Problem 1

$\{x_i\} \in \{-1, 0, 1, 2\}$

a) $\sum x_i = 20$

b) $\sum x_i^2 = 120$

c) $\sum x_i^3$ → Find max
→ Find min

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of -1 a
of 1 b
# of 2 c

step 1

① $b - a + 2c = 20$
② $b + a + 4c = 120$
③ $b - a + 8c = 20 + 6c$

step 2

$\boxed{c = 0}$ For min
$\Rightarrow b = 70 \quad a = 50$

step 3

From ①+②
$2b + 6c = 140$
$3c = 70 - b$

step 4

$3c \leq 70 \quad c \leq 23$

For max

$\boxed{c = 23 \quad b = 1 \quad a = 27}$

Problem 2

$$1 = \frac{a+b+c+d+e}{abcde} = \frac{1}{bcde} + \frac{1}{acde} + \frac{1}{abde} + \frac{1}{abce} + \frac{1}{abcd}$$

$$1 \le \frac{1}{de} + \frac{1}{de} + \frac{1}{de} + \frac{1}{e} + \frac{1}{d}$$

$$1 \le \frac{3+e+d}{de}$$

example:

$$de - e - d + 1 \le 4$$

$\{a,b,c,d,e\} = \{5,2,1,1,1\}$

$$(d-1)(e-1) \le 4 \Rightarrow e \le 5$$

$$1 = \frac{x_1 + x_2 \cdots x_n}{x_1 x_2 \cdots x_n} = \frac{1}{x_2 x_3 \cdots x_n} + \frac{1}{x_1 x_3 \cdots x_n} \cdots + \frac{1}{x_1 x_2 \cdots x_{n-1}}$$

$$1 \le \frac{1}{x_n x_{n-1}} + \frac{1}{x_n x_{n-1}} \cdots + \frac{1}{x_n x_{n-1}} + \frac{1}{x_n} + \frac{1}{x_{n-1}}$$

$$1 \le \frac{n-2+x_n+x_{n-1}}{x_n x_{n-1}}$$

example:

$$x_n x_{n-1} \le x_n + x_{n-1} + n - 2$$

$\{x_1, x_2 \cdots x_n\} = \{n, 2, 1, 1 \cdots 1\}$

$$x_n x_{n-1} - x_n - x_{n-1} + 1 \le n-1$$

$$(x_n - 1)(x_{n-1} - 1) \le n-1 \Rightarrow x_n \le n$$

Problem 3

$$F(x) = \frac{ax+b}{cx+d}$$

$F(20) = 2005$

$F(22) = 1983$

$F(F(x)) = x$

$\{F(x): \quad cx+d \neq 0\}$

$$\frac{a\frac{ax+b}{cx+d}+b}{c\frac{ax+b}{cx+d}+d} = \frac{a(ax+b)+b(cx+d)}{c(ax+b)+d(cx+d)} = x$$

$$(ac+cd)x^2 + (d^2-c^2)x - (ab+bd) \equiv 0$$

$\Rightarrow \quad a+d=0$

$$F(x) = \frac{ax+b}{cx-a}$$

$a = 1$

$$\frac{x+\frac{b}{a}}{\frac{c}{a}x-1} = \frac{x+b'}{c'x-1}$$

$$= \frac{x+\frac{b}{a}}{\frac{c}{a}x-1} = \frac{x+b'}{c'x-1}$$

$$F(x) = \frac{x+b}{cx-1}$$ (2 variables)

$\Rightarrow$ this can be solved!

Problem 4

$$1 - x + x^2 - x^3 + x^4 \cdots - x^{19} + x^{20} - x^{21}$$

$$= a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \cdots + a_{21} y^{21}$$

$\boxed{y = x+1}$

Find a_3

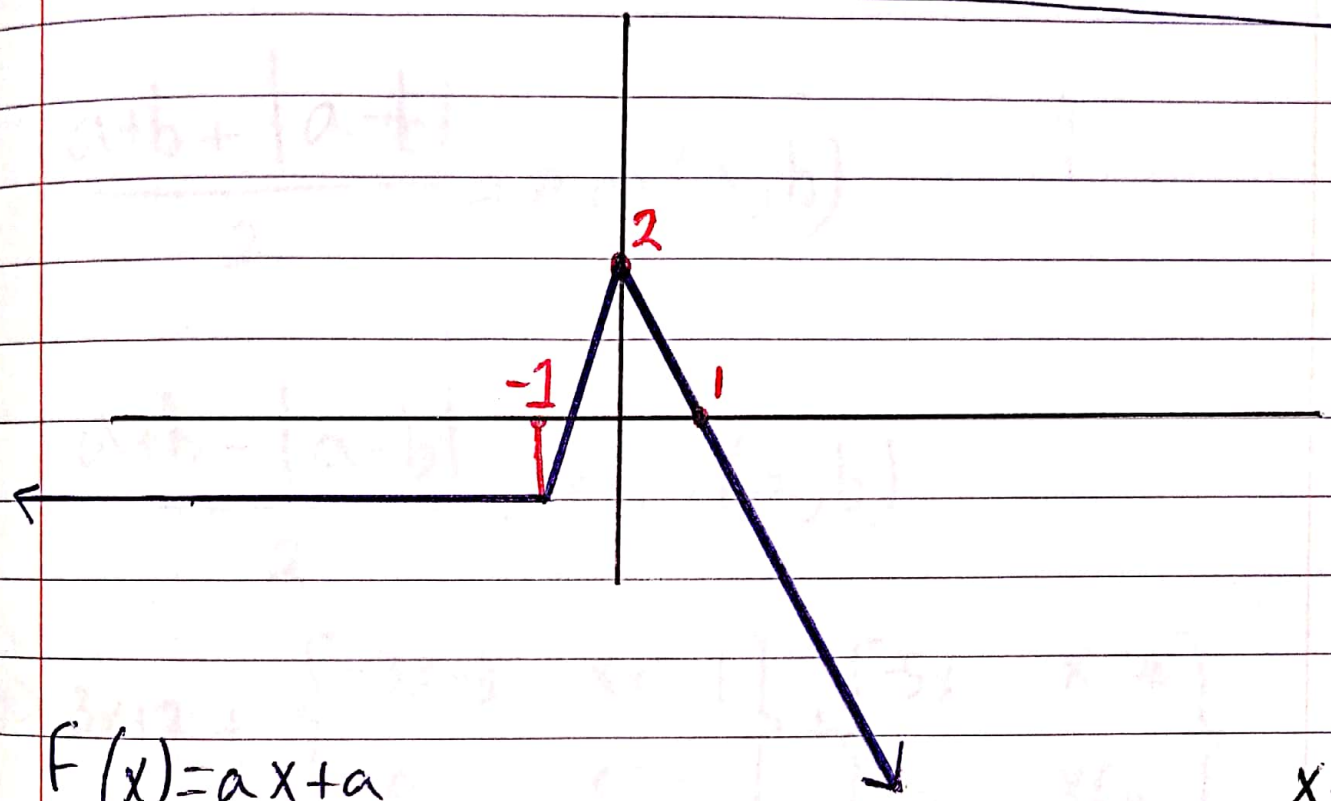
$$LHS = \frac{1 - x^{22}}{x+1} = \frac{1-(y-1)^{22}}{y}$$

$$\Rightarrow 1 - (y-1)^{22} = a_0 y + \cdots a_{21} y^{22}$$

$$\Rightarrow \text{coefficent of } y^4 = -\binom{22}{4}$$

Problem 5 Sol 1

$$|F(x)| - |G(x)| + |H(x)| = \begin{cases} -1 & x < -1 \\ 3x+2 & -1 \le x \le 0 \\ -2x+2 & x > 0 \end{cases}$$



$F(x) = ax + a$

$G(x) = bx$

$$|ax+a| - |bx| + H(x) = \begin{cases} -ax-a+bg+H(x) & x \le -1 \\ ax+a+by+H(x) & -1 \le x \le 0 \\ ax+a-by+H(x) & x > 0 \end{cases}$$

$a = \frac{3}{2} \quad b = \frac{5}{2} \quad H = \frac{1}{2} - x$

$F(x) = \frac{3}{2}x + \frac{3}{2} \quad G(x) = \frac{5}{2}x \quad H(x) = \frac{1}{2} - x$

Problem 5 Sol 2

$$|F(x)| - |G(x)| + H(x) = \begin{cases} -1 & x < -1 \\ 3x+2 & -1 \le x \le 0 \\ -2x+2 & x > 0 \end{cases}$$

$$\frac{a+b+|a-b|}{2} = \max(a,b)$$

$$\frac{a+b-|a-b|}{2} = \min(a,b)$$

$$\rightarrow 3x+2 + \begin{Bmatrix} -3x-3 & x < -1 \\ 0 & x \ge -1 \end{Bmatrix} + \begin{Bmatrix} -5x & x > 0 \\ 0 & x \le 0 \end{Bmatrix}$$

$$= 3x+2 + \max(-3x-3, 0) + \min(-5x, 0)$$

$$= -\left|\frac{5x}{2}\right| + \frac{-5}{2} + \left|\frac{3x+3}{2}\right| - \frac{3x+3}{2} + 3x+2$$

$$= \left|\frac{3}{2}(x+1)\right| - |2.5x| + \frac{1}{2} - x$$

$$F(x) = \frac{3}{2}(x+1) \quad G(x) = 2.5x \quad H(x) = \frac{1}{2} - x$$

Problem 6

$F(x) = \frac{2}{4^x+2}$ Find $F\left(\frac{1}{2022}\right) + F\left(\frac{2}{2022}\right) \cdots + F\left(\frac{2021}{2022}\right)$

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idea: $F(x) + F(1-x) = 1$

Proof: $\frac{2}{4^x+2} + \frac{2}{4^{1-x}+2} = 1$

$\Rightarrow F\left(\frac{1}{2022}\right) \longleftrightarrow F\left(\frac{2021}{2022}\right)$

$F\left(\frac{x}{2022}\right) \longleftrightarrow F\left(\frac{2022-x}{2022}\right)$

$1010 + F\left(\frac{1011}{2022}\right)$

$= 1010 + F\left(\frac{1}{2}\right)$

$= 1010 + \frac{2}{4}$

$= \frac{2021}{2}$

Problem 7 Sol 1

$a \geq 0 \quad \sqrt{a^2+a+1} - \sqrt{a^2-a+1}$

$\sqrt{a^2+a+1} - \sqrt{a^2-a+1} = X$

$\sqrt{a^2+a+1} = X + \sqrt{a^2-a+1}$

$a^2+a+1 = X^2 + a^2 - a + 1 + 2X\sqrt{a^2-a+1}$

$2a - X^2 = 2X\sqrt{a^2-a+1}$

$4a^2 + X^4 - 4aX^2 = 4X^2a^2 - 4X^2a + 4X^2$

$4a^2 + X^2 = 4X^2 + 4X^2a^2$

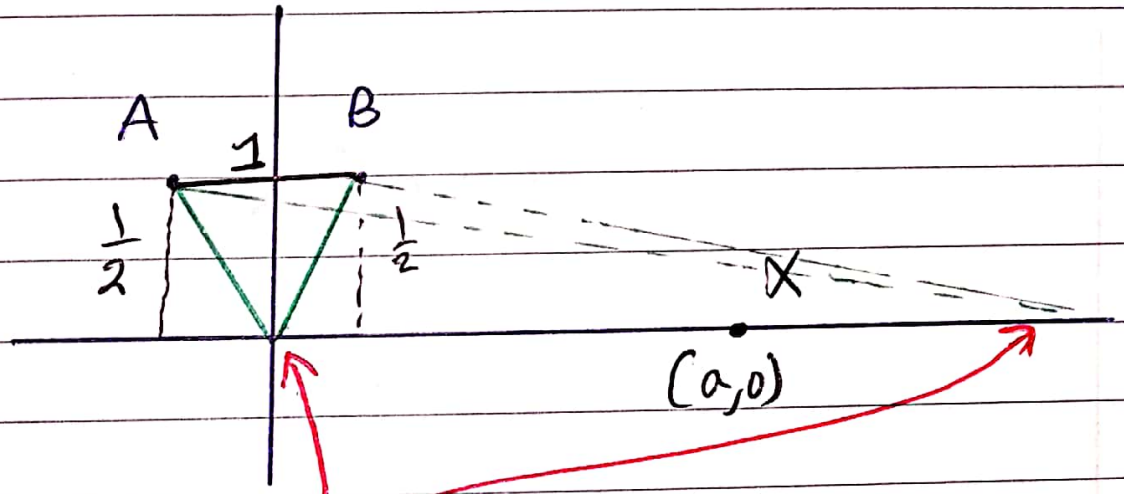
$a^2 = \dfrac{4X^2 - X^4}{4 - 4X^2} = \dfrac{X^2(4-X^2)}{4(1-X^2)}$

$1 - X^2 > 0 \quad 0 \leq X < 1$

Problem 7 Sol 2

$a \geq 0 \quad \sqrt{a^2+a+1} - \sqrt{a^2-a+1}$

$$\sqrt{\left(a+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \sqrt{\left(a-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = AX - BX$$



$AX - BX = [0, 1]$

Problem 8

$$(1+x)(1+2x)(1+4x)\dots(1+2^n x) = 1 + a_1 x + a_2 x^2 \dots$$

find a_2

$2^1, 2^2 \dots 2^n$	$2^{n+1} - 2$
$2^3, 2^4 \dots 2^{n+1}$	$2^{n+2} - 2^3$
$2^5, 2^6 \dots 2^{n+2}$	$\vdots$
$\vdots$	$2^{2n-1} - 2^{2n-3}$
$2^{2n-3}, 2^{2n-2}$	$2^{2n} - 2^{2n-1}$
2^{2n-1}	

Sol 1

basis solution

$$= \frac{2^{2n+2} - 3 \cdot 2^{n+2} + 2}{3}$$

$$\sum_{\substack{i,j=0 \\ i \geq j}}^{n} 2^{i+j} = \frac{1}{2} \sum_{\substack{i,j=0 \\ i \neq j}}^{n} 2^{i+j} = \frac{1}{2}\left(\sum_{i,j=0}^{n} 2^{i+j} - \sum_{i=0}^{n} 2^{2i}\right)$$

Sol 2

$$= \frac{1}{2}\left(\left(\sum_{i=0}^{n} 2^i\right)\left(\sum_{j=0}^{n} 2^j\right) - \sum_{i=0}^{n} 4^i\right) = \frac{(2^{n+1}-1)^2 - \frac{4^{n+1}+1}{3}}{2}$$

$$F_{n+1}(x) = F_n(x)(1+2^{n+1}x) = (1 + b_n x + c_n x^2 + \dots)(1 + 2^{n+1}x)$$

Sol 3

$$b_{n+1} = b_n + 2^{n+1} \qquad b_n = 2^{n+1} - 1$$

$$c_{n+1} = c_n + 2^n b_n = c_n + 2^{2n+1} - 2^n \qquad c_n = \frac{2^{2n+2} - 3 \cdot 2^{n+2} + 2}{3}$$

Problem 9

$m, n \geq 1 \quad 0 \leq x \leq 1$ Find $\max |x^m - x^n|$

$n > m \quad x^m - x^n = x^m (1 - x^{n-m})$

Sol 1

$= y^{\frac{m}{n-m}} \cdot (1-y) = \left(y(1-y)^{n-m}\right)^{\frac{1}{n-m}} y$

$= \underbrace{\alpha y \cdot \alpha y \cdot \alpha y \cdots \alpha y}_{m} \left(\underbrace{(1-y)(1-y)\cdots(1-y)}_{n-m}\right)$

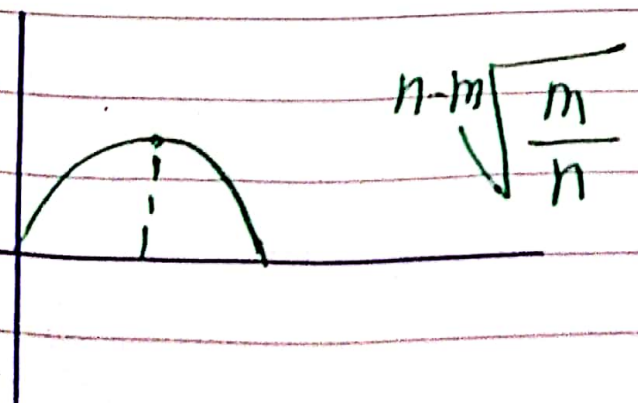
$\Rightarrow \alpha m y + n - m - y(n-m)$

$\alpha = \frac{n-m}{m} \qquad \frac{m^m \cdot (n-m)^{n-m}}{n^n}$

$f(x) = x^m - x^n$

$f'(x) = m x^{m-1} - n x^{n-1} = x^{m-1}(m - n x^{n-m})$

Sol 2



Problem 10

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$ Find x

$$\frac{1 + \left(\frac{3}{2}\right)^{3x}}{\left(\frac{3}{2}\right)^{x} + \left(\frac{3}{2}\right)^{2x}} = \frac{7}{6}$$

$$\left(\frac{3}{2}\right)^{x} = a \qquad \frac{1 + a^3}{a + a^2} = \frac{7}{6}$$

$$\frac{a^2 - a + 1}{a} = \frac{7}{6}$$

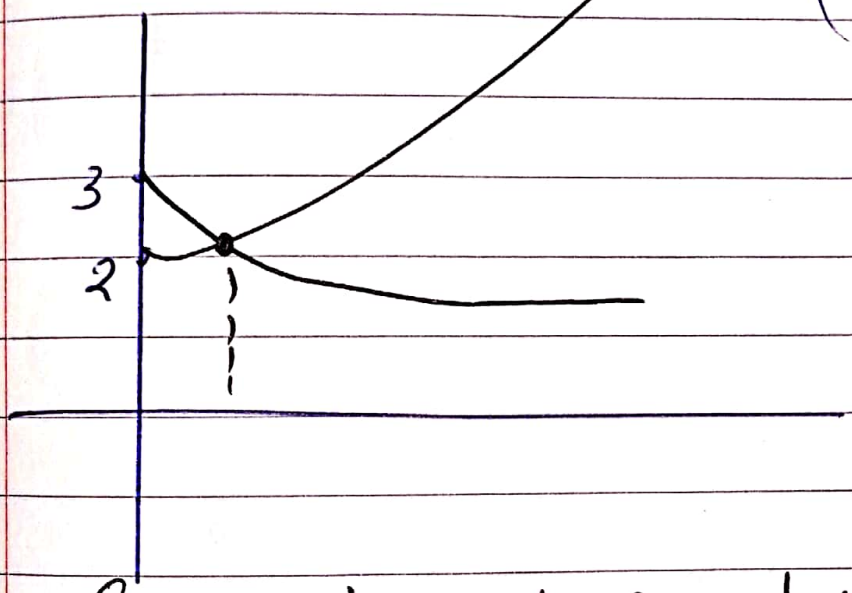
$$a = \frac{3}{2}, \frac{2}{3}$$

$$\Rightarrow \boxed{x = 1, -1}$$

Problem 11

$$2021^x + 2022^x = 2000^x + 2001^x + 2002^x$$

$$1 + \left(\frac{2022}{2021}\right)^x = \left(\frac{2000}{2021}\right)^x + \left(\frac{2001}{2021}\right)^x + \left(\frac{2002}{2021}\right)^x$$



at most 1 solution

$\Downarrow$

$\rightarrow$ exactly 1 solution

Problem 12

$\boxed{\text{Solve}}$ $2x^2(2^x-1)+x(2^{x^2}-2)=2^{x+1}-2$

$(x^2-1)(2^x-1)+x(2^{x^2}-1)=0$

iff $x(x^2-1)\neq 0$

$$\frac{2^x-1}{x}+\frac{2^{x^2-1}-1}{x^2-1}=0 \Rightarrow f(x)+f(x^2-1)=0$$

$$f(a)=\frac{2^a-1}{a} \Rightarrow f(a)\geq 0 \qquad \boxed{f(a)=0 \text{ if } a=0}$$

$f(x)=f(x^2-1)=0$

$x=x^2-1=0$

$\Rightarrow \boxed{x=1,-1,0}$

Problem 13

$$(x-a_1)(x-a_2)\cdots(x-a_n) = p(x)\cdot q(x) + 1$$

$p(a_i)\,q(a_i) = -1$ $\Rightarrow$ one of them $= 1$ and the other one $= -1$

$H(x) = p(x) + q(x)$

$\Rightarrow H(a_1) = H(a_2) = \cdots = H(a_n) = 0$

$H(x) \equiv 0$

$p(x) = -q(x)$

$$\underbrace{(x-a_1)(x-a_2)\cdots(x-a_n)}_{\text{(increasing)}} - 1 = p(x)\,q(x) = \underbrace{-p^2(x)}_{\text{negative (decreasing)}}$$

$\Rightarrow\!\Leftarrow$

$\nexists\ p(x), q(x)$

Problem 14

$a, b, c > 0$

① $a+b+c \leq 4$

② $ab+bc+ca \geq 4$

$$\begin{cases} |a-b| \leq 2 \\ |b-c| \leq 2 \\ |c-a| \leq 2 \end{cases}$$

at least 2 are correct

$$\Updownarrow$$

$$(a-b)^2+(b-c)^2+(c-a)^2 \leq 8$$

$$2a^2+2b^2+2c^2-2(ab+bc+ac) \leq 8$$

$$a^2+b^2+c^2-ab-bc-ac \leq 4$$

$$①^2 - 3② \leq 4$$

$$16 - 12 \leq 4$$

$$4 \leq 4$$

Problem 15

$$\sum_{k=0}^{n} \frac{(2n)!}{(n-k)!\,(n+k)!} = S \quad \text{find } S$$

$$\frac{1}{(2n)!} \sum_{k=0}^{n} \binom{2n}{n-k}$$

$$= \frac{1}{(2n)!} \sum_{k=0}^{n} \binom{2n}{k} = \frac{1}{(2n)!} \cdot \frac{2^{2n} + \binom{2n}{n}}{2}$$

$$\boxed{\binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} \cdots + \binom{2n}{2n} = 2^{2n}}$$

Problem 16

Solve For (x,y) $\begin{cases} (1+x)(1+x^2)(1+x^4)=y^7 \\ (1+y)(1+y^2)(1+y^4)=x^7 \end{cases}$

$\boxed{x<y<-1}$

a) $xy=0$ ↯

b) $x=y \Rightarrow \dfrac{x^7-1}{x-1}=0 \quad x\neq 1$ ↯

c) $x>y>0 \quad (1+x)(1+x^2)(1+x^4)>1+x^7>1+y^7$ ↯

d) $x<y<0 \quad \underbrace{(1+x)}_{\leq 0}\underbrace{(1+x^2)}_{\geq 0}\underbrace{(1+x^4)}_{\geq 0}=y^7<0 \Rightarrow x<y<-1$

$2+(x+y)+(x^2+y^2)+(x^3+y^3)+(x^4+y^4)+(x^5+y^5)+(x^6+y^6)=0$ ↯

e) $x<0<y \quad (1+y)(1+y^2)(1+y^4)>0>x^7$ ↯

Problem 17

$a+b+c+d+e=8 \qquad a^2+b^2+c^2+d^2+e^2=16$

$\min(a)? \qquad \max(a)?$

$\boxed{\max(a)} \qquad b^2+c^2+d^2+e^2 \geq \frac{(b+c+d+e)^2}{4} \geq \frac{(8-a)^2}{4}$

$4a^2+(8-a)^2=64$

$5a^2-16a \leq 0 \qquad a \in \left[0; \frac{16}{5}\right] \qquad a=\frac{16}{5} \quad b,c,d,e=\frac{6}{5}$

$\boxed{\min(a)} \qquad \frac{b+c+d+e}{4}=x \qquad x=\frac{8-a}{4}$

$b=x+b_1$

$c=x+c_1 \qquad b_1+c_1+d_1+e_1=0$

$d=x+d_1$

$e=x+e_1$

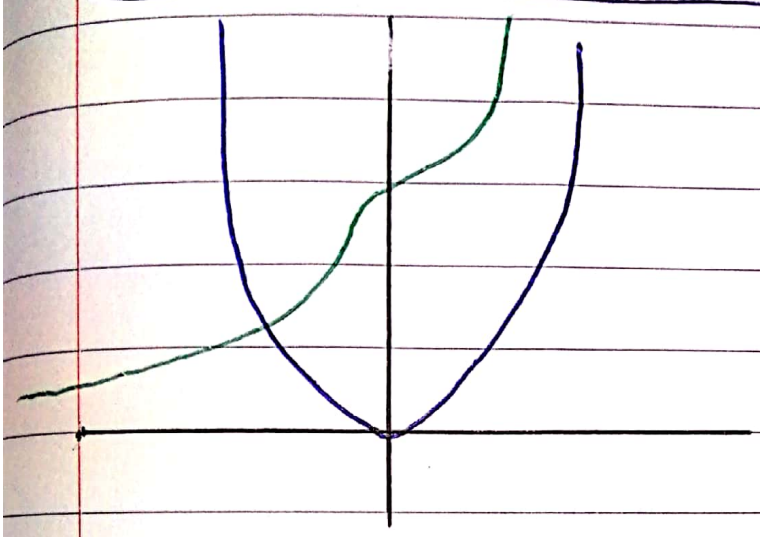
$16=a^2+x^2+(\ldots)+x^2+(\ldots)+x^2+(\ldots)+x^2+(\ldots) > a^2+4x^2$

$a^2+4\left(\frac{8-a}{4}\right)^2$

$a \geq 0 \qquad a=0 \quad b,c,d,e=2$

Problem 18

$x^2 = 2^x + 3^x + 6^x$ Find all real x



$= 2^x + 3^x + 6^x$

$= x^2$

$\boxed{x < 0}$ at most 1 solution

because $2^x + 3^x + 6^x = 3$ when $x = 0$ and is decreasing and x^2 is 0 when $x = 0$ and is increasing

Solution: $x = -1$

$\boxed{0 \le x \le 1}$ $x^2 \le 1$ $2^x + 3^x + 6^x \ge 3$ no solution

$\boxed{1 \le x \le 3}$ $x^2 \le 9$ $2^x + 3^x + 6^x \ge 11$ no solution

$\boxed{3 \le x \le 6}$ $x^2 \le 9 \cdot 4$ $2^x + 3^x + 6^x \ge 256$

$\boxed{6 \le x \le 12}$

Problem 19

numbers 1000, 1001, 1002, ... 2999 are written on the board in each step Hamza erases two numbers (a,b) are writes $\frac{\min(a,b)}{2}$. after 1999 moves there will be one number remaining, Prove that this number is always ~~bigger~~ smaller than 1 no matter what hamza does.

$\boxed{\text{Step 1}}$ take the sum of it's inverses $\left(\frac{1}{1001}, \frac{1}{1002}, \frac{1}{1003} \ldots \frac{1}{2999}\right)$

at each step it increases because if we have a, b

$a \geq b \quad \frac{1}{a} + \frac{1}{b} \leq \frac{1}{b} + \frac{1}{b} \leq \frac{2}{b} = \frac{1}{\frac{b}{2}}$

$\boxed{\text{Step 2}}$ in the beging their sum is > 1 because

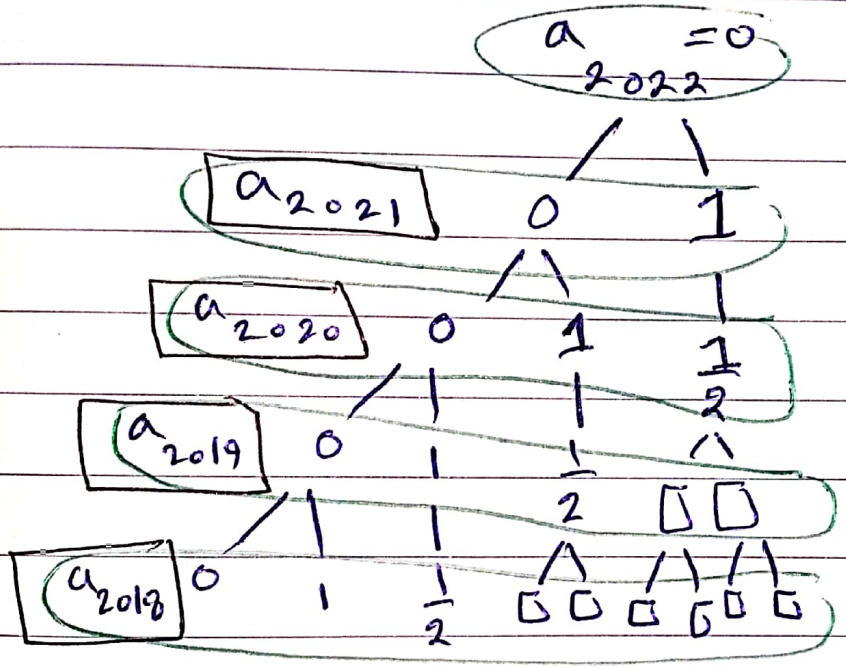
$$\frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \ldots + \frac{1}{2999} > \frac{2000^2}{1000 + 1001 + 1002 \ldots 2999} > 1$$

$\boxed{\text{Step 3}}$ From step 1, step 2 we can conclude that the finale number is > 1 so $\underbrace{\text{its inverse}}_{\text{the original number}}$ is < 1

Problem 20

$a_{n+1} = 4a_n(1-a_n)$ if $a_{2022} = 0$

Find all posible values for a_1



Sol 1

X 2 -1

times 2 minus 1

$a_0 = 2^{2020} + 1$

Sol 2

$a_n = \sin^2 X_n$

$\sin^2 X_{n+1} = 4\sin^2 X_n \cdot \cos^2 X_n = \sin^2 2X_n$

$X_{n+1} = 2X_n$

$X_{2022} = X_1 \cdot 2^{2021}$

$\sin(2^{2021} X_1) = 0$ $\quad 2^{2021} X_1 = \pi K$ $\quad X_1 = \frac{\pi K}{2^{2021}}$

$a_n = \sin^2 \frac{K\pi}{2^{2021}}$ $\quad 0 \le K \le 2^{2020}$ $\Rightarrow$ $\boxed{K \text{ has } 2^{2020}+1 \text{ values}}$