Number Theory

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Problems – April 8

1. If a, b, c and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ are integers, prove that abc is a perfect cube.
2. If $p > 3$ is a prime number, prove that $\frac{2^{2p}+1}{5}$ is a composite number.
3. Let $a > 1$ be an integer. Prove that:
(a) $a^n - 1 \mid a^m - 1$ if and only if $n \mid m$;
(b) $a^n + 1 \mid a^m + 1$ if and only if $n \mid m$ and $\frac{m}{n}$ is odd.
4. Prove that $5^{2^n} - 1$ is divisible by 2^{n+2}, but not by 2^{n+3}.
5. Prove that there are infinitely many positive integers n for which $n \mid 2^n + 1$.
6. Prove that every multiple of $2^n - 1$ has at least n binary units.
7. Integers $a > b > 1$ are such that $a^2 + b - 1$ is divisible by $b^2 + a - 1$.Prove that $b^2 + a - 1$ cannot be a power of a prime.
8. Find all integers x and y that satisfy $x^2 + x = y^3 + y^2 + y$.
9. Let a and b be integers of different parity. Prove that there exists an integer c such that the three numbers $c + a$, $c + b$ and $c + ab$ are perfect squares.
10. Let $a, b \in \mathbb{N}$ be such that $a!b!$ is divisible by $a! + b!$. Prove that $3a \geqslant 2b + 2$.
11. Find all prime numbers p such that $\frac{p^2-p-2}{2}$ is a perfect cube.