Email training, N4

Level 2, October 4-10

Problem 4.1. Find the maximum possible value of $x^6 + y^6$ if it's known that $x^2 + y^2 = 1$.

Solution 4.1.

$$x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = (x^2 + y^2)^2 - 3x^2y^2 = 1 - 3x^2y^2$$

So we need to estimate the minimal possible value of x^2y^2. Since $x^2y^2 \geq 0$, one has

$$x^6 + y^6 = 1 - 3x^2y^2 \leq 1$$

and the equality holds when $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$.

Answer: 1.

Problem 4.2. Let a, b, c, d be real numbers such that

$$a^4 + b^4 + c^4 + d^4 = 16.$$

Prove the inequality

$$a^5 + b^5 + c^5 + d^5 \leq 32.$$

Solution 4.2. We have $a^4 \leq a^4 + b^4 + c^4 + d^4 = 16$, i.e. $a \leq 2$ from which it follows that $a^5 \leq 2a^4$. Similarly we obtain $b^5 \leq 2b^4$, $c^5 \leq 2c^4$ and $d^5 \leq 2d^4$. Hence

$$a^5 + b^5 + c^5 + d^5 \leq 2(a^4 + b^4 + c^4 + d^4) = 32.$$

equality holds if one of a, b, c, d is equal 2 and the rest are equal 0.

Problem 4.3. Let $S(n)$ be the sum of divisors of n (for example $S(6) = 1 + 2 + 3 + 6 = 12$). Find all n for which $S(2n) = 3S(n)$.

Solution 4.3. Let $d_1, d_2, \ldots, d_k$ are all divisors of n. Then all divisors of $2n$ are from the multi-set $\{2d_1, 2d_2, \ldots, 2d_k, d_1, d_2, \ldots, d_k\}$ and the sum of all elements from this multi-set is equal to $3S(n)$. So to have $S(2n) = 3S(n)$ one needs to have all numbers from the multi-set pairwise different. It's possible if and only if all d_i's are odd, otherwise it would appear twice - as $2\frac{d_i}{2}$ and as d_i.

Answer: n is odd.

Problem 4.4. Find all pairs of possitive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{lcm(x,y)} + \frac{1}{gcd(x,y)} = \frac{1}{2}.$$

Solution 4.4. We put $x = du$ and $y = dv$ where $d = gcd(x, y)$. So we have $(u, v) = 1$. From the conclusion of the problem we have

$$\frac{1}{du} + \frac{1}{dv} + \frac{1}{d} + \frac{1}{duv} = \frac{1}{2},$$

$$u + v + uv + 1 = \frac{duv}{2},$$

or

$$2(u+1)(v+1) = duv.$$

Since $gcd(v, v+1) = 1$ therefore v divides $2(u+1)$.

Case 1. $u = v$. Then $u = v = 1$ and we get $d = 2(1+1)(1+1) = 8$ which leads $x = y = 8$.

Case 2. $u < v$. Then $u + 1 \leq v$ so $2(u+1) \leq 2v$ so $\frac{2(u+1)}{v}$ is equal either 1 or 2.

If $\frac{2(u+1)}{v} = 1$ then we have $(d-2)u = 3$ which means $(d, u) = (3, 3)$ or $(d, u) = (5, 1)$. So we get $(x, y) = (9, 24)$ or $(x, y) = (5, 20)$.

If $\frac{2(u+1)}{v} = 2$ then we have $(d-2)u = 4$ which means $(d,u) = (3,4)$ or $(d,u) = (4,2)$ or $(d,u) = (6,1)$. So we get $(x,y) = (12,15)$ or $(x,y) = (8,12)$ or $(x,y) = (6,12)$.

Case 3. $u > v$. This is identical to the case 2.

Answer: $(8,8), (9,24), (24,9), (5,20), (20,5), (12,15),$
$(15,12), (8,12), (12,8), (6,12), (12,6)$.

Problem 4.5. Ali has chosen 8 cells of the chessboard 8×8 such that no any two lie on the same line or in the same row (we call it general configuration). On each step Baba chooses 8 cells in general configuration and puts coins on them. Then Ali shows all coins that are out of cells chosen by Ali. If Ali shows even number of coins then Baba wins, otherwise Baba removes all coins and makes the next move. Find the minimal number of moves that Baba needs to guarantee the win.

Solution 4.5. Now lets show how Baba can win in 2 moves. On first step he puts 8 coins on the diagonal of the board. If he does not win then on that diagonal there are odd number of cells chosen by Ali. Let on the diagonal cell A is chosen and the diagonal cell B is not chosen. On next step Baba puts coins on 6 diagonal cells (but A and B) as well puts 2 coins on two other vertices of the rectangle with vertices A and B. Note, that these 2 cells are not chosen, since they are on the same row or column with the cell A which is chosen. So the answer will be even. It's obvious that he can't win in 1 move, since we may assume that Ali chooses his cells after Baba makes his first move.

Answer: 2 moves.

Problem 4.6. Let $1 \leq r \leq n$. We consider all r-element subsets of $(1, 2, \ldots, , n)$. Each of them has a minimum. Prove that the average of these minima is $\frac{n+1}{r+1}$.

Solution 4.6. There are exactly $\binom{n-k}{r-1}$ subsets with minimal element equal k (chose k and the rest $r-1$ elements arbitrary from the set $\{k+1, \ldots, n\}$. So the total sum of minimal elements is equal

$$\sum_{k=1}^{n} k\binom{n-k}{r-1} = \sum_{k=1}^{n} \binom{k}{1}\binom{n-k}{r-1}$$

Let there are $n+1$ ball among a line and we need to chose any $r+1$ of them. For some value of k between 1 and n, inclusive, we say that the second ball will occur in the $(k+1)$th place. Clearly, there are $\binom{k}{1}$ ways to arrange the bits coming before the second 1, and $\binom{n-k}{r-1}$ ways to arrange the bits after the second 1. So there are $\sum_{k=1}^{n} \binom{k}{1}\binom{n-k}{r-1}$ ways to chose any $r+1$ balls, which is eventually equal to $\binom{n+1}{r+1}$. So the average is equal

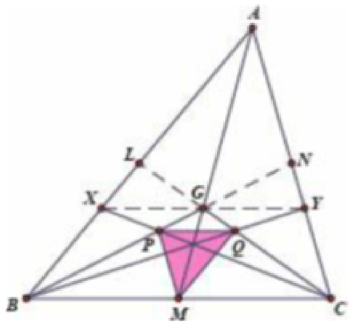
$$\frac{\binom{n+1}{r+1}}{\binom{n}{r}} = \frac{n+1}{r+1}.$$

Problem 4.7. Let G be the centroid of ΔABC. Draw a line $XY \parallel BC$ passing through G, intersecting AB, AC at X, Y respectively. Let BG and CX intersect at P, as well as CG and BY intersect at Q. Let M is the midpoint of BC. Prove that $\Delta ABC \sim \Delta MQP$.

Solution 4.7. Let L, N be the midpoints of AB and AC, respectively. Since $XY \parallel BC$ and G is the centroid, it is easy to see that $GX = GY$.

Hence, $\frac{PX}{CP} = \frac{GX}{BC} = \frac{GY}{BC} = \frac{QY}{BQ}$ and by the Intercept theorem $PQ \parallel BC$.

One also sees that $\frac{GY}{CM} = \frac{AG}{AM} = \frac{2}{3}$. Hence $\frac{GQ}{CQ} = \frac{GY}{BC} = \frac{1}{3}$.



Since $\frac{CG}{CL}=\frac{2}{3}$, we have $\frac{CQ}{CL}=\frac{CQ}{CG}\cdot\frac{CQ}{CL}=\frac{3}{4}\cdot\frac{2}{3}=\frac{1}{2}$, i. e. Q is the midpoint of CL. Hence $MQ \parallel AB$ by the Midpoint theorem.