ALGEBRA

$a_1, a_2, \dots, a_n$ then

$$AM := \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$GM := \sqrt[n]{a_1 a_2 \dots a_n}$$

$$HM := \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}$$

$$QM := \sqrt{\frac{a_1^2 + a_2^2 \dots a_n^2}{n}}$$

$$PM_t = \sqrt[t]{\frac{a_1^t + a_2^t + \dots + a_n^t}{n}} \quad t \geq 2 \text{ integer}$$

$$= \left(\frac{a_1^t + \dots + a_n^t}{n}\right)^{\frac{1}{t}} \quad t > 2 \text{ real}$$

$t > 2$ then

$$\boxed{PM_t \geq QM \geq AM \geq GM \geq HM}$$

Problem 1 $a, b, c > 0$,

$$(a+b)(b+c)(c+a) \geq 8abc.$$

$$\underset{2\sqrt{ab}}{\geq} \quad \underset{2\sqrt{bc}}{\geq} \quad \underset{2ca}{\geq}$$

Problem 2 Prove $x_1, x_2, \dots, x_n \geq 0$ reals:

$$x_1 + 2x_2 + \dots + nx_n \leq \frac{n(n-1)}{2} + x_1 + x_2^2 + \dots + x_n^n.$$

Problem 3 For given $n \geq 1$ find smallest value of

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \dots + \frac{x_n^n}{n}$$

if $x_i > 0$

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = n$$

Second

$$x_k^k + \underbrace{1+1+1+\dots+1}_{k-1} \geq k\cdot\sqrt[k]{x_k^k\cdot 1\cdot 1\cdots 1} = k\cdot x_k$$

sum over $k = 1, \dots n$.

$$x^k + \underbrace{\frac{1}{x}+\frac{1}{x}+\dots+\frac{1}{x}}_{k} \geq (k+1)\sqrt[k]{x^k\cdot\frac{1}{x}\cdot\frac{1}{x}\cdots\frac{1}{x}} = k+1 \quad \text{so}$$

$$x^k + \frac{k}{x} \geq k+1 \rightsquigarrow \frac{x^k}{k} + \frac{1}{x} \geq 1+\frac{1}{k}$$

$$\sum_{k=1}^{n}\left(\frac{x_k^k}{k}+\frac{1}{x_k}\right) \geq n + 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$$

$$\sum_{k=1}^{n}\frac{x_k^k}{k} + \cancel{\sum_{k=1}^{n}\frac{1}{x_k}} \geq \cancel{n} + 1+\frac{1}{2}+\dots+\frac{1}{n}$$

$\boxed{x_k = 1}$

$$\frac{x_k^{k}}{k} + \frac{1}{x_k} \geq 1 + \frac{1}{k} \qquad k = 1 \ldots n$$

$$x_1 + \frac{x_2^{2}}{2} + \ldots + \frac{x_n^{n}}{n} + \cancel{\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)} \geq \cancel{n} + \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

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so :

$$x_1 + \frac{x_2^{2}}{2} + \ldots + \frac{x_n^{n}}{n} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} .$$

$$= \quad \text{iff} \quad \underline{x_k = 1} .$$

$$\cancel{x^{k} + \underbrace{1 + 1 + \ldots + 1}_{} \geq (k+1) \sqrt[k+1]{}}$$

$$x^{k} + \underbrace{\frac{1}{x} + \frac{1}{x} + \ldots + \frac{1}{x}}_{k} \geq (k+1) \cdot \sqrt[k+1]{x^{k} \cdot \frac{1}{x} \cdot \ldots \cdot \frac{1}{x}} = (k+1)$$

$$+ \frac{1}{x_k^{k}}$$

$x^k + \frac{k}{x} \geq k+1 \quad |:k$

$\frac{x^k}{k} + \frac{1}{x} \geq 1 + \frac{1}{k}$ AM $\geq$ GM $=$ $x^k = \frac{1}{x}$ $\rightsquigarrow$ $x = 1$

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PROBLEM 4

$a, b, c > 0$ reals $\therefore$ $a+b+c = 1$ $\left(\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right)$

Prove $a^2 + b^2 + c^2 + 2\sqrt{3abc} \leq 1$

$\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + 2 \cdot \frac{1}{3} = 1$

PROBLEM 5 $a, b, c \in (0, 1)$

$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$

TRICK (Make the same order of expression LHS, RHS)

$a+b+c=1$

$$a^2+b^2+c^2+2\sqrt{3abc} \le 1$$

$$a^2+b^2+c^2+2\sqrt{3abc} \le 1$$

$2 \quad 2 \quad 2 \qquad \frac{3}{2}$

$$\cancel{a^2+b^2+c^2}+2\sqrt{3abc(a+b+c)} \le (a+b+c)^2$$

$2 \qquad 2 \qquad 1 \qquad 1$

$= \cancel{a^2+b^2+c^2}+2ab+2bc+2ca$

$\Downarrow$

$$\sqrt{3abc(a+b+c)} \le ab+bc+ca$$

$x=ab$
$y=bc$
$z=ca$

$$\sqrt{3(xy+yz+zx)} \le x+y+z$$

$$3(xy+yz+zx) \le x^2+y^2+z^2+2xy+2yz+2zx \iff$$

$$x^2+y^2+z^2 \ge xy+yz+zx \iff (x-y)^2+(y-z)^2+(z-x)^2 \ge 0$$