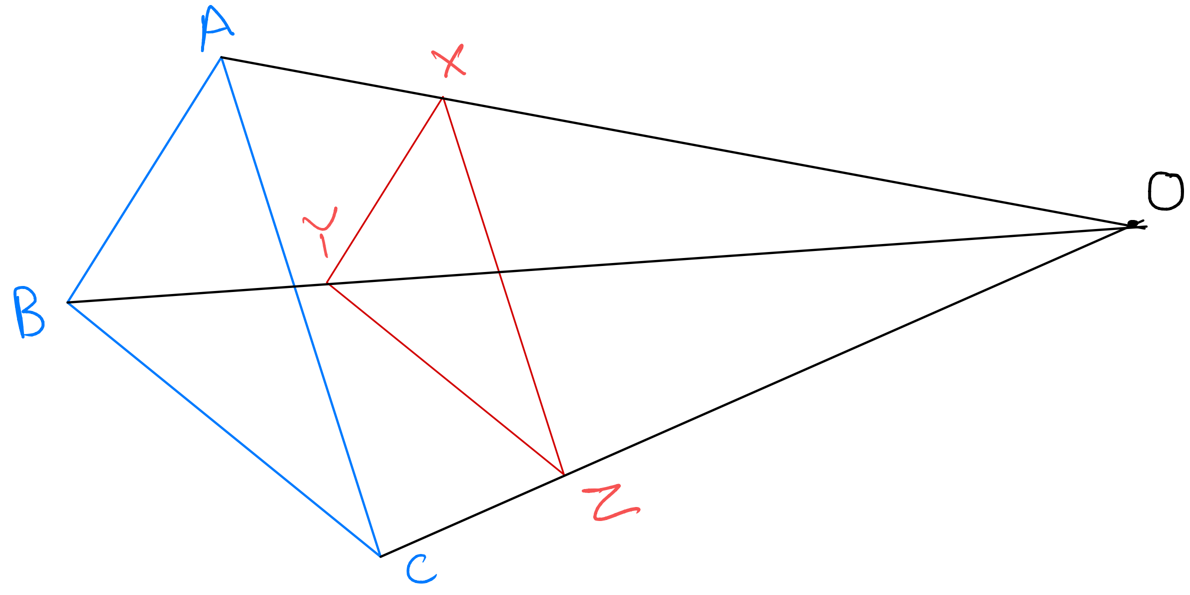
Intensive Training

Geometry

Day 9

19 April 2021

P1) 3.1.3. COROLLARY. Let ABC and XYZ be non-congruent triangles such that $\overline{AB} \parallel \overline{XY}$, $\overline{BC} \parallel \overline{YZ}$, and $\overline{CA} \parallel \overline{ZX}$. Then lines AX, BY, CZ concur at some point O, and O is a center of a homothety mapping $\triangle ABC$ to $\triangle XYZ$.



Let $O = AX \cap BY$. O is a center of homothety that sends X to A and Y to B.

Assume that C' is the image of Z

$\Rightarrow$ $\triangle ABC' \sim \triangle XYZ$ (by homothety)

However $\triangle ABC \sim \triangle XYZ$ $\Rightarrow$ $\triangle ABC \sim \triangle ABC'$

$\Rightarrow$ $C = C'$ $\square$

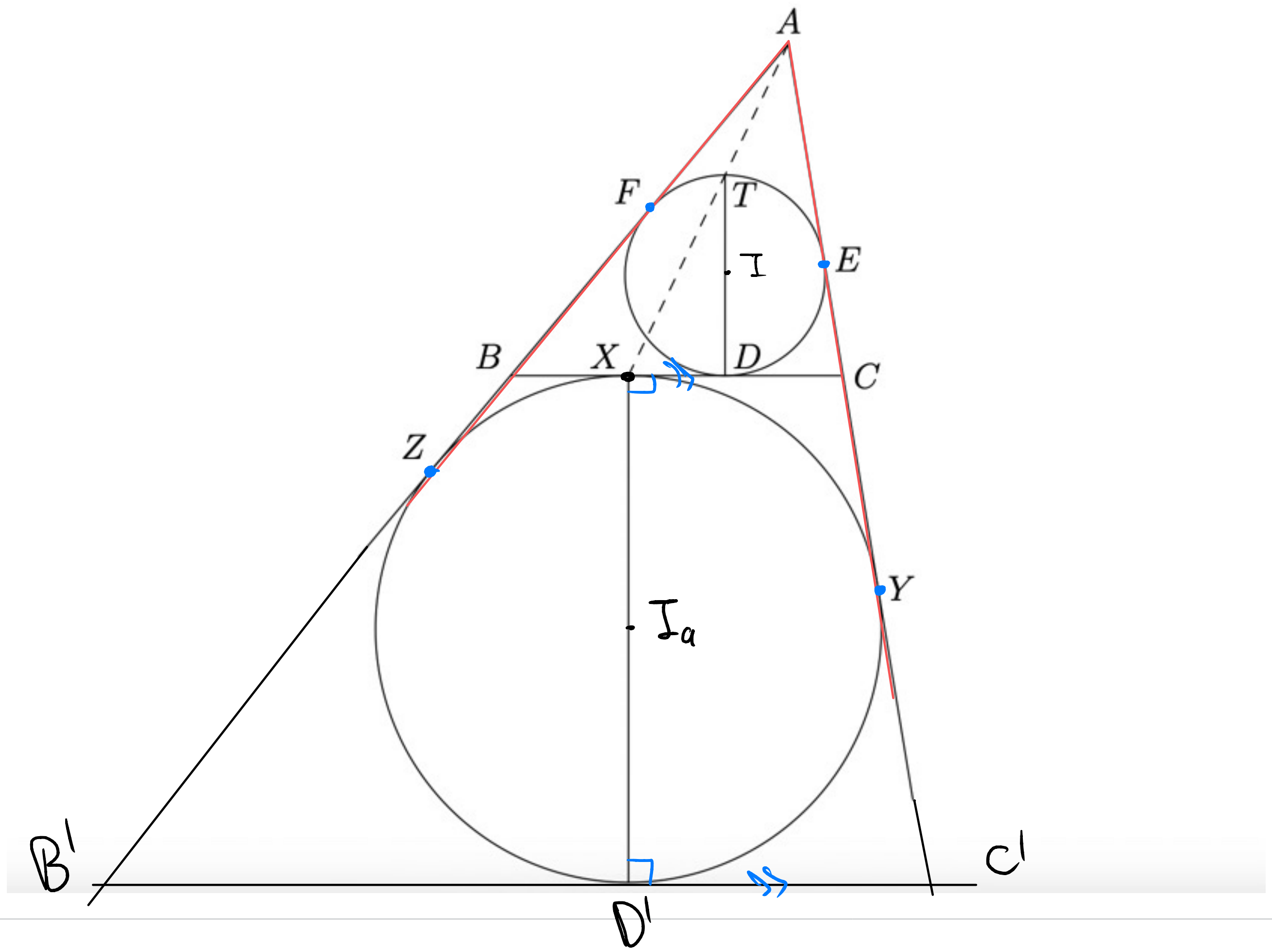
Another way

$AC' \parallel XZ \Rightarrow AC' \parallel AC$

$BC' \parallel YZ \Rightarrow BC' \parallel BC$ $\Rightarrow C = C'$ $\square$

P2)

27. EXERCISE. Let ABC be a triangle. The incircle touches $\overline{BC}$ at D, while the A-excircle touches BC at X. Show that $\overline{AX}$ passes through the antipode of D on the incircle. (So TD is a diameter in (I))



Let h be a homothety at A that sends I to I_a (We know that A, I, I_a are collinear). We also know that

$$\frac{AE}{AY} = \frac{AF}{AZ} = \frac{AI}{AI_a} = \frac{r}{r_a}$$

$\Rightarrow$ h sends E to Y, F to Z, (I) to (I_a)

We want to show that h sends T to X.

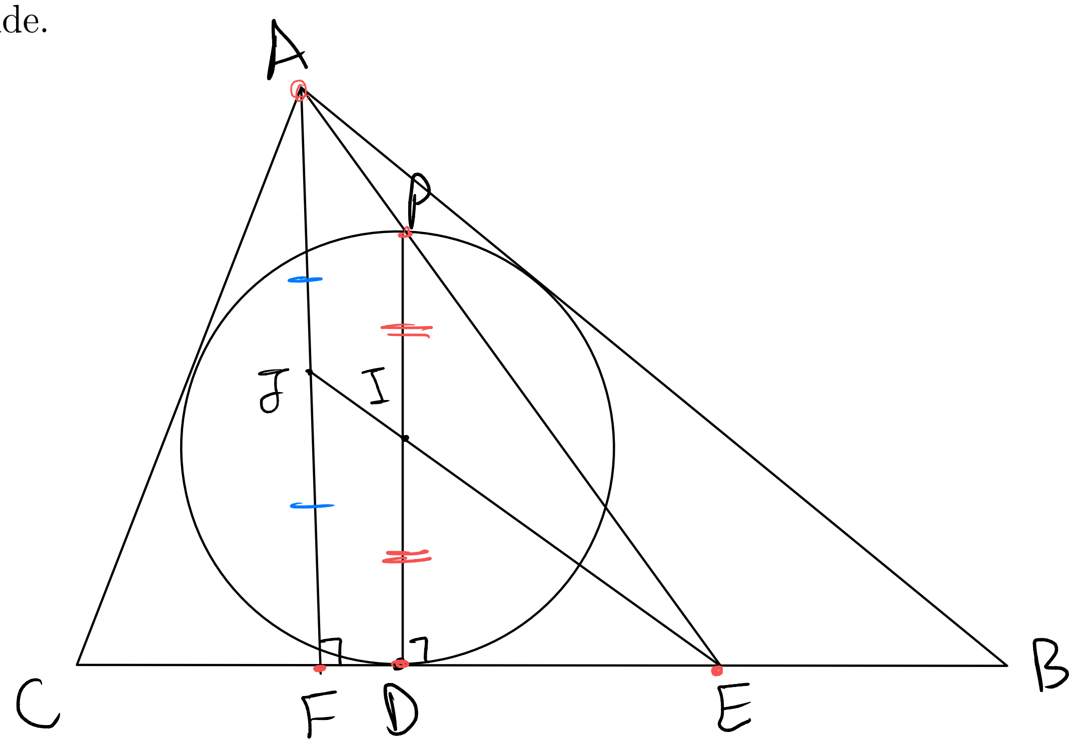
Assume that $h(C) = C'$, $h(B) = B'$, $h(D) = D'$ $\Rightarrow$ $B'C' \parallel BC$, $D' \in B'C'$

$I_aX \perp BC$, $I_aD' \perp B'C'$, $BC \parallel B'C'$ $\Rightarrow$ X, I_a, D' are collinear

$\Rightarrow$ XD' diameter in I_a $\Rightarrow$ $X = h(T)$
$\Rightarrow$ A, T, X collinear $\square$

27. EXERCISE. Let ABC be a triangle. The incircle touches $\overline{BC}$ at D, while the A-excircle touches BC at E. Show that $\overline{AE}$ passes through the antipode of D on the incircle.

28. EXERCISE. In the notation of the previous exercise, show that $\overline{EI}$ bisects the A-altitude.



$ID \perp BC \Rightarrow ID \parallel AF$

Let $P = AE \cap (I)$, $J = EI \cap AF$

From P27, PD is a diameter $\Rightarrow$ P, I, D collinear and I is the midpoint of PD

Since $PD \parallel AF$, then J is the midpoint of AF (can be seen as a homothety with center E) $\square$

30. Exercise (*). In triangle ABC with contact triangle DEF, point M is the midpoint of $\overline{BC}$. Prove that the lines AM, EF, DI are concurrent. (Hint: