May Camp - 2021
NT L3 LTE Lemma Part 1

Lemma 1. *For any integers x, y and positive integers n and p, where p is a prime such that $p \nmid x$, $p \nmid y$ and $p \mid x-y$, the following identities hold:*

- *If p is odd, then*
$$\nu_p(x^n - y^n) = \nu_p(x-y) + \nu_p(n),$$
- *If $p = 2$ and $4 \mid x-y$, then*
$$\nu_2(x^n - y^n) = \nu_2(x-y) + \nu_2(n),$$
- *If $p = 2$ and n is even, then*
$$\nu_2(x^n - y^n) = \nu_2(x-y) + \nu_2(x+y) + \nu_2(n) - 1.$$

For example, let's prove that 2 is a primitive root module 3^k for all positive k.

Assume that $2^d \equiv 1 \pmod{3^k}$. Obviously, d must be even, otherwise $2^d \equiv \equiv (-1)^d \equiv -1 \pmod 3$. So $d = 2d_1$ and $2^d - 1 = 4^{d_1} - 1^{d_1}$. By LTE lemma,
$$\nu_3(4^{d_1} - 1^{d_1}) = \nu_3(4-1) + \nu_3(d_1) \geqslant k.$$
Hence $d_1 = 3^{k-1}d_2$ and $d = 2 \cdot 3^{k-1} d_2 = \varphi(3^k)d_2$, and we are done.

Problem list

1. Let n be a square-free integer (that is n is not divisible by any perfect square $b^2 > 1$). Find all pairs of coprime positive integers (x, y) such that $x^n + y^n$ is divisible by $(x+y)^3$.
2. Find the smallest positive integer n such that 3^n ends with 01 when written in base 143.
3. Prove that $n^7 + 7$ is not a perfect square for any positive integer n.
4. Let $a \geqslant 3$ be an integer. Prove that there exist infinitely many numbers n such that $a^n - 1$ is divisible by n.
5. Let $a \geqslant 3$ be an integer. Prove that there exist infinitely many numbers n such that $a^n - 1$ is divisible by n^2.

Homework

1. Show that if $p > 2$ is a prime, g is a primitive root $\pmod p$, and $p^2 \nmid g^{p-1} - 1$, then g is a primitive root $\pmod{p^n}$ for any $n \in \mathbb{N}$.
2. Let $a \geqslant 4$ be an even integer. Prove that there exist infinitely many square-free numbers n such that $a^n - 1$ is divisible by n^2.