Email training, N5

Level 3, October 11-17

Problem 5.1. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.

Problem 5.2. Prove the identity

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \ldots + (-1)^n \frac{\binom{n}{n}}{x+n}.$$

Problem 5.3. Prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Problem 5.4. Let $x, y, z \geq 0$ and $x + y + z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + xz + zx.$$

Problem 5.5. Let $a, b, c > 0$. Prove that

$$\frac{a+b}{a^2+b^2} + \frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Problem 5.6. Let $n > 3$, $x_1, x_2, \ldots, x_n > 0$ and $x_1 x_2 \ldots x_n = 1$. Prove that

$$\frac{1}{1 + x_1 + x_1 x_2} + \frac{1}{1 + x_2 + x_2 x_3} + \ldots + \frac{1}{1 + x_n + x_n x_1} > 1.$$

Problem 5.7. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

Solution submission deadline October 17, 2021

Submit single PDF file in filename format L3_YOURNAME_week5.pdf

submission email **imo20etraining@gmail.com**