Email training, N12

December 8-17, 2019

Problem 12.1. Find all pairs (x, y) of non-negative integers such that $x^2 + 3y$ and $y^2 + 3x$ are simultaneously perfect squares.

Solution 12.1. Without lose of generality one may assume that $y \geq x$. If $y = 0$, then it's obvious that $x = 3k^2$ for some non-negative integer k. Otherwise, from the inequality

$$x^2 < x^2 + 3y < x^2 + 4x + 4 = (x+2)^2$$

follows that $x^2 + 3y = (x+1)^2$. Hence $3y = 2x + 1$. Then $x = 3k + 1$ and $y = 2k + 1$ for some integer $k \geq 0$ and so $y^2 + 3x = 4k^2 + 13k + 4$. If $k > 5$ then

$$(2k+3)^2 < 4k^2 + 13k + 4 < (2k+4)^2,$$

so y^2+3x cannot be a square. By checking cases $k = 0, 1, 2, 3, 4, 5$ one concludes that the only solutiona are $x = y = 1$ (case $k = 0$) and $x = 16, y = 11$ (case $k = 5$).

Answer: $(0, 3k^2)$, $(3k^2, 0)$, $(11, 16)$, $(16, 11)$ and $(1, 1)$.

Problem 12.2. Prove that for any sequence of positive integers $a_1, a_2, \ldots, a_n$ there exists a positive integer k, for which

$$s(ka_1) = s(ka_2) \ldots = s(ka_n),$$

where $s(m)$ is the sum of digits of m.

Solution 12.2. First of all lets prove that for any positive integer b having at most p digits one has

$$s(\underbrace{99\ldots99}_{p \text{ times}} \cdot b) = 9p.$$

Without lose of generality we may assume that the last digit of b isn't 0. Let $b = \overline{b_1 b_2 \ldots b_n}$. Then

$$\underbrace{99\ldots99}_{p \text{ times}} \cdot \overline{b_1 b_2 \ldots b_n} = 10^p \cdot \overline{b_1 b_2 \ldots b_n} - \overline{b_1 b_2 \ldots b_n} =$$

$$\overline{b_1 b_2 \ldots b_{n-1}(b_n - 1)\underbrace{99\ldots99}_{p-n \text{ times}}(9-b_1)(9-b_2)\ldots(9-b_{n-1})(10-b_n)}.$$

Therefore

$$s(\underbrace{99\ldots99}_{p \text{ times}} \cdot b) = 9p.$$

Now, lets choose p enough big, to satisfy conditions $a_i < 10^p$ for all $1 \leq i \leq n$. Then the number $k = 10^p - 1$ satisfies to the condition of the problem.

Problem 12.3. Find all real solutions (a, b, c, d) to the equations

$$\begin{cases} a + b + c = d, \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d}. \end{cases}$$

Solution 12.3. Take a and b arbitrary. Then one has: $-c + d = a + b$ and $cd = -ab$. So $-c$ and d are the roots of the quadratic $x^2 - (a+b)x + ab = 0$. Solving, the roots are a, b. So either $c = -a$, $d = b$, or $c = -b, d = a$.

Answer: a, b arbitrary, $c = -a, d = b$..

Problem 12.4. Find the maximal possible value of the expression

$$|...||x_1{-}x_2|{-}x_3|{-}x_4| - \ldots |{-}x_{1990}|,$$

where $x_1, x_2, ..., x_{1990}$ is the permutation of numbers $1, 2, 3, \ldots, 1990$.

Solution 12.4. Note, that $|a - b|$ and $a + b$ have the same parity, so the answer has parity of

$$1 + 2 + \ldots + 1990 \equiv 1[2]$$

. So the answer is odd. Since for any $a, b \geq 0$ one has

$$|a - b| \leq max\{a, b\},$$

then we conclude that the answer is at most 1990. Since the answer is odd, we conclude that the value of expression can't be bigger than 1989. Let give example for 1989. Since

$$||||4k + 2| - (4k + 4)| - (4k + 5)| - (4k + 3)| = 0,$$

therefore

$$|...|||2 - 4| - 5| - 3| - \ldots - (4k + 2)| - (4k + 4)| - (4k + 5)| - (4k + 3)| -$$
$$\ldots - 1986| - 1988| - 1989| - 1987| - 1990| - 1| = ||0 - 1990| - 1| = 1989.$$

Problem 12.5. On birthday party several married couples came, each of them with from 1 to 10 children. The total number of triples { *father, mother, kid*}, where no two of them are from the same family is exactly 3630. Find the number of children participating to the party.

Solution 12.5. Let p married couples and d children participated to the party. Then each kid consists in exactly $(p–1)(p–2)$ triples. Therefore, the total number of triples is $d(p - 1)(p - 2)$ which is equal to 3630. Since $d \leq 10p$ it follows that $3630 \leq 10p^3$ from which follows that $p \geq 8$.

Note that $3630 = 2 \cdot 3 \cdot 5 \cdot 11^2$ has two factors $p–1$ and $p–2$ that differ by 1. If one of them is divisible by 11 then another one gives residue 1 or 10 and is divisor of $2 \cdot 3 \cdot 5$. The only possibility is $p - 2 = 10$, from which follows, that $p = 12$ and $d = 33$.

If $p - 1$ and $p - 2$ are co-prime with 11 then they are divisors of $2 \cdot 3 \cdot 5 = 30$ which isn't possible, since $p \geq 8$ and $lcm(p - 1, p - 2) = (p - 1)(p - 2) \geq 7 \cdot 6 > 30$.

Answer: 33 children.

Problem 12.6. Let 20 black and 20 white balls are placed among the line. On each step player is allowed to swap 2 neighbor balls. Find the minimal number of moves player needs to guarantee position with 20 consecutive black balls.

Solution 12.6. To guarantee 20 consecutive black balls one should have all white balls at the corners. Lets consider the leftmost and rightmost white balls. One of them is closer to the corner, so it can be moved to the corner by at most 10 moves. From the remaining 19 white balls consider leftmost and rightmost balls. One of them can be moved to the corner by at most 10 moves and so on. Therefore, after $20 \cdot 10 = 200$ moves one can guarantee all white ball be at the corners.

Now consider the following configuration: 10 black balls, then 20 white balls and after that 10 black balls. For each white ball at least 10 moves needed to move to the corner.

Answer: 200 moves.