$$p^2 - p + 1 = a^3 \Rightarrow p(p-1) = (a-1)(a^2+a+1)$$

$$\Rightarrow p \mid a-1 \text{ or } p \mid a^2+a+1 \Big\} \text{ ① point}$$

Clearly $a > 1$

Case 1: $p \mid a-1$

then $a \geq p+1$

$\frac{a-1}{p} \in \mathbb{Z} \Rightarrow \frac{p-1}{a^2+a^2+1} \in \mathbb{Z} \Rightarrow a^2+a+1 \mid p-1$

but $a^2+a+1 > p^2+p+1 > p-1$ ② points

$\Rightarrow\Leftarrow$ No solutions

Case 2: $p \mid a^2+a+1$

then $\frac{a^2+a+1}{p} = \frac{p-1}{a-1} = k \in \mathbb{Z}$

$$\Rightarrow \begin{cases} a^2+a+1 = pk \\ p-1 = (a-1)k \end{cases} \Big\} \text{ ① point}$$

$$\Rightarrow a^2+a+1 = ((a-1)k+1)k$$

$$\Rightarrow a^2 + (1-k^2)a + (k^2-k+1) = 0 \Big\} \text{ ① point}$$

$$\Rightarrow \sqrt{\Delta} \in \mathbb{Z} \Rightarrow \sqrt{(1-k^2)^2 - 4(k^2-k+1)} \in \mathbb{Z}$$

$\Rightarrow k^4 - 6k + 4k - 3$ is a perfect square } ① point

but when $k \geq 4$ we have

$k^4 - 6k^2 + 4k - 3 > (k^2-3)^2 \Leftrightarrow 4k-3 > 9 \Leftrightarrow k > 3$ ✓

$k^4 - 6k^2 + 4k - 3 < (k^2-2)^2 \Leftrightarrow 2k^2 - 4k > 1 \Leftrightarrow 2k(k-2) > 1$ ✓ ③ points

① point { Checking $k = 1, 2, 3$ and finding $k = 3 \Rightarrow (p, a) = (19, 7)$