Email training, N3

September 25-October 1

Problem 3.1. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.

Problem 3.2. Prove the identity

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \ldots + (-1)^n \frac{\binom{n}{n}}{x+n}.$$

Problem 3.3. Prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Problem 3.4. Let $x, y, z \geq 0$ and $x + y + z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + xz + zx.$$

Problem 3.5. Let $a, b, c > 0$. Prove that

$$\frac{a+b}{a^2+b^2} + \frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Problem 3.6. Let $n > 3$, $x_1, x_2, \ldots, x_n > 0$ and $x_1 x_2 \ldots x_n = 1$. Prove that

$$\frac{1}{1 + x_1 + x_1 x_2} + \frac{1}{1 + x_2 + x_2 x_3} + \ldots + \frac{1}{1 + x_n + x_n x_1} > 1.$$

Problem 3.7. -

Let M be the midpoint of the side AC of a triangle ABC and let H be the foot point of the altitude from B. Let P and Q be the orthogonal projections of A and C on the bisector of angle B. Prove that the four points M, H, P and Q lie on the same circle.

Problem 3.8. -

ABCD is a trapezium , $AD \parallel BC$. P is the point on the line AB such that $\angle CPD$ is maximal. Q is the point on the line CD such that $\angle BQA$ is maximal. Given that P lies on the segment AB, prove that $\angle CPD = \angle BQA$.

Solution submission deadline October 1, 2022