# A bit more just about primes

## Lesson by Senya, group L4+

---

**Problem 1.** Just an exercise: Positive integer numbers $a, b, c, d$ are such that $ab = cd$. Is it possible that $a + b + c + d$ is a prime?

> Prime numbers are mysterious and many simple-sounding facts are still open problems. However, we did manage to get a good estimate on $p_n$, i.e $n-$th prime number, namely $p_n \approx n \cdot ln(n)$. While this fact is not that easy to prove, it is rather elementary to get to a result about $p_n$ that is very close to this fact.

**Problem 2.** Let $p$ be prime and $p^a | \binom{n}{k}$. Prove that $p^a \leq n$.

**Problem 3.** Let $p_k$ be $k-th$ largest prime number and let $p = 2m + 1$

i) Prove that $\binom{m}{p} > \frac{2^p}{p}$;

ii) Prove that $\binom{m}{p} \leq p^k$;

Thus we can conclude that $p_k^{k+1} > 2^{p_k}$. Finally, prove that there is a constant $C$ such that $p_k < C \cdot k \cdot ln(k)$.

**Problem 4.** As before let $p_k$ be the $k-$th prime number. Prove that are infinitely many $k$ such that

a) $2p_k < p_{k+1} + p_{k-1}$

b) $2p_k \geq p_{k+1} + p_{k-1}$

c) $p_k^2 > p_{k-1}p_{k+1}$

d) $p_k - p_{k-1} < \sqrt{k-1}$

**Problem 5.** There is a finite set of prime numbers $P$. Prove that there is a number $x$ such that it is representable in the form $a^p + b^p$ (where $a, b$ are integers) if and only if $p \in P$.

**Problem 6.** There is a prime number $p$. Prove that there is a prime number $q$ such that $n^p - p$ is not divisible by $q$ for any positive integer $n$.

**Problem 7.** Is it possible to place positive integers into the cells of a $2019 \times 2019$ board in such a way that the ratio of any two neighbouring numbers (larger number divided by the smaller) is an integer not larger than 2019?