Email training, N10

November 17 - 23, 2019

When we are examining a process that involves repeated transformations, we should explore what does not change. This is often critical to understanding the properties of the process. We have a starting state S we need to find some argument to answer the questions

- Is it possible to reach a given state?
- Find all reachable states.
- Find all reachable end states (states where no more transformations are possible).
- Will the process inevitably converge to some end state?
- Is there periodicity?

If we want to answer any of these questions, it's likely that we will make use of invariants or monoinvariants. For example when repeatedly sum the digits of a number to get a new number then the invaiant is the value of the number mod 9, since

$$27708 \to 24 \to 6 \to 6 \dots.$$

Example 1. A dragon has 100 heads. A knight can cut off 15, 17, 20, or 5 heads, respectively, with one blow of his sword. In each of these cases, 24, 2, 14, or 17 heads grow on its shoulders. If all heads are blown off, the dragon dies. Can the dragon ever die?

Once we hit upon the idea of using invariants, however, it becomes trivial. We note:

$$(24 - 15) \equiv (2 - 17) \equiv (14 - 20) \equiv (17 - 5) \equiv 0[3].$$

The knight's gallantry can never change the number of heads of the dragon mod 3. Since we start at $100 \equiv 1(mod 3)$, we can never get to 0. So dragon will never die.

Many problems require a generalization of the idea of an "invariant." Even if we cannot identify some function of the state of a process that never changes, we may be able to identify a function that always changes in the same direction. In fact, this information can be invaluable. Consider the following fact:

If there is some positive integral function that decreases at each step of a process, the process must eventually terminate.

This is trivial: if a positive integral function starts at n and decreases with each step, the process certainly cannot continue for more than $n-1$ steps. Yet despite its apparent obviousness, this principle is perhaps the most powerful way for us to draw conclusions about how a process behaves.

Example 2. In the Senate of Kazakhstan, each member has at most three enemies. A member cannot be his own enemy, and enmity is mutual. Prove that the Senate can be divided into two factions such that each Senator has at most one enemy within his faction.

First, we separate the members arbitrarily into two factions. Let H be the sum of all the enemies each member has in his own faction. Suppose one member (let's call him Bob) has at least two enemies in his own faction. Then if Bob switches factions, H will decrease. Let this process continue for all members in the same situation as Bob. Since H is a positive integral function that decreases at each step of the process, the process must terminate. At this point, no Senator can have more than one enemy in his own faction, because otherwise the process (by definition) would not have terminated. Thus we have found the desired division of Senators.

The extremal principle rests on three important facts (from Engel):

1. Every finite nonempty set A of nonnegative integers or real numbers has a minimal element $\min A$ and a maximal element $\max A$, which need not be unique.
2. Every nonempty subset of positive integers has a smallest element.

Problem 10.1. Let a set of integers a, b, c during the one minute changes to the set of integers $a + b - c, b + c - a, c + a - b$. Originally we have the set $2000, 2002, 2003$. Is it possible that after some time we get the set $2001, 2002, 2003$.

Problem 10.2. Let 4 corner cells of the board 8×8 are coloured black and other 60 cells are coloured black. At each step one allowed to choose a row or column and color it (it means change colours of all cells in that row or column). Is it possible after some steps get a board where all cells are white.

Problem 10.3. Countries Alistan and Babastan have their own currencies ali and baba. In Alistan 1 ali can be exchanged to 10 baba, however in Babastan 1 baba can be exchanged to 10 ali. Omar may travel between two countries and originally has 1 ali. Prove that his amount of alis will never be equal to the amount of babas (only integer amounts are available).

Problem 10.4. There is a pile containing 1001 coins. At each step one allowed to choose a pile containing more than 2 coins, throw out one coin from the pile and divide the pile into two non-empty piles. Is it possible that after some steps we achieve a situation that all piles have exactly 3 coins?

Problem 10.5. There are 13 gray, 15 red and 17 blue chameleons living on the island. Whenever two chameleons of different colors meet, they change their color to the third one (if gray and blue meet then both of them become red). Is it possible that at some moment all chameleons on the island have the same color?

Problem 10.6. Positive integers 1 and 2 are written on the board in laboratory. Every morning professor Ali erases the written numbers from the board and writes their arithmetical and harmonic means. It occurs that

a) At some moment $\frac{941664}{665857}$ was written on the board. Determine either it is written as arithmetical or harmonic mean and determine the other number written on the board.

b Determine if some moment the number $\frac{35}{24}$ can be written on the board.

Problem 10.7. The rectangle is covered by bricks 2×2 and 1×4. Haider looses on peace 2×2. Then he replaces it by a peace 1×4. Prove, that he can't cover the original rectangle.xed

Problem 10.8. Let a convex 10-gon is given, where no any three diagonals intersect at one point. At each vertex of the 10-gon and at each intersection point of two diagonals the numbers $+1$ are written. At each step one may choose any diagonal and any side and change change all signs on that diagonal/side. Is it possible that after some steps all numbers are equal -1.

Problem 10.9. Let several numbers are written on the board (not all of them equal 0). At each step one may choose two numbers a and b and replace them by $a - \frac{b}{2}$ and $b + \frac{a}{2}$. Prove that one can't achieve a situation when all numbers are equal 0.

Problem 10.10. Let non-regular polygon is inscribed to the circle. Each step one may choose a vertex A which divides the arc between two neighbour to the A vertices into non equal parts and move the point A to the midpoint of that arc. Is it possible that after 100 moves one gets a polygon which is equal to the original one?

Solution submission deadline November 23, 2019