Test Algebra:

Find all $f: \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ s.t.

Remi of Induction

$$f(f(n)) + f(n) = 2n$$

[20]

Solution

$f(n) = n$ is a solution.

$\boxed{n=1}$ $\underbrace{f(f(1))}_{>0} + \underbrace{f(1)}_{>0} = 2 \leadsto \boxed{f(1)=1}$

Induction

Assume $f(n) = n$ for any $n < k$.

we will prove $\boxed{f(k)=k}$.

Take $n := k$

$$f(f(k)) + f(k) = 2k.$$

- Suppose $\boxed{f(k) < k}$ then

$$f(f(k)) = f(k)$$

$$f(k) + f(k) = 2k \leadsto f(k) = k \quad ↯$$

- Suppose $f(k) > k$.

$f(f(k)) = 2k - f(k) < 2k - k = k$

$f(f(f(k))) = f(f(k))$

$f(f(f(k))) + f(f(k)) = 2f(k)$

$2f(f(k)) = 2f(k) \rightsquigarrow$ $k > f(f(k)) = f(k) > k$

contradiction !

$\boxed{f(k)} = k$.

Induction complete. !

$\boxed{23}$ $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers ≥ 1

Suppose $|x_i - x_{i+1}| < 1$ for any $i \in \{1, \ldots, n-1\}$

$$|x_1 - x_2| < 1$$
$$|x_2 - x_3| < 1$$
$$\vdots$$
$$|x_{n-1} - x_n| < 1$$

Prove that:

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} < 2n-1$$

Do it by induction!

$\boxed{\text{How to start?}}$

$\boxed{n=2}$ $|x_1 - x_2| < 1.$

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} < 2 \cdot 2 - 1 = 3$$

$$x_1^2 + x_2^2 < 3x_1x_2 \quad \Big| - 2x_1x_2$$

$$(x_1 - x_2)^2 < x_1 x_2 \quad \rightarrow \text{true because}$$

$$(x_1-x_2)^2=(|x_1-x_2|)^2<\boxed{1<a_1a_2}$$ ✓

Suppose for n, we are looking $n+1$.

Have:
$$\overbrace{\frac{x_1}{x_2}+\frac{x_2}{x_3}+\dots\frac{x_{n-1}}{x_n}}^{M}+\frac{x_n}{x_1}<2n-1$$

WANT:
$$\underbrace{\frac{x_1}{x_2}+\frac{x_2}{x_3}+\dots+\frac{x_{n-1}}{x_n}}_{M}+\frac{x_n}{x_{n+1}}+\frac{x_{n+1}}{x_1}<2(n+1)-1$$

$\downarrow$

. . . .

Have: $M+\frac{x_n}{x_1}<2n-1$

WANT: $M+\frac{x_n}{x_{n+1}}+\frac{x_{n+1}}{x_1}<2(n+1)-1$

$\swarrow$

$$M+\frac{x_n}{x_{n+1}}+\frac{x_{n+1}}{x_1}<(2n-1)-\frac{x_n}{x_1}+\frac{x_n}{x_{n+1}}+\frac{x_{n+1}}{x_1}\overset{?}{<}2(n+1)-1$$

$$\cancel{(2n-1)} - \frac{x_n}{x_1} + \frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_1} < \cancel{2(n+1)} - 1$$

Enough
$$\frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_1} - \frac{x_n}{x_1} < 2$$

$$x_1 x_n + x_{n+1}^2 - x_{n+1} \cdot x_n < 2 x_1 x_{n+1}$$

$$x_n\left(x_1 - x_{n+1}\right) + x_{n+1}\left(x_{n+1} - x_1\right) < x_1 x_{n+1}$$

Enough:
$$\frac{\left(x_n - x_{n+1}\right)\left(x_1 - x_{n+1}\right)}{x_1 x_{n+1}} < 1$$

$$\frac{\left(x_n - x_{n+1}\right)\left(x_1 - x_{n+1}\right)}{x_1 x_{n+1}} \le \left|\frac{\left(x_n - x_{n+1}\right)\left(x_1 - x_{n+1}\right)}{x_1 x_{n+1}}\right|$$

$$\le \frac{\left|x_1 - x_{n+1}\right|}{x_1 x_{n+1}} = \left|\frac{1}{x_{n+1}} - \frac{1}{x_1}\right| <$$

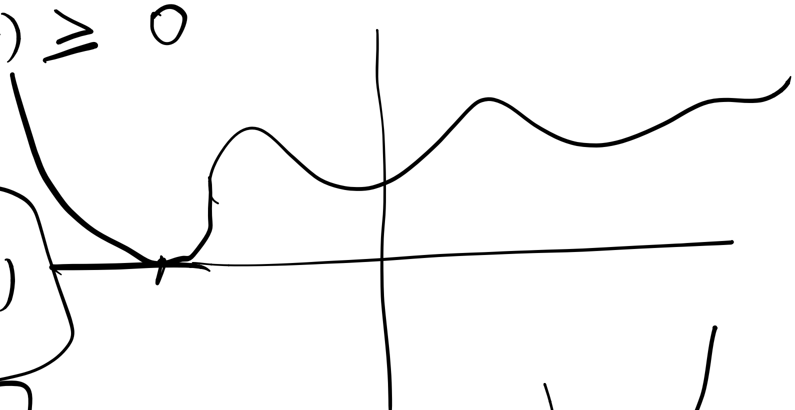
$$< \max\left(\frac{1}{x_{n+1}}, \frac{1}{x_1}\right) \le 1$$

$\boxed{24}$

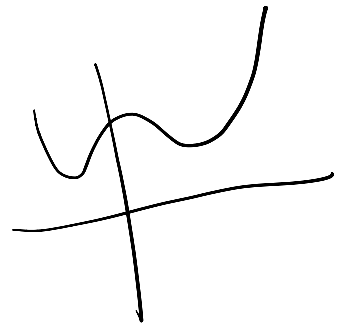
$P(x) \geq 0$

$\parallel$

$A^2(x) + B^2(x)$



$\boxed{P = A^2 + B^2}$



x^2+1

x^2+1^2

even

$P(x) = (x-x_1)^{\alpha_1}(x-x_2)^{\alpha_2} \cdots (x-x_k)^{\alpha_k}(x^2+p_1x+q_1)$

$(x^2+p_2x+q_2)\cdots(x^2+p_\ell x+q_\ell)$

real roots

$\Delta < 0$

$p_1^2 - 4q_1 < 0$

$p_2^2 - 4q_2 < 0$

$\vdots$

$p_\ell^2 - 4q_\ell < 0$

x^2+1

$\boxed{\Delta = 0-4 = -4 < 0}$

the sum

The sae trick as in $\boxed{116}$ $\Longrightarrow$ α_i are even.

$$P(x) = (x-x_1)^{2\alpha_1}(x-x_2)^{2\alpha_2}\cdots(x-x_k)^{2\alpha_k}(x^2+p_1x+q_1)$$
$$\cdot (x^2+p_2x+q_2)\cdots(x^2+p_\ell x+q_\ell)$$

$$\boxed{C^2 = C^2 + 0^2}$$

$$C^2(A^2+B^2) = (AC)^2 + (BC)^2$$

If 1 prove

$$P_1(x) = A_1^2(x) + B_1^2(x)$$
$$P_2(x) = A_2^2(x) + B_2^2(x)$$
$$\Downarrow$$

P_1P_2 is also sum of two square ?

$$(A^2+B^2)(C^2+D^2) = (AC-BD)^2 + (AD+BC)^2$$

It is enough to prove that

$x^2 + px + q$ with $\Delta < 0$

$p^2 - 4q < 0$

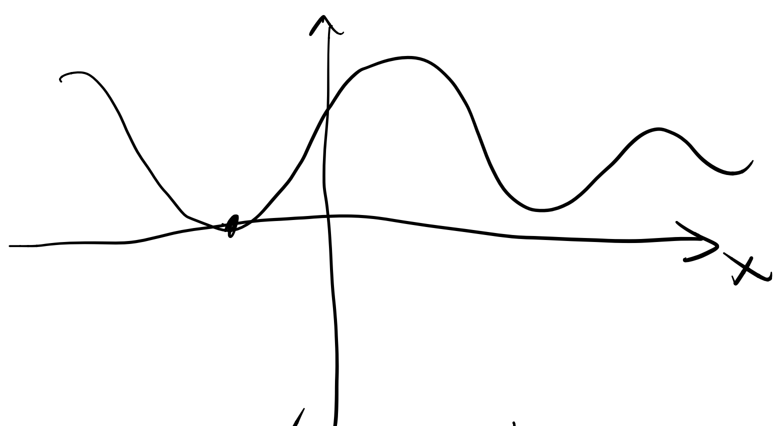
is sum of two squares.

$$(x-e)^2 + r = \boxed{(x-e)^2 + (\sqrt{r})^2} \qquad \square$$

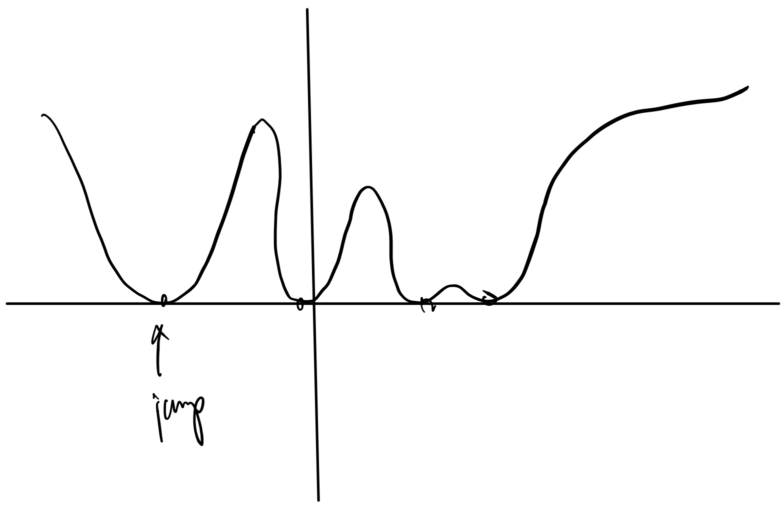
$$e = -\frac{p}{2}$$

$$r = -\frac{\Delta}{4} > 0$$

1) if $P(x) \geq 0$ for any x



multiplicity of any real root is even.



2) $(A^2+B^2)(C^2+D^2)$ = sum of two squares!

3) Don't forget about canonical expression of quadratic function

$$f(x) = ax^2 + bx + c$$
$$\shortparallel$$
$$a(x-p)^2 + q$$

where

$$p = -\frac{b}{2a}$$

$$q = \frac{-\Delta}{4a}$$

$\Delta = b^2 - 4ac$ – discriminant.

At home

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$$P(x) = U_1(x)^2 + \dots + U_k(x)^2$$

$$\Downarrow$$

$$P(x)^2 = V_1(x)^4 + \dots + V_m(x)^4$$

the same idea! revision

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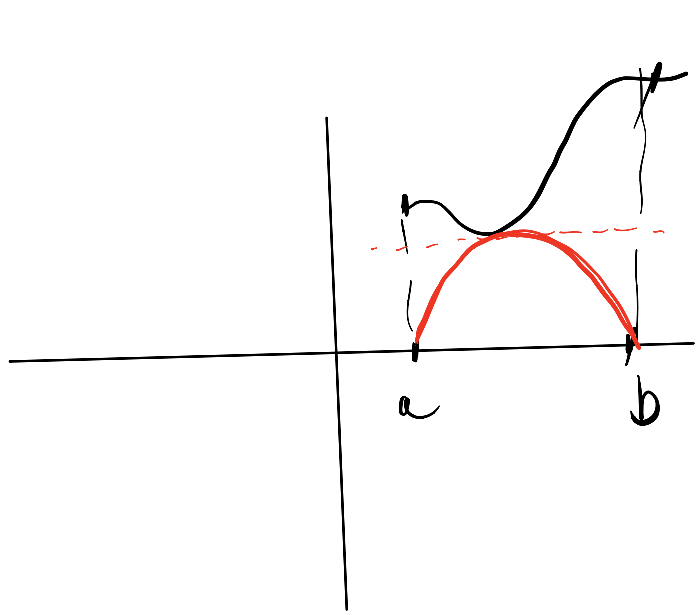

$$p(x) = A^2(x) + (x-a)(b-x)\cdot\left(\sum \text{square}\right)$$

$$\frac{P(x)}{(x-a)(b-x)} \geq c \quad \text{for some } c > 0$$

a_n - sequen in $\mathbb{N}_{>0}$

$a_{n+1} = f(a_n)$

$$\boxed{a_{n+1} + a_n = 2n}$$

$a_2 + a_1 = 2$

$a_3 + a_2 = 4$

$a_4 + a_3 = 6$

$\vdots$

$a_{n+1} + a_n = 2n$

$a_2 = 2 - a_1$

$a_3 = 4 - 2 + a_1 = 2 + a_1$

$a_4 = 6 - (4 - 2 + a_1)$

$= 4 - a_1$

$a_5 = 8 - (4 - a_1) =$

$= 4 + a_1$

$$\underset{}{a_1}, \quad \underset{\substack{\| \\ a_2}}{2 - a_1}, \quad \underset{\substack{\| \\ a_3}}{2 + a_1}, \quad \underset{\substack{\| \\ a_4}}{4 - a_1}, \quad \underset{\substack{\| \\ a_5}}{4 + a_1} \quad \ldots$$

$a_n =$

$a_{n+1} + a_n = 2n$

$\boxed{b_n = a_n - n}$

$b_{n+1} + \cancel{n} + 1 + b_n + \cancel{n} = 2n$

$b_{n+1} + b_n + 1 = 0$

$b_2 + b_1 + 1 = 0$

$b_3 + b_2 + 1 = 0$ . . .

$\vdots$

$a_n =$