NT Equations

Lesson by Senya, groups L4 & L4+

Following up on the success and beauty of the problem 5 from the IMO 2022, let's do a bit of technical Number Theory around Diophantine equations. Well, mostly technical.

Problem 1. i) Let $\frac{m}{n} = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{p-1}$ where $p > 2$ is prime. Show that m must be divisible by p;

ii) Let $\frac{m}{n} = \frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{\cancel{p^2}\,(p-1)^2}$ where $p > 3$ is prime. Show that m must be divisible by p;

Problem 2. Find all a, b, c such that $2^{a!} + 2^{b!} + 2^{c!}$ is a perfect cube.
positive integers

Problem 3. Do there exist integers x, t, s such that

$$(x^2 + y)(y^2 + t)(t^2 + \cancel{s}\,x) = 20152015?$$

Problem 4. Find all integers a, b, c such that

$$a^5 + b^5 + c^5 + 5 = d^5.$$

Problem 5. There are two positive integers a and b such that $ab(a + b)$ is divisible by $a^2 + ab + b^2$. Prove that $|a - b|^3 > ab$.
different

Problem 6. For a given positive integer n find all pairwise different integers $x_1, x_2, ..., x_n$ such that $x_1^3 + ... + x_n^3 = (x_1 + ... + x_n)^2$.
positive

Problem 7. Solve in integers: $a^b + 1 = (a + 1)^c$.

Problem 8. Find all non-negative integers x and y such that

$$(xy - 7)^2 = x^2 + y^2.$$

Problem 9. For any given prime p prove that the equation $x^2 + y^2 \equiv -1 \pmod{p}$ has an integer solution.

Problem 10. All of the positive divisors of some integer N are written in a row: $1 = d_1 < d_2 < ... < d_s = N$. It turned out that $(d_1, d_2) + (d_2, d_3) + ... + (d_{s-1}, d_s) = N - 2$. What possible values can N take?

Problem 11. Find all triples (p, x, y) where p is prime and x and y are positive integers such that both of the numbers $x^{p-1} + y$ and $x + y^{p-1}$ are powers of p.