Email training, N6

September 29-October 5

Problem 6.1. Find 8 positive integers $n_1, n_2, \ldots, n_8$ such that we can express every integer n with $|n| < 2019$ as $a_1n_1 + \ldots + a_8n_8$ with each $a_i = 0, \pm 1$.

Solution 6.1. Put $f(n) = 1 + 3 + 3^2 + \ldots + 3^n$. We can prove by induction that any number in the range $[-f(n), f(n)]$ can be represented as $a_0 + a_1 \cdot 3 + \ldots + a_n 3^n$ with each $a_i =$ 0or ± 1. It is evidently true for $n = 1$. So suppose it is true for n. Note that $f(n) = (3^{n+1} - 1)/2$, so $f(n) + 1 - 3^{n+1} > -(3^{n+1} - 1)/2 = -f(n)$. So if $f(n) < m \leq f(n+1)$, then $-f(n) < m - 3^{n+1} \leq f(n)$. By induction $m - 3^{n+1} = a_0 + \ldots + a_n 3^n$, and hence $m = a_0 + \ldots + a_{n+1}3^{n+1}$. Similarly for $-f(n+1) \leq m < -f(n)$, whilst for $-f(n) \leq m \leq f(n)$, the result follows immediately. That completes the induction.

Other solution. There are $n + 1$ coefficients each of which can take 3 values, so there are 3^{n+1} possibilities. No two possibilities can give the same number, or, equating them and moving negative coefficients to the opposite side, we would have two different representations for the same number in base 3. But obviously all the values lie in the range $[-f(n), f(n)]$, and that range contains just 3^{n+1} values. So each value is achievable and in only one way.

Problem 6.2. Find the minimum value of the expression

$$|\, x - 1 \,| + |\, 2x - 1 \,| + |\, 3x - 1 \,| + \ldots + |\, 119x - 1 \,| \,.$$

Solution 6.2. Suppose that $\frac{1}{n} \leq x \leq \frac{1}{n-1}$. Then the expression inside the absolute value for the first $n - 1$ terms is at most 0, and the expression inside for remaining terms is at least 0. This means that

$$f(x) = \sum_{i=1}^{n-1}(1 - ix) + \sum_{i=n}^{119}(ix - 1).$$

Calculation gives

$$f(x) = (2n - 121) + x\Big(60 \cdot 119 - n(n-1)\Big).$$

Therefore, on each closed interval $\left[\frac{1}{n}, \frac{1}{n-1}\right]$, f is a linear function. As x increases, n decreases, so the slope of the linear function increases, crossing from nonpositive to nonnegative at some value of n. The minimum value of f is at the point(s) where this slope changes sign, since f is decreasing before the sign change and increasing after it. It turns out that at $n = 85$, we have

$$60 \cdot 119 = 85 \cdot 84.$$

This means the slope of the line is 0 for $x \in \left[\frac{1}{85}, \frac{1}{84}\right]$, and f obtains the minimum value anywhere on this interval. Letting $n = 85$ we get

$$f(x) = (2 \cdot 85 - 121) + x \cdot 0 = 49$$

which is the required minimum.

Problem 6.3. Find all primes p such that $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

Solution 6.3. For $p \neq 3$, one has $3|p^2-1$, and so $3|(p^2+11)$. Similarly, for $p \neq 2$, one has $4|p^2-1$, and so $4|(p^2+11)$. Except in these two cases, then, $12|(p^2+11)$; since 12 itself has six divisors $(1, 2, 3, 4, 6, 12)$ and $p^2+11 > 12$ for $p > 1$, we conclude that p^2+11 must have more than six divisors. The only cases to check are $p = 2$ and $p = 3$.

If $p = 2$, then $p^2 + 11 = 15$, which has only four divisors $(1, 3, 5, 15)$, while if $p = 3$, then $p^2 + 11 = 20$, which indeed has six divisors $(1, 2, 4, 5, 10, 20)$. Hence $p = 3$ is the only solution.

Problem 6.4. Find the number of odd coefficients of the polynomial $(x^2 + x + 1)^{33}$.

Solution 6.4. Note, that we may work on modulo 2 when dealing with the coefficients of the polynomials. Then

$$(x^2 + x + 1)^2 \equiv (x^2 - x + 1)(x^2 + x + 1)[2] \equiv (x^4 + x^2 + 1)[2].$$

So, repeating this 4 more times we get

$$(x^2 + x + 1)^{32} \equiv (x^{64} + x^{32} + 1)[2].$$

Finally

$$(x^2 + x + 1)^{33} \equiv (x^{64} + x^{32} + 1)(x^2 + x + 1)[2] \equiv$$
$$(x^{66} + x^{65} + x^{64} + x^{34} + x^{33} + x^{32} + x^2 + x + 1)[2],$$

which means that $(x^2 + x + 1)^{33}$ contains 9 odd coefficients.

Problem 6.5. The integers a and b have the property that for every nonnegative integer n, the number $2^n a + b$ is a perfect square. Show that $a = 0$.

Solution 6.5. Let $x_n^2 = 2^n a + b$ for any positive integer n. Obviously $a \geq 0$ and if $a > 0$ then x_n is strictly increases. Note, that for any positive integer n we have

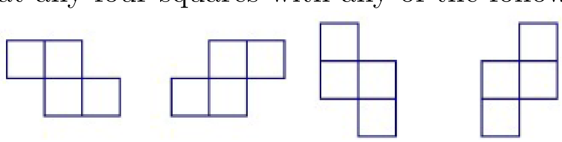
$$x_{n+2}^2 = 2^{n+2} a + b,$$
$$4x_n^2 = 4 \cdot 2^n a + b.$$

Subtracting these equations we get

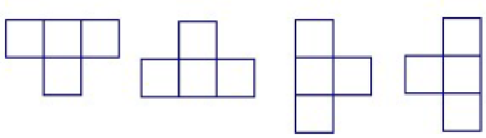
$$(2x_n - x_{n+2})(2x_n + x_{n+2}) = 3b.$$

In the left side the first multiplier is positive integer and the second one is increasing and will be bigger than $3b$ for big enough n. So we get that the left side is bigger than $3n$, which is contradiction. So $a = 0$.

Problem 6.6. Can we number the squares of an 8×8 board with the numbers $1, 2, \ldots, 64$ so that any four squares with any of the following shapes



have sum divisible by 4? Can we do it for the following shapes?



Solution 6.6. Part a). it's possible. Since only divisibility by 4 is improtant, we may assume that we have 16 values of each 0, 1, 2 and 3. Put the numbers in the board like this

$$\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 \\
2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 \\
0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 \\
2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\
2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3
\end{array}$$

Note, that this construction satisfies to the conditions of the problem. **Part b).** Assume its possible. Consider a part of the board of dimension 3×3.

$$\begin{array}{ccc}
a & b & c \\
m & n & k \\
x & y & z
\end{array}$$

Since $(m+n+k+b)-(b+n+y+k) = m-y$ is divisible by 4 then $m \equiv y[4]$. Analogously $y \equiv k \equiv b \equiv m[4]$. So all of them have the same residue base 4. Now, paint the table as the chessboard. Then we may state that all 30 non-corner black cells have the same residue base 4. However, for each residue we have only 16 numbers. Contradiction.

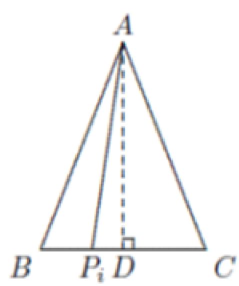
Problem 6.7. Let ABC is an isosceles triangle with $AB = AC = 2$. There are 100 points $P_1, P_2, \ldots, P_{100}$ on the side BC. Denote $m_i = AP_i^2 + BP_i \cdot CP_i$. Find the value of $m_1 + m_2 + \ldots + m_{100}$.

Solution 6.7. -

From A introduce $AD \perp BC$ at D. Then $BD = DC$. Let $BD = DC = x$ and $DP_i = x_i$.

By Pythagoras' Theorem, for $1 \leq i \leq 100$,

$$\begin{aligned}
m_i &= AP_i^2 + BP_i \cdot P_iC \\
&= AP_i^2 + (x - x_i)(x + x_i) \\
&= AP_i^2 - x_i^2 + x^2 \\
&= AD^2 + x^2 = AB^2 = 4.
\end{aligned}$$



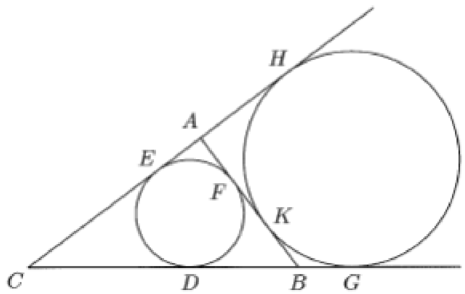
Thus,

$$m_1 + m_2 + \cdots + m_{100} = 400.$$

Problem 6.8. Let a and b be two sides of a triangle. How should the third side c be chosen so that the points of contact of the incircle and excircle with side c divide that side into three equal segments?(The excircle corresponding to the side c is the circle which is tangent to the side c and the extensions of the sides a and b .)

Solution 6.8. -

Let the triangle be ABC with $BC = a, CA = b$ and $AB = c$.



We cannot have $a = b$ as otherwise both circles will be tangent to AB at its midpoint. We may assume that $a > b$. Let the incircle be tangent to BC at D, CA at E and AB at F. Then $AF = \frac{c}{3}$ and $BF = \frac{2c}{3}$. It follows that

$$a - b = (BD + CD) - (AE + CE) = BD - AE = BF - AF = \frac{c}{3}.$$

Hence we should choose $c = 3(b - a)$. We still have to check that for this choice, AB is also trisected by its point of tangency with the excircle opposite C. Let this circle be tangent to the extension of CB at G, the extension of CA at H and AB at K. Then $AK + BK = c$ while

$$\begin{aligned} AK - BK &= AH - BG \\ &= (CH - CA) - (CG - CB) \\ &= CB - CA \\ &= a - b \\ &= \frac{c}{3}. \end{aligned}$$

Thus we indeed have $AK = \frac{2c}{3}$ and $BK = \frac{c}{3}$.