Around prime numbers

Lesson by Senya, group L3

All you need to know for today is what a prime number is (and some basic properties of it). While the first problems are easy, the problems closer to the end of the problem sheet are not that obvious.

By the way, when it comes to "basic properties" of prime numbers: there is one that says that prime $p|ab$ then $p|a$ or $p|b$. Sounds obvious, right? But by just using the definition of a prime number and what "being divisible by" means, can you prove it?

Problem 1. Just an exercise: Positive integer numbers a, b, c, d are such that $ab = cd$. Is it possible that $a + b + c + d$ is a prime?

Problem 2. Is it true that the quadratic polynomial $n^2 + n + 41$ takes only prime values for positive integers n?

Problem 3. Show that $n-$th prime number is smaller than 2^{2^n}.

Problem 4. Prove that for any positive integer n there is always a prime number between n and $n!$.

(Of course you are not allowed to use Bertrands Postulate, or something like it :))

Despite the definition of a prime number being simple, mathematicians still do not know quite a lot about them. Many of the results about them only sound easy, but the proofs are either non-existent or very tough. But soon we will prove a rather strong result about them and the proof will be completely elementary.

Problem 5. Let p be prime and $p^a|\binom{n}{k}$. Prove that $p^a \leq n$.

Problem 6. Let p be $k-th$ largest prime number and let $p = 2m + 1$

i) Prove that $\binom{p}{m} > \frac{2^p}{p}$;

ii) Prove that $\binom{p}{m} \leq p^k$;

Thus we can conclude that $p_k^{k+1} > 2^{p_k}$. Finally, if you know what a log is, prove that $p_k < 2 \cdot k \cdot log_2(k)$.

When it comes to olympiads though, the most common use of a prime number is via the idea of considering prime factorization and using the basic properties of prime numbers.

Problem 7. There are 101 integer numbers written in a circle. Is it possible that the ratio of any two neighbours is a prime number (when the larger number is divided by the smaller of course)?

Problem 8. Find all triples of prime numbers p, q, r such that $p^2 + pq + q^2 = r^2$.

Problem 9. Find all primes p and q such that $p^3 - q^5 = (p + q)^2$.

Problem 10. Find all odd integers $n > 1$ such that for any two of its divisors a, b that are co-prime, the numbers $a + b - 1$ is also a divisor of n.

Problem 11. Let A be some infinite set containing only prime numbers. It turned out that for any $p_1, ..., p_k$ from the set A, all of the prime divisors of the number $p_1p_2...p_k - 1$ are also in A. Prove that A must be precisely the set of all prime numbers.