Bertrand's postulate

Lesson by Eyad, group L4

$\mathbb{P}$

Problem 1. The Bertrands postulate states that for any n there is a prime number between n and $2n$. This is a cool result, let's prove it:

i) Show that no prime $\frac{2n}{3} < p < n$ divides $\binom{2n}{n}$;

ii) Consider $\binom{2n}{n}$. Then for any prime p we have $v_p(\binom{2n}{n}) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right)$, right? Thus show that

   (a) For any prime p we have $v_p(\binom{2n}{n}) \leq log_p(2n)$. Thus if p^a divides $\binom{2n}{n}$ then $p^a \leq 2n$;

   (b) Moreover, prove that if $p > \sqrt{2n}$ then p^2 does not divide $\binom{2n}{n}$, i.e for such p it is at most p that divides $\binom{2n}{n}$;

iii) Show that $\binom{2t+1}{t} < 4^t$, and then show that product of all the primes between 1 and N is at most 4^N;

iv) Show that $\binom{2n}{n} > \frac{4^n}{2n+1}$;

v) The important ideas are all above, now we are just left with estimating the $\binom{2n}{n}$ in two different ways and get a contradiction if we assume that there are no primes between n and $2n$. So, by assuming that there are no primes between n and $2n$ and by noting that

$$\frac{4^n}{2n+1} < \binom{2n}{n} < \left(\prod_{1<p\leq\sqrt{2n}} p \right) \cdot \left(\prod_{\sqrt{2n}<p\leq 2n/3} p \right) \cdot \left(\prod_{2n/3<p\leq n} p \right)$$

and using the things proved above, get to a contradiction (well, you will only get a contradiction for n big enough, but this is good enough as we can consider n small enough by hand)

Problem 2. Prove that for any n it is not possible to divide the numbers $1, 2, ..., n$ into two groups in such a way that the product of the numbers in the first group is equal to the product of the numbers in the second group.

Problem 3. Prove that in fact, there are at least $\sqrt{n}$ primes between n and $2n$ for any $n > 5$.