Problem 15 $f:\mathbb{R}\to\mathbb{R}$

$$f(f(f(x)))\geq f(x+y)+y$$

$\forall x,y\in\mathbb{R}$

Solution

$P(x, f(f(x))-x) \Longrightarrow$

$0\geq f(f(x))-x \quad \forall x$

$\Longleftrightarrow \boxed{x\geq f(f(x)) \quad \forall x \ (*)}$

$\overset{(*)}{\Longrightarrow} \underline{f(x)}\geq f(f(\underline{f(x)}))\geq f(x+y)+y$

So $f(x) \geq f(\underbrace{x+y}_{z}) + \underbrace{y}_{z-x}$

$\forall x, y \in \mathbb{R}$

or $f(x) \geq f(y) + y - x$

$\Longleftrightarrow f(x) + x \geq f(y) + y$

$\forall x, y \in \mathbb{R}$

$\Longrightarrow f(x) + x = c \quad \forall x$

where c is constant

[Check:

$f(f(f(x))) = f(f(c-x))$

$= f(x) = c-x$

$f(x+y)+y = c-x-y+y$

$= c-x$ ✓]

Thus, $\boxed{f(x) = c-x \quad \forall x}$

is the only solution ▮

Problem 16 $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n) = 2f(f(n))$ $\forall n \in \mathbb{Z}$

Solution

$$f(f(n)) = 2f(f(f(n)))$$

$$f(f(f(n))) = 2f(f(f(f(n))))$$

$$\vdots$$

$$\vdots$$

$$\Longrightarrow \quad f(n) = 2^m \underbrace{f(f\cdots(f}_{m+1}(n))\cdots))$$

$$\Rightarrow 2^m \mid f(n) \quad \forall m \in \mathbb{N}$$

$$\Rightarrow \boxed{f(n)=0} \quad \forall n \in \mathbb{Z}$$

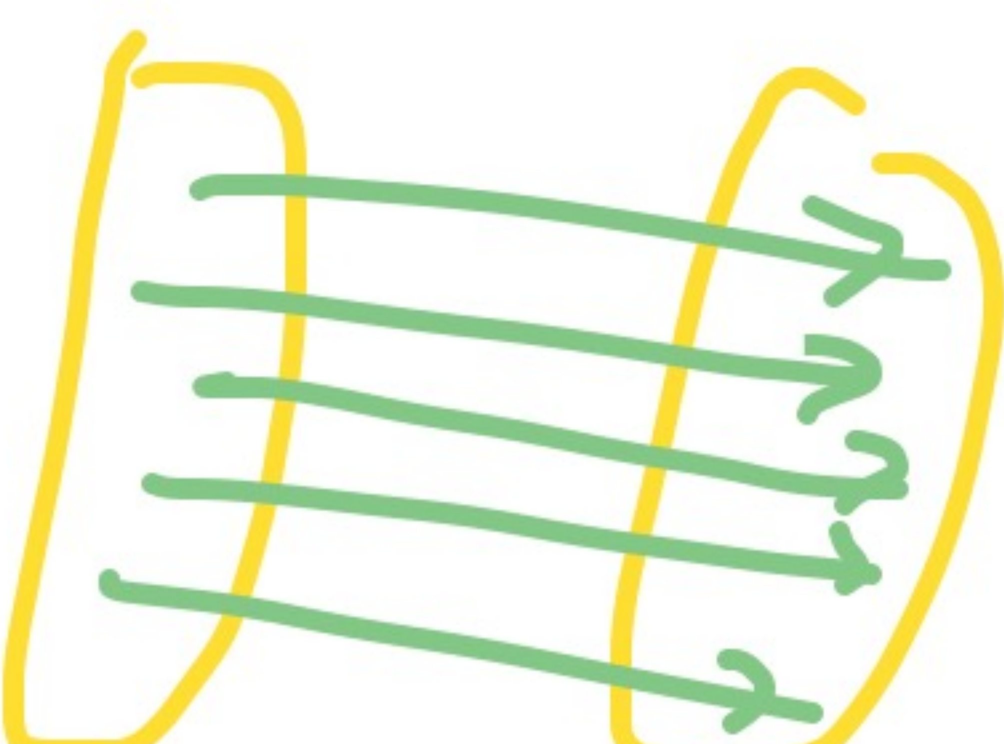
is the only solution ■

Problem 17 | $f:\mathbb{R}\to\mathbb{R}$

$$f(x+f(x)+y) = x+f(x)+f(y)$$

$\overset{??}{\Longrightarrow}$ f bijective

(تقابل)



$P(x, -f(x)) \Rightarrow$

$f(x) = x + f(x) + f(-f(x))$

$\Leftrightarrow f(-f(x)) = -x$

$\forall x \in \mathbb{R}$

$\Rightarrow f$ is clearly $\boxed{\text{surjective}}$

and if $f(a) = f(b)$

$\Rightarrow -a = f(-f(a)) = f(-f(b)) = -b$

$\Rightarrow f$ $\boxed{\text{injective}}$ $\blacksquare$

$\boxed{\text{Problem 18}}$ $f:\mathbb{R}\to\mathbb{R}$

$$f(x+f(x)+2y)=x+f(f(x))+2f(y)$$

$$P\left(x,-\frac{x}{2}\right)\implies$$

$$\cancel{f(f(x))}=x+\cancel{f(f(x))}+2f\left(-\frac{x}{2}\right)$$

$$\iff f\left(-\frac{x}{2}\right)=-\frac{x}{2}\quad\forall x\in\mathbb{R}$$

$$\iff \boxed{f(x)=x\quad\forall x\in\mathbb{R}}$$