Lecture 3

Winter Camp

Number Theory

4 Jan 2021

Level 2

Find all integers x, y such that $x^3(y+1) + y^3(x+1) = 19$.

WLOG assume that $x \geq y$

Case 1: $x, y > 0$

$$x^3(y+1) + y^3(x+1)$$

$$\left.\begin{array}{l} x^3(y+1) \geq y^3(y+1) \\ y^3(x+1) \geq y^3(y+1) \end{array}\right\} \Rightarrow 19 \geq 2y^3(y+1)$$

$$\Rightarrow 9 \geq y^3(y+1)$$

$$\Rightarrow 2 > y$$

$$\Rightarrow \boxed{y=1} \Rightarrow x=2$$

$(x,y) = (1,2), (2,1)$

Case 2: $x=0$ or $y=0$

if $x=0 \Rightarrow y^3 = 19 \Rightarrow \Leftarrow !!$ No solutions

Find all integers x, y such that $x^3(y+1) + y^3(x+1) = 19$.

<u>Case 2</u>: $x, y < 0$

Let $x = -a$, $y = -b$ $\Rightarrow$ $a, b > 0$ and $b > a$

$$a^3(b-1) + b^3(a-1) = 19$$

$$\Rightarrow \quad 2a^3(a-1) \leq 19$$

$$\Rightarrow \quad a^3(a-1) \leq 9$$

$$\Rightarrow \quad a \leq 2 \Rightarrow a = 1, 2$$

When $a = 2$, $b(b^2+8) = 27$ which has no solutions

When $a = 1$, $b = 20$

$$\Rightarrow \quad (x, y) = (-20, -1), (-1, -20)$$

Find all integers x, y such that $x^3(y+1) + y^3(x+1) = 19$.

Case 3: $x > 0 > y$

$$\left.\begin{array}{l} x^3(y+1) \leq 0 \\ y^3(x+1) < 0 \end{array}\right\} \Rightarrow \quad 19 < 0 \quad \Rightarrow\Leftarrow !!$$

$$(x,y) = \{(1,2), (2,1), (-20,-1), (-1,-20)\}$$

Find all ordered pairs of positive integers (m, n) such that :
$125 * 2^n - 3^m = 271$

$-3^m \equiv 1 \pmod 5 \Rightarrow 3^m \equiv -1 \pmod 5 \Rightarrow m \equiv 2 \pmod 4$

* if $n \geq 4$:

$-3^m \equiv -1 \pmod{16} \Rightarrow 3^m \equiv 1 \pmod{16}$

$\Rightarrow \mathrm{ord}_{16} 3 \mid m \Rightarrow 4 \mid m$

but $m \equiv 2 \pmod 4 \Rightarrow \Leftarrow !!$

* if $n = 1 \longrightarrow$ no solutions

$n = 2 \longrightarrow$ no solutions

$n = 3 \longrightarrow n = 6$

or $-2^n \equiv 1 \pmod 9$

$\Rightarrow n \equiv 3 \pmod 6 \Rightarrow n = 3$

$(n, m) = (3, 6)$

Find all pairs of positive integers (x, y) such that $2^x + 3^y$ is a perfect square. IBMO SL 2017

Assume that $2^x + 3^y = z^2$

* (mod 3):

$2^x \equiv z^2 \pmod 3$, $3 \nmid z$ $\Rightarrow$ $2^x \equiv 1 \pmod 3$ $\Rightarrow$ $\boxed{x \text{ is even}}$

$\Rightarrow x \geq 2$

* (mod 4)

$3^y \equiv z^2 \pmod 4$, $2 \nmid z$ $\Rightarrow$ $3^y \equiv 1 \pmod 4$ $\Rightarrow$ $\boxed{y \text{ is even}}$

$y = 2y_1$ $\Rightarrow$ $2^x = z^2 - 3^{2y_1}$

$= (z - 3^{y_1})(z + 3^{y_1})$

$\gcd(z - 3^{y_1}, z + 3^{y_1}) = \gcd(z - 3^{y_1}, 2z) = \gcd(z - 3^{y_1}, 2) = 2$

since $z - 3^{y_1} < z + 3^{y_1}$ and both are powers of 2

the $z - 3^{y_1} = 2$, $z + 3^{y_1} = 2^{x-1}$

Find all pairs of positive integers (x, y) such that $2^x + 3^y$ is a perfect square.

Notes about the GCD trick

$$2^a = (a-b)(a+b)$$

if $\gcd(a,b)=1$

$$\gcd(a-b,\ a+b) = \gcd(a-b,\ 2a)$$
$$= \gcd(a-b,\ 2) \boxed{= 2}$$

$$a-b = 2^{\alpha} = 2$$
$$a+b = 2^{\beta}$$

$$3^{2y_1} = \frac{(z-2^{x_1})}{1} \frac{(z+2^{x_1})}{3^{2y_1}}$$

Find all pairs of positive integers (x, y) such that $2^x + 3^y$ is a perfect square.

$$\begin{cases} z - 3^{y_1} = 2 \\ z + 3^{y_1} = 2^{x-1} \end{cases} \Rightarrow 3^{y_1} = \frac{2^{x-1} - 2}{2} = 2^{x-2} - 1$$

if $x \geq 5$: $3^{y_1} \equiv -1 \pmod 8$

but $3^y \equiv 1, 3 \pmod 8 \quad \forall\ y \in \mathbb{Z}$ $\Rightarrow \Leftarrow$

for $x = 1, 2, \rightarrow 4$ we get

$$(x, y) = (2, 3)$$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

① when $t=0$

$$5^t + 4^y = z^2$$
$$5^t = z^2 - 2^{2y}$$
$$5^t = (z-2^y)(z+2^y) \Rightarrow z \text{ odd} \Rightarrow \gcd(z-2^y, z+2^y) = 1$$

$$\Rightarrow \begin{cases} z - 2^y = 1 \\ z + 2^y = 5^t \end{cases}$$

$$\Rightarrow z = 2^y + 1 \Rightarrow 2^y + 2^y + 1 = 5^t$$

$$\Rightarrow \boxed{2^{y+1} + 1 = 5^t}$$

if $t=0 \Rightarrow 2^{y+1} = 0 \Rightarrow \Leftarrow !!$

if $t>0$:

$$2^{y+1} = 5^t - 1$$
$$v_2(5^t - 1) = y+1$$
$$y+1 = v_2(5-1) + v_2(5+1) + v_2(t) - 1 \Rightarrow y+1 = \underline{\underline{v_2(t)}} + 2$$
$$\Rightarrow y = v_2(t) + 1 \Rightarrow \boxed{2^{y-1} \mid t}$$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

$$2^{y-1} \mid t \Rightarrow t \geq 2^{y-1}$$

$$2^{y+1} + 1 = 5^t > 2^{2t} \Rightarrow 2^{y+1} + 1 > 2^{2t}$$

$$\Rightarrow 2^{y+1} \geq 2^{2t} \Rightarrow y+1 \geq 2t$$

$$\Rightarrow t \geq 2^{2t-2}$$

$$\Rightarrow t \geq 2^{2(t-1)} \Rightarrow t \leq 1$$

$$\Rightarrow t = 1$$

$$\Rightarrow y = 1$$

$$\Rightarrow z = 3$$

$$(x, y, z, t) = (0, 1, 3, 1)$$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

Assume that $x > 0$

② t is even

Since $x > 0$

$5^t + 0 \equiv z^2 \pmod 3$, $3 \nmid z$

$\Rightarrow 5^t \equiv 1 \pmod 3$

$\Rightarrow (-1)^t \equiv 1 \pmod 3 \Rightarrow t$ is even, $t = 2t'$

③ When $y = 0$

$5^{2t'} + 3^x = z^2$

$3^x = (z - 5^{t'})(z + 5^{t'}) \Rightarrow z$ even

$\gcd(z - 5^{t'}, z + 5^{t'}) = 1$

$$\Rightarrow \begin{cases} z - 5^{t'} = 1 \\ z + 5^{t'} = 3^x \end{cases} \Rightarrow z = 5^{t'} + 1 \Rightarrow 2 \cdot 5^{t'} + 1 = 3^x$$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

$$2 \cdot 5^{t'} + 1 = 3^x$$

$$3^x \equiv 1 \pmod 5 \Rightarrow x \equiv 0 \pmod 4 \Rightarrow 4 \mid x$$

$$x = 4x'$$

$$2 \cdot 5^{t'} = 3^{4x'} - 1 = (3^{2x'}+1)(3^{2x'}-1)$$

$$2 \parallel \text{L.H.S}, \quad 4 \mid \text{R.H.S.} \quad \Rightarrow \Leftarrow !!$$

Now assume that $y > 0$ too

(4) When $t = 0$

$$1 + 3^x 4^y = z^2 \Rightarrow 3^x \cdot 4^y = (z-1)(z+1)$$

Since $\gcd(z-1, z+1) = 2$ we get

	case 1	case 2	case 3	case 4
$z-1 =$	2	$2^{2y-1} \cdot 3^x$	$2 \cdot 3^x$	2^{2y-1}
$z+1 =$	$\underbrace{2^{2y-1} \cdot 3^x}_{\text{مستحيلة}}$	$\underbrace{2}_{\text{مستحيلة}}$	$\underline{\underline{2^{2y-1}}}$	$\underline{\underline{2 \cdot 3^x}}$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

Case 3: $\begin{cases} z-1 = 2\cdot 3^x \\ z+1 = 2^{2y-1} \end{cases}$ $\Rightarrow$ $2\cdot 3^x + 2 = 2^{2y-1}$

$\Rightarrow$ $3^x + 1 = 2^{2(y-1)}$ (note that $y > 1$)

$\Rightarrow 3^x = 2^{2(y-1)} - 1 \Rightarrow 3^x = (2^{y-1}-1)(2^{y-1}+1)$

but $\gcd(2^{y-1}-1, 2^{y-1}+1) = 1$

$\Rightarrow \begin{cases} 2^{y-1}-1 = 1 \\ 2^{y-1}+1 = 3^x \end{cases} \Rightarrow \begin{matrix} y = 2 \\ x = 1 \end{matrix}$ $(x, y, z, t) = (1, 2, 7, 0)$

Case 4: $\begin{cases} z-1 = 2^{2y-1} \\ z+1 = 2\cdot 3^x \end{cases}$ $\Rightarrow$ $2^{2y-1} + 2 = 2\cdot 3^x \Rightarrow 2^{2y-2} + 1 = 3^x$

$\Rightarrow 2^{2(y-1)} + 1 = 3^x$

$1 \equiv (-1)^x \pmod 4 \Rightarrow x$ is even, $x = 2x'$

$\Rightarrow 1 = 3^{2x'} - 2^{2(y-1)} \Rightarrow 1 = (3^{x'} - 2^{(y-1)})(3^{x'} + 2^{(y-1)})$

but $3^{x'} + 2^{(y-1)} > 1$ $\Rightarrow \Leftarrow$!!

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

(5) When $x, y, t > 0$

$$5^{2t'} + 3^x 4^y = z^2$$

$$3^x 4^y = (z - 5^{t'})(z + 5^{t'})$$

Since $\gcd(z - 5^{t'}, z + 5^{t'}) = 2$

$$\begin{cases} z - 5^{t'} = 2, & 2 \cdot 3^x, & 2^{2y-1} \\ z + 5^{t'} = 3^x 2^{2y-1}, & 2^{2y-1}, & 3^x \end{cases}$$

<u>Case 1</u>: $5^{t_1} + 1 = 3^x \cdot 2^{2(y-1)}$

<u>Case 2</u>: $5^{t'} + 3^x = 2^{2(y-1)}$

<u>Case 3</u>: $5^{t'} + 2^{2(y-1)} = 3^x$

Check if there exists positive integers a, b and prime number p such that $a^3 - b^3 = 4p^2$