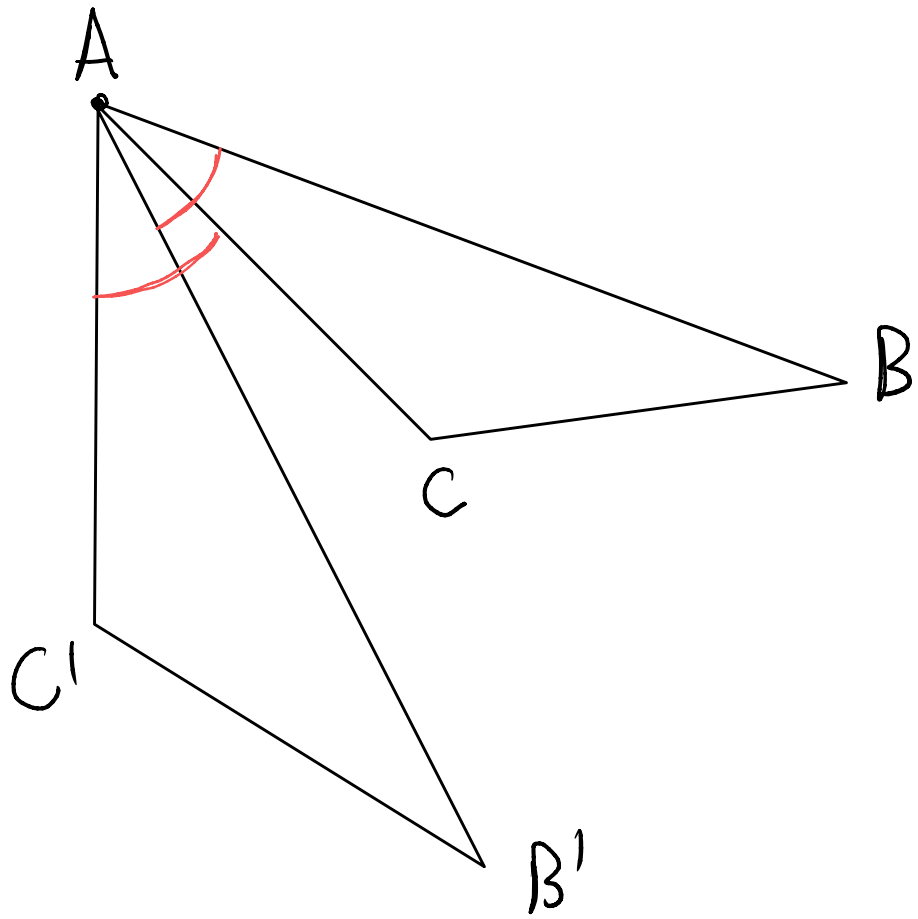
Intensive Training

Geometry

Day 8
18 April 2021

Spiral Similarity

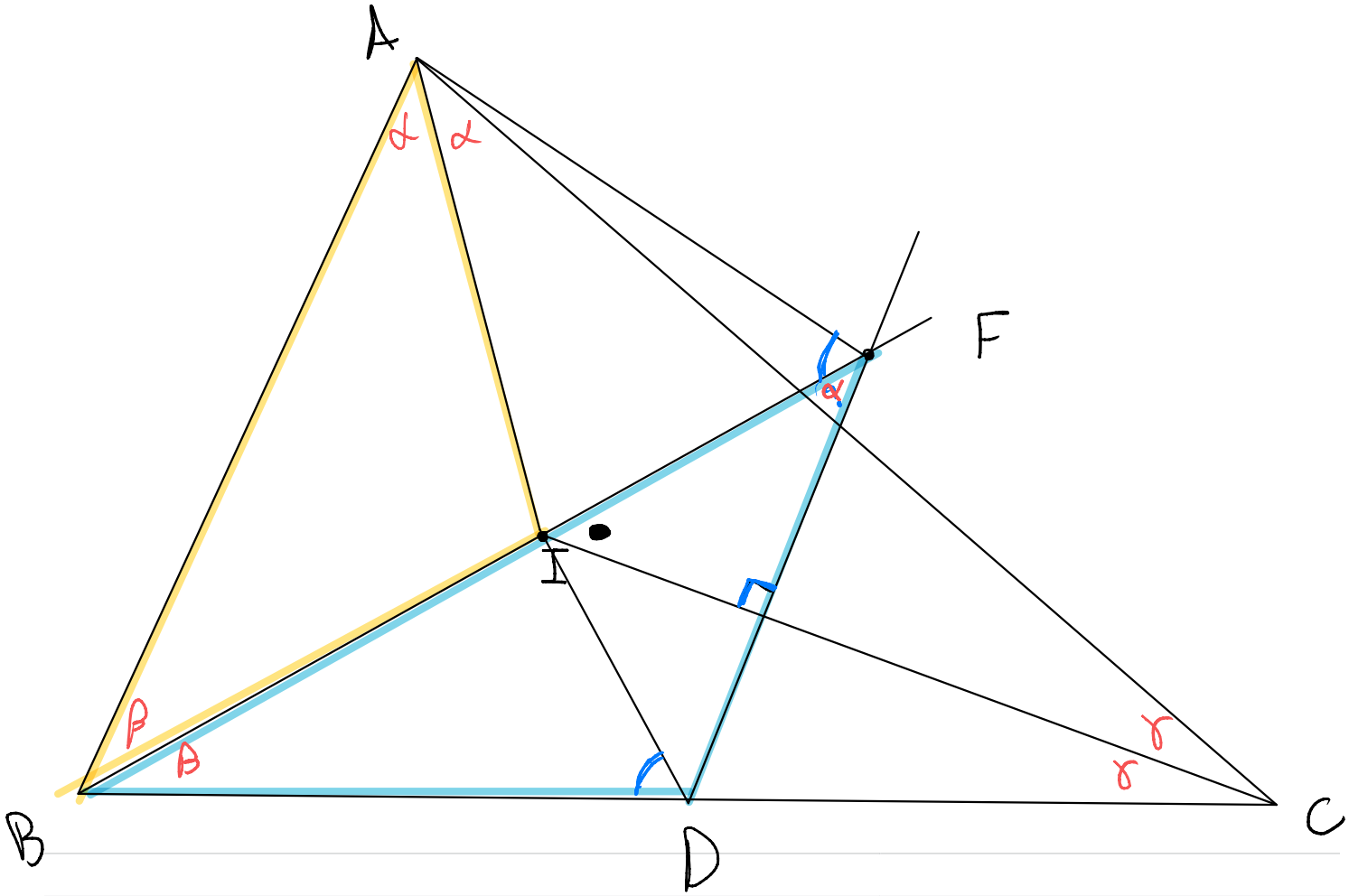
<u>Lemma:</u> If $\triangle ABC \sim \triangle AB'C'$, then $\triangle ABB' \sim \triangle ACC'$



$$\angle BAC = \angle B'AC' \implies \angle BAB' = \angle BAC + \angle CAB'$$
$$= \angle B'AC' + \angle CAB'$$
$$= \angle CAC'$$

$$\frac{BA}{CA} = \frac{B'A}{C'A}, \quad \angle BAB' = \angle CAC' \xRightarrow{S.A.S.} \triangle ABB' \sim \triangle ACC'$$

Problem 4. Let ABC be a triangle with incentre I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A in the line EF lies on the line BC.



We want to show that $\angle AFI = \angle IDB$

$\iff \triangle BAF \sim \triangle BID$

by lemma $\iff \triangle BAI \sim \triangle BFD$

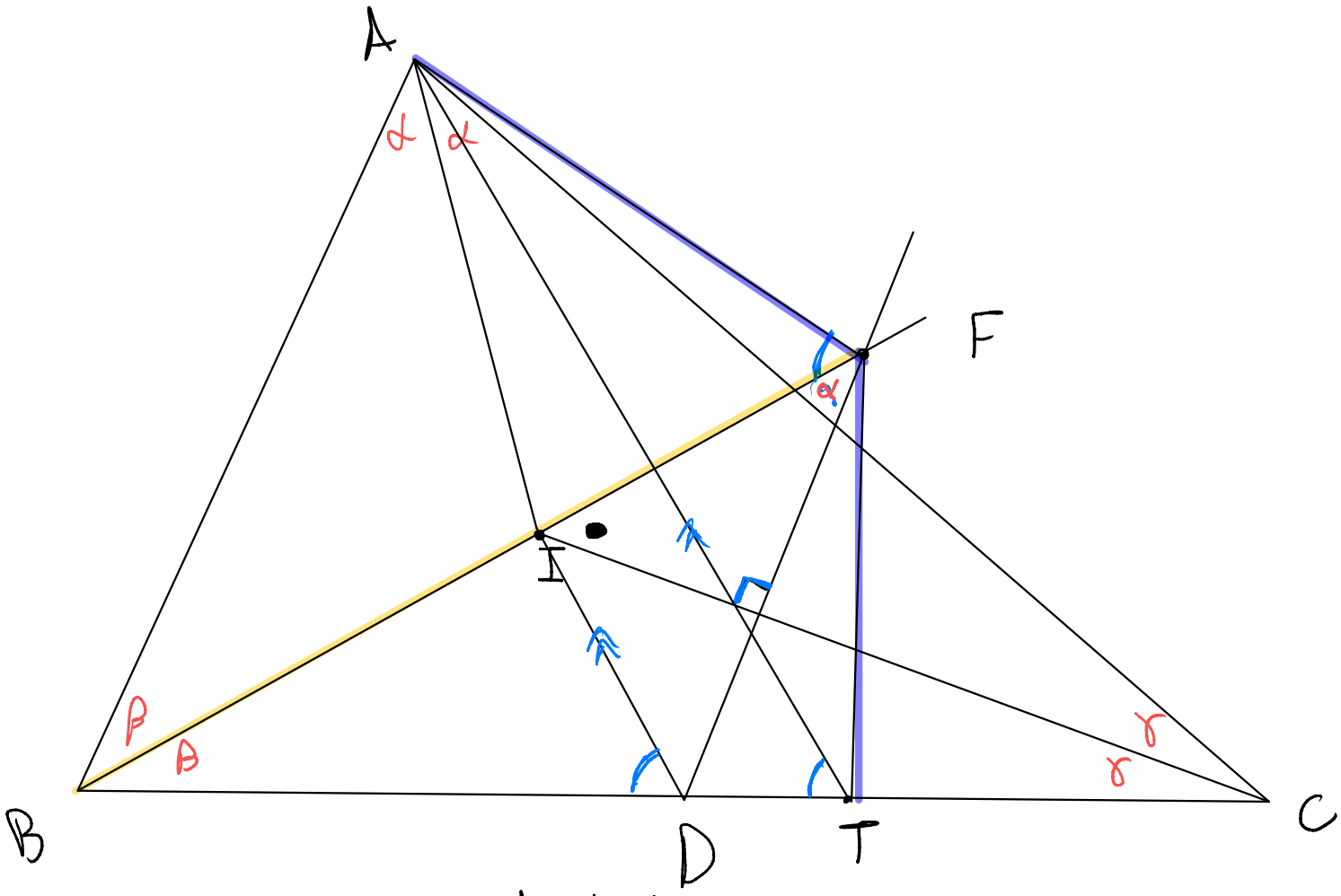
However, $\angle DFB = 90 - \angle FIC = 90 - (\beta + \gamma)$

$= \alpha = \angle BAI$

and $\angle ABI = \angle FBD = \beta$ $\Rightarrow \triangle BAI \sim \triangle BFD$

$\Rightarrow \angle AFI = \angle IDB$

Problem 4. Let ABC be a triangle with incentre I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A in the line EF lies on the line BC.



If $T \in BC$ such that $AT \parallel ID$

$AF = FT \iff ABTF$ cyclic (see lemma on 1/4)

$\iff \angle AFB = \angle ATB$

$\iff \angle ADB = \angle ATB$

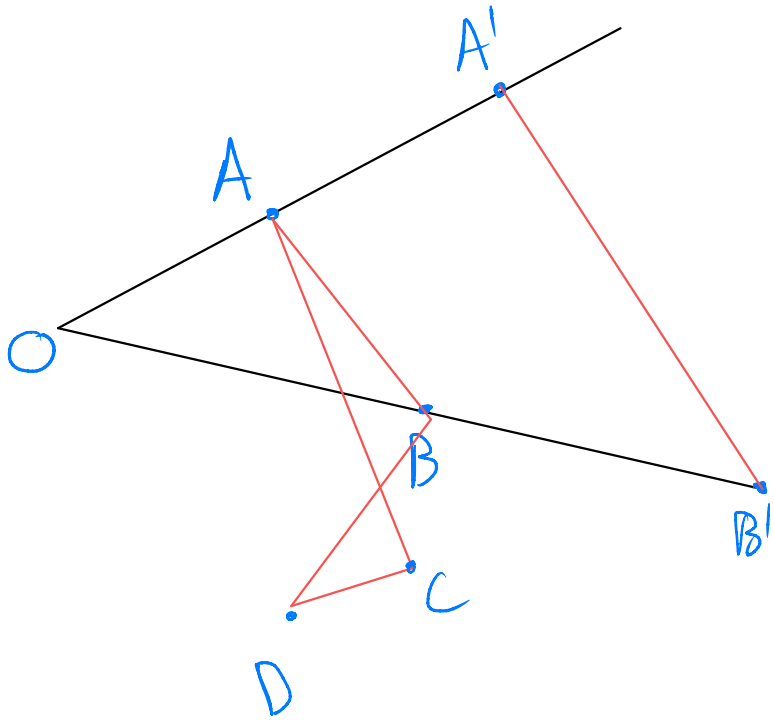
$\iff ID \parallel AT$

$\Rightarrow AF = FT$,

Similarly, $AE = ET$. $\square$

Homothety

* center of homothety : O
* scale factor : k



A' = image A , O, A, A' collinear , $\frac{OA'}{OA} = k$

B' = image B , O, B, B' collinear $\frac{OB'}{OB} = k$

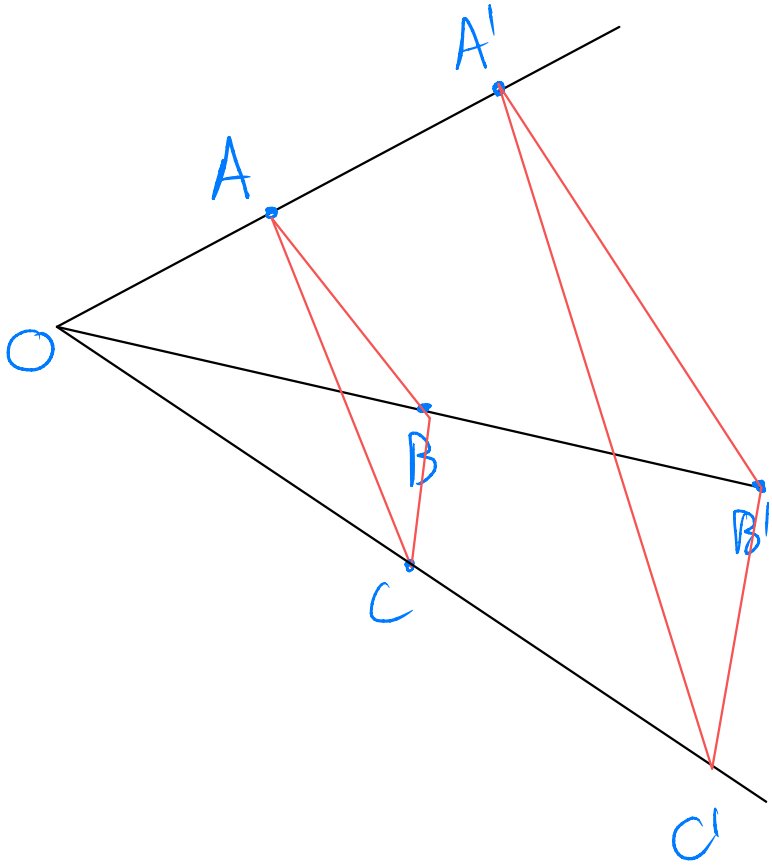
$\Rightarrow \triangle OAB \sim \triangle OA'B'$, $\frac{A'B'}{AB} = k$

$AB \parallel A'B'$

Moreover, if C' is the image of C,

$\triangle ABC \sim \triangle A'B'C'$ with ratio k

$A_1 A_2 \text{—} A_n \sim A_1' A_2' \text{—} A_n'$ with ratio k



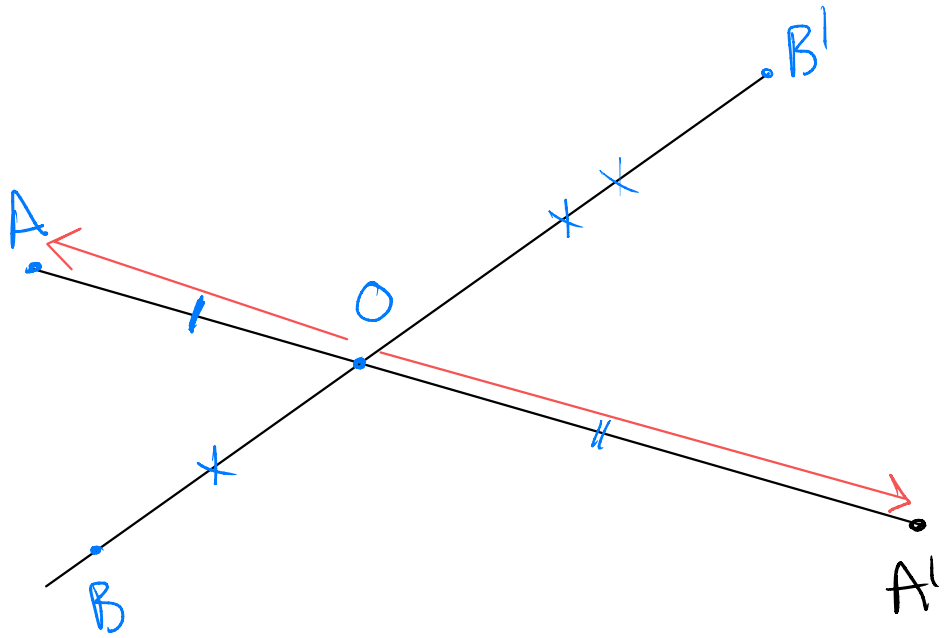
If O_1 is the center (ABC), O_2 is the center $(A'B'C')$
then O_2 is the image of O_1

Similarly I_2 is the image of I_1

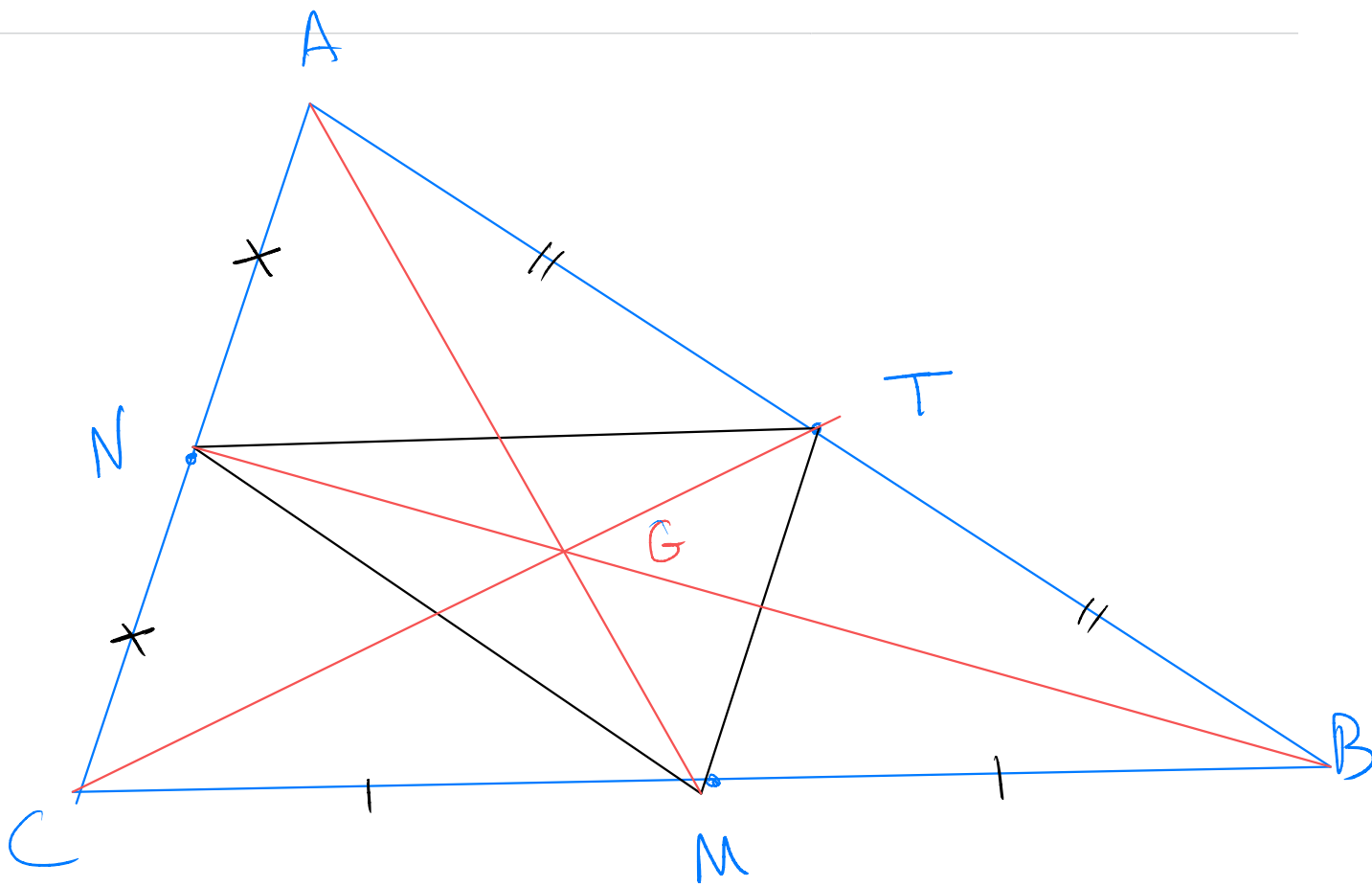
H_2 is the image of H_1

Works for any points defined in a unique way.

Example when $k = -2$



$$\frac{OA'}{OA} = -2 \Rightarrow OA' = -2\,OA$$



G is a center of homothety with ratio -2

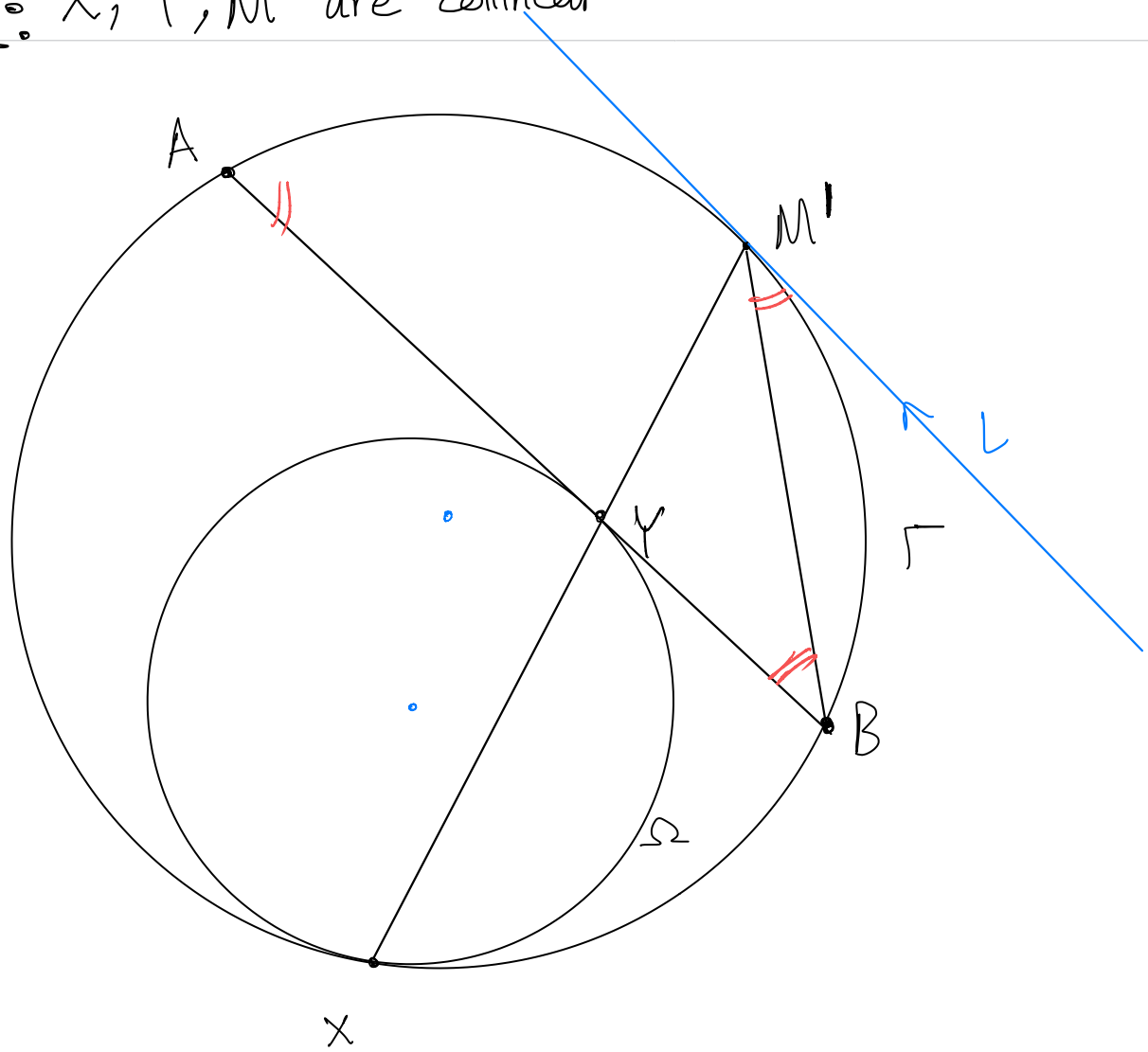
that sends M to A

T to C

N to B

$\frac{AG}{GM} = 2$

Lemma: X, Y, M are collinear



Consider, h, homothety with center X and ratio

$\frac{XO_1}{XO_2}$ or $\frac{r_1}{r_2}$ $\Longrightarrow$ h sends Ω to Γ

Let M' be the image of Y

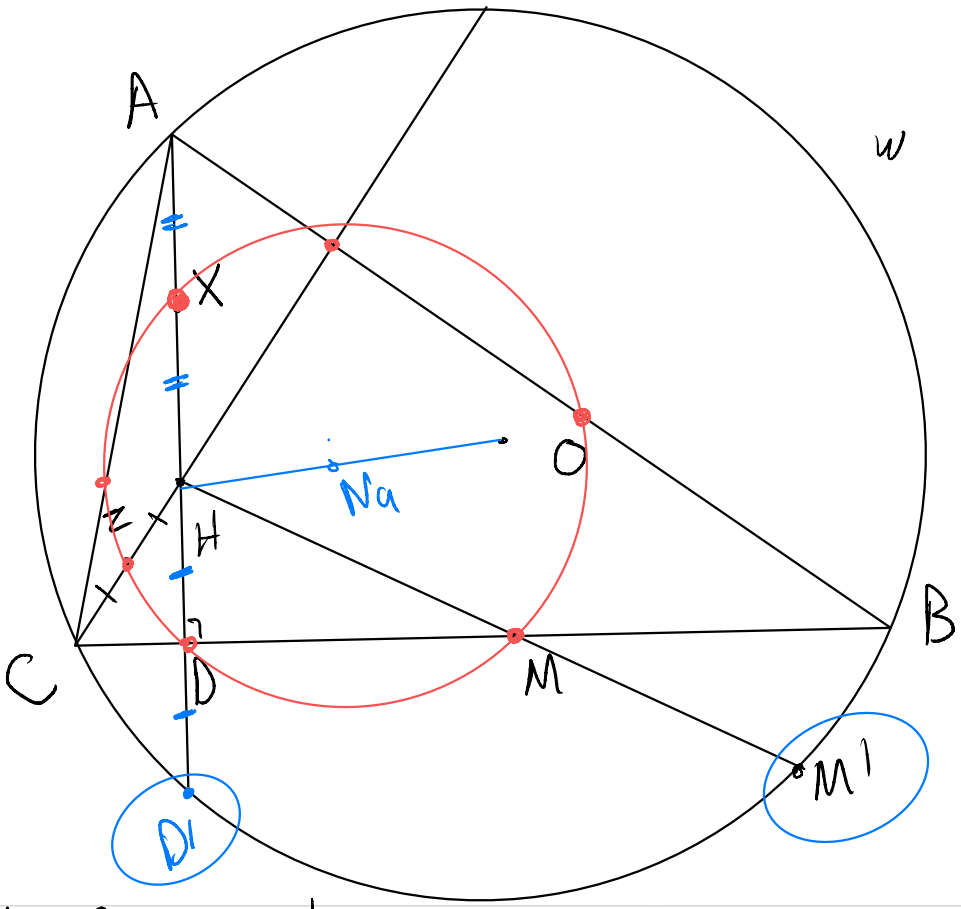
The X, Y, M' are collinear.

$Y \in \Omega \Rightarrow M' \in \Gamma$

$\left\{ \begin{array}{l} AB \text{ is the tangent to } \Omega \text{ at } Y \\ L \text{ is the tangent to } \Gamma \text{ at } M' \end{array} \right. \Rightarrow$ L is the image of AB $\Rightarrow L \parallel AB$

$\Rightarrow \angle M'AB = \angle M'BA \Rightarrow M' = M$ (midpoint of AB) □

Lemma 3.11 (Nine-Point Circle). *Let ABC be a triangle with circumcenter O and orthocenter H, and denote by N_9 the midpoint of $\overline{OH}$. Then the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, $\overline{AH}$, $\overline{BH}$, $\overline{CH}$, as well as the feet of the altitudes of $\triangle ABC$, lie on a circle centered at N_9. Moreover, the radius of this circle is half the radius of (ABC).*



Let $D' = AH \cap w$, $M' = HM \cap w$

Consider a homothety, h, with center H and ratio 2

$HD = DD' \Rightarrow h(D) = D'$, $h(E) = E'$, $h(F) = F'$

Also $XH = \frac{1}{2}AH \Rightarrow h(X) = A$, $h(Y) = B$, $h(Z) = C$

$HM = \frac{1}{2}HM' \Rightarrow h(M) = M'$, $h(N) = N'$, $h(T) = T'$

Note that $D', E', F', A, B, C, M', N', T'$ are all on (ABC)

$\Rightarrow$ by homothety $D, E, F, X, Y, Z, M, N, T$ are on one circle S with center N. $\Rightarrow$ O is the image of N

$\Rightarrow$ H, N, O are collinear, $\frac{HO}{HN} = 2 \Rightarrow N = N_9$ $\square$