Power of point to circle + Radical axis

For students of level 3

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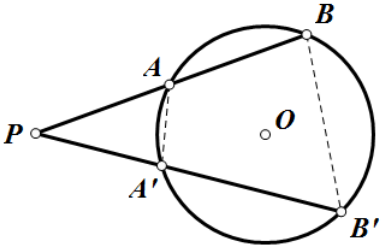
Outline

The power of a point to a circle and the radical axis between two circles are the importance tool in Olympiad geometry. With the simple and easy-to-understand definition, we can use them to resolve a large number of problem, even the hard ones. In this lecture, we will review the base knowledge related to the power of a point to a circle, the radical axis and the radical center.

Then, by applying them, we will work on some interesting or challenging problems. Finally, we will consider the special case when one of two circle has the radius of 0, which called degenerate circle.

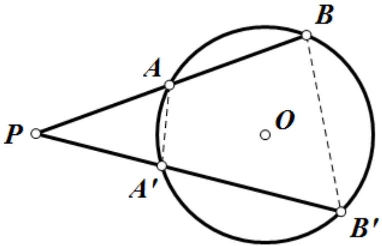
Power of a point to a circle

Let be given a circle (O) and a point P. An arbitrary line d passes through P and cuts (O) at two points A, B.



Then, the value $\mathscr{P}_{P/(O)} = PA \cdot PB$ is independent of the choice of the line d. Indeed, if we draw another line d' passes through P and cuts (O) at two points A', B' and we have $PA \cdot PB = PA' \cdot PB'$.

This called the **power of point** P **to the circle** (O).



Question

How to prove this fact?

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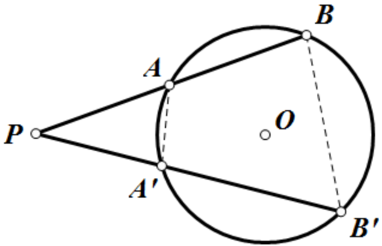

Question

How to prove this fact?

It is easy to use the similarity of two triangles. Note that in the given figure, two triangles PAA' and $PB'B$ are similar.

Question

Is the definition true for the case P lies inside the circle?

They also extend the length of segment to the algebra length, i.e $\overline{PA}$ is the length of segment PA with the direction. If PA, PB have the opposite direction, then $\overline{PA} \cdot \overline{PB} = -PA \cdot PB < 0$.

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So $\mathscr{P}_{P/(O)} = \overline{PA} \cdot \overline{PB}$, this is the formal definition of the power of a point to a circle.

If P lies inside the circle, then PA, PB have the opposite direction. We can conclude that:

1. P lies outside the circle $(O) \Leftrightarrow \mathscr{P}_{P/(O)}$ is positive.
2. P lies inside the circle $(O) \Leftrightarrow \mathscr{P}_{P/(O)}$ is negative.
3. P lies on the circle $(O) \Leftrightarrow \mathscr{P}_{P/(O)}$ is zero.

Radical axis

Now let consider the locus of point that have the same power to the two given circles. This locus is exist and is a line perpendicular to the line connects two centers, this called radical axis. We have 3 cases:

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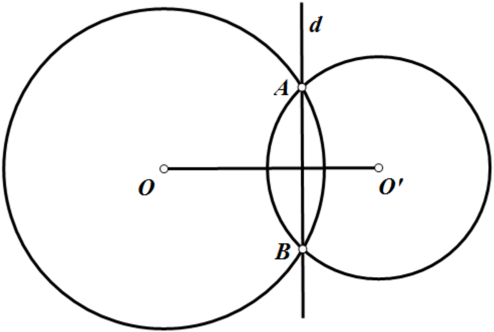
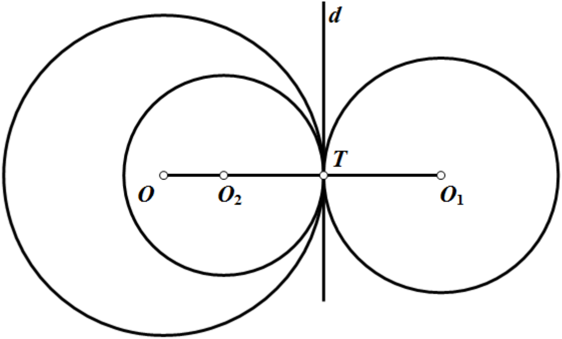
(1) Two circles are intersect.



The radical axis is the line passes through two common points.

Radical axis (cont.)

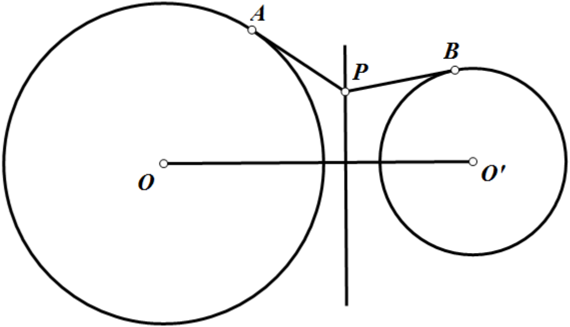
(2) Two circles are tangent.



The radical axis is the internal tangent line of two circles.

This also true for both cases internal tangent and external tangent of two circles.

(3) Two circles are apart.



The radical axis is a line divide the plane into two separate sub planes that contain the circles. The way to determine this line will be discussed later. Let's use Pythagoras's theorem to prove that the radical axis always perpendicular to the line passes through two centers.

Question

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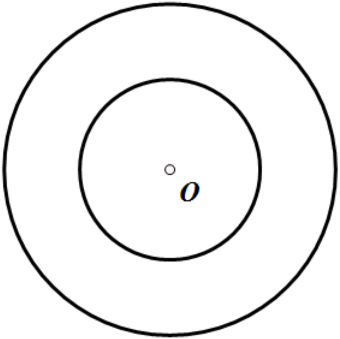
The answer is no. Let consider two circle with the same center (but two difference values of radius)



Radical center

Question

With three distinct circles, is there any point have the same power to these circles?

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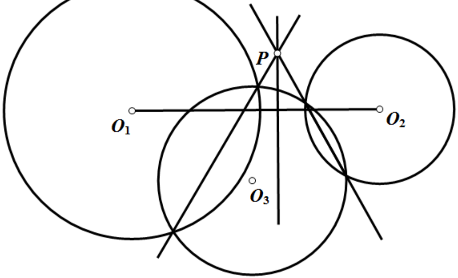
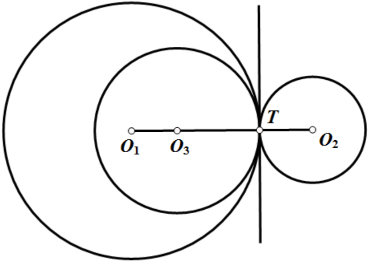
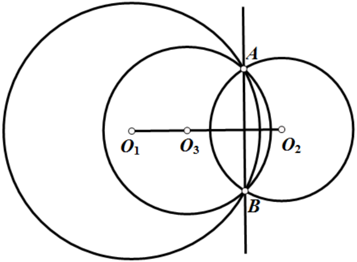
The answer is depend on the relative location of these circles. The number if point can be:

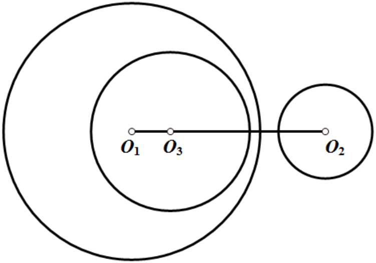
1. infinite many if these circles share one or two common point(s).
2. zero if three centers are collinear and these circles don't share any common point.
3. one, otherwise (almost case).

In the case this point exist, we have the radical center of three distinct circles.


P
O_1
O_2
O_3


T
O_1
O_3
O_2


A
B
O_1
O_3
O_2


O_1
O_3
O_2

Degenerate circle

The circle with radius equal 0 is the **degenerate circle**, which contains only one point as center. All properties of circle can be applied into this case.

Question

If the center of degenerate circle (A) lies on the circle (O), what is the radical axis of them?

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Question

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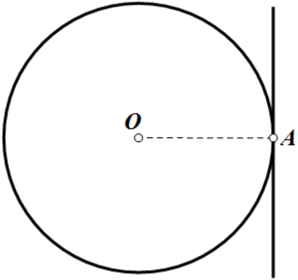
This is the line perpendicular to AO at A.



Question

But when the point A not lies on the circle (O), how to construct the radical axis?

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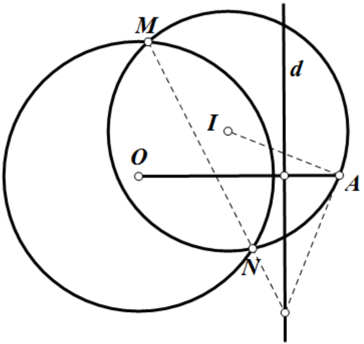
The following is the answer



Problem

Let ABC be a triangle with AD, BE, CF are the altitudes that concurrent at H. The lines passes through A and parallel to BE cuts CF at P, the lines passes through A and parallel to CF cuts BE at Q. Prove that PQ is perpendicular to the median respect to vertex A of triangle ABC.

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Problem

(Iran TST 2011) Let ABC be a triangle with $\angle B > \angle C$ and M is the midpoint of BC, altitude CD, BE. Let K, L are the midpoints of ME, MD. Suppose KL cuts the line that passes through A and parallel to BC at T. Prove that $TA = TM$.

Problem

Let M be the point lies in interior of triangle ABC. Suppose that tangent line of circle (MBC) at M cuts BC at X. Define the points Y, Z similarly. Prove that X, Y, Z are collinear.

Problem

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Problem

Let ABC be an acute triangle and AD, BE, CF are the altitudes that concurrent at H. The line passes through H and parallel to EF cuts BC at X. The points Y, Z are defined similarly.

1. *Prove that X, Y, Z are collinear.*
2. *Let I is the incenter of triangle ABC. The circle (IAD) cuts BC at M differs to D. The points N, P are defined similarly. Prove that M, N, P are collinear.*

Problem

Let two circles (O_1, R_1) and (O_2, R_2) tangent at M, given that $R_1 < R_2$. The point A moves on the circle (O_2) and doesn't belong to the line AO_1. Let AB, AC are two tangent lines of circle (O_1). The lines CM, BM cut (O_2) at F, E. Suppose that D is the intersection of EF and the tangent line of (O_2) at A. Prove that D belongs to the common tangent of two circle (O_1) and (O_2).

Problem

(China TST 2011) Let P is one of two intersections of two circles (O_1) and (O_2), the line AB is common tangent line of two circles. The line passes through A and is perpendicular to BP cuts the line O_1O_2 at C. Prove that AP is perpendicular to PC.

Problem

Let ABC be an triangle with incircle I and circumcircle (O). The line AI cut (O) at D and the line passes through I and perpendicular to AD cuts BC at K. Let H be the projection of K on line OI. Prove that K, H, A, D belongs to a same circle.

Problem

Let ABC be a triangle with $\angle B < \angle A$. On the segment BC, take D such that $\angle CAD = \angle ABC$. The circle of center O passes through B, D and cuts AB, AD at E, F respectively. Suppose that BF cuts DE at G and M is the midpoint of AG. Prove that CM is perpendicular AO.

Problem

Let $ABCD$ be a parallelogram with T as the intersection of two diagonals. Take (ω) as a circle of center O, passes through D and tangent to BD. Suppose that this circle cuts segment CD at E and cuts ray AD at F in such a way that B, E, F are collinear. Prove that $\angle ATD = \angle DOB$.

Problem

(China TST 2008) Let A be a point lying outside circle (O) and AB, AC are tangent line of (O). Take D, E, M on (O) such that $MD = ME$. Suppose that MB, MC cut DE at R, S respectively and take $X \in OB,\ Y \in OC$ such that $RX \perp DE,\ SY \perp DE$. Prove that $XY \perp AM$.

Problem

Let AB be a chord of circle (O) and M be midpoint of the arc AB. A circle (I) lying on the different side with M, respect to AB that tangent to AB and internally tangent to (O), given that . The lines pass through M and perpendicular to AI, BI intersect AB at C, D respectively. Prove that $AB = 2CD$.

Problem

(G3 IMO Shortlist 2009) Let ABC be an acute triangle with Z, Y are the tangent point of incircle to segment AB, AC. Let G is the intersection of BY, CZ. Construct point R, S in such a way that two quadrilateral $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.