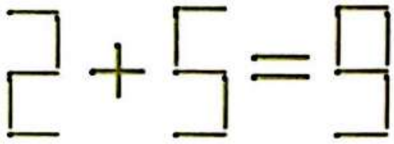
Combinatorial Number Theory

Lesson by Senya, groups L4 & L4+

It is time to accept the reality which is that nowadays at many international olympiads the NT is very likely to have combinatorial flavour (if NT is present at all...). The reason for this is that we are not that good at coming up with some pure NT that has some interesting idea and is elementary at the same time.

Problem 1. Move just one match to obtain a correct equality.



Problem 2. Set of all positive integers is split into several (finitely many) subsets. Prove that one of these subsets has the following property: for any positive integer m there are infinitely many numbers in this set that are divisible by m.

Problem 3 (IMO 2020, P5). A deck of $n > 1$ cards is given. A positive integer is written on each card. The deck has the property that the arithmetic mean of the numbers on each pair of cards is also the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that the numbers on the cards are all equal?

Problem 4. There are n pairwise co-prime numbers, each being between 1 and $(2n-1)^2$. Prove that one of the must be prime.

Problem 5. Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

Problem 6. Is it possible to write positive integers in the cells of a 2022×2022 board in such a way that for any rectangle on this board, the sum of the numbers placed in it is a perfect square if and only if the rectangle is square itself?

Problem 7. Determine all integers $n2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.