Preparation for Saudi Arabia Team 2021

May/June Session: Level 3

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Lesson 1

Plane and space colorings

Problems:

1. A plane is colored in n colors. Is it possible to find two equally colored points a unit distance apart if:
   (a) $n = 3$,
   (b) $n = 7$?
2. A plane is colored in k colors. Show that for each n and m there exist m mutually congruent n-gons whose vertices are all of the same color.
3. A plane is colored in k colors. Show that there exists a rectangle whose vertices are all of the same color.
4. Each point of a three-dimensional space is colored with one of 3 colors. Prove that we can select a color c such that for each positive real number r there is a triangle with area r whose vertices are all colored with color c.
5. Each point of a three-dimensional space is colored with one of two colors such that whenever an isosceles triangle ABC with $AB = AC$ has vertices of the same color c it follows that the midpoint of BC also is colored with c. Prove that there exists a perpendicular square prism with all vertices of equal color.
6. Is it possible to color the plane in exactly 3 colors (not less) such that each line in the plane is colored in only 2 colors?
7. A plane is colored in 2 colors. Is it always possible to find an equilateral unit triangle whose vertices are of one color?
8. Each point of a three-dimensional space is colored in one of two colors. Prove that there exists a triangle congruent to a given triangle whose vertices are all of one color.