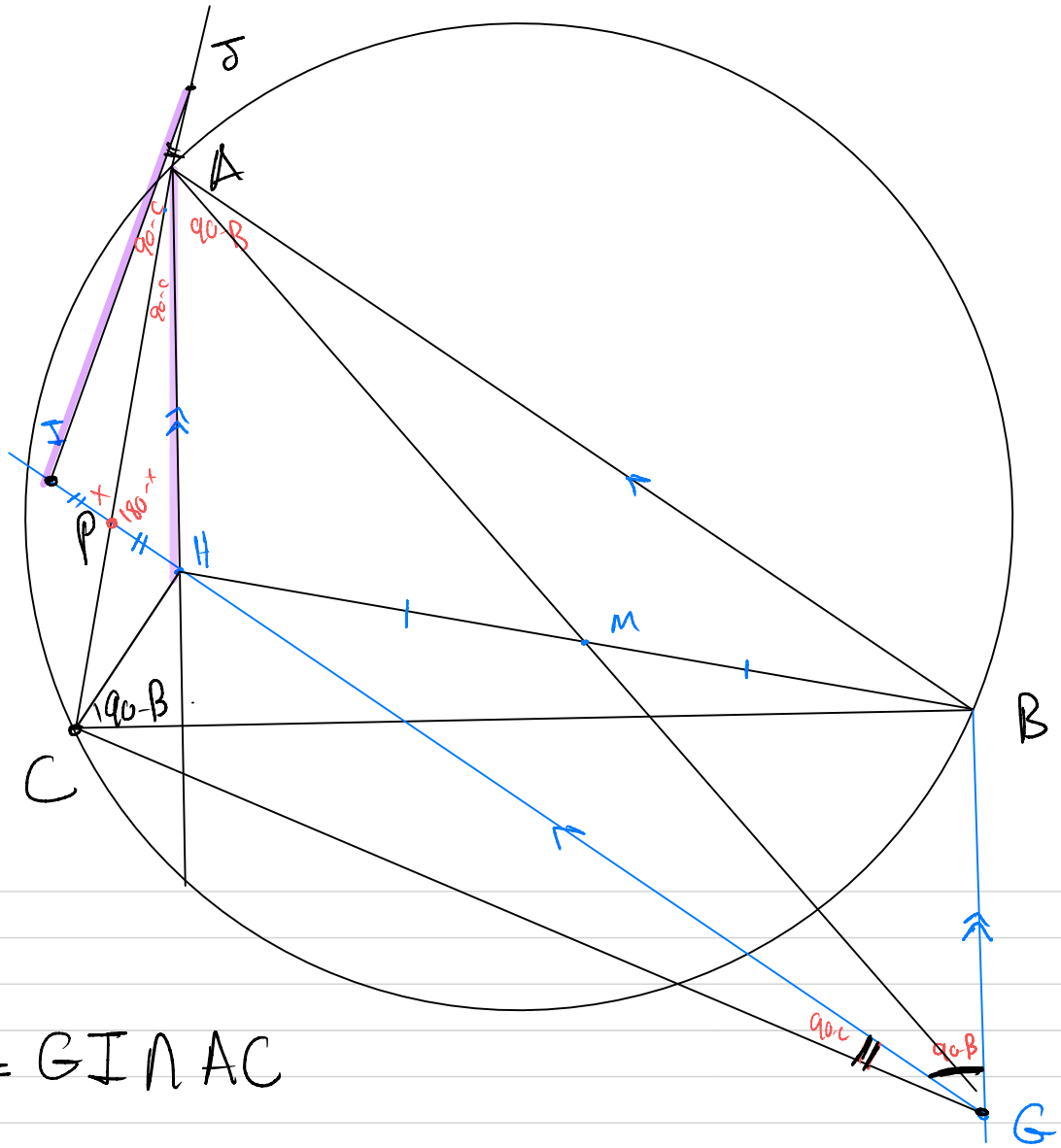
Intensive Training
Geometry

Day 12
29 April 2021

Includes solutions for:

IMO 2015 Shortlist G1
BMO 2016 P2
IMO 2016 P4

38. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$. (IMO 2015 Shortlist G1)



Let $P = GI \cap AC$

Note that $\angle BGH = \angle BAH = \angle HCB = 90 - B$

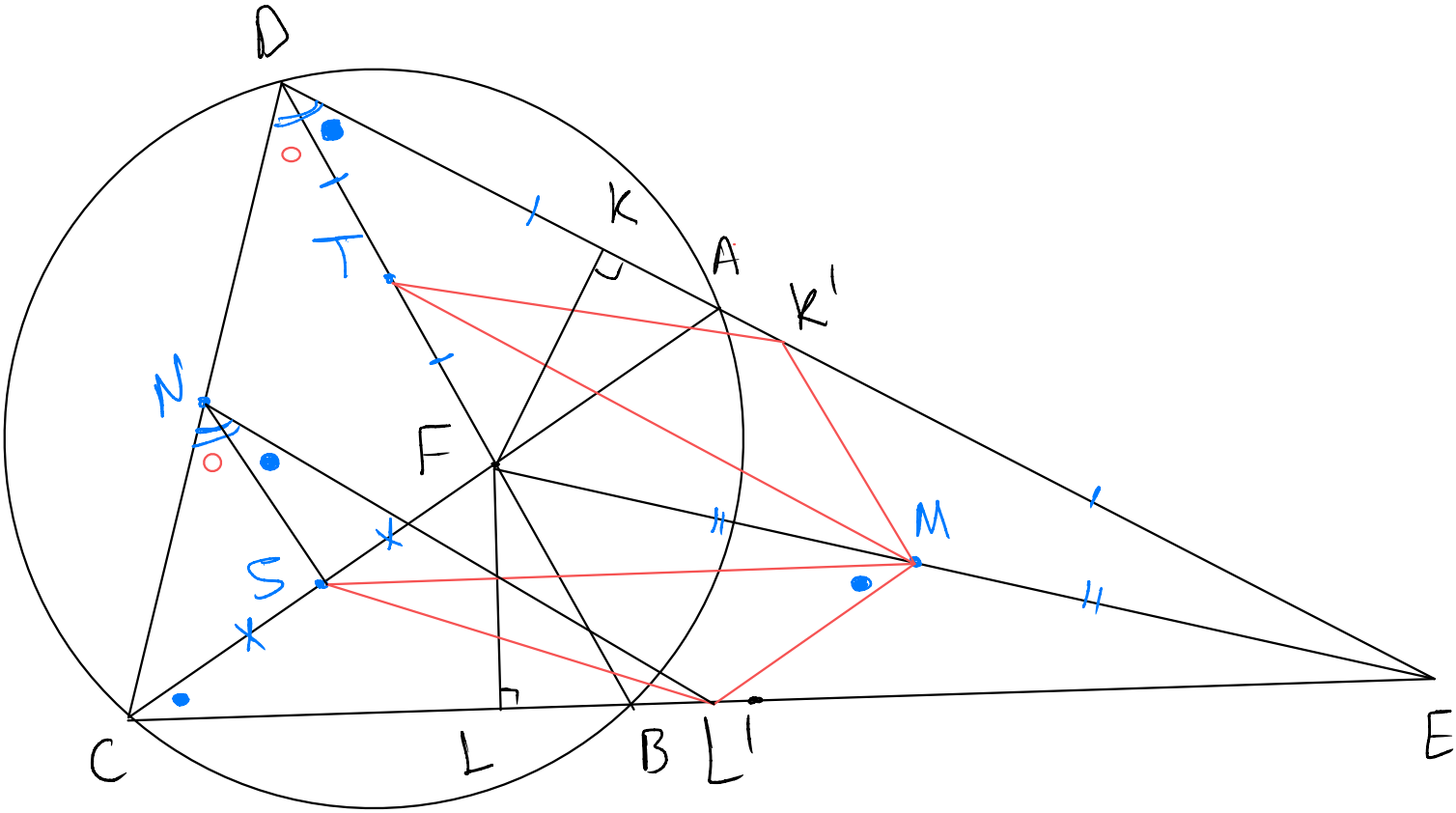
$\Rightarrow HCGB$ cyclic $\Rightarrow \angle CGH = \angle CBH = 90 - C$

But $JICG$ cyclic $\Rightarrow \angle IJP = \angle IJC = \angle IGC = 90 - C$

In $\Delta IJP = \dfrac{IJ}{IP} = \dfrac{\sin \angle IPJ}{\sin 90 - C}$, in ΔAHP: $\dfrac{AH}{HP} = \dfrac{\sin \angle APH}{\sin 90 - C}$

but $\sin(180 - x) = \sin x \Rightarrow \dfrac{IJ}{IP} = \dfrac{AH}{HP} \Rightarrow IJ = AH$ $\square$

39. Let $ABCD$ be a cyclic quadrilateral with $AB < CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.



(KTM) is the Oq circle in $\triangle DFE$

Let K' be the midpoint of DE, L' midpoint of CE

N midpoint DC.

$ML'CS$ is a parallelogram $\Rightarrow \angle SML' = \angle SCL'$

$= \angle TDK'$

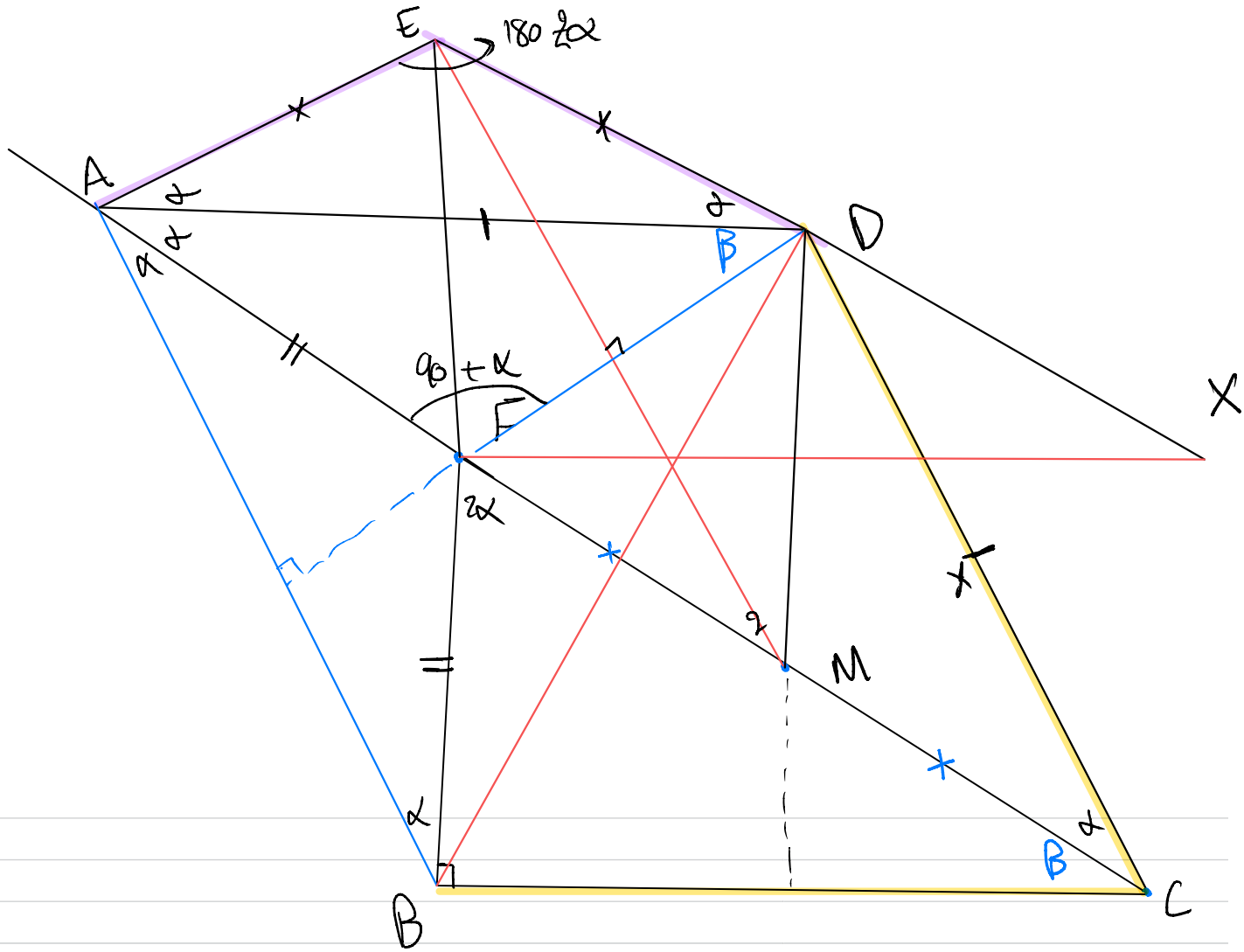
$NS \parallel DF$, $NL' \parallel DE \Rightarrow \angle SNL' = \angle FDE$

$\Rightarrow \angle SNL' = \angle SML' \Rightarrow N \in (MSL')$

$\Rightarrow N \in (MLS)$

Similarly, $N \in (MKT) \Rightarrow N$ is the intersection point $N \in CD$ $\square$

40. Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.



E, D, X collinear because $ED \parallel AC$

$\triangle ADC \sim \triangle AFB \Longrightarrow$ by spiral similarity lemma

$\triangle ADF \sim \triangle ACB$

$\Rightarrow \angle ADF = \angle ACB = \beta = 90 - 2\alpha$ (1)

$\angle AFD = 180 - (\alpha + \beta) = 180 - (90 - \alpha) = 90 + \alpha$

$\Rightarrow$ E is the center of (AFD) (2)

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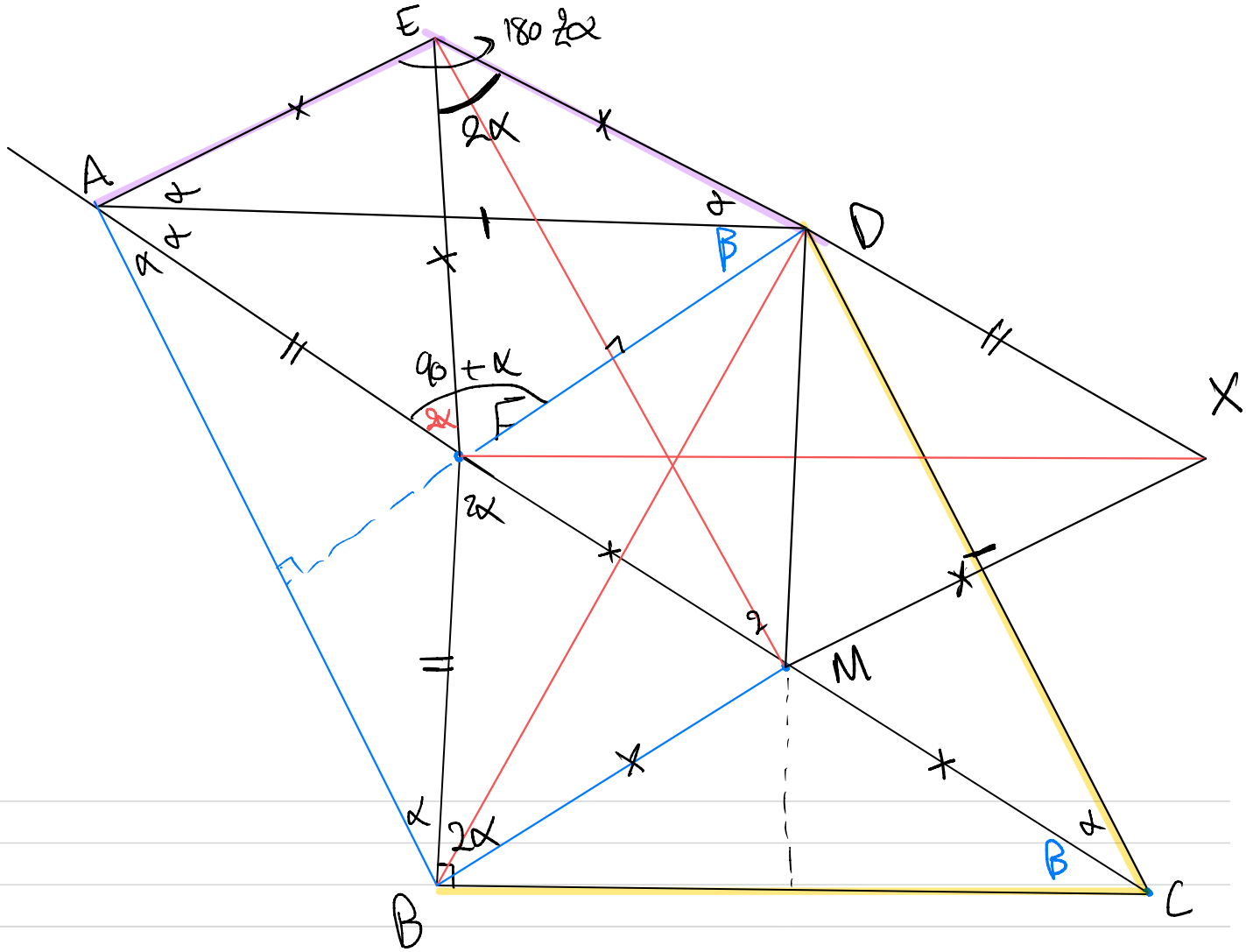


$\angle FED = 2\angle FAD = 2\alpha$ and $EA = EF = ED$

$\angle EFA = \angle EAF = 2\alpha \Rightarrow \angle EFA = \angle CFB = 2\alpha$

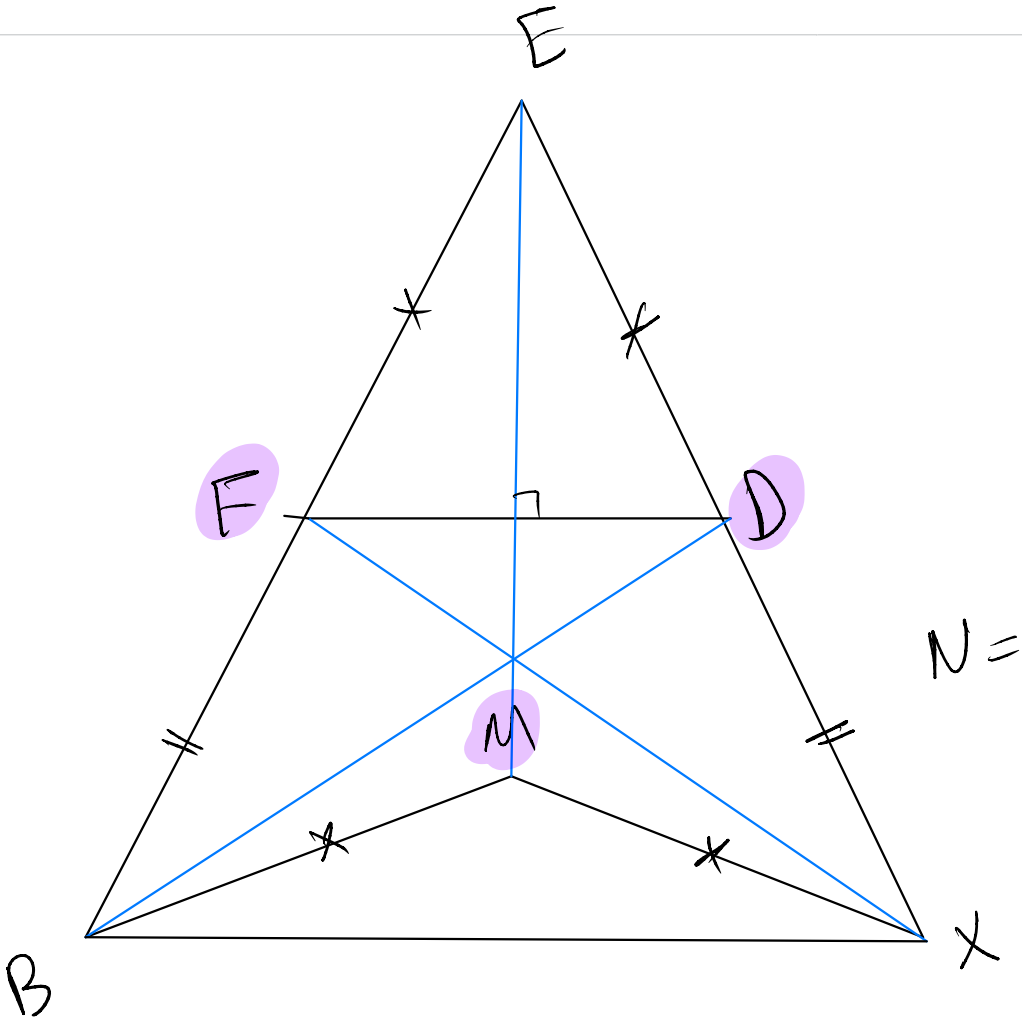
$\Rightarrow E, F, B$ are collinear. 3

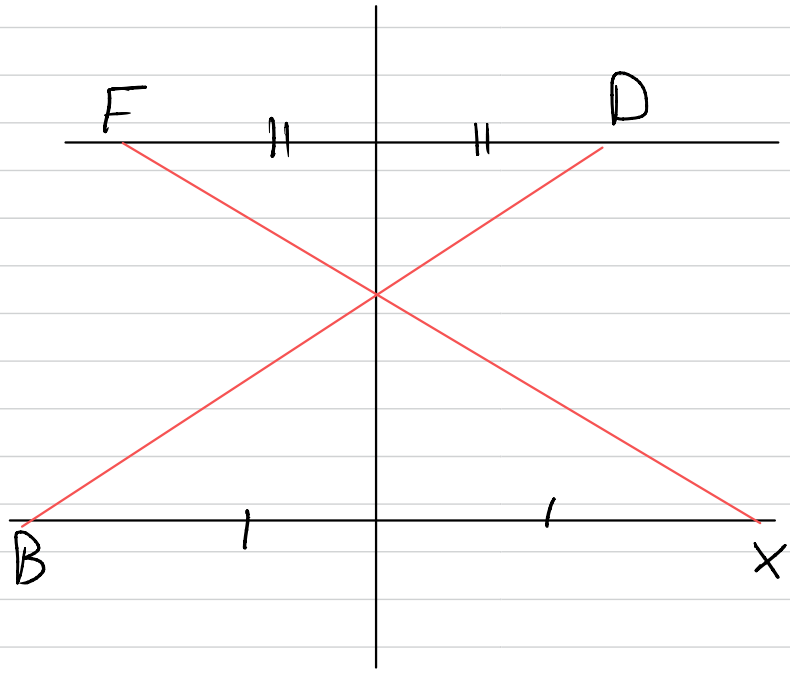
Also $MF = MB = MC$ because $\angle FBC = 90$.

$\triangle AEF \sim \triangle FMB$ by angles $\Rightarrow \triangle AEF \equiv \triangle FMB$

because $FA = FB \Rightarrow AE = FM$

Since $AM = EX$, $DX = AF$



$N = FX \cap BD$

$EM \perp BX$, by symmetry $XF \cap DB$ lies on EM $\square$