Email training, N11

November 26 - December 7, 2019

Problem 11.1. Find all positive integers n such that the products of its digits is equal $n^2 - 10n - 22$.

Problem 11.2. Prove there exist infinitely many positive integers divisible by 2017 and each of them containing the same number of digits 0, 1, ..., 9.

Problem 11.3. Solve equation, where x and y are positive integers.

$$x! + 13 = y^2.$$

Problem 11.4. Let $S(n)$ be a sum of divisors of n (for example $S(9) = 1 + 3 + 9 = 13$). Find all positive integers n satisfying condition $S(2n) = 3S(n)$.

Problem 11.5. Find all values of a for which the equation $x^3 + ax^2 + 56x - 4 = 0$ has 3 roots forming consecutive terms of a geometric progression.

Problem 11.6. Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate the sum

$$\sum_{k=0}^{2019} f\left(\frac{k}{2019}\right).$$

Problem 11.7. Find the number of real solutions of equation $x^3 + 2x^2 - 6 = 0$.

Problem 11.8. Prove that for any real numbers a, b and c the following inequality holds

$$ab + bc + ca + |a - b| \leq 1 + \frac{1}{3}(a + b + c)^2.$$

Problem 11.9. One cuts a grid of size 8×8 by a straight line. Find the maximal possible number of cells that are cut by the line.

Problem 11.10. In the cells of the grid 10×10 are written positive integers, all of them less than 11. It is known that the sum of 2 numbers written in the cells having common vertex is a prime number. Prove that there are 17 cells containing the same number.

Problem 11.11. Let 16 football teams participate to the tournament, where each 2 teams play exactly once. Winning team gets 3 points, in the case of draw each team gets 1 point and losing team gets 0 point. The team is called "lucky" if it earns more than the half of maximum possible points. Find the maximum possible number of "lucky" teams.

Problem 11.12. Each cell of the board $n \times n$ is painted either white or black. Any row and column, that intersect at white cell, have together at least n black cells. Prove that the total number of black cells is at least $\frac{n^2}{2}$.

Solution submission deadline December 7, 2019