Problem 8

$$2x^6 + y^7 = 11.$$

$1^k, 2^k, \dots, (p-1)^k \rightsquigarrow$ $\boxed{\dfrac{p-1}{\gcd(p-1,k)}}$ – small.

$\boxed{k = 6, 7 \quad \boxed{6 \cdot 7 + 1 = 43,}}$ $\boxed{43 \in \mathbb{P}.}$

$\dfrac{42}{\gcd(42,6)} = \dfrac{42}{6} = 7$ non zero residues of $x^6 \bmod 43$

$\dfrac{42}{\gcd(42,7)} = \dfrac{42}{7} = 6$ non zero residues of $x^7 \bmod 43$.

$\boxed{1, 21, 41, 11, 16}$

10 $\boxed{x^2+y^2=3z^2}$

11 $x^2+y^2+z^2-2xyz=0$

10 $x^2+y^2=3z^2 \Rightarrow 3 \mid \boxed{x^2+y^2} \Rightarrow 3 \mid x^2,\ 3 \mid y^2 \Rightarrow$

$x=3x_1,\ y=3y_1$

x^2	0	1	2
$x^2 \bmod 3$	0	1	1

$= 0, 1$

$x^2+y^2 \bmod 3$ | $1+1$, $\boxed{0+0}$, $0+1$ | $\rightsquigarrow$ | 2, $\boxed{0}$, 1

$9x_1^2+9y_1^2=3z^2 \mid :3$

$3x_1^2+3y_1^2=z^2 \Rightarrow 3 \mid z^2 \Rightarrow z=3z_1.$

$3x_1^2+3y_1^2=9z_1^2 \mid :3 \Rightarrow \boxed{x_1^2+y_1^2=3z_1^2}$

By Fermat descent

$x=y=z=0$

$x^4 + y^4 + z^4 = 9u^4$

$0, -1, 1$

$\cancel{n^4 = 0, -1, 1}$

~~mod 9~~

$\boxed{\text{modulo } 5}$

Little Fermat Theorem.

p - prime

$x^{p-1} \equiv 1 \pmod p$

if $\gcd(p, x) = \boxed{1}$

if $5 \nmid x$ then.

$x^4 \equiv 1 \pmod 5$

Suppose $5 \nmid u$.

$9u^4 \equiv 4 \mod 5$

$\equiv 1$

$\boxed{x^4 + y^4 + z^4 \equiv 4 \mod 5}$

$\equiv$

$0,1 \quad 0\,1\,2\,3$

$\lightning$

$\Leftarrow$ so $5 \mid u^4$

$5 \mid x^4 \quad 5 \mid y^4 \quad 5 \mid z^4 \Rightarrow$

$\Uparrow$

$0,1 \quad 0,1 \quad 0,1$

$5 \mid x^4 + y^4 + z^4$

[13] $x^2 + y^2 + z^2 = x^2 y^2$

9] $k^2 + 4m$, $m^2 + 5k$ are square.

find all $k, m > 0$.

$$\boxed{x^2+y^2+z^2-2xyz=0}$$

$|x|+|y|+|z|$.

or.

$\boxed{2^m \mid x}$

Hint. mod 2

$x = 2x_1$
$y = 2y_1$
$z = 2z_1$

$x_2^2+y_2^2+z_2^2-\boxed{8}x_2y_2z_2=0$

$(0, 0, 0)$

12 $x^4+y^4+z^4=9u^4$

$4x_1^2+4y_1^2+4z_1^2-16x_1y_1z_1=0 \mid :4$

$$\boxed{x_1^2+y_1^2+z_1^2-4x_1y_1z_1=0}$$

$2 \mid x^2+y^2+z^2 \Rightarrow$

I x, y, z cannot bye odd all so at least 1 number is even.

this means that $4 \mid 2xyz \Rightarrow 4 \mid \boxed{x^2+y^2+z^2}$

$\boxed{0 \quad 0 \quad 0}$
1 1 0
1 0 0

x^2 mod 4

x	0	1	2	3
x^2 mod 4	0	1	0	1

$= 0, 1$

$\Downarrow$

$2 \mid x,\ 2 \mid y,\ 2 \mid z$

$$a^2+b^2+c^2=a^2b^2 \quad (*)$$

1) a, b, c are odd. Then $(2n+1)^2 \equiv 1 \pmod 4$, so

$a^2+b^2+c^2 \equiv 3 \mod 4$ ↯

$\quad \parallel$

$(ab)^2$

2) Two are odd and one is even. then

$a^2+b^2+c^2 \equiv 2 \mod 4$ ↯

$\quad \parallel$

$(ab)^2$

$\quad \equiv$

$0, 1$.

3) Two are ~~..~~ even and one is odd

$a^2+b^2+c^2 \equiv 1 \pmod 4$ ↯

$\quad \parallel$

$(ab)^2$

$\quad \equiv$

0

a, b, c

ab is even

$2 \mid a,\ 2 \mid b,\ 2 \mid c \Rightarrow$

$a = 2a_1,\ b = 2b_1,\ c = 2c_1$.

$\Downarrow (*)$

$$4a_1^2+4b_1^2+4c_1^2 = 16\, a_1^2 b_1^2 \;\Big|\, :4$$

$$a_1^2+b_1^2+c_1^2 = 4a_1^2b_1^2$$

Fent den.

$\boxed{a=b=c=0}$

$x=5x_1,\quad y=5y_1,\quad z=5z_1,\quad u=5u_1$

$\cancel{5^4}x_1^4+\cancel{5^4}y_1^4+\cancel{5^4}z_1^4=9\cdot\cancel{5^4}u_1^4 \quad \Big| :5^4$

$x_1^4+y_1^4+z_1^4=9u_1^4 \xrightarrow{\text{Fermat desc.}} x=y=z=u=0$.

$x^2+y^2+z^2=x^2y^2$ $\quad\boxed{\text{modulo } 4}$ $\quad$ Hint : $\quad\boxed{\text{USAMO}}$

(14) $2a^2+1$

$2b^2+1$ are not all perfect square.

$2(ab)^2+1$

$\underline{k > m}$

squ.

$$k^2 < k^2 + 4m < k^2 + 4k < k^2 + 4k + 4 = (k+2)^2$$

$$\Downarrow$$

$$\cancel{k^2} + 4m = (k+1)^2 = \cancel{k^2} + 2k + 1$$

$$\underset{\text{even}}{4m} = \underset{\text{odd}}{2k+1} \quad \lightning$$

No solution here.

Finally $(1,2)$, $(9,22)$, $(8,9)$ are only solution.

OK?

9] k^2+4m , m^2+5k are squares. find all $k, m > 0$.

"Square between squares".

$\boxed{\text{HINT}}$

I $m \geq k$ then

$$(m+3)^2 = m^2 + 6m + 9 > m^2 + 5m \geq m^2 + 5k > m^2$$

$(\dots)$.

• $m^2+5k = (m+1)^2 = \cancel{m^2} + 2m + 1$

$5k = 2m+1 \Rightarrow 2m = 5k-1$ (*)

$k^2+4m = k^2 + 2\cdot 2m = k^2 + 2\cdot(5k-1) = k^2+10k-2$ — square

$\underbrace{k^2+10k-2}_{\boxed{\text{square}}} < k^2+10k+25 = (k+5)^2$

$\Downarrow$!

$\cancel{k^2}+10k-2 \leq (k+4)^2 = \cancel{k^2}+8k+16$

$2k \leq 18 \Rightarrow k \leq 9$ but from (*) k is odd

$k = 1, 3, 5, 7, 9$, but then $k^2+10k-2$ is $\boxed{9}, 37, 73, 117, \boxed{169}$
$9 = 3^2$, $169 = 13^2$

$$\boxed{\begin{array}{cc} \multicolumn{2}{c}{\text{so}} \\ k=1, & k=9 \\ \Downarrow & \Downarrow \\ m=2 & m=22 \end{array}}$$

$(1,2)$, $(9,22)$

• $\cancel{m^2}+5k=(m+2)^2=\cancel{m^2}+4m+4$

$5k=4m+4 \Rightarrow 4m=5k-4$, $k^2+4m=k^2+5k-4$

$k^2+5k-4<k^2+6k+9=(k+3)^2$

$\Downarrow$!

$\cancel{k^2}+5k-4 \le (k+2)^2=\cancel{k^2}+4k+4$

$k \le 8$. $m=\frac{5}{4}k-1 \Rightarrow 4|k$

$k=4, 8$, $k^2+5k-4=32$ or 100
$100 = 00^2$

$m=\frac{5}{4}\cdot 8-1=9$.

$(8, 9)$

(14) Prove there is no $a, b > 0$ integers s.t

$$2a^2+1$$
$$2b^2+1$$
$$2(ab)^2+1$$

are all square.

Hint: Suppose a, b exist. $a, b > 1$

WLOG $a \geq b$ then

$4(2a^2+1)(2(ab)^2+1)$ is square ...

$$\| $$

$$(4a^2b+b)^2 < \underbrace{(4a^2b+b)^2 + 8a^2-b^2+4}_{\text{square}} < (4a^2b+b)^2 + 8a^2b+2b+1$$

$$= (4a^2b+b+1)^2$$

$\lightning$

Problem $ab+1 \mid a^2+b^2$ Prove $\dfrac{a^2+b^2}{ab+1}$ is square.

posit. integer

Vieta Jumping

$\boxed{\text{Vieta Jumping}}$

Viet formulas?

Reason $ax^2+bx+c=0$ (*) , $a \neq 0$, $\Delta = b^2-4ac \geq 0$

$\Rightarrow$ x_1, x_2 - solution of (*)

$$\boxed{\begin{aligned} x_1+x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned}}$$

Example

$x^2+x+1=0$

$\begin{cases} x_1+x_2=-1 \\ x_1x_2=1. \end{cases}$

$$\begin{cases} x_1 + x_2 = kb \\ x_1 x_2 = b^2 - k \end{cases}$$

$\boxed{x_1 = a}$

$a + x_2 = kb$

$a \cdot x_2 = b^2 - k$

$\Rightarrow$ $\boxed{\begin{array}{l} x_2 = \underbrace{k}_{\mathbb{Z}}\underbrace{b}_{\mathbb{Z}} - \underbrace{a}_{\mathbb{Z}} \\ x_2 = \dfrac{b^2 - k}{a} \neq 0 \end{array}}$ $\in \mathbb{Z}$.

<u>Moreover</u> $x_2 \neq 0$, We prove $x_2 > 0$. By contradiction

Suppose $\boxed{x_2 \leq -1}$ $\quad x_2 = \dfrac{b^2 - k}{a} \leq -1 \Rightarrow b^2 - k \leq -a \Rightarrow$

$\Rightarrow \boxed{k \geq b^2 + a}$

BUT

$k = \dfrac{a^2 + b^2}{ab + 1} = k \geq b^2 + a \Rightarrow a^2 + b^2 \geq (ab+1)(b^2 + a) = ab^3 + a^2 b + b^2 + a$

↯

So $x_2 \in \mathbb{Z}$, $\neq 0$, > 0 $x_2 > 0$ positive integer.

$\boxed{\text{Now}}$ Go back to beginning and assume $a+b$ has minimum sum.
but (x_2, b) is also solution, so (x_2, b) has greater sum

$$x_2 + \cancel{b} \geq a + \cancel{b} \ldots$$

(a, b)
(x_2, b)

$$\frac{b^2}{a} > \frac{b^2 - k}{a} = x_2 \geq a$$

$$\frac{b^2}{a} > a \Longrightarrow b^2 > a^2$$

$a \geq b$

$x_2 \neq 0$

$k = b^2$

$\|$

square.

Vieta jumping.

(a, b) $\quad a \geq b.$

$\downarrow$ produce quadratic equation in a

$\downarrow$ Assume $a+b$ smallest sum.

$f(X)$

$\downarrow$ take another root of q. eq. x_2.

proof that pair (x_2, b) has the same property as (a, b)

$\downarrow$ $x_2 + b \geq a + b$

$\downarrow$ Contradiction

$ab+1 \mid a^2+b^2 \Rightarrow \dfrac{a^2+b^2}{ab+1}$ is square

(a,b) $\quad \boxed{\dfrac{a^2+b^2}{ab+1} = k}$ $\quad k \in \mathbb{Z}_+$.

We need to prove k is square. $\quad$ Suppose k is not square

WLOG $\boxed{a} \geq b$ $\quad (a, \boxed{b})$

"Viet part" $\quad a^2+b^2 = k \cdot ab + k$ $\quad (a,b)$

$a^2 - a(k \cdot b) + b^2 - k = 0$

Assume (a,b) has $\boxed{\text{minimal sum}}$ $a+b$

$X^2 - X \cdot kb + b^2 - k = 0$

$X_1 = \boxed{a}$, $\quad \boxed{\boxed{X_2}\ ?}$ $\quad (a,b)$ $\quad (X_2, b)$.

$b^2 - bkb + b^2 - k \neq 0$