Intensive Training

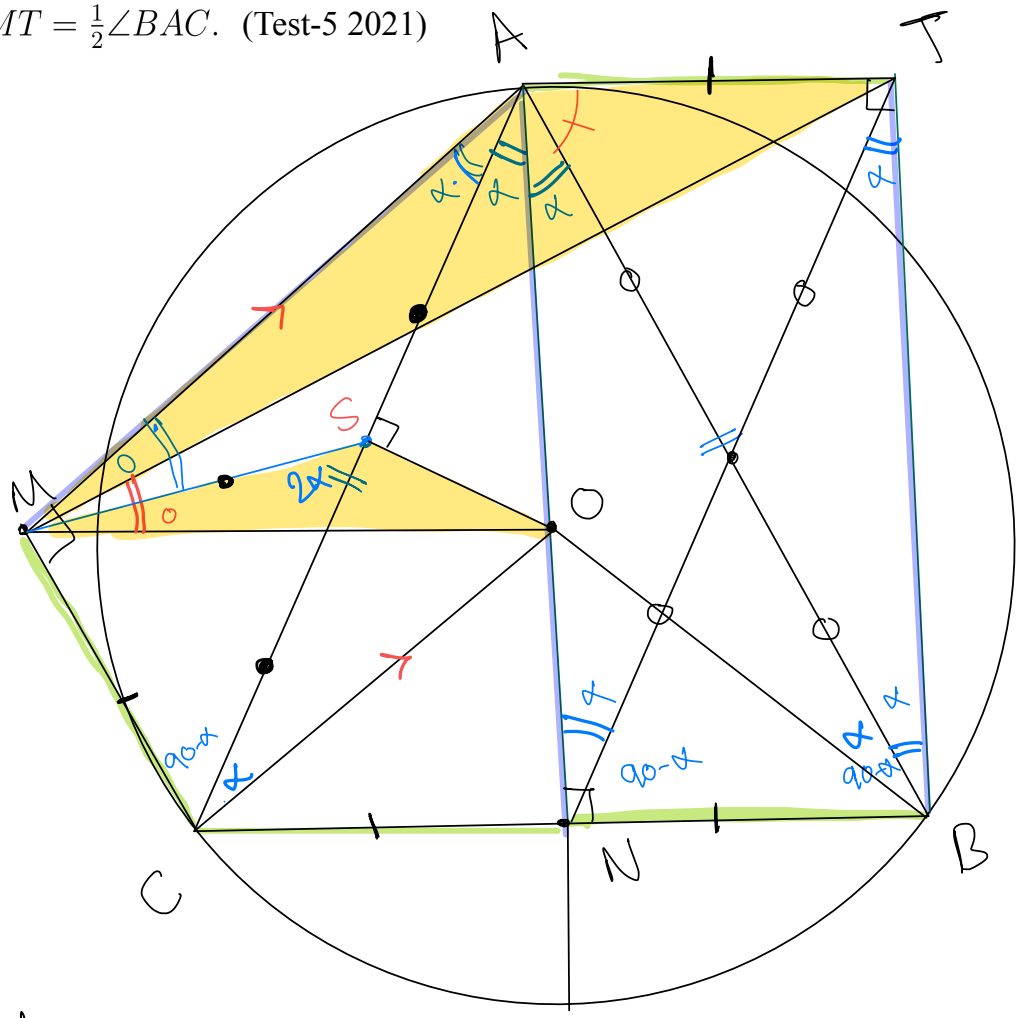
Geometry

Day 2
25 March 2021

Includes solutions for:

Exam-5 Problem 4
EGMO 2017 P1
EGMO 2018 P1
JBMO 2013 P2
JBMO 2014 P2

Problem 4. Let ABC be an isosceles triangle ($AB = AC$) with its circumcenter O. Point N is the midpoint of the segment BC and point M is the reflection of the point N with respect to the side AC. Suppose that T is a point so that $ANBT$ is a rectangle. Prove that $\angle OMT = \frac{1}{2}\angle BAC$. (Test-5 2021)



Solution 1:

Let S be the midpoint of AC and $\angle BAC = 2\alpha$.

Note that $\angle SMA = \alpha$

We want to show that $\triangle TAM \sim \triangle OSM$

Since $\angle AMC = 90$, and S is a midpoint, $\angle MSC = 2\angle SAM$

$= 2\angle SAN = 2\alpha \Rightarrow \angle MSO = 90 + 2\alpha = \angle MAT$ (1)

$\triangle ASO \sim \triangle ANC$: $\dfrac{SO}{AT} = \dfrac{SO}{CN} = \dfrac{AO}{AC} = \dfrac{AS}{AN}$

$AS = SC = MS \Rightarrow \dfrac{MS}{AM} = \dfrac{AS}{AM} = \dfrac{AS}{AN}$

$\Rightarrow \dfrac{SO}{AT} = \dfrac{MS}{AM}$ (2)

From (1), (2), $\triangle TAM \sim \triangle OSM \Rightarrow \angle OMS = \angle TMA$

$\Rightarrow \angle OMT = \angle SMA = \angle SAM = \alpha = \frac{1}{2}\angle BAC$ $\square$

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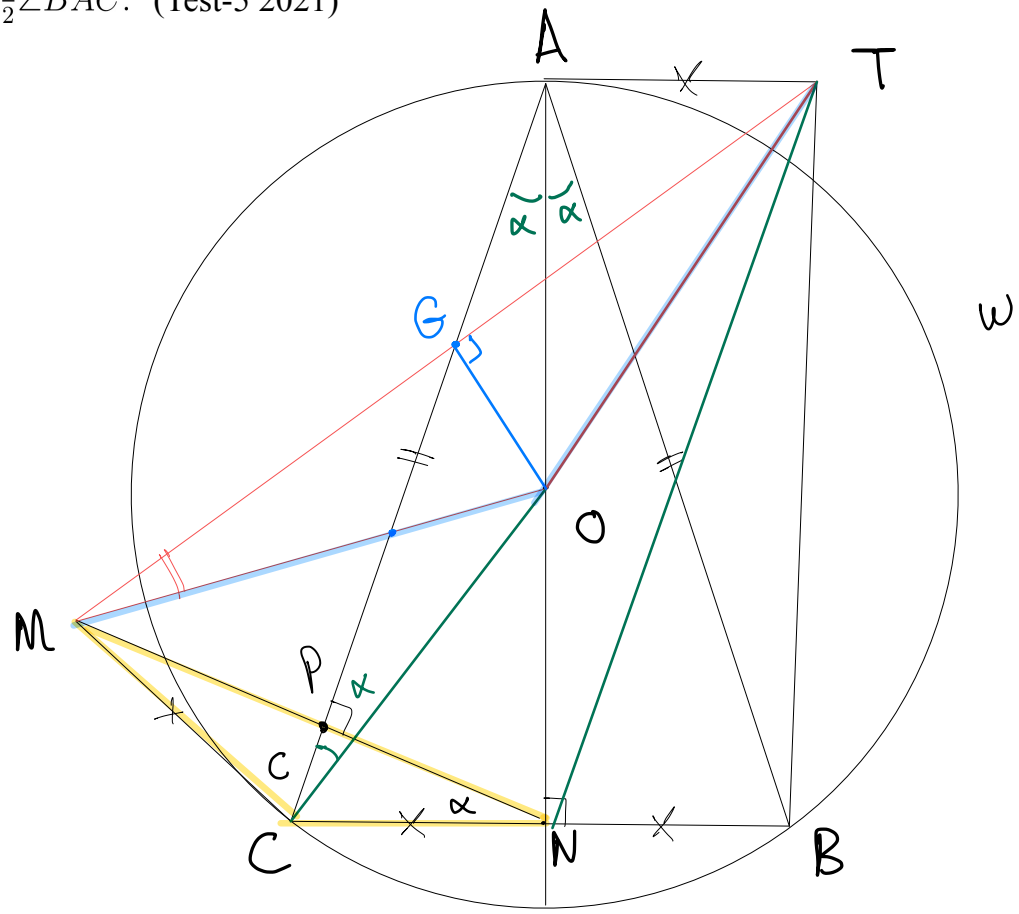


Solution 2: Let $G = TM \cap AC$ and $P = MN \cap AC$.

We want to show that $OGMC$ is cyclic

Note that A, O, N are collinear $OA = OC \Rightarrow \angle OCA = \alpha$

Also note that $CM = CN = NB = AT$. Since $AT = CN$ and $AT \parallel CN$, $ATNC$ is a parallelogram $\Rightarrow TN \parallel AC$

In $\triangle MTN$: P midpoint of MN, $PG \parallel TN \Rightarrow$ G midpoint of MT

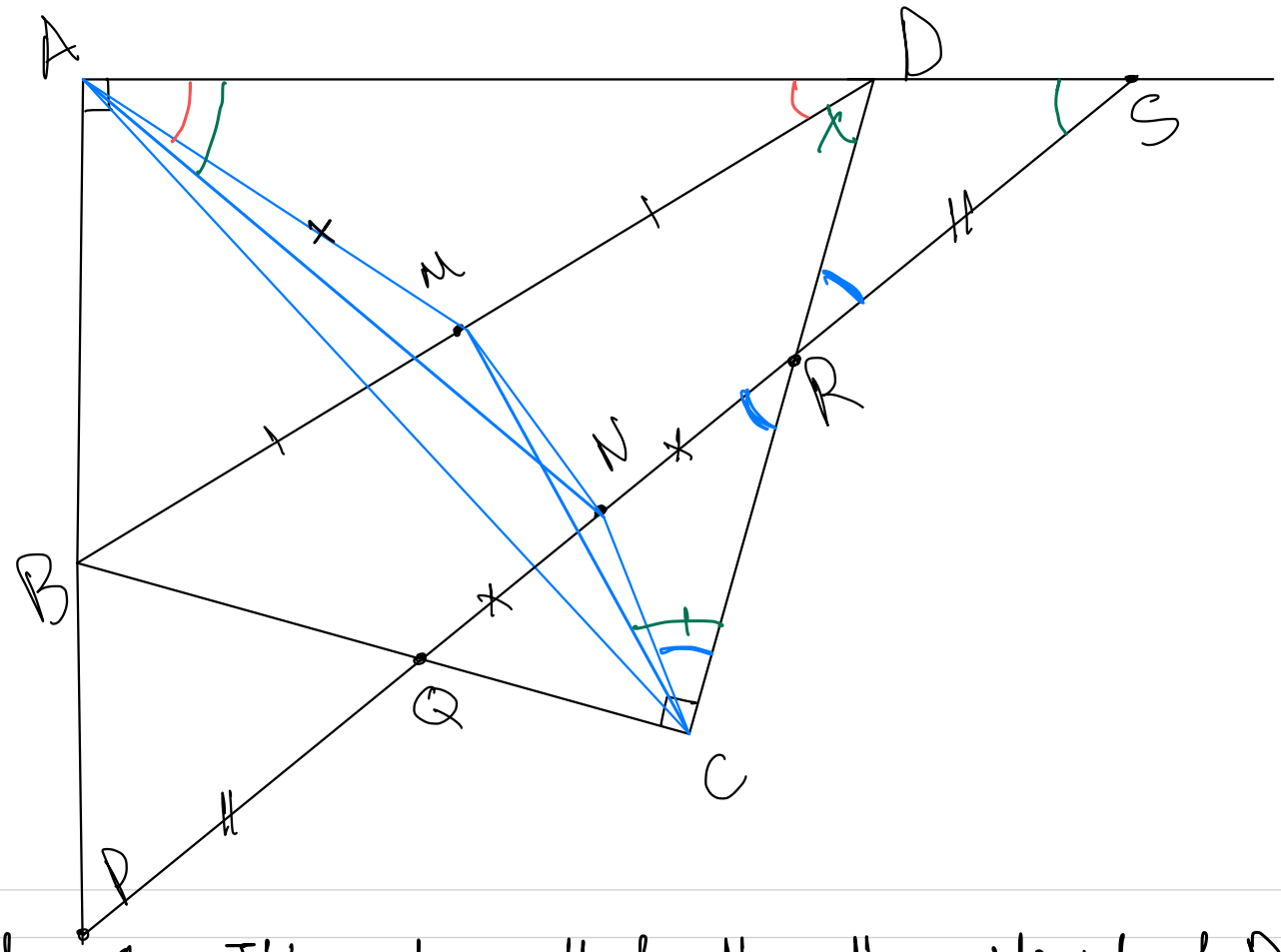
$$\left.\begin{array}{l} MC \text{ is a tangent to } \omega \Rightarrow \mathcal{P}_\omega(M) = MC^2 \\ TA \text{ is a tangent to } \omega \Rightarrow \mathcal{P}_\omega(T) = MT^2 \end{array}\right\} OM = OT$$

Finally, in $\triangle OMT$, OG is a median $\Rightarrow OG \perp TM$

$\Rightarrow \angle OGT = \angle OCM = 90$ so $OGMC$ is cyclic

$\Rightarrow \angle GMO = \angle GCO = \alpha$ $\square$

5. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90°$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$.Let the midpoint of BD be M and the midpoint of QR be N.Prove that the points M, N, A and C lie on a circle. (EGMO 2017)



Solution 1: It's clear that N is the midpoint of RQ

In $\triangle ABD$: $AM = MD = MB \Rightarrow \angle MAD = \angle MDA$

In $\triangle APS$: $AN = NS = NP \Rightarrow \angle NAS = \angle NSA$

In $\triangle QRC$: $QN = NR = NC \Rightarrow \angle NCR = \angle CRN$

In $\triangle BCD$: $BM = CM = MD \Rightarrow \angle MCD = \angle CDM$

$\angle NAM = \angle NAS - \angle MAD = \angle NSA - \angle MDA$

$\angle NCM = \angle MCD - \angle NCR = \angle CDM - \angle CRN$

However, in $\triangle DSR$: $\angle ADR = \angle DSR + \angle DRS$

$\angle ADM + \angle BDC = \angle ASN + \angle NRC \Rightarrow \angle NAM = \angle NCM$

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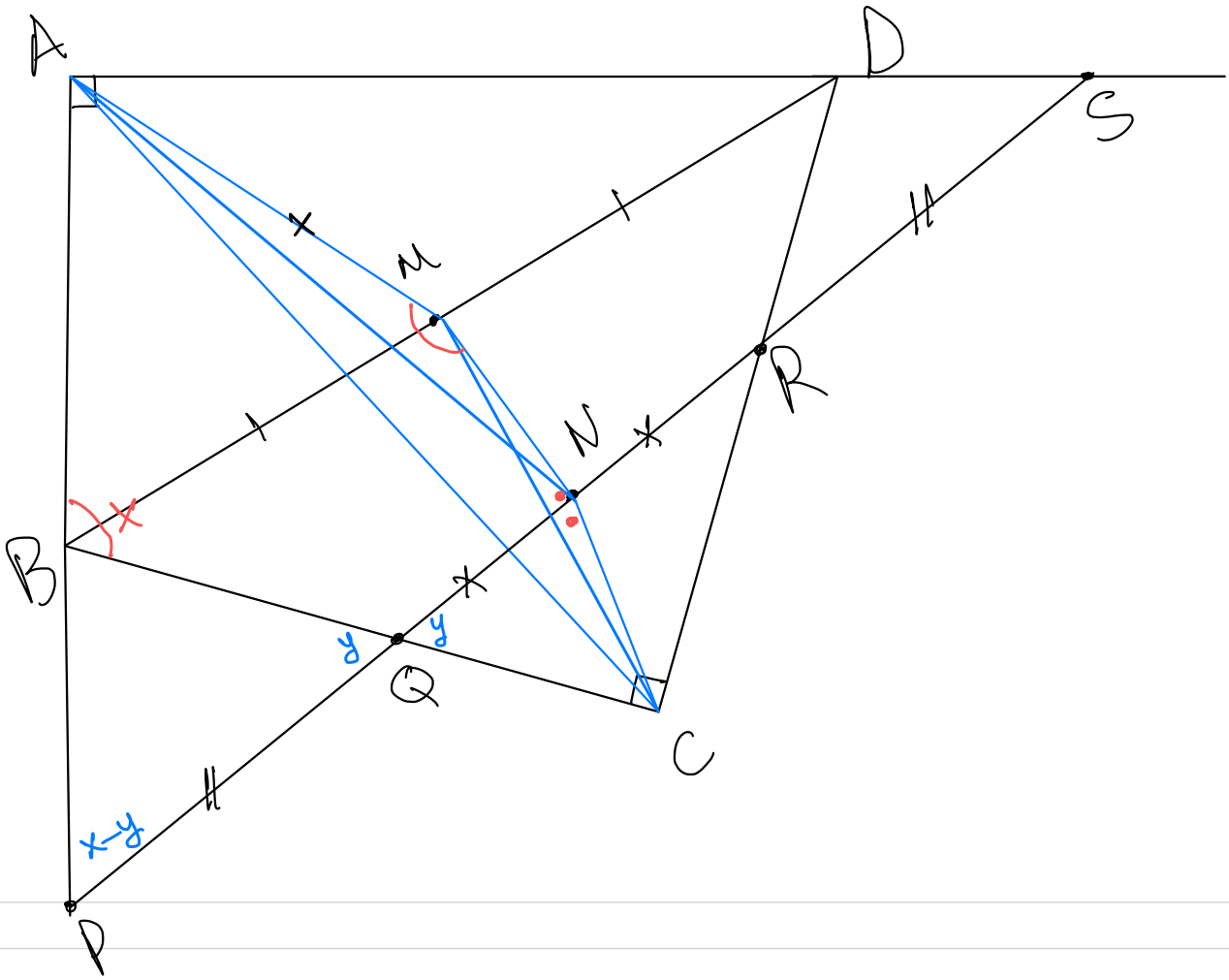


Solution 2: It's clear that N is the midpoint of RQ

Let $\angle ABC = x$, $\angle RQC = y$.

M is the center of $(ABCD)$; therefore, $\angle AMC = 360 - 2x$ (1)

We want to show that $\angle ANC = \angle AMC$

Note that $\angle QNC = 180 - 2\angle NQC$ (because NC is a median)

$= 180 - 2y$ (2)

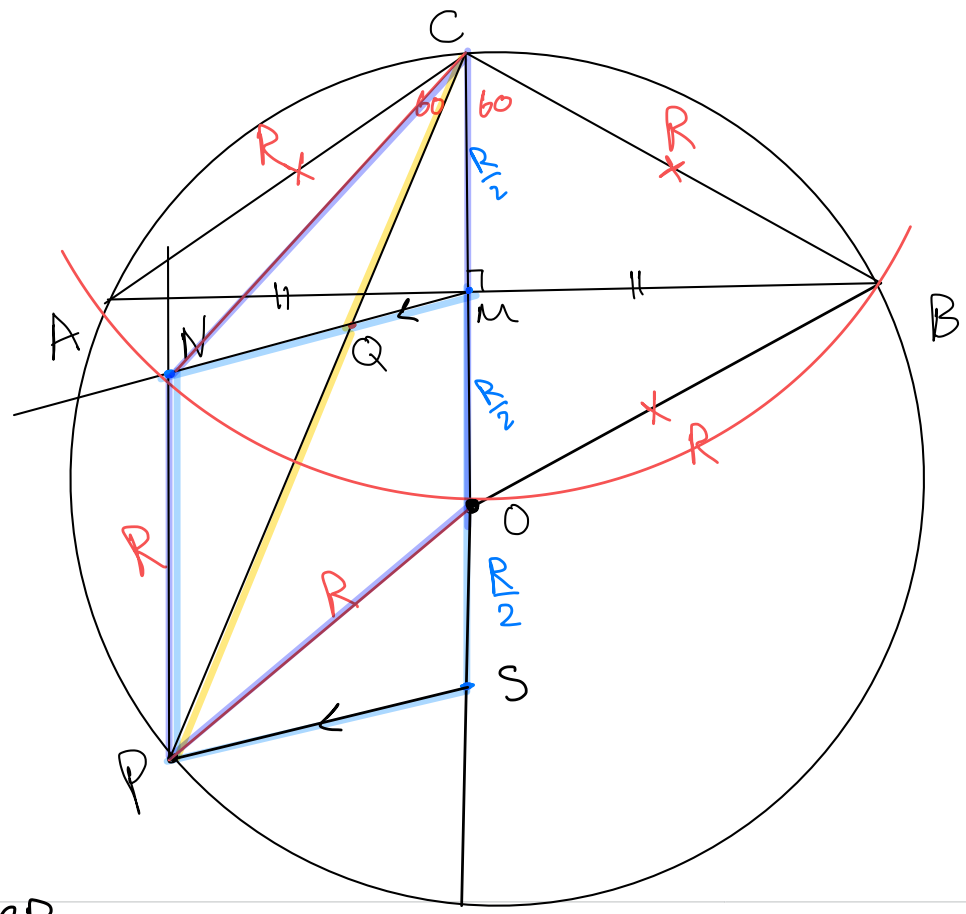
$\angle APN = \angle BPQ = \angle ABQ - \angle BQP = x - y$

$\Rightarrow \angle ANQ = 180 - 2\angle NPA = 180 - (x-y)$ (3)

From (2) and (3), $\angle ANC = \angle ANQ + \angle QNC = 360 - 2x$

From (1) $\angle ANC = \angle AMC \Rightarrow AMNC$ is a cyclic. $\square$

6. Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point of the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N. Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P. (EGMO 2018)



In $\triangle OCB$:

$\angle OCB = 60$, $OC = OB \Rightarrow OC = OB = CB$

$\Rightarrow$ BM is a median $\Rightarrow$ $OM = MC = \frac{1}{2}R$

Let S be on the line OC such that S is not on the segment OC and $OS = \frac{R}{2}$.

In CSP: $\frac{CM}{CS} = \frac{1}{3} = \frac{CQ}{CP} \Rightarrow MQ \parallel PS$ *

Since $MN \parallel PS$, $MS \parallel NP$, $MNPS$ is a parallelogram

$\Rightarrow NP = SM = R$

Therefore, $NP = R = CO$, $NP \parallel CO \Rightarrow CNPO$ is a parallelogram

$\Rightarrow CN = OP = R$, so C is the center of the circle. $\square$

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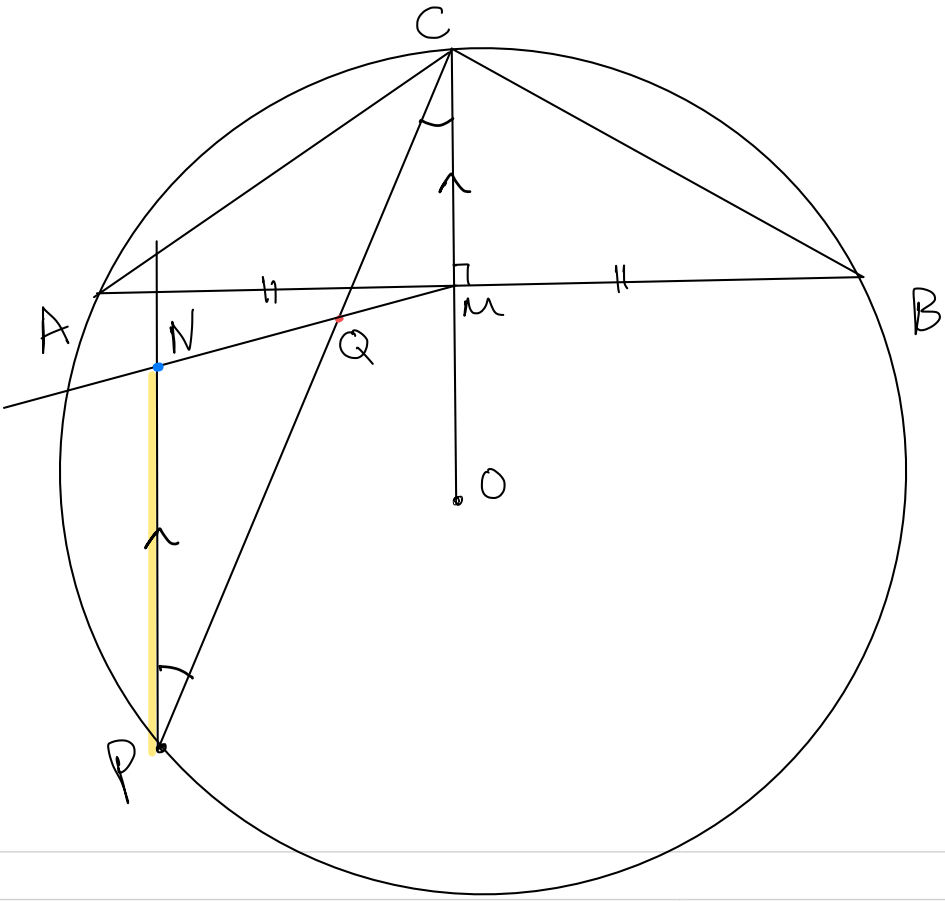


Solution 2:

$\triangle PNQ \sim \triangle CMQ \Rightarrow \dfrac{PN}{CM} = \dfrac{PQ}{CQ} = 2$

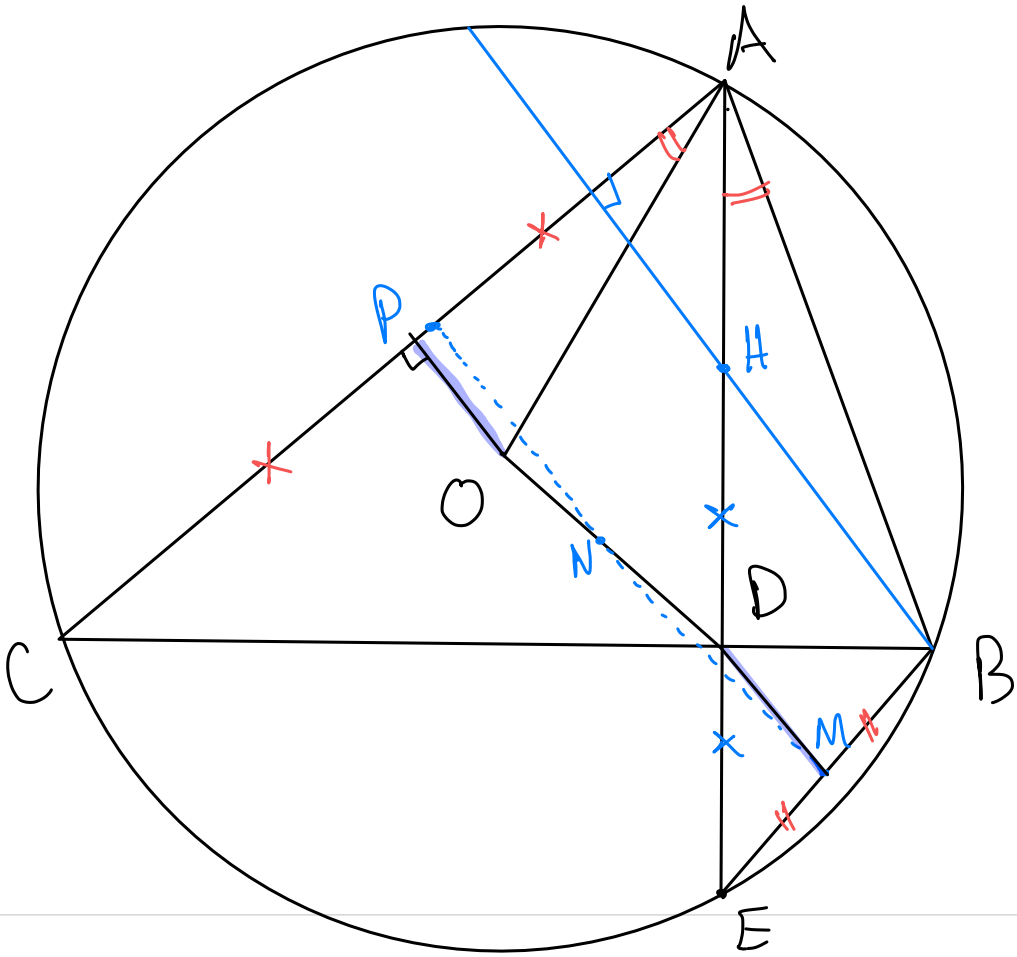
$\Rightarrow PN = 2CM$ constant

$PN \parallel CM$, PN is constant

N is on a circle because

P is on a circle $\square$

7. Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear. (JBMO 2013)



Since $\angle OAC = 90 - B \Rightarrow AD \perp BC \Rightarrow$ the orthacenter H lies on AD.

<u>lemma 1:</u> $HD = DE$ <u>lemma 2:</u> $PO = \frac{1}{2} BH$

Using lemma 1,2 we get $OP = \frac{1}{2} BH = \frac{1}{2} BE = MD$

$\Rightarrow MD = OP$ (1)

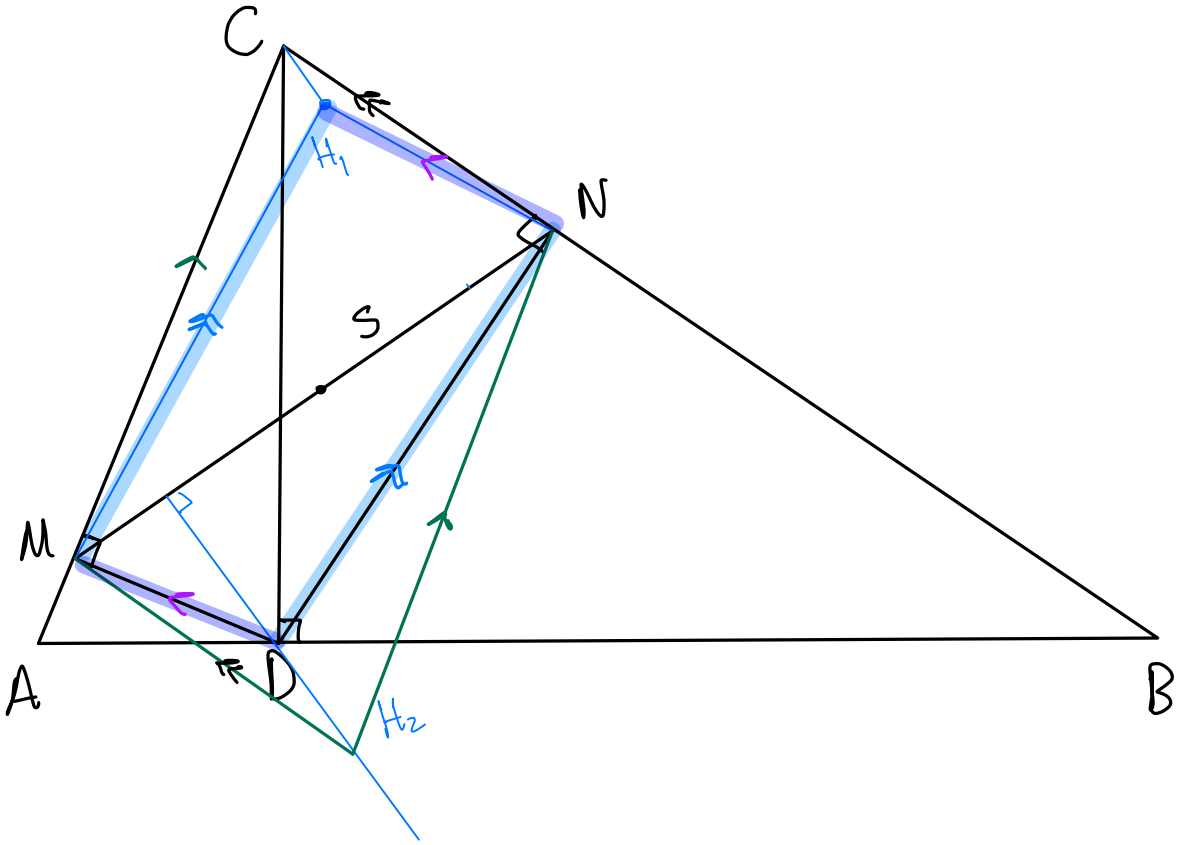
D is the midpoint of $HE \Rightarrow MD \parallel BH \Rightarrow MD \parallel OP$ (2)

From (1) and (2): $POMD$ is a parallelogram

Let $N' = MP \cap OD \Rightarrow N'$ is the midpoint of OD

$\Rightarrow N' = N \Rightarrow M, N, P$ are collinear $\square$

8. Consider an acute triangle ABC of area S. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocentres of the triangles MNC, respectively MND. Find the area of the quadrilateral AH_1BH_2 in terms of S. (JBMO 2014)



Let S be the midpoint of MN.

We will show that CH_1H_2D is a parallelogram

Note that $CH_1 \perp MN$, $H_2D \perp MN \Rightarrow CH_1 \parallel DH_2$

$NH_1 \perp AC$, $DM \perp AC \Rightarrow NH_1 \parallel DM$

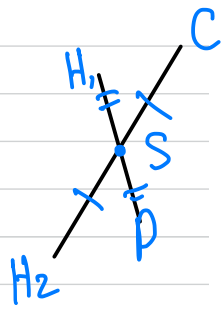
$MH_1 \perp NC$, $DN \perp NC \Rightarrow MH_1 \parallel DN$

$H_2N \perp MD$, $CA \perp MD \Rightarrow H_2N \parallel CA$

$H_2M \perp DN$, $CB \perp DN \Rightarrow H_2M \parallel CB$

In the parallelogram H_1NDM: S mid point H_1D

In the " CNH_2M: S is the mid point of H_2C

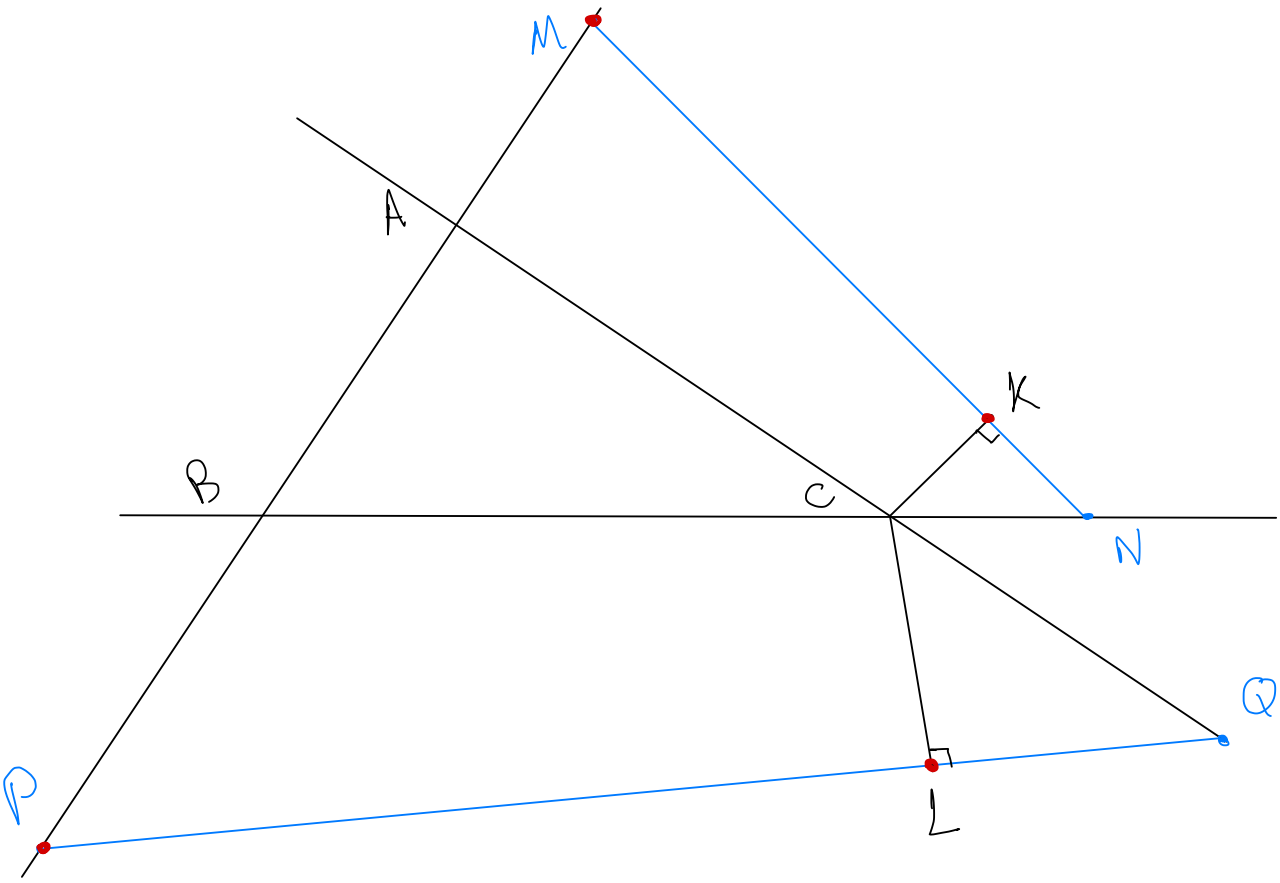


Therefore, CDH_2H_1 is a parallelogram

$H_1H_2 \perp AB$, $H_1H_2 = CD$

In H_1BH_2A, $[AH_1BH_2] = \frac{1}{2} AB \cdot H_1H_2 = \frac{1}{2} AB \cdot CD = S$ $\square$

9. In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic. (BMO 2013)



(left as a homework)