Email training, N6

Level 4, October 18-24

Problem 6.1. Prove the inequality

$$\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\leq 12.$$

Solution 6.1. By using Cauchy inequality we get

$$1\cdot\sqrt{a+1}+1\cdot\sqrt{2a-3}+1\cdot\sqrt{50-3a}\leq$$

$$\sqrt{1^2+1^2+1^2}\cdot\sqrt{\sqrt{a+1}^2+\sqrt{2a-3}^2+\sqrt{50-3a}^2}=12.$$

Problem 6.2. Let the parabola $y=x^2+px+q$ is given, which intersects coordinate axes in 3 different points. Consider the circumcircle of the triangle having vertices these 3 points. Prove that there is a point that belongs to that circle, regardless of values p and q. Find that point.

Solution 6.2. For some p and q draw the requested circle and take the intersection point of the circle and y-axis, namely $(0,a)$ (different from $(0,q)$). The circle has 2 chords (axes) that intersect at point $(0,0)$. By applying the chord rule we get $x_1x_2=qa$, where x_1 and x_2 are intersection point coordinates of the circle and x-axis. According to the Viet theorem we have $x_1x_2=q$, so $a=1$. We conclude that the circle passes the point $(0,1)$, independently from the values of p and q.

Problem 6.3. Find all integer polynomials P for which $(x^2+6x+10)P^2(x)-1$ is the square of an integer polynomial.

Solution 6.3. By denoting $P(x-3)=R(x)$ and switching from x to $y=x+3$ as the new independent variable we reformulate the problem in the following form

$$(x^2+6x+10)P^2(x)-1=\Big((x+3)^2+1\Big)P^2(y-3)-1=(y^2+1)R^2(y)-1.$$

Note, that according to the problem conditions we have that $R(y)$ is integer polynomial. So we get

$$(y^2+1)R^2(y)-1=Q^2(y).$$

From this relation follows that R and Q are co-prime and have no common root. Note, that if $degQ=n$ then $degR=n-1$. After differentiation we get

$$2yR^2+2RR'(y^2+1)=2QQ'.$$

From this follows, that R divides $2QQ'$. Since R and Q are co-prime, then we conclude that R divides $2Q'$. Since $degR=degQ-1=degQ'$ we conclude that $R=aQ'$ for some number a. By putting this relation into our equation we get

$$a^2(y^2+1)Q'^2=Q^2+1.$$

Equating the leading coefficients we get $a^2=\frac{1}{n^2}$. Therefore, we get

$$(y^2+1)Q'^2=n^2(Q^2+1).$$

Let $Q(y)=a_ny^n+\ldots+a_1y+a_0$. Let $k<n$ is the greatest number for which $x_k\neq 0$. By calculating the coefficients of y^{n+k} in both sides we conclude that

$$2nka_na_k=2n^2a_na_k,$$

which means $n=k$, which is impossible. So $Q(y)=a_ny^n$. From the equation $(y^2+1)Q'^2=n^2(Q^2+1)$ follows that $n=1$ and $a_1=1$. So $Q(y)=y$. It means $R(y)=\pm1$ and so on is $P(y)$ and $P(x)$.

Answer: $P(x)=\pm1$.

Problem 6.4. a) Find the minimum number of elements that must be deleted from the set $\{1, 2 \ldots, 2018\}$ such that the set of the remaining elements does not contain two elements together with their product. b) Does there exist, for any k, an arithmetic progression with k terms in the infinite sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots.$$

Solution 6.4. a) If we extract the numbers $1, 2, 3, ..., 44$ and then choose $x, y \in \{45, ..., 2005\}$, then $xy \geq 45^2 > 2018$, and thus it cannot belong to the given set.

Therefore the number of elements to be deleted is at most 43. Consider now the triples $(1, 1000)$, $(2, 87, 2 \cdot 87)$, $(3, 86, 3 \cdot 86)$, $\ldots$, $(44, 45, 44 \cdot 45)$ consisting of elements of the given set. We must delete at least one number from each triple in order that the remaining set be free of products. Thus the minimum number of elements to be deleted is 44.

b). Yes, it does. Take for instance $\frac{1}{k!}, \frac{2}{k!}$, . . . , $\frac{k}{k!}$.

Problem 6.5. Prove that not all zeros of a polynomial of the form $x^n+2nx^{n-1}+2n^2x^{n-2}+\ldots$ can be real.

Solution 6.5. Suppose that all its zeros $x_1, x_2, \ldots, x_n$ are real. They satisfy

$$\sum_i x_i = -2n, \quad \sum_{i<j} x_i x_j = 2n^2.$$

However, we have

$$\sum_{i<j} x_i x_j = \frac{1}{2}\Big(\sum_i x_i\Big)^2 - \frac{1}{2}\sum_i x_i^2 = 2n^2 - \frac{1}{2}\sum_i x_i^2 < 2n^2,$$

a contradiction.

Problem 6.6. Let the polynomial $P(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \ldots + a_1x + a_0$ with all coefficients $a_i \in [100, 101]$ is given. Find the minimal possible value of n for which $P(x)$ has a root.

Solution 6.6. Example for $n = 100$. Consider the polynomial

$$P(x) = 100(x^{200} + x^{198} + ... + x^2 + 1) + 101(x^{199} + x^{197} + ... + x)$$

which has root –1. So $n = 100$ satisfies.

Let $n < 100$. Assume x is a root. Then $x < 0$. Denote $x = -t$. It's enough to show, that

$$100(t^{2n} + t^{2n-2} + \ldots + t^2 + 1) > 101(t^{2n-1} + t^{2n-3} + \ldots + t)$$

for all $t > 0$. By multiplying both sides by $t + 1$ we get an equivalent inequality

$$100(t^{2n+1} + t^{2n} + ... + 1) > 101(t^{2n} + t^{2n-1} + ... + t),$$

or

$$100(t^{2n+1} + 1) > t^{2n} + t^{2n-1} + ... + t.$$

If we prove $t^{2n+1} + 1 \geq t^{2n} + t$ and $t^{2n+1} + 1 \geq t^{2n-1} + t^2$ and so on, then by taking their sum we get

$$100(t^{2n+1} + 1) > n(t^{2n+1} + 1) > t^{2n} + t^{2n-1} + ... + t.$$

It remains to prove that for $k = 1, ..., n$ we have $t^{2n+1} + 1 > t^{2n+1-k} + t^k$. It follows from the simple relation $(t^k–1)(t^{2n+1-k}–1) \geq 0$ since both breckets have the same sign for $t > 0$.

Answer: $n = 100$.

Problem 6.7. Let AD is the altitude of the triangle ABC. Let J, K be the incenters of the triangles ABD, ACD respectively. Let JK intersects AB, AC at E, F respectively. Prove that $AE = AF$ if and only if $AB = AC$ or $\angle A = 90^o$.

Solution 6.7. -

Let $CK \cap BJ = \{I\}$, hence I is the incenter of the triangle ABC.

We have $\angle KAC = \frac{1}{2}(90^\circ - \angle C) = 45^\circ - \frac{\angle C}{2}$,

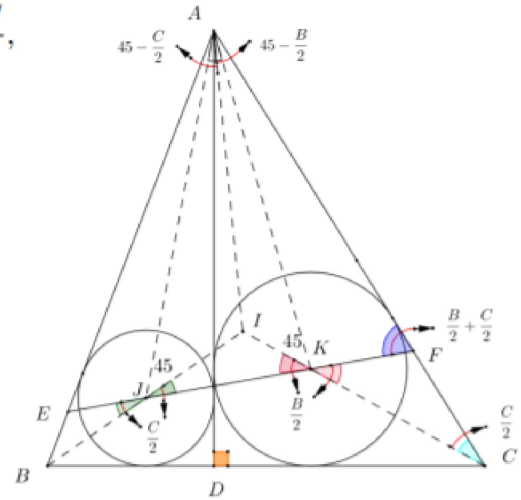
$\angle JAB = 45^\circ - \frac{\angle B}{2}$.

So $\angle KAI = \frac{\angle A}{2} - (45^\circ - \frac{\angle C}{2}) = 45^\circ - \frac{\angle B}{2}$,

$\angle JAI = 45^\circ - \frac{\angle C}{2}$.

Now $AE = AF$ is equivalent to

$$\angle AFE = 90^\circ - \frac{1}{2}\angle A = \frac{\angle B}{2} + \frac{\angle C}{2}.$$



It is equivalent to $\angle FKC = \angle IKJ = \frac{\angle B}{2}, \angle EJB = \angle IJK = \frac{\angle B}{2}$, and $\angle AKI = \angle AJI = 45^\circ$ as well. From trigonometric Ceva in the triangle AJK and the point I, it is equivalent to:

$$\frac{\sin 45}{\sin B/2} \cdot \frac{\sin C/2}{\sin 45} \cdot \frac{\sin(45 - C/2)}{\sin(45 - B/2)} = 1 \Leftrightarrow$$

$$2\sin C/2\sin(45 - C/2) = 2\sin B/2\sin(45 - B/2) \Leftrightarrow$$

$$\cos(B - 45) - \cos 45 = \cos(C - 45) - \cos 45 \Leftrightarrow$$

$$\cos(B - 45) = \cos(C - 45) \Leftrightarrow$$

Its equivalent to: $B - 45 = C - 45 \Leftrightarrow \angle B = \angle C \Leftrightarrow AB = AC$, OR $B - 45 = -(C - 45) \Leftrightarrow \angle B + \angle C = 90^\circ \Leftrightarrow \angle A = 90^\circ$. And we are done.