $LN = NA$. (JBMO 2010 P3)



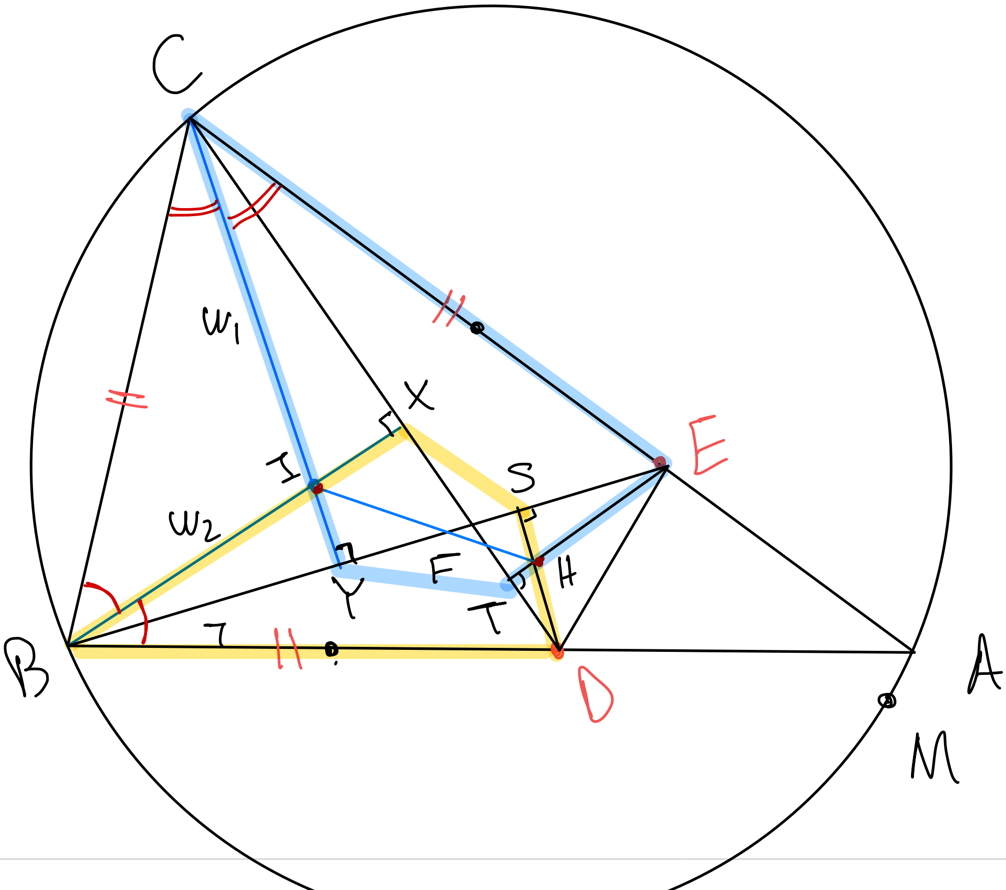

From lemma 3, in $\Delta AKB$, AKMB is a cyclic

$\Rightarrow \angle BAM = \angle BKM = \angle BNL$

$\Rightarrow \begin{cases} ABLN \text{ is a cyclic} \\ N = \text{angle bisector of } \angle ABL \cap (ABL) \end{cases}$

$\Rightarrow NA = NL$ $\square$

14. Let $D$ and $E$ be points in the interiors of sides $AB$ and $AC$, respectively, of a triangle $ABC$, such that $DB = BC = CE$. Let the lines $CD$ and $BE$ meet at $F$. Prove that the incentre $I$ of triangle $ABC$, the orthocentre $H$ of triangle $DEF$ and the midpoint $M$ of the arc $BAC$ of the circumcircle of triangle $ABC$ are collinear. (EGMO 2014 P2)




$CI \perp BE$, $BI \perp CD$

$\Rightarrow$ $I$ is the orthacenter in $\triangle BCF$

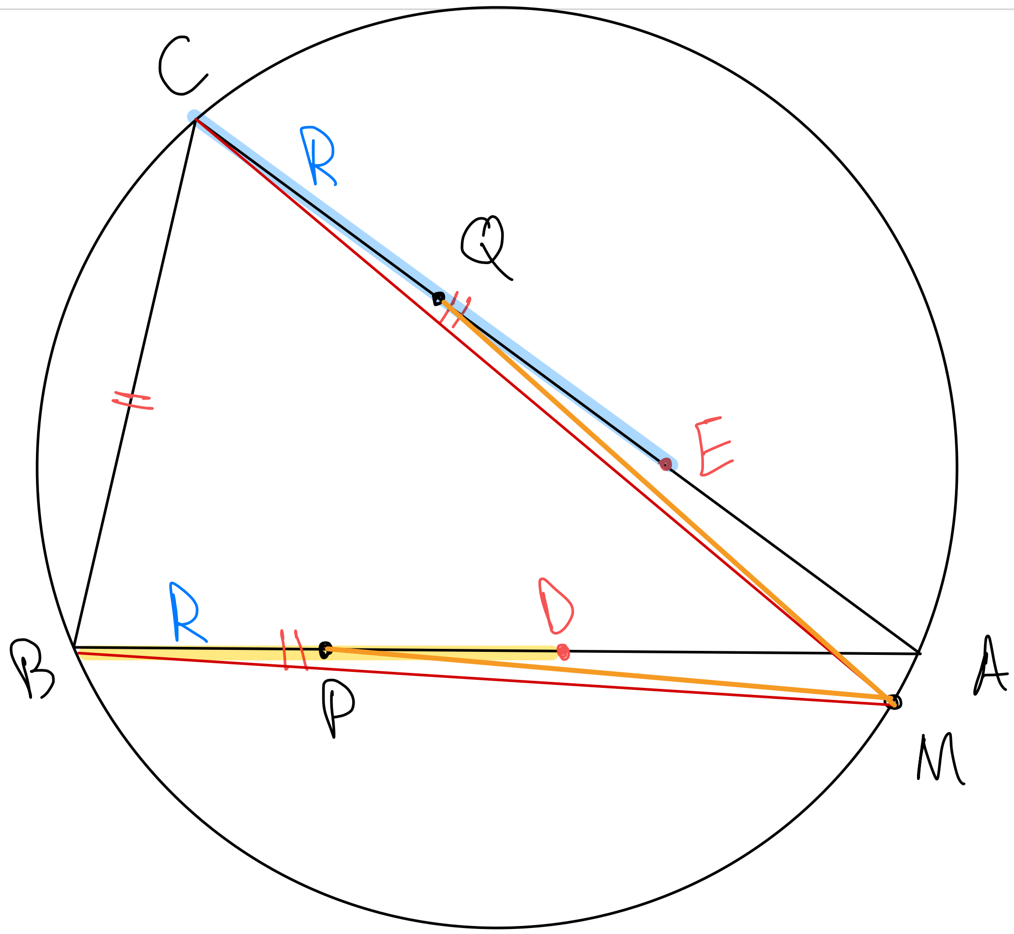
$\begin{cases} EYTC \text{ is a cyclic with circle } \omega_1 \\ BXSD \text{ is a cyclic with circle } \omega_2 \end{cases}$

$P_{\omega_1}(H) = TH \cdot HE$, $P_{\omega_2}(H) = SH \cdot HD$

However $TH \cdot HE = SH \cdot HD$ because $DTSE$ is a cyclic

so $P_{w_1}(H) = P_{w_2}(H)$. Similarly $P_{w_1}(I) = P_{w_2}(I)$

Therefore, $HI$ is the radical axis of $\omega_1$ and $\omega_2$.




Let P and Q be the midpoints of CE and BD.

$\Longrightarrow \begin{cases} P \text{ is the center of } w_2 \\ Q \text{ is the center } w_1 \end{cases}$

We want to show that M is on the radical axis of $w_1$ and $w_2$.

$\mathcal{P}_{w_1}(M) = MQ^2 - R^2$ , $\mathcal{P}_{w_2}(M) = MP^2 - R^2$ , $R = \frac{BC}{2}$

$\mathcal{P}_{w_1}(M) = \mathcal{P}_{w_2}(M) \iff MQ = MP.$

Since $\begin{cases} MC = MB \\ BP = CQ \end{cases}$, $\{ MQ = MP \iff \Delta MQC \equiv \Delta MPB \}$

$\iff \angle MBP = \angle MCQ$

$\iff \angle MBA = \angle MCA$ true (cyclic) $\square$