Email training, N10

November 17 - 26, 2019

Problem 10.1. Let a set of integers a, b, c during the one minute changes to the set of integers $a+b-c, b+c-a, c+a-b$. Originally we have the set $2000, 2002, 2003$. Is it possible that after some time we get the set $2001, 2002, 2003$.

Solution 10.1. Note that

$$(a+b-c)+(b+c-a)+(c+a-b)=a+b+c,$$

which means that the total sum of numbers doesn't change, ie its INVARIANT. In the beginning we have the sum equal to $2000+2002+2003=6005$ and at the end we expect it to be equal $2001+2002+2003=6006$ which is impossible.

Answer: Not possible.

Problem 10.2. Let 4 corner cells of the board 8×8 are coloured black and other 60 cells are coloured white. At each step one allowed to choose a row or column and color it (it means change colours of all cells in that row or column). Is it possible after some steps get a board where all cells are white?

Solution 10.2. Consider the up-left square of size 2×2. It contains 3 white and 1 black cells. At each step either we change nothing in that square either we change exactly 2 cells which means that the number of white cells is always odd, so it can't be equal 4. This means that even in top-left square of size 2×2 we can't have everything white, so we can't have the entire board white as well.

Alternative way is to correspond -1 to white cells and $+1$ to black cells. Then in top-left square of size 2×2 we have product equal $(-1)^3 \cdot (+1) = -1$. Each step we either change nothing in that square, either change two signs, which means we multiply two numbers by -1, so the total product will not change, since $(-1)^2 = +1$.

Problem 10.3. Countries Alistan and Babastan have their own currencies ali and baba. In Alistan 1 ali can be exchanged to 10 baba, however in Babastan 1 baba can be exchanged to 10 ali. Omar may travel between two countries and originally has 1 ali. Prove that his amount of alis will never be equal to the amount of babas (only integer amounts are available).

Solution 10.3. Let at some moment we have A ali and B baba and we change x ali's to baba. Then we will have $A-x$ ali and $B+10x$ baba. Note that

$$B+10x-(A-x)=(B-A)+11x,$$

which means that value of baba-ali in terms of mod 11 doesn't change. The same is true when changing from baba to ali. So the difference of baba and ali is invariant mod 11. In the beginning we have it equal to $0-1=-1 \equiv 10[11]$. The condition that ali=baba means that the difference must be equal 0, which is NOT POSSIBLE.

Problem 10.4. There is a pile containing 1001 coins. At each step one allowed to choose a pile containing more than 2 coins, throw out one coin from the pile and divide the pile into two non-empty piles. Is it possible that after some steps we achieve a situation that all piles have exactly 3 coins?

Solution 10.4. Note, that at each step we lose one coin and get one more pile, which allows to state that the amount COINS + PILES is invariant. In the beginning this amount is equal to $1 + 1001 = 1002$. Assume we get the desired configuration. Then we have x piles which of them containing $3x$ coins. So COINS + PILES $= 3x + x = 4x$ which must be equal 1002. But 1002 isn't divisible by 4, so we can't get desired configuration. NOT POSSIBLE.

Problem 10.5. There are 13 gray, 15 red and 17 blue chameleons living on the island. Whenever two chameleons of different colors meet, they change their color to the third one (if gray and blue meet then both of them become red). Is it possible that at some moment all chameleons on the island have the same color?

Solution 10.5. Consider chameleons of any two colors, for example gray and red. Consider the value GRAY - RED. Note, that at each step either both of them decrease by 1 (in this case their difference doesn't change), either one of them decreases by 1 and another one increases by 2 (then the difference will change by 3). So we may state that GRAY-RED is invariant mod 3. In the beginning we have this value equal to $13 - 15 \equiv 1[3]$. If we assume that all chameleons have the same color then we should have GRAY-RED $= 0 - 0 \equiv 0[3]$ or $0 - 45 \equiv 0[3]$. So NOT POSSIBLE.

Problem 10.6. Positive integers 1 and 2 are written on the board in laboratory. Every morning professor Ali erases the written numbers from the board and writes their arithmetical and harmonic means. It occurs that

a) At some moment $\frac{941664}{665857}$ was written on the board. Determine either it is written as arithmetical or harmonic mean and determine the other number written on the board.

b Determine if some moment the number $\frac{35}{24}$ can be written on the board.

Solution 10.6. a) Assume at some moment we have number a and b on the board. Then next day we will have numbers

$$\frac{a+b}{2} \quad \text{and} \quad \frac{2}{1/a + 1/b} = \frac{2ab}{a+b}.$$

Note that

$$\frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab,$$

which means that the product of numbers is invariant, so the product is equal $1 \cdot 2 = 2$. If at some moment one number is equal to $\frac{941664}{665857}$ then the another number is $2 : \frac{941664}{665857} = \frac{665857}{470832}$.

b) We have

$$MIN < HARMONIC\ MEAN < ARITHMETICALMEAN < MAXIMUM,$$

so the biggest number on the board decreases and the minimal number increases. Note, that after second day we have arithmetical means equal to $\frac{17}{12}$ which is less than $\frac{35}{24}$. So after second day all number written on the board will be always less than $\frac{35}{24}$. Not possible.

Problem 10.7. The rectangle is covered by bricks 2×2 and 1×4. Haider looses on peace 2×2. Then he replaces it by a peace 1×4. Prove, that he can't cover the original rectangle.

Solution 10.7. Color the rectangle in 4 colors by the following pattern

$$\begin{array}{c} 1\,0\,1\,0\,1\,0\,1\,0\,\ldots \\ 0\,0\,0\,0\,0\,0\,0\,0\,\ldots \\ 1\,0\,1\,0\,1\,0\,1\,0\,\ldots \\ 0\,0\,0\,0\,0\,0\,0\,0\,\ldots \\ 1\,0\,1\,0\,1\,0\,1\,0\,\ldots \\ 0\,0\,0\,0\,0\,0\,0\,0\,\ldots \\ \ldots\ldots\ldots\ldots \end{array}$$

Assume that we have X bricks of size 2×2 and Y bricks of size 1×4. Note that every brick of size 2×2 covers exactly one cell of color 1 and every brick of size 1×4 covers either 0 or 2 cells of color 1. It allows to state the number of 1's has the same parity as X. After changing one brick Haider has $X - 1$ bricks of size 2×2 and $Y + 1$ bricks of size 1×4. Whatever he does, he can cover cells of color 1 of parity $X - 1$. Since the number of 1's has parity of X, it means we can't cover all cells colored in 1. NOT POSSIBLE.

Problem 10.8. Let a convex 10-gon is given, where no any three diagonals intersect at one point. At each vertex of the 10-gon and at each intersection point of two diagonals the numbers $+1$ are written. At each step one may choose any diagonal and any side and change change all signs on that diagonal/side. Is it possible that after some steps all numbers are equal -1.

Solution 10.8. Consider 3 diagonals, such that any 2 of them intersect inside the polygon (for example A_1A_5, A_2A_6 and A_3A_8). Consider their intersection points X, Y and Z. On that points we have $+1$'s. At each step we either change non of them, either we change exactly 2 of them. So we may state, that the product of numbers written on these points doesn't change. Originally this product is equal $+1$, so we can't have all of them equal -1, since the product can't be equal -1. Since we can't get these three numbers equal -1, then we can't make equal -1 all numbers. NOT POSSIBLE.

Problem 10.9. Let several numbers are written on the board (not all of them equal 0). At each step one may choose two numbers a and b and replace them by $a - \frac{b}{2}$ and $b + \frac{a}{2}$. Prove that one can't achieve a situation when all numbers are equal 0.

Solution 10.9. Note that

$$\left(a - \frac{b}{2}\right)^2 + \left(b + \frac{a}{2}\right)^2 = 1.25(a^2 + b^2) \geq a^2 + b^2,$$

which means the sum of squares of numbers written on the board increases(or stays the same). Originally this value is positive, so it will never become 0. This means we will always have some number which is not equal 0. NOT POSSIBLE.

Problem 10.10. Let non-regular polygon is inscribed to the circle. Each step one may choose a vertex A which divides the arc between two neighbour to the A vertices into non equal parts and move the point A to the midpoint of that arc. Is it possible that after 100 moves one gets a polygon which is equal to the original one?

Solution 10.10. This problem is similar to the previous problem. Consider the sum of the arc length squares. Since for $a \neq b$ one has

$$a^2 + b^2 > \left(\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2,$$

then the sum of squares is strictly decreasing, so after 10 steps it can't be equal to the original value.