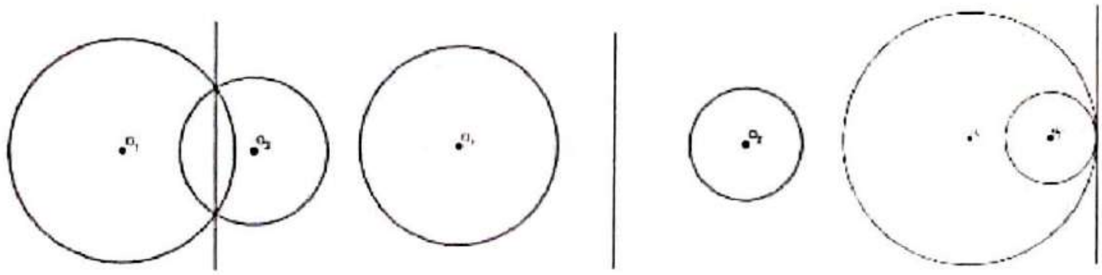
Saudi Arabia 2022 – Math Camp

Level 3

Geometry – Power of a Point and Radical Axis

Instructor: Regis Barbosa

Theorem 4. (Radical Axis) Given two circles Γ_1 and Γ_2 with different centers, the locus of the points P on the plane such that the power of P with respect to Γ_1 is equal the power of P with respect to Γ_2 ($Pot_{\Gamma_1}P = Pot_{\Gamma_2}P$) is a line perpendicular to the line through the centers of Γ_1 and Γ_2.



Proof. Let O_1 and r_1 be the center and the radius of Γ_1 and O_2 and r_2 be the center and the radius of Γ_2. Consider Cartesian coordinates where $O_1(0,0)$ and $O_2(k,0)$ with $k \neq 0$ because the circles are non-concentric.

The point $P(x,y)$ have the same power when

$$Pot_{\Gamma_1}P = Pot_{\Gamma_2}P \Leftrightarrow PO_1^2 - r_1^2 = PO_2^2 - r_2^2$$
$$\Leftrightarrow x^2 + y^2 - r_1^2 = (x-k)^2 + y^2 - r_2^2 \Leftrightarrow -r_1^2 = -2kx + k^2 - r_2^2$$
$$\Leftrightarrow 2kx = k^2 + r_1^2 - r_2^2 \Leftrightarrow x = \frac{k^2 + r_1^2 - r_2^2}{2k}$$

As x has a fixed value, we conclude that the points lie on a line perpendicular to the x axis.

Problems

13. Let ω and γ be two circles intersecting at P and Q. Let their common external tangent touch ω at A and γ at B. Prove that PQ passes through the midpoint M of AB.

14. (USAMO/2009) Given circles ω_1 and ω_1 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

15. (Russia/2014) A trapezoid $ABCD$ with bases AB and CD is inscribed into circle Ω. A circle ω passes through the point C and D, and intersects the segments CA and CB at $A_1 \neq C$ and $B_1 \neq D$, respectively. The points A_2 and B_2 are symmetric to A_1 and B_1 with respect to the midpoints of CA and CB, respectively. Prove that the points A, B, A_2 and B_2 are concyclic.

16. (AIME II/2019) In acute triangle ABC points P and Q are the feet of the perpendiculars from C to AB and from B to AC, respectively. Line PQ intersects the circumcircle of ΔABC in two distinct points, X and Y and. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

17. (Japan/2011) Let ABC be a given acute triangle and let M be the midpoint of BC. Draw the perpendicular HP from the orthocenter H of ABC to AM. Show that $AM \cdot PM = BM^2$.

18. (Iran TST/2011) In acute triangle ABC angle B is greater than angle C. Let M is midpoint of BC. Let D and E are the feet of the altitude from C and B, respectively. Let K and L are midpoint of ME and MD, respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

19. (IMO Shortlist/1995) ABC is a triangle. A circle through B and C meets the side AB again at C' and meets the side AC again at B'. Let H be the orthocenter of ABC and H' the orthocenter of $AB'C'$. Show that the lines BB', CC' and HH' are concurrent.
17. (Japan/2011) Let ABC be a given acute triangle and let M be the midpoint of BC. Draw

20. (IMO/2013) Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Saudi Arabia 2022 – Math Camp

Level 3

Geometry – Power of a Point and Radical Axis

Instructor: Regis Barbosa

Theorem 5. (Radical Center) Given three circles, no two concentric, the three pairwise radical axes are either concurrent or all parallel. In the last case, the three centers are collinear.

Proof. Let Γ_1, Γ_2 and Γ_3 be the three circles with centers O_1, O_2 and O_3 respectively. Let r_{12} be radical axis of Γ_1 and Γ_2 and r_{23} be the radical axis of Γ_2 and Γ_3.
If r_{12} and r_{23} are parallel, then O_1O_2 and O_2O_3 are parallel. The point O_2 is common and the three centers are collinear. The third radical axis r_{13} is perpendicular to the line through the centers and parallel to the other radical axes.
If r_{12} and r_{23} are not parallel, then they meet at point C.

$$Pot_{\Gamma_1}C = Pot_{\Gamma_2}C = Pot_{\Gamma_3}C \Rightarrow Pot_{\Gamma_1}C = Pot_{\Gamma_3}C \Rightarrow C \in r_{13}$$

Problems

21. (USAMO/1997) Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent (or parallel).

22. (IberoAmerican/1999) An acute triangle ΔABC is inscribed in a circle with center O. The altitudes of the triangle are AD, BE and CF. The line EF cut the circumference on P and Q.
a) Show that OA is perpendicular to PQ.
b) If M is the midpoint of BC, show that $AP^2 = 2 \cdot AD \cdot OM$.

23. (AIME I/2016) Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$ and $XD = 37$. Find AB^2.

24. (IMO 1995) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.

25. (IMO Shortlist/2009) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

26. (IMO Shortlist/2011) Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.