$a, b \in \mathbb{Z}_+$

$$2a^2 + a = 3b^2 + b$$

$a-b$, $2a+2b+1$ are squares.

P4

p, q — consecutive odd prime numbers.

$\Downarrow$

$p+q$ is a product of ≥ 3 natural number > 1.

Proof: Between $p < q$ are not primes,

p $\qquad$ q

$(2, 3)$ $\quad$ $\boxed{2+3=p}$

$3, 5$ $\quad$ $8 = 2^3$

$\frac{p+q}{2}$ is integer

$p \qquad \frac{p+q}{2} \qquad q$

$\Downarrow$

$a, b > 1$.

$\frac{p+q}{2}$ is not prime. $\Longrightarrow$ $\frac{p+q}{2} = a \cdot b$

$\Downarrow$

$p+q = 2 \cdot a \cdot b$

P5)

$a, b \in \mathbb{Z}_+$

$$2a^2 + a = 3b^2 + b$$

$a-b$, $2a+2b+1$ are squares.

HINT| try to do as follows:

$$X \cdot Y = Z^2 \qquad GCD(X, Y) = 1$$

$p_1^{\alpha_1} \cdots p_k^{\alpha_k} q_1^{\beta_1} \cdots q_l^{\beta_l}$ $\Downarrow$

X is square Y square

$$(a-b)(2a+2b+1) = b^2$$

claim

$$\gcd(a-b, 2a+2b+1) = 1.$$

$p \mid a-b$, $p \mid 2a+2b+1$

$\boxed{p \mid 1}$ ↯

$\Downarrow$ $\Downarrow$

$p \mid b \Rightarrow p \mid a$

$\boxed{P6}$ Find all pairs (a,b) :

$$a-b=p \text{ - prime}$$

$$ab = \text{square}$$

$$\gcd(a,b) = 1 \vee p$$

$a-b=2$

$ab = \text{square}$

$$\boxed{\begin{array}{l}\gcd(a,b)=1\\ \gcd(a,b)=p\end{array}}$$

$$a-b=p$$

1° Look at $\gcd(a,b)$

denote by $d = \gcd(a,b)$

$$a = a_1 \cdot d \quad , \quad b = d \cdot b_1$$

$$\gcd(a_1, b_1) = 1$$

$$a_1 d - b_1 d = p \rightsquigarrow (a_1 - b_1) d = p$$

$$d = 1 \vee d = p$$

$1^\circ)$ $\boxed{d = p}$ $\rightsquigarrow$ $(a_1 - b_1)\cancel{p} = \cancel{p}$

$$a_1 - b_1 = 1$$

$$b_1 = a_1 - 1$$

$$ab = d \cdot a_1 \cdot d \cdot b_1 = d^2 \cdot a_1 \cdot (a_1 + 1) = \text{perfect square}$$

$$a_1(a_1 - 1) = \text{perf square.}$$

$$(a_1 - 1)^2 < a_1(a_1 + 1) < a_1^2$$

2° $\gcd(a, b) = 1$.

$$ab = \text{perfect square}$$

$$a = x^2 \qquad b = y^2$$

$$p = a - b = x^2 - y^2 = (x - y)(x + y) = 2$$

$\begin{cases} x - y = 1 \\ x + y = 2 \end{cases}$ no sol.

so $+\begin{cases} x - y = 1 \\ x + y = p \end{cases}$

$p > 2$

$$2x = p + 1 \rightsquigarrow x = \frac{p+1}{2}$$

$$y = \frac{p-1}{2}$$

$$\left(\left(\frac{p+1}{2}\right)^2, \left(\frac{p-1}{2}\right)^2\right) \qquad \underline{p>2}$$

$$\left(\frac{p+1}{2}\right)^2 - \left(\frac{p-1}{2}\right)^2 = \frac{p^2+2p+1}{4} - \frac{p^2-2p+1}{4}$$

$$= p$$

<u>P7</u> Find all $n \in \mathbb{Z}_+$ which has $\sqrt{n}$ divisors.

$\underline{n=1}$

$n>1$

$$n = p_1^{2\alpha_1} \cdots p_k^{2\alpha_k} \rightsquigarrow \text{has } (2\alpha_1+1)(2\alpha_2+1)\cdots(2\alpha_k+1)$$

<u>Reminder</u> $N = q_1^{s_1} \cdots q_n^{s_n} \rightsquigarrow q_1^{x_1} \cdots q_n^{x_n}$

How many divisors N has?

$x_1 \in \{0, \dots, s_1\}$ s_1+1

$\vdots$

$x_n \in \{0, \dots, s_n\}$ s_n+1

$$(s_1+1) \cdot \ldots \cdot (s_n+1)$$

$$p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k} = (2\alpha_1+1)(2\alpha_2+1)\cdots(2\alpha_k+1)$$

$$\boxed{p_i \geq 3}$$

$$p_i^{\alpha_i} \geq 3^{\alpha_i} \geq 2\alpha_i + 1$$

$$3^x \geq 2x+1 \quad \text{for } x \geq 1$$

"=" $x=1$

Induction

$x=1$ ✓

$x \rightsquigarrow x+1$

Induction assumption

$$3^{x+1} = 3 \cdot 3^x \geq 3 \cdot (2x+1) = 6x+3$$

$$\geq 2(x+1)+1 \quad ✓$$

$$p_i^{\alpha_i} = 2\alpha_i + 1$$

$p_i = 3$, $\boxed{\alpha_i = 1}$

$$n = 3^2 = 9$$

$\sqrt{n} = 3$ ✓ $\qquad \underbrace{1,\ 3,\ 9}_{3 \text{ divisors}}$ $\qquad \square$