# Competition Preparation for Saudi Arabia Team 2021: Level 4

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## Lesson 1

# Markings, Colorings and Numberings

**Problems:**

1. Let there be $n$ light-bulbs in one row and let the light-bulb in the $k$-th position be ON and all others OFF. In one move, we're allowed to select three consecutive light-bulbs and change their states (turn them ON if they're OFF, and turn them OFF if they're ON). For which values of $n$ and $k$ can we reach a state where all the light-bulbs are OFF?

2. On a $29 \times 29$ board 99 $2 \times 2$ squares are placed without overlap. Prove that at least one more $2 \times 2$ square can be placed without overlap.

3. Let $n$ be a positive integer. A spider travels on a $(2n+1) \times (2n+1)$ board. In every move, it can travel two squares vertically (up or down), two squares diagonally or one square horizontally (left or right). Find the minimum number of moves the spider needs to pass through every single square. The spider is considered to have visited the starting square, the ending square after every move and the square between the starting and ending square of a move when the spider moves two squares. The starting point can be arbitrarily chosen.

4. An $n \times n$ grid is the collection of edges of all unit squares on an $n \times n$ board. Let an $n \times n$ grid be tiled by unit angles (two unit edges meeting at a $90^o$ angle) oriented up-left, up-right, down-left or down-right (thus, a unit square is tiled by two unit angles). Prove that the number of up-left unit angles is equal to the number of down-right unit angles.

5. (IMO 1999-P3) Let $n$ be a positive integer. On an $2n \times 2n$ certain squares are marked such that each square is adjacent to at least one marked square. Two squares are adjacent if they share exactly one side (Two diagonally connected squares are not adjacent and a square is not adjacent to itself). Find the smallest possible number of marked squares.

6. Let $n$ and $k$ be positive integers such that $1 < k \leq n^2$. Assuming we label the centers of squares with integer coordinates $(x, y)$, let an $n$-diamond be a collection of squares for which $|x|+|y| \leq n$ (For example, the $X$ pentomino is a 1-diamond). Show that the number of ways of placing $k$ dominoes ($2 \times 1$ or $1 \times 2$) on an $n$-diamond is larger than the number of ways of placing $k$ $2 \times 2$ squares on a $(2n+1) \times (2n+1)$ square, in both cases without any overlap.

7. On an infinite board two players take turns writing in $X$ and $O$ (respectively) with the winner being one who connects nine consecutive symbols belonging to them (horizontally, vertically or diagonally). Prove that the second player can prevent the first player from ever winning.

8. A queen sits at the center of a $(2n+1) \times (2n+1)$ board. Find the number of ways for the queen to reach the edge so that in each move the queen must move closer to the edge. The distance to the edge is the distance from the center of the square to the center of the closest square adjacent to the edge.

9. A king starts from one square of an $8 \times 8$ board and visits each square exactly once before returning to the original square without ever moving diagonally. Prove that the number of horizontal moves is not equal to the number of vertical moves (Hint: Can a $6 \times 6$ board be covered with an equal number of horizontal and vertical dominoes?)