P1

$$x^3+y^3+K(x^2+y^2)=2020$$

Sol1

$$\underbrace{(x^3+Kx^2)}_{X}+\underbrace{(y^3+Ky^2)}_{Y}=2020$$

$$X+Y=Y+Z=Z+X=2020$$

$$\Rightarrow X=Y=Z=1010$$

$\Rightarrow$ $\boxed{x, y, z \text{ are the zeros of } f(t)=t^3+Kt^2-1010}$

$$\Rightarrow xyz=1010$$

Sol 2

$$x^3+y^3+K(x^2+y^2)=y^3+z^3+K(y^2+z^2)$$

$$\Longleftrightarrow \quad x^2+xz+z^2+Kx+Kz=0$$

$$y^2+yz+z^2+Ky+Kz=0$$

$$x+y+z+K=0$$

$$\Longleftrightarrow \quad \boxed{K=-(x+y+z)}$$

$$y^2+yz+z^2=(x+y+z)(y+z)$$

$$\Longleftrightarrow \quad \boxed{xy+yz+zx=0}$$

$$x^3+y^3=2020+(x+y+z)(x^2+y^2)$$

$$\Leftrightarrow -2020 = z(x^2+y^2) + xy(x+y)$$

$$= x^2(y+z) + y^2(x+z)$$

$$= x(xy+xz) + y(xy+yz)$$

$$= -xyz - xyz$$

$$= -2xyz$$

$$\Leftrightarrow \boxed{xyz = 1010}$$

P2) $2^a \cdot 3^b + 9 = c^2$

Sol) clearly, $3 \mid c$

$\Rightarrow$ we may write $c = 3c_1$
(and so $b \geq 2$)

$$\Rightarrow 2^a \cdot 3^{b-2} = (c_1 - 1)(c_1 + 1)$$

$\gcd(c_1 - 1, c_1 + 1) = 2$

since $c_1 + 1 > c_1 - 1$, we have
3 cases:

Case 1:

$c_1 - 1 = 2, \quad c_1 + 1 = 2^{a-1} \cdot 3^{b-2}$

$$\Rightarrow \boxed{(a, b, c) = (3, 2, 9)}$$

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Case 2:

$c_1 - 1 = 2 \cdot 3^{b-2}, \quad c_1 + 1 = 2^{a-1}$

$$\Rightarrow \boxed{3^{b-2} + 1 = 2^{a-2}} \quad \boxed{\begin{array}{l} 3^0 + 1 = 2^1 \\ 3^1 + 1 = 2^2 \end{array}}$$

Case 2.1: $b - 2 = 0$

it is same with Case 1

Case 2.2: $b - 2 > 0$

now, $\pmod 3 \Rightarrow a-2$ even

$\Rightarrow 3^{b-2} = \underbrace{\left(2^{\frac{a-2}{2}}-1\right)}_{x}\underbrace{\left(2^{\frac{a-2}{2}}+1\right)}_{y}$

$\gcd(x, x) = 1$, $x, y \mid 3^{b-2}$

$\Rightarrow 2^{\frac{a-2}{2}}-1 = 1$, $2^{\frac{a-2}{2}}+1 = 3$

$\Rightarrow \boxed{(a,b,c) = (4,3,21)}$ (2)

Case 3: $c_1 - 1 = 2^{a-1}$, $c_1+1 = 3^{b-2}\cdot 2$

$\Rightarrow \boxed{2^{a-2}+1 = 3^{b-2}}$ $2^1+1 = 3^1$, $2^3+1 = 3^2$

Clearly, $a-2>0$

<u>subcase 1</u>: $a-2=1$

$\Rightarrow$ $\boxed{(a,b,c)=(3,3,15)}$ 3

<u>subcase 2</u>: $a-2\geq 2$

(mod 4) $\Rightarrow$ $b-2$ even

$\Rightarrow$ $2^{a-2} = \underbrace{(3^{\frac{b-2}{2}}-1)}_{x}\underbrace{(3^{\frac{b-2}{2}}+1)}_{y}$

$(x,y)=2$, $x,y \mid 2^{a-2}$

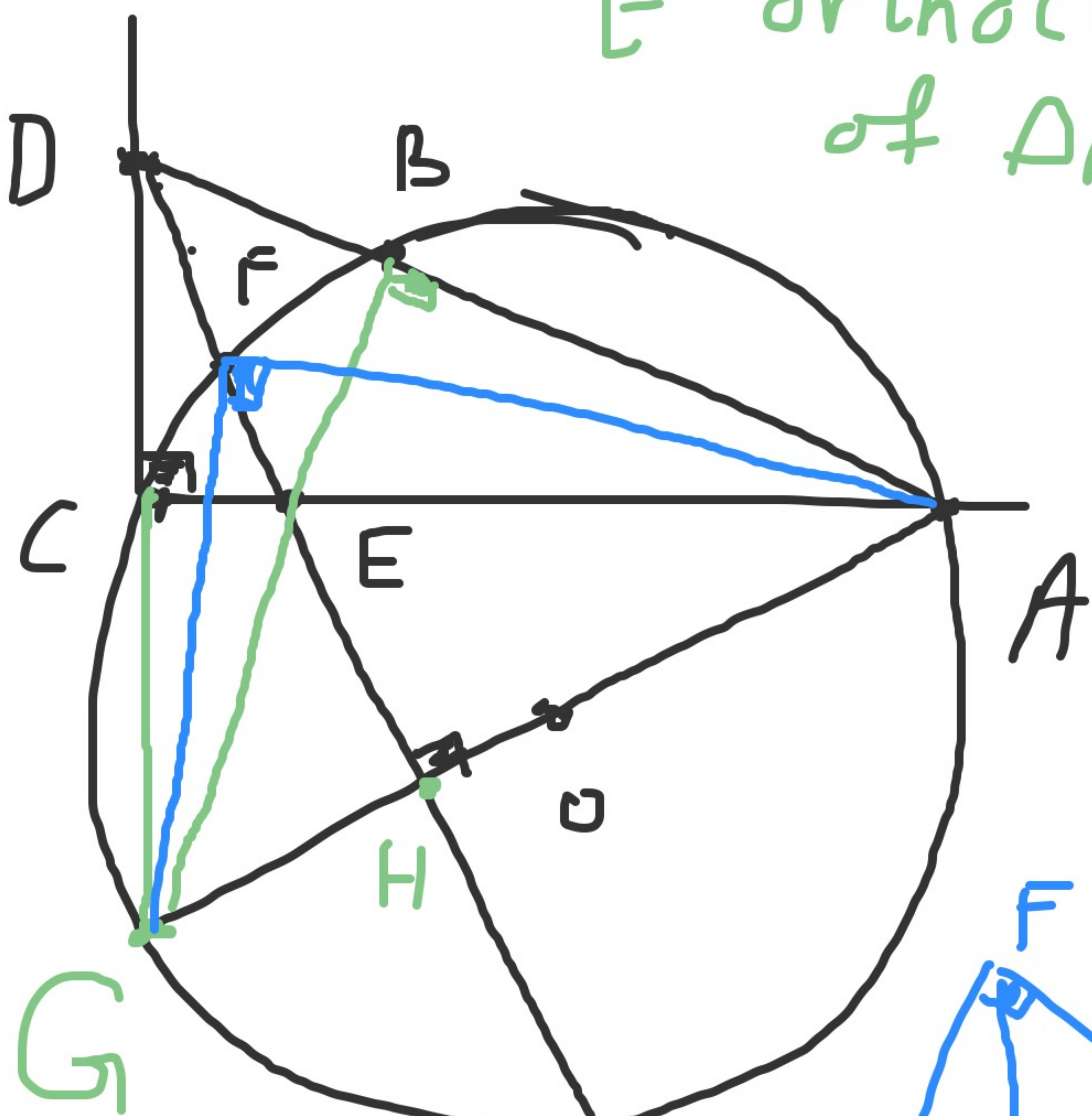
$\Rightarrow$ $x=2$, $y=4$ $\Rightarrow$ $3^{\frac{b-2}{2}}-1=2$

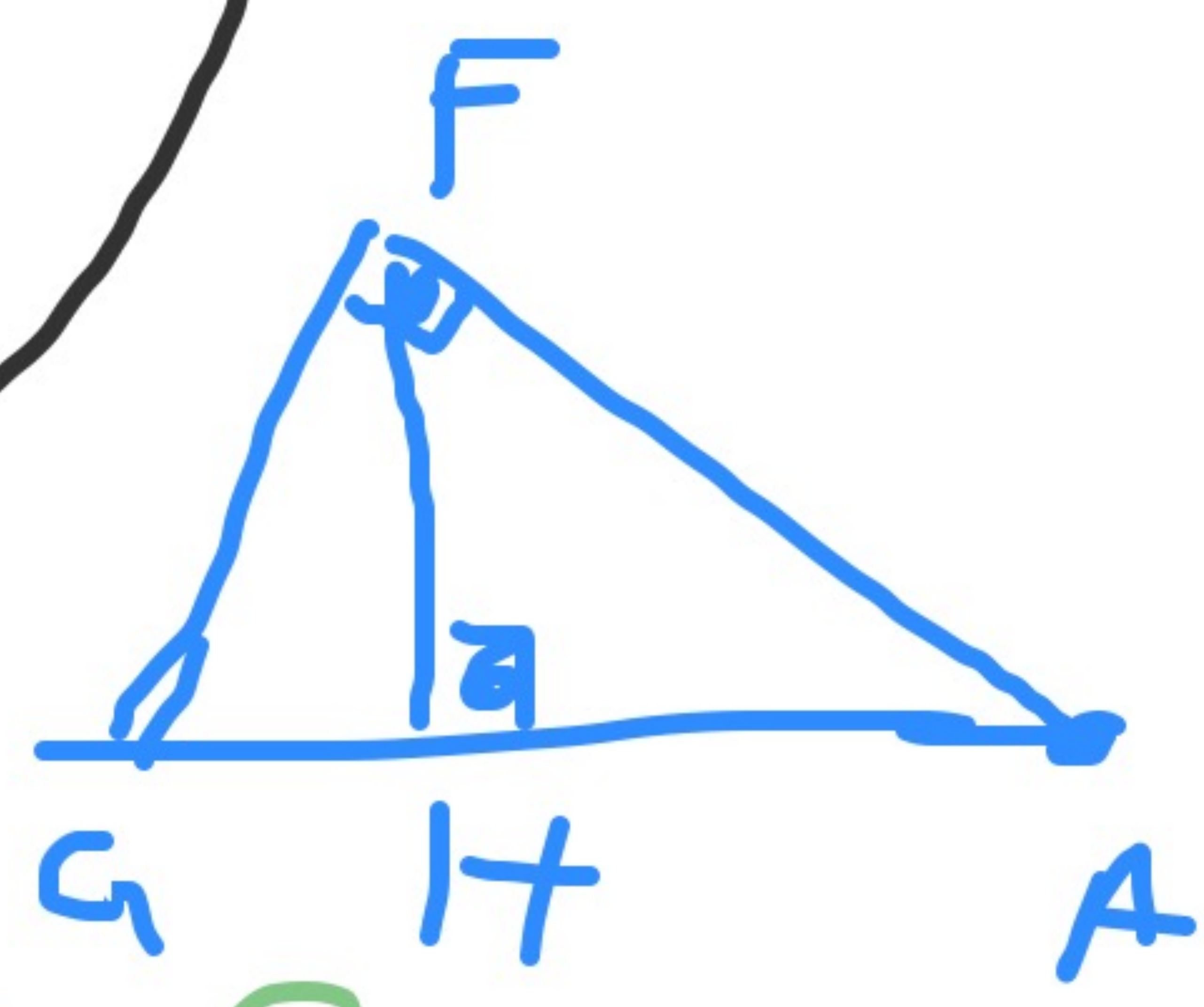
$\Rightarrow$ $\boxed{(a,b,c)=(5,4,51)}$ 4

P3

(DFC) tgt (BFE)

E orthocentre of ΔADG





$GF^2 = GH \cdot GA$

$= GE \cdot GB = GC \cdot GD$

$\Rightarrow$ GF tangent to (BFE)

$\Rightarrow$ GF tangent to (DFC)

P4 maximal # of non-attacking knights in a 6 × 6

Sol Ans: 18

Example:

Color like chessboard and place 18 knights at the white squares.

Now: Consider the numbering as below:

17	18	16	13	14	15
10	11	17	15	16	13
18	7	8	9	12	14
11	10	12	6	5	9
7	8	1	2	3	4
1	2	3	4	6	5

for this pairing, every 2 squares having the same # can attack each other.

$\Rightarrow$ we can't place more than 18 knights

P 5 $\quad 3 \leq a+b+c \leq 6$

Prove:

$$\frac{a}{2+bc} + \frac{b}{2+ca} + \frac{c}{2+ab} \geq 1$$

Sol By Cauchy:

$$LHS = \sum \frac{a^2}{2a+abc} \geq$$

$$\frac{(a+b+c)^2}{2(a+b+c)+3abc}$$

AM-GM

$$\geq \frac{(a+b+c)^2}{2(a+b+c)+\frac{1}{9}(a+b+c)^3} \overset{?}{\geq} 1$$

call $s = a+b+c$

$\Rightarrow \quad 3 \leq s \leq 6$

and it is enough to verify that

$$\frac{s}{2 + \frac{1}{9}s^2} \overset{?}{\geq} 1$$

$$\Longleftrightarrow \quad 0 \overset{?}{\geq} (s-3)(s-6) \quad ✓$$

"$=$" $\quad a=b=c=1$ or

$a=b=c=2$