$\xrightarrow[\text{start}]{NT}$

$(x+1)^2+(x+2)^2+\dots+(x+2001)^2=y^2$

$\cancel{2001\left(x^2+2002x+\frac{2002\cdot 4003}{6}\right)=y^2}$

$\cancel{\geq x^2+2002x+\frac{2002\cdot 4003}{6}\geq (x+1001)^2}$

WLOG $x\equiv 0[3]$

$(x+1)^2\equiv 1[3]$, $(x+2)^2\equiv 1$, $(x+3)^2\equiv 0$ ... $(x+2001)^2\equiv 0[3]$

$\Rightarrow (x+1)^2+\dots+(x+2001)^2\equiv 2[3]$ but $y^2\equiv 0,1[3]$

$p^3-p^2=q^2+q^5+2pq$

$\Rightarrow p^2(p-1)=q^2(q+1)(p^2+q^2-pq)+2pq$

$\Rightarrow p(p^2-p-2q)=q^2(q+1)(p^2+q^2-pq)$

if $q\neq 0[3]$

$p^3\equiv 0,1,2$, $q^5\equiv 1,2$

p	q
2	1 ✓
1	1 ✓
2	2 ✓
1	2
2	1
0	1 ✓
1	0 ✓

$\Rightarrow p^3\equiv 2[3], q^5\equiv 1[3]$ or $p^3\equiv 1[3], q^5\equiv 1$ or $p^3\equiv 2[3], q^5\equiv 2[3]$, $p^3\equiv 1, q^5\equiv 2$, $p^3\equiv 0, q^5\equiv 1$

$\Downarrow$ $(p+q)^2\equiv 0$ ✗ $\Downarrow$ $(p+q)^2\equiv 1$ ✗ $\Downarrow$ $(p+q)^2\equiv 1$ ✗ $\Downarrow$ $(p+q)^2\equiv 0$ ✗ $\Downarrow$ $(p+q)^2\equiv 1$

$\Rightarrow q=3 \Rightarrow p^3-243=p^2+6p+9$,

$p(p^2-p-6)=252=2^2\cdot 7\cdot 9 \Rightarrow p=2,7,9$, $p=2\Rightarrow p^2-p-6=-4$

$p=9\Rightarrow 68=p^2-p-6$

$x^5 - y^2 = 4$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad [11]$

$x^5 \equiv 0 \quad 1, -1, 1, 1, 1, -1, -1, -1, 1, -1 \; [11] \Rightarrow x^5 \equiv 0, 1, 10$

$y^2 \equiv 0, 1, 4, 9, 3, 5 \; [11]$

$x^5 - 4 \equiv 7, 8, 6 \not\equiv y^2 \; [11]$

$p = 2x^2 - 1$

$p^2 = 2y^2 - 1$

$p(p-1) = 2(y-x)(y+x)$

$\Rightarrow p \mid x+y$ or $y-x$ but $p > y > x \Rightarrow p > y-x \Rightarrow p \nmid y-x \Rightarrow p \mid x+y$ but

$p > x, p > y \Rightarrow 2p > x+y$ so $p = x+y \Rightarrow p-1 = 2(y-x) \Rightarrow 2p-(p-1) = 2(x+y) - 2(y-x)$

$\Leftrightarrow p+1 = 4x$, $p = 2x^2 - 1 \Rightarrow 2x^2 = 4x \Leftrightarrow x = 0, 2 \Rightarrow p = -1 \times \quad p = 7 \checkmark$

$p = 2 \cdot (2)^2 - 1$, $p^2 = 2(5)^2 - 1$

8)

$$\cancel{4xy-x-y=z^2}$$

$$\cancel{(2x-1)(2y-1)=z^2-x-y}$$

$$4xy-x-y=z^2$$

$$16xy-4x-4y=(2z)^2$$

$$16xy-4x-4y+1=(2z)^2+1$$

$$(4x-1)(4y-1)=(2z)^2+1$$

$4x-1\equiv 3[4]$ then $a\mid 4x-1$ s.t $a\equiv 3[4]$ $\Rightarrow a\mid (2z)^2+1 \Rightarrow a\mid 2z, 1$

$\Rightarrow a=1$ ×

Lemma: $p=4k+3$, $a^2+b^2\equiv 0[p]$ $\overset{?}{\Rightarrow}$ $p\mid a,b$

$-a^2\equiv b^2[p]$

$a^{p-1}\equiv b^{p-1}[p]$

$(a^2)^{\frac{p-1}{2}}\equiv (b^2)^{\frac{p-1}{2}}[p]$

$(a^2)^{\frac{p-1}{2}}\equiv (-a^2)^{\frac{p-1}{2}}[p]$

$2(a^2)^{\frac{p-1}{2}}\equiv 0[p]$

$a\equiv 0[p] \Rightarrow b\equiv 0[p]$

$p\mid a,b$

غيرها

$x^2 \equiv -1 \; [p]$

$x^{p-1} \equiv 1 \; [p]$

$(x^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} [p]$

$\Rightarrow 1 \equiv (-1)^{\frac{p-1}{2}} \; [p]$

$\frac{p-1}{2}$ is even $\Rightarrow p = 4k+1$

$\frac{1}{pq} = \frac{1}{x} + \frac{1}{y}$ $\quad$ p q primes $\quad x, y \in \mathbb{Z}^+$

$pqx + pqy = xy = pq(x+y)$

$\frac{1}{x} < \frac{1}{pq} \Rightarrow x > pq$

$\frac{1}{y} < \frac{1}{pq} \Rightarrow y > pq$

$(x-pq)(y-pq) = xy \quad pq^2 + pqx + pqy$

$xy = pq(x+y)$

$xy - (x+y)pq = 0$

$pq^2 + xy - pqx - pqx = pq^2$

$(x-pq)(y-pq) = pq^2$

$x - pq = 1, p, q, p^2q^2, pq, p^2q, pq^2, p^2q^2$

$y - pq = 1, q, p, q^2, p^2, pq, q, p, 1$

$$x^2y^2 - x^2y^2 + 14xy - 49 = 0$$

$$x^2y^2 - x^2 - y^2 + 1 = 14xy - 48$$

$$x^2y^2 - 12xy + 36 = x^2 + y^2 - 2xy - 13$$

$$13 = (x+y)^2 - (xy - 6)^2$$

$$x^2(y-1) + y^2(x-1) = 1$$

$$x^2y + xy^2 - x^2 - y^2 = 1$$

$$sp - s^2 + 2p = 1$$

$$s^2 - sp - 2p + 1 = 0$$

$$(x+1)^2 y + (y+1)^2 x = 1$$

$$x^2y + 2xy + y + xy^2 + 2xy + x = 1$$

$$xy(x+y+4)\ x+y+4 = 5$$

$$\underline{(xy+1)(x+y+4) = 5}$$

$$a = x-1$$
$$b = y-1$$

$$\underline{(a+b+4)(ab+1) = 5}$$

$$a^3+b^3+c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = n$$

$$\frac{1}{2}(a-b)^2+(b-c)^2+(c-a)^2$$

$$\underbrace{(a+b+c)}_{n}\underbrace{(a^2+b^2+c^2-ab-bc-ca)}_{-m} = n \quad = 3$$

case: $3n \neq 9+n$

$\Rightarrow 3 \mid n$ or $3 \mid n^2 - 3m$

$\Downarrow$

$3^2 \mid n(n^2-3m) \quad \Rightarrow v_3(n) = 2k$

case

$n = 9k \Rightarrow (x,y,z) = (k-1, k, k+1)$

$$\underbrace{(x+y+z)}_{3k}\frac{1}{2}\underbrace{\left((x-y)^2+(y-z)^2+(z-x)^2\right)}_{b}$$

$\cancel{x^2(y^2-2y+1)+y^2}$

$\cancel{2x^2y^2-x^2y^2-x^2-xy^2+y^2=p}$

$\cancel{x+y+1 \mid x+y+1}$

$\cancel{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=p}$

$\cancel{x+y+z=1,\ x^2y^2+z^2-xy-yz-zx=p}$

case: $3k+1=n$

$k,\ k+1,\ k+1$

$(x+y+z)=3k+1$

$\gcd(x+y,\ x-y)$ 1 or 2

$\frac{1}{2}((x-y)^2+(y-z)^2+(z-x)^2)=1$

case: $3k+2=n$

$k,\ k+1,\ k+1$

Q.8

$x^4+4\cdot 1^4=(x^2+2+2x)(x^2+2-2x)=py^4$

case x odd

$2u:\ x\ \ 4(2u^2+2u+1)(2u^2+2u+1)=py^4$

x even

case 2.1 $\quad p \mid x^2+2x+2$

$x^3+y^3+z^3-3xyz=(x+y+z)\frac{1}{2}((x-y)^2+(y-z)^2+(z-x)^2)=p$

$(x+y+z)((x-y)^2+(y-z)^2+(z-x)^2)=2p$

case 1 $x+y+z=p$ and $(x-y)^2+(y-z)^2+(z-x)^2=2$

case 2 or $=2p$ and $=1$

case 2 $x=y=z \Rightarrow =0+2$

case 1

W.L.O.G $x=y$, $|y-z|=|z-x|=1$

$1=x-z=z-x$ or $x-z=x-z=1$ or -1

$x-z=z-x \Rightarrow z=x$ ✗

$x=\frac{p+1}{3}$ $1=x-z=1$ or $x-z=-1$ $\Rightarrow$ $x=\frac{p-1}{3}$

$(x,y,z)=(k,k,k-1)$ / $(x,y,z)=(k,k,k-1)$

if $p=3k-1$ | if $p=3k+1$

$3(xy+yz+zx)=4xyz$

$4 \mid xy+yz+zx \Rightarrow 2 \mid x, y, z$

$\Rightarrow x=y=z=0$

$xy+yz+zx-xyz=2$

$(1-x)(1-y)(1-z)=3-(x+y+z)=(1-x)+(1-y)+(1-z)$

$(x-1)(y-1)(z-1)=\underset{a}{(x-1)}+\underset{b}{(y-1)}+\underset{c}{(z-1)}$

$3a \geq a+b+c=abc$

if $a \neq 0$

$3 \geq bc$

b	c	a
1	2	3
2	1	3
3	1 = 2	
1	3 = 2	
1	1	x

if $a=0 \Rightarrow x=1 \Rightarrow y+z=2$

$\Rightarrow y=z=1$

$\Rightarrow (a,b,c)=(3,1,2)$ and تبديلاتها , $(1,0,0)$

$\Rightarrow (x,y,z)=(4,2,3)$ and تبديلاتها , $(2,1,1)$

Q.8

case 3 $p \mid x^2+2x+2$

$$\gcd\left(\frac{x^2+2x+2}{p}, x^2+2x+2\right)=1$$

لكن حاصل ضربهم مربع (y^4) لأن كلاهما مربع

$$x^2+2x+2=a^2$$

$$x^2+2x+1=a^2-1$$

$$\Rightarrow (x+1)^2=(a-1)(a+1)$$

$x=2z+1$

$$4(z)^2=(a-1)(a+1)$$

كلاهما مربع

$$\Rightarrow a-1=0 \quad , a+1=2$$

$$a=1 \Rightarrow x=1 \Rightarrow p=5$$

case 4 $p \mid x^2-2x+2$

$$\Rightarrow x^2-2x+2=b^2$$

$$(x+1)^2=(b-1)(b+1) \Rightarrow b=1 \Rightarrow x=-1 \Rightarrow p=5$$

$\gcd(a+b, a\cdot b) = \begin{cases} \gcd(a,b) \\ 2\gcd(a,b) \end{cases}$

$\gcd(x^2+2, 2x) = 4, 2, 1$

$3(xy+yz+zx) = 4xyz$

$\Leftrightarrow 4 \mid xy+yz+zx$

WLOG $x = 2x_1$

$4 \mid 2x_1y + yz + 2zx_1$

$\Rightarrow$ WLOG $y = 2y_1$

$\frac{3}{x} + \frac{3}{y} + \frac{3}{z} = \frac{4}{3}$

$\frac{3}{x} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{4}{3}$

$x = 1, 2$

$\Rightarrow (x,y,z) = (1,6,6), (1,4,12), (2,2,3)$

$(x+y)^2 + 3x+y+1 = z^2$

$x^2+2xy+y^2+3x+y+1=z^2$

$x^2+y^2-z^2+2xy+3x-y+1=0$

$\Rightarrow$ L.H.S. $> 4y^2+4y+1$

$z^2 > (2y+1)^2$

$\cancel{x \geq y \Rightarrow 3x^2+x = 2x^2+x^2+x > 2xy+y^2+y}$

$(x+y)^2 < (x+y)^2+3x+y+1 \leq (x+y+2)^2$

$\Rightarrow 2 \geq x+y+1 \Rightarrow$

$2xy+\cancel{x^2+y^2}+3x+y+1 = \cancel{x^2+y^2}+2xy+2x+2y+1$

$x-y=0 \Rightarrow x=y$

$x = 3k^{15}$ $\quad 3k^{15}$

$y = 2k^{10}$ $\quad 2k^{10}$

$z = 1k^{6}$ $\quad k^{6}$

$x^2 = y^3 + z^9$

$3^2 = 2^3 + 1^9$

$9k^{30} = 8k^{30} + k^{30}$

$(x^2+y^2) = z(z^4+1)$

lemma: if $p = 4k+1$ then p can be presented as x^2+y^2

$(a^2+c^2)(b^2+d^2) = (ab+cd)^2 + (ad-bc)^2$

$(z^4+1) \quad z \quad x \quad y$

z is a sum of squares?

is $x^2 \equiv -1\,[p]$?

$x = \left(\frac{p-1}{2}\right)!$

$(p-1)! \equiv -1\,[p]$ wilson

$\Rightarrow -1$

$z = p$

$p = a^2 + b^2$

$\Rightarrow p^5 + p = x^2 + y^2$

$\gcd(x, y, p) = 1$ or p if $p \Rightarrow p^2 / p^5 + p \Rightarrow p^2 / p$ ↯

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

$z = \frac{xy}{x+y}$ $\quad \gcd(x,y) = d$, $da = x$, $db = y$

$z = \frac{dab}{a+b}$

$d \mid a+b \Rightarrow (a+b)k = d$

$\Rightarrow z = kab$, $x = ka(a+b)$, $y = kb(a+b)$

2) $\cancel{k^2a^4b^2 + k^2a^4 + 2k^2a^3b + k^2a^2b^2 + k^2b^4 + 2k^2ab^3 + k^2a^3b^2}$

$\cancel{k^2(3a^4 + b^4 + a^3b + b^3a + a^2b^2 + a^2b^2)} = ((a+b)^4 - 2ab(a^2+b^2+2ab) + ab) = (a+b)^4 - 2ab(a+b)^2 + a^2b^2)$

$= k^2(a^2b^2 + a^2(a+b)^2 + b^2(a+b)^2)$

$\|$

$((a+b)^4 - 2ab(a+b)^2 + 2ab) = (a+b-ab)^2$

$x^3+y^3+z^3 = x^2+y^2+z^2$

$x^3y^2 = 13^2$

$x^3+y^3+z^3 = x^2+y^2+z^2$

$z = -y$

$\Rightarrow x^3 = x^2+2y^2$ $\qquad y = kx$

$x^3 = x^2+2k^2x^2$

$\div x^2$

$x = 2k^2+1$

$x = 2k^2+1,\ y = 2k^3+k,\ z = -2k^3-k$

$x^2+y^2 = (m^3+n^2)^2 = ((m+n)^2-mn)^2$

x	y	x^2+y^2
$(m+n)$	$(m+n)$	$2(m+n)^2$
$m(m+n)$	$n(m+n)$	$(m+n)^2(m^2+n^2)$
$m(m^2+n^2)$	$n(m^2+n^2)$	$(m^2+n^2)^3$

$x = m(m^2+n^2)^a,\ y = n(m^2+n^2)^a,\ z = 2a+1$

a) $(m^3+n^2)^2 = \underbrace{(m^3n^2)}_{c^2}{}^{2k}(m^2+n^2) = c^3n^2 \cdot n^2c^2 \Rightarrow x = (m^3+n^2)^k m,\ y = n(m^3+n^2)^k,\ z$

9 (i)

$(m^2+n^2)^{2k} = (m^2+n^2)(m^2+n^2)^{2k-1}$ lagrange

$z = 2k$, $x = (m^2+n^2)^{k-1}(m^2-n^2)$ $y = (m^2+n^2)^{k-1}(-2mn)$

$x^3 + y^4 = 7$

but $x^3 y^4 \not\equiv 7$ [13]

$x^2 - y! = 2001$

$x^2 \equiv 2001 - y!$ [7]

Check for $y \geq 7$

$2001 + y! \equiv 6$ [7] $\not\equiv x^2$

$y = 1, 2, 3, 4, 5, 6 \Rightarrow y = 4, x = 45$

$y > 2$ $3^x - 2^y = 7$

$3^x \equiv -1$ [8] but $3^x \equiv 1, 3$ [8]

$\Rightarrow 3^x = 9 \Rightarrow x = 2$

$x^3 - 4xy + y^3 = -1$

$x^3 + y^3 + 1 - 3xy - xy = 0$

$(x+y+1)((x-y)^2 + (x-1)^2 + (y-1)^2) = 2xy$

$x^3 + y^3 + 4^3 - 4xy = -63$

$(3x)^3 + (3y)^3 + 4^3 - 4 \cdot 3x \cdot 3y \cdot 3 = 37$ $\times 27$

$(3x + 3y + 4)(9x^2 + 9y^2 + 16 - 12x - 12y - 9xy) = 37$

end

NT

$3x+3y+4=1$ or $3x+3y+4=37$

$x+y=11$

$x-y$ is odd

$9x^2+9y^2+16-12x-12y-9xy=1 \Leftrightarrow (3x-3y)^2+(3x-4)^2+(3y-4)^2=2$

odd

$\Rightarrow$ x, y have same parity $\lightning$

$3x+3y+4=17$