Training Session in Mathematics February-March 2022

Level 4

Nikola Petrović

Lesson 1

Tables and Arrays

Problems:

1. An even number of people are seated around a table. After a break, they are again seated around the same table, not necessarily in the same places. Prove that at least two persons have the same of number persons between them as before the break.

2. Let M be a finite set of lattice points and n be a positive integer. A *mine-avoiding path* is a path of lattice points with length n, beginning at $(0,0)$ and ending at a point on the line $x+y=n$, that does not contain any point in M. Prove that if there exists a mine-avoiding path, then there exist at least $2^{n-|M|}$ mine-avoiding paths.

3. Let $mn+1$ students of differing heights be arranged in a row. Show that we can pick either $m+1$ students out of their row whose heights form an increasing sequence, or $n+1$ students whose heights form a decreasing sequence.

4. There are n boys and n girls in a school class, where n is a positive integer. The heights of all the children in this class are distinct. Every girl determines the number of boys that are taller than her, subtracts the number of girls that are taller than her, and writes the result on a piece of paper. Every boy determines the number of girls that are shorter than him, subtracts the number of boys that are shorter than him, and writes the result on a piece of paper. Prove that the numbers written down by the girls are the same as the numbers written down by the boys (up to a permutation).

5. A snake is a set of fields on a board such that they can be arranged in a sequence such that each field of the snake is to the right or above the previous field. Find the number of ways to tile an $m \times n$ board with snakes.

6. Some squares of a $n \times n$ board ($n > 2$) are black, the rest are white. In every white square we write the number of all the black squares having at least one common vertex with it. Find the maximum possible sum of all these numbers.

7. From an $n \times n$ sheet of stamps, we must tear out blocks of three adjacent stamps in a single row or column. (We can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece.) Let $b(n)$ be the smallest number of blocks we can tear out and make it impossible to tear out any more blocks. Prove that there are real constants c and d such that
$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$
for all $n > 0$.

8. Let a, b, c be fixed positive integers. There are $a+b+c$ ducks sitting in a circle, one behind the other. Each duck picks either rock, paper, or scissors, with a ducks picking rock, b ducks picking paper, and c ducks picking scissors. A move consists of an operation of one of the following three forms:

   - If a duck picking rock sits behind a duck picking scissors, they switch places.
   - If a duck picking paper sits behind a duck picking rock, they switch places.

- If a duck picking scissors sits behind a duck picking paper, they switch places.

Determine, in terms of a, b, and c, the maximum number of moves which could take place, over all possible initial configurations.

9. Let n be an odd integer. Consider a square lattice of size $n \times n$, consisting of n^2 unit squares and $2n(n+1)$ edges. All edges are painted in red or blue so that the number of red edges does not exceed n^2. Prove that there is a cell that has at least three blue edges.

10. Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is reachable from a string A if it is possible to insert some amount of $\leftarrow$'s in A, such that typing the resulting characters produces B. So, our example shows that "faecdb" is reachable from "abcdef".

    Prove that for any two strings A and B, A is reachable from B if and only if B is reachable from A.

11. Let n be a positive integer. Players A and B are given a circular necklace with $3n$ blue beads and $3n$ green beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

    - Player A must remove three consecutive beads which are blue, green and blue, in that order, on each of his turns.
    - Player B must remove three consecutive beads which are green, blue and green, in that order, on each of his turns. They win if all the beads are removed in $2n$ turns.

    Prove that if they can win with player A going first, they can also win with player B going first.

12. Ana and Banana are playing a game. First Ana picks a word, which is defined to be a nonempty sequence of capital English letters. (The word does not need to be a valid English word.) Then Banana picks a nonnegative integer k and challenges Ana to supply a word with exactly k subsequences which are equal to Ana's word. The subsequences of a string of length n are the 2^n strings which are formed by deleting some of its characters, possibly all or none, while preserving the order of the remaining characters. Ana wins if she is able to supply such a word, otherwise she loses.

    For example, if Ana picks the word "TST", and Banana chooses $k = 4$, then Ana can supply the word "TSTST" which has 4 subsequences which are equal to Ana's word.

    Which words can Ana pick so that she wins no matter what value of k Banana chooses?

13. Let n be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X , there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

14. A traveler visits a conference of wise men on an island solely inhabited by liars and truth-tellers. Liars always lie, while truth-tellers always tell the truth. At this conference, n wise men are seated at a round table with seats labeled from 1 to n clockwise (from above). The traveler goes around and interviews the wise men. For each i, $i = 1, 2, \ldots, n$, the wise man at the i-th seat told the traveler how the wise men seated in the next i seats in the clockwise direction were all liars.

    (a) For which values of n was it possible for the wise men to give out these statements.

    (b) Among the solutions to the previous part, for which values of n will the reported still be unable to determine with certainty for each wise man whether they are a liar or a truth-teller.

    (c) Let $n = 2022$. Identify the seats where truth-tellers sit.

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Lesson 2

Combinatorial geometry

Problems:

1. There are 100 circles of radius one in the plane. A triangle formed by the centres of any three given circles has area at most 2022. Prove that there is a line intersecting at least three of the circles.

2. Does there exist a set S of 100 points in a plane such that the center of mass of any 10 points in S is also a point in S?

3. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

4. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and $60°$ rotational symmetry (that is, there is a point O such that a $60°$ rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2 + 3n + 1$ citizens at $3n^2 + 3n + 1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

5. Consider a rectangle R partitioned into 2016 smaller rectangles such that the sides of each smaller rectangle is parallel to one of the sides of the original rectangle. Call the corners of each rectangle a vertex. For any segment joining two vertices, call it basic if no other vertex lie on it. (The segments must be part of the partitioning.) Find the maximum/minimum possible number of basic segments over all possible partitions of R.

6. To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon $\mathcal{P}_8$, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is greater than $\frac{1}{3}$, for all $n \geq 6$.

7. There are 2015 distinct circles in a plane, with radius 1. Prove that you can either select a subset of 27 circles such that each pair of circles within the subset intersects each other or that you can select a subset of 27 circles such that no two circles within the subset intersect each other.

8. A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that:

   - No three points in P lie on a line, and
   - No two points in P lie on a line through the origin.

A triangle with vertices in P is fat if O is strictly inside the triangle. Find the maximum number of fat triangles.

9. A collection of n squares on the plane is called tri-connected if the following criteria are satisfied:

   - All the squares are congruent.
   - If two squares have a point P in common, then P is a vertex of each of the squares.
   - Each square touches exactly three other squares.

   How many positive integers n are there with $2018 \leq n \leq 3018$, such that there exists a collection of n squares that is tri-connected?

10. There is a grid of equilateral triangles with a distance 1 between any two neighboring grid points. An equilateral triangle with side length n lies on the grid so that all of its vertices are grid points, and all of its sides match the grid. Now, let us decompose this equilateral triangle into n^2 smaller triangles (not necessarily equilateral triangles) so that the vertices of all these smaller triangles are all grid points, and all these small triangles have equal areas. Prove that there are at least n equilateral triangles among these smaller triangles.

11. There are 2019 points given in the plane. A child wants to draw k (closed) discs in such a manner, that for any two distinct points there exists a disc that contains exactly one of these two points. What is the minimal k, such that for any initial configuration of points it is possible to draw k discs with the above property?

12. Let n be a positive integer. There are given n lines such that no two are parallel and no three meet at a single point.

    (a) Prove that there exists a line such that the number of intersection points of these n lines on both of its sides is at least $\left\lfloor \frac{(n-1)(n-2)}{10} \right\rfloor$. Notice that the points on the line are not counted.

    (b) Find all n for which there exists a configurations where the equality is achieved.

13. In the coordinate plane are finitely many walls; which are disjoint line segments, none of which are parallel to either axis. A bulldozer starts at an arbitrary point and moves in the $+x$ direction. Every time it hits a wall, it turns at a right angle to its path, away from the wall, and continues moving. (Thus the bulldozer always moves parallel to the axes.)

    Prove that it is impossible for the bulldozer to hit both sides of every wall.

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Lesson 3

Generating functions

Problems:

1. Find the generating function for the following sequences:
   (a) $a_n = \lfloor \frac{n+2}{2} \rfloor$
   (b) $a_n = 3^n + n$
   (c) $a_n = \binom{n}{k}$ as a function of k

2. Prove that the number of ways of representing n as a sum of distinct natural numbers is equal to the number of ways of representing n as a sum of odd numbers.

3. Let p be an odd prime number. Use generating functions to find the number of subsets of $\{1, 2, \ldots, p\}$ whose sum of elements is divisible to p.

4. Catalan numbers are numbers c_n satisfying $c_0 = 1$ and for $n > 0$, $c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}$. Find the generating function for the Catalan numbers and use it to find the formula for them as a function of n.

5. Show that each of the following numbers satisfies the recurrence relation for Catalan numbers $c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}$:
   (a) The number of binary trees with $n+1$ leaves. (A binary tree, inductively defined, consists of either a lone root point, called a leaf, or a root point connected to two roots of two, smaller, binary trees.)
   (b) The number of ordered trees with $n+1$ vertices. (An ordered tree, inductively defined, consists of either a lone root point, or a root point connected to several, i.e. one or more, root points of ordered trees which are arranged in a specific order. For example, if there are three branches from the root, and one further branch, the cases where the extra branch is on the left, middle and right branch of the root will be regarded as three different ordered trees.)
   (c) The number of triangulations of an $(n+2)$-gon.
   (d) The number of ways to tile a stair-step shape of size n ($n, n-1, n-2, \ldots, 1$ fields respectively in each row starting from the same vertical line) with n rectangles.
   (e) The number of ways to pair $2n$ points on the circle into n chords so that no two chords intersect.
   (f) The number of ways to decompose the set of vertices of a regular n-gon into subsets whose convex hulls do not intersect each other.

6. Let n be positive integer and fix $2n$ distinct points on a circle. Determine the number of ways to connect the points with n arrows (oriented line segments) such that all of the following conditions hold: each of the $2n$ points is a startpoint or endpoint of an arrow; no two arrows intersect; and there are no two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that A, B, C and D appear in clockwise order around the circle (not necessarily consecutively).

7. Find the generating function of the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and use it to find the closed formula (not depending on other terms) for F_n using the roots of $x^2 + x - 1 = 0$.

8. For any positive n prove that the number of natural numbers whose digits are from the set $\{1, 3, 4\}$ and whose sum of digits is $2n$ is a perfect square. For example for $n = 3$ there are $9 = 3^2$ such numbers: 111111, 1113, 1131, 1311, 3111, 33, 114, 141, 411. (Hint: $x^4 + x^3 + x - 1 = (x^2 + 1)(x^2 + x - 1)$.

9. A sequence is defined by $a_1 = 1$, and $a_{n+1} = 2a_n + n$. Using a generating function, find the formula for a_n.

10. Find the sum of the number of non-negative integer solutions of each of the following equations: $x + 2y = n$, $2x + 3y = n - 1, \ldots$, $nx + (n+1)y = 0$.

11. Let $m, n \geq 2$ be positive integers, and let $a_1, a_2, \ldots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \ldots, e_n$, not all zero, with $|\, e_i \,| < m$ for all i, such that $e_1 a_1 + e_2 a_2 + \ldots + e_n a_n$ is a multiple of m^n.

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Lesson 4

Set Theory

Problems:

1. You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestants' guesses before writing down your own. For each question, after all guesses are submitted, the host announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, since you hacked the scoring system. After announcing the correct answer, the host proceeds to read the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

2. Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the symmetric difference of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

3. Let A be a finite set of positive numbers , $B = \{\frac{a+b}{c+d} | a, b, c, d \in A\}$. Show that: $|B| \geq 2|A|^2 - 1$, where $|X|$ be the number of elements of the finite set X.

4. Suppose $A_1, A_2, \cdots, A_n \subseteq \{1, 2, \cdots, m\}$ and $|A_i| = 2, i = 1, 2, \cdots, n$, satisfying that $A_i + A_j$, $1 \leq i \leq j \leq n$, are distinct from each other, where $A+B = \{a + b | a \in A,\ b \in B\}$. Determine the maximal value of n.

5. Bonus problem: A rectangle R is partitioned into smaller rectangles whose sides are parallel with the sides of R. Let B be the set of all boundary points of all the rectangles in the partition, including the boundary of R. Let S be the set of all segments whose points belong to B. Let a maximal segment be a segment in S which is not a proper subset of any other segment in S. Let an intersection point be a point in which 4 rectangles of the partition meet. Let m be the number od maximal segments, i the number od intersection points and r the number of rectangles. Prove that $m + i = r + 3$.