January Camp 2022
Problems

Geometry – L2

Angle chasing

Problem 1. Let ABC be a scalene triangle. Prove that the angle bisector of angle BAC and the perpendicular bisector of the side BC intersect at the point on the circumcircle of ABC.

Problem 2. Let the altitudes AD, BE, CF of triangle ABC intersect at H. Prove that H is the incenter of triangle DEF.

Problem 3. Let I, O, H be the incenter, circumcenter, and orthocenter of an acute triangle ABC, respectively. Prove that if the points B, C, H, I lie on a single circle, then O lies on this circle too.

Problem 4*. Let $ABCD$ be a cyclic quadrilateral and let I_1, I_2, I_3 and I_4 be the incenters of triangles ABC, ABD, CDA and BCD, respectively.

- Prove that AI_1I_2B is cyclic.
- Prove that $I_1I_2I_3I_4$ is a rectangle.

Problem 5. (Miquel Theorem) Let P, Q, R be arbitrary points on the sides BC, CA, AB of a triangle ABC. Show that the circumcircles of triangles ARQ, BPR, and CQP pass through a common point.

Problem 6. Let $ABCD$ be a convex quadrilateral inscribed in a circle. Lines AB and CD meet at point P, and lines BC and DA meet at point Q. Prove that the bisectors of angles BPC and AQB are perpendicular.

Problem 7. Let $ABCD$ be a cyclic quadrilateral and denote by P its intersection of diagonals. Let circle ω passing through A and B intersect segments PC, PD at X, Y, respectively. Prove that XY is parallel to CD.

Problem 8*. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Problem 9. Let H and O denotes orthocenter and circumcenter of triangle ABC. Let M be the midpoint of side BC. Prove that AO and HM intersects on circumcircle of triangle ABC.

Problem 10. (Miquel Theorem) Four lines determine four triangles. Prove that their circumcircles intersect at one point.

Problem 11. Let ABC be an acute-angled triangle with $\sphericalangle BAC = 60^\circ$ and $AB > AC$. Let I, H denote its incenter and orthocenter, respectively. Prove that

$$2\sphericalangle AHI = 3\sphericalangle ABC.$$

Problem 12. (Simson line) Let P lies on circumcircle of triangle ABC. Prove that projections of P on sides of a triangle lie on one line.

Problem 13*. (Steiner line) Let P lies on circumcircle of triangle ABC. Prove that reflection of P wrt sides of a triangle lie on one line containing orthocenter of triangle ABC.

Problem 14. Let ABC be a triangle with orthocenter H. Let ℓ be a line passing through H. Prove that reflections of ℓ wrt AB and AC intersect on circumcircle of ABC.

Problem 15*. Point O is the circumcenter of triangle ABC. Circle passing through points A, O intersects lines AB, AC at P, Q, respectively. Prove that the orthocenter of triangle OPQ lies on BC.

Power of a point

Problem 16. Let ABC be a non-right triangle with orthocenter H and let M, N be points on its sides AB and AC. Prove that the common chord of circles with diameters CM and BN passes through H.

Problem 17. Let ABC be a triangle. A circle tangent to BC and AC intersects AB at K and L. Prove that

$$|AK - BL| \leq |AC - BC|.$$

Problem 18. Let ω be the incircle of square $ABCD$. Diagonal AC meets circle ω at E, such that $AE < EC$. Segment BE intersects ω at point F. Prove that $EF = 2\cdot BF$.

Problem 19. Let o_1 and o_2 be disjoint circles, one lying outside of the other. Consider variable lines that intersect o_1 and o_2 at points A, B, and C, D, respectively, such that $AB = CD$ and A, B, C, D lie in that order. Prove that the midpoints of segments BC lie on a fixed line.

Problem 20. Let ABC be an equilateral triangle. A circle intersects segments AB, BC, CA at points K, L; M, N; P, Q, where points K, L, M, N, P, Q lie on the circle in this order. Prove that

$$AK + BM + CP = BL + CN + AQ.$$

Problem 21. Line k is tangent to circle o at point A. Line segment CD is a chord of circle o parallel to line k. The tangent line to circle o at point D meets line k at point B. Line segment BC intersects circle o for the second time at point E. Prove that line DE passes through the midpoint of segment AB.

Problem 22. Let A, B, C, D lie on the circle Ω. Lines AB and CD intersect at P, lines AD and BC intersect at Q. Prove that

$$\mathrm{Pow}(P, \Omega) + \mathrm{Pow}(Q, \Omega) = PQ^2.$$

Problem 23. Convex hexagon $ABCDEF$ satisfies $AB = BC$, $CD = DE$, $EF = FA$. Show that lines containing altitudes of triangles BCD, DEF and FAB from vertices C, E, A, respectively, are concurrent.

Problem 24. Let $ABCDEF$ be a convex hexagon. It is known that the quadrilaterals $ABCD$, $CDEF$, and $EFAB$, are cyclic. Prove that hexagon $ABCDEF$ is cyclic.

Problem 25. Let the incircle ω of triangle ABC touches BC, CA, and AB at D, E, and F, respectively. Let Y_1, Y_2, Z_1, Z_2, and M be the midpoints of BF, BD, GE, CD, and BC, respectively. Let $Y_1Y_2 \cap Z_1Z_2 = X$. Prove that $MX \perp BC$.

Problem 26. Let $ABCDEF$ be a convex hexagon in which $AB = AF$, $BC = CD$, $DE = EF$ and $\measuredangle ABC = \measuredangle EFA = 90°$. Prove that $AD \perp CE$.

Problem 27. Let $ABCD$ be a cyclic quadrilateral with AB and CD not parallel. Let M be the midpoint of CD. Let P be a point inside $ABCD$ such that $PA = PB = CM$. Prove that AB, CD and the perpendicular bisector of MP are concurrent.

Problem 28. Given is triangle ABC. Points P and Q were chosen such that

$$\measuredangle PBC = \measuredangle QCB = 90° \quad \text{and} \quad AP = PB, \;\; AQ = CQ.$$

Tangent to circumcircle of ABC passing through point A intersects BC at R. Prove that P, Q and R are collinear.

Problem 29. Let $ABCD$ be a cyclic quadrilateral ($AB \neq CD$). Quadrilaterals $AKDL$ and $CMBN$ are rhombi with equal sides. Prove that points K, L, M, N lie on a single circle.

Problem 30. Let circles ω_1 and ω_2, with centres in O_1, O_2, respectively, intersect at two distinct points P and Q. Their common tangent, closer to point P, touches the circles at A, B respectively. Let the perpendicular from A to the line BP meet O_1O_2 at C. Prove that $\measuredangle APC = 90°$.

Problem 31. Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively, such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either triangle ABM or triangle ACM is 2.

Problem 32. Let $ABCD$ be a circumscribed quadrilateral with $BC = 2AB$. Suppose that perpendicular bisector of BC and bisector of angle DCB intersect at X. Prove that AX and BD are perpendicular.

Problem 33. Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Problem 34. Let ABC be an acute angled triangle with orthocenter H and circumcenter O. Let P and Q lie on AC and BC, respectively such that $HPCQ$ is a parallelogram. Prove that $OP = OQ$.

Problem 35. An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle circumscribed on ABC crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.