$$f(x)+yf(y)=x+f(y^2-x)+f(f(x)) \quad \forall x,y$$

$P(0,0) \Rightarrow f(f(0))=0$

$P(0,1) \Rightarrow f(0)+\cancel{f(1)}=\cancel{f(1)}+\cancel{f(f(0))}$

$\boxed{f(0)=0}$ +1

$P(0,x) \Rightarrow xf(x)=f(x^2)$ —— ①

$P(x,0) \Rightarrow f(x)=x+f(-x)+f(f(x))$ —— ② +1

① $\Rightarrow (x \longleftrightarrow -x)$ $\boxed{f(-x)=-f(x) \quad [f \text{ odd}] \quad ③}$ +1

②, ③ $\rightarrow 2f(x)=x+f(f(x))$ +1

$P(x,y) \Longleftrightarrow yf(y)=f(x)+f(y^2-x)$

$\overset{①}{\Longleftrightarrow} f(y^2)=f(x)+f(y^2-x)$

$\Rightarrow f(a)+f(b)=f(a+b) \quad \forall a+b \geq 0$ +2

if $a+b<0$: $f(a+b)=-f(-a-b)=-f(-a)-f(-b)=f(a)+f(b)$

$$\Longrightarrow \boxed{f(x+y) = f(x) + f(y) \quad \forall x, y}$$

+1

① $\quad x f(x) = f(x^2) \quad \forall x$

---

①: $\quad x \longrightarrow x+1$

$$LHS = (x+1) f(x+1) = \cancel{(x+1)} f(x) + \boxed{\boxed{(x+1)} f(1)}$$

$$RHS = f((x+1)^2) = \cancel{f(x^2)} + \cancel{2} \boxed{f(x)} + \cancel{f(1)}$$

+3

$$\Longrightarrow \boxed{f(x) = x f(1) \quad \forall x}$$

$$\| $$

$$K$$

$$xK + y^2 K = x + (y^2 - x)K + K^2 x$$

$$2xK = (K^2+1)x \longleftrightarrow (K-1)^2 x = 0 \quad \forall x$$

$$\Longleftrightarrow K = 1$$

$$\Longrightarrow f(x) = x \quad \forall x \qquad \blacksquare$$

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$

$$f(m) + f(n) \mid m+n \qquad \forall\, m,n \in \mathbb{N}$$

$$m = n \Rightarrow f(n) \mid n \qquad \forall\, n$$

$$\Rightarrow f(1) = 1 \quad ; \quad f(p) \in \{1, p\} \ \forall p \in \mathbb{P}$$

$$m = 1 \Rightarrow f(n)+1 \mid n+1,$$

$$\downarrow$$

$$p \in \mathbb{P} \qquad n := p-1 \Rightarrow f(p-1)+1 \mid p$$

$$\Rightarrow f(p-1) = p-1 \quad \forall p \in \mathbb{P}$$

$$m := p-1 \Rightarrow f(n) + p-1 \mid n+p-1 \qquad \forall\, p \in \mathbb{P}$$

$$\Leftrightarrow f(n) + p-1 \mid n - f(n)$$

$$\Longrightarrow \boxed{f(n) = n \quad \forall n \in \mathbb{N}} \checkmark$$

$f: \mathbb{N} \longrightarrow \mathbb{N}$ (NG - IMO SL 2016)

$$f(m)+f(n)-mn \mid mf(m)+nf(n)$$

$$f(m)+f(n) \neq mn$$

$$\forall m,n \in \mathbb{N}$$

$m=n=1 \Longrightarrow 2f(1)-1 \mid 2f(1)$

$\Longleftrightarrow 2f(1)-1 \mid 1 \Rightarrow \boxed{f(1)=1}$

$m=1 \Longrightarrow f(n)-n+1 \mid nf(n)+1$

$(f(n) \equiv n-1)$ $\quad \equiv n(n-1)+1$

$\Longleftrightarrow \boxed{f(n)-n+1 \mid n^2-n+1}$

$m=n \Longrightarrow 2f(n)-n^2 \mid 2nf(n) \Rightarrow \boxed{2f(n)-n^2 \mid n^3}$

$2f(n) \equiv n^2$ $\quad \equiv n^3$

$\Longrightarrow 2f(p)-p^2 \in \{-p,-1,1,p,p^2,p^3\}$ (*)

$$\Longrightarrow f(p) \in \left\{ \underset{1}{\frac{p^2-p}{2}}, \underset{2}{\frac{p^2-1}{2}}, \underset{3}{\frac{p^2+1}{2}}, \underset{4}{\frac{p^2+p}{2}}, \underset{5}{p^2}, \underset{6}{\frac{p^2+p^3}{2}} \right\}$$

$\forall\, p \in \mathbb{P}_{\geq 7}$

1) $\frac{p^2-p}{2} - p + 1 \mid p^2 - p + 1$

$\equiv p^2 - \frac{p^2-p}{2}$ $\Rightarrow \frac{p^2-p}{2} - p + 1 \mid p - \frac{p-1}{2} = \frac{p+1}{2}$ ↯

2) $\frac{p^2-1}{2} - p + 1 \mid p^2 - p + 1$

$p - 1 \mid \frac{p^2-1}{2} - p + 1 = (p-1)\left(\frac{p+1}{2} - 1\right)$ ↯

3) $\frac{p^2+1}{2} - p + 1 \mid p^2 - p + 1 \Leftrightarrow \frac{p^2+1}{2} - p + 1 \mid p - 2$ ↯

$= p^2 - 2p + 3$

4) $\frac{p^2+p}{2} - p + 1 \mid p^2 - p + 1 \Rightarrow \frac{p^2+p}{2} - p + 1 \mid p - \frac{p+1}{2} = \frac{p-1}{2}$ ↯

6) $\frac{p^2+p^3}{2} - p + 1 > p^2 - p + 1$ ↯.

$$\Longrightarrow \boxed{f(p) = p^2 \quad \forall\, p \geq 7}$$

$m = p \in \mathbb{P}_{\geq 7} \Longrightarrow$

$$f(n) + p^2 - np \mid nf(n) + p^3$$

$$= p(np - f(n))$$

$$= np^2 - pf(n)$$

$$= n^2 p - nf(n) - pf(n)$$

$$\Downarrow$$

$$f(n) + p^2 - np \mid p(n^2 - f(n))$$

$$p > f(n) \Rightarrow f(n) + p^2 - np \mid n^2 - f(n)$$

$$P(x) = x^2 - nx + f(n)$$

$$p \to \infty, \quad f(n) + p^2 - np \to \infty$$

$$> |n^2 - f(n)|$$

$$\Longrightarrow \boxed{f(n) = n^2 \quad \forall n}$$

$f : \mathbb{N} \longrightarrow \mathbb{N}$ $\qquad m^2 + f(n) \mid m f(m) + n \quad \forall m, n \in \mathbb{N}$

Sol 1 $\quad m := f(n) \Rightarrow f(n)^2 + f(n) \mid f(n) f(f(n)) + n$

$$\Rightarrow \boxed{f(n) \mid n \quad \forall n \qquad (1)}$$

$n = 1 \Rightarrow m^2 + f(1) \mid m f(m) + 1$

$\Rightarrow m f(m) + 1 \geq m^2 + f(1) \geq m^2 + 1$

$$\Rightarrow \boxed{f(m) \geq m \quad \forall m \qquad (2)}$$

$$\Rightarrow \boxed{f(n) = n \quad \forall n \in \mathbb{N}}$$

Sol 2

$m = n = p \in \mathbb{P}$

$\rightarrow p^2 + f(p) \mid p(f(p)+1)$

if $p \nmid p^2 + f(p) \rightarrow p^2 + f(p) \mid f(p)+1$ ↯

$\rightarrow p \mid p^2 + f(p) \Leftrightarrow p \mid f(p) \quad \forall p \in \mathbb{P}$ (1)

- $m = f(n) \Longrightarrow f(n) \mid n \quad \forall n$ (2)

(1), (2) $\rightarrow f(p) = p \quad \forall p \in \mathbb{P}$

$m = p \rightarrow p^2 + f(n) \mid p^2 + n$

$\Leftrightarrow p^2 + f(n) \mid f(n) - n \quad \forall p \in \mathbb{P}$

$\rightarrow f(n) = n \quad \forall n \in \mathbb{N}$

Sol 3

Pick $d \mid f(1)$

$n=1$, $m=d \Rightarrow d^2+f(1) \mid d f(d)+1$

$\Rightarrow d=1 \Longrightarrow \boxed{f(1)=1}$

$n=1 \longrightarrow m^2+1 \mid m f(m)+1 \Longrightarrow \boxed{f(m) \geq m}$

$m=1 \longrightarrow f(n)+1 \mid n+1 \Longrightarrow \boxed{f(n) \leq n}$

$f(n)=n \quad \forall\, n \in \mathbb{N}$. $\boxtimes$

Sol 4

$m=n \Rightarrow n^2+f(n) \mid nf(n)+n$

$\Leftrightarrow n^2+f(n) \mid n^3-n$

$n=2 \Rightarrow 4+f(2) \mid 6 \Leftrightarrow f(2)=2$

$m=2,\ n=1 \Rightarrow 4+f(1) \mid 5 \Leftrightarrow f(1)=1$

$m=1,\ n=p-1 \Rightarrow 1+f(p-1) \mid p \Leftrightarrow f(p-1)=p-1$

$m=p-1 \Rightarrow (p-1)^2+f(n) \mid (p-1)^2+n$

$\Leftrightarrow (p-1)^2+f(n) \mid f(n)-n \quad \forall p \in \mathbb{P}$

$\Rightarrow \boxed{f(n)=n \quad \forall n \in \mathbb{N}}$

$f: \mathbb{Z} \longrightarrow \mathbb{Z}$

$$f(m)-f(n) \mid m^3-n^3 \quad \forall m,n \in \mathbb{Z}$$

WLOG assume $f(0)=0$ (by shifting)

(if $f(0)\neq 0$; let $g(n)=f(n)-f(0)$
$\Rightarrow g(m)-g(n) \mid m^3-n^3$)

$P(0,1) \Rightarrow f(1) \mid 1$

$\Rightarrow f(1)=\pm 1$

WLOG assume $f(1)=1$
(by taking the convenient sign)

(if $f(1)=-1$; let $g(n)=-f(n)$)
$P(1,-1)$, $P(0,-1) \Rightarrow f(-1)=-1$

$P(p,0) \Rightarrow f(p) \mid p^3 \Rightarrow f(p) \in \{\pm p^3, \pm p^2, \pm p, \pm 1\}$

$P(p,1) \Rightarrow f(p)-1 \mid p^3-1$; $P(p,-1) \Rightarrow f(p)+1 \mid p^3+1$

1 ✗ $\quad -1$ ✗ $\quad p$ ✓ $\quad -p$ ✗ $\quad p^2$ ✗ $\quad -p^2$ ✗

$0 \mid p^3-1 \quad 0 \mid p^3+1 \quad p^3$ ✓ $\quad p+1 \mid p^3-1$ $\quad p^2-1 \mid p^3-1$ $\quad p^2+1 \mid p^3+1$

$2 \quad p+1 \mid p^2-p+1$ $\quad p+1$

$2-p$ $\quad -p^3$ ✗

$f(p) = p$ or p^3 $\forall\, p \in \mathbb{P}_{>2}$

$\exists\, \infty$ly many primes p such that

$$f(p) = p^d \qquad (d = 1 \text{ or } 3)$$

$$m = p \implies p^d - f(n) \mid p^3 - n^3$$

$$p^d - f(n) \mid p^3 - f(n)^{\frac{3}{d}}$$

$$\implies p^d - f(n) \mid n^3 - f(n)^{\frac{3}{d}} \qquad \forall\, p \in \mathbb{P}_{>2}$$

$$\iff n^3 = f(n)^{\frac{3}{d}} \iff f(n) = n^d \quad \forall\, n \in \mathbb{Z}$$

Thus;

1. $f(n) = n + C$ $\forall n$
2. $f(n) = -n + C$ $\forall n$
3. $f(n) = n^3 + C$ $\forall n$
4. $f(n) = -n^3 + C$ $\forall n$

$\blacksquare$

Test 2 (Problem 3)

$(a_n)_{n\geq 0}$; $a_1 = 1$; $a_{n+1} = a_n + \frac{1}{a_n}$

$$a_{2021} = a_{2020} + \frac{1}{a_{2020}}$$

$$a_{2021}^2 > a_{2020}a_{2021} = a_{2020}^2 + 1 = a_{2019}^2 + \frac{1}{a_{2019}^2} + 3$$

$$\vdots$$

$$a_{n+1}^2 = a_n^2 + 2 + \frac{1}{a_n^2}$$

$$> a_n^2 + 2$$

$$= a_1^2 + \sum_{i=1}^{2019} \frac{1}{a_i^2} + 4039$$

$$> 4040 > 60^2$$

BMO 2019 | $f: \mathbb{P} \longrightarrow \mathbb{P}$

$$f(p)^{f(q)} + q^{p} = p^{q} + f(q)^{f(p)} \quad \forall\, p,q \in \mathbb{P}$$

---

(1) $P(2,7)$ | $f(2)^{f(7)} - f(7)^{f(2)} = -1$

$f(2) = 2 \quad f(7) = 7$

$$f(p)^{2^{f(p)}} - 2^{p} = p^{2} - 2^{p}$$

$f : \mathbb{N} \longrightarrow \mathbb{N}$

$n \mid m \iff f(n) \mid f(m) - n$

$P(n,n)$; $\boxed{f(n) \mid n \quad \forall\, n \in \mathbb{N} \quad (1)}$

$\boxed{n \mid m \iff f(n) \mid f(m) \quad (2)}$

$p \in \mathbb{P}$; $f(p) \mid p$

if $f(p) = 1$, $f(p) = 1 \mid f(m) \ \forall m$

$\rightarrow p \mid m \quad \forall m$ ↯

$\Rightarrow \boxed{f(p) = p \quad \forall\, p \in \mathbb{P}}$

Claim: $f(p^k) = p^k \quad \forall\, k \in \mathbb{N}$.

suppose $f(p^l) = p^l \quad \forall\, 0 \leq l < k$

$$p^{k-1} \mid p^k \overset{②}{\Longrightarrow} p^{k-1} = f(p^{k-1}) \mid f(p^k)$$

① : $f(p^k) \mid p^k$

$$\Rightarrow f(p^k) = p^{k-1} \text{ or } p^k$$

if $f(p^k) = p^{k-1}$

$$\Rightarrow f(p^k) \mid p^{k-1} = f(p^{k-1}) \overset{②}{\Longrightarrow} p^k \mid p^{k-1} \quad ↯$$

Thus; $\boxed{f(p^k) = p^k \quad \forall p \in \mathbb{P} \quad k \in \mathbb{N}_0}$

Now, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$

① : $f(n) \mid n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$

② : $p_i^{\alpha_i} \mid n \Rightarrow p_i^{\alpha_i} = f(p_i^{\alpha_i}) \mid f(n) \quad \forall 1 \leq i \leq s$

$$\Rightarrow n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \mid f(n)$$

$$\Longrightarrow \boxed{f(n) = n \quad \forall n \in \mathbb{N}}$$

$f : \mathbb{N} \longrightarrow \mathbb{N}$

$$n + f(m) \mid f(n) + n f(m) \qquad \forall m, n \in \mathbb{N}$$

$m = n = 1 \Longrightarrow 1 + f(1) \mid 2f(1)$

$\Longleftrightarrow 1 + f(1) \mid 2 \Longleftrightarrow \boxed{f(1) = 1}$

$m = 1 \Longrightarrow n + 1 \mid f(n) + n$

$\longleftrightarrow \boxed{n + 1 \mid f(n) - 1 \quad (*)}$

if $f(n_0) = 1$

$\Rightarrow n_0 + f(m) \mid 1 + n_0 f(m)$

$\longleftrightarrow f(m)^2 - 1 \longleftrightarrow n_0^2 - 1$

<u>Case 1</u> : $U = \{ n \mid f(n) = 1 \}$ is infinite
$n_1 < n_2 < \cdots$

$\forall u \in U$; $u + f(m) \mid f(m)^2 - 1 \quad \forall m \in \mathbb{N}$

$\longrightarrow \boxed{f(m) = 1 \quad \forall m \in \mathbb{N}}$

Case 2: U is finite

$\exists N$ such that $f(n) \geq 1 \quad \forall n \geq N$

$\left((*) \; ; \; n+1 \mid f(n) - 1 \right) \Rightarrow f(n) > n+1 \; \forall \; n \geq N$

$\Rightarrow f$ is unbounded

Now $\quad n + f(m) \mid f(n) + n f(m)$

$\longleftrightarrow n + f(m) \mid f(n) - n^2$

$\forall \; n, m \in \mathbb{N}$

since $f(m) \longrightarrow \infty$; $f(n)$ must be n^2

Thus; $\boxed{f(n) = n^2 \quad \forall n \in \mathbb{N}}$ (sh)

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$

$$f(m)^2 + f(n) \mid (m^2+n)^2 \qquad \forall m,n \in \mathbb{N}$$

$$f(1)^2 + f(1) \mid 4 \implies f(1) = 1$$

$$f(p-1)+1 \mid p^2 \qquad (m=1,\ n=p-1)$$

$$\Rightarrow f(p-1) = p^2-1 \text{ or } p-1$$

if $f(p-1) = p^2-1$

$m = p-1,\ n=1$: $(p^2-1)^2+1 \mid ((p-1)^2+1)^2$

$$(p^2-1)^2+1 \overset{?}{>} \left((p-1)^2+1\right)^2 = (p-1)^4+2(p-1)^2+1$$

$$\Longleftrightarrow \quad (p-1)^2(p+1)^2 \overset{?}{>} (p-1)^4+2(p-1)^2$$

$$\Longleftrightarrow \quad (p+1)^2 \overset{?}{>} (p-1)^2+2$$

$$\Longleftrightarrow \quad 2p+1 \overset{?}{>} -2p+3 \Longleftrightarrow 4p \overset{?}{>} 2 \checkmark$$

a contradiction!!

So $f(p-1) = p-1 \quad \forall p \in \mathbb{P}$

Now ; $m = p-1$:

$$(p-1)^2 + f(n) \mid \left((p-1)^2+n\right)^2$$

$$\Longleftrightarrow \quad \left(f(n)-n\right)^2 \quad \forall p \in \mathbb{P}$$

$$\Longrightarrow \quad \boxed{f(n) = n \quad \forall n}$$

$f : \mathbb{N} \longrightarrow \mathbb{N}$

$f(f(n)) + f(n) = 2n \quad \forall n \in \mathbb{N}$

Sol 1 | $n = 1 \Rightarrow f(f(1)) + f(1) = 2$

$\Rightarrow \boxed{f(1) = 1}$

if $f(a) = f(b) \Rightarrow$

$2a = f(f(a)) + f(a) = f(f(b)) + f(b) = 2b$

$\Rightarrow f$ injective

By induction: $n = 1$ ✓

Assume $f(k) = k \quad \forall \; k < n \quad ; n \geq 2$

$f(n) \notin \{f(1), f(2), \dots, f(n-1)\} = \{1, 2, \dots, n-1\}$

$\Rightarrow f(n) \geq n \quad ;$ call $m = f(n)$

Since $m \notin \{1, 2, \dots, n-1\}$

$$f(m) \notin \{f(1), \dots, f(n-1)\}$$

$$\Rightarrow \quad f(f(n)) = f(m) \geq n$$

$$\Rightarrow 2n = f(f(n)) + f(n) \geq n+n$$

$$\Rightarrow \boxed{f(n) = n \quad \forall n \in \mathbb{N}}$$

$$f : \mathbb{R}^* \longrightarrow \mathbb{R}^*$$

$(*)$ $\quad f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}^*$

$(**)$ $\quad f(1) = 1$

$(***)$ $\quad f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \quad \forall x \in \mathbb{R}^*$

Prove that $f(x) = x \quad \forall x \in \mathbb{R}^*$

$$\boxed{f(x)+1 = f(x+1) = (x+1)^2 f\left(\frac{1}{x+1}\right)}$$

$$f\left(\frac{1}{x}\right)+1 = f\left(\frac{x+1}{x}\right) = \frac{(x+1)^2}{x^2} f\left(\frac{x}{x+1}\right)$$

$$x^2 f\left(\frac{1}{x}\right) + x^2 = (x+1)^2 f\left(\frac{x}{x+1}\right)$$

$$= \boxed{f(x) + x^2}$$

$$+ \Rightarrow 2f(x) + x^2 + 1 = (x+1)^2 = 2x + x^2 + 1$$

$$\Leftrightarrow f(x) = x \quad \forall x \in \mathbb{R}^* \setminus \{-1\}$$

$$\& \quad f(-1) = f(-2) + f(1) = -2+1 = -1$$

$$\Rightarrow f(x) = x \quad \forall x \in \mathbb{R}^*$$

Does there exist a non-constant $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x+y) \geq f(xy) \quad \forall x, y \in \mathbb{R}$$

?

(1)

---

(2)

Find all $f : \mathbb{N} \longrightarrow \mathbb{N}$

s.t

(i) $f(mn) = f(m) f(n)$

(ii) $m+n \mid f(m) + f(n)$

$\forall m, n \in \mathbb{N}$

(1) $y=0 \Rightarrow \underline{f(x) \geq f(0)} \quad \forall x \in \mathbb{R}.$

$x=-y \Rightarrow f(0) \geq f(-x^2) \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x) = f(0) \quad \forall x \leq 0.$

Now $x = y \leq 0$

$\Rightarrow \underbrace{f(2x)}_{\substack{\| \\ f(0)}} \geq f(x^2) \quad \forall x \leq 0$

$\Rightarrow f(x) = f(0) \quad \forall x \geq 0$

So the answer is $\underline{\underline{\text{NO}}}$ ∎

2] $f(1)=1$; $f(n^k)=f(n)^k$

$$f(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) = f(p_1)^{\alpha_1}f(p_2)^{\alpha_2}\cdots f(p_s)^{\alpha_s}$$

(so f is determined by the primes)

$\rightarrow$ $n \mid f(n)$ $\forall n \in \mathbb{N}$ $\quad q \mid f(q)$ $\forall q \in \mathbb{P}$

$n+1 \mid f(n)+1$

$p, q \in \mathbb{P}$ $p \neq q$. $\exists\, m, n \in \mathbb{N}$ s.t

$mp = nq+1$ $\quad (mp \equiv 1 \,[q])$

$mp = nq+1 \mid f(nq)+1 = f(n)f(q)+1$

$\Rightarrow \gcd(p, f(q)) = 1$.

$\Rightarrow$ $\boxed{f(p) \text{ is a power of } p \ \forall p \in \mathbb{P}}$

write $f(2) = 2^k$; $f(2^n) = 2^{kn}$

$$\Rightarrow 3 \mid f(2)+1 = 2^k + 1$$

$$\Rightarrow k \text{ odd}$$

Now ; let $p \in \mathbb{P}$

$$p+2^n \mid f(p) + 2^{kn}$$

$$p+2^n \mid p^k + 2^{kn}$$

TST

Turkey

$$\Rightarrow p+2^n \mid f(p) - p^k \quad \forall n \in \mathbb{N}$$

$$\Rightarrow f(p) = p^k \quad \forall p \in \mathbb{P}$$

$$\Rightarrow f(n) = n^k \quad \forall n \in \mathbb{N}$$

where k is odd

Find all $f: \mathbb{N} \longrightarrow \mathbb{N}$ s.t

$$\left(f(m)+n\right) f\left(m+f(n)\right) = \left(m+f(n)\right)^2$$

$\forall\ m, n \in \mathbb{N}$

$m = n$: $\boxed{f\left(n+f(n)\right) = n+f(n) \quad \forall n \in \mathbb{N}}$

$F = \{ k \mid f(k) = k \}$; $|F| = \infty$

$\left(\because \quad n+f(n) \in F \quad \forall n \right)$

Now ; $f(m)+n \mid \left(m+f(n)\right)^2 \quad \forall m, n$

fix $m \in \mathbb{N}$ and take $n \in F$

$\Rightarrow \quad f(m)+n \mid (m+n)^2 \Leftrightarrow f(m)+n \mid (m-f(m))^2$

$\forall\ n \in F$

$\Rightarrow \boxed{f(m) = m \quad \forall m \in \mathbb{N}}$

$f: \mathbb{R} \longrightarrow \mathbb{Z}$ s.t $\forall\, x, y \in \mathbb{R}$:

$$f(f(y)-x)^2 + f(x)^2 + f(y)^2 = f(y)\left(1+2f(f(y))\right)$$

$x = y = 0$:

$$f(f(0))^2 + 2f(0)^2 = f(0) + 2f(0)f(f(0))$$

$$\Leftrightarrow \left(f(f(0)) - f(0)\right)^2 = f(0) - f(0)^2$$

$f(0) \geq f(0)^2 \Leftrightarrow 1 \geq f(0)$

$\Rightarrow f(0) = 0$ or 1 $\quad \left(f(0) = f(0)^2 = f(f(0))\right)$

$(x, y) \longrightarrow (0, x)$:

$$f(f(x))^2 + f(x)^2 + f(0) = f(x) + 2f(x)f(f(x))$$

$$\Leftrightarrow \boxed{\left(f(x) - f(f(x))\right)^2 = f(x) - f(0)}$$

$$\Leftrightarrow \quad f(f(x)) = f(x) \pm \sqrt{f(x)-f(0)}$$

$$\Leftrightarrow \quad \underbrace{f(f(x))-f(0)}_{m^2} = \underbrace{f(x)-f(0)}_{k^2} \pm \underbrace{\sqrt{f(x)-f(0)}}_{k}$$

$$\Leftrightarrow \quad m^2 = n^2+n \quad \text{for some } m,n \in \mathbb{Z}_{\geq -1}$$

(whether n^2+n or n^2-n, it is always the product of two consecutive integer; that is if $m^2 = k^2-k$, choose $n=k-1$
if $m^2 = k^2+k$, choose $n=k$)

if $n \geq 1 \Rightarrow n^2 < m^2 < (n+1)^2$ ↯.

$\Rightarrow n = \pm 1$; $m = 0$

$\Rightarrow f(f(x)) = f(0) \quad \forall x$

$\Rightarrow (f(x)-f(0))^2 = f(x)-f(0) \quad \forall x.$

$\Rightarrow f(x) = f(0)$ or $f(0)+1$

$\forall x \in \mathbb{R}$

<u>Case 1</u>: $f(0) = 1 \Rightarrow f(x) = 1$ or 2

$f(f(x)) = 1 \quad \forall x$.

$$f(f(y)-x)^2 + f(x)^2 + f(y)^2 = 3f(y)$$

$$2f(y)^2 \leq f(f(y)-y)^2 + 2f(y)^2 = 3f(y)$$

$\Rightarrow f(y) \leq \frac{3}{2} < 2 \quad \forall y \Rightarrow \boxed{f(x) = 1 \ \forall x}$ (1)

<u>Case 2</u>: $f(0) = 0 \Rightarrow f(x) = 0$ or $1 \quad \forall x$

$f(f(x)) = 0 \quad \forall x$

$$0 \leq f(f(y)-x) + f(x) + \cancel{f(y)} = \cancel{f(y)}$$

$\Rightarrow \boxed{f(x) = 0 \quad \forall x \in \mathbb{R}}$ (2)