Email training, N1

Level 3, September 13-19, 2021

1. Problems with Solutions

Problem 1.1. Characterize all positive integers $n > 1$ for which the expression

$$(n-1)! \cdot \Big(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\Big) \tag{1}$$

is not divisible by n.

Solution 1.1. Let's notice that for odd number n one has

$$(n-1)! \cdot \left(\frac{1}{i} + \frac{1}{n-i}\right) = n \cdot \frac{(n-1)!}{i(n-i)}$$

is divisible by n so for all odd numbers n the expression (1) is divisible by n. Let us consider the case when n is even number $n = 2k$.
In this case again by summing terms like $\frac{1}{i}$ and $\frac{1}{n-i}$ one concludes that the expression (1) is not divisible by n if and only if the expression

$$(2k-1)! \cdot \frac{1}{k}$$

is not divisible by n. So one needs to find positivie integers k for which one has

$$\frac{(2k-1)!}{2k^2} \notin N.$$

It is easy to check that this happens when $k = 1$, $k = 4$ or k is prime number.

So the answer is $n = 2, 4, 8$ or $n = 2p$ where p is odd prime.

Problem 1.2. Let $x(n)$ be the biggest prime divisor of n. Prove that there exist infinitely many number n such that $x(n) < x(n+1) < x(n+2)$.

Solution 1.2. Let $p > 2$ be a prime number. Denote $x_k = p^{2^k} + 1$: My simple manipulation one can conclude that

$$x_k = (p^{2^k} - 1) + 2 = (p-1)(p+1)(p^2+1)\ldots(p^{2^{k-1}} + 1) + 2$$

which means, that

$$x_k = (p-1)x_0x_1\ldots x_{k-1} + 2.$$

From this equation follows, that for arbitrary indices $i \neq j$ one has $gcd(x_i, x_j) = 2$. It means that numbers in the sequence x_k must contain members with arbitrary big prime multipliers, which means that the sequence $P(x_k)$ is not bounded. Let s be the biggest index, for which $P(x_i) < p$ for all $0 \leq i \leq s$. Then one has the following

$$P(x_{s+1} - 2) = P\bigg((p-1)x_0x_1\ldots x_s\bigg) = max(P(p-1), P(x_0), P(x_1), \ldots, P(x_s)) < p$$

as well as $P(x_{s+1}-1) = P(p^{2^{s+1}}) = p$ and $P(x_s) > p$ (the last one follows from the definition of s. So one has that

$$P(x_{s+1}-2) < P(x_{s+1}-1) < P(x_{s+1}).$$

So we have a triple of consecutive integers that satisfy to the condition of the problem. This triple is corresponding to the prime p. By changing p one can get infinitely many triples. The statement of the problem is proved.

Problem 1.3. Let the sequence a_1, a_2, ..., a_{20} is the permutation of integers 1, 2, ..., 20. Find the maximum possible value of

$$\min\{|a_2 - a_1|, |a_3 - a_2|, \ldots, |a_{20} - a_{19}|, |a_1 - a_{20}|\}.$$

Solution 1.3. Let us consider neighbors of number 10. One of them is not equal 20 and the difference with 10 by absolute value will be at most 9.So the answer to the problem can't be bigger than 9: Below is the example for 9.

$$1, 10, 19, 8, 17, 6, 15, 4, 13, 2, 11, 20, 9, 18, 7, 16, 5, 14, 3, 12$$

(1 and 12 are neighbors as well).