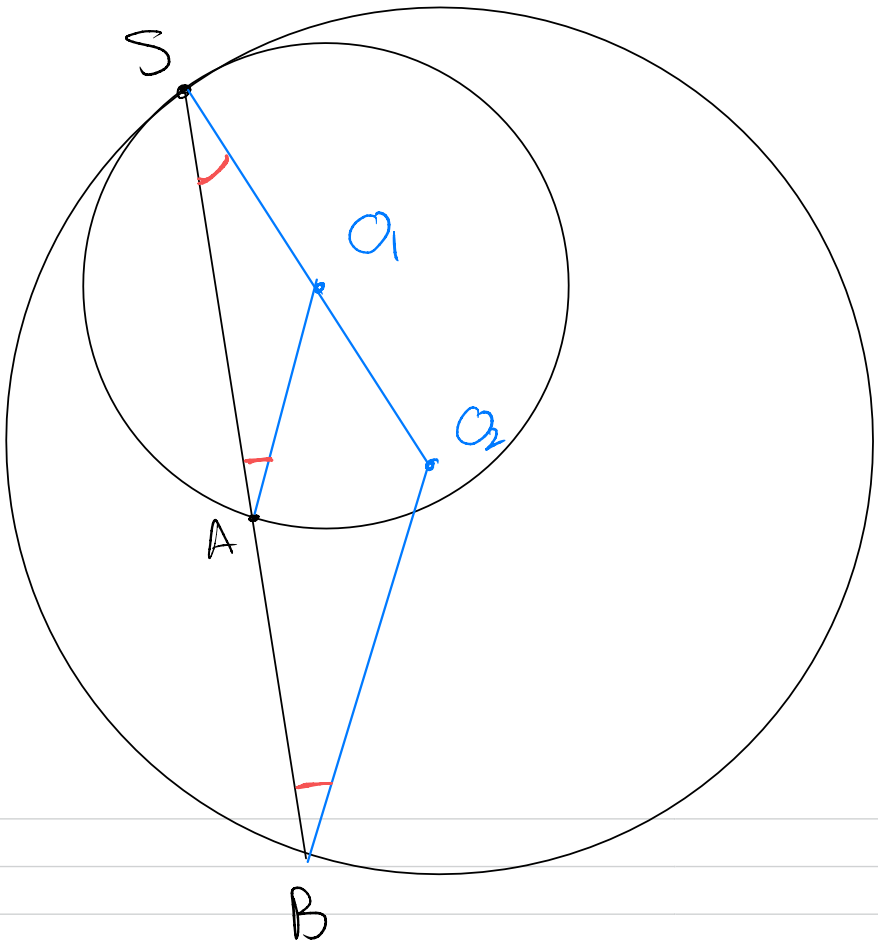
Intensive Training
Geometry

Day 6
8 April 2021

Includes solutions for:

EGMO 2016 – P1
BMO 2020 – P1
APMO 2014 – P1
Exam 7 – P4

Lemma: If S is a common tangent and S, A, B are collinear then $\frac{SA}{SB} = \frac{r}{R}$

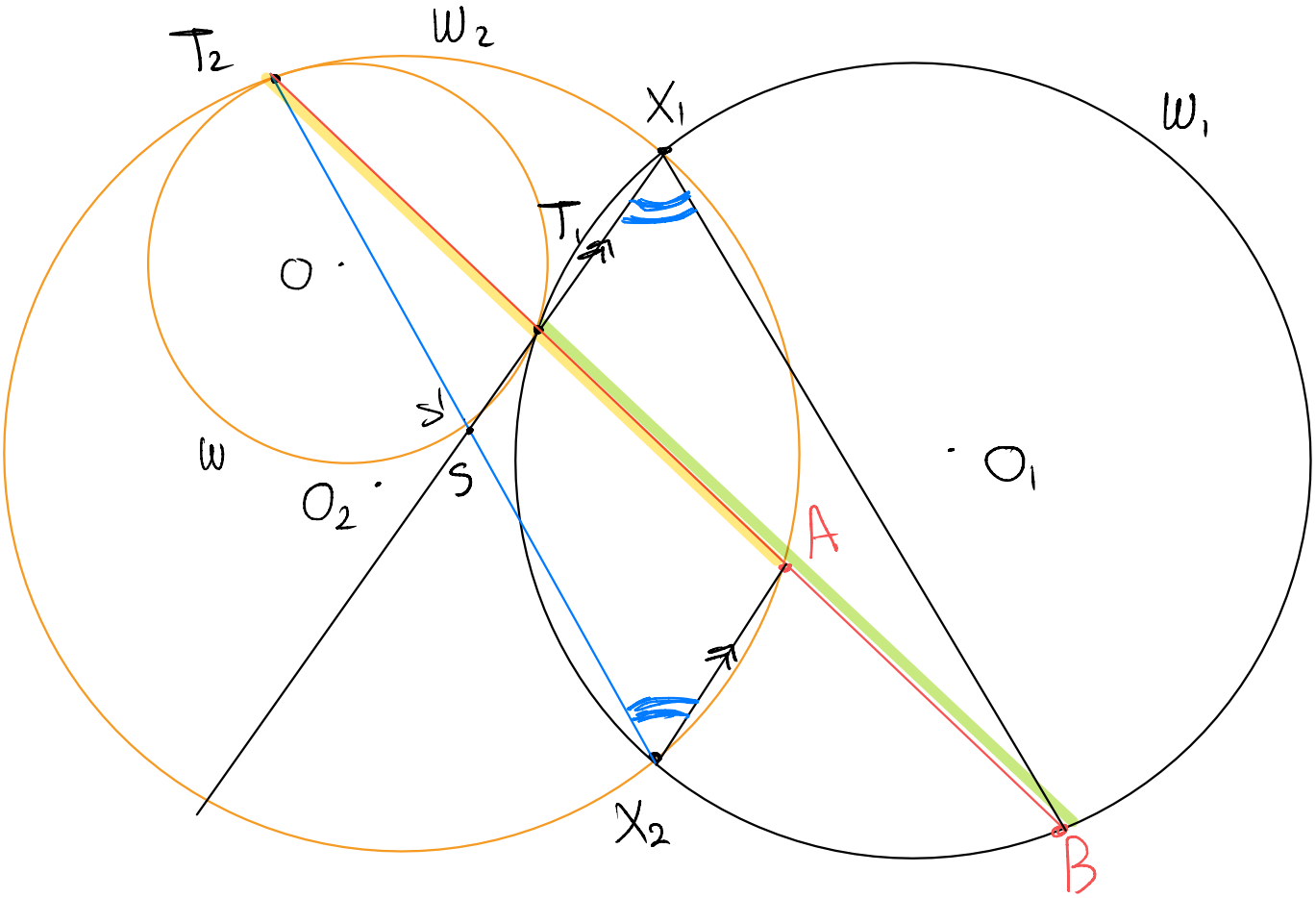


Proof:

$\Delta S O_1 A \sim \Delta S O_2 B$

$\Rightarrow \frac{SA}{SO_1} = \frac{SB}{SO_2} \Rightarrow \frac{SA}{SB} = \frac{r}{R}$

17. Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω. (EGMO 2016 P1).



Let r, R be the radius of ω, ω_1 respectively

From Lemma on ω, ω_2: $\dfrac{T_2T_1}{T_2A} = \dfrac{r}{R}$

From lemma on ω, ω_1: $\dfrac{T_2T_1}{T_1B} = \dfrac{r}{R}$

$\Big\}\ T_2A = T_1B$

Since ω_1, ω_2 have the same radius and $T_2A = T_1B$

$\Rightarrow \angle T_2X_2A = \angle T_1X_1B$

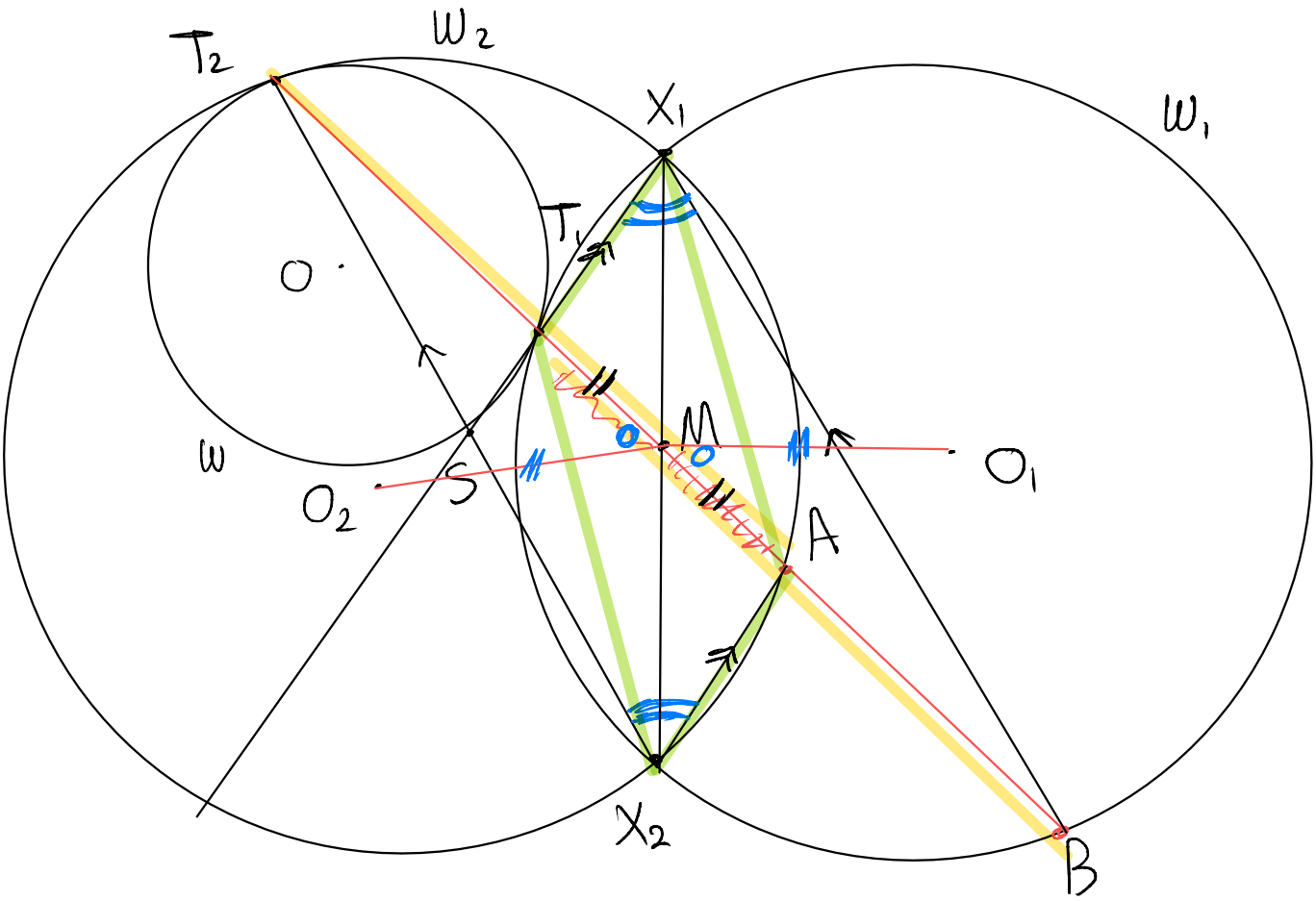
If $S = X_1T_1 \cap X_2T_2$, $S' = T_2X_2 \cap \omega$

* If $X_1T_1 \parallel X_2A \Rightarrow \dfrac{T_2S}{T_2X_2} = \dfrac{T_2T_1}{T_2A}$.

* From lemma on $\omega, \omega_2 \Rightarrow \dfrac{T_2S'}{T_2X_2} = \dfrac{r}{R} = \dfrac{T_2T_1}{T_2A}$

$\Big\}\ T_2S = T_2S' \Rightarrow S = S'$

17. Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω. (EGMO 2016 P1).



We want to show that $T_1X_1AX_2$ is a parallelogram

Let M be the midpoint of T_1A.

$\mathcal{P}_{\omega_2}(M) = MA \cdot MT_2$, $\mathcal{P}_{\omega_1}(M) = MT_1 \cdot MB$

However, $MA = MT_1$ and $MT_2 = T_2A - MT_1$

$= T_1B - MA = MB$

Therefore, $\mathcal{P}_{\omega_2}(M) = \mathcal{P}_{\omega_1}(M) \Rightarrow M \in X_1X_2$

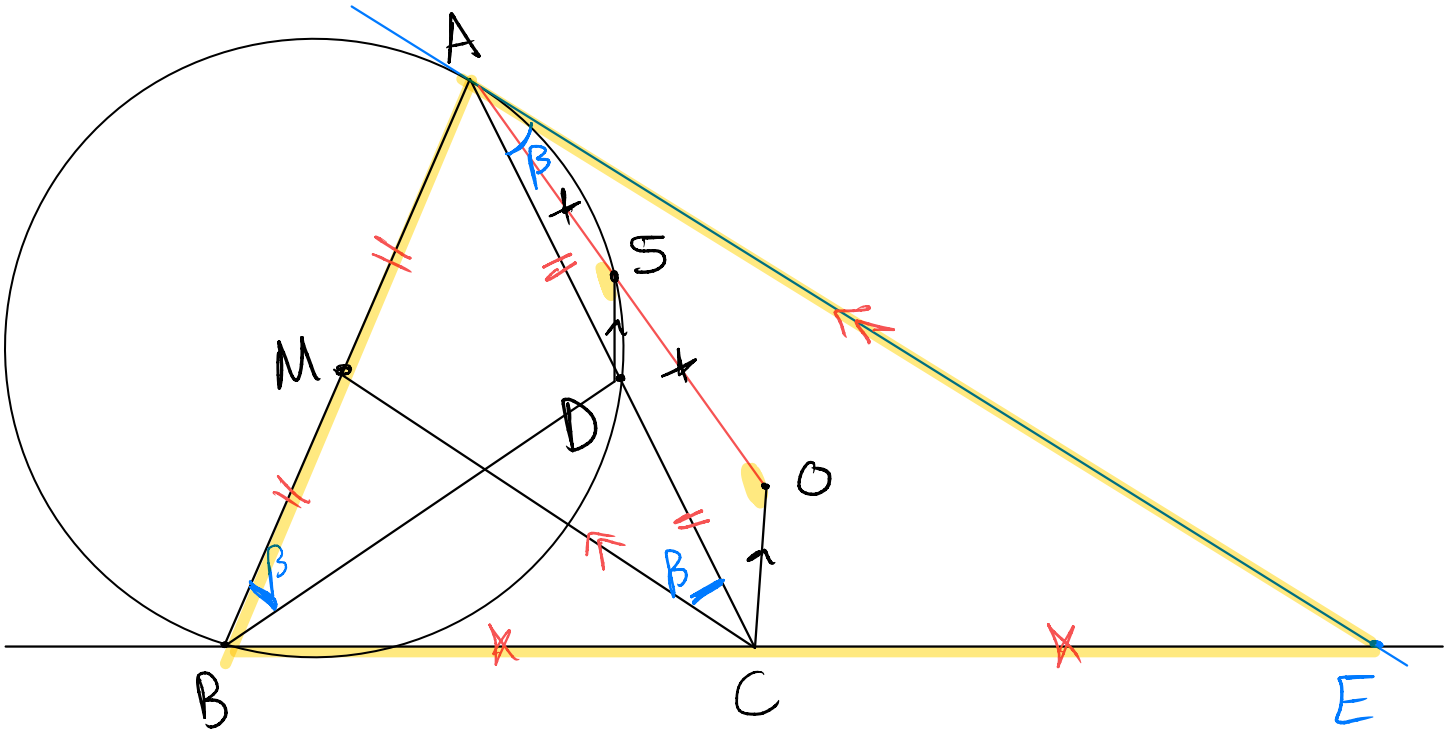
$\Rightarrow MO_1 = MO_2$ (M on the radical axis and $R(\omega_1) = R(\omega_2)$)

so $\Delta MO_2T_2 \equiv \Delta MO_1B$ (SSS)

$\Rightarrow \angle T_2MO_2 = \angle BMO_1 \Rightarrow O_1, M, O_2$ collinear

$\Rightarrow M$ midpoint X_1X_2 $\square$

18. Let ABC be an acute triangle with $AB = AC$, let D be the midpoint of the side AC, and let γ be the circumcircle of the triangle ABD. The tangent of γ at A crosses the line BC at E. Let O be the circumcenter of the triangle ABE. Prove that midpoint of the segment AO lies on γ. (BMO 2020 P1)



* C is the midpoint of BE

proof. Let M be the midpoint of AB

$\angle MCA = \angle DBA$ (by symmetry)

$\angle EAC = \angle ABD$ (tangent)

$\Rightarrow$ $AE \parallel MC$, M midpoint BE $\xRightarrow{\Delta ABE}$ C midpoint BE

Let S be the midpoint of AO. Then in ΔAOC we have

$SD \parallel OE \Rightarrow \angle ASD = \angle AOC$

$S \in (ABD) \Leftrightarrow ASDB$ cyclic

$\Leftrightarrow \angle ASD = 180 - \angle ABD = 180 - \beta$

$\Leftrightarrow \angle AOC = 180 - \beta$ *

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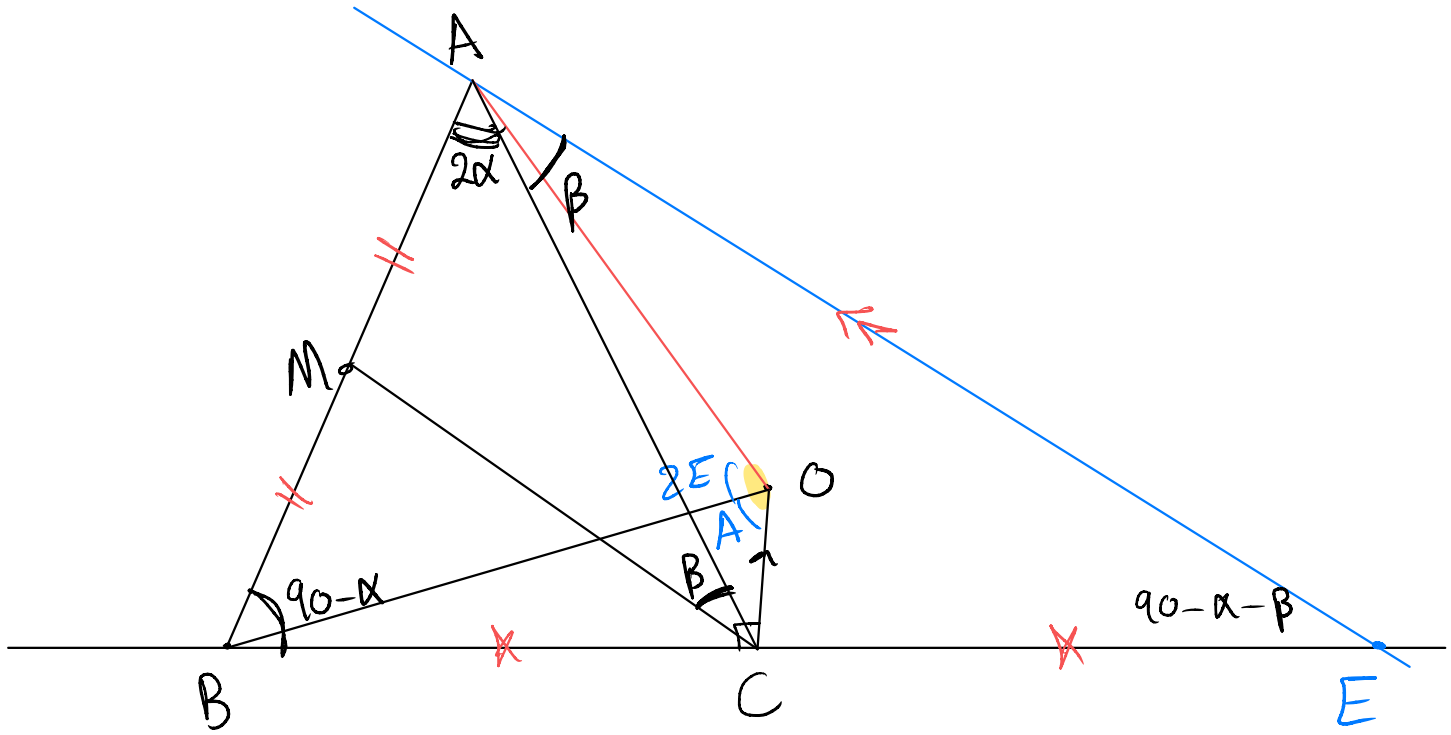


If $\angle BAC = 2\alpha$, then $\begin{cases} \angle ABC = 90-\alpha \\ \angle BAE = 2\alpha+\beta \end{cases} \Rightarrow \angle AEB = 90-\alpha-\beta$ ✱

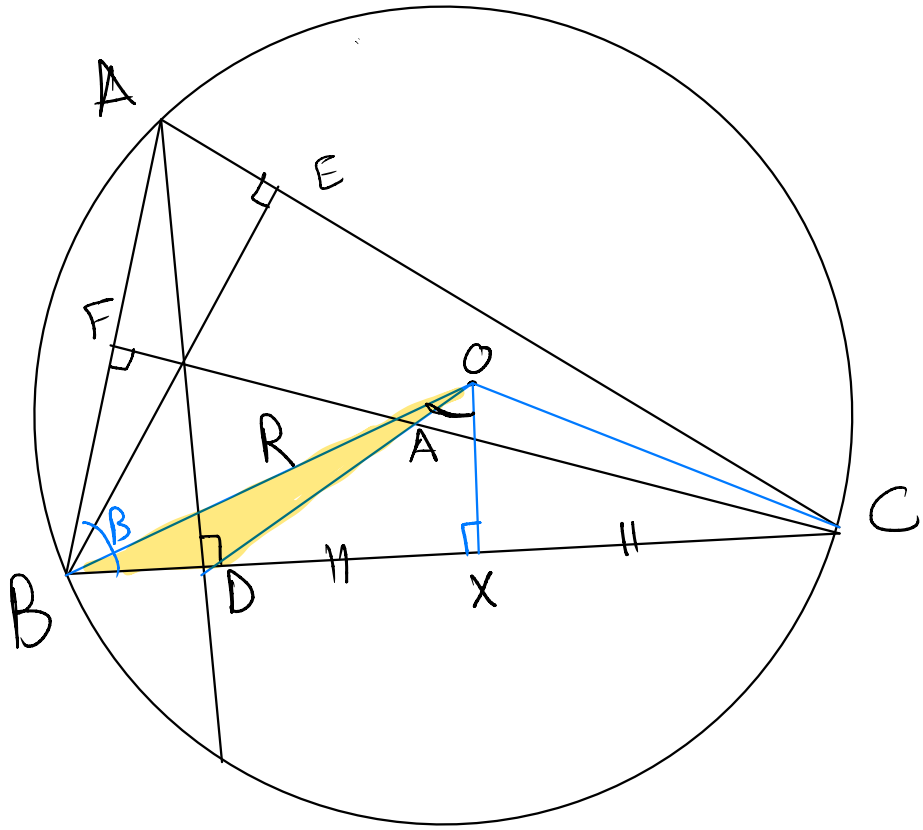
$\left.\begin{array}{l} \angle MOC = 180 - \angle B \\ \angle AOM = \angle E \end{array}\right\} \Rightarrow \angle AOC = 180 - \angle B + \angle E$

or $\angle AOC = 2\angle E + \angle A$

From ✱: $\begin{array}{l} \angle E = 90-\alpha-\beta \\ \angle A = 2\alpha+\beta \end{array} \Rightarrow \angle AOC = 2(90-\alpha-\beta)+2\alpha+\beta = 180-\beta$

$\Rightarrow \angle AOC = 180-\beta$, from ✱, $S \in (ABD)$ $\square$

19. Let ABC be an acute triangle with altitudes AD, BE, and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas. (APMO 2014 P1)



Let X be the midpoint of BC, $OX \perp BC$.

ΔOBD :

$$[\Delta OBD] = \frac{1}{2} BD \cdot OX$$

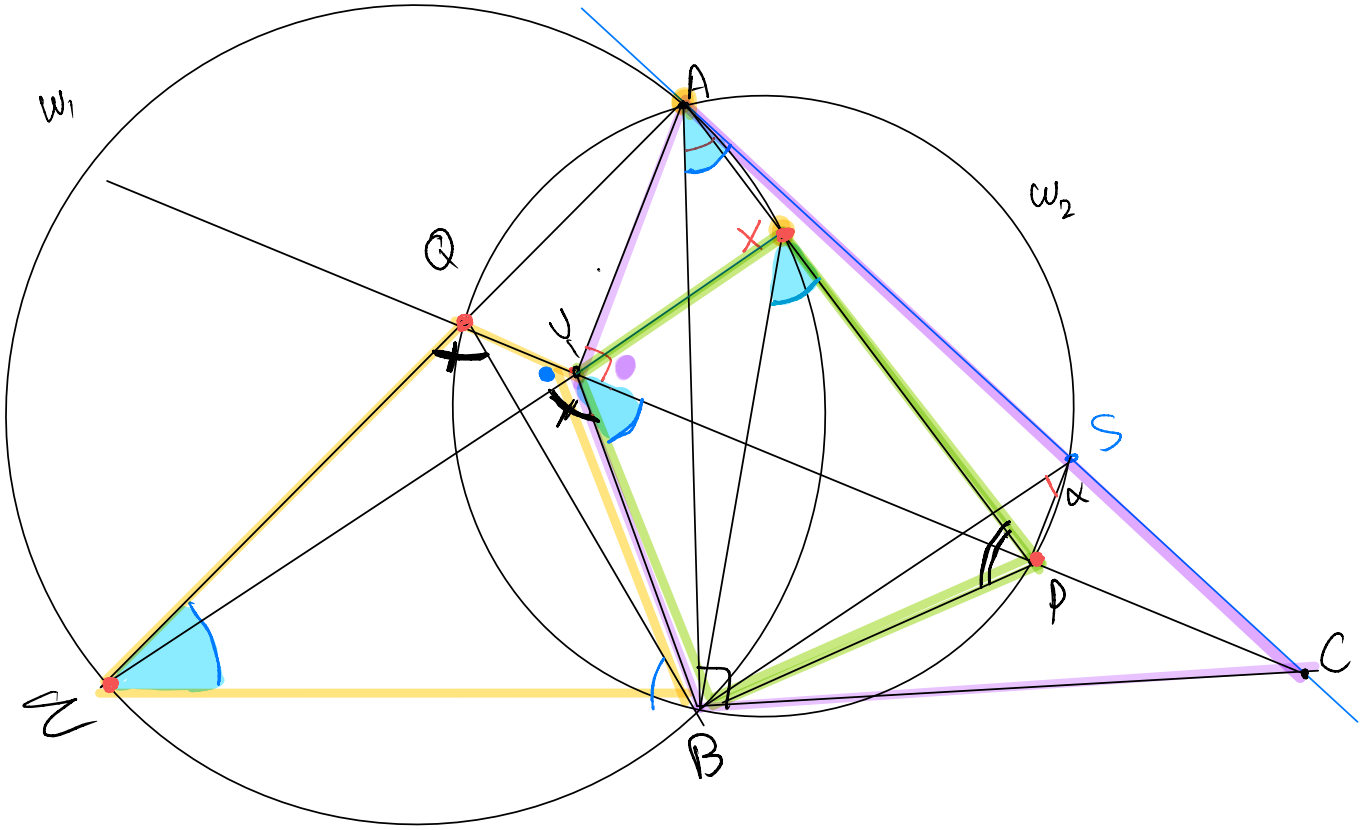
$$OX = R \cos A \longleftarrow \Delta OBX$$

$$BD = c \cos B \longleftarrow \Delta BAD$$

$$\Rightarrow \begin{cases} [\Delta OBD] = \frac{1}{2} OX \cdot BD = \frac{1}{2} R \cos A (c \cos B) = R \cos A \cos B \cos C \cdot \frac{c}{2 \cos C} \\ [\Delta ODC] = \frac{1}{2} OX \cdot DC = \frac{1}{2} R \cos A (b \cos C) = R \cos A \cos B \cdot \cos C \cdot \frac{b}{2 \cos B} \end{cases}$$

$[\Delta OBD] = [\Delta OAE]$ and similarly for the others. $\square$

Problem 4. Circles ω_1 and ω_2 intersect each other at points A and B. Point C lies on the tangent line from A to ω_1 such that $\angle ABC = 90°$. Arbitrary line ℓ passes through C and cuts ω_2 at points P and Q. Lines AP and AQ cut ω_1 for the second time at points X and Z respectively. Let Y be the foot of altitude from A to ℓ. Prove that points X, Y and Z are collinear.



$\angle AZB = \angle CAB = \alpha$

$\angle AYC = \angle ABC = 90 \Rightarrow AYBC$ cyclic

$\Rightarrow \angle BYC = \angle BAC = \alpha$

$\Rightarrow \angle BYC = \angle BZQ = \alpha$

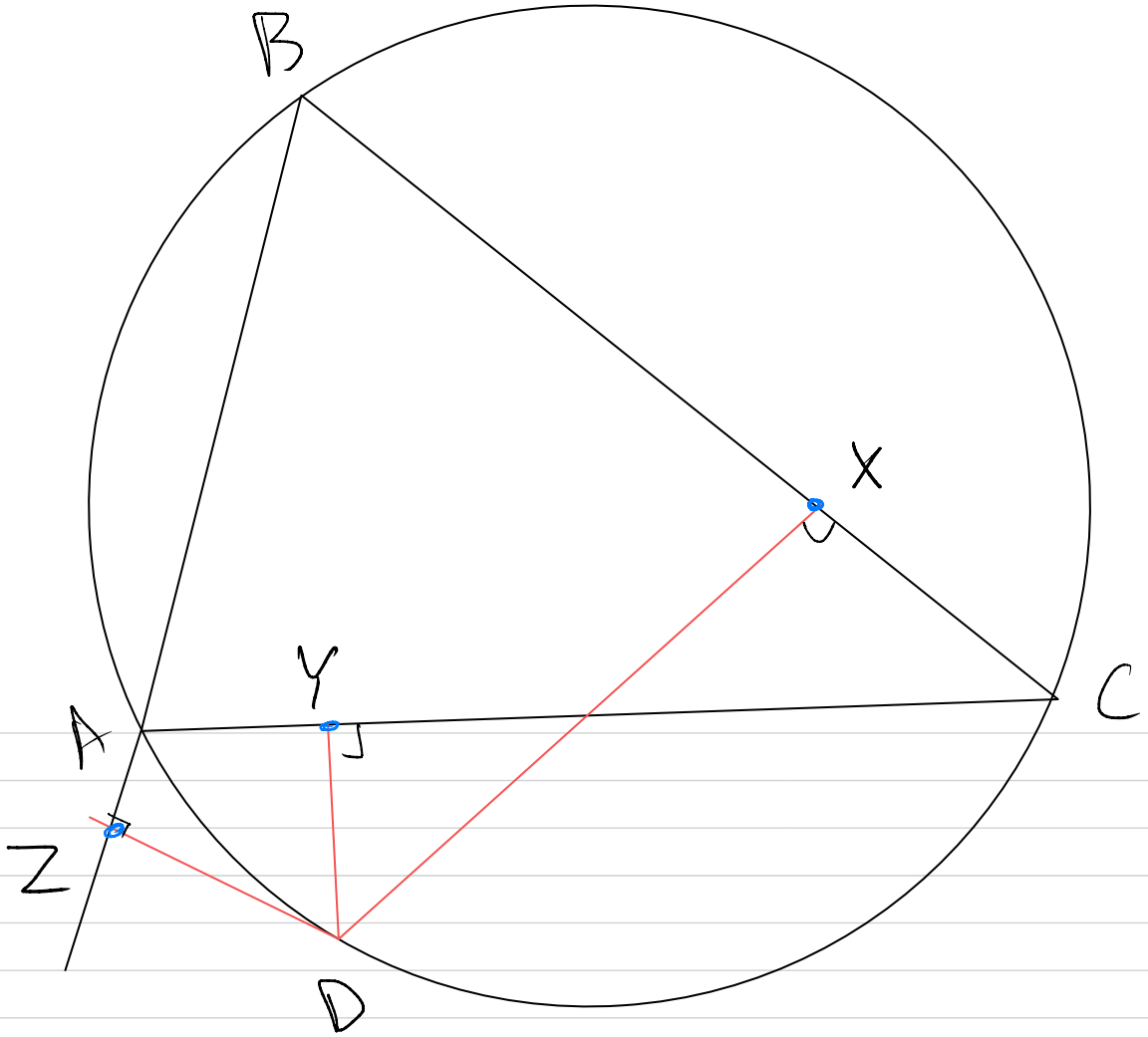
$\Rightarrow BYQZ$ cyclic

Note that $\angle BXP = \angle BZA = \alpha$ ($AZBX$ cyclic (ω_1))

$\Rightarrow \angle BXP = \angle PYB = \alpha \Rightarrow PBYX$ cyclic

On the other hand $AQBP$ cyclic $= (\omega_2)$

$\Rightarrow \angle ZQB = \angle APB \Rightarrow \angle ZYB = \angle XPB$ but $PBYX$ cyclic

$\Rightarrow X, Y, Z$ are collinear $\square$

Exercise: If $ABCD$ is cyclic and X, Y, Z are points on BC, AC, AB respectively such that $\begin{cases} DX \perp BC \\ DY \perp AC \\ DZ \perp BA \end{cases}$ then X, Y, Z are collinear.