Email training, N7-8

October 8 - 19, 2019

Problem 7.1. Let d be any positive integer not equal to $2, 5$ or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Solution 7.1. Consider residues mod 16. A perfect square must be 0, 1, 4 or 9 (mod 16). d must be $1, 5, 9$, or 13 for $2d-1$ to have one of these values. However, if d is 1 or 13, then $13d-1$ is not one of these values. If d is 5 or 9, then $5d-1$ is not one of these values. So we cannot have all three of $2d-1$, $5d-1$, $13d-1$ perfect squares.

Problem 7.2. Let x, y, z be nonnegative real numbers with $x + y + z = 1$. Show that $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$.

Solution 7.2.

$$(1-2x)(1-2y)(1-2z) = 1 - 2(x+y+z) + 4(yz+zx+xy) - 8xyz =$$

$$4(yz+zx+xy) - 8xyz - 1.$$

Hence $yz+zx+xy-2xyz = 1/4(1-2x)(1-2y)(1-2z)+1/4$. By the arithmetic/geometric mean theorem

$$(1-2x)(1-2y)(1-2z) \leq ((1-2x+1-2y+1-2z)/3)^3 = 1/27.$$

So $yz + zx + xy - 2xyz \leq \frac{1}{4} \cdot \frac{28}{27} = 7/27$.

Problem 7.3. Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Solution 7.3. By clearing denominators and regrouping, we see that the given equation is equivalent to

$$(3a - 2018)(3b - 2018) = 2018^2.$$

Each of the factors is congruent to $1 (mod 3)$. There are 6 positive factors of $2018^2 = 2^2 \cdot 1009^2$ that are congruent to $1 (mod 3)$. These numbers are $1, 2^2, 1009, 2^2 \cdot 1009, 1009^2, 2^2 \cdot 1009^2$. These lead to the 6 possible pairs:

$(a, b) = (673, 1358114), (674, 340033), (1009, 2018), (2018, 1009), (340033, 674), (1358114, 673).$

As for negative factors, the ones that are congruent to $1 (mod 3)$ are $-2, -2 \cdot 1009, -2 \cdot 1009^2$. However, all of these lead to pairs where $a \leq 0$ or $b \leq 0$.

Problem 7.4. Given a set M of 2019 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4-th power of an integer.

Solution 7.4. Suppose we have a set of at least $3 \cdot 2^n + 1$ numbers whose prime divisors are all taken from a set of n. So each number can be written as $p_1^{r_1}, \ldots, p_n^{r_n}$ for some non-negative integers r_i, where p_i is the set of prime factors common to all the numbers. We classify each r_i as even or odd. That gives 2^n possibilities. But there are more than $2^n + 1$ numbers, so two numbers have the same classification and hence their product is a square. Remove those two and look at the remaining numbers. There are still more than $2^n + 1$, so we can find another pair. We may repeat to find $2^n + 1$ pairs with a square product. [After removing 2^n pairs, there are still $2^n + 1$ numbers left, which is just enough to find the final pair.] But we may now classify these pairs according to whether each exponent in the square root of their product is odd or even. We must find two pairs with the same classification. The product of these four numbers is now a fourth power.

Applying this to the case given, there are 9 primes less than or equal to 23 (2, 3, 5, 7, 11, 13, 17, 19, 23), so we need at least $3 \cdot 512 + 1 = 1537$ numbers for the argument to work (and we have 2019).

Problem 7.5. Let $p_n(k)$ be the number of permutations of the set $\{1, \dots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

Solution 7.5. For any k, if there are $p_n(k)$ permutations that have k fixed points, then we know that each fixed point is counted once in the product $k \cdot p_n k$. Therefore the given sum is simply the number of fixed points among all permutations of $\{1, \dots, n\}$. However, if we take any x such that $1 \leq x \leq n$ and x is a fixed point, there are $(n-1)!$ ways to arrange the other numbers in the set. Therefore our desired sum becomes $n \cdot (n-1)! = n!$, so we are done.

Problem 7.6. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 2019$ for every n.

Solution 7.6. We prove that if $f(f(n)) = n + k$ for all n, where k is a fixed positive integer, then k must be even. If $k = 2h$, then we may take $f(n) = n + h$.

Suppose $f(m) = n$ with $m \equiv n \mod k$. Then by an easy induction on r we find $f(m + kr) = n + kr$, $f(n + kr) = m + k(r + 1)$. We show this leads to a contradiction. Suppose $m < n$, so $n = m + ks$ for some $s > 0$. Then $f(n) = f(m + ks) = n + ks$. But $f(n) = m + k$, so $m = n + k(s - 1) \geq n$. Contradiction. So we must have $m \geq n$, so $m = n + ks$ for some $s \geq 0$. But now $f(m + k) = f(n + k(s + 1)) = m + k(s + 2)$. But $f(m + k) = n + k$, so $n = m + k(s + 1) > n$. Contradiction.

So if $f(m) = n$, then m and n have different residues $\pmod k$. Suppose they have r_1 and r_2 respectively. Then the same induction shows that all sufficiently large $s \equiv r_1 \pmod k$ have $f(s) \equiv r_2 \pmod k$, and that all sufficiently large $s \equiv r_2 \pmod k$ have $f(s) \equiv r_1 \pmod k$. Hence if m has a different residue $r \mod k$, then $f(m)$ cannot have residue r_1 or r_2. For if $f(m)$ had residue r_1, then the same argument would show that all sufficiently large numbers with residue r_1 had $f(m) \equiv r \pmod k$. Thus the residues form pairs, so that if a number is congruent to a particular residue, then f of the number is congruent to the pair of the residue. But this is impossible for k odd.

Problem 7.7. Let all vertices of a polygon are lattice points (point with integer coefficients) in a coordinate plane. Prove that the area of the polygon is equal to

$$I + \frac{1}{2}B - 1,$$

where I is the number of lattice points in the interior and B being the number of lattice points on the boundary.

Solution 7.7. If a triangle on the lattice points with no point in its interior or on its edges, it has an area of $\frac{1}{2}$. Such triangle must contain two lattice points distance 1 from each other and one lattice point on a line parallel to the opposite edge distance 1 apart. The minimum distance between two distinct lattice points is 1. If no two lattice points have distance 1, by $\frac{1}{2}bh$ the area is more than 1 and similarly for the height. Removing 1 of the mentioned triangles either removes 1 boundary point, turns 1 interior point into a boundary point, accounting for the $I + \frac{1}{2}B$ part. The -1 part is accounted for by looking at the area of the unit triangle with 3 boundary points, 0 interior points, and $\frac{1}{2}$ area.

Problem 7.8. Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

Solution 7.8. We prove that it does. Suppose otherwise, that E is the disjoint union of e and e' with no right-angled triangles in either set. Take points X, Y, Z two-thirds of the way along BC, CA, AB respectively (so that $BX/BC = 2/3$ etc). Then two of X, Y, Z must be in the same set. Suppose X and Y are in e. Now YX is perpendicular to BC, so all points of BC apart from X must be in e'. Take W to be the foot of the perpendicular from Z to BC. Then B and W are in e', so Z must be in e. ZY is perpendicular to AC, so all points of AC apart from Y must be in e'. e' is now far too big. For example let M be the midpoint of BC, then AMC is in e' and right-angled.

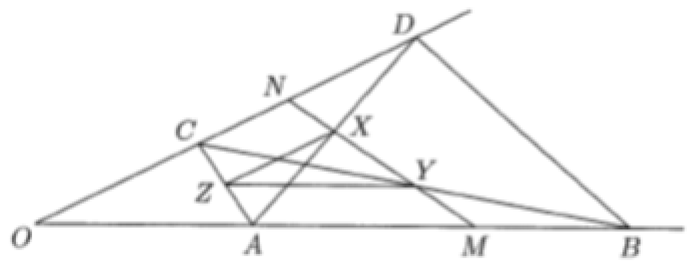
Problem 7.9. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Solution 7.9. Let O be the center of the circle mentioned in the problem. Let T be the second intersection of the circumcircle of CDO with AB. By measures of arcs, $\angle DTA = \angle DCO = \frac{\angle DCB}{2} = \frac{\pi}{2} - \frac{\angle DAB}{2}$. It follows that $AT = AD$. Likewise, $TB = BC$, so $AD + BC = AB$, as desired.

Problem 7.10. Let AB and CD are segments lying on the two sides of an angle whose vertex is O, such that A is between O and B, as well as C is between O and D. The line connecting the midpoints of the segments AD and BC intersects AB at M and CD at N. Prove that $\frac{OM}{ON} = \frac{AB}{CD}$.

Solution 7.10. -

Let X, Y and Z be the midpoints of AD, BC and AC respectively.



Then $XZ||ON$ and $YZ||OM$. Hence XYZ and NMO are similar triangles. Therefore

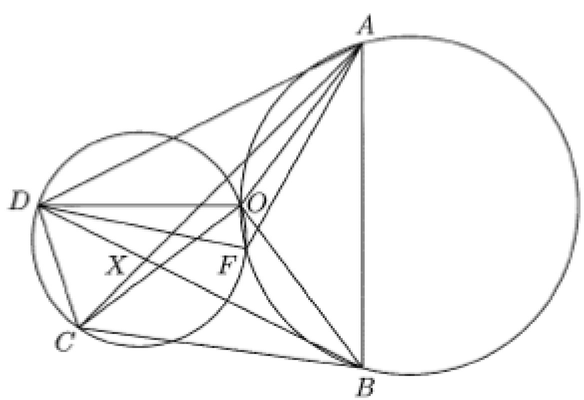
$$\frac{OM}{ON} = \frac{ZY}{ZX}.$$

Since $ZX = \frac{1}{2}CD$ and $ZY = \frac{1}{2}AB$, we obtain

$$\frac{OM}{ON} = \frac{AB}{CD}.$$

Problem 7.11. Let all vertices of convex quadrilateral $ABCD$ lie on the circumference of a circle with center O. Let F be the second point of intersection of the circumcircles of the triangles ABO and CDO. Prove that the circle passing through the points A, F and D also passes through the point of intersection of the segments AC and BD.

Solution 7.11. -



Let $\angle ADB = \alpha$ and $\angle DAC = \beta$. Then $\angle AOB = 2\alpha$. Hence $\angle ABO = 90° - \alpha$. Hence $\angle AFO = 90° - \alpha$. Similarly $\angle DFO = 90° - \beta$. Hence $\angle AFD = 180° - \alpha - \beta$. Let the point where AC and BD meet be X. Then $\angle AXD = \angle 180° - \alpha - \beta$. Thus $\angle AFD = \angle AXD$, which means that $AFXD$ is a cyclic quadrilateral.