Email training, N3

September 25-October 1

Problem 3.1. Prove that for all $n \geq 4$ the following inequalities hold $n! > 2^n$ and $2^n \geq n^2$.

Solution 3.1. For $n > 3$ one has

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot n > 1 \cdot 2 \cdot 2 \cdot 2^2 \cdot 2 \cdot \ldots \cdot 2 = 2^n.$$

For $n = 4$ one has $2^4 = 16 = 4^2$. By induction, for $n \geq 4$ one has

$$2^{n+1} \geq 2n^2 = (n+1)^2 + (n–1)^2 - 2 > (n+1)^2.$$

Problem 3.2. It is known that $a < 1$, $b < 1$ and $a + b \geq 0.5$. Prove that $(1-a)(1-b) \leq \frac{9}{16}$.

Solution 3.2. From the conditions of the problem follows that $1-a \geq 0$ and $1-b \geq 0$. By using the AM-GM inequality one gets

$$\sqrt{(1-a)(1-b)} \leq \frac{(1-a)+(1-b)}{2} = 1 - \frac{a+b}{2} \leq \frac{3}{4}.$$

By taking the square one gets the desired inequality.

Problem 3.3. Let a and b are divisors of n with $a > b$. Prove that $a > b + \frac{b^2}{n}$.

Solution 3.3. Since a and b are divisors of n, therefore $\frac{n}{a}$ and $\frac{n}{b}$ are divisors of n as well. So

$$1 \leq \frac{n}{b} - \frac{n}{a} = \frac{(a-b)n}{ab} < \frac{(a-b)n}{b^2}.$$

After multiplication by $\frac{b^2}{n}$ one gets

$$\frac{b^2}{n} < a - b.$$

Problem 3.4. Read the proof of Bernouli inequality. Conclude that $8^{91} > 7^{92}$ and for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

(https://www.youtube.com/watch?v=7BZWeWZoVcY).

Solution 3.4. First part.

$$\frac{8^{91}}{7^{91}} = \left(1 + \frac{1}{7}\right)^{91} > 1 + \frac{91}{7} > 7,$$

therefore
$$8^{91} > 7 \cdot 7^{91} = 7^{92}.$$

Second part.
$$\left(1 + \frac{5}{n}\right)^n > 1 + n \cdot \frac{5}{n} = 6,$$
therefore
$$1 + \frac{5}{n} > 6^{1/n}.$$
Also
$$\left(1 + \frac{-5}{6n}\right)^n > 1 + n \cdot \frac{-5}{6n} = \frac{1}{6},$$
$$\left(\frac{6n-5}{6n}\right)^n > \frac{1}{6},$$
$$6 > \left(\frac{6n}{6n-5}\right)^n,$$
$$6^{1/n} > \frac{6n}{6n-5} = 1 + \frac{5}{6n-5}.$$

Problem 3.5. Prove that for any positive integer $n \geq 3$ the following inequality holds
$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{3}{5}.$$

Solution 3.5. Let's prove by induction. For $n = 3$ one has
$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} > \frac{3}{5}.$$
For $n \geq 3$ one has

When moving from n to $n+1$ the left side increases by
$$\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0,$$
by positive number, so the inequality will still hold.

Problem 3.6. Let a, b, c are positive and less than 1. Prove that
$$1 - (1-a)(1-b)(1-c) > k,$$
where $k = max(a, b, c)$.

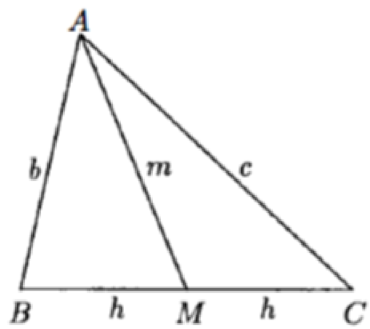
Solution 3.6. Since $0 < 1-a, 1-b, 1-c < 1$ therefore one may state that
$$1 - k > (1-a)(1-b)(1-c),$$
since in right side one multiplier is equal to $1-k$ and two others are positive and less than one. From that inequality immediately follows that
$$1 - (1-a)(1-b)(1-c) > k.$$

Problem 3.7. -

In the triangle ABC the median AM is drawn. Is it possible that the radius of the circle inscribed in the triangle ABM could be twice as large as the radius of the circle inscribed in the triangle ACM?

Solution 3.7. -

Let b, c, m and $2h$ be the lengths of AB, AC, AM respectively, and let r_B and r_C be the radii of the inscribed circles for triangles ABM, ACM.



Since the area of a triangle is given by half the circumference times the in-radius, and since triangles ABM, ACM have equal area (equal base and height) we have

$$\frac{1}{2}(b+h+m)r_B = \frac{1}{2}(c+h+m)r_C.$$

So, if $r_B = 2r_C$ then

$$b+h+m = \frac{1}{2}(c+h+m),$$

leading to

$$h+m+2b = c.$$

But h, m and c are sides of ΔAMC so $c \leq h+m$. Hence $b = 0$. and ΔABC is degenerate with $A = B$.

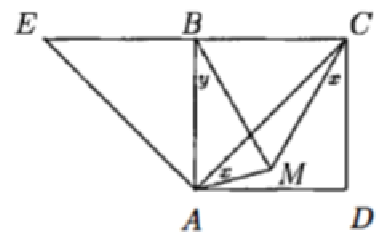
So the required solution is impossible unless both radii are zero.

Problem 3.8. -

A point M is chosen inside the square ABCD in such a way that $\angle MAC = \angle MCD = x$. Find $\angle ABM$.

Solution 3.8. -

Extend the line CB to E, as shown, with $BC = BE$, and construct the line AE.



Since $\angle ACM = (45 - x)^\circ$, and $\angle CAM = x^\circ$,

$$\angle AMC = (180 - x - (45 - x))^\circ = 135^\circ.$$

Furthermore, since $\angle AEB = 45^\circ$, quadrilateral $ECMA$ is cyclic. We now note that $\angle EAC = 90^\circ$, and so EC is a diameter of this exscribed circle. Therefore $BA = BM = BC$ (all radii of the exscribed circle). Thus ΔBAM is isosceles and $y = 180 - 2\angle BAM = 180 - 2(45 + x) = 90 - 2x$.