Competition Preparation for Saudi Arabia Team: Level 4

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Lesson 9

More double counting problems

Problems:

1. At a party attended by n people each person has exactly k friends, each two people who are friends have exactly l (other) friends in common and each two people who are not friends have exactly m friends in common. Prove that: $m(n-k)-k(k-l)+k-m=0$.

2. Let n and k be the positive integers and let S be a set of n points in the plane such that no three points are collinear and for every point P of S there are at least k points in S equidistant from P. Prove that $k<\frac{1}{2}+\sqrt{2n}$.

3. Suppose that c contestants took part in a mathematics competition with p problems and that each problem was solved by at least k contestants. Show that there exists a pair of contestants who, between them, solved all p problems if: $p<\dfrac{c(c-1)}{(c-k)(c-k-1)}$.

4. Suppose that a boys and a girls took part in a competition, that each contestant solved at most n problems, where $n>1$, and that for each pair of one boy and one girl there was a problem solved by both of them. Prove that there exists a problem solved by at least $\left\lceil \dfrac{a}{2(n-1)} \right\rceil$ girls and at least $\left\lceil \dfrac{a}{2(n-1)} \right\rceil$ boys.

5. Let A be a family of subsets of $\{1,2,\ldots,n\}$. We will call A an anti-chain if no two elements of A exist such that one is the subset of the other. Prove that if A is an anti-chain then:

$$\sum_{S\in A}\frac{1}{\binom{n}{|S|}}\leq 1.$$