THE Mathematical Battle

December 6, year 2022, KAUST

You have three hours of time for solving the problems without any disturbance. Enjoy :)

Problem 1. There is an acute-angled triangle ABC where CC_1 is its altitude. There is a line passing through C, and let A_1 and B_1 be the projections of A and B on this line correspondingly. Assume C lies on the interval A_1B_1. Prove that the center of the circumcirle of ABC lies on the Euler's circle of the triangle ABC.

Problem 2. Given infinite sequences $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ of real numbers satisfying $a_{n+1}+b_{n+1} = \frac{a_n + b_n}{2}$ and $a_{n+1}b_{n+1} = \sqrt{a_n b_n}$ for all $n \geq 1$. Suppose $b_{2016} = 1$ and $a_1 > 0$. Find all possible values of a_1.

Problem 3. Prove that for any positive integer a there is a prime number $b > a$ such that $1 + 2^b + 3^a$ is divisible by $1 + 2^a + 3^a$.

Problem 4. Find all polynomials $P(x)$ with integer coefficients such that $P(P(n)+n))$ is a prime number for infinitely many integers n.

Problem 5. Prove that the sum of the inversed values of any set of pairwise different palindromes is smaller than 20. (Palindrome is a number that remains the same when its digits are reversed. E.g 16461)

Problem 6. There is a 2015×2015 board and there are 1800 chess pieces placed on it, each piece is either a rook or a queen. It turned out that all of the cells of the board are under attack by one of the pieces. Prove that there must be at least 214 queens among the pieces.

Problem 7. Let $ABCDEF$ be a convex hexagon with pairwise equal inner angles. Prove that for any point P of the plane the following inequality holds:

$$PA + PB + PC + PD + PE + PF \geq AB + BC + CD + DE + EF + FA.$$

Problem 8. Darboux has two identical boxes, each of which is a chock-a-block (i.e no space left in between) filled with toys, each being some tetrahedron. On each toy from the first box he has written $h_1 \cdot V_1$ where V_1 is the volume of the toy and h is the shortest distance from any of the vertices of this toy to the bottom of the box. Whereas on each toy from the second box he has written $V_2 \cdot h$ where V_2 is the volume of the toy and h is the largest of the distances from the vertices of this brick to the bottom of the box. Prove the sum of the numbers written on the toys in the first box is less than or equal to the sum of the numbers written on the toys from the second box.

Problem 9. Find all pairs of prime p and an integer x such that

$$p^3 = 5^x + 2.$$

Problem 10. Call a graph $a-good$ if it does not contain a complete graph on 100 vertices and the degree of each of the vertices is at most a. Given two positive integers $d \geq 100$ and k such that for any $d-good$ graph there exists a proper k-colouring of its vertices. Prove that for any $d^2 - 2$ good graph there exists a proper $(d-1)k$ colouring of its vertices.

(A "proper m-colouring of vertices" is a colouring of the vertices in m colours in such a way that no two vertices of the same colour are connected.)