Number Theory

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Problems – April 6

1. Can all numbers greater than 10^{100} be written as the sum of a prime and a perfect square?
2. Find all positive integers n for which (a) $n(n-10)$; (b) $n^3 - n$ is a perfect square.
3. Find all positive integers n for which $n \cdot 2^n + 4$ is a perfect square.
4. Determine all positive integers n for which $1! + 2! + \cdots + n!$ is a perfect square.
5. Find all positive integers n such that the sum of digits of $n!$ is equal to 9.
6. If primes p and q satisfy $p \mid q-1$ and $q \mid p^3 - 1$, prove that $q = p^2 + p + 1$.
7. Suppose each of the positive integers a, b, c, d is divisible by $ad - bc$. How much is $|ad - bc|$?
8. If $n > 1$, prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is never an integer.
9. If n is a positive integer, prove that at least one of the numbers $n, n+1, \ldots, n+5$ is coprime with each of the remaining five numbers.
10. If a and b are positive integers and $a^2 + b^2$ is divisible by ab, prove that $a = b$.
11. If $n^2 + 1$ has a divisor $d > n$, where $n > 1$, prove that in fact $d > n + \sqrt{n}$.
12. Find all pairs (n, d) of positive integers such that d is a divisor of n and $n^2 + d^2$ is divisible by $nd + 1$.
13. Find all pairs (a, b) of positive rational numbers with $a < b$ such that $a^b = b^a$.