Email training, N4

October 2-8

Problem 4.1. Solve the system of equations

$$\begin{cases} (x-1)(y-1)(z-1) = xyz - 1 \\ (x-2)(y-2)(z-2) = xyz - 2. \end{cases}$$

Solution 4.1. Substitute $x = a+1$, $y = b+1$, $z = c+1$. Then

$$(a+1)(b+1)(c+1) = abc + 1,$$
$$(a+1)(b+1)(c+1) = (a-1)(b-1)(c-1) + 2.$$

In particular

$$(a-1)(b-1)(c-1) = abc - 1.$$

These equations give

$$(ab+bc+ca) + (a+b+c) = 0$$
$$-(ab+bc+ca) + (a+b+c) = 0$$

so $a+b+c = ab+bc+ca = 0$. Thus $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 0$ and so $a = b = c = 0$, From this follows $x = y = z = 1$.

Problem 4.2. Prove that there are infinitely many positive integers m such that the number of odd distinct prime factors of $m(m+3)$ is a multiple of 3.

Solution 4.2. Let $f(m) = m(m+3)$ and $d(m)$ be the number of odd distinct prime factors of $f(m)$. We have that $f(m)f(m+1) = f(m^2+4m)$; furthermore we see that the only common prime factor of $f(m)$ and $f(m+1)$ is 2. Indeed, suppose that p is an odd prime that divides both $f(m)$ and $f(m+1)$: then p divides also $f(m+1) - f(m) = 2(m+2)$, so p divides $m+2$. But $GCD(m+2, m+3) = 1$, so p must divide both m and $m+2$: contradiction. Hence, $d(m^2+4m) = d(m) + d(m+1)$. We see that if the remainders of the division by 3 of $d(n)$ and $d(n+1)$ are distinct and both non-zero, $d(n^2+4n) = d(n) + d(n+1)$ is divisible by 3.

We want to prove that, chosen any $n > 2$, there exists an integer $m \geq n$ so that $d(m)$ is divisible by 3.

Suppose by contradiction that $d(m)$ is never a multiple of 3 for any $m \geq n$. Then, for any $m \geq n$ we see that the remainder of $d(m)$ must be equal to the remainder of $d(m+1)$, otherwise $d(m^2+4m) = d(m) + d(m+1)$ would be a multiple of 3. Hence, $d(m)$ has always the same remainder for any $m \geq n$. This, however, contradicts the fact that $d(n^2+4n) = d(n) + d(n+1) = 2d(n)$ has 2 as remainder if $d(n)$ has 1 as remainder and vice versa, so our thesis is proven.

Problem 4.3. Let $1 = d_0 < d_1 < \ldots < d_m = 4k$ be all positive divisors of $4k$, where k is a positive integer. Prove that there exists $i, 1 \leq i \leq m$ such that $d_i - d_{i-1} = 2$.

Solution 4.3. Suppose not. Let $n = 4k$ for convenience and consider the largest even d for which d and $d+2$ both divide n. (This exists as $d = 2$ works.) Then $d+1$ also divides n. If $d \equiv 0 \pmod 4$ then $2d+2$ and $2d+4$ both divide n. If $d \equiv 2 \pmod 4$ then $2d$ and $2d+2$ both divide n. In both cases, we find a larger pair of consecutive evens dividing n, contradiction.

Problem 4.4. Let k be a positive integer such that $p = 8k+5$ is a prime number. The integers $r_1, r_2, \ldots, r_{2k+1}$ are chosen so that the numbers $0, r_1^4, r_2^4, \ldots, r_{2k+1}^4$ give pairwise different remainders modulo p. Prove that the product

$$\prod_{1 \leq i < j \leq 2k+1} (r_i^4 + r_j^4)$$

is congruent to $(-1)^{k(k+1)/2}$ modulo p.

Solution 4.4. Let g be a generator modulo p; assume that $r_i \equiv g^{i-1}$ for all $1 \leq i \leq 2k+1$. Since $p \equiv 5 \pmod 8$, $\{r_1^8, \ldots, r_{2k+1}^8\}$ is also the set of nonzero eighth powers modulo p. Then the product

$$\begin{aligned}\prod_{1 \leq i < j \leq 2k+1} (r_i^4 + r_j^4) &\equiv \prod_{1 \leq i < j \leq 2k+1} (r_i^8 - r_j^8) \div \prod_{1 \leq i < j \leq 2k+1} (r_i^4 - r_j^4) \\ &\equiv \prod_{0 \leq i < j \leq 2k} (g^{8i} - g^{8j}) \div \prod_{0 \leq i < j \leq 2k} (g^{4i} - g^{4j}) \\ &\equiv \prod_{0 \leq i < j \leq 2k} (h^{2i} - h^{2j}) \div \prod_{0 \leq i < j \leq 2k} (h^{i} - h^{j})\end{aligned}$$

where $h = g^4$. This is equal to the sign of the permutation $x \to 2x$ $(0, 2, \ldots, 2k, 1, 3, \ldots, 2k-1)$ on $\mathbb{Z}/(2k+1)\mathbb{Z}$. This permutation has $\frac{k(k+1)}{2}$ inversions, which solves the problem.

Problem 4.5. Given n, find a finite set S consisting of natural numbers larger than n, with the property that, for any $k \geq n$, the $k \times k$ square can be tiled by a family of $s_i \times s_i$ squares, where $s_i \in S$.

Solution 4.5. Each of the sets $S = \{s \in Z | n \leq s \leq n^2\}$, and $S = \{s \in Z | s \text{ is prime}, n^2 < s < 2n^2 + n\}$ is convenient. We will prove this by induction on the size of the square, for the second set S. Consider a $k \times k$ square, where $k \geq n^2$. By hypothesis, the $m \times m$ square is tiled by squares from S, for any m with $n \leq m < k$. If k is composite, then write $k = pq$, where $k > p \geq n$. We cover the $k \times k$ square by means of q^2 numbers of $p \times p$ squares and we proceed inductively. If k is prime, then $k > 2n^2 + n$. We divide the square into two squares, one $m \times m$ and the other $(k-m) \times (k-m)$, and two $m \times (k-m)$ rectangles, where $m = n(n+1)$. We have $k - m > n^2$ and thus each $m \times (k-m)$ rectangle can be covered with $m \times n$ or $m \times (n+1)$ pieces and each of these pieces can be further divided into squares.

Problem 4.6. Ann and Max play a game on a 100×100 board. First, Ann writes an integer from 1 to 10000 in each square of the board so that each number is used exactly once. Then Max chooses a square in the leftmost column and places a token on this square. He makes a number of moves in order to reach the rightmost column. In each move the token is moved to a square adjacent by side or by vertex. For each visited square (including the starting one) Max pays Ann the number of coins equal to the number written in that square. Max wants to pay as little as possible, whereas Ann wants to write the numbers in such a way to maximise the amount she will receive. How much money will Max pay Ann if both players follow their best strategies?

Solution 4.6. The answer is 500,000.

At least 500,000. Ann writes numbers as follows:

$$\begin{bmatrix} 1 & 200 & 201 & 400 & \dots & 9801 & 10000 \\ 2 & 199 & 202 & 399 & \dots & 9802 & 9999 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 99 & 102 & 299 & 302 & \dots & 9899 & 9902 \\ 100 & 101 & 300 & 301 & \dots & 9900 & 9901 \end{bmatrix}$$

Split the board into 50 100×2 blocks; it is clear that Max must pay at least $200(2a-1)$ coins in the a^{th} block.

At most 500,000. The sum of all the numbers is 50,005,000. Hence there is a 2×100 block with sum at most 1,000,100.

Let the top and bottom rows be $a_1, \dots, a_{100}$ and $b_1, \dots, b_{100}$ in order. Then Max can guarantee $\sum \min(a_k, b_k)$. Since all 200 numbers are distinct, this pays at most

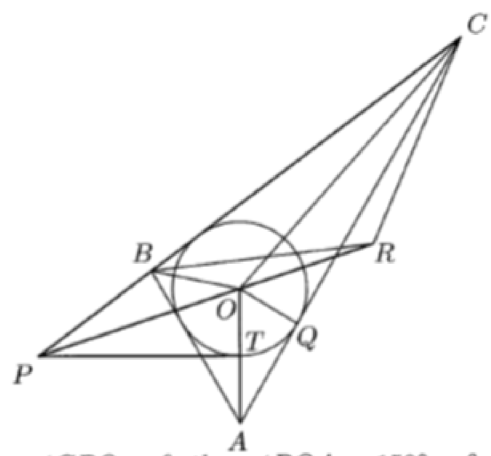
$$\sum \min(a_k, b_k) \leq \sum \frac{a_k + b_k - 1}{2} \leq \frac{1{,}000{,}100 - 100}{2} = 500{,}000$$

coins.

Problem 4.7. A circle with center O is inscribed in an angle. Let A be the reflection of O across one side of the angle. Tangents to the circle from A intersect the other side of the angle at points B and C. Prove that the circumcenter of triangle ABC lies on the bisector of the original angle.

Solution 4.7. -

Let T be the midpoint of AO and P be the vertex of the given angle. Let P be closer to B than to C. Let Q be the point of contact of the circle with AC. In triangle OQA, $\angle OQA = 90°$ and $OA = 2OQ$. Hence $\angle OAQ = 30°$. Therefore $\angle OAB = 30°$.



Let $\angle ABO = \angle CBO = \theta$, then $\angle BOA = 150° - \theta$. Since PT is perpendicular to AO, we have

$$\angle BPT + 90° + 150° - \theta + 180° - \theta = 360°.$$

Hence $\angle BPT = 2\theta - 60°$, which means that $\angle BPO = \theta - 30°$. Therefore $\angle BOP = 30°$. Since O is the incentre of triangle ABC,

$$\angle BOC = 90° + \frac{1}{2}\angle BAC = 120°.$$

Now let R be the circumcentre of triangle ABC, then

$$\angle BRC = 2\angle BAC = 120°.$$

Hence $BORC$ is a cyclic quadrilateral. In triangle BRC, $\angle BRC = 120°$ and $BR = RC$. Hence $\angle BCR = 30°$. Therefore $\angle BOR = 150°$. Thus

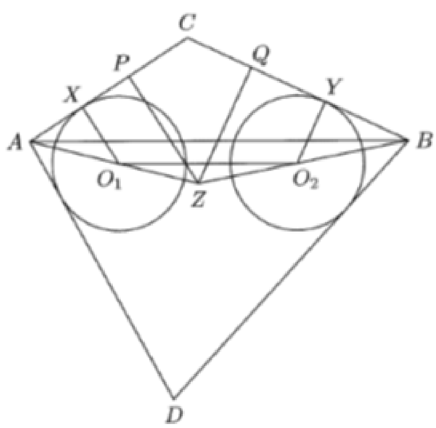
$$\angle POB + \angle BOR = 180°,$$

and therefore P, O, R lie on a straight line.

Problem 4.8. AB intersects two equal circles, is parallel to the line joining their centres, and all the points of intersection of the segment and the circles lie between A and B. From the point A tangents to the circle nearest to A are drawn, and from the point B tangents to the circle nearest to B are also drawn. It turns out that the quadrilateral formed by the four tangents extended contains both circles. Prove that a circle can be drawn so that it touches all four sides of the quadrilateral.

Solution 4.8. -

Let C and D be the other two vertices of the quadrilateral. Let X and Y be the points of contact of the two circles with AC and BC respectively. We see that O_1X is perpendicular to AC and O_2Y is perpendicular to BC where O_1 and O_2 are the centres of the corresponding circles. Let AO_1 and BO_2 extended meet in Z. Also let P and Q be the feet of the perpendiculars from Z to AC and BC respectively.



Since APZ and AXO_1 are similar triangles, we have $\frac{PZ}{XO_1} = \frac{AZ}{AO_1}$.

Since ABZ and O_1O_2Z are similar triangles, $\frac{AZ}{O_1Z} = \frac{AB}{O_1O_2}$ holds.

Hence

$$\frac{PZ}{XO_1} = \frac{AZ}{AO_1} = \left(1 - \frac{ZO_1}{AZ}\right)^{-1} = \left(1 - \frac{O_1O_2}{AB}\right)^{-1}.$$

Similary,

$$\frac{QZ}{YO_2} = \frac{BZ}{BO_2} = \left(1 - \frac{ZO_2}{BZ}\right)^{-1} = \left(1 - \frac{O_1O_2}{AB}\right)^{-1}.$$

Since $XO_1 = YO_2$, we have $PZ = QZ$ which means that CZ bisects $\angle ACB$. Thus the angle bisectors of $\angle A$, $\angle B$ and $\angle C$ in the quadrilateral $ABCD$ meet at one point. Hence there exists a circle which touches all four sides of the quadrilateral.