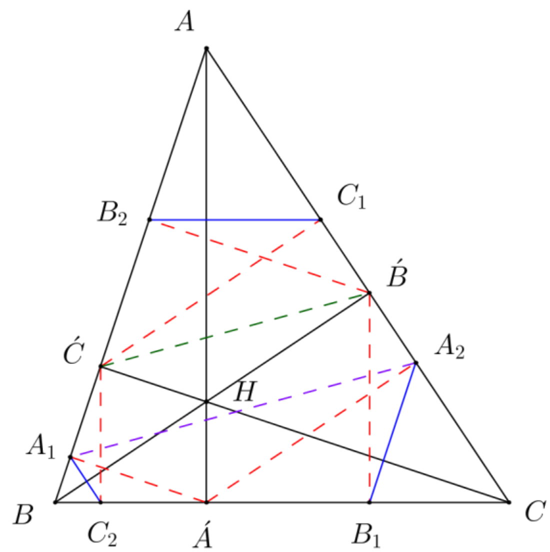
Problem 1.7. Let AA', BB' and CC' are the altitudes of the triangle ABC. Let A_1 and A_2 are the projections of A' on AB and AC, respectively, B_1 and B_2 are the projections of B' on BC and BA, as well as C_1 and C_2 are the projections of C' on CA and CB. Prove that:

- $B_2C_1 \parallel BC$,
- The hexagon $A_1B_2C_1A_2B_1C_2$ is cyclic.

Sol: We have $C'C_1 \parallel A'A_2 \perp AC$, similarly $B'B_2 \parallel A'A_1, C'C_2 \parallel B'B_1$.



(1) Let us prove $B_2C_1 \parallel BC$. Since $\angle BB'C = \angle BC'C = 90°$, hence $BC'B'C$ is cyclic, and moreover $\angle AB'C' = \angle ABC, \angle AC'B' = \angle ACB$.

Hence $\triangle AB'C' \sim \triangle ABC$. In the $\triangle AB'C'$, the line segments $B'B_2, C'C_1$ are altitudes. Therefore, repeating the above argument we can assert that $\angle AB_2C_1 = \angle AB'C'$. Consequently $\angle AB_2C_1 = \angle ABC$ and $B_2C_1 \parallel BC$. We then prove that $A_1C_2 \parallel AC, A_2B_1 \parallel AB$.

(2) To prove that the vertices of the hexagon $A_1B_2C_1A_2B_1C_2$ lie in a circle, look at the points A_1, B_2, C_1 which are not in a straight line, hence there is a circle (say ω) passes through all of them.

To prove that ω passes through A_2, take the obvious proportion (where H is the orthocenter of $\triangle ABC$)

$$\frac{AC'}{AA_1} = \frac{AH}{AA'} = \frac{AB'}{AA_2}$$

It follows that $A_1A_2 \parallel C'B'$ (similarly $B_1B_2 \parallel A'C', C_1C_2 \parallel A'B'$). Therefore,

$\angle C_1A_2A_1 = \angle AB'C' = \angle ABC = \angle AB_2C_1$

Which means that the points A_1, B_2, C_1, A_2 lie in one circle, so ω passes through A_2

Now we will prove that B_1 lies on ω . $\angle AC_1B_2 = \angle ACB$ (since $B_2C_1 \parallel BC$) ,

$\angle A_2B_1B_2 = \angle BC'A' = \angle ACB$ (since $B_1A_2 \parallel BA, B_1B_2 \parallel A'C'$).

Hence $\angle AC_1B_2 = \angle A_2B_1B_2$, and consequently $B_2C_1A_2B_1$ is cyclic, which means that B_1 lies on ω. In the same way we can prove that C_2 lies on ω, and we are done.