TEST Problem

$k, n > 1$, $\boxed{p = 2k-1}$ prime:

$$p \mid \binom{n}{2} - \binom{k}{2} \implies p^2 \mid \binom{n}{2} - \binom{k}{2}$$

$$\binom{n}{2} - \binom{k}{2} = \frac{n(n-1)}{2} - \frac{k(k-1)}{2} =$$

$$= \frac{(4n^2-4n+1) - (4k^2-4k+1)}{8} = \frac{(2n-1)^2 - p^2}{8}$$

p is odd divisor of $(2n-1)^2 - p^2$

$$p \mid 2n-1 \leadsto p^2 \mid (2n-1)^2$$

$\boxed{8}$ Find all $n \geq 1$

$$\underbrace{\overbrace{1+2^n+4^n}^{A} \mid \overbrace{1+2^{n+1}+4^{n+1}}^{B}}$$

$$1+2+4 \mid 1+2^2+4^2$$

$$\cancel{7 \mid 21}$$

Why?

$$1+2^n+4^n < 1+2^{n+1}+4^{n+1} <$$

$$< 4(1+2^n+4^n)$$

$$1+2^{n+1}+4^{n+1} < 4+2\cdot 2^{n+1}+4^{n+1}$$

$A \mid B$

$$\boxed{A < B < 4A}$$

$\dfrac{B}{A}$ 2, 3

$\boxed{1^\circ}$ $\dfrac{B}{A} = 2$

$$1+2^{n+1}+4^{n+1} \neq 2+2^{n+1}+2\cdot 4^{n}$$

2° $\dfrac{B}{A} = 3$

$$1+2^{n+1}+4^{n+1} = 3+3\cdot 2^{n}+3\cdot 4^{n}$$

$$2^n+2 = 4^n$$

$$2^{n-1}+1 = 2^{2n-1}$$

$n-1=0$ $n=1$

If $n-1>0$ then LHS is odd
RHS is even ↯

$\boxed{9}$

$$v_p\left(\frac{\mathrm{lcm}(a,b,c)^2}{\mathrm{lcm}(a,b)\,\mathrm{lcm}(b,c)\,\mathrm{lcm}(c,a)}\right) = v_p\left(\frac{\gcd(a,b,c)^2}{\gcd(a,b)\,\gcd(b,c)\,\gcd(c,a)}\right)$$

$$a = p_1^{\alpha_1} \cdots\cdots p_k^{\alpha_k} \qquad \alpha_i$$
$$b = p_1^{\beta_1} \cdots\cdots p_k^{\beta_k} \qquad \beta_i \geq 0$$
$$c = p_1^{\gamma_1} \cdots\cdots p_k^{\gamma_k} \qquad \gamma_i$$

$$\mathrm{lcm}(a,b) = p_1^{\max(\alpha_1,\beta_1)} \cdots\cdots p_k^{\max(\alpha_k,\beta_k)}$$

$$\mathrm{lcm}(a,b,c) = \qquad 5^2 \cdot \underline{2}$$

$$\mathrm{lcm}(5,10) = 10 = 5^1 \cdot 2^1$$
$$\begin{array}{cc} \shortparallel & \shortparallel \\ 5^1 & 5^1 \cdot 2^1 \end{array} \qquad 5^2$$

$$v_5(10) = 1$$

$$a = p^k \cdot b \qquad p \nmid b$$

$$\boxed{v_p(a) = k}$$

$$\forall p: \quad v_p(\mathrm{lcm}(a,b)) = \max\{v_p(a), v_p(b)\}.$$

$$v_p(\gcd(a,b)) = \min\{v_p(a), v_p(b)\}.$$

$$a = b$$

$$\forall_{p \in \mathbb{P}} \; v_p(a) = v_p(b)$$

$$N = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

$$v_{p_1}(N) = \alpha_1$$

$$v_{p_2}(N) = \alpha_2$$

$$\vdots$$

$$v_{p_k}(N) = \alpha_k$$

$$\boxed{v_5(100) = 2}$$

$$5^2 \cdots$$

Properties

$$v_p(\mathrm{lcm}(a,b)) = \max\{v_p(a), v_p(b)\}$$

$$v_p(\gcd(a,b)) = \min\{v_p(a), v_p(b)\},$$

if $a \mid b$ then $v_p(a) \leq v_p(b)$

if we want check if $a = b$

it suffices to prove that

$$v_p(a) = v_p(b)$$

if $B \mid A$

$$\boxed{v_p\left(\frac{A}{B}\right) = v_p(A) - v_p(B)}$$

$$v_{p_i}\left(\frac{\mathrm{lcm}(a,b,c)^2}{\mathrm{lcm}(a,b)\,\mathrm{lcm}(b,c)\,\mathrm{lcm}(c,a)}\right) =$$

$a = p_1^{\alpha_1} \cdots\cdots p_k^{\alpha_k}$

$b = p_1^{\beta_1} \cdots\cdots p_k^{\beta_k}$

$c = p_1^{\gamma_1} \cdots\cdots p_k^{\gamma_k}$

$$= 2\max\{\alpha_i, \beta_i, \gamma_i\} - \max\{\alpha_i, \beta_i\} - \max\{\beta_i, \gamma_i\} - \max\{\gamma_i, \alpha_i\}$$

$$v_{p_i}\left(\frac{\gcd(a,b,c)^2}{\gcd(a,b)\gcd(b,c)\gcd(c,a)}\right) =$$

$$2\min\{\alpha_i,\beta_i,\gamma_i\} - \min\{\alpha_i,\beta_i\} - \min\{\beta_i,\gamma_i\} - \min\{\gamma_i,\alpha_i\}$$

WLOG $\alpha_i \geq \beta_i \geq \gamma_i$

$$2\max\{\alpha_i,\beta_i,\gamma_i\} - \max\{\alpha_i,\beta_i\} - \max\{\beta_i,\gamma_i\} - \max\{\gamma_i,\alpha_i\} =$$

$$= 2\alpha_i - \alpha_i - \beta_i - \alpha_i = \boxed{-\beta_i}$$

$$2\min\{\alpha_i,\beta_i,\gamma_i\} - \min\{\alpha_i,\beta_i\} - \min\{\beta_i,\gamma_i\} - \min\{\gamma_i,\alpha_i\} =$$

$$= 2\gamma_i - \beta_i - \gamma_i - \gamma_i = \boxed{-\beta_i}$$

17, 18 at Hoe

$$n^3 - 7n = x^2$$

n

$$n(n^2-7) = x^2$$

1° $\boxed{7 \nmid n}$

$$\gcd(n, n^2-7) = d$$

p - ~~prim~~ $p \mid d$,

$p \mid n$

$p \mid n^2 - 7$

$\Downarrow$

$p \mid 7 \rightsquigarrow p = 7$

$$\boxed{\gcd(n, n^2-7) = 1}$$

$$4^3 - 7 \cdot 4 = 64 - 28 = \boxed{36} = 6^2$$

$n = a^2$

$n^2 - 7 = b^2$

$$a^4 - 7 = b^2 \Rightarrow a^4 - b^2 = 7$$

$$= \underbrace{(a^2 - b)}_{1}\underbrace{(a^2 + b)}_{7} = 7$$

$2a^2 = 8$ $\quad a = 2$

$$\boxed{n = 4}$$

2° $\boxed{7 \mid n}$ $\quad n = 7m$

$$7m\left((7m)^2 - 7\right) = x^2$$

$$49m(7m^2-1) = x^2$$

$$7 \mid x \qquad x = 7y$$

$$\cancel{49}\, m(7m^2-1) = \cancel{49} y^2$$

$$m(7m^2-1) = y^2$$

$$m = c^2$$

$$7m^2-1 = d^2$$

$$7c^4 - 1 = d^2 \rightsquigarrow \boxed{7 \mid d^2+1}$$

$$d^2 \equiv -1 \pmod 7 \;?$$

$$0, 1, 4, 2,$$

$\lightning$

$\boxed{17, 18, 11, 12, 13}$ $\quad$ $\boxed{\text{Hazevale}}$!