Competition Preparation for Saudi Arabia Team 2021: Level 4

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Lesson 7

Double counting

Problems:

1. A group of n students attends several classes such that if two students attend the same two classes, then the number of students in the two classes is different from each other. Prove that the largest possible number of classes is $(n-1)^2$.

2. There are n people at a party. Prove that there are two people at the party such that at least $\lfloor \frac{n}{2} \rfloor - 1$ people know either both of them or neither of them. (If person A knows person B, then B knows A.)

3. At a competition of n students, for each group of 3 students there is a student they are conspiring to get rid of. Any student who conspiring to get rid of student A in some group of three is an *enemy* of student A. Prove that there is a student with at least $\sqrt[3]{(n-1)(n-2)}$ enemies.

4. Show that among n points there are at least $\sqrt{n - \frac{3}{4}} - \frac{1}{2}$ different distances between pairs of points.

5. At a bookstore 10 people bought 3 books each such that each two persons have at least one book in common that they bought. Let M be the largest number of people that purchased the same book. Find the smallest possible value of M.

6. Let $n > 1$ be a positive integer and let a set P of n^2 points be given that form an grid of $(n-1) \times (n-1)$ unit squares. Let S be the set of all squares whose vertices are in P and let a_k be the number of unordered pairs of different points of P such that both of them together are vertices of exactly k squares in S. Prove that $a_0 = a_2 + 2a_3$.

7. At a contest there are m candidates and $n \geq 3$ judges where n is an odd integer. Each candidate is evaluated by a judge as either passing or failing. Suppose that each pair of judges agrees on at most k candidates. Prove that: $\dfrac{k}{m} \geq \dfrac{n-1}{2n}$.

8. At a social gathering of students, every boy knows at least one girl. Prove that there exists a set S of at least half the students at the gathering such that every boy in S knows an odd number of girls in S.