Lecture 4

Winter Camp

Number Theory

6 Jan 2021

Level 2

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

In the last lecture we showed that

① When $x=0$ the only solution is $\rightarrow (0,1,3,1)$

② When $x>0$, t is even $\quad t = 2t'$

③ When $x>0$ and $y=0$ $\rightarrow$ no solutions

④ When $x>0$, $y>0$ and $t=0$ $\rightarrow (1,2,7,0)$

⑤ When $x, y, t > 0$ we have three cases:

<u>Case 1</u>: $\quad 5^{t_1} + 1 = 3^x \cdot 2^{2(y-1)}$

<u>Case 2</u>: $\quad 5^{t'} + 3^x = 2^{2(y-1)}$

<u>Case 3</u>: $\quad 5^{t'} + 2^{2(y-1)} = 3^x$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

<u>Case 1</u>: $5^{t'} + 1 = 3^x \cdot 2^{2(y-1)}$

$(\text{mod } 2) \longrightarrow 2 \mid 3^x 2^{2(y-1)} \Rightarrow y \geq 2$

$(\text{mod } 4) \longrightarrow 1+1 \equiv 0 \pmod 4 \quad \Rightarrow\Leftarrow !!$

<u>Case 2</u>: $5^{t'} + 3^x = 2^{2(y-1)}$

$(\text{mod } 2) \longrightarrow 2 \mid 2^{2(y-1)} \Rightarrow y \geq 2$

$(\text{mod } 5) \longrightarrow 3^x \equiv 4^{y-1} \pmod 5$

$3^x \equiv (-1)^{y-1} \pmod 5 \Rightarrow x$ is even

$(\text{mod } 4) \longrightarrow 1 + (-1)^x \equiv 0 \pmod 4 \Rightarrow x$ is odd

$\Rightarrow\Leftarrow !!$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

<u>Case 3</u>: $5^{t'} + 2^{2(y-1)} = 3^x$

$\pmod 2 \longrightarrow y \geq 2$

$\pmod 4 \longrightarrow 1 + 0 \equiv (-1)^x \pmod 4 \longrightarrow$ x is even

$x = 2x'$

$$5^{t'} = 3^{2x'} - 2^{2(y-1)}$$

$$5^{t'} = \underbrace{\left(3^{x'} - 2^{(y-1)}\right)}_{5^{\alpha}} \underbrace{\left(3^{x'} + 2^{(y-1)}\right)}_{5^{\beta}}$$

$\alpha + \beta = t'$

$\gcd\left(3^{x'} - 2^{y-1}, 3^{x'} + 2^{y-1}\right) = 1 \Rightarrow \gcd(5^{\alpha}, 5^{\beta}) = 1$

$\Rightarrow \alpha = 0$ or $\beta = 0$

but $\beta > \alpha \Rightarrow \alpha = 0$

$\Rightarrow \boxed{3^{x'} - 2^{(y-1)} = 1}$

Solve in nonnegative integers the equation $5^t + 3^x 4^y = z^2$.

$$3^{x'} - 2^{(y-1)} = 1$$

when $y=2$: $\quad x'=1 \longrightarrow (2,2,2,13)$

$$5^2 + 9 \cdot 16 = 13^2$$

when $y>2$:

$(\text{mod } 4) \longrightarrow (-1)^{x'} \equiv 1 \pmod 4 \Rightarrow x'$ is even

$x' = 2x''$

$$3^{2x''} - 1 = 2^{(y-1)}$$

$$(3^{x''}-1)(3^{x''}+1) = 2^{(y-1)}$$

$\gcd(3^{x''}-1, 3^{x''}+1) = 2$ and $3^{x''}+1 > 2$

$\Rightarrow 3^{x''} - 1 = 2 \Rightarrow x'' = 1 \Rightarrow x' = 2$

$3^{x'} + 2^{y-1} = 5^{t'} \Rightarrow 3^2 + 8 = 5^{t'}$

$\Rightarrow x$

Find all integer solutions to the equation $x^2 = y^2(x + y^4 + 2y^2)$.

When $x=0 \Rightarrow y=0 \longrightarrow (0,0)$

$y=0 \Rightarrow x=0$

Assume that $x, y \neq 0$

Note that $y^2 | x^2 \Rightarrow y | x \Rightarrow x = ym$

$(ym)^2 = y^2(ym + y^4 + 2y^2) \Rightarrow m^2 = ym + y^4 + 2y^2$

Let $\gcd(y, m) = d$ and $y = dz$, $m = dn$, $\gcd(z, n) = 1$

$\Rightarrow d^2n^2 = d^2zn + d^4z^4 + 2d^2z^2$

$\Rightarrow n^2 = zn + d^2z^4 + 2z^2$

$\Rightarrow z | n^2 \Rightarrow z = 1$ because $\gcd(z, n) = 1$

$n^2 = n + d^2 + 2$

Find all integer solutions to the equation $x^2 = y^2(x + y^4 + 2y^2)$.

$$n^2 - n = d^2 + 2$$

$$\Rightarrow \quad 4n^2 - 4n + 1 = 4(d^2+2) + 1$$

$$\Rightarrow \quad (2n-1)^2 = 4d^2 + 9$$

$$\Rightarrow \quad (2n-1)^2 - 4d^2 = 9$$

$$\Rightarrow \quad \underbrace{(2n-1-2d)}_{a'=2a-1}\underbrace{(2n-1+2d)}_{b'=2b-1} = 9$$

$$\begin{cases} 2n-2d = 2a \\ 2n+2d = 2b \end{cases} \Rightarrow \begin{cases} n-d = a \\ n+d = b \end{cases} \Rightarrow \begin{cases} n = \frac{b+a}{2} \\ d = \frac{b-a}{2} \end{cases}$$

$a \equiv b \pmod 2 \Rightarrow a' \equiv b' \pmod 4$, $\quad a'b' = 9$

$$\begin{cases} 9, 1 \\ -9, -1 \\ 1, 9 \\ -1, -9 \end{cases} \Rightarrow \begin{cases} b=5, a=1 \\ b=-4, a=0 \\ b=1, a=5 \\ b=0, a=-4 \end{cases} \Rightarrow \begin{cases} n=3, d=2 \\ n=-2, d=-2 \\ n=3, d=-2 \\ n=-2, d=2 \end{cases}$$

$y = d$, $x = d^2 n$

$$\Rightarrow \quad (x,y) = (0,0), (12,2), (-12,2), (-8,2), (-8,-2)$$

Prove that there are not intgers a and b with conditions,

i) $16a - 9b$ is a prime number.

ii) ab is a perfect square.

iii) $a + b$ is also perfect square.

Assume that there exists $a, b \in \mathbb{Z}$ such that

$$\begin{cases} 16a - 9b = p & \text{where } p \text{ is prime} \\ ab = n^2 \\ a+b = m^2 \end{cases}$$

Since $ab > 0$, then $a, b > 0$ or $a, b < 0$

but $a+b > 0$ so $a, b > 0$

Assume that $\gcd(a,b) = d$, $\begin{matrix} a = dx' \\ b = dy' \end{matrix}$, $\gcd(x', y') = 1$

$\Rightarrow d^2 x' y' = n^2 \Rightarrow x' y' = \left(\frac{n}{d}\right)^2 \in \mathbb{Z}$

$\Rightarrow \begin{matrix} x' = x^2 \\ y' = y^2 \end{matrix}$, $x, y \in \mathbb{Z}_{\geq 0}$

$\Rightarrow a = dx^2, b = dy^2$

Prove that there are not intgers a and b with conditions,

i) $16a - 9b$ is a prime number.

ii) ab is a perfect square.

iii) $a + b$ is also perfect square.

$$16(dx^2) - 9(dy^2) = p \Rightarrow d \mid p \Rightarrow d = 1 \text{ or } p$$

<u>Case 1</u>: $d = p$

$$16x^2 - 9y^2 = 1$$

$$(4x - 3y)(4x + 3y) = 1$$

$$4x + 3y > 0 \Rightarrow 4x - 3y > 0$$

$$\Rightarrow \begin{cases} 4x - 3y = 1 \\ 4x + 3y = 1 \end{cases} \Rightarrow y = 0 \Rightarrow x = \frac{1}{4}$$

$$\Rightarrow \Leftarrow !!$$

Prove that there are not intgers a and b with conditions,

i) $16a - 9b$ is a prime number.

ii) ab is a perfect square.

iii) $a + b$ is also perfect square.

<u>Case 2</u>: $d = 1$

$a = x^2$
$b = y^2$ $\Longrightarrow \begin{cases} 16x^2 - 9y^2 = p \\ x^2 + y^2 = m^2 \end{cases}$

$(4x)^2 - (3y)^2 = p \Rightarrow (4x-3y)(4x+3y) = p$

$4x+3y > 0 \Rightarrow 4x-3y > 0$ and $4x+3y \geq 4x-3y > 0$

$\Rightarrow \begin{cases} 4x-3y = 1 \longrightarrow 4x = 3y+1 \\ 4x+3y = p \end{cases}$

$\Rightarrow x \equiv 1 \pmod 3$

$\Rightarrow x = 3k+1$, $k \geq 0$

$\Longrightarrow y = 4k+1$

$x^2 + y^2 = (3k+1)^2 + (4k+1)^2 = 25k^2 + 14k + 2 = m^2$

Prove that there are not intgers a and b with conditions,

i) $16a - 9b$ is a prime number.

ii) ab is a perfect square.

iii) $a + b$ is also perfect square.

$$25\left(25k^2 + 14k + 2\right) = 25m^2$$

$$(25k)^2 + 14(25k) + 50 = (5m)^2$$

$$(25k+7)^2 + 1 = (5m)^2$$

$$\Rightarrow \quad 1 = (5m)^2 - (25k+7)^2$$

$$\Rightarrow \quad 1 = \underbrace{(5m - 25k - 7)}_{1}\underbrace{(5m + 25k + 7)}_{1}$$

$$\Rightarrow \quad 25k + 7 = 0 \quad \Rightarrow \quad \Leftarrow \; !!$$

$$5m = 1$$

Prove that there are not intgers a and b with conditions,

i) $16a - 9b$ is a prime number.

ii) ab is a perfect square.

iii) $a + b$ is also perfect square.

Another way to solve $25k^2 + 14k + 2 = m^2$

$$(5k+1)^2 = 25k^2 + 10k + 1 < m^2$$

$$(5k+2)^2 = 25k^2 + 20k + 4 > m^2$$

$$(5k+2)^2 > m^2 > (5k+1)^2 \quad \Rightarrow \Leftarrow !!$$

Determine all triples (m, n, p) satisfying :

$$n^{2p} = m^2 + n^2 + p + 1$$

where m and n are integers and p is a prime number.

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.