Lecture 4

Training Camp

22 November 2020

Level 2

Example 3.3.5 (IMO 1990). *Find all natural n such that $\frac{2^n+1}{n^2}$ is an integer.*

First $n=1$ is a solution. Assume $n \neq 1$, let p be the smallest prime divisor of n.

① The smallest prime divisor of n is 3 $(p=3)$

p is odd since 2^n+1 is odd

$p \mid 2^n+1 \Rightarrow p \mid 2^{2n}-1$ or $2^{2n} \equiv 1 \pmod{p}$

$\Rightarrow \operatorname{ord}_p(n) \mid \gcd(2n, p-1)$

since p is the smallest prime divisor of n then $\gcd(n, p-1)=1$

$\Rightarrow \operatorname{ord}_p(n) \mid 2$

$\Rightarrow 2^2 \equiv 1 \pmod{p}$

$\Rightarrow p \mid 4-1 \Rightarrow \boxed{p=3}$ ✓

Example 3.3.5 (IMO 1990). *Find all natural n such that $\frac{2^n+1}{n^2}$ is an integer.*

② $v_3(n)=1$

Since n is an odd number and $3 \mid 2+1$ we can use LTE

$$v_3(2^n+1) = v_3(2+1) + v_3(n)$$

$$v_3(2^n+1) = 1 + v_3(n)$$

$$v_3(n^2) = 2\, v_3(n)$$

but $n^2 \mid 2^n+1$ so $v_3(n^2) \leqslant v_3(2^n+1)$

$$\Rightarrow \quad 2\, v_3(n) \leqslant 1 + v_3(n)$$

$$\Rightarrow \quad v_3(n) \leqslant 1$$

$$\Rightarrow \quad v_3(n) = 1 \checkmark$$

if $n=3$ we are done; otherwise $n=3m$, $m>1$

Example 3.3.5 (IMO 1990). *Find all natural n such that $\frac{2^n+1}{n^2}$ is an integer.*

$n = 3m$, $\gcd(3, m) = 1$

Let q be the smallest prime divisor of m

$2^{3m} \equiv -1 \pmod{q} \Rightarrow 8^m \equiv -1 \pmod{q}$

$\Rightarrow \operatorname{ord}_q(8) \mid \gcd(2m, q-1)$

$\Rightarrow \operatorname{ord}_q(8) \mid 2$

$\Rightarrow 8^2 \equiv 1 \pmod{q}$

$\Rightarrow q \mid 63 \Rightarrow \boxed{q = 7}$ (since $\gcd(m, 3) = 1$)

but $8^m \equiv -1 \pmod{q}$

so $8^m \equiv -1 \pmod{7} \Rightarrow 7 \mid 1-(-1)$

$\Rightarrow\Leftarrow !!$

Solutions are $n = 1, 3$

Problem 7 (Russia 1996). Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

Since $\gcd(x,y) = 1$, $\gcd(x,3) = \gcd(y,3) = 1$

① k is odd

mod 3:

$$x^k \equiv -y^k \pmod 3 \Rightarrow (xy^{-1})^k \equiv -1 \pmod 3$$

$$\Rightarrow k \text{ is odd}$$

② $3 \mid x+y$

Since k is odd

$$x+y \mid x^k+y^k \Rightarrow \boxed{x+y \mid 3^n} \Rightarrow \boxed{3 \mid x+y}$$

$x+y \geq 2$

$$\boxed{x+y = 3^m}$$

Problem 7 (Russia 1996). Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

③ k can be written as $3^{n-m} a$, $a \in \mathbb{Z}^+$

Using LTE on $x^k + y^k$ since $3 \mid x+y$ and k is odd

$$v_3(x^k + y^k) = v_3(x+y) + v_3(k)$$

$$n = v_3(x+y) + v_3(k)$$

Since $x+y = 3^m$

$$\Rightarrow \quad v_3(k) = n - m$$

$$\Rightarrow \quad k = 3^{n-m} a, \quad a \in \mathbb{Z}, \gcd(3, a) = 1$$

Problem 7 (Russia 1996). Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

(4) $m+1 \geq n$

Assume that (WLOG) $x > y$, since $x+y = 3^m$ $\Rightarrow$ $x > \frac{1}{2} 3^m > 3^{m-1}$

$$3^n = x^{3^{n-m} a} + y^{3^{n-m} \cdot a} \Rightarrow 3^n > x^{3^{n-m} \cdot a}$$

but $x > 3^{m-1}$

$$3^n > x^{(3^{n-m}) a} > \left(3^{m-1}\right)^{3^{n-m} a} > \left(3^{m-1}\right)^{3^{n-m}}$$

$$\Rightarrow 3^n > 3^{(m-1)(n-m)}$$

$$\Rightarrow n > (m-1)(n-m)$$

$$\Rightarrow n > (m-1) n - m(m-1)$$

$$\Rightarrow m(m-1) > (m-2) n$$

Problem 7 (Russia 1996). Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

$$m(m-1) > (m-2)\,n$$

$$\Rightarrow \quad \frac{m(m-1)}{m-2} > n$$

but $\quad (m+2)(m-2) > m(m-1)$

$$\Rightarrow \quad m+2 > n \quad \Rightarrow \quad m+1 \geq n \quad *$$

Since $\quad x+y = 3^m, \qquad x^k + y^k = 3^n$

and $k > 1$

$$x^k + y^k > x + y$$

$$3^n > 3^m \quad \Rightarrow \quad n > m \Rightarrow n \geq m+1 \; **$$

Problem 7 (Russia 1996). Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

(5) k is less than 5, (or $k=3$)

Using * and ** we conclude that $n = m+1$

$$3(x+y) = x^k + y^k$$

if $k \geq 5$:

$$x^k + y^k > x^4 + y^4$$

$$(1+1)(x^4+y^4) \geq (x^2+y^2)^2$$

$$(1+1)(x^2+y^2) \geq (x+y)^2$$

$$\Rightarrow \quad 2(x^4+y^4) \geq \left(\frac{(x+y)^2}{2}\right)^2 = \frac{(x+y)^4}{4}$$

$$\Rightarrow \quad x^4+y^4 \geq \frac{(x+y)^4}{8}$$

$$x^k + y^k \geq \frac{(x+y)^4}{8}$$

Problem 7 (Russia 1996)**.** Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

but since $x^k + y^k = 3(x+y)$

$$3(x+y) > \frac{(x+y)^4}{8}$$

$$\Rightarrow \quad 24 > (x+y)^3 \quad \Rightarrow \quad 2 \geq x+y$$

but $x+y$ is odd

$$x+y = 1 \quad \Rightarrow \Leftarrow$$

Therefore $k = 3$ since k is odd

Problem 7 (Russia 1996). Find all positive integers n for which there exist positive integers x, y and k such that $\gcd(x, y) = 1, k > 1$ and $3^n = x^k + y^k$.

6 $n = 2$ is the only solution

$$\Rightarrow 3(x+y) = x^3 + y^3$$

$$\Rightarrow 3(x+y) = (x+y)(x^2 - xy + y)$$

$$\Rightarrow 3 = \underline{x^2 - xy + y^2} > (x-y)^2$$

$$\Rightarrow 3 > (x-y)^2 \Rightarrow x - y = 1 \quad (\text{since } \gcd(x,y) = 1,\ x \neq y)$$

$$\Rightarrow y = x - 1$$

$$3 = x^2 - xy + y^2 = (x-y)^2 + xy$$

$$3 = 1 + x(x-1) \Rightarrow x(x-1) = 2$$

$$\Rightarrow x = 2$$

$$\Rightarrow 3^n = 2^3 + 1^3 \Rightarrow \boxed{n = 2}$$

ex $k = 3$, $x, y = 2, 1$

Problem 8 (Russia 1996). Let x, y, p, n, k be positive integers such that n is odd and p is an odd prime. Prove that if $x^n + y^n = p^k$, then n is a power of p.

$$x^n + y^n = p^k$$

Since n is odd $x+y \mid x^n + y^n \Rightarrow x+y \mid p^k$

$$\Rightarrow x+y = p^m, \; m \leq k$$

since $x, y \geq 1$, $x+y > 2$ and so $m \geq 1$

Using LTE (n odd, $p \mid x+y$)

$$\underline{v_p(x^n + y^n)} = v_p(x+y) + v_p(n)$$

$$k = m + v_p(n)$$

Problem 8 (Russia 1996). Let x, y, p, n, k be positive integers such that n is odd and p is an odd prime. Prove that if $x^n + y^n = p^k$, then n is a power of p.

$k = m + v_p(n)$

let $v_p(n) = b$

lemma

$$\begin{cases} v_p(x^n + y^n) = v_p(x^{p^b} + y^{p^b}) \\ \text{if } v_p(n) = b \end{cases}$$

again using LTE

$$\underline{v_p(x^{p^b} + y^{p^b})} = v_p(x+y) + v_p(p^b)$$

$$= m + b$$

$$= m + v_p(n)$$

$$= v_p(x^n + y^n)$$

but $x^{p^b} + y^{p^b}$ and $x^n + y^n$ are all powers of p (since $p^b | n$)

$\Rightarrow$ $x^{p^b} + y^{p^b} = x^n + y^n \Rightarrow \boxed{p^b = n}$

More Number Theory Problems ☺

. Let $\mathcal{F}$ be a set of subsets of the set $\{1, 2, \ldots, n\}$ such that

(1) if A is an element of $\mathcal{F}$, then A contains exactly three elements;

(2) if A and B are two distinct elements in $\mathcal{F}$, A and B share at most one common element.

Let $f(n)$ denote the maximum number of elements in $\mathcal{F}$. Prove that

$$\frac{(n-1)(n-2)}{6} \leq f(n) \leq \frac{(n-1)n}{6}.$$

Determine all positive integers k such that

$$\frac{\tau(n^2)}{\tau(n)} = k,$$

for some n.

Let n be a positive integer greater than two. Prove that the Fermat number f_n has a prime divisor greater than $2^{n+2}(n+1)$.

Find all positive integers x, y, z and t such that $2^x 3^y + 5^z = 7^t$.