Around divisors

Lesson by Senya, group L3

For the problems below we assume that divisors are positive divisors unless otherwise specified (which will not happen...).

Before we proceed to the problems, let's do/recall (if you already know) some basics to warm-up: Suppose $p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ is the unique prime factorization of some number n. How many positive divisors does n have? What is the sum of all the positive divisors of n?

Problem 1. Let $d_1, d_2, ..., d_s$ be the set of all the positive divisors of $n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ (where the RHS of the last equation is a prime factorization of n). What is the value of $\frac{1}{d_1} + ... + \frac{1}{d_s}$?

Problem 2. There is a house with 100 flats numbered $1, 2, ..., 100$, initially all of the flats have light working. On day k the administrator called Mr. Fun changes the state of lights (i.e if it was working then he makes it not working, and if it was not working then he makes it working) in flats with numbers divisible by k. After 100 days of having "fun" he was fired. Flats with which numbers still have the light working?

Problem 3. Prove that a number less than a million has at most 2000 divisors.

Problem 4. Find all primes p such that $p^2 + 11$ has exactly 6 divisors.

Problem 5. Prove that $n!$ can be expressed as the sum of n of its different divisors (for any positive integer n).

Problem 6. The largest proper divisor of n is equal to d. Is it possible that the largest possible divisor of $n + 2$ is equal to $d + 2$?

A proper divisor of a number is any of its divisors besides 1 and the number itself.

Problem 7. Is it possible to divide all of the divisors of 100! into two groups of equal sizes such that the product of the numbers in the first group is equal to the product of the numbers in the second group?

Problem 8. There is a number 2022 written on the board. Ahmad and Khalid play a game: if at some moment there is a number n written on the board, then it is possible to replace with $n - d$ where d is a divisor of n. They take turns, Ahmad goes first. Who has a winning strategy?

Problem 9. A perfect number is the one that is equal to the sum of all of its divisors not equal to itself.

i) Let $2^p - 1$ be a prime number. Prove that $(2^p - 1) \cdot 2^{p-1}$ is a perfect number;

ii) Prove that if an even number is perfect then it has to be of the form $(2^p - 1) \cdot 2^{p-1}$ where $2^p - 1$ is prime;

iii) Does there exist an odd perfect number?