Email training, N7-8

October 8 - 19, 2019

Problem 7.1. Find all positive integers n and k for which

$$n^2 + 7 = 2 \cdot 3^k.$$

Solution 7.1. Note that $n^2 \equiv 0[3]$ or $n^2 \equiv 1[3]$, so $n^2 + 7 \equiv 1$ or $2[3]$. Since the right side is always divisible by 3, it means the equation has no solution.

Problem 7.2. There are two irreducible rational numbers with denominators 600 and 700. Find the minimal possible value of the denominator of their sum.

Solution 7.2. Let we have two irreducible rational numbers $\frac{x}{600}$ and $\frac{y}{700}$. Then

$$\frac{x}{600} + \frac{y}{700} = \frac{7x + 6y}{2^3 \cdot 3 \cdot 5^2 \cdot 7}.$$

Since $gcd(y, 7) = 1$, therefore the numerator isn't divisible by 7. By the similar argument for x it isn't divisible by 2 or 3. Note, that we may get numerator divisible by 25, for example when we choose $x = 1$, $y = 3$. So, the smallest possible valur for denominator is $2^3 \cdot 3 \cdot 7 = 168$.

Problem 7.3. Solve in integers

$$x^2 + y^2 + z^2 = 2xyz.$$

Solution 7.3. Note, that in the case if x, y, z are all odd, then the left side will be odd and the right side will be even, contradiction. So, at least one number from x, y, z is even, so the right side of the equation is divisible by 4.

Since the square of the number gives residue 0 or 1 when divided by 4, then we conclude that all x, y and z must be even to get residue 0 on the left side. So $x = 2x_1$, $y = 2y_1$ and $z = 2z_1$. By putting in the equation we get

$$x_1^2 + y_1^2 + z_1^2 = 4x_1y_1z_1.$$

Analogously, since the right side is divisible by 4 then x_1, y_1 and z_1 are even. Let $x_1 = 2x_2$, $y_1 = 2y_2$ and $z_1 = 2z_2$. Again, by putting in the equation we get

$$x_2^2 + y_2^2 + z_2^2 = 8x_2y_2z_2.$$

By continuing n times we get $x = 2^n x_n$, $y = 2^n y_n$ and $z = 2^n z_n$. So x, y, z are divisible by any power of 2. It's possible only when $x = y = z = 0$. Obviously, it is solution for the equation. **Answer:** $x = y = z = 0$.

Problem 7.4. Prove that $40^{1963} + 1963^{40}$ is composite number.

Solution 7.4. According to the Fermat's little theorem we have

$$1963^{40} \equiv 1(mod\ 41).$$

Also, since $40 \equiv -1(mod\ 41)$, therefore $40^{1963} \equiv (-1)^{1963} = -1(mod\ 41)$. By taking the sum we get

$$1963^{40} + 40^{1963} \equiv 1 - 1 = 0(mod\ 41).$$

So, we get that the expression is divisible by 41. Since it's bigger than 41, so it's composite.

Problem 7.5. Let n and q are positive integers, such that all prime divisors of q are greater than n. Show that

$$(q-1)(q^2-1)\dots(q^{n-1}-1) \equiv 0[n!].$$

Solution 7.5. Let $p \leq n$, prime. Since p is not a divisor of q^k, we have $q^{k(p-1)} \equiv 1[p]$, for all k. Therefore at least $\left[\frac{n-1}{p-1}\right]$ factors from the left-hand side are divisible by p.

Let us now compute the exponent of p in $n!$, which is

$$v_p(n!) = \sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right] < \frac{n}{p-1}.$$

Since $v_p(n!)$ is an integer, therefore we may state that

$$v_p(n!) \leq \left[\frac{n-1}{p-1}\right].$$

Thus, for every prime p the power in the left side is at least as the power of p in $n!$. So the statement of the problem is proved.

Problem 7.6. Find all pairs of integers (m,n) such that

$$\binom{n}{m} = 1984.$$

Solution 7.6. We consider $0 \leq n \leq \frac{m}{2}$. We have $1984 = 26 \cdot 31$. Since

$$1984 = \frac{(n+1)(n+2)\dots(m-1)m}{(m-n)!}$$

, then 31 divides $(n+1)(n+2)\dots(m-1)m$, so that $m \geq 31$.

If $n \geq 3$, then $C_m^n \geq C_m^3 \geq C_{31}^3 > 1984$. Thus $n = 0, 1$ or 2. Obviously $n = 0$ has no solution.

If $n = 1$, then $m = 1984$, while $C_m^2 = 1984$ does not have any solutions in natural numbers.

Thus, the solutions are $(1984, 1)$ and $(1984, 1983)$.

Problem 7.7. Prove that for any positive integer n the following identity holds

$$\frac{2n-1}{2} - \frac{2n-2}{3} + \dots - \frac{2}{2n-1} + \frac{1}{2n} = \frac{1}{n+1} + \frac{3}{n+2} + \dots + \frac{2n-1}{2n}.$$

Solution 7.7. We use the following two identities:

$$\frac{2n-k}{k+1} = \frac{2n+1}{k+1} - 1,$$

and

$$\frac{2l-1}{n+l} = 2 - \frac{2n+1}{n+l}.$$

Hence the problem statement transforms to the following identity.

$$\left(\frac{2n+1}{2} - 1\right) - \left(\frac{2n+1}{3} - 1\right) + \dots - \left(\frac{2n+1}{2n-1} - 1\right) + \left(\frac{2n+1}{2n} - 1\right) =$$

$$\left(2 - \frac{2n+1}{n+1}\right) + \left(2 - \frac{2n+1}{n+2}\right) + \dots + \left(2 - \frac{2n+1}{2n-1}\right) + \left(2 - \frac{2n+1}{2n}\right),$$

or

$$(2n+1)\Big(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots-\frac{1}{2n-1}+\frac{1}{2n}\Big)-1=$$
$$2n-(2n+1)\Big(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\Big).$$

This is equivalent to the

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}.$$

This is Catalan's identity. To prove this, lets add $2(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n})$ to both sides. Then we get

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}=$$
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}.$$

Problem 7.8. Let a, b, c be a positive real numbers such that $abc = 8$. Prove that

$$\frac{ab+4}{a+2}+\frac{bc+4}{b+2}+\frac{ca+4}{c+2}\geq 6.$$

Solution 7.8. We have $ab+4=\frac{8}{c}+4=\frac{4(c+2)}{c}$ and similarly $bc+4=\frac{4(a+2)}{a}$ and $ca+4=\frac{4(b+2)}{b}$. It follows that

$$(ab+4)(bc+4)(ca+4)=\frac{64}{abc}(a+2)(b+2)(c+2)=8(a+2)(b+2)(c+2),$$

so by applying AM-GM we get

$$\frac{ab+4}{a+2}+\frac{bc+4}{b+2}+\frac{ca+4}{c+2}\geq 3\sqrt[3]{\frac{(ab+4)(bc+4)(ca+4)}{(a+2)(b+2)(c+2)}}=6.$$

Problem 7.9. Let a sequence of positive integers $a_1, a_2, \ldots$ is given with $a_1 = 1$ and

$$a_{n+1}\leq 1+a_1+a_2+\ldots+a_n.$$

Prove that any positive integer N can be written as a sum of distinct terms of the sequence $\{a_n\}$.

Solution 7.9. Actually, we will prove by induction on n that each positive integer not exceeding $a_l + a_2 + \ldots + a_n$ can be expressed as a sum of distinct terms chosen from $a_1, a_2, \ldots, a_n$. This is obvious for $n = 1$. Assume that it holds for some $n > 1$ and let k be a positive integer such that $k \leq a_1 + a_2 + \ldots + a_n + a_{n+l}$. The case $k \leq a_1 + a_2 + \ldots + a_n$ is directly settled by the induction hypothesis, so we may assume that

$$a_1+a2+\ldots+a_n<k\leq a_l+a_2+\ldots+a_n+a_{n+l}.$$

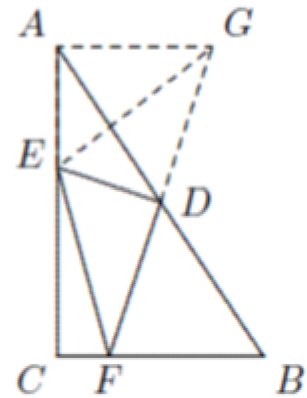
By hypothesis, the left-hand side is greater than or equal to a_{n+l}, thus $0 \leq k - a_{n+l} \leq a_l + a_2 + \ldots + a_n$. We are done if $k - a_{n+l} = 0$. And, if $k - a_{n+l} > 0$, then by the induction hypothesis, $k - a_{n+l}$ is expressible as a sum of distinct terms among $a_l, a_2, \ldots, a_n$. So k can be expressed as a sum of distinct terms chosen from $a_1, a_2, \ldots, a_{n+1}$, and the induction is complete.

Problem 7.10. In triangle ABC let $\angle C = 90^o$ and let D is the midpoint of AB. Let E and F are two points on AC and BC respectively, such that DE and DF are perpendicular. Prove that $EF^2 = AE^2 + BF^2$.

Solution 7.10. -

From A introduce $AG \parallel CB$, intersecting the extension of FD at G, connect EG.

By symmetry, we have $DG = DF$, $AG = BF$, so ED is the perpendicular bisector of FG. Thus,

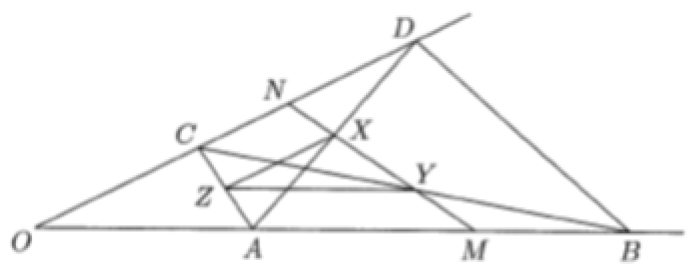


$$\begin{aligned} EF^2 &= EG^2 \\ &= AE^2 + AG^2 \\ &= AE^2 + BF^2. \end{aligned}$$

Problem 7.11. Let AB and CD are segments lying on the two sides of an angle whose vertex is O, such that A is between O and B, as well as C is between O and D. The line connecting the midpoints of the segments AD and BC intersects AB at M and CD at N. Prove that $\frac{OM}{ON} = \frac{AB}{CD}$.

Solution 7.11. -

Let X, Y and Z be the midpoints of AD, BC and AC respectively.



Then $XZ \| ON$ and $YZ \| OM$. Hence XYZ and NMO are similar triangles. Therefore

$$\frac{OM}{ON} = \frac{ZY}{ZX}.$$

Since $ZX = \frac{1}{2}CD$ and $ZY = \frac{1}{2}AB$, we obtain

$$\frac{OM}{ON} = \frac{AB}{CD}.$$

Problem 7.12. -

(1) لدينا ABC مثلث متطابق الأضلاع النقطتان E, D تقعان على الضلعين AC, BC على الترتيب بحيث $AE = DC$. النقطة Q تقع على القطعة المستقيمة AD وبحيث $\angle AQB = 100^\circ$. أوجد قياس $\angle QBE$.

Problem 7.13. -

(2) لدينا $ABCD$ شكل رباعي مرسوم داخل دائرة قطرها AC وطول نصف قطرها 10، وكذلك $BA = BD$.
إذا كان $BC = 6$ فأوجد طول CD .