Competition Preparation for Saudi Arabia Team 2021: Level 4

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Lesson 3

Combinatoral ideas in geometry

Problems:

1. Let $n \geq 3$ be an integer and let f be a real valued function on a plane such that $f(A_1)+f(A_2)+\cdots+f(A_n)=0$ for every regular n-gon $A_1A_2\ldots A_n$. Prove that f is zero everywhere on the plane.

2. The faces of a polyhedron are colored in either black or white so that there are more black faces than white faces and no two black faces share an edge. Prove that it is impossible to inscribe a sphere inside the polyhedron.

3. Each point in space is colored with one of the three colors. Prove that there exists a color c such that for every positive real number r there is a triangle of area r such that all the vertices of that triangle are colored with c.

4. Each point in space is colored with one of two colors such that whenever an isosceles triangle ABC with $AB = AC$ has vertices of the same color c it follows that the midpoint of BC also is colored with c. Prove that there exists a perpendicular rectangular prism with all vertices of equal color.

5. Let there be given in a plane a collection of line segments whose sum of lengths is less than $\sqrt{2}$. Prove that there exists a unit infinite lattice which doesn't intersect any segment.

6. Let n blue points and n red points be given, no three of which are collinear. Prove that there exist n segments whose one end-point is red and the other is blue, such that no two segments share a common point.

7. Let $n > 3$ (straight and infinitely long) lines, no two of which are parallel and no three of which intersect in one point, divide the plane into sectors. Let the rank of a sector be the number of lines it touches. Let m be the maximum rank among all sectors. Let S_k be the number of sectors of rank k. Prove that $S_3 \geq 4 + S_5 + S_6 + S_7 + \cdots + S_m$. Can you find examples where equality holds and both sides are larger than 4?

8. Let n points be selected inside a unit square such that no three points are collinear and no two points are on a line parallel to the sides of the square. Let a partition of the square into rectangles be given so that each selected point is on the edge of a rectangle. Prove that there are at least $n + 1$ rectangles in the partition.

9. Finitely many unit circles are given in a plane such that the area of their union is S. Prove that it is possible to select a subset of these circles such that no two circles in the subset intersect with each other and the area of these circles is at least $2S/9$.