Email training, N2

Level 4, September 20-26

Problems with Solutions

Problem 2.1. Let polynomial

$$P(x) = \underbrace{((\ldots((}_{k} x-2)^2-2)^2-\ldots)^2-2)^2$$

is given. Find coefficient at x^2.

Solution 2.1. Let

$$P_k(x) = \underbrace{((\ldots((}_{k} x-2)^2-2)^2-\ldots)^2-2)^2 = \ldots + a_kx^2 + b_kx + c_k.$$

One has $a_1 = 1$, $b_1 = 4$ and $c_1 = 2$.
Since $P_k(x) = P_{k-1}(x) - 2)^2$, therefore

i) $c_k = c_{k-1}^2 - 2$,
ii) $b_k = 2b_{k-1}c_{k-1}$,
iii) $a_k = 2a_{k-1}c_{k-1} + b_{k-1}^2$.

Its obvious, that $c_k = 2$, from which immediately follows, that $b_k = 4^k$. By putting those values into *iii*) on gets

$$a_k = 4a_{k-1} + 4^{2k-2}.$$

Lets calculate a_k. Denote $x_k = \frac{a_k}{4^{k-1}}$. Then one has $x_1 = 1$ and

$$x_{k+1} = \frac{a_{k+1}}{4^k} = \frac{4a_k + 4^{2k}}{4^k} = \frac{a_k}{4^{k-1}} + 4^k = x_k + 4^k.$$

From this one immediately obtains, that

$$x_k = x_1 + 4 + 4^2 + ... + 4^{k-2} + 4^{k-1} = \frac{4^k - 1}{3}.$$

So $a_k = 4^{k-1}x_k = \frac{4^{k-1}(4^k-1)}{3}$.

Problem 2.2. Let the sequence $a_1, a_2, \ldots, a_n$ is such that $a_1 = 0$, $|a_2| = |a_1+1|$, $|a_3| = |a_2+1|$, $\ldots$, $|a_n| = |a_{n-1}+1|$: Prove that

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq -\frac{1}{2}.$$

Solution 2.2. Let prove by induction on n. The cases $n = 1$ and $n = 2$ are obvious: If all terms of the sequence are non-negative then we are done. Otherwise there exists an index k such that $a_k \geq 0$ and $a_{k+1} < 0$. Then $a_{k+1} = -(a_k + 1)$, so

$$\frac{a_1 + a_2 + \ldots + a_n}{n} = \frac{a_1 + a_2 + \ldots + a_{k-1} + a_{k+2} + \ldots + a_n - 1}{n}. \qquad 2.2.1$$

$|a_{k+2}| = |a_{k+1} + 1| = |-(a_k + 1) + 1| = |a_k| = |a_{k-1} + 1|$, which means that terms a_{k-1} and a_{k+2} could be considered as consecutive terms of our sequence, so according to our induction hypothesis one has

$$\frac{a_1 + a_2 + ... + a_{k-1} + a_{k+2} + ... + a_n}{n-2} \geq -\frac{1}{2}. \qquad 2.2.2$$

By combining (2.2.1) and (2.2.2) one gets

$$\frac{a_1+a_2+\ldots+a_n}{n}=\frac{a_1+a_2+\ldots+a_{k-1}+a_{k+2}+\ldots+a_n-1}{n}\geq\frac{-\frac{n-2}{2}-1}{n}=-\frac{1}{2}.$$

Problem 2.3. Prove that for any 2 positive integers m and n with $m>n$ holds the following inequality

$$lcm(m,n)+lcm(m+1,n+1)>\frac{2mn}{\sqrt{m-n}} \tag{2.4.1}$$

where $lcm(a,b)$ is the least common multiplier of a and b (for example $lcm(6,8)=24$).

Solution 2.3. Since $lcm(a,b)\cdot gcd(a,b)=ab$ one can rewrite the inequality (2.4.1) in the following form

$$\frac{mn}{gcd(m,n)}+\frac{(m+1)(n+1)}{gcd(m+1,n+1)}>\frac{2mn}{\sqrt{m-n}}$$

$$\frac{mn}{gcd(m-n,n)}+\frac{(m+1)(n+1)}{gcd(m-n,n+1)}>\frac{2mn}{\sqrt{m-n}} \tag{2.4.2}$$

Lets prove the following inequality, which is stronger than (2.4.2)

$$\frac{mn}{gcd(m-n,n)}+\frac{mn}{gcd(m-n,n+1)}>\frac{2mn}{\sqrt{m-n}}$$

$$\frac{1}{gcd(m-n,n)}+\frac{1}{gcd(m-n,n+1)}>\frac{2}{\sqrt{m-n}} \tag{2.4.3}$$

Let $gcd(m-n,n)=x$ and $gcd(m-n,n+1)=y$. Since n and $n+1$ are coprime, then x and y are coprime as well. Also, lets note that x and y are divisors of $m-n$, which means $xy|(m-n)$, therefore $m-n\geq xy$. Now, let back to inequality (2.4.3).

$$\frac{1}{(m-n,n)}+\frac{1}{(m-n,n+1)}=\frac{1}{x}+\frac{1}{y}>2\sqrt{\frac{1}{x}\cdot\frac{1}{y}}\geq\frac{2}{\sqrt{m-n}}.$$

Problem 2.4. Do there exist an infinite sequence p_1, p_2, p_3, $\ldots$ of prime numbers such that for any positive integer n the following condition holds

$$|p_{n+1}-2p_n|=1.$$

Solution 2.4. Answer: NO. Assume such a sequence exists. Without lose if generality one can assume that $p_1>3$. Since p_1 is prime then it gives residue 1 or 2 mod 3. Consider the case when $p_1=1[3]$ (other case is identical). One has $p_2=2p_1+1$ or $p_2=2p_1-1$. Since $2p_1-1$ is divisible by 3 and bigger then 3 then it is not prime, and therefore $p_2=2p_1-1$. The same arguments are correct for $2p_2-1$. So $p_{n+1}=2p_n-1$ for all n. By induction it's easy to prove that

$$p_n=2^{n-1}(p_1-1)+1.$$

Lets consider p_{p_1}. It is prime and bigger than p_1, so it is not divisible by p_1. But, according to Fermat's theorem one has $2^{p_1-1}=1[p_1]$, and therefore $2^{p_1-1}(p_1-1)+1$ is divisible by p_1. We got contradiction.
The case $p_1=2[3]$ can be prove in the same way.

Problem 2.5. Let convex s-gon is divided to q quadrilaterals such that b of them are not convex. Prove that

$$q\geq b+\frac{s-2}{2}.$$

Solution 2.5. Let p be the number of vertices inside the s-gon. The total sum of angles of all quadrilaterals is $180(s-2)+360p$ which is equal $360q$. So

$$180(s-2)+360p=360q$$

by dividing both side by 180 one gets

$$q=p+\frac{s-2}{2}.$$

To complete the proof one needs to show that $p \geq b$.
Non-convex quadrilateral can't have angle bigger 180^o at the vertex belonging to s-gon, therefor that point should be one of the p points inside the s-gon. Also 2 different non-covex quadrilaterals can have angle bigger than 180^o on the same vertex inside the polygon $(180+180>360)$. It means, that the number of vertices inside the polygon p can't be less than the number of non-convex quadrilaterals b.

Problem 2.6. Let positive numbers are written along the circle, such that all of them are less than 1. Prove that one can split the circle to 3 parts such that for each two arcs the sums of numbers written on them differs by at most 1.

Solution 2.6. By weight of arc lets denote the sum of numbers written on it. So we have three arcs and three numbers. By variance of partition of the circle by 3 arcs we will denote the difference between the highest and lowest weights. Consider the partition having the minimum variance. Lets prove that the variance is at most 1 (it will solve the problem).
Let 3 weights are $a \leq b \leq c$ and $c-a>1$. Take the number r from arc c (on the border with a) and move it to arc a. We will have new particion with weights $a+r, b, c-r$. Then

$$-1 \leq -r \leq b-a-r=b-(a+r)<b-a \leq c-a,$$

$$-1 \leq -r \leq (c-r)-b=c-b-r<c-b \leq c-a,$$

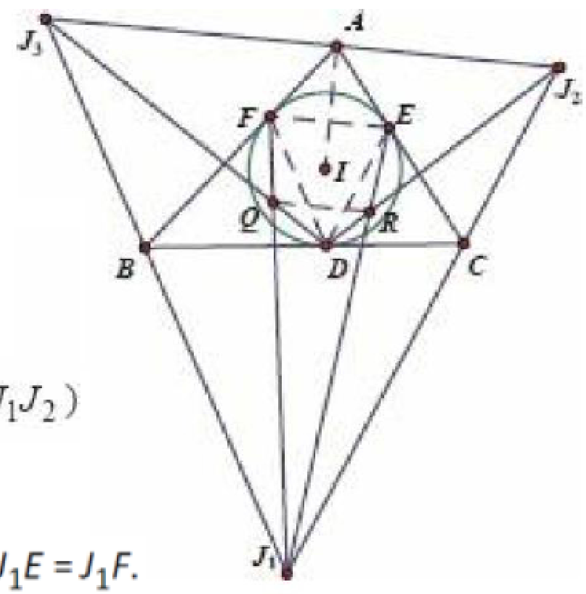
$$-1 \leq (c-a)-2 \leq (c-a)-2r-(c-r)-(a+r)<c-a.$$

So the new variance of new partition is less than $c-a$, which contradicts to the definition of a, b and c. So $c-a \leq 1$.

Other solution. Consider partition having least some $a^2+b^2+c^2$. Again, if $c>a+1$ then $(a+r)^2+b^2+(c-r)^2<a^2+b^2+c^2$.

Problem 2.7. Let incircle of triangle ABC has center I and touches sides BC, AC and AB at points D, E, F respectively. Let J_1, J_2, J_3 be te ex-centres opposite A, B, C respectively. Let J_2F and J_3E intersect at P, J_3D and J_1F intersect at Q, J_1E and J_2D intersect at R. Show that I is the circumcenter of PQR.

Solution 2.7. -



Recall that J_2J_3 // EF because both are perpendicular to AI.

Similarly, J_1J_2 // DE and J_1J_3 // DF.

It follows that $\Delta DEF \sim \Delta J_1J_2J_3$.

Now $\dfrac{J_1Q}{FQ} = \dfrac{DF}{J_1J_3}$ (since $J_1J_3 \,//\, DF$)

$= \dfrac{DE}{J_1J_2}$ ($\Delta DEF \sim \Delta J_1J_2J_3$) $= \dfrac{J_1R}{ER}$ ($DE \,//\, J_1J_2$)

Hence, QR // EF. Notice that AJ_1 is the perpendicular bisector of EF and hence, $J_1E = J_1F$.

It follows that AJ_1 is also the perpendicular bisector of QR. Since I lies on AJ_1, we must have $QI = RI$. Similarly, $PI = QI$ and the conclusion follows.