— Geometry for L4 —

— November 27, 2021 — Hello Inversion! —

Given a circle ω of radius r centered at S, *inversion in* ω is a transformation mapping each point $X \neq S$ to such X^* that $SX \cdot SX^* = r^2$. The center of inversion is considered to be mapped onto an extra point ∞ belonging to all lines of the plane.

Recall the basic properties of inversion:

- changing the radius of inversion gives homothetic maps;
- inversion maps the set of (lines & circles) to itself;
- circles passing through S are interchanged with lines not passing through S;
- inversion preserves angles between curves (e.g. tangencies);
- inversion fixes circles perpendicular to the inversion circle and lines passing through the inversion center.

7. Circle ω lies inside circle Ω and they are internally tangent at S. Segments SA and SB are diameters of ω and Ω, respectively. A circle o tangent to segment AB, ω and Ω touches AB at C. Prove that SC is the harmonic mean of SA and SB.

8. Circle ω lies inside circle Ω and they are disjoint. Consider all pairs of circles o_1, o_2 externally tangent to ω and internally tangent to Ω. Prove that the radical axes of all such pairs of circles have a common point.

9. Given a triangle ABC with incenter I let D, E, F be the tangency points of the incircle with sides BC, CA, AB, respectively and let K, L, M be symmetrical to I with respect to EF, FD, DE, respectively. Prove that the circumcenters of ABC, DEF and KLM are collinear.

10. In an acute triangle ABC segment AD is an altitude and M, N are projections of D onto AB, AC, respectively. Lines MN and AD intersect circumcircle of ABC at P, Q and A, R, respectively. Prove that D is the incenter of PQR.

11. Given a non-isosceles triangle ABC with incenter I let D, E, F be the tangency points of the incircle with sides BC, CA, AB, respectively. Circumcenters of EAF, FBD, DCE intersect the circumcenter of ABC again at X, Y, Z, respectively. Prove that DX, EY, FZ are concurrent.