Email training, N3

September 8-14

Problem 3.1. Find all triples (a, b, c) such that $a = (b + c)^2$, $b = (c + a)^2$ and $c = (a + b)^2$.

Solution 3.1. Since a, b and c are squares of real numbers it means all of them are non negative. Assume $a \neq b$. Then

$$a - b = (b + c)^2 - (c + a)^2 = (b - a)(b + 2c + a).$$

Since $a - b \neq 0$, so $b + 2c + a = -1$, which is impossible since $b + 3c + a \geq 0$. So $a = b$ and in the same way $a = b = c$. Putting in the equation we get $a = 4a^2$, so $a = 0$ either $a = \frac{1}{4}$.

Answer: $a = b = c = 0$ and $a = b = c = \frac{1}{4}$.

Problem 3.2. Find all integer solutions of the equation

$$1 + x + x^2 + x^3 + x^4 = y^4.$$

Solution 3.2. Note that

$$(2y^2)^2 = (2x^2 + x + 1)^2 - (x^2 - 2x - 3) < (2x^2 + x + 1)^2$$

when $x < -1$ or $x > 3$ and

$$(2y^2)^2 = (2x^2 + x)^2 + 3x^2 + 4x + 4 > (2x^2 + x)^2$$

always. So we get $(2x^2 + x)^2 \leq (2y^2)^2 < (2x^2 + x + 1)^2$ whenever $x \notin [-1, 3]$. By considering cases $x = -1, 0, 1, 2, 3$ separately we conclude that the only solutions are $(x, y) = (-1, \pm 1)$ and $(x, y) = (0, \pm 1)$.

Problem 3.3. Three prime numbers p, q, r and a positive integer n are given such that the numbers

$$\frac{p + n}{qr}, \frac{q + n}{rp}, \frac{r + n}{pq}$$

are integers. Prove that $p = q = r$.

Solution 3.3. Assume without lose of generality that $p \geq q \geq r$. Since p divides $(q + n) - (r + n) = q - r$, it follows $q = r$ because $0 \leq q - r < p$.

Then q divides $r + n$ implies q divides n. But we have q divides $p + n$, too, so we must in fact have $q \mid p$. Since p is prime, we must have $p = q = r$.

Problem 3.4. $a_1, a_2, ..., a_{100}$ are permutation of $1, 2, ..., 100$. $S_1 = a_1, S_2 = a_1 + a_2, ..., S_{100} = a_1 + a_2 + ... + a_{100}$. Find the maximum number of perfect squares from S_i

Solution 3.4. Let n be the number of squares among those numbers. As there is 71 possible squares less than $5050 = 1 + 2 + \cdots + 100$, we get trivial bound $n \leq 71$. We put $S_0 = 0$ and consider the subsequence that is perfect square. $S_{m_1} = k_1^2$, $S_{m_2} = k_2^2$, ..., $S_{m_n} = k_n^2$ with $m_1 < m_2 < \ldots < m_n$.

Whenever $S_{m_{i+1}}$ and S_{m_i} have different parities then

$$S_{m_{i+1}} - S_{m_i} = a_{m_i+1} + a_{m_i+2} + \ldots + a_{m_{i+1}-1},$$

contains odd summander. Since there are only 50, which means in the sequence S_{m_i} there are at most 50 consecutive squares. So, in total can be at most $50 + \frac{71-50}{2} < 61$ perfect squares, so $n \leq 60$.

It remains to give example for 60. Consider the sequence $a_i = 2i - 1$ for $1 \leq i \leq 50$. Then we get all $S_i = i^2$ for $1 \leq i \leq 50$. It remains to construct 10 more squares. Consider $a_{51+4i} = 2 + 8i$, $a_{52+4i} = 100 - 4i$, $a_{53+4i} = 4 + 8i$ and $a_{54+4i} = 98 - 4i$ for $0 \leq i \leq 7$. Then we get $S_{54+4i} = (52 + 2i)^2$. To get the last 2 square we arrange the remaining number in the following order

$$30, 40, 64, 66, 68, 6, 8, 14, 16, 32, 38, 46, 54, 62, 22, 24, 48, 56.$$

Then we get $S_{87} = 66^2 + 2 \cdot 134 = 68^2$ and $S_{96} = 70^2$. **Answer:** 60.

Problem 3.5. Is it possible to put positive integers in the cells of the table 7×7 such that the sum of number in any square 2×2 and any square 3×3 is an odd number.

Solution 3.5. Assume it's possible. Consider a square 6×6. It can be divided into 4 square of size 3×3 so the total sum of number in the square 6×6 will be a sum of 4 odd numbers, which is even. On other size the square 6×6 can be divided into 9 squares of size 2×2 which means the total sum of numbers in the square 6×6 will be a sum of 9 odd numbers, which is odd. We got contradiction.

Answer: Not possible.

Problem 3.6. The natural numbers from 1 to 50 are written down on the blackboard. At least how many of them should be deleted, in order that the sum of any two of the remaining numbers is not a prime?

Solution 3.6. Notice that if the odd (respectively even), numbers are all deleted, then the sum of any two remaining numbers is even and exceeds 2, so it is certainly not a prime. We prove that 25 is the minimal number of deleted numbers. To this end, we group the positive integers from 1 to 50 in 25 pairs, such that the sum of the numbers within each pair is a prime:

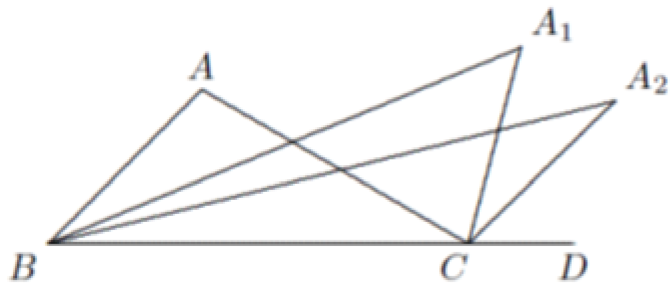
$$(1,2), (3,4), (5,6), (7,10), (8,9), (11,12), (13,16), (14,15), (17,20),$$
$$(18,19), (21,22), (23,24), (25,28), (26,27), (29,30), (31,36), (32,35),$$
$$(33,34), (37,42), (38,41), (39,40), (43,46), (44,45), (47,50), (48,49).$$

Since at least one number from each pair has to be deleted, the minimal number is 25. **Answer:** 25.

Problem 3.7. In the triangle ABC one has $\angle A = 96^o$. The segment BC is extended to an arbitrary point D. The angle bisectors of angles ABC and ACD intersect at A_1, and the angle bisectors of A_1BC and A_1CD intersect at A_2 and so on... the angle bisectors of A_4BC and A_4CD intersect at A_5. Find the size of BA_5C in degrees.

Solution 3.7. -

Since A_1B and A_1C bisect $\angle ABC$ and $\angle ACD$ respectively, $\angle A = \angle ACD - \angle ABC = 2(\angle A_1CD - \angle A_1BC) = 2\angle A_1$, therefore $\angle A_1 = \frac{1}{2}\angle A$.



Similarly, we have $A_{k+1} = \frac{1}{2}A_k$ for $k = 1, 2, 3, 4$. Hence

$$A_5 = \frac{1}{2}A_4 = \frac{1}{4}A_3 = \frac{1}{2^3}A_2 = \frac{1}{2^4}A_1 = \frac{1}{2^5}A = \frac{96^\circ}{32} = 3^\circ.$$

Problem 3.8. Let $ABCD$ is a parallelogram. A point M is drawn on the line AB such that $\angle MAD = \angle AMO$, where O is the point of intersection of the diagonals of the parallelogram. Prove that $MD = MC$.

Solution 3.8. -

Extend MO to cut CD at N. Since $\angle MAD = \angle AMN$, $AMND$ is an isosceles trapezoid. By symmetry, $AM = NC$ so that $AMCN$ is a parallelogram. Hence $\angle MDC = \angle AND = \angle MCD$ and therefore $MC = MD$.