# INTRODUCTION TO FUNCTIONAL EQUATION

**Problem 1.**

a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x-f(x))=x-f(x)$ for all $x$. Prove that $f(0)=0$ and $x=0$ is the unique solution of equation $f(x)=0$.

b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x-1)f(y^2)=y(f(xy)-f(y))$ for all $x,y$ and $f(2022)\neq 0$, calculate $f(2022)$.

**Problem 2.**

a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(xy)=xf(y)+yf(x)$ and $f(x^2)=f(f(x))$ for all $x,y\in\mathbb{R}$. Prove that $f(x)=0$ for all $x$.

b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x+y)=f(x)+f(y)$ and $2f(x)^2=f(x^2)+xf(x)$ for all $x$. It is given that $f(2022)\neq 0$, prove that $f(x)=x$ for all $x$.

**Problem 3.**

a) Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ such that $f(1)=\dfrac{1}{2}$ and

$$f(xy)=f(x)f(4/y)+f(y)f(4/x) \text{ for all } x,y>0.$$

b) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x+y)-f(x-y)=2y(3x^2+y^2) \text{ for all } x,y\in\mathbb{R}.$$

**Problem 4.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

a) $f(y^2)=f(x+y)f(y-x)+x^2$ for all $x,y\in\mathbb{R}$.

b) $f(x^2+xy+f(y))=f(x)^2+xf(y)+y$ for all $x,y\in\mathbb{R}$.

**Problem 5.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

a) $f(x^2+f(y))=xf(x)+y$ for all $x,y\in\mathbb{R}$.

b) $f(x^2+xy)=f(x)f(y)+yf(x)+xf(x+y)$ for all $x,y\in\mathbb{R}$.

**Problem 6.** Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

a) $f(f(x+y))=f(x+y)+f(x)f(y)-xy$ for all $x,y\in\mathbb{R}$.

a') $f(f(x-y))=f(x)-f(y)+f(x)f(y)-xy$ for all $x,y\in\mathbb{R}$.

b) $f((x+y)^2)=f(x)f(x+2y)+yf(x)$ for all $x,y\in\mathbb{R}$.

b') $f((x+y)^2)=f(x)f(x+2y)+yf(y)$ for all $x,y\in\mathbb{R}$.

**Problem 7*.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(0)=0$ and

$$2f(-\frac{1}{2}xy+f(x+y))=xf(y)+yf(x) \text{ for all } x,y\in\mathbb{R}.$$

Prove that $f(-f(2))=f(2)$.