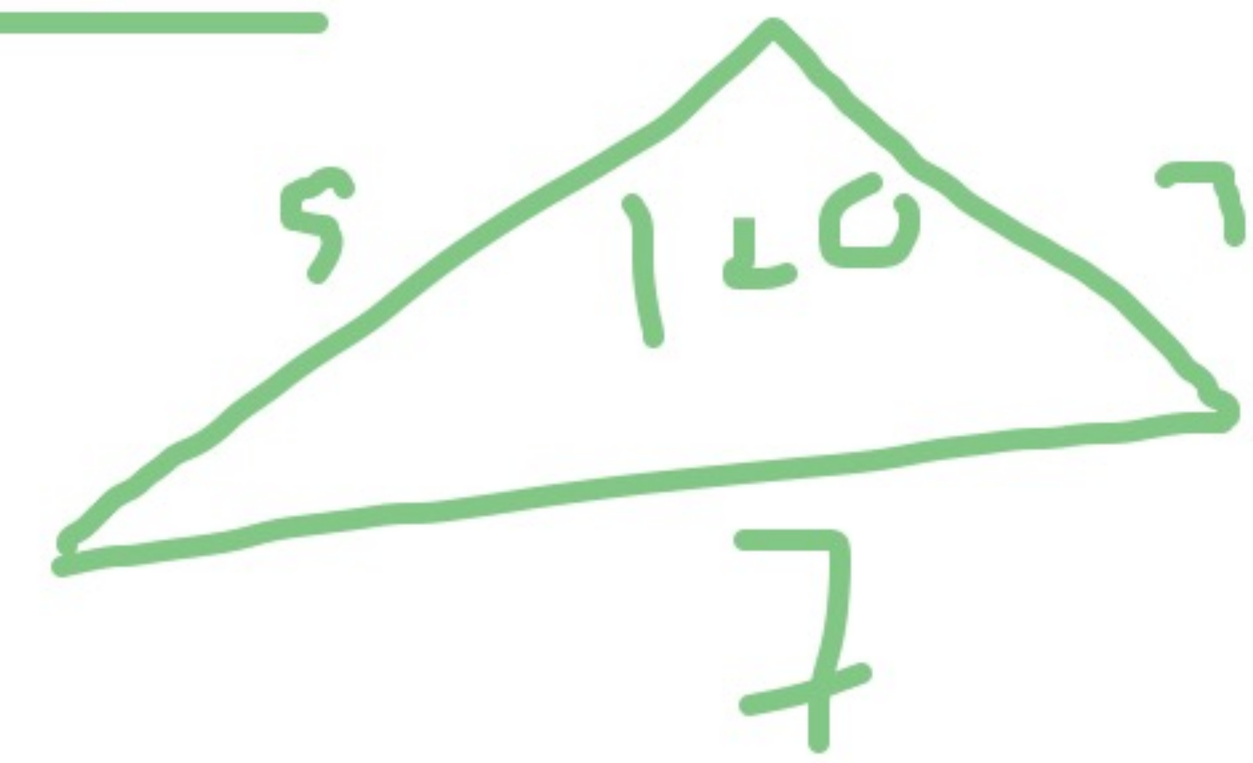
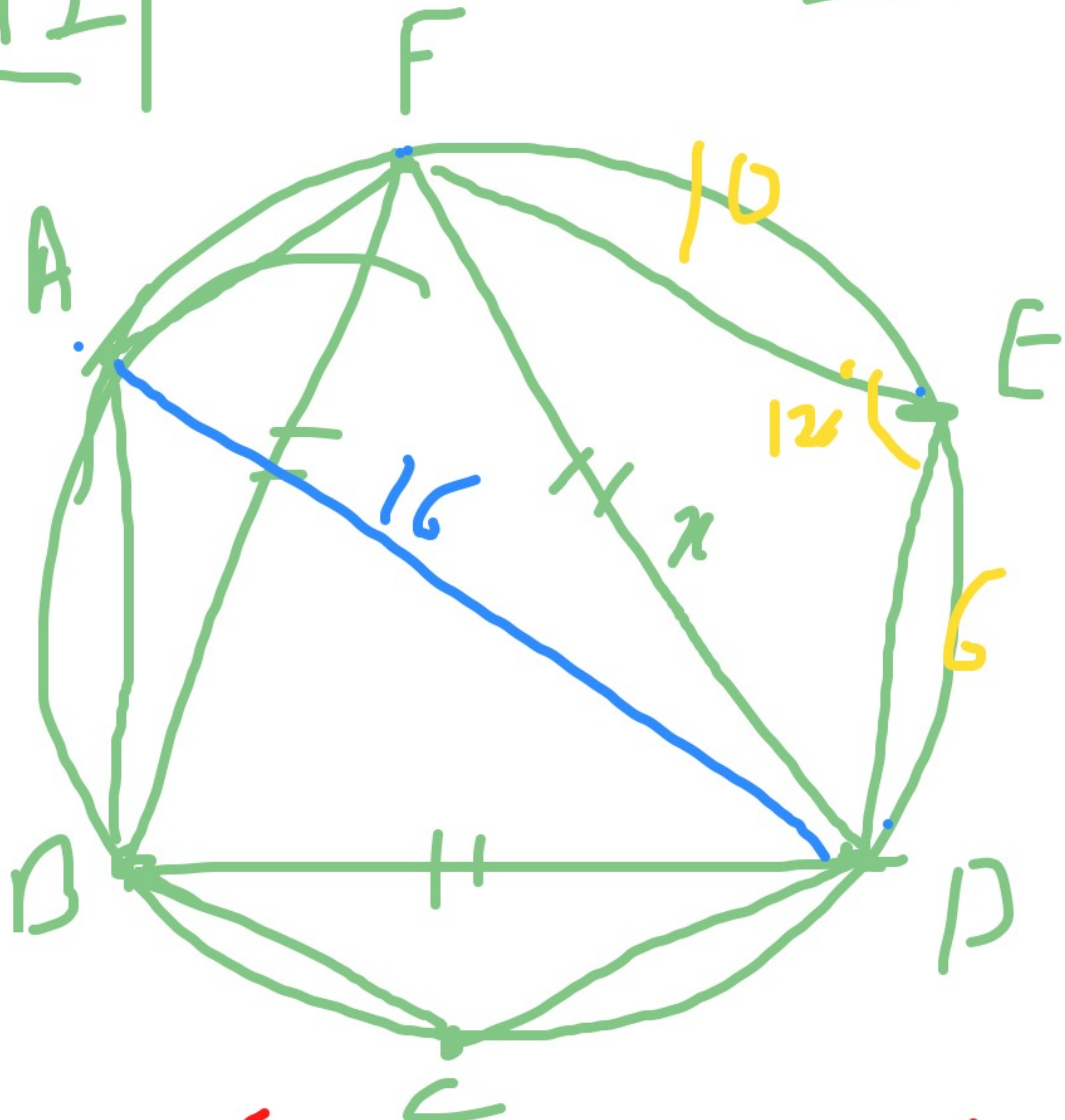
Problem1

Ans: $94\sqrt{3}$

Solution1





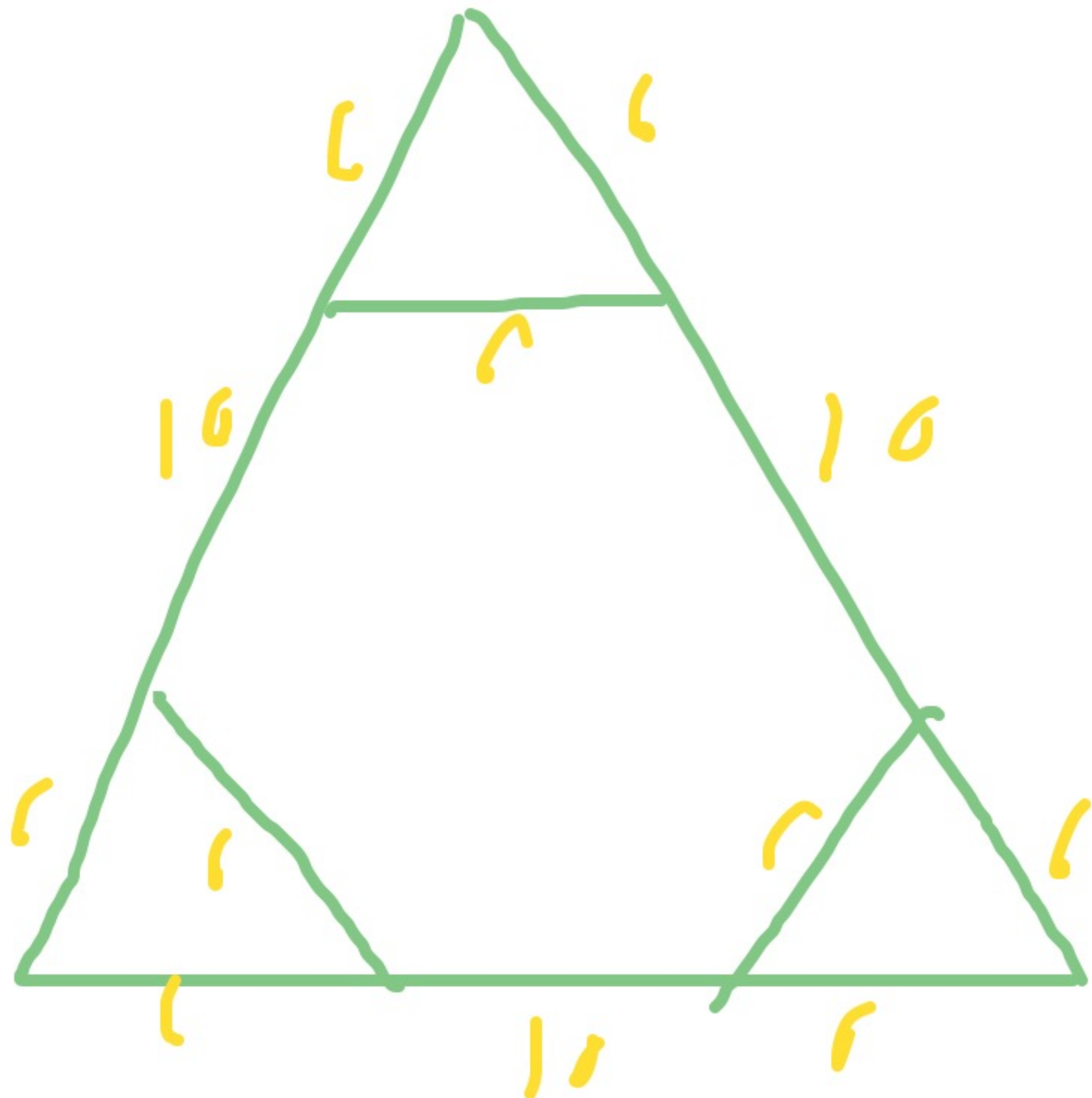
$- \angle A = 120^\circ$ (common in all solution)

$$x^2 = 16^2 + 6^2 + 10 \times 6 = 14^2$$

$$\Rightarrow x = 14$$

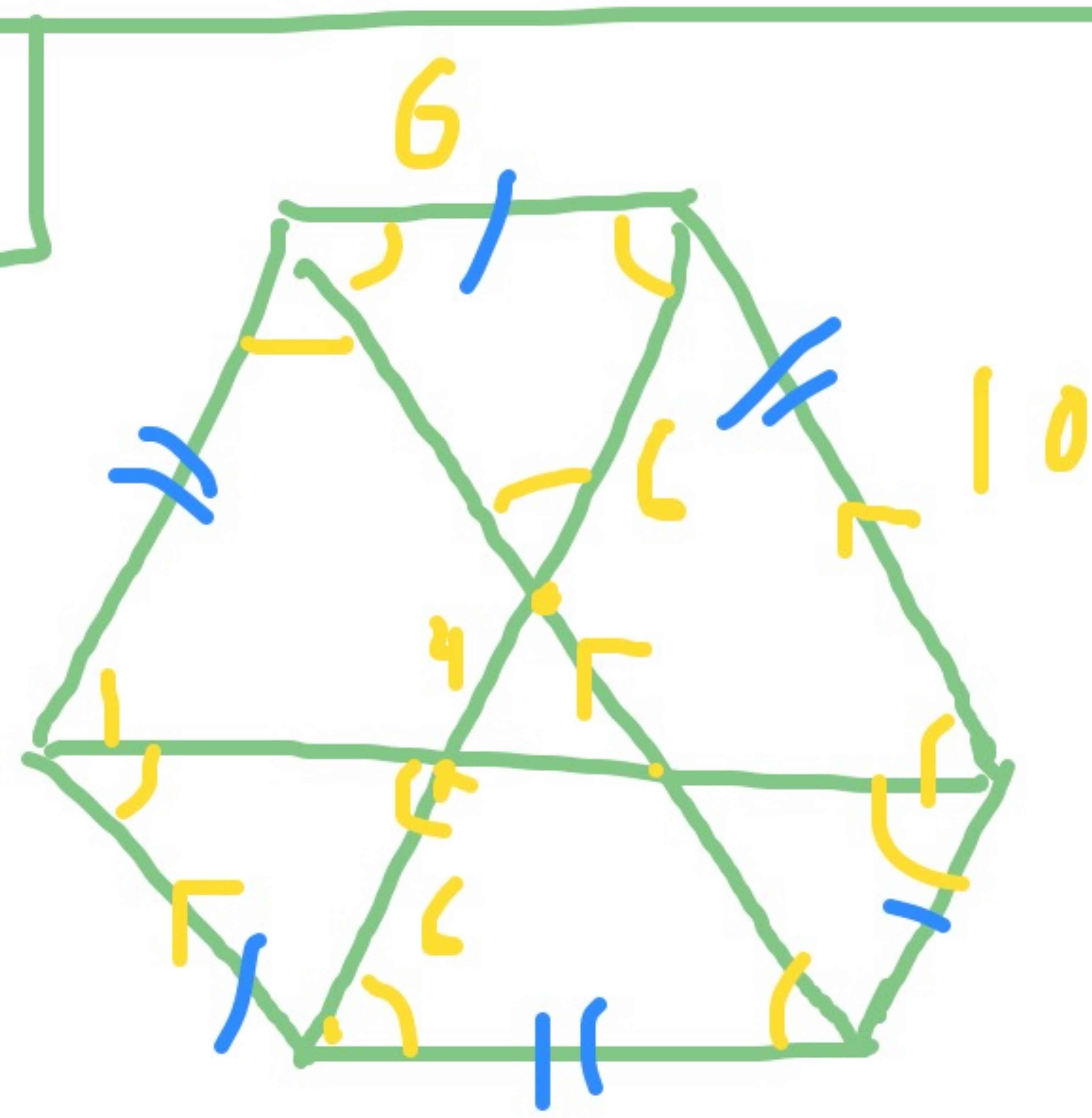
$$\frac{\sqrt{3}}{4}\left(14^2 + 6\times 10 + 6\times 10 + 6\times 10\right) = 94\sqrt{3}$$

Sol 2



$$\text{Area} = \frac{\sqrt{3}}{4}\left(26^2 - 3\times 6^2\right) = 94\sqrt{3}$$

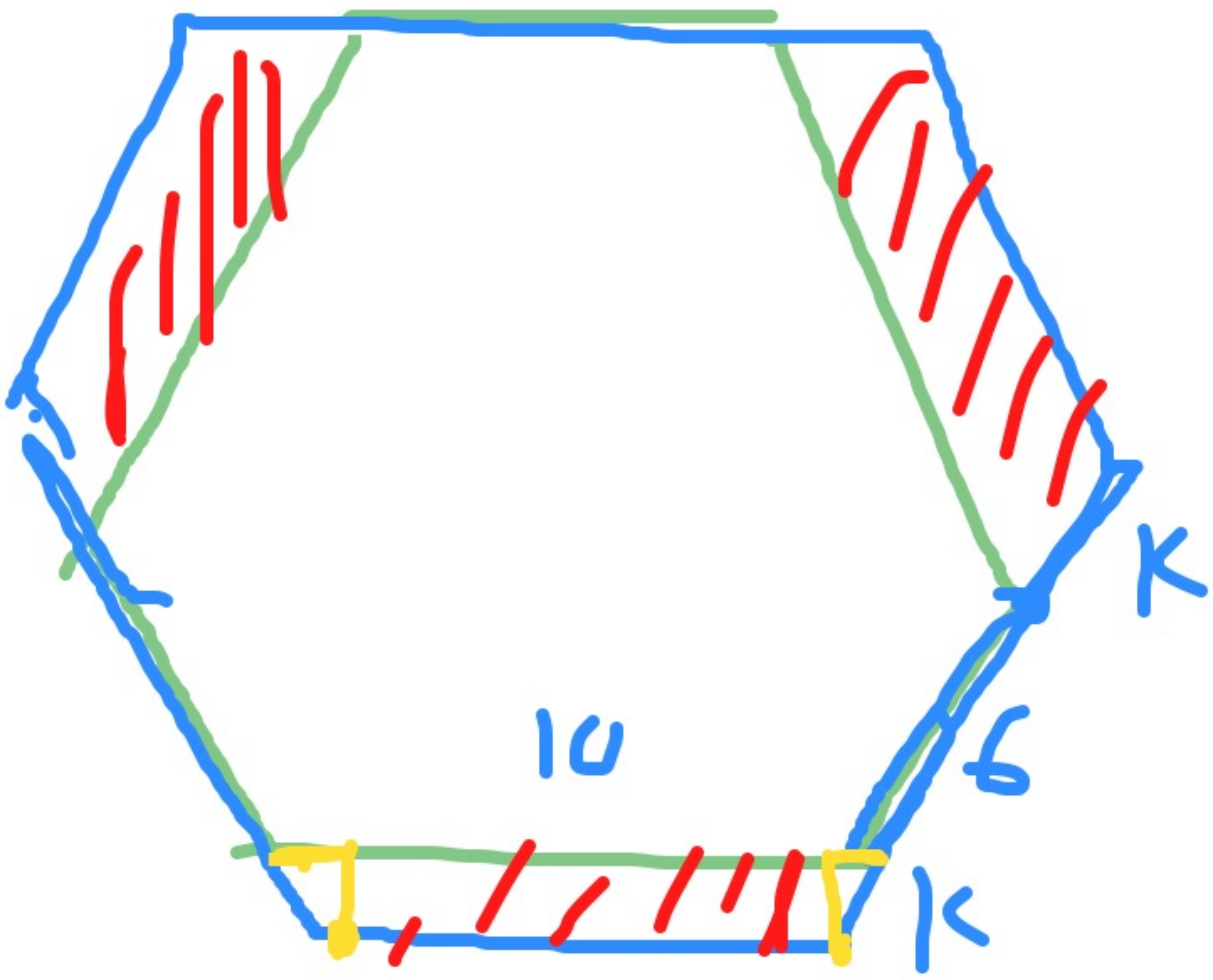
Sol 3



$$\text{Area} = \frac{\sqrt{3}}{4}\left(3\times 6^2 + 3\times 10^2 - 2\times 4^2\right)$$

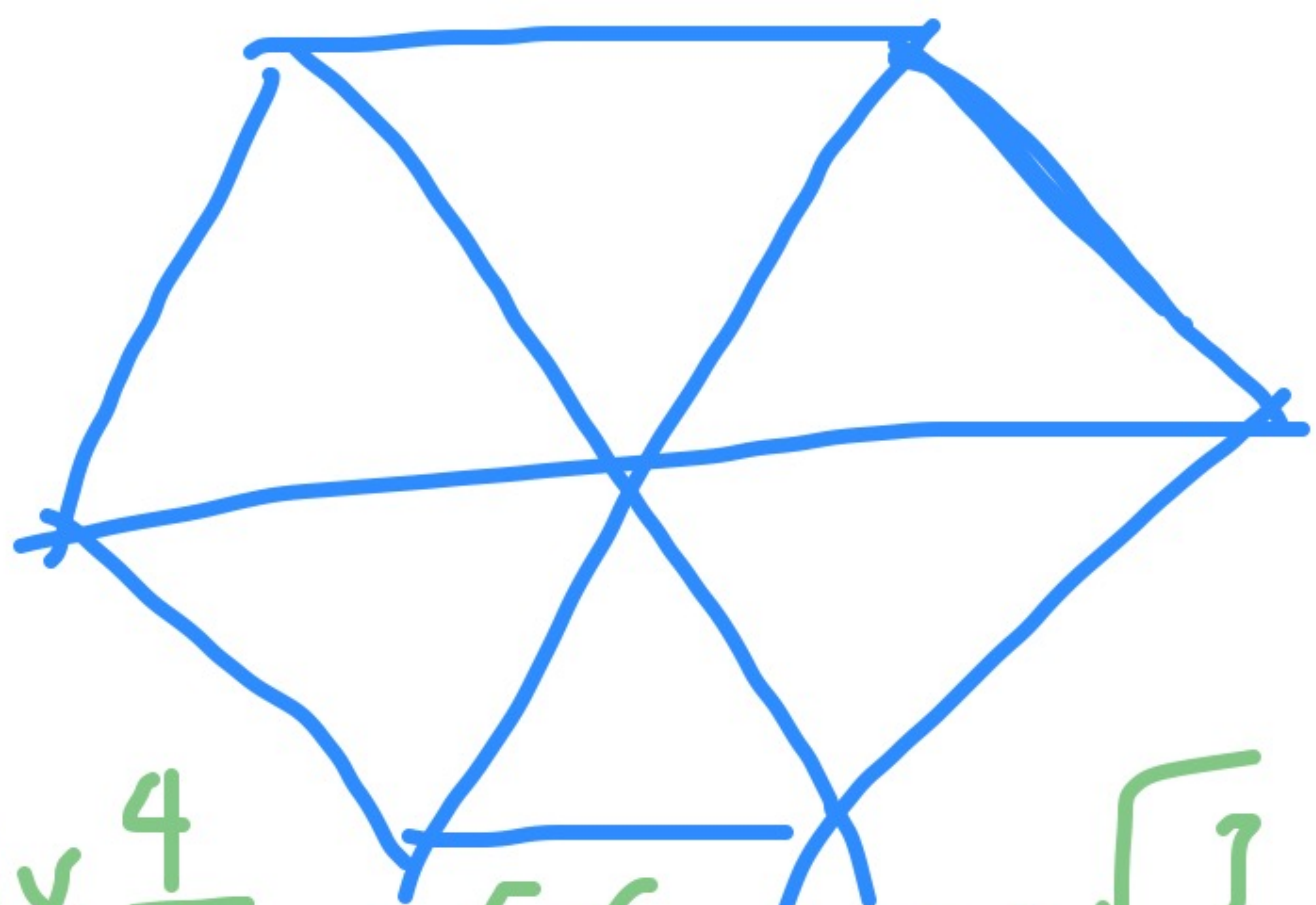
$$= 94\sqrt{3}$$

$\boxed{\text{Sol 4}}$



$$6+2K = 10 - 2\cdot\frac{K}{2} \Rightarrow K = \frac{4}{3}$$

$$6+2K = \frac{26}{3}$$



$\Rightarrow$ Area $=$

$$6\cdot\frac{\sqrt{3}}{4}\times\left(\frac{26}{3}\right)^2 - 3\times\frac{4}{\sqrt{3}}\times\frac{56}{3} = 94\sqrt{3}$$

Problem 2

$n > 1$

$$(n^3+2n^2+n)^n > 4^n (n!)^3$$

Sol 1

Since $n > 1$: by AM-GM

$$\frac{1^3+2^3+\cdots+n^3}{n} > \sqrt[n]{(n!)^3}$$

$$\Leftrightarrow \frac{n^2(n+1)^2}{4n} > \sqrt[n]{(n!)^3}$$

$$\Leftrightarrow (n^3+2n^2+n)^n > 4^n (n!)^3 \quad \blacksquare$$

Sol2

$$\left(\frac{n+1}{2}\right)^n = \left(\frac{1+2+\cdots+n}{n}\right)^n > n!$$

$$\left(\frac{n(n+1)^2}{4}\right)^n = \left(\frac{n+1}{2}\right)^{2n}\cdot n^n > (n!)^2\cdot n^n > (n!)^3 \qquad \blacksquare$$

Sol 3 By induction on $n \geq 2$

$n=2$ is obvious. ✓

Suppose

$$n^n(n+1)^{2n} > 4^n(n!)^3$$

we need

$$(n+1)^{n+1}(n+2)^{2n+2} > 4^{n+1}((n+1)!)^3$$

by induction hypothesis, it is enough to prove

$$(n+1)^{n+1}(n+2)^{2n+2} \geq 4(n+1)^3 n^n (n+1)^{2n}$$

$$\iff (n+2)^{2n+2} \geq 4\,(n+1)^{n+2} \cdot n^n$$

but $(n+2)^{n+2} > (n+1)^{n+2}$ 1

and by AM-GM:

$$(n+2)^n = \Big(2+2+\underbrace{1+\cdots+1}_{n-2}\Big)^n$$

$$\geq n^n \cdot 4 \qquad 2$$

Now, 1, 2 give the desired result

Problem 3

$P_1 = \{1, 25\}$

$P_2 = \{2, 26\}$

$\vdots \qquad \vdots$

$\{24, 48\}$

$\{49, 73\}$

$\vdots \qquad \vdots$

$P_{1008} = \{1992, 2016\}$

$P_{1009} = \{2017\}$

$P_{1010} = \{2018\}$

$P_{1011} = \{2019\}$

$P_{1012} = \{2020\}$

such a set can take ≤ 1 element of each of the P_i's

$\Rightarrow |S| \leq 1012$

$\{1, 2, \ldots, 24,$
$49, \ldots, 72$
$\vdots$
$1969, \ldots, 1992$
$2017, 2018, 2019, 2020\}$

5.12

$S_1 = \{24, 48, \ldots, 2016\}$

$S_r = \{ n \mid 1 \le n \le 2020, n \equiv r [24] \}$

$|S_0| = |S_5| = |S_6| = \cdots = |S_{23}| = 84$

$|S_1| = |S_2| = |S_3| = |S_4| = 85$

we pick ≤ 42 from S_i

we pick ≤ 43 from S_i

$\Rightarrow$ we pick $\le 84 \times 24 + 4$
$= 1012$

Problem 4

$$\frac{1}{x^2}+\frac{y}{xz}+\frac{1}{z^2}=\frac{1}{1441}$$

sol 1 $1441 = 11\cdot 131$

$$1441(x^2+xyz+z^2)=x^2z^2$$

Let $d=\gcd(x,z)$, $x=dx_1$, $z=dz_1$

$$\Rightarrow 1441(x_1^2+x_1yz_1+z_1^2)=d^2x_1^2z_1^2$$

Now, $\gcd(x_1^2+x_1yz_1+z_1^2,\ x_1z_1)=1$

$$\Rightarrow x_1^2z_1^2 \mid 1441 \Rightarrow x_1=z_1=1$$

$\Rightarrow 1441(y+2) = d^2$

$\Rightarrow y+2 = 1441 m^2$

$\Rightarrow x = z = d = 1441 m$

$\Rightarrow$

$(x, y, z) = (1441m, 1441m^2 - 2, 1441 m)$

(easily checked).

$\boxed{\text{Sol 2}}$ $1441(x^2 + 2xyz + z^2) = x^2 z^2$

$\Rightarrow 11 \mid x^2 z^2 \Rightarrow 11 \mid xz$, wlog

$11 \mid x \Rightarrow 11 \mid z$, similarly, $131 \mid x, z$

$\Rightarrow x = 1441a, \; z = 1441b$

$$a^2 + aby + b^2 = 1441a^2b^2$$

$\Rightarrow ab \mid a^2 + b^2$

$a = da_1, \; b = db_1$

$\left[a_1b_1 \mid a_1^2 + b_1^2 \Rightarrow a_1 \mid b_1^2, \; b_1 \mid a_1^2 \right.$

$\left. \Rightarrow a_1 = b_1 = 1 \right] \Rightarrow a = b$

$\Rightarrow y + 2 = 1441a^2$

$(x, y, z) = (1441a, 1441a^2 - 2, 1441a)$ ■

Problem 5

2

$\square \Rightarrow BP = b - c$

$CD = s - b$

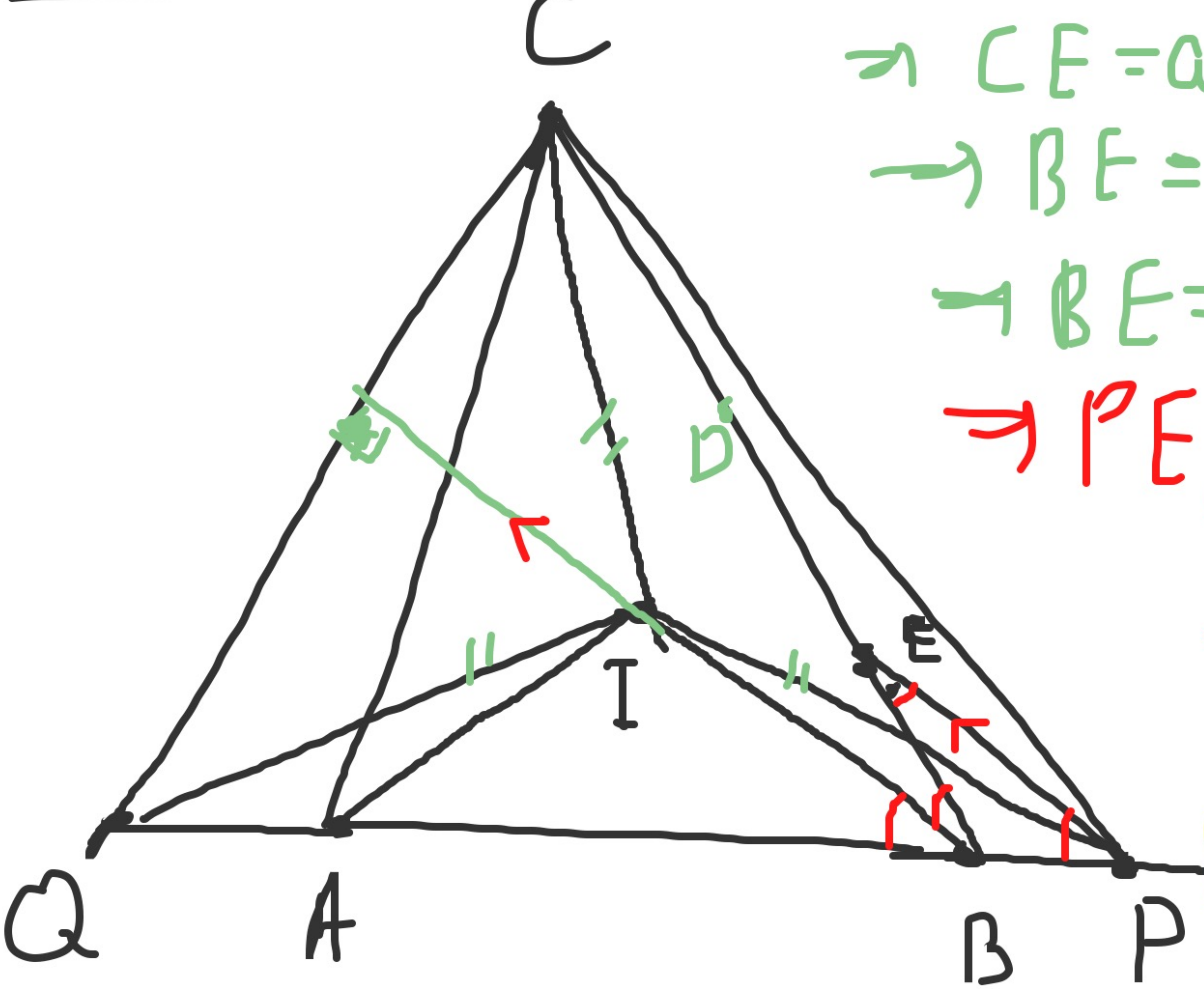
$\rightarrow CE = a + c - b$

$\rightarrow BE = b - c$

$\rightarrow BE = BP$

$\Rightarrow PE \parallel BI$

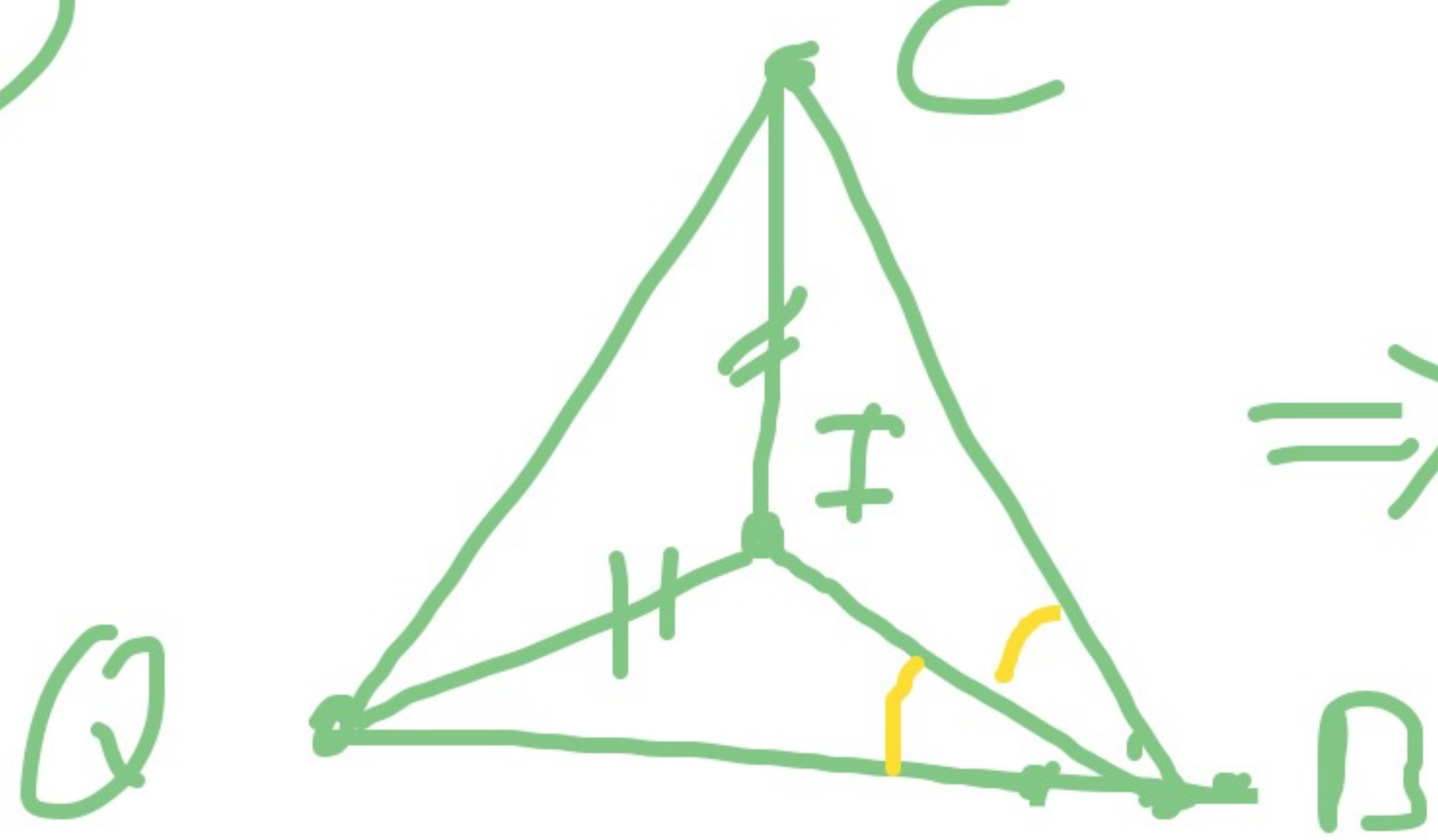
$\perp CQ$



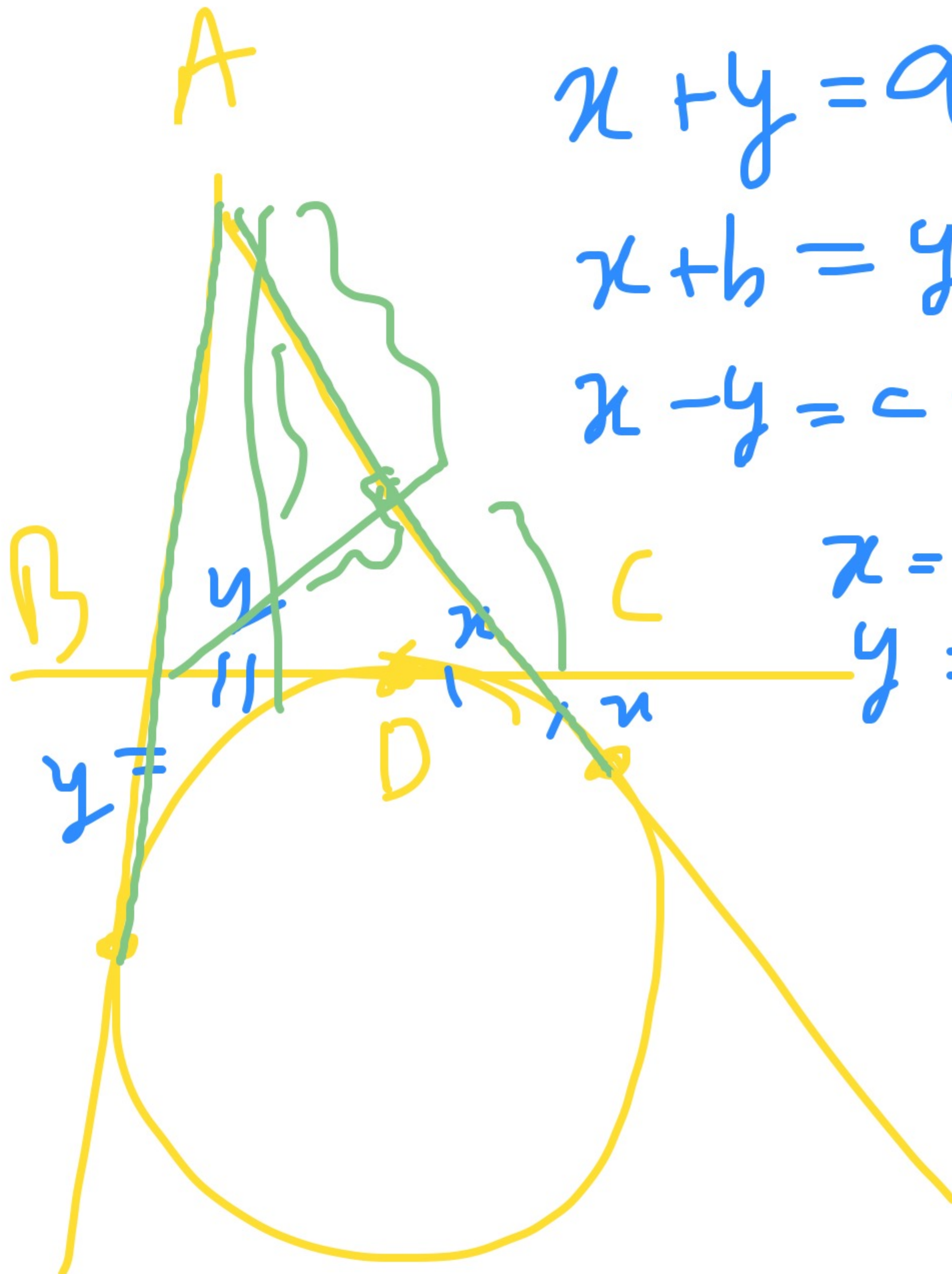
$BI \perp CQ$

$\Uparrow$

1



$\Longrightarrow$ $\boxed{BC = BQ}$, $\boxed{AP = AC}$



$x + y = a$

$x + b = y + c$

$x - y = c - b$

$x = s - b$

$y = s - c$

$CD = s - b$

$BD = s - c$