Email training, N11

November 27 - December 7, 2019

Problem 11.1. Construct a polynomial with integer coefficients with root $\sqrt{2}+\sqrt{3}$.

Solution 11.1. Let $a=\sqrt{2}+\sqrt{3}$. Then

$$a-\sqrt{2}=\sqrt{3}.$$

By taking the square we get

$$a^2-2a\sqrt{2}+2=3,$$
$$a^2-1=2a\sqrt{2}.$$

Again by taking the square we get

$$a^4-2a^2+1=8x^2,$$
$$a^4-10a^2+1=0.$$

So a is the root of equation $x^4-10x^2+1=0$.

Problem 11.2. Find the coefficient of x^{100} after opening the brackets and grouping terms of $(1+x+x^2+\ldots+x^{100})^3$.

Solution 11.2.

$(1+x+x^2+\ldots+x^{100})^3=(1+x+x^2+\ldots+x^{100})\cdot(1+x+x^2+\ldots+x^{100})\cdot(1+x+x^2+\ldots+x^{100})$.

Actually the question is in how many way we may choose x^a from the first bracket, x^b from the second one, and x^c from the third bracket, such that $a+b+c=100$. Note also that $a,b,c\geq 0$. This question has answer $\binom{100+3-1}{3-1}=\binom{102}{2}=5151$.

Problem 11.3. Five numbers are written on the board. Mohammed calculates the sum of all pairs and gets the results 110, 112, 113, 114, 115, 116, 117, 118, 120 and 121. Find the numbers written on the board.

Solution 11.3. Note that all numbers are different, since no number appears twice in the result list. denote the number by x_1,x_2,x_3,x_4 and x_5, by having $x_1<x_2<x_3<x_4<x_5$.

The total sum of numbers is the sum of all pairs divided by 4, so

$$x_1+x_2+x_3+x_4+x_5=\frac{1156}{4}=289.$$

Also we have $x_1+x_2=110$, $x_1+x_3=112$, $x_3+x_5=120$, $x_4+x_5=121$. From here we get $x_1+x_2+x_4+x_5=231$, so $x_3=289-231=58$. The we find all other numbers $x_1=112-x_3=54$, $x_2=110-x_1=56$, $x_5=120-x_3=62$, $x_4=121-x_5=59$.

Problem 11.4. Find all positive integers m and n for which $1!+2!+3!+\cdots+n!=m^2$.

Solution 11.4. $1!=1^2$ gives a solution $(1,1)$, $1!+2!=3$ which is not a square, $1!+2!+3!=3^2$ gives a solution $(3,3)$, $1!+2!+3!+4!=33$ which is not a square.

Since $10|k!$ for all $k\geq 5$,

$$1!+2!+\ldots+n!\equiv 3 \mod 10$$

for all $n\geq 4$. But there are no perfect squares congruent to 3 modulo 10; hence the only solutions are those already found.

Problem 11.5. Let a, b, c, be a positive integer such that $a^2+b^2=c^2$. Prove that $\frac{1}{2}(c-a)(c-b)$ is a perfect square.

Solution 11.5. Note that

$$2(c-a)(c-b) = 2c^2 + 2ab - 2bc - 2ac = \\ = a^2 + b^2 + c^2 + 2ab - 2bc - 2ac = (a+b-c)^2, \tag{1}$$

hence

$$\frac{1}{2}(c-a)(c-b) = \left(\frac{a+b-c}{2}\right)^2.$$

It remains to show that $a+b-c$ is even. Since $(a+b-c)^2$ is equal to $2(c-a)(c-b)$ is even, so is $a+b-c$.

Problem 11.6. Are there exists integers a, b such that a^2+b and $a+b^2$ are consecutive integers?

Solution 11.6. If so, then $(a^2+b)-(a+b^2) \in \{-1, 1\}$ but

$$(a^2+b)-(a+b^2) = a(a-1) - b(b-1)$$

is an even number – contradiction.

Problem 11.7. Let numbers 1, 2, 3, ..., 19, 20 are written on the board. At each step one may erase any two numbers a and b and write the number $a+b$–1. Which number will be written on the board after 19 steps.

Solution 11.7. Note that after each step the total sum of the numbers decreases by 1. At the beginning the total sum is $1+2+\ldots+19+20 = 210$. After 19 steps will remain one number and it will be equal $210-19 = 191$.

Problem 11.8. Let 100 points are drawn on the plane such that the distance between 2 any points is less than 1. Also it's known that for any three points A, B and C the triangle ABC is not acute. Prove that there exists a circle of radius 0.5 which contains in it's interior all 100 points.

Solution 11.8. Let AB is the longest segment. Let's draw circle having diameter AB. For any point C, in the triangle ABC exists a not acut angle, so it should be opposite of the longest segment which is AB. So $\angle C \geq 90^o$, so C is on or inside of our circle. So all points are either on or inside the circle. The radius of circle is less then 0.5. So if we draw a circle with radius 0.5 and having the same center, then all points will be inside the circle.

Problem 11.9. 10 players participate to chess tournament. Each day they play 5 games - one game each. After 9 days all chess players have been played with each other. It occurs that the most of games are played between the players from Jeddah. Prove that every day at least two players from Jeddah have been played with each other.

Solution 11.9. In total $\binom{10}{2} = 45$ games are played. Let a players are from Jeddah. Then players from Jeddah have played between them

$$\binom{a}{2} = \frac{a(a-1)}{2} \geq \frac{45}{2}$$

games, so $a(a-1) \geq 45$. From this we get $a \geq 8$. So at least 8 players are from Jeddah, therefore, by pigeonhole principle every day there are 2 players from Jeddah on one of 5 boards.

Problem 11.10. The numbers 1, 1 are written on the board. At each step between two neighbor numbers Aziz writes their sum. Below are the results after first three steps.

$$1, 2, 1$$

$$1, 3, 2, 3, 1$$

$$1, 4, 3, 5, 2, 5, 3, 4, 1$$

Find the total sum of numbers written after 100-th step.

Solution 11.10. Note, that at each step corner number 1's have influence in it's neighbor,and non corner numbers have influence in it's 2 neighbors. So, if we denote by x_n the sum after n'th step, then we have the following recurrent relation

$$x_{n+1} = 3x_n - 2, \quad x_0 = 2.$$

We need to find x_{100}. Note that

$$x_{n+1} - 1 = 3(x_n - 1)$$

so for $z_n = x_n - 1$ we have

$$z_{n+1} = 3z_n, \quad z_0 = 1.$$

So $z_n = 3^n$ and therefore $x_{100} = z_{100} + 1 = 3^{100} + 1$.