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FUNCTIONAL EQUATIONS and more

Small reminder(?):
Function f is injective if equation $f(x) = f(y)$ implies $x = y$ (for different arguments f attains different values.)
Function f is periodic if there exists t s.t. for all x holds $f(x + t) = f(x)$.

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1. Determine if there exists injective function $f : \mathbb{R} \to \mathbb{R}$ such that for all real x holds $f(x^2) - (f(x))^2 \geqslant \frac{1}{4}$.

2. Function $f : \mathbb{R} \to \mathbb{R}$ for each real x satisfy $f(x) = f(2x) = f(1 - x)$. Prove that f is periodic.

3. Let $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ satisfy for all $x \in \mathbb{R}$ equation $f(x + 2) = f(x - 1)f(x + 5)$. Prove that f is periodic.

4. Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying for all real x equality $\frac{1}{x} f(-x) + f\left(\frac{1}{x}\right) = x$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}$ equation $4f(f(x)) = 2f(x) + x$. Prove that $f(a) = 0$ if and only if $a = 0$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying for all real x, y equality $xf(x) - yf(y) = (x - y)f(x + y)$.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying for each $x, y \in \mathbb{R}$

$$f(xf(x) + f(y)) = (f(x))^2 + y.$$

8. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(1) = \frac{1}{2}$ and for all $x, y \in \mathbb{R}_+$ holds

$$f(xy) = f(x)f\left(\frac{3}{y}\right) + f(y)f\left(\frac{3}{x}\right).$$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real a, b holds $|f(a) - f(b)| \leqslant |a - b|$. Prove that if $f(f(f(0))) = 0$, then $f(0) = 0$.

Functional Equations 2 and more

1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying for all reals x, y

$$f(x) \leqslant x \quad \text{and} \quad f(x+y) \leqslant f(x) + f(y).$$

2. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ holds

$$f(x^2 + y) = f(x^{27} + 2y) + f(x^4).$$

3. Determine all injective functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ holds

f(x)+S

$$f(f(x) + y) = f(x+y) + 1.$$

4. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ holds

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2).$$

5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(2) = 2$ and for all reals x, y holds

$$f(xy) = x^2 f(y) + y f(x).$$

6. Find all $f\colon \mathbb{R} \to \mathbb{R}$ such that for all reals x, y holds

$$f(y - xy) = f(x)y + (x-1)^2 f(y).$$

7. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying conditions

a) $f(x) + f(y) \geqslant xy$ for all reals x, y;

b) for each $x \in \mathbb{R}$ exists $y \in \mathbb{R}$ such that $f(x) + f(y) = xy$.

8. Find all pairs of nonconstant functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying for all reals x, y

$$f(x)f(y) = g(x)g(y) + g(x) + g(y).$$

L3 — FUNCTIONAL EQUATIONS 3 and more

Discussion of solutions of problems:

1. (**7 from 1PS**) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying for each x, $y \in \mathbb{R}$

$$f(xf(x) + f(y)) = (f(x))^2 + y.$$

2. (**9 from 1PS**) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that for all real a, b holds $|f(a) - f(b)| \leqslant |a - b|$. Prove that if $f(f(f(0))) = 0$, then $f(0) = 0$.

3. (**5 from 2PS**) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(2) = 2$ and for all reals x, y holds

$$f(xy) = x^2 f(y) + yf(x).$$

4. (**6 from 2PS**) Find all $f\colon \mathbb{R} \to \mathbb{R}$ such that for all reals x, y holds

$$f(y - xy) = f(x)y + (x-1)^2 f(y).$$

5. (**7 from 2PS**) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying conditions

a) $f(x) + f(y) \geqslant xy$ for all reals x, y;

b) for each $x \in \mathbb{R}$ exists $y \in \mathbb{R}$ such that $f(x) + f(y) = xy$.

6. (**8 from 2PS**) Find all pairs of nonconstant functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying for all reals x, y

$$f(x)f(y) = g(x)g(y) + g(x) + g(y).$$

7. (**Test 2 Problem**) Find all real a, b, c such that

$$\begin{cases} x^4 + y^2 + 4 = 5yz \\ y^4 + z^2 + 4 = 5zx \\ z^4 + x^2 + 4 = 5xy. \end{cases}$$