1 Completing the square and quadratic equations

1. Find the greatest integer n for which the equation

$\frac{1}{x-1}-\frac{1}{nx}+\frac{1}{x+1}=0$ has real solutions.

2. Solve the system of equations $\begin{cases} x-y=3 \\ x^2+(x+1)^2=y^2+(y+1)^2+(y+2)^2 \end{cases}$

3. Evaluate $\frac{1}{\sqrt{x+2\sqrt{x-1}}}+\frac{1}{\sqrt{x-2\sqrt{x-1}}}$, *where* $1 \leq x < 2$.

4. Solve the system of equations $\begin{cases} x+\frac{1}{y}=-1 \\ y+\frac{1}{z}=\frac{1}{2} \\ z+\frac{1}{x}=2. \end{cases}$

5. Solve the equation $\frac{1}{3x-1}+\frac{1}{4x-1}+\frac{1}{7x-1}=1$.

6. Find all pairs (a,b) of positive real numbers such that

$4a+9b=\frac{9}{a}+\frac{4}{b}=12$

7. Let a,b,c be real numbers such that $a \geq b \geq c$. Prove that

$a^2+ac+c^2 \geq 3b(a-b+c)$

8. Prove that : $3(x+y+1)^2+1 \geq 3xy$ for all $x,y \in \mathbb{R}$.

9. Find all pairs $(x,y) \in \mathbb{R}$ such that : $4x^2+9y^2+1=12(x+y-1)$.

10. Prove that if $a \geq b > 0$,then $\frac{(a-b)^2}{8a} \leq \frac{a+b}{2}-\sqrt{ab} \leq \frac{(a-b)^2}{8b}$.

11. Simplify the expression $\frac{4}{4x^2+12x+9}-\frac{12}{6x^2+5x-6}+\frac{9}{9x^2-12x+4}$

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12. Solve in real numbers the equation $x^4 + 16x - 12 = 0$.

13. The equation $x^4 - 4x = 1$ has two real roots. Find their product.

14. Find all $n > 1$ such that $x_1^2 + x_2^2 + ... + x_n^2 \geq x_n(x_1 + x_2 + ... + x_{n-1})$ for all real numbers $x_1, ..., x_n$.

2 Factorization and algebraic identities

1. Simplify $\frac{1}{ab}+\frac{1}{a^2-ab}+\frac{1}{b^2-ba}$.

2. Simplify $\frac{a^2+2a-80}{a^2-2a-120}$.

3. Solve in real numbers the equation $x^3-3x^2+3x+3=0$.

4. Factor x^4+x^2+1.

5. Factor $(x^4-6x^2+1)(x^4-7x^2+1)$.

6. Prove that for all real numbers a,b,c:

$$(\frac{2a+2b-c}{3})^2+(\frac{2b+2c-a}{3})^2+(\frac{2c+2a-b}{3})^2=a^2+b^2+c^2.$$

7. Let m and n be positive integers. Prove that $(m^3-3mn^2)^2+(3m^2n-n^3)^2$ is a perfect cube.

8. Solve in integers the equation $\frac{x^3-3x+2}{x^2-3x+2}=y$.

9. Simplify $\frac{x^2+4y^2-z^2+4xy}{x^2-4y^2-z^2+4yz}$.

10. Solve in real numbers the system of equations $\begin{cases} x+y=2z \\ x^3+y^3=2z^3 \end{cases}$.

11. Solve in real numbers the system of equations $\begin{cases} x+y=xy-5 \\ y+z=yz-7 \\ z+x=zx-11 \end{cases}$.

12. Let a,b,c be real numbers such that $\begin{cases} a+b+2ab=3 \\ b+c+2bc=4 \\ c+d+2cd=-5 \end{cases}$ find $d+a+2da$.

13. Solve in positive integers the system of equations $\begin{cases} x-y-z=40 \\ x^2-y^2-z^2=2012 \end{cases}$.

14. Factor $x^4+2x^3+2x^2+2x+1$.

15. a) Factor x^5+x+1.

b) Find the prime factorization of 100011.

16. Solve in real numbers the equations $\frac{1}{x+1}+\frac{1}{x^2-x}+\frac{1}{x^3-x}=1$.

17. Let a,b,c be real numbers such that $ab+bc+ca=1$. Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}=\frac{2}{(a+b)(b+c)(c+a)}.$$

18. If a,b,c,d are real numbers such that $a^2+b^2+c^2+d^2\le 1$, find

the maximal value of

$$(a+b)^4+(a+c)^4+(a+d)^4+(b+c)^4+(b+d)^4+(c+d)^4$$

3 Factoring expression involving $a-b, b-c, c-a$

1. Prove that for all pairwise distinct real numbers a, b, c

$$\frac{bc}{(\mathrm{a}-\mathrm{b})(\mathrm{a}-\mathrm{c})}+\frac{ca}{(\mathrm{b}-\mathrm{c})(\mathrm{b}-a)}+\frac{ab}{(\mathrm{c}-a)(\mathrm{c}-b)}=1.$$

2. Prove that if a, b, c are pairwise distinct real numbers, then

$$\frac{a^2}{(\mathrm{b}-\mathrm{c})^2}+\frac{b^2}{(\mathrm{c}-a)^2}+\frac{c^2}{(a-b)}\geq 2.$$

3. Prove that if a, b, c are pairwise distinct real numbers, then

$$\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}\neq 0.$$

4. Prove that for all real numbers a, b, c we have

$$(a-b)^5+(b-c)^5+(c-a)^5=5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca).$$

5. Positive real numbers a, b, c satisfy $\frac{a(b-c)}{b+c}+\frac{b(c-a)}{c+a}+\frac{c(a-b)}{a+b}=0$

   Prove that $(a-b)(b-c)(c-a)=0$.

6. Prove that for all positive real numbers a, b, c

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2}\geq a+b+c.$$

4 Factoring $a^3+b^3+c^3-3abc$

1. Prove that for all real numbers a,b,c we have

$(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-a)(c-a)$.

2. Let a,b,c be real numbers such that $(a-b)^2+(b-c)^2+(c-a)^2=6$

Prove that $a^3+b^3+c^3=3(a+b+c+abc)$.

3. Let a,b,c be real numbers such that $a+b+c=1$

Prove that $a^3+b^3+c^3-1=3(abc-ab-bc-ca)$.

4. Let a,b,c be real numbers such that

$$(-\frac{a}{2}+\frac{b}{3}+\frac{c}{6})^3+(\frac{a}{3}+\frac{b}{6}-\frac{c}{2})^3+(\frac{a}{6}-\frac{b}{2}+\frac{c}{3})^3=\frac{1}{8}$$

Prove that $(a-3b+2c)(2a+b-3c)(-3a+2b+c)=9$.

5. Prove that if a,b,c,d are real numbers such that $a+b+c+d=0$ then

$a^3+b^3+c^3+d^3=3(abc+bcd+cda+dab)$.

6. Let a,b,c are nonzero real numbers, not all equal, such that

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ and $a^3+b^3+c^3=3(a^2+b^2+c^2)$

Prove that $a+b+c=3$.