Winter Camp 2021

Algebra Level 2

Day 1 (1)

31 December 2020

Problem 1. Find all functions $f : \mathbb{R}/\{0\} \to \mathbb{R}$ satisfying the conditions:

1:05

$\downarrow$

1:20

- $f(1) = 1$
- $f(\frac{1}{x+y}) = f(\frac{1}{x}) + f(\frac{1}{y})$
- $(x+y)f(x+y) = xyf(x)f(y)$

for all x, y with $xy(x+y) \neq 0$.

$$(x+1)f(x+1) = xf(x) \Longrightarrow f(n) = \frac{1}{n} \qquad n \in \mathbb{Z}^+$$

$$f\left(\frac{1}{x+1}\right) = f\left(\frac{1}{x}\right) + 1 \Rightarrow f\left(\frac{1}{n+1}\right) = n+1 \qquad n \in \mathbb{Z}^+$$

$$z = \frac{1}{2x} : \underbrace{f(z) = 2f(2z)}$$

$$\forall z \in \mathbb{R}/\{0\}$$

$$x = y : \begin{cases} f\left(\frac{1}{2x}\right) = 2f\left(\frac{1}{x}\right) \\ 2x\, f(2x) = x^2 f(x)^2 \end{cases} \to \underbrace{f(x)^2 = \frac{2f(2x)}{x}}$$

$$\Rightarrow \underbrace{f(x)^2 = \frac{f(x)}{x}} \qquad f(x) \neq 0 \Rightarrow f(x) = \frac{1}{x}.$$

$\boxed{\exists c : f(c) = 0}$

$y = c \quad : \quad (x+c)\, f(x+c) = 0$

$\Rightarrow \boxed{f(x+c) = 0} \quad \forall\, x \neq -c$

$\boxed{f(1) = 1}$ $\quad \Rightarrow \quad f(x) = 0 \quad \forall\, x \in \mathbb{R}, \; x \neq 0$

$\Rightarrow\Leftarrow \quad \Rightarrow \nexists\, c$

$\Rightarrow \quad f(x) = \frac{1}{x} \quad \forall\, x \in \mathbb{R} / \{0\}$

1:36 $\downarrow$ 1:51

Problem 2. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ satisfying the conditions:

- $f(x+1) = f(x) + 1$ for all x from $\mathbb{Q}^+$
- $f(x^2) = \underbrace{f^2(x)}_{f(x)\cdot f(x)}$ for all x from $\mathbb{Q}^+$

$\boxed{f(x+n) = f(x)+n}$ $\quad n \in \mathbb{Z}^+$

$x = \frac{p}{q} \qquad p, q \in \mathbb{Z}^+ \qquad f\left(\frac{p}{q}+n\right) = f\left(\frac{p}{q}\right)+n$

$$\cdot f\left(\left(\frac{p}{q}+n\right)^2\right) = f^2\left(\frac{p}{q}+n\right) = \underbrace{f^2\left(\frac{p}{q}\right) + 2n f\left(\frac{p}{q}\right) + n^2}$$

$$= \underbrace{f\left(\frac{p^2}{q^2} + 2n\frac{p}{q} + n^2\right)}$$

$n = q$: $f\left(\left(\frac{p}{q}+q\right)^2\right) = \cancel{f^2\left(\frac{p}{q}\right)} + 2q f\left(\frac{p}{q}\right) + \cancel{q^2}$

$$= \underbrace{f\left(\frac{p^2}{q^2}\right)} + 2p + q^2 = \cancel{f^2\left(\frac{p}{q}\right)} + 2p + \cancel{q^2}$$

$\Rightarrow f\left(\frac{p}{q}\right) = \frac{p}{q} \qquad \forall\, p, q \in \mathbb{Z}^+$

Problem 3. Cauchy Equation (additive function) with monotonicity.
For a function $f : \mathbb{R} \to \mathbb{R}$. If $f(x+y) = f(x) + f(y)$ and f is increasing, then prove $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

$f(x) \geq f(y) \quad \forall x \geq y$

$$\boxed{f(x) = x f(1) \quad \forall x \in \mathbb{Q}}$$

$1, 2, \ldots$

$f(x+1) = f(x) + f(1)$

$f(n) = n f(1)$ استقراء

$f(n+1) = (n+1) f(1)$

$$\boxed{f(x) = x f(1) \quad \forall \; x \in \mathbb{Z}^+}$$

$x = y = 0: \quad f(0) = 0$

$y = -x: \quad f(0) = 0 = f(x) + f(-x) \to \boxed{f(-x) = -f(x)} \Rightarrow f$ odd

$x \in \mathbb{Z}^+$

$f(-x) = -f(x) = -x f(1)$

$$\Rightarrow \boxed{f(x) = x f(1) \quad \forall \; x \in \mathbb{Z}}$$

$$x=\frac{p}{q}>0:\ f\left(\frac{p}{q}\right)=f\left(\frac{p-1}{q}+\frac{1}{q}\right)=f\left(\frac{p-1}{q}\right)+f\left(\frac{1}{q}\right)$$
$$=p\,f\left(\frac{1}{q}\right)$$

- $f(nx)=nf(x) \quad n\in\mathbb{Z}$

$$f(nx)=f((n-1)x)+f(x)$$
$$=f((n-2)x)+2f(x)$$
$$=\dots$$
$$=nf(x)$$

$$f\left(n\cdot\frac{p}{q}\right)=nf\left(\frac{p}{q}\right)$$

$$n=q:\ f(p)=q\,f\left(\frac{p}{q}\right)$$
$$\Rightarrow f\left(\frac{p}{q}\right)=\frac{p}{q}f(1)$$
$$\Rightarrow \boxed{f(x)=xf(1) \quad \forall x\in\mathbb{Q}}$$

$\exists x_0:\ f(x_0)\neq x_0 f(1)$

if $f(x_0)>x_0 f(1)$ ($0<f(1)$ افترض)

$$\frac{f(x_0)}{f(1)}>x_0 \quad \Rightarrow \exists r\in\mathbb{Q}:\ \frac{f(x_0)}{f(1)}>r>x_0$$

$$\overset{\text{increasing}}{\Longrightarrow} \underbrace{f(r)}_{=rf(1)}\geq f(x_0)$$

$$f(x_0)>rf(1)$$

$$\Rightarrow\Leftarrow$$

$$\Rightarrow \boxed{f(x)=xf(1) \quad \forall x\in\mathbb{R}}$$

$$f(x+y)=f(x)+f(y)$$

Problem 4. $f:\mathbb{R}\to\mathbb{R}$. If f is additive and $f(x^2)=xf(x)$, then prove $f(x)=f(1)x$ for all $x\in\mathbb{R}$.

$$\boxed{x \to x+1}$$

$$f(x^2+2x+1) = (x+1)f(x+1) = (x+1)\left(f(x)+f(1)\right)$$

$$= \cancel{f(x^2)} + 2f(x) + \cancel{f(1)} = \cancel{xf(x)} + xf(1) + f(x) + \cancel{f(1)}$$

$$\Leftrightarrow \quad f(x) = xf(1)$$

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Problem 5. $f : \mathbb{R} \to \mathbb{R}$. If f is additive and $f(x^2) = f^2(x)$, then prove $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

$x \to x+1$: $f(x^2+2x+1) = f^2(x+1) = (f(x)+f(1))^2$

$$\cancel{f(x^2)} + 2f(x) + \cancel{f(1)} = \cancel{f^2(x)} + 2f(x)f(1) + \cancel{f(1)^2}$$

$\longrightarrow f(1) = 0$ or 1

$$f(x^2) = f^2(x) \geq 0$$

$a = x^2$ $\quad \boxed{f(a) \geq 0} \quad \forall a \geq 0$

Hint f increasing $\Leftrightarrow f(x) \geq f(y)$ $\quad x \geq y$

$x = y+t$, $t \geq 0$

$$f(x) = f(y+t) = f(y) + \underbrace{f(t)}_{\geq 0} \geq f(y) \longrightarrow f \text{ increasing}$$

Prob.3 $\longrightarrow f(x) = f(1)x \quad \forall x \in \mathbb{R}$.

Problem 6. $f : \mathbb{R}/\{0\} \to \mathbb{R}$. If f is additive and $f(x) = x^2 f(\frac{1}{x})$, then prove $f(x) = f(1)x$ for all $x \in \mathbb{R}/\{0\}$.

Problem 7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition:

$$f(x^2 - y) = xf(x) - f(y)$$

for all x, y from $\mathbb{R}$

Problem 8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^4 + f(y)) = y + f^4(x) \qquad \forall x, y \in \mathbb{R}$$

Problem 9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(2x^2 + y + f(y)) = 2y + 2f^2(x) \qquad \forall x, y \in \mathbb{R}$$

Problem 10. Find all functions $f : \mathbb{R}/\{0\} \to \mathbb{R}/\{0\}$ such that

$$f(x+y) = x^2 f(\frac{1}{x}) + y^2 f(\frac{1}{y})$$

Problem 11. $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y + xy) = f(x) + f(y) + f(xy)$$

Prove that f is additive.