Email training, N3

September 8-14

Problem 3.1. Let (a_n) be sequnce of positive integers such that first k members $a_1, a_2, ..., a_k$ are distinct positive integers, and for each $n > k$, number a_n is the smallest positive integer that can't be represented as a sum of several (possibly one) of the numbers $a_1, a_2, ..., a_{n-1}$. Prove that $a_n = 2a_{n-1}$ for all sufficently large n.

Solution 3.1. Consider the numbers $a_1 < a_2 < \cdots a_k$, we let a few more terms to fill up the numbers from 1 to a_k, untill eventually a term a_{m+1} is greater than a_k. At this point every number from 1 to $a_k > a_m$ is sum of several numbers in a_1 to a_k. So WLOG that the initial set of numbers $a_1, \cdots a_k$ are such that every number from 1 to a_k is sum of several from $a_1, \cdots, a_k$.

Consider the expression $x_n = a_1 + \cdots + a_n$ for every $n \geq k$, clearly x_n is the largest number than is the sum of several of $a_1, \cdots, a_n$. So $x_n + 1$ is not representable by sum of a several of $a_1, \cdots a_n$, so $a_{n+1} \leq x_n + 1$. On the other hand, since every number from 1 to $a_{n+1} - 1$ is sum of several of $a_1, \cdots, a_n$, then every number from 1 to $2a_{n+1} - 1$ is sum of several of $a_1, \cdots, a_{n+1}$. So $a_{n+2} \geq 2a_{n+1}$, and so

$$x_n - a_{n+1} = x_{n+1} - 2a_{n+1} \geq x_{n+1} - a_{n+2}$$

That is the difference $x_{n+1} - a_{n+2}$ is non increasing. So since its a integer sequence at least -1, it is eventually constant.

Take such M such that $x_{n+1} - a_{n+2} = x_n - a_{n+1}$ for all $n > M$, then $a_{n+2} = x_{n+1} - x_n + a_{n+1} = 2a_{n+1}$ for all $n > M$, as desired.

Problem 3.2. Let p be a prime and let $f(x)$ be a polynomial of degree d with integer coefficients. Assume that the numbers $f(1), f(2), \ldots, f(p)$ leave exactly k distinct remainders when divided by p, and $1 < k < p$. Prove that

$$\frac{p-1}{d} \leq k - 1 \leq (p-1)\left(1 - \frac{1}{d}\right).$$

Solution 3.2.

Problem 3.3. Three prime numbers p, q, r and a positive integer n are given such that the numbers

$$\frac{p+n}{qr}, \frac{q+n}{rp}, \frac{r+n}{pq}$$

are integers. Prove that $p = q = r$.

Solution 3.3. Assume without lose of generality that $p \geq q \geq r$. Since p divides $(q+n) - (r+n) = q - r$, it follows $q = r$ because $0 \leq q - r < p$.

Then q divides $r + n$ implies q divides n. But we have q divides $p + n$, too, so we must in fact have $q \mid p$. Since p is prime, we must have $p = q = r$.

Problem 3.4. $a_1, a_2, ..., a_{100}$ are permutation of $1, 2, ..., 100$. $S_1 = a_1, S_2 = a_1 + a_2, ..., S_{100} = a_1 + a_2 + ... + a_{100}$. Find the maximum number of perfect squares from S_i

Solution 3.4. Let n be the number of squares among those numbers. As there is 71 possible squares less than $5050 = 1 + 2 + \cdots + 100$, we get trivial bound $n \leq 71$. We put $S_0 = 0$ and consider the subsequence that is perfect square. $S_{m_1} = k_1^2$, $S_{m_2} = k_2^2$, ..., $S_{m_n} = k_n^2$ with $m_1 < m_2 < \ldots < m_n$.

Whenever $S_{m_{i+1}}$ and S_{m_i} have different parities then

$$S_{m_{i+1}} - S_{m_i} = a_{m_i+1} + a_{m_i+2} + \ldots + a_{m_{i+1}-1},$$

contains odd summander. Since there are only 50, which means in the sequence S_{m_i} there are at most 50 consecutive squares. So, in total can be at most $50 + \frac{71-50}{2} < 61$ perfect squares, so $n \leq 60$.

It remains to give example for 60. Consider the sequence $a_i = 2i - 1$ for $1 \leq i \leq 50$. Then we get all $S_i = i^2$ for $1 \leq i \leq 50$. It remains to construct 10 more squares. Consider $a_{51+4i} = 2 + 8i$, $a_{52+4i} = 100 - 4i$, $a_{53+4i} = 4 + 8i$ and $a_{54+4i} = 98 - 4i$ for $0 \leq i \leq 7$. Then we get $S_{54+4i} = (52 + 2i)^2$. To get the last 2 square we arrange the remaining number in the following order

$$30, 40, 64, 66, 68, 6, 8, 14, 16, 32, 38, 46, 54, 62, 22, 24, 48, 56.$$

Then we get $S_{87} = 66^2 + 2 \cdot 134 = 68^2$ and $S_{96} = 70^2$. **Answer:** 60.

Problem 3.5. There are 100 students taking an exam. The professor calls them one by one and asks each student a single person question: "How many of 100 students will have a "passed" mark by the end of this exam?" The students answer must be an integer. Upon receiving the answer, the professor immediately publicly announces the student's mark which is either "passed" or "failed."

After all the students have got their marks, an inspector comes and checks if there is any student who gave the correct answer but got a "failed" mark. If at least one such student exists, then the professor is suspended and all the marks are replaced with "passed." Otherwise no changes are made.

Can the students come up with a strategy that guarantees a "passed" mark to each of them?

Solution 3.5. There is a strategy using induction. Indeed for $n = 2$ first says 1 and if he passes the second says 1 as well. Otherwise the second says 0. Now, let A_n be the strategy for a particular n. We will find one for $n + 1$ as well. Indeed, the first says n. If he fails we can apply on the remaining n people the strategy A_n found above since the goal for the teacher is the same (not to pass all of the remaining ones). If he passes we apply A_n with each answer increased by 1 (taking into account the first student). Since, the goal for the teacher is still the same, this will work too.

Problem 3.6. In a social network with a fixed finite setback of users, each user had a fixed set of followers among the other users. Each user has an initial positive integer rating (not necessarily the same for all users). Every midnight, the rating of every user increases by the sum of the ratings that his followers had just before midnight.

Let m be a positive integer. A hacker, who is not a user of the social network, wants all the users to have ratings divisible by m. Every day, he can either choose a user and increase his rating by 1, or do nothing. Prove that the hacker can achieve his goal after some number of days.

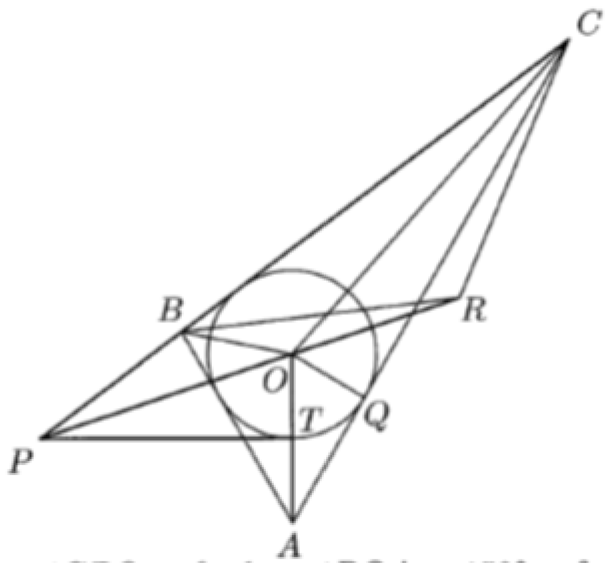
Solution 3.6.

Problem 3.7. A circle with center O is inscribed in an angle. Let A be the reflection of O across one side of the angle. Tangents to the circle from A intersect the other side of the angle at points B and C. Prove that the circumcenter of triangle ABC lies on the bisector of the original angle.

Solution 3.7. -

Let T be the midpoint of AO a
angle. Let P be closer to B tha

Let T be the midpoint of AO and P be the vertex of the given angle. Let P be closer to B than to C. Let Q be the point of contact of the circle with AC. In triangle OQA, $\angle OQA = 90°$ and $OA = 2OQ$. Hence $\angle OAQ = 30°$. Therefore $\angle OAB = 30°$.



Let $\angle ABO = \angle CBO = \theta$, then $\angle BOA = 150° - \theta$. Since PT is perpendicular to AO, we have

$$\angle BPT + 90° + 150° - \theta + 180° - \theta = 360°.$$

Hence $\angle BPT = 2\theta - 60°$, which means that $\angle BPO = \theta - 30°$. Therefore $\angle BOP = 30°$. Since O is the incentre of triangle ABC,

$$\angle BOC = 90° + \frac{1}{2}\angle BAC = 120°.$$

Now let R be the circumcentre of triangle ABC, then

$$\angle BRC = 2\angle BAC = 120°.$$

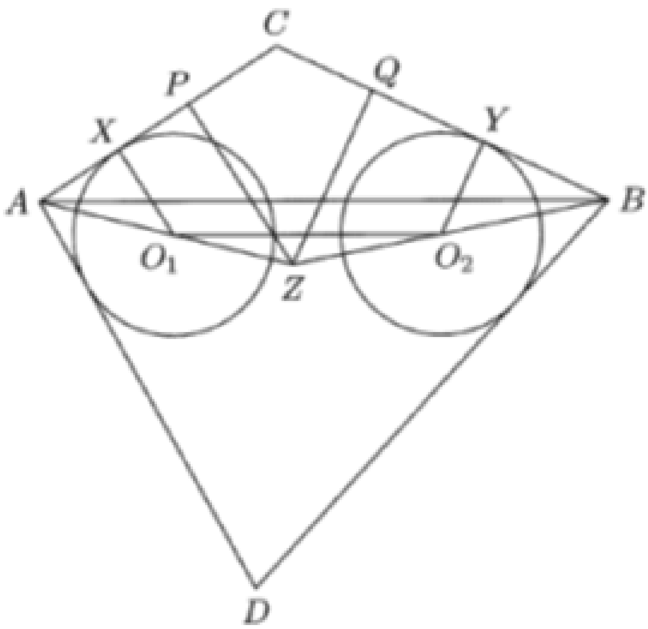
Hence $BORC$ is a cyclic quadrilateral. In triangle BRC, $\angle BRC = 120°$ and $BR = RC$. Hence $\angle BCR = 30°$. Therefore $\angle BOR = 150°$. Thus

$$\angle BOP + \angle BOR = 180°$$

Problem 3.8. The segment AB intersects two equal circles at 4 points and is parallel to the line joining their centres. From the point A tangents to the circle nearest to A are drawn, and from the point B tangents to the circle nearest to B are drawn. The tangent lines form a quadrilateral which contains both circles. Prove that a circle can be drawn that touches all four sides of the quadrilateral.

Solution 3.8. -

Let C and D be the other two vertices of the quadrilateral. Let X and Y be the points of contact of the two circles with AC and BC respectively. We see that O_1X is perpendicular to AC and O_2Y is perpendicular to BC where O_1 and O_2 are the centres of the corresponding circles. Let AO_1 and BO_2 extended meet in Z. Also let P and Q be the feet of the perpendiculars from Z to AC and BC respectively.



Since APZ and AXO_1 are similar triangles, we have $\dfrac{PZ}{XO_1} = \dfrac{AZ}{AO_1}$.

Since ABZ and O_1O_2Z are similar triangles, $\dfrac{AZ}{O_1Z} = \dfrac{AB}{O_1O_2}$ holds.

Hence

$$\frac{PZ}{XO_1} = \frac{AZ}{AO_1} = \left(1 - \frac{ZO_1}{AZ}\right)^{-1} = \left(1 - \frac{O_1O_2}{AB}\right)^{-1}.$$

Similary,

$$\frac{QZ}{YO_2} = \frac{BZ}{BO_2} = \left(1 - \frac{ZO_2}{BZ}\right)^{-1} = \left(1 - \frac{O_1O_2}{AB}\right)^{-1}.$$

Since $XO_1 = YO_2$, we have $PZ = QZ$ which means that CZ bisects $\angle ACB$. Thus the angle bisectors of $\angle A$, $\angle B$ and $\angle C$ in the quadrilateral $ABCD$ meet at one point. Hence there exists a circle which touches all four sides of the quadrilateral.