— ALGEBRA FOR L4 —

— FEBRUARY CAMP, 2022 — PROBLEMS —

Each problem is assigned (sometimes a bit artificially, as it comprises various areas and approaches) one of the following categories:

M — manipulations, identities, (systems of) equations;

I — inequalities, optimization;

F — functions (including trigonometric), functional equations;

Q — (ir)rational numbers;

P — polynomials (including Vieta, interpolation);

S — sequences and progressions.

Moreover there are special categories **W** for warm-up problems (usually short or known, used to present or revise some technique), and **K** for particularly challenging problems (so that everyone had something to do if they had finished the 'regular' problem long before its discussion).

The problems had been selected at *random*, where the distribution was not uniform — it was created accordingly with the group's preferences. Below the statements (and solution sketches for exemplary methods) can be found. Enjoy!

Note: This is a working document. If you spot a mistake or have a suggestion, contact me directly (preferably via WhatsApp).

Łukasz

K1. Let $a_1 < a_2 < a_3 < \ldots$ be a sequence consisting of all positive integers of the form x^y where $x, y \geq 2$ are positive integers. Prove that

$$\sum_{i=1}^{\infty} \frac{1}{a_i - 1} = 1.$$

SOLUTION. By the infinite geometric progression sum formula, we have

$$S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{a_i^j}.$$

Now for each $k > 1$ there are equally many ways to write k as a_i^j with $i, j \geq 1$ and as x^y with $x, y \geq 2$. Indeed: a_i^j is of the form x^y unless $j = 1$, and x^y is of the form a_i^j unless x is the smallest possible. So

$$S = \sum_{x=2}^{\infty} \sum_{y=2}^{\infty} \frac{1}{x^y} = \sum_{x=2}^{\infty} \frac{1}{x^2(1-\frac{1}{x})} = \sum_{x=2}^{\infty} \frac{1}{x-1} - \frac{1}{x} = 1.$$

REMARK. This is the GOLDBACH-EULER THEOREM.

W1.

(a) Suppose that a, b, c are real numbers such that
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}.$$
Prove that for every odd positive integer n
$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$$

(b) Suppose that real numbers a, b, c satisfy
$$\sqrt[3]{a-b}+\sqrt[3]{b-c}+\sqrt[3]{c-a}=0.$$
Prove that at least two of the numbers a, b, c are equal.

(c) Real numbers a, b satisfy $a^3+b^3=3ab-1$. Find all possible values of $a+b$.

(d) Factor $(x+y+z)^3-x^3-y^3-z^3$.

(e) Factor $(x+y+z)^5-x^5-y^5-z^5$.

Solution. (a) The assumption can be rewritten equivalently as
$$(a+b)(b+c)(c+a)=0,$$
meaning some two of a, b, c are of sum 0, so the same can be said about the triple a^n, b^n, c^n (as n is odd).

(b) By $x^3+y^3+z^3=3xyz$ if $x+y+z=0$ (here $x=\sqrt[3]{a-b}$ and so on), we have $xyz=0$.

(c) By $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ with $c=1$ we have that either $a+b=-1$, or $a=b=1$ and so $a+b=2$.

(d) We know each of $x+y$, $y+z$, $z+x$ has to be a factor, and the degree of our expression is 3, so it for sure equals
$$c(x+y)(y+z)(z+x),$$
where the coefficient c is readily seen to be $c=3$.

(e) Similarly we get that it's $c(x+y)(y+z)(z+x)Q(x,y,z)$, where Q is of degree 2, so each monomial does not contain at least one of the three variables. Therefore to see what they are it remains to plug $x=0$, $y=0$, $z=0$ in the form before factorization and get
$$5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx).$$

Remark. We know that $a^3+b^3+c^3=3abc$ is equivalent to $a+b+c=0$ or $a=b=c$. It turns out that
$$x^5+y^5+z^5+5xyz(xy+yz+zx)=0$$
is equivalent to $x+y+z=0$, as seen from the following factorization of the left-hand side:
$$\tfrac{1}{4}(x+y+z)(2x^2(x-y-z)^2+2y^2(y-z-x)^2+2z^2(z-x-y)^2+2(xy+yz+zx)^2$$
$$+(x^2-y^2)^2+(y^2-z^2)^2+(z^2-x^2)^2).$$

P1. For every $n \geq 1$ determine the minimum value attained by the polynomial
$$P_n(x) = x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \ldots + (2n-1)x^2 + 2nx,$$
where $x \in \mathbb{R}$.

Solution. Answer $P_n(-1) = -n$. Because

$$P_n(x) = -n + (x+1)^2(x^{2n-2} + 2x^{2n-4} + \ldots + (n-1)x^2 + n).$$

P2. Given is an integer $n \geq 2$ and a polynomial P of degree n with integer coefficients satisfying
$$P(P(k)) = P(k) + 1$$
for $k = 1, 2, \ldots, n-1$. Find all possible values of $P(n+1)$.

Solution. For polynomials over $\mathbb{Z}[x]$ we have $x - y \mid P(x) - P(y)$, where $x, y \in \mathbb{Z}$. Note that $P(k) \neq k$ and use this fact to show that $P(k) - k = \pm 1$. In particular $P(1) = 0$ or $P(1) = 2$. Take $Q(x) = P(x) - x - 1$; Q is of degree n and has integer coefficients.
If $P(1) = 0$, then $P(0) = 1$, $P(2) = 3$ (1 leads to contradiction), and $Q(k) = 0$ for $k = 0, 2, 3, \ldots, n$, implying

$$P(x) = ax(x-2)(x-3)\ldots(x-n) + x + 1$$

for some non-zero integer a. Plugging $x = 1$ gives $n = 2$ or $n = 3$ and $P(x) = ax^2 - (a+1)x + 1$ or $P(x) = -x^3 + 5x^2 - 5x + 1$, respectively. So $P(n+1) \in \{6a - 2, -3\}$, $a \neq 0$.
If $P(1) = 2$, then by similar reasoning we have $Q(k) = 0$ for $k = 1, 2, \ldots, n$, so for every real x

$$P(x) = ax(x-1)(x-2)\ldots(x-n) + x + 1$$

for some non-zero integer a. This polynomial satisfies given conditions and $P(n+1) = an! + n + 2$ (here 10 for $(a, n) = (3, 2)$ is included, but -3 is not). Finally the answer is $an! + n + 2$ where a is a non-zero integer, and moreover for $n = 3$ one extra value -3.

F1. Find all functions $f \colon \mathbb{Q}_+ \to \mathbb{Q}_+$ such that for every $x \in \mathbb{Q}_+$
$$f(x+1) = f(x) + 1 \quad \text{and} \quad f(x^3) = (f(x))^3.$$

Solution. By easy induction we get that if $x \in \mathbb{Q}_+$ and $m \in \mathbb{Z}_+$, then $f(x+m) = f(x) + m$. Now take $x = p/q$ and $m = q^2$. We have

$$(f(x+m))^3 = f(x^3 + 3p^2 + 3pq^3 + q^6) = f(x^3) + 3p^2 + 3pq^3 + q^6$$

because $3p^2 + 3pq^3 + q^6 \in \mathbb{Z}$. On the other hand,

$$(f(x) + m)^3 = (f(x))^3 + 3q^2(f(x))^2 + 3q^4 f(x) + q^6.$$

The left-hand sides of the two above are equal, so so are the right-hand sides, i.e.

$$(f(x))^2 + q^2 f(x) = x^2 + q^2 x \iff (f(x) - x)(f(x) + x + n^2) = 0.$$

The second factor is positive, so $f(x) = x$ for every x.

Q1. Can every positive rational be expressed as
$$\frac{a^2+b^3}{c^5+d^7},$$
where a, b, c, d are positive integers?

SOLUTION. E.g. $(a,b,c,d)=(p^3q^2,p^5q^2,pq,p^2q)$ gives p/q.

K2. Nonzero real numbers x, y, z satisfy
$$x+\frac{y}{z}=y+\frac{z}{x}=z+\frac{x}{y}=2.$$
Find all possible values of $x+y+z$.

SOLUTION. Denote $s=x+y+z$. We easily get
$$s=xy+yz+zx, \quad xyz=7-2s, \quad 3xyz=-s^2+4s,$$
so $s^2-10s+21=0$ hence $s\in\{3,7\}$. Now $x=y=z=1$ easily gives 3 and we need to proof that the three roots of $P(t)=t^3-7t^2+7t+7$ satisfy the given system of equations (in some order). First of all we observe all roots are real and then play (a lot) with Vieta and symmetric polynomials.

F2. Let $f(x)=x^2-2$. Prove that for every positive integer n the equation
$$\underbrace{f(f(\dots f}_{n}(x)\dots))=x$$
has 2^n different real solutions.

SOLUTION. Note that if $|x|>2$, then
$$|f(x)|=x^2-2>2|x|-2>|x|,$$
so x cannot be a solution. So we can write $x=2\cos\alpha$, where $\alpha\in[0,\pi]$. As $\cos 2\varphi=2\cos^2\varphi-1$, the equation is of the form
$$0=\cos\alpha-\cos 2^n\alpha=2\sin\frac{2^n+1}{2}\alpha\sin\frac{2^n-1}{2}\alpha,$$
so $\alpha=2k\pi/(2^n+1)$ or $\alpha=2k\pi/(2^n-1)$, where k is an integer. It remains to note that if $0\le k\le 2^{n-1}-1$, then $0\le 2k\pi/(2^n-1)\le\pi$ and if $0\le k\le 2^{n-1}$, then $0\le 2k\pi/(2^n+1)\le\pi$ and the solutions are different as long as $k\neq 0$. In total we get $2^{n-1}+2^{n-1}+1-1=2^n$ distinct solutions.

F3. Find all functions $f\colon[0,\infty)\to[0,\infty)$ satisfying for every $x\in[0,\infty)$
$$f(x)=\sqrt{1+xf(x+1)}.$$

SOLUTION. We easily obtain the bound
$$f(x)\le\prod(k+x)^{2^{-k}}<4x$$
for $x\ge 1$, so $f(x)<1+2x$ for all $x\ge 0$. Then from $f(x)\le 1+ax$ we get $f(x)\le 1+\frac{a+1}{2}x$, for $a=2,\frac{3}{2},\frac{7}{4},\dots\to 1$ (so for a arbitrarily close to 1). Consequently $f(x)\le 1+x$. Similarly, starting with $f(x+1)\ge f(x)$ we get $f(x)\ge 1+\frac{x}{2}$ and then if $f(x)\ge 1+ax$ then $f(x)\ge 1+\sqrt{a}x$,

so for $a = \frac{1}{2}, \frac{1}{\sqrt{2}}, \ldots \to 1$ we have $f(x) \geq 1 + ax$ (with a arbitrarily close to 1). Therefore $f(x) \geq 1 + x$ and finally $f(x) = 1 + x$.

REMARK. This functional equation is closely related with RAMANUJAN'S results on nested radicals; in particular it could be used to verify that the following is a reasonable "equality" (to be formal, one should think of a limit of a sequence):

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}}.$$

S1. Sequence (a_n) is given by $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \sqrt{\frac{1 - \sqrt{1 - a_n}}{2}} \quad \text{for } n \geq 1.$$

Prove that $a_1 + a_2 + a_3 + \ldots < 1.03$.

SOLUTION. Note that $a_1 = \sin \frac{\pi}{6}$ and all $a_n < 1$. If $a_n = \sin \alpha$, then

$$a_{n+1} = \sqrt{\frac{1 - \cos \alpha}{2}} = \sin \frac{\alpha}{2},$$

so $a_n = \sin \frac{\pi}{3 \cdot 2^n}$. Now using $\sin x < x$ we get

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \sum_{n=2}^{\infty} \sin \frac{\pi}{3 \cdot 2^n} < \frac{3 + \pi}{6} < \frac{6.18}{6} = 1.03.$$

REMARK. This sum is approximately 1.0202.

K3. Find all two-variable polynomials $P(x,y)$ such that for every $a, b, c \in \mathbb{R}$

$$P(ab, c^2 + 1) + P(bc, a^2 + 1) + P(ca, b^2 + 1) = 0.$$

SOLUTION. Plugging $(0,0,0)$ gives $P(0,1) = 0$. Plugging $(0,0,\sqrt{y-1})$ for $y \geq 1$ gives $P(0,y) = 0$, so $P(0,y) = 0$ for each $y \in \mathbb{R}$ and therefore $x \mid P(x,y)$. Plugging $(a,b,0)$ gives $P(x,1) = 0$ for all $x \in \mathbb{R}$, so $y - 1 \mid P(x,y)$. If $P(x,y) = x(y-1)Q(x,y)$, then the given equality can be rewritten as

$$cQ(ab, c^2 + 1) + aQ(bc, a^2 + 1) + bQ(ca, b^2 + 1) = 0,$$

for all $abc \neq 0$ hence all a, b, c. For $(0,0,c)$ we get (similarly as above) $Q(0,y) = 0$ for $y > 1$ and hence for all y and $x \mid Q(x,y)$. Now $P(x,y) = x^2(y-1)R(x,y)$, where R satisfies the same equation as P. Repeating this argument we prove P is divisible by $x^{2n}(y-1)^n$ for arbitrarily large n, so $P(x,y) \equiv 0$.

W3.

(a) Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$:

$$f(-x) = -f(x), \quad f(x+1) = f(x) + 1, \quad f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \text{ (if } x \neq 0).$$

(b) Find all pairs of functions $f, g \colon \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$:

$$f(x)f(y) = g(x)g(y) + g(x) + g(y).$$

SOLUTION. (a) The idea behind the proof is that our assumptions can be used to express $f(x)$ in terms of $f(-x)$ or $f(1+x)$ or $f(\frac{1}{x})$ only. In particular, we can make a cycle

$$x \rightsquigarrow x+1 \rightsquigarrow \frac{1}{x+1} \rightsquigarrow -\frac{1}{x+1} \rightsquigarrow \frac{x}{x+1} \rightsquigarrow 1+\frac{1}{x} \rightsquigarrow \frac{1}{x} \rightsquigarrow x.$$

Performing the actual manipulations gives

$$\begin{aligned}
f(x) &= f(x+1)-1 = (x+1)^2 f\left(\frac{1}{x+1}\right)-1 = -(x+1)^2 f\left(-\frac{1}{x+1}\right)-1 \\
&= -(x+1)^2\left(f\left(\frac{x}{x+1}\right)-1\right)-1 = -(x+1)^2\left(f\left(\frac{x+1}{x}\right)\cdot\frac{x^2}{(x+1)^2}-1\right)-1 \\
&= -x^2 f\left(\frac{x+1}{x}\right)+x^2+2x = -x^2\left(1+f\left(\frac{1}{x}\right)\right)+x^2+2x = -x^2 f\left(\frac{1}{x}\right)+2x \\
&= -f(x)+2x,
\end{aligned}$$

so $f(x)=x$ for $x \notin \{0,-1\}$. Moreover $f(0)=0$ and $f(-1)=-1$ follow directly from $f(0)=-f(0)$ and $f(-1)=f(0)-1$. Finally we check that $f(x)=x$ satisfies all given conditions.
(b) Take $h(x)=g(x)+1$, then

$$f(x)f(y)+1 = h(x)h(y).$$

In particular $f(x)^2+1=h(x)^2$, so

$$(f(x)f(y)+1)^2 = h(x)^2h(y)^2 = (f(x)^2+1)(f(y)^2+1),$$

which simplifies to $(f(x)-f(y))^2=0$. Therefore f is constant, and consequently so are h and g.

S2. We start with the sequence $(1,1)$. In each step between every pair of consecutive terms of this sequence we sandwich their sum. So after the first step we have $(1,2,1)$, then after the second step $(1,3,2,3,1)$, etc. For every $n \geq 1$ compute the sum of cubes of terms after the n-th step.

SOLUTION. Let A_n be the sum of cubes of terms after the n-th step, so $A_1=10$, $A_2=64$. Suppose that after the $(n-1)$-th step we have the sequence $(a_1,a_2,\ldots,a_k)$. Then after the n-th step we have $(a_1,a_1+a_2,a_2,\ldots,a_{k-1}+a_k,a_k)$, so

$$A_n = 3A_{n-1}-2+3\sum_{i=1}^{k-1} a_ia_{i+1}(a_i+a_{i+1}),$$

and after the $(n+1)$-st step we have $(a_1,2a_1+a_2,a_1+a_2,a_1+2a_2,a_2,\ldots)$, so

$$A_{n+1} = 21A_n-20+21\sum_{i=1}^{k-1} a_ia_{i+1}(a_i+a_{i+1}) = 7A_n-6.$$

Hence

$$A_{n+1}-1 = 7(A_n-1) = 7^2(A_{n-1}-1) = \ldots = 7^n(A_1-1) = 9\cdot 7^n,$$

so finally $A_n = 9\cdot 7^{n-1}+1$.

F3. Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ s.t. $f(-1)\neq 0$ and for every $x,y\in\mathbb{R}$

$$f(1+xy)-f(x+y)=f(x)f(y).$$

SOLUTION. Cf. A5/IMOSL 2012 e.g on AoPS.

F4. Let $c\geq 1$ be an integer. Consider a graph on vertex set $\mathbb{Z}_+$, in which an edge $\{a,b\}$ exists iff $a+b+c\mid a^2+b^2+c$. Prove that this graph is connected.

SOLUTION. It is enough to prove that for an appropriately defined function $f\colon \mathbb{Z}_+\to\mathbb{Z}_+$ we have edges $\{n,f(n)\}$ and $\{f(n),f(n+1)\}$ for every n. The most natural candidate (following directly from the divisibility assumption) is

$$f(n)=n^2+n(c-1)+\frac{1}{2}c(c-1),$$

and it turns out to work: $f(n)+f(n+1)+c=n^2+(n+c)^2+c$ and

$$f(n)^2-n^2+f(n+1)^2-(n+c)^2=(f(n)+n)(n^2+(n+c)^2+c).$$

I1. Maximize

$$(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2),$$

where a, b, c are non-negative real numbers satisfying $a+b+c=3$.

SOLUTION. Say $a\leq b\leq c$. By $a\geq 0$ we have

$$a^2-ab+b^2\leq b^2 \quad\text{and}\quad c^2-ca+a^2\leq c^2.$$

Moreover, by AM-GM and $b+c\leq 3$:

$$b^2c^2(b^2-bc+c^2)=\frac{4}{9}\cdot\frac{3bc}{2}\cdot\frac{3bc}{2}\cdot\left((b+c)^2-3bc\right)\leq\frac{4}{9}\cdot\left(\frac{(b+c)^2}{3}\right)^3=12.$$

It remains to note that for $(a,b,c)=(0,1,2)$ we get exactly 12.

P3. Find all polynomials $P\in\mathbb{R}[x]$ of odd degree such that $P(x^2-1)=(P(x))^2-1$ for every $x\in\mathbb{R}$.

SOLUTION. We'll prove $P(x)=x$ is the only such polynomial. First of all note that $|P(x)|=|P(-x)|$ for all x. If $P(x)=P(-x)$ for infinitely any x, then $P(x)=P(-x)$ for all x which is impossible since x has an odd degree. Therefore $P(x)=-P(-x)$ for infinitely many x hence for all x. Every $x\geq -1$ is of the form y^2-1, so

$$P(x)=(P(y))^2-1\geq -1.$$

We'll construct an infinite sequence of fixed points of P, therefore proving that $P(x)=x$. First, $P(1)=1$ (because $P(-1)=-1$). Then if z is a fixed point (i.e. $P(z)=z$), then so is $\sqrt{z+1}$:

$$(P(\sqrt{z+1}))^2=P(z)+1=z+1\implies |P(\sqrt{z+1}|=\sqrt{z+1}.$$

But $-\sqrt{z+1}$ is not in range of P (smaller than -1). Therefore

$$1,\quad \sqrt{2},\quad \sqrt{1+\sqrt{2}},\quad \ldots$$

is a strictly increasing sequence of fixed points of P.

W4.

(a) Non-negative real numbers a, b, c satisfy $a+b+c=1$. Prove that

$$ab+bc+ca-2abc\le\frac{7}{27}.$$

(b) Non-negative real numbers a, b, c, d satisfy

$$a\le 1,\quad a+b\le 5,\quad a+b+c\le 14,\quad a+b+c+d\le 30.$$

Prove that $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\le 10$.

Solution. (a) Consider the polynomial $P(x)=(x-a)(x-b)(x-c)=x^3-x^2+(ab+bc+ca)x-abc$. What we want to prove is equivalent to $P(\frac{1}{2})\le\frac{1}{216}$. Note that at most one root of P is greater than $\frac{1}{2}$ (as they are all non-negative and their sum is 1). If this is the case, the inequality above holds trivially, as the left-hand side is negative. If a, b, c are all less than $\frac{1}{2}$, then just use AM-GM to get

$$\left(\frac{1}{2}-a\right)\left(\frac{1}{2}-b\right)\left(\frac{1}{2}-c\right)\le\left(\frac{\frac{3}{2}-(a+b+c)}{3}\right)^3=\frac{1}{216}.$$

(b) We would like to prove $(a,b,c,d)=(1,4,9,16)$ is the equality case, or equivalently $(a/1,b/2,c/3,d/4)=(1,2,3,4)$, so let's write

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}=\sqrt{\frac{a}{1}\cdot 1}+\sqrt{\frac{b}{2}\cdot 2}+\sqrt{\frac{c}{3}\cdot 3}+\sqrt{\frac{d}{4}\cdot 4}\le\frac{1}{2}\left(\frac{a}{1}+1+\frac{b}{2}+2+\frac{c}{3}+3+\frac{d}{4}+4\right),$$

so it's enough to prove $a+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}\le 10$. And that is indeed true, as we can appropriately combine our four assumptions to get it:

$$a+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}=\frac{a}{2}+\frac{a+b}{6}+\frac{a+b+c}{12}+\frac{a+b+c+d}{4}\le\frac{1}{2}+\frac{5}{6}+\frac{14}{12}+\frac{30}{4}=10.$$

K4. Initially all positive integers are colored black. We repeatedly perform the following procedure: in the n-th step we recolor the smallest black number c_n to blue and simultaneously recolor the number c_n+n to red. Find the exact formula for the sequence of all blue numbers $(c_n)=(1,3,4,6,8,9,\dots)$.

Solution. Denote $a_n=\lfloor\varphi n\rfloor$ and $b_n=a_n+n=\lfloor(\varphi+1)n\rfloor$, where φ is the positive number satisfying $\frac{1}{\varphi}+\frac{1}{\varphi+1}=1$ (the so called golden ratio, $\varphi=\frac{1}{2}(1+\sqrt{5})$. We will prove that each positive integer appears in exactly one of the sequences (a_n), (b_n). Note that it will yield $a_n=c_n$ as the sequence (c_n) is uniquely defined and (a_n) will agree with its definition (with (b_n) being the sequence of red numbers, i.e. all other than blue). For $\gamma>0$ denote

$$S_\gamma=\{\lfloor\gamma n\rfloor\mid n\in\mathbb{Z}_+\}.$$

Note that if $\gamma\notin Q$, then $m\in S_\gamma$ if and only if the interval $(\frac{m}{\gamma},\frac{m+1}{\gamma})$ contains an integer (note it's left-open as γ is irrational). Now if $m\in S_\varphi\cap S_{\varphi+1}$, then there exist $n_1,n_2\in\mathbb{Z}_+$ such that $n_1\in(\frac{m}{\varphi},\frac{m+1}{\varphi})$ and $n_2\in(\frac{m}{\varphi+1},\frac{m+1}{\varphi+1})$, so $n_1+n_2\in(m,m+1)$ — contradiction. Similarly if $m\notin S_\varphi\cup S_{\varphi+1}$, then there exist $n_1,n_2\in\mathbb{Z}_+$ such that $n_1\in(\frac{m+1}{\varphi},\frac{m}{\varphi}+1)$ and $n_2\in(\frac{m+1}{\varphi+1},\frac{m}{\varphi+1}+1)$, so $n_1+n_2\in(m+1,m+2)$ — contradiction.

REMARK. What we proved is a special case of one direction of the BEATTY-BANG THEOREM: The sets S_α, S_β form a partition of $\mathbb{Z}_+$ (i.e. $S_\alpha \cup S_\beta = \mathbb{Z}_+$ and $S_\alpha \cap S_\beta = \varnothing$) if and only if α, β are positive irrational numbers satisfying $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

W5.

(a) Find all triples (a,b,c) of non-negative integers such that $x^a+x^b+x^c$ is divisible by x^2+x+1.

(b) Find all triples (a,b,c) of distinct integers for which there exists $P(x)\in\mathbb{Z}[x]$ such that $P(a)=b$, $P(b)=c$, $P(c)=a$.

SOLUTION. (a) All triples with $\{a \bmod 3, b \bmod 3, c \bmod 3\}=\{0,1,2\}$. Identity $x^2+x+1 \mid x^3-1$ may be used to repeatedly decrease exponents by 3, so (a,b,c) is a solution if and only if $(a \bmod 3, b \bmod 3, c \bmod 3)$ is and it's easy to see that the only subquadratic polynomial divisible by x^2+x+1 (with all non-zero coefficients equal to 0) is x^2+x+1 itself.
(b) From the classical divisibility lemma (P has integer coefficients) we get that $|a-b|\le |b-c|\le|c-a|\le|a-b|$, so no such triple (of distinct numbers) exists.

S3. Sequence (a_n) is defined by $a_1=a_2=1$ and

$$a_{n+1}=a_n+a_{\lfloor n/2\rfloor}, \quad \text{for } n\ge 2.$$

Prove that in this sequence there are infinitely many terms divisible by 7.

SOLUTION. Suppose that k is such that $a_k\equiv 0$ (from here on all the congruences are understood modulo 7). At least one such k exists, as $a_7=7$. We will show that there is $k'>k$ such that $a_{k'}$ is divisible by 7 as well. Note that $a_{2k+1}=a_{2k}+a_k$ and $a_{2k+2}=a_{2k+1}+a_k$, so $a_{2k}\equiv a_{2k+1}\equiv a_{2k+2}\equiv r$. If $r\equiv 0$, then $k'=2k$. If $r\not\equiv 0$, then note that

$$a_{4k+i+1}=a_{4k+i}+a_{2k+\lfloor i/2\rfloor}$$

for $i=0,1,2,3,4,5$, which means that $a_{4k+i+1}\equiv a_{4k+i}+r$ for $i=0,1,2,3,4,5$. In other words, (a_{4k+i} mod 7) for $i=0,1,2,3,4,5,6$ is a 7-term arithmetic sequence (mod 7) whose difference is non-zero (mod 7) — thus one of its terms, $a_{4k+i}\equiv 0$ and we take $k'=4k+i$.

F5. Let $f\colon[0,1]\to\mathbb{R}$ be a function satisfying $f(\frac{1}{n})=(-1)^n$ for every integer $n\ge 1$. Prove that f cannot be expressed as $g(x)-h(x)$, where $g,h\colon[0,1]\to\mathbb{R}$ are strictly increasing.

SOLUTION. Suppose that such functions g, h exist. For every $k\ge 1$ we have

$$g\left(\frac{1}{2k+1}\right)=-1+h\left(\frac{1}{2k+1}\right)<-1+h\left(\frac{1}{2k}\right)=-2+g\left(\frac{1}{2k}\right)<-2+g\left(\frac{1}{2k-1}\right),$$

so for every $n\ge 1$

$$g(1)>2+g\left(\frac{1}{3}\right)>4+g\left(\frac{1}{5}\right)>\ldots>2n+g\left(\frac{1}{2n+1}\right)>2n+g(0),$$

and therefore $g(1)-g(0)>2n$ for arbitrarily large n — contradiction.

K5. Prove that there are no such polynomials $f_1,f_2,f_3,f_4\in\mathbb{Q}[x]$ that for every $x\in\mathbb{R}$

$$x^2+7=(f_1(x))^2+(f_2(x))^2+(f_3(x))^2+(f_4(x))^2.$$

Solution. First of all note that $f_i(x)=a_ix+b_i$ for some $a_i,b_i\in\mathbb{Q}$, where $i=1,2,3,4$ — if at least one of these functions had degree greater than 1, then the degree of the right-hand side of the given equation would be strictly greater than the degree of the left-hand side. This leads to the system of equations

$$\sum_{i=1}^{4} a_i^2=1, \quad \sum_{i=1}^{4} a_ib_i=0, \quad \sum_{i=1}^{4} b_i^2=7.$$

In particular, we get that

$$\sum_{i=1}^{4}(a_i+b_i)^2=8, \quad \sum_{i=1}^{4}(a_i-b_i)^2=8, \quad \sum_{i=1}^{4}(a_i+b_i)(a_i-b_i)=6.$$

Put $a_i+b_i=\frac{x_i}{m}$, $a_i-b_i=\frac{y_i}{m}$, where $x_i,y_i,m\in\mathbb{Z}$ and not all these numbers are even (as if they were, they could all be halved). Then the new system becomes

$$\sum_{i=1}^{4} x_i^2=8m^2, \quad \sum_{i=1}^{4} y_i^2=8m^2, \quad \sum_{i=1}^{4} x_iy_i=6m^2.$$

Now quadratic residues modulo 8 tell us that all x_i's are even (the first equation), all y_i's are even (the second equation), and finally m is even (the third equation) — contradiction!