Lecture2

Winter Camp

Number Theory

3 Jan 2021

Level 2

Let m and n be integers greater than 1 such that $\gcd(m, n-1) = \gcd(m, n) = 1$. Prove that the first $m-1$ terms of the sequence $n_1, n_2, \ldots,$ where $n_1 = mn + 1$ and $n_{k+1} = n \cdot n_k + 1$, $k \geq 1$, cannot all be primes.

Note that $n_k = n^k m + \underbrace{n^{k-1} + n^{k-2} + \cdots + n + 1}$

$= n^k m + \dfrac{n^k - 1}{n-1}$

$m \mid \dfrac{n^k - 1}{n-1}$ if and only if $m \mid n^k - 1$ since $\gcd(m, n-1) = 1$

and since $\gcd(m, n) = 1$ $\boxed{m \mid n^{\varphi(m)} - 1}$ $\Rightarrow$ $m \mid \dfrac{n^{\varphi(m)} - 1}{n-1}$

$m \mid n^{\varphi(m)} m + \dfrac{n^{\varphi(m)} - 1}{n-1} = n_{\varphi(m)}$ and $\varphi(m) < m-1$

$n^{\varphi(m)} > m \implies n_{\varphi(m)}$ is not a prime

$\square$

JBMO Shortlist 2019

N1 SL 2019

Find all prime numbers p for which there exist positive integers x, y, and z such that the number

$$x^p + y^p + z^p - x - y - z$$

is a product of exactly three distinct prime numbers.

① When $p=2$ taking $(x,y,z)=(2,4,4)$ gives solution

② When $p=3$ 〃 $(x,y,z)=(1,2,3)$

③ When $p=5$ 〃 $(x,y,z)=(1,1,2)$

④ When $p>5$

$A = x^p + y^p + z^p - x - y - z$

- $2 \mid A$,
- $3 \mid A$ p is odd $\Rightarrow$ $x^p \equiv x \pmod 3$
- $p \mid A$ because $p \mid x^p - x,\ y^p - y,\ z^p - z$

$\Bigg\}\ 6p \mid A$

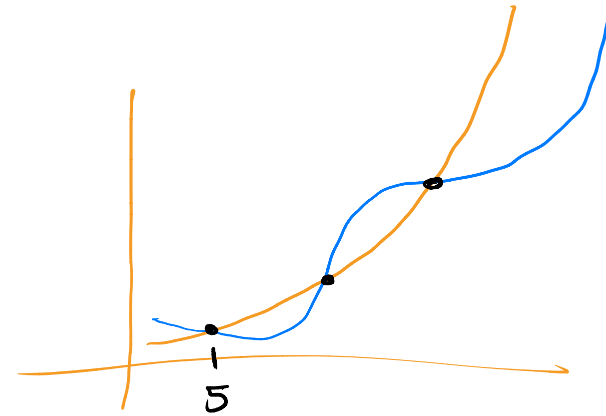
since A is a product of three primes, $A = 6p$

Find all prime numbers p for which there exist positive integers x, y, and z such that the number

$$x^p + y^p + z^p - x - y - z$$

is a product of exactly three distinct prime numbers.

$$x^p - x + y^p - y + z^p - z = 6p$$



one of x, y, z must be larger than 1

$$\Rightarrow \quad A \geq 2^p - 2$$

but in general $2^n > 6n+2$ for $n \geq 6$

, we can prove it by induction.

Base case: $n=6$ $\quad 2^6 > 6.6+2$

Induction step: if $2^{n-1} > 6(n-1)+2$ then $2^n > 12(n-1)+4$

$$\Rightarrow \quad 2^n > 12n-8 = 6n+6n-8 > 6n+6.6-8 > 6n+2$$

$\Rightarrow \quad A \geq 2^p-2 > 6p \Rightarrow$ no solutions when $p>5$.

Find all triples (p, q, r) of prime numbers such that all of the following numbers are integers

$$\frac{p^2+2q}{q+r}, \frac{q^2+9r}{r+p}, \frac{r^2+3p}{p+q}$$

① $q+r$ is even

$2 \mid q+r$, $q+r \mid p^2+2q$ $\Rightarrow$ $2 \mid p^2+2q$ $\Rightarrow 2 \mid p \Rightarrow \boxed{p=2}$

$\frac{4+2q}{q+r}$, $\frac{q^2+9r}{r+2}$, $\frac{r^2+6}{p+2}$ are integers

$4+2q < 3(q+r)$ $\Rightarrow$ $\frac{2q+4}{q+r} = 2, 1$

$\Rightarrow$ $2q+4 = 2q+2r,\ q+r$

$\Rightarrow$ $r=2$ or $q = r-4$

$\Rightarrow$ $q, r = 2$ or $q = r-4$

but $(2,2,2)$ is not a solution $\Rightarrow$ $q = r-4$

Find all triples (p, q, r) of prime numbers such that all of the following numbers are integers

$$\frac{p^2+2q}{q+r}, \frac{q^2+9r}{r+p}, \frac{r^2+3p}{p+q}$$

$$\frac{4+2q}{q+r}, \quad \frac{q^2+9r}{r+2}, \quad \frac{r^2+6}{q+2}$$

$$r+2 \mid q^2+9r$$

$$q+6 \mid q^2+9(q+4)$$

$$q+6 \mid q^2+9q+36 \qquad q \equiv -6 \pmod{q+6}$$

$$q+6 \mid (-6)^2+9(-6)+36$$

$$q+6 \mid 36-54+36 \Rightarrow q+6 \mid 18$$

$$\Rightarrow q+6 = 18, 9$$

$$\Rightarrow \boxed{q=3}$$

and $(2,3,7)$ is a solution.

Find all triples (p, q, r) of prime numbers such that all of the following numbers are integers

$$\frac{p^2+2q}{q+r}, \frac{q^2+9r}{r+p}, \frac{r^2+3p}{p+q}$$

② $q+r$ is odd

$\Rightarrow$ $q=2$ or $r=2$

- if $q=2$ and r is odd

  $r+p \mid 4+9r$ but $4+9r$ is odd

  $\Rightarrow$ $r+p$ is odd $\Rightarrow$ p is even $\Rightarrow$ $p=2$

  $\Rightarrow$ $2+2 \mid r^2+6$ $\Rightarrow$ $2 \mid r$ $\Rightarrow$ $r=2$ $\Rightarrow\Leftarrow$!

- if $r=2$ and q is odd

  $p+q \mid 4+3p$ $\Rightarrow$ $\begin{cases} p=2 \rightarrow \text{2nd fraction} \ldots \\ p+q \text{ is odd} \Rightarrow q=2 \rightarrow \leftarrow !! \end{cases}$

Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

Let $\gcd(x, y) = d$

$x = da$, $y = db$ $\Rightarrow \gcd(a, b) = 1$

$$d(a^4 - b^4) = p(a^3 - b^3)$$

$$\Rightarrow d(a^2 - b^2)(a^2 + b^2) = p(a-b)(a^2 + ab + b^2)$$

$$\Rightarrow d(a-b)(a+b)(a^2+b^2) = p(a-b)(a^2+ab+b^2)$$

$$\Rightarrow d(a+b)(a^2+b^2) = p(a^2+ab+b^2)$$

$\gcd(a+b, a) = 1$, $\gcd(a+b, b) = 1$ $\Rightarrow \gcd(a+b, ab) = 1$ *

if $\gcd(a+b, a^2+ab+b^2) = \gcd(a+b, (a+b)^2 - ab) = \gcd(a+b, ab)$
$= 1$

but $a+b \mid p(a^2+ab+b^2) \Rightarrow a+b \mid p \Rightarrow a+b = 1, p$

Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

Case 1 $\quad a+b=1$

$a, b \geq 0 \Rightarrow \begin{cases} a=1, b=0 \\ a=0, b=1 \end{cases}$ and $d \cdot 1 = p$

$\Rightarrow (x, y) = (0, p), (p, 0)$

Case 2 $\quad a+b=p$

$d(a^2+b^2) = a^2+ab+b^2$

$\gcd(a^2+b^2, a) = \gcd(b^2, \underline{a}) = 1$ and $\gcd(a^2+b^2, \underline{b}) \Rightarrow$

$\Rightarrow \gcd(a^2+b^2, ab) = 1$

$\gcd(a^2+b^2, a^2+ab+b^2) = \gcd(a^2+b^2, ab) = 1$

but $a^2+b^2 \mid a^2+ab+b^2 \Rightarrow a^2+b^2 \mid 1 \Rightarrow a^2+b^2 = 1$

$\Rightarrow \begin{cases} a=0, b=1 \\ a=1, b=0 \end{cases}$ ...

Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

another way to solve $d(a^2+b^2) = a^2+ab+b^2$

$(d-1)(a^2+b^2) = ab$

but $a^2+b^2 \geqslant 2ab \Rightarrow d=1$

$\Rightarrow ab = 0 \Rightarrow a=0$ or $b=0$

$\Rightarrow (0,p), (p,0)$

but $\gcd(a,b)=1 \Rightarrow$ no solutions

Find all integers x, y such that $x^3(y+1)+y^3(x+1)=19$.