Email training, N12

December 8-17, 2019

Problem 12.1. Find the minimal possible value of the expression

$$|a-1|+|b-2|+|c-3|+|3a+2b+c|.$$

Solution 12.1. Let's apply the inequality $|x|+|y| \geq |x-y|$ several times:

$$|a-1|+|b-2|+|c-3|+|3a+2b+c| \geq$$
$$|a-1|+|b-2|+|3a+2b+c-(c-3)| =$$
$$|a-1|+|b-2|+|3a+2b+3| \geq$$
$$|a-1|+|b-2|+|\frac{3a+3}{2}+b| \geq$$
$$|a-1|+|\frac{3a+3}{2}+b-(b-2)| =$$
$$|a-1|+|\frac{3a+7}{2}| \geq |a-1|+|\frac{a+7}{3}| \geq$$
$$|a-1-(a+\frac{7}{3})| = \frac{10}{3}.$$

The equality holds when $c=3, b=2, a=-\frac{7}{3}$.

Problem 12.2. Let s is the number of divisors of positive integer n. Evaluate the product of divisors of n in terms of s and n.

Solution 12.2. Note, that if d is divisor of n, then so is also $\frac{n}{d}$. By grouping all pairs in this way we get pairs, where geometrical mean in each pair is equal $\sqrt{n}$. If $\sqrt{n}$ is a divisor of n then put him in separate group alone. So the product of all divisors will be equal $\sqrt{n}^s = n^{s/2}$.

Problem 12.3. Find the smallest integer n such that 2019! is not divisible by n^n.

Solution 12.3. We use the formula to calculate the power of integer $x \geq 2$ in $n!$, which is

$$V_x(n!) \geq [\frac{n}{x}]+[\frac{n}{x^2}]+[\frac{n}{x^3}]+\ldots+[\frac{n}{x^n}].$$

Note that when x is prime we have equality.

So

$$V_{47}(2017) = [\frac{2017}{47}]+[\frac{2017}{47^2}]+[\frac{2017}{47^3}]+\ldots = 42.$$

So 2017! isn't divisible by 47^{47}. Let's prive that for $n < 47$, we have 2017! is divisible by n^n.

For $n \leq 44$ we have $V_n(2017!) \geq [\frac{2017}{n}]+[\frac{2017}{n^2}] \geq [\frac{2017}{44}]+[\frac{2017}{44^2}] \geq 46$ and 2017! is divisible by n^n.

For $n=45$ we have $v_3(2017!) > [\frac{2017}{3}] > 90$ and $v_5(2017!) > [\frac{2017}{5}] > 45$, so 2017! is divisible by 45^{45}.

In the case $n=46$ we have $v_2(2017!) > [\frac{2017}{2}] > 46$ and $v_{23}(2017!) > [\frac{2017}{23}] > 46$, so 2017! is divisible by 46^{46}. We conclude that for $n \leq 46$ we have 2017! is divisible by n^n.

Problem 12.4. Find the maximal possible value of the expression

$$|...||x_1 - x_2| - x_3| - x_4| - \ldots | - x_{1990}|,$$

where $x_1, x_2, ..., x_{1990}$ is the permutation of numbers $1, 2, 3, \ldots, 1990$.

Solution 12.4. Note, that $|a-b|$ and $a+b$ have the same parity, so the answer has parity of

$$1+2+\ldots+1990 \equiv 1[2]$$

. So the answer is odd. Since for any $a, b \geq 0$ one has

$$|a-b| \leq max\{a, b\},$$

then we conclude that the answer is at most 1990. Since the answer is odd, we conclude that the value of expression can't be bigger than 1989. Let give example for 1989. Since

$$||||4k+2|-(4k+4)|-(4k+5)|-(4k+3)|=0,$$

therefore

$$|...|||2-4|-5|-3|-\ldots-(4k+2)|-(4k+4)|-(4k+5)|-(4k+3)|-$$
$$\ldots-1986|-1988|-1989|-1987|-1990|-1|=||0-1990|-1|=1989.$$

Problem 12.5. The numbers $12, 1, 10, 6, 8, 3$ (in written order) are written on the verticies of regular hexagon. On each step one allowed to choose any side of the hexagon and either increase by 1 values written on the vertices of the side, either dicrease both of them by 1. Is it possible to get the following configurations?

a) $14, 6, 13, 4, 5, 2$;
b) $6, 17, 14, 3, 15, 3$.

Solution 12.5. a) Yes. For example

$$(12, 1, 10, 6, 8, 3) \to (12, 1, 10, 6, 7, 2) \to$$
$$(14, 3, 10, 6, 7, 2) \to (14, 3, 10, 4, 5, 2) \to (14, 6, 13, 4, 5, 2).$$

b) Note that the sum of original number is equal 40 which is even. At each step the sum increases by 2, so it will always stay even. But $6+17+14+3+15+3=57$ is odd. So , not possible.

Problem 12.6. Let there are n regions on the plane, part of them are red, and the rest are blue. At each step one allowed to choose a region X, such that most of it's neighbour regions have different from X color and paint region X by opposite color. Prove, that this process can't be processed infinitely long time.

Solution 12.6. Let's we change the color of region A. Assume it has k neighbours having color of A and m neighbours having different from A color (we have $k < m$). Let's calculate the number of neighbour pairs having the same color. When we change color of A then k pairs will disappear and m new pairs will appear, so the number of pairs will increase by $m-k$. So this value is mono-invariant and is increasing number. Since the number of all neighbour pairs is finite, this process can't continue infinitely long.