Number Theory

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Problems – April 29

1. Given 2000 positive integers whose all prime divisors are less than 25, prove that one can always find four distinct numbers whose product is a fourth power.
2. At most how many numbers can we choose from the set $\{1, 2, \dots, 15\}$ so that the product of no three of them is a square?
3. At most how many numbers can we choose from the set $\{105, 106, \dots, 210\}$ so that every two are coprime?
4. Let Abe an 8-element subset of the set $\{1, 2, \dots, 17\}$. Prove that there exists $k > 0$ such that the equation $x - y = k$ has at least three solutions in A.
5. At most how many numbers from the set $\{1, 2, \dots, 2000\}$ can we choose so that no two differ by 4 or 7?
6. Can all positive integers be divided into 100 nonempty classes so that there are no three different numbers a, b, c from three different sets that satisfy $a + 99b = c$?
7. Is it possible to write the numbers $1, 2, \dots, 101$ around a circle in some order so that for every two adjacent numbers we have $25 \leqslant |x - y| \leqslant 49$?
8. Denote $N = 1000!$. Is it possible to order the numbers $1, 2, \dots, N$ around a circle so that every number equals the previous number plus 17 or 28 modulo N?
9. Prove that there is a permutation $a_1, a_2, \dots, a_{2021}$ of the numbers $1, 2, \dots, 2021$ such that there are no indices $i < j < k$ for which $a_i - a_j = a_j - a_k$?
10. Solve the equation $2^a - 5^b = 7$ in positive integers a, b.
11. Solve the equation $2^a - 5^b = 3$ in positive integers a, b.
12. Let a and b be different positive integers. Prove that there is a positive integer n such that $a^n - b^n$ is not a perfect power.
13. Let p be a prime number. Prove that there exists a prime number q such that $x^p \equiv p \pmod{q}$ has no solutions.