KSA intensive training

March-April Camp 2021 – LEVEL 4

GEOMETRY

Trainer: Lê Phúc Lữ (Vietnam)

Part 1. SELECTED PROBLEMS.

Problem set 1. Overview

Problem 1. Let ABC be an acute, non isosceles triangle with G is its centroid. Take D,E on AB, AC respectively such that G is the orthocenter of ADE. Denote M as the midpoint of segment DE. Prove that GM is perpendicular to BC.

Problem 2. Let ABC be an acute, non-isosceles triangle with AD, BE, CF are altitudes and d is the tangent line of the circumcircle of triangle ABC at A. The line throught H and parallel to EF cuts DE, DF at Q, P respectively. Prove that d is tangent to the ex-circle respect to vertex D of triangle DPQ.

Problem 3. Let ABC be an acute, non isosceles triangle with the orthocenter H, circumcenter O and AD is the diameter of (O). Suppose that the circle (AHD) meets the lines AB, AC at F, E respectively. Denote J, K as orthocenter and nine-point center of AEF. Prove that HJ parallel to BC and KO=KH.

Problem set 2. Cyclic quadrilateral

Problem 1. Let ABC be an acute, non-isosceles triangle inscribed in (O) and BB',CC' are altitudes. Denote E,F as the intersections of BB',CC' with (O) and D,P,Q are projections of A on BC,CE,BF. Prove that the perpendicular bisectors of PQ bisects two segments AO,BC.

Problem 2. Let ABC be a triangle with incircle (I), tangent to BC,CA,AB at D,E,F respectively. On the line $DF,$ take points M,P such that $CM \parallel AB,$ $AP \parallel BC.$ On the line $DE,$ take points N,Q such that $BN \parallel AC,$ $AQ \parallel BC.$ Denote X as intersection of PE,QF and K as the midpoint of $BC.$ Prove that if $AX = IK$ then $\angle BAC \le 60°.$

Problem 3. Let ABC be an acute, non-isosceles triangle inscribed in (O) and M,N are midpoints of segments AB,AC. Denote D as the midpoint of the arc BAC of (O). Take a point K inside triangle ABC such that $\angle KAB = 2\angle KBA$, $\angle KAC = 2\angle KCA$. Prove that $KA = KD$.

Problem 4. Let $ABCD$ be a rectangle with P lies on the segment AC. Denote Q as a point on minor arc PB of (PAB) such that $QB = QC$. Denote R as a point on minor arc PD of (PAD) such that $RC = RD$. The lines CB,CD meet (CQR) again at M,N respectively. Prove that $BM = DN$.

Problem 5. Let ABC be a non isosceles triangle with M,N,P are midpoints of the segments BC,CA,AB respectively. Suppose that I as the intersection of the angle bisectors of $\angle BPM$, $\angle MNP$ and J as the intersection of bisectors of $\angle CNM, \angle MPN$. The circle of center I that tangent to MP at D, the circle of center J that tangent to MN at E. Prove that $DE \parallel BC$ and the radical axis of $(I),(J)$ bisects DE.

Problem set 3. Shooting lemma

Problem 1. Let A be a point lies outside circle (O) and tangent lines AB,AC of (O). Consider points D,E,M on (O) such that $MD = ME$. The line DE cuts MB,MC at R,S. Take $X \in OB, Y \in OC$ such that $RX, RY \perp DE$. Prove that $XY \perp AM$.

Problem 2. Let ABC be an non-isosceles triangle with incenter I, circumcenter O and a point D on segment BC such that (BID) cut segments AB at $E \neq B$ and (CID) cuts segment AC at $F \neq C$. Circle (DEF) cuts segments AB,AC again at M,N. Let $P = IB \cap DE$ and $Q = IC \cap DF$. Prove that EN, FM, PQ are parallel and the median of vertex I in triangle IPQ bisects the arc BAC of (O).

Problem 3. Let ABC be a triangle inscribed in the circle (O) with incenter I and ex-circle J respect to A. The line passes through O, perpendicular to AI cuts (O) at M (on the same side with O respect to AI). Suppose that MI, MJ cut (O) at P,Q. Take $R,S \in BC$ such that PR,QS are tangent to (O). Prove that PQ, AI, BC are concurrent and (ARS) are tangent to (O).

Problem 4. Let ABC be a triangle inscribed in circle (O) with diamter KL passes through the midpoint M of AB such that L,C lie on the different sides respect to AB. A circle passes through M,K cuts LC at P,Q (point P lies between Q,C). The line KQ cuts (LMQ) at R. Prove that $ARBP$ is cyclic and AB is the symmedian of triangle APR.

Problem 5. Let AB be a chord of the circle (O). Denote M as the midpoint of the minor arc AB. A circle (O') tangent to segment AB and internally tangent to (O). A line passes through M, perpendicular to $O'A, O'B$ and cuts AB respectively at C,D. Prove that $AB = 2CD$.

Problem 6. Consider circles $(O_1),(O_2),(O_3)$ are tangent to d at A,B,C and (O_2) is the biggest circle, externally tangent to $(O_1),(O_3)$. Let BD be the diameter of (O_2). The external tangent line (differs from d) of $(O_1),(O_3)$ cuts (O_2) at X,Y. Let K be the midpoint of the arc XBY of (O_2).

a) Prove that DK bisects segment AC.

b*) Prove that the circle of diameter AC touches DX, DY.

Problem 7*. Let ABC be a triangle with circumcenter O and incenter I, ex-center in angle A is J. Denote D as the tangent point of là (I) on BC and the angle bisector of angle A cuts $BC,(O)$ respectively at E,F. The circle (DEF) meeds (O) again at T. Prove that AT passes through an intersection of (J) and (DEF).

Problem set 4. Symmedian & Isogonal conjugate

Problem 1. Let ABC with median AD. Denote DM, DN, AE as the symmedian of triangles ABD, ACD, ABC. Prove that $MNED$ is the isosceles trapezoid.

Problem 2. Let $ABCD$ be a cyclic quadrilateral with $AB \cap CD = E$, $AD \cap BC = F$. Denote M,N are midpoints of segments AC, BD. Prove that the angle bisectors of angle E,F meet on the line MN.

Problem 3. Let ABC be a non-isosceles triangle with $AB^2 + AC^2 = 2BC^2$. Denote AD, AM as the internal angle bisector and the median of triangle. Take E such that D is the incenter of triangle AME. Prove that $MA = 3ME$.

Problem 4. Let ABC be a triangle inscribed in circle (O) and the symmedian AK with $K \in (O)$. Suppose that KB cuts (ADB) at X and KC cuts (ADC) at Y. Prove that $OK \perp XY$ and D is the midpoint of XY.

Problem 5. Let $ABCD$ be a parallelogram with $\angle CAD = 90°$ and H is the projection of A on CD. Suppose that the tangent line of circle (ABD) at D cuts AC at K. Prove that $\angle KBA = \angle HBD$.

Problem 6. Let ABC be a triangle with altitudes AD, BE intersect at H. Denote M, N as the midpoints of segments AH, BC. BM cuts AN at T. Prove that $\angle TED = 90°$.

Problem 7. Let $ABDC$ be a rhombus and P lies inside triangle ABC such that $\angle BPC = 180° - \angle A$. Suppose that $BP \cap AC = E$, $CP \cap AB = F$ and X, Y are excenters respect to angles B, C in triangles ABE, ACF. Circles (AEF) cuts XY at T. Prove that TP passes through D and the center of (AEF) is equidistance to X, Y.

Problem 8. Let ABC be an acute, non-isosceles triangle inscribed in circle (O) and I is the midpoint of BC, D is the midpoint of minor arc BC. Take K on line AI such that OK || AD. The line through K, perpendicular to AO cuts AB, AC at E, F respectively. Take G on KD such that AG perpendicular to BC. Prove that A lies on the line joining midpoints of segments EF, DG.

Problem 9. Let ABC be an acute, non-isosceles triangle inscribed in circle (O) and M is the midpoint of BC. Tangent lines of (O) from B, C intersect at T. Denote H, K as projections of T on AB, AC. Take A' on (O) such that AA' || BC and L is the projection of A' on BC. Prove that H, K, L, M are concyclic.

Problem 10. Let ABC be a triangle inscribed in a fix circle (O) with BC is fix and A vary on (O). Denote H as the orthocenter of triangle ABC and take D, E on AB, AC respectively such that H is the midpoint of DE. Prove that when A moves on (O), the center of (ADE) belongs a fixed circle.

Problem set 5. Advanced angle chasing

Problem 1. Let $ABCD$ be a quadrilateral such that

$$\begin{cases} \angle ADC = 135° \\ \angle ADB - \angle ABD = 2.\angle DAB = 4.\angle CBD. \\ BC = \sqrt{2}.CD \end{cases}$$

Prove that $AB = BC + AD$.

Problem 2. Let $ABCD$ be a quadrilteral with $\angle A = \angle B = 90°$, $AB = AD$. Denote E as the midpoint of AD, suppose that $CD = BC + AD, AD > BC$. Prove that $\angle ADC = 2.\angle ABE$.

Problem 3. Let ABC be a triangle. Take points E, F on CA, AB respectively such that $BE \perp CF$. Denote O as the intersection of BE, CF and X lies inside triangle ABC such that $\angle ACO = \angle XCB, \angle ABO = \angle XBC$. Prove that $\angle EXF + \angle BAC = 90°$.

Problem 4. Let ABC be a triangle with $\angle C = \angle A + 90°$. Take D on the opposite ray of the ray BC such that $AC = AD$. Take E such that E, A lie on different side respect to BC and $\angle EBC = \angle A; \angle EDC = \frac{1}{2}\angle A$. Prove that $\angle CED = \angle ABC$.

Problem 5. Let ABC be a triangle with $\angle B = 2.\angle C$. Take P, Q on the perpendicular bisector of BC such that

$$\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}.$$

Prove that Q is the circumcenter of triangle CPB.

Problem 6. Let ABC be convex quadrilateral and X lying inside it such that

$$XA \cdot XC^2 = XB \cdot XD^2 \text{ and } \angle AXD + \angle BXC = \angle CXD.$$

Prove that $\angle XAD + \angle XCD = \angle XBC + \angle XDC.$

Problem set 6. Harmonic technique

Problem 1. Let ABC be an acute, non-isosceles triangle with BD, CE are altitudes. Take points $M, N \in BC$ such that $(ADM), (ADN)$ are tangent to BC. Prove that M, N, A, E are concyclic.

Problem 2. Let ABC be an acute, non-isosceles triangle with circumcenter O. Tangent lines to (O) at B, C meet at T. A line passes through T cuts segments AB at D and cuts ray CA at E. Take M as midpoint of DE and suppose that MA cuts (O) again at K. Prove that (MKT) is tangent to (O).

Problem 3. Let ABC be an acute, non-isosceles triangle with circumcenter O, incenter I and (I) tangent to BC, CA, AB at D, E, F respectively. Suppose that EF cuts (O) at P, Q. Prove that (PQD) bisects segment BC.

Problem 4. Let ABC be a triangle with $AB < AC$ and incircle (I) tangent to BC at D. Take K on AD such that $CD = CK$. Suppose that AD cuts (I) at G and BG cuts CK at L. Prove that K is the midpoint of CL.

Problem 5. Let ABC be a triangle with $AB < AC$ inscribed in (O). Tangent line at A of (O) cuts BC at D. Take H as the projection of A on OD and E, F as projections of H on AB, AC. Suppose that EF cuts (O) at R, S. Prove that (HRS) is tangent to OD.

Problem 6. Let ABC be an acute, non-isosceles triangle with altitude AD $(D \in BC)$, M is the midpoint of AD and O is the circumcenter. Line AO meets BC at K and circle of center K, radius KA cuts AB, AC at E, F respectively. Prove that AO bisects EF.

Problem 7. Let $ABCD$ be a cyclic quadrilateral with O is circumcenter and AC meets BD at I. Suppose that rays DA, CD meet at E and rays BA, CD meet at F. The Gauss line of $ABCD$ meets AB, BC, CD, DA at points M, N, P, Q respectively. Prove that the circle of diameter OI is tangent to two circles $(ENQ), (FMP)$.

Problem set 7. Point circle

Problem 1. Let ABC be a triangle with BB' is the diameter of (O), AB' cuts BD at T. MT cuts DE at K. Prove that: AK || BC.

Problem 2. Let ABC be a triangle with circumcenter O, incenter I. AI cuts (O) againt at D. Take E on BC such that ∠AIE = 90, H is projection of E on IO. Prove that A, H, E, D are concyclic.

Problem 3. Circle (O1), (O2) meet at P. AB is the common tangent line of these circles with A on (O1), B on (O2). Take C on O1O2 such that AC ⊥ BP. Prove that AP ⊥ PC.

Problem 4. Let ABC be a triangle with ∠B < ∠A. Take D on BC such that ∠CAD = ∠B. A random circle (O) passes through B, D cuts AB, AD at E, F respectively. BF cuts DE at G, M is the midpoint of AG. Prove that CM ⊥ AO.

Problem 5. Let ABC be an acute triangle with O lies inside it. AO, BO, CO cut BC, CA, AB at D, E, F respectively. M, N, P are the reflections of O w.r.t EF, DF, DE. Prove that AM, BN, CP are concurrent.

Problem 6. Let ABC be an acute triangle with X, Y, Z are tangent points of incircle with BC, CA, AB. Denote G as intersection of BY, CZ. Take R, S such that BCYR and BCSZ are parallelograms and denote (J) as the excircle of ABC with respect to angle A. Prove that BY is the radical axis of (J), point circle R and then, prove that GR = GS.

Problem 7. Let ABCD be a parallelogram and AC cuts BD at T. A circle of center O, radius OD is tangent to BD at D, and cuts segment CD at E, cuts ray AD at F. Suppose that B, E, F are collinear. Prove that <ATD = <DOB.

Problem set 8. Completed quadrilateral

Problem 1. Let ABC be an acute, non isosceles triangle with orthocenter H, altitude AD and M is the midpoint of BC. On the ray HM, take K such that AK is tangent to (HKM). Prove that the circle of center A, radius AK is tangent to (KBC).

Problem 2. Let ABC be a triangle inscribed in circle (O). On the segments AB, AC, take D, E such that AD=AE. Suppose that BE cuts CD at I, AI cuts BC at K and EF cuts BC at T. Circles (TBD), (TCE) meet again at S. On the ray TS, take G such that GK perpendicular to BC. Prove that SO bisects GK.

Problem 3. Let ABC be a triangle inscribed in circle (O), M is the midpoint of BC and point D on (O) such that AD perpendicular to BC. (AOD) cuts AB, AC again at E, F respectively. Prove that the midpoint of EF is equidistance to O, M.

Problem 4. Let ABCD be a quadrilateral inscribed in circle (O) with AC perpendicular to BD at K. Denote E, F, G, H as projections of K onto AB, BC, CD, DA and EG cuts FH at L. Prove that O, K, L are collinear.

Problem 5. Let ABC be a triangle inscribed in circle (O) with M is the midpoint of BC. Take D on BC, differs from M, such that (ABD) cuts segment AC at E, circle (ACD) cuts segment AB at F.

a) Prove that if AD is the symmedian then EF || BC and DE = DF.

b) Suppose that AD is not the symmedian, (AEF) cuts (O) again at P and AP cuts BC at S. BE cuts CF at K and AK cuts EF at R. Segment RM cuts (AEF) at T. Prove that (SD,BC)=-1 and (AEF) is tangent to (BTC).

Problem set 9. Overall review

Problem 1. Let ABC be a non-isosceles triangle with altitudes AD, BE, CF with orthocenter H. Suppose that $DF \cap HB = M, DE \cap HC = N$ and T is the circumcenter of triangle HBC. Prove that $AT \perp MN$.

Problem 2. Let ABC be triangle with the symmedian point L and circumradius R. Construct parallelograms $ADLE, BHLK, CILJ$ such that $D, H \in AB; K, I \in BC; J, E \in CA$. Suppose that DE, HK, IJ pairwise intersect at X, Y, Z. Prove that inradius of XYZ is $\frac{R}{2}$.

Problem 3. Let ABC be triangle with M is the midpoint of BC and X, Y are excenters with respect to angle B, C. Prove that MX, MY intersect AB, AC at four points are vertices of circumscribe quadrilateral.

Problem 4. Let ABC be a triangle with $AB = AC$ and M is the midpoint of the altitude AD. Consider (ω) as the circle of center M and tangent to AB, AC. From some point T on the line BC (outside triangle ABC), construct two tangents of (ω) and they cut AB at P, Q, cut AC at R, S. Prove that $PQ = RS$.

Problem 5. Let ABC be a triangle with D, E are tangent points of incircle and excircle respect to vertex A with the segment BC. Suppose that $BC = 2|AB - AC|$, prove that $\angle BAC = 2\angle DAE$.

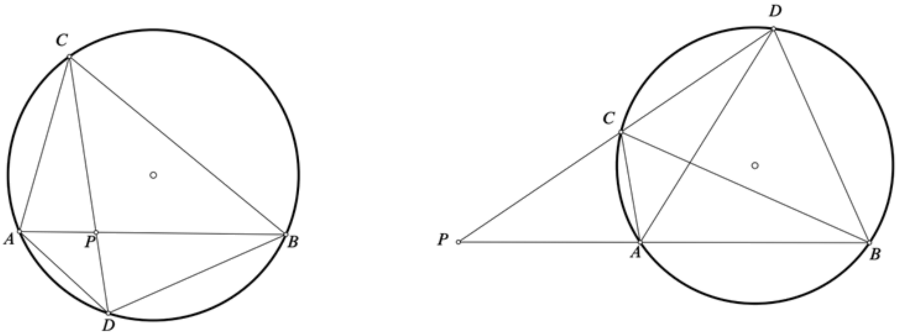
Problem 6. Let $ABCD$ be a convex quadrilateral and non-cyclic. Denote X_a as the product of $\mathcal{P}_{A/(BCD)}$ with S_{BCD}. Defind X_b, X_c, X_d similarly. Prove that

$$|X_a| = |X_b| = |X_c| = |X_d|.$$

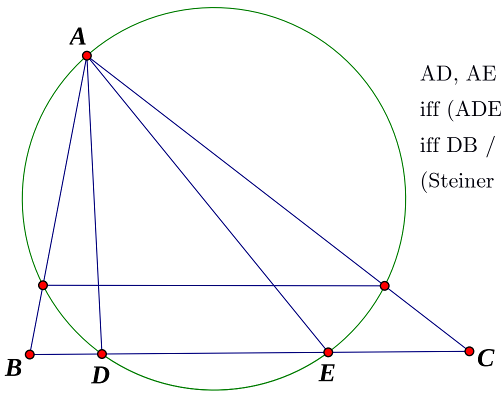
Part 2. SOLUTION DISCUSSED WITH CLASS

Problem set 1. Overview.

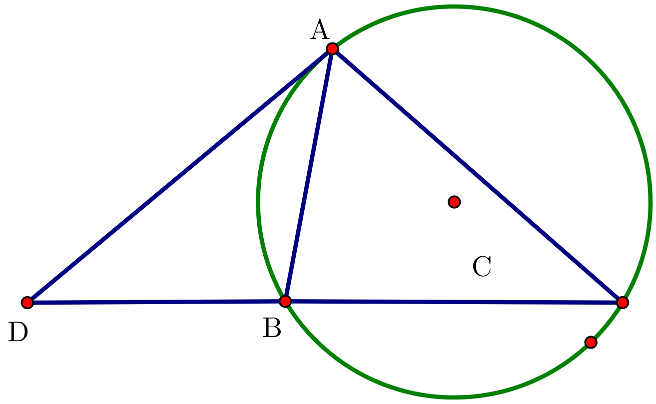
ratio lemma $\frac{PA}{PB} = \frac{CA}{CB} \cdot \frac{DA}{DB}$.



PA/PB = [CAP]/[CBP] = [ADP] / [BDP] = [ACD] / [BCD]
= (CA.AD.sin (CAD)) / (BC.BD.sin (CBD))
= CA/CB.DA/DB.



AD, AE are isogonal conjugate
iff (ADE) are tangent to (ABC)
iff DB / DC . EB / EC = AB^2 / AC^2
(Steiner theorem)



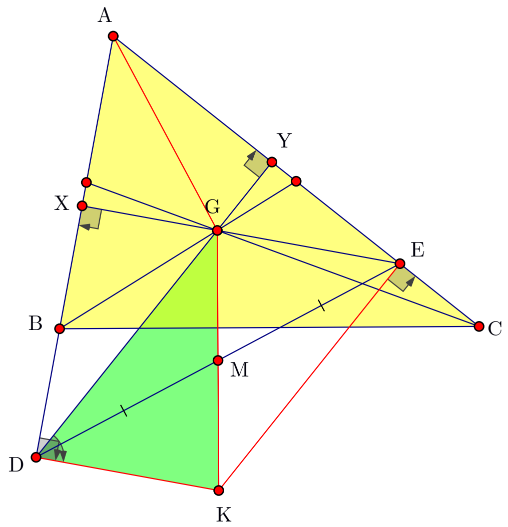
AD is tangent to (ABC)
∠DAB = ∠C.
two triangles are similar: DAB ~ DCA.

$$\frac{DA}{DC} = \frac{AB}{AC} = \frac{DB}{DA}$$

--> $\frac{DB}{DC} = \frac{DB}{DA} \cdot \frac{DA}{DC} = \frac{AB^2}{AC^2}$

(important for calculation).

Problem 1. Let ABC be an acute, non isosceles triangle with G is its centroid. Take D,E on AB, AC respectively such that G is the orthocenter of ADE. Denote M as the midpoint of segment DE. Prove that GM is perpendicular to BC.



Solution 1. four point theorem:

$GB^2 - GC^2 = MB^2 - MC^2$.

Solution 2. construct parallelogram DGEK.

$\angle GDK = 180 - \angle DGE = \angle BAC$.

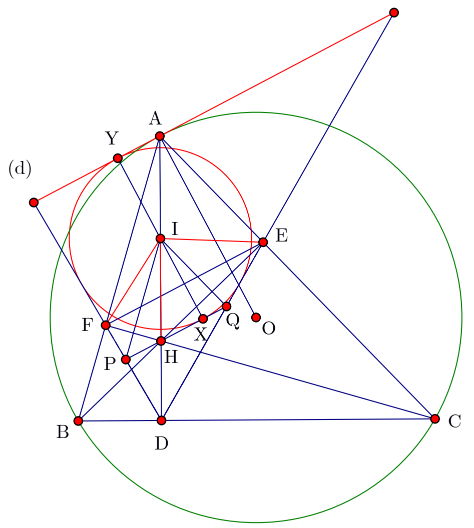
$DG \perp AC$, $DK \perp AB$. So to prove: $GK \perp BC$, we need to prove what?

$AG \perp DM$.

Triangles DGK, ACB are similar?

$GX.AB = GY.AC = 2/3.$ [ABC].

$AC / AB = GX / GY = GD / GE = GD / DK$.



Problem 2. Let ABC be an acute, non-isosceles triangle with AD, BE, CF are altitudes and d is the tangent line of the circumcircle of triangle ABC at A. The line throught H and parallel to EF cuts DE, DF at Q, P respectively. Prove that d is tangent to the ex-circle respect to vertex D of triangle DPQ.

DA is the angle bisector of $\angle PDQ$ so ex-center I lying on AD.

1st way: We need to prove: I is the midpoint of AH?

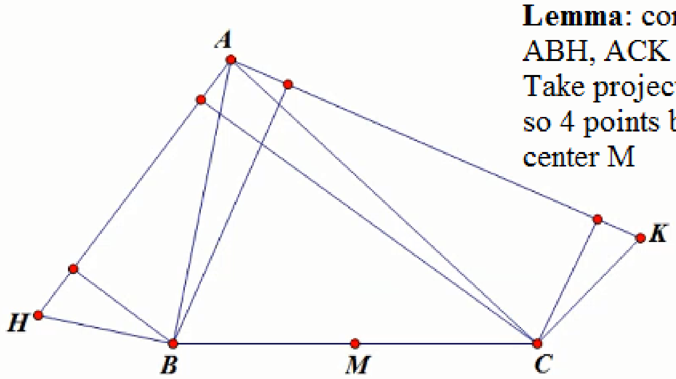
<=> I, F, D, E are cyclic <=> IF = IE.

AH, AO isogonal, but $AO \perp EF$ --> AH passes through the circumcenter (AEF).

2nd way: I is midpoint of AH --> I is the ex-center of DPQ.

IF = IH. Angle chasing (H is the incenter) --> PH = PF. So IP is the perpendicular bisector of HF, then IP is external angle bisector of $\angle DPQ$ --> I is the ex-center.

Since (d) // PQ and IA = IH then distance from I to (d) and PQ are equal.



Lemma: construct two right triangles ABH, ACK with AH, AK are isogonal. Take projections from B, C to AH, AK so 4 points belongs to a same circle of center M

Problem 3. Let ABC be an acute, non isosceles triangle with orthocenter H, circumcenter O and AD is the diameter of (O). Suppose that the circle (AHD) meets the lines AB, AC at F, E respectively. Denote J, K as orthocenter and nine-point center of AEF. Prove that HJ parallel to BC and KO=KH.

Part 1. 1st way.

EHDF is isosceles trapezoid and BC bisects HD.

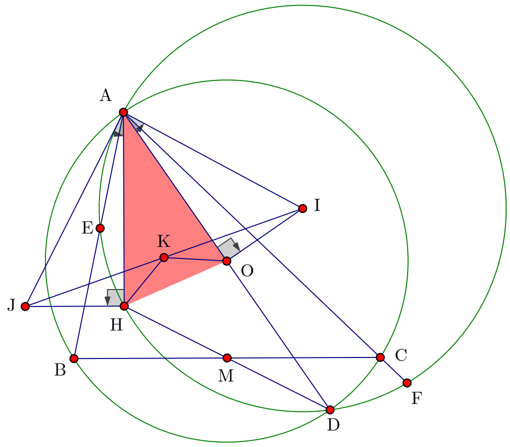
JEF and BDH are homothety (M, T are midpoint of BC, EF) --> JT // BM. Also BHC ~ HEF and HM // EF so HT // BC. Then JH // BC.

2nd way. $\angle HAJ = \angle OAI$. HA = 2R.cosA, AJ = 2AI.cosA

HA / AJ = AO / AI so AHJ ~ AOI, hence

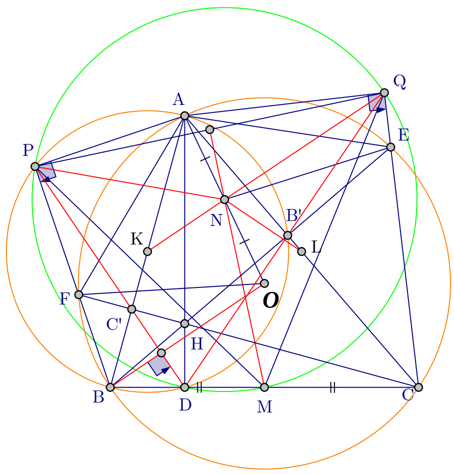
$\angle AHJ = \angle AOI = 90$ --> JH $\perp$ AH --> JH // BC.

3rd way: BC is the Simson line of D in (AEF) --> Steiner line (d) of D in (AEF) is image of BC in homothety ratio 2 and J lies on (d), and H lies on (d) and (d) || BC so JH || BC.



Lemma. In triangle AHO we have AJ, AI isogonal and $\angle AHJ = \angle AOI = 90$, K is midpoint IJ --> KO=KH.

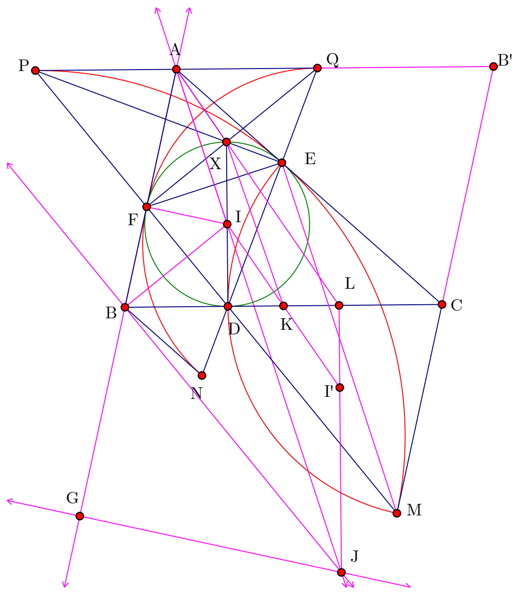
Problem set 2. Cyclic quadrilateral.



Problem 1. Let ABC be an acute, non-isosceles triangle inscribed in (O) and BB', CC' are altitudes. Denote E, F as the intersections of BB', CC' with (O) and D, P, Q are projections of A on BC, CE, BF. Prove that the perpendicular bisectors of PQ bisects two segments AO, BC.

APB and AQC are similar --> AP, AQ are isogonal, and applying the lemma (previous problem) to get MP = MQ.
OE = OF and AE = AF (angle chasing) so NE = NF.

APBB' is a kite. K, L is midpoint of AB, AC.
Angle chasing --> D, C', P and D, B', Q are collinear.
NK = NL = R/2.
OB ⊥ DC' and OB || NK --> NK ⊥ DP, but KP=KD so NP = ND. Similarly, NQ = ND --> NP = NQ.



Problem 2. Let ABC be a triangle with incircle (I), tangent to BC, CA, AB at D, E, F respectively. On the line DF, take points M, P such that $CM \| AB$, $AP \| BC$. On the line DE, take points N, Q such that $BN \| AC$, $AQ \| BC$. Denote X as intersection of PE, QF and K as the midpoint of BC. Prove that if $AX = IK$ then $\angle BAC \le 60°$.

Prove that: AX is the radical axis of (PEM), (QFN).

EF is the anti parallel of PQ in triangle DPQ. AP = AQ = AE = AF. So X is the orthocenter of DPQ.

Part 1. *AX is the radical axis of (PEM), (QFN).*

Power of X from two circles = XF.XQ = XE.XP.

Way 1. Angle chasing (C is the center of DEM ...) --> AC is tangent to (PEM) --> power of A to (PEM) = AE^2. Similarly to power of A to (QFN).

Way 2. extend lines AQ, MC meet at B' --> (PEM) is the excircle of triangle ACB' and E is the tangent point of this circle to AC.

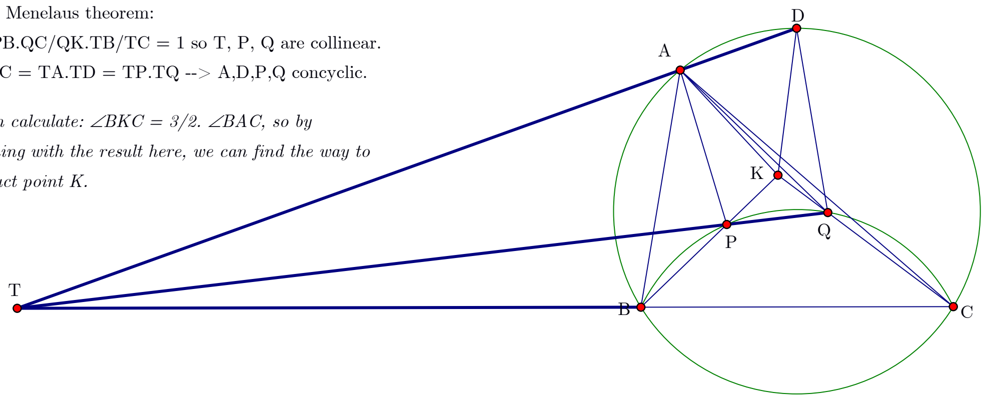
Part 2. *AX = IK ...*

AX passes through L on BC such that KD = KL (can be proved by homothety)
If AX = IK --> AXKI is the parallelogram --> AX = IK = XL / 2.
Thales: 1/3 = AX / AL = AI / AJ = AF / AG. So:
s = 3(s-a) --> 3a = 2s --> 2a = b+c and cos A = (b^2+c^2-a^2)/(2bc) >= 1/2.

Problem 3. Let ABC be an acute, non-isosceles triangle inscribed in (O) and M, N are midpoints of segments AB, AC. Denote D as the midpoint of the arc BAC of (O). Take a point K inside triangle ABC such that $\angle KAB = 2\angle KBA$, $\angle KAC = 2\angle KCA$. Prove that $KA = KD$. (no need M, N)

AP, AQ are angle bisectors. PA = PB, QA = QC.
KA^2 = KP.KB = KQ.KC --> BPQC cyclic.
AD: external angle bisector of BAC
AD meets BC at T --> TB / TC = AB/AC.
PK / PB = AK / AB and QK / QC = AK / AC
So by Menelaus theorem:
PK/PB.QC/QK.TB/TC = 1 so T, P, Q are collinear.
TB.TC = TA.TD = TP.TQ --> A,D,P,Q concyclic.

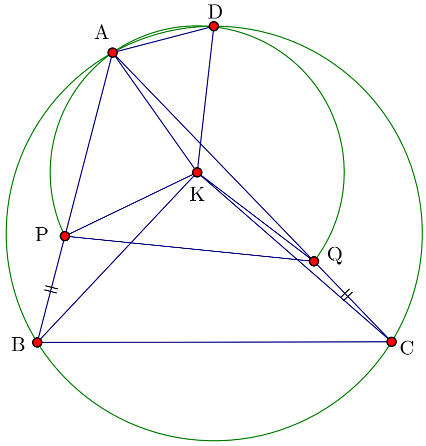
We can calculate: $\angle BKC = 3/2.\ \angle BAC$, so by combining with the result here, we can find the way to construct point K.

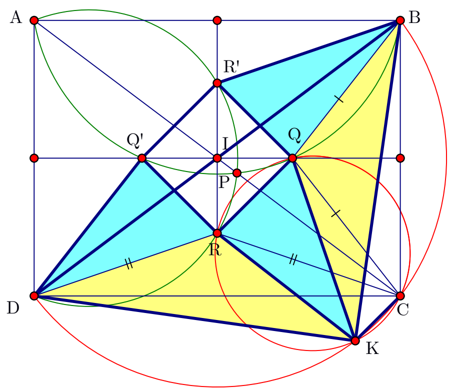


Problem 3. Let ABC be an acute, non-isosceles triangle inscribed in (O) and M,N are midpoints of segments AB,AC. Denote D as the midpoint of the arc BAC of (O). Take a point K inside triangle ABC such that $\angle KAB = 2\angle KBA$, $\angle KAC = 2\angle KCA$. Prove that $KA = KD$. (no need M, N)

2nd idea: take P, Q such that:
KA = KP = KQ.
∠KPA = ∠KAB = 2∠KBP --> PK = BP.
Similarly, KQ = QC so
BP = QC. Applying the lemma, (APQ)
passes through D.
K is the center (APQ) --> KA = KD.

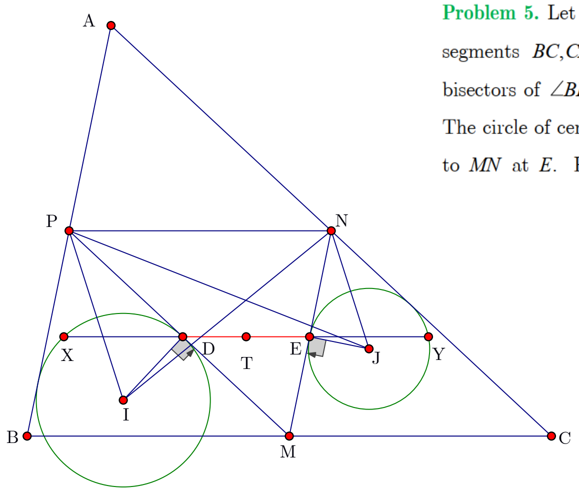
<u>The construction of K</u>: from angle
chasing to get ∠BKC = 3∠BAC/2
and from this problem, KA = KD.
∠KBC = ∠B - ∠KBA
∠KCB = ∠C - ∠KCA
∠KBA + ∠KCA = ∠A/2.





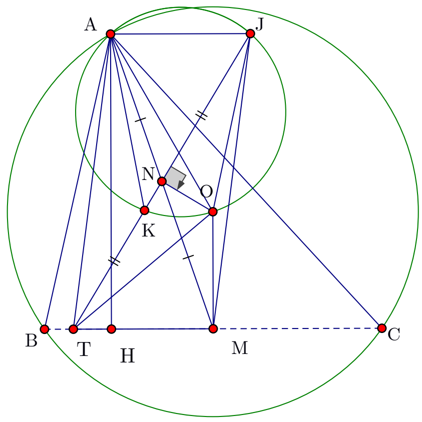
Problem 4. Let $ABCD$ be a rectangle with P lies on the segment AC. Denote Q as a point on minor arc PB of (PAB) such that $QB = QC$. Denote R as a point on minor arc PD of (PAD) such that $RC = RD$. The lines CB,CD meet (CQR) again at M,N respectively. Prove that $BM = DN$.

By lemma, BM = DN iff (CMN) cut (CBD) at K as the midpoint of the arc BD.
So we need to prove: C, K, R, Q concyclic (we can forget M, N from now on).
I is center symmertric --> IQ = IQ', IR = IR'
so QRQ'R' is rhombus.
IA.IP = IQ.IQ' = IR.IR' --> IQ^2 = IR^2 then IQ = IR, implies that RQR'Q' is square. Take CK' || RQ (K' lies on (CRQ) then:
K'Q = CR = DR, K'R = CQ = DQ' and ∠K' = ∠C = ∠D -->
△K'RQ = △DQ'R. Similarly: △K'RQ = △BQR'.
Need to prove: DK'I = IK'B.
△RDK' = △QK'B (side.angle.side) --> K'D = K'B.
CD || QQ' and RQ || KC --> ∠KCD = ∠RQQ' = 45 --> K lies on (BCD).
Thus K is the midpoint of the arc BD of (BCD).



Problem 5. Let ABC be a non isosceles triangle with M,N,P are midpoints of the segments BC,CA,AB respectively. Suppose that I as the intersection of the angle bisectors of $\angle BPM$, $\angle MNP$ and J as the intersection of bisectors of $\angle CNM, \angle MPN$. The circle of center I that tangent to MP at D, the circle of center J that tangent to MN at E. Prove that $DE \| BC$ and the radical axis of $(I),(J)$ bisects DE.

△NJM ~ △PIM since:
∠IPM = ∠MNJ = 1/2. ∠A and
MN / MP = AB/AC = c/b.
From sine law --> IP / NJ = b/c. Thus from similarity, we get △PID ~ △NJE --> PD / NE = PI/NJ = PM/NM
--> DE || PN || BC.
Extend DE cuts (I) and (J) at X, Y. We need to prove:
DX = EY
iff ID.cosin ∠XDI= IE.cosin ∠YEJ
iff ID.sin ∠MDE = IE. sin ∠MED
iff ID / IE = sin ∠MED / sin ∠MDE = MD / ME = b/c.
Take T as the midpoint of DE --> TD.TX = TE.TY.

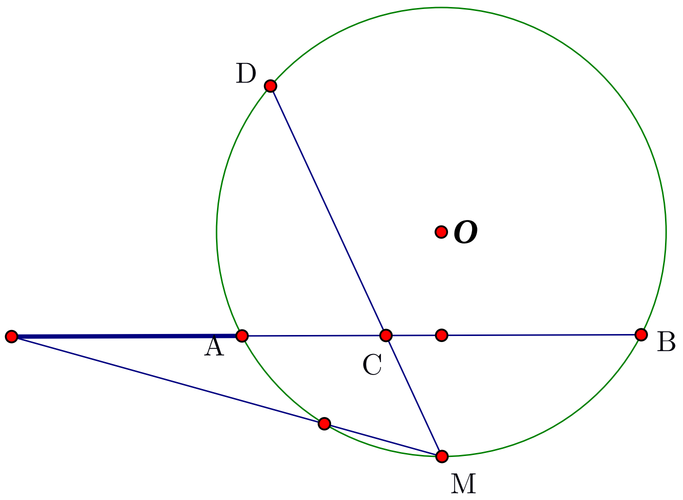


Problem 6. Let ABC be a triangle with circumcenter O, the median AM. Let N be the midpoint of the segments AM and take J such that AJ || BC, NJ $\perp$ ON. Circle (AOJ) meets JN again at K. Prove that $\angle$KAB = $\angle$MAC.

Idea: prove that tangent from B, C of (O) and AK are concurrent.

Extend NK cuts BC at T then two triangles ANJ, MNT are congruent --> AJMT is parallelogram.
ON is the perpendicular bisector of TJ.
$\angle$NOT = $\angle$NMT (ONTM cyclic).
= $\angle$NOJ
$\angle$AMO = $\angle$NMO = $\angle$NTO = $\angle$NJO = $\angle$OAK.
$\angle$AMO = $\angle$NAH (AH || OM).
--> $\angle$OAK = $\angle$HAM and done since AH, AO are isogonal conjugate.

Problem set 3. Shooting lemma

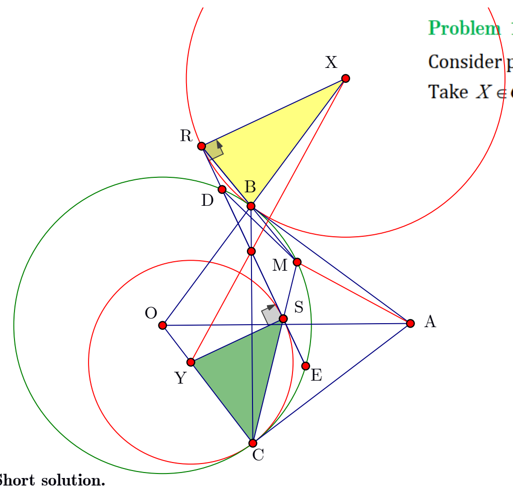


The inversion center M, power k:

(O) <--> AB.

This "shoot" any point from line AB to the circle (O).

MA^2 = MB^2 = MC.MD = k



Problem 1. Let A be a point lies outside circle (O) and tangent lines AB, AC of (O). Consider points D, E, M on (O) such that $MD = ME$. The line DE cuts MB, MC at R, S. Take $X \in OB, Y \in OC$ such that $RX, \cancel{RY} \perp DE$. Prove that $XY \perp AM$.
SY

We have XR = XB, YC = YS.

MS.MC = MB.MR = MD^2 = ME^2 (by shooting lemma).

△XRB, △YSC by Desargues's theorem.

Four-point theorem:

MX^2 - MY^2 = AX^2 - AY^2.

R.H.S = (BX^2 + AB^2) - (AC^2 + YC^2)

= BX^2 - YC^2 = XR^2 - YS^2.

triangle XRM, applying cosine law:

XR^2 + RM^2 - XM^2 = 2.XR.RM.cos(XRM)

YS^2 + SM^2 - YM^2 = 2.YS.SM.cos(YSM).

cos(XRB) = RB/(2XR) --> 2XR.RM.cos(XRM) = - RB.RM.

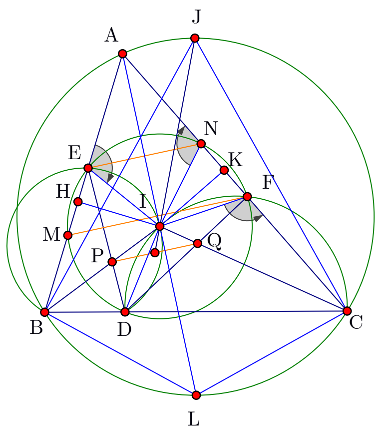
... 2.YS.SM.cos(YSM) = -SC.SM.

(XR^2 - YS^2) = (XM^2 - YM^2)+ SC.SM - RB.RM + (-RM^2 + SM^2) = (XM^2 - YM^2)+ MC.SM - BM.RM

Short solution.

AC^2 = AB^2 --> A lies on the radical of (X), (Y)

MB.MR = MS.MC --> M lies on the radical of (X), (Y).



Problem 2. Let ABC be an non-isosceles triangle with incenter I, circumcenter O and a point D on segment BC such that (BID) cut segments AB at $E \neq B$ and (CID) cuts segment AC at $F \neq C$. Circle (DEF) cuts segments AB, AC again at M, N. Let $P = IB \cap DE$ and $Q = IC \cap DF$. Prove that EN, FM, PQ are parallel and the median of vertex I in triangle IPQ bisects the arc BAC of (O).

BI is angle bisector --> ID = IE, similarly, ID = IF so I is the circumcenter of (DEF). We need to prove: AI is perpendicular bisector of EN, MF.

We have: AH = AK, IH = IK.

AEI and ANI are congruent: IE = IN, AI commont side (∠AEI = ∠IFC = ∠ANI since AEIF cyclic). Thus EN ⊥ AI, similarly, MF ⊥ AI --> EN || MF.

Need to prove: AI ⊥ PQ.

Shooting lemma --> IP.IB = ID^2 = IQ.IC

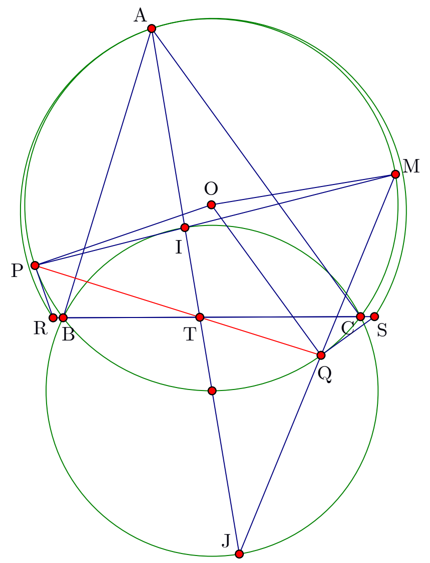
so B, P, Q, C concyclic --> ∠IPQ = ∠ICB (since ∠AIB = 90 + ∠ACB/2 = 90 + ∠ICB).

Last part: median of triangle IPQ bisects arc BAC (?)

IJ is median of IPQ iff IJ is symmedian of IBC (PQ is antiparallel).

L is the center of (IBC) --> ∠JBL = ∠JCL = 90 --> JB, JC tangent to (IBC)

so IJ is the symmedian.



Problem 3. Let ABC be a triangle inscribed in the circle (O) with incenter I and ex-circle J respect to A. The line passes through O, perpendicular to AI cuts (O) at M (on the same side with O respect to AI). Suppose that MI, MJ cut (O) at P, Q. Take $R, S \in BC$ such that PR, QS are tangent to (O). Prove that PQ, AI, BC are concurrent and (ARS) are tangent to (O).

We have: I, P, J, Q concyclic by shooting lemma:

MI.MP = MJ.MQ = MA^2. Consider 3 circles:

(B,I,C,J), (C,P,Q,B) <-> (O), (P,I,J,Q) by radical axis --> PQ, AI, BC.

Second part: (ARS) tangent to (O) <-> AR, AS are isogonal conjugate <->

RB/RC.SB/SC = AB^2 / AC^2 (*)

RB/RC = RB^2/(RB.RC) = RB^2/PR^2.

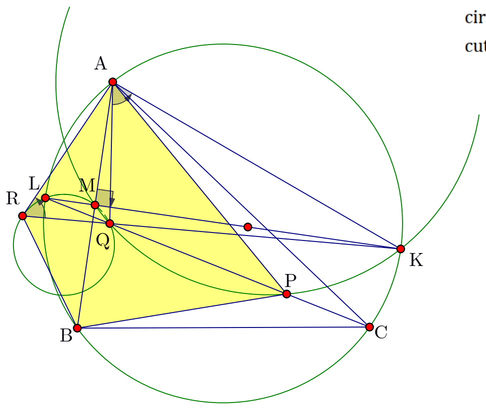
RB/PR.SQ/SC = AB/AC; sin(BPR) / sin(PBR). sin(SCQ)/sin(QSC)?

* PR is the tangent of (O) --> RB/RC = PB^2 / PC^2

SB/SC = QB^2/QC^2 so substitute to (*)

(*) iff PB/PC.QB/QC = AB/AC, true by applying the **ratio lemma**

PB/PC.QB/QC = TB/TC and TB/TC = AB/AC since AT is angle bisector.



Problem 4. Let ABC be a triangle inscribed in circle (O) with diamter KL passes through the midpoint M of AB such that L, C lie on the different sides respect to AB. A circle passes through M, K cuts LC at P, Q (point P lies between Q, C). The line KQ cuts (LMQ) at R. Prove that $ARBP$ is cyclic and AB is the symmedian of triangle APR.

1st part. ARBP cyclic. K, L are midpoints of the arc AB.

KA^2 = KM.KL = KQ.KR --> **KAQ, KRA** are similar.

--> ∠KRA = ∠KAQ. Similarly: ∠KRB = ∠KBQ.

--> ∠BRA = ∠KAQ + ∠KBQ.

LA^2 = LM.LK = LP.LQ

--> ∠LPA = ∠LAQ, ∠LPB = ∠LBQ --> ∠APB = ∠LAQ + ∠LBQ.

Thus: ∠ARB + ∠APB = ∠KAL+∠KBL=180.

2nd part. AB is symmedian of APR iff ARPB is harmonic iff RA/RB = PA/PB (since this quadrilarteral is cycle).

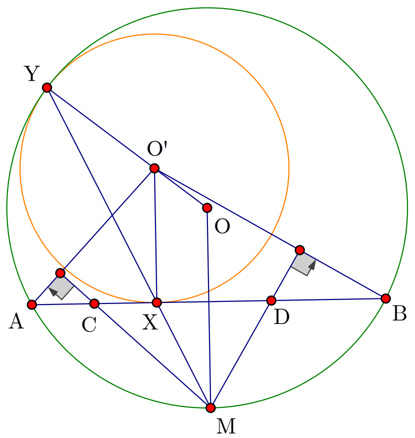
From KAQ ~ KRA --> RA / AQ = KA / KR.

And similarly: RB / BQ = KB / KR --> **RA/RB = AQ/BQ**.

Similarly: **PA/PB = AQ/BQ** --> done.

Sharygin geometry contest from Russia (2018).

Problem 5. Let AB be a chord of the circle (O). Denote M as the midpoint of the minor arc AB. A circle (O') tangent to segment AB and internally tangent to (O). A line passes through M, perpendicular to $O'A, O'B$ and cuts AB respectively at C, D. Prove that $AB = 2CD$.



We have: X, Y, M are collinear by homothety.

We will prove that: C, D are midpoints of XA, XB.

Consider: (O'), (A;0), (B;0).

By shooting lemma: MA^2 = MB^2 = MX.MY

--> M is radical center of 3 circles.

MC ⊥ AO' --> MC is the radical axis --> power of C from (O'), (A) are equal --> CA^2 = CX^2 then CA = CX. Similarly: DB = DX.

Problem 6. Consider circles $(O_1),(O_2),(O_3)$ are tangent to d at A,B,C and (O_2) is the biggest circle, externally tangent to $(O_1),(O_3)$. Let BD be the diameter of (O_2). The external tangent line (differs from d) of $(O_1),(O_3)$ cuts (O_2) at X,Y. Let K be the midpoint of the arc XBY of (O_2).

a) Prove that DK bisects segment AC.

b*) Prove that the circle of diameter AC touches DX,DY.

E, F are tangent points then D, E, A and D, F, C are collinear --> DE.DA = DB^2 = DF.DC --> D lies on the radical axis of (O1), (O3).

K, E, Z and K, F, T are collinear. Shooting lemma:

KE.KZ = KX^2 = KY^2 = KF.KT

so K also lies the radical axis of (O1), (O3) --> KD bisects AC.

DH, DL are tangent of (AC) with H, L lie on (AC).

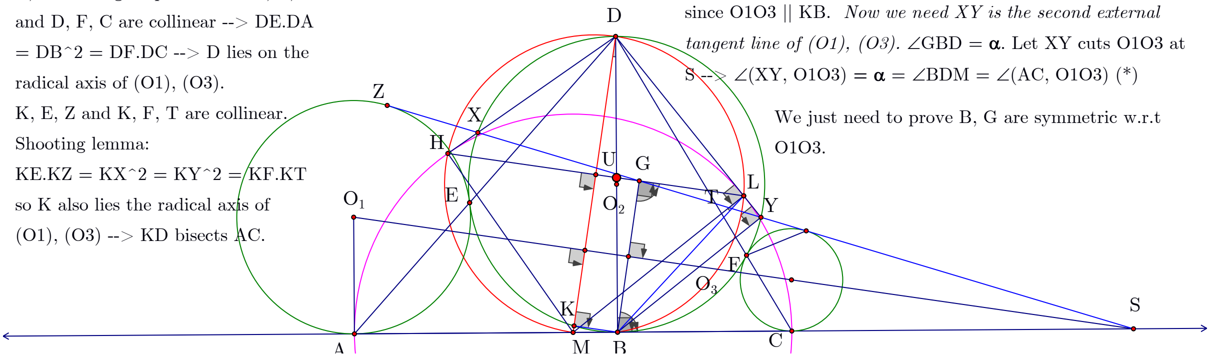
Redefine: *X, Y as the intersections of DH, DL with (O2).*

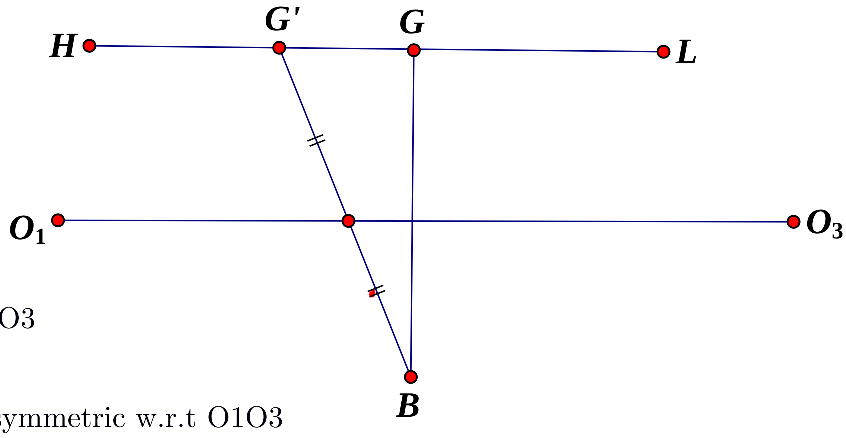
HL cuts XY at G. HL ⊥ DM and DM ⊥ O1O3

--> **HL || O1O3**. KB || O1O3 || HL -->

∠GYB = ∠XYB = ∠XDB = ∠HDB = ∠HLB --> B, G, L, Y concyclic --> BG ⊥ HL and **∠LGY = ∠LBY = α** and ∠LBY = ∠BLM (since ML || BY) = ∠BDM = ∠KBM = ∠(AC, O1O3) since O1O3 || KB. *Now we need XY is the second external tangent line of (O1), (O3).* ∠GBD = **α**. Let XY cuts O1O3 at S --> ∠(XY, O1O3) **= α** = ∠BDM = ∠(AC, O1O3) (*)

We just need to prove B, G are symmetric w.r.t O1O3.



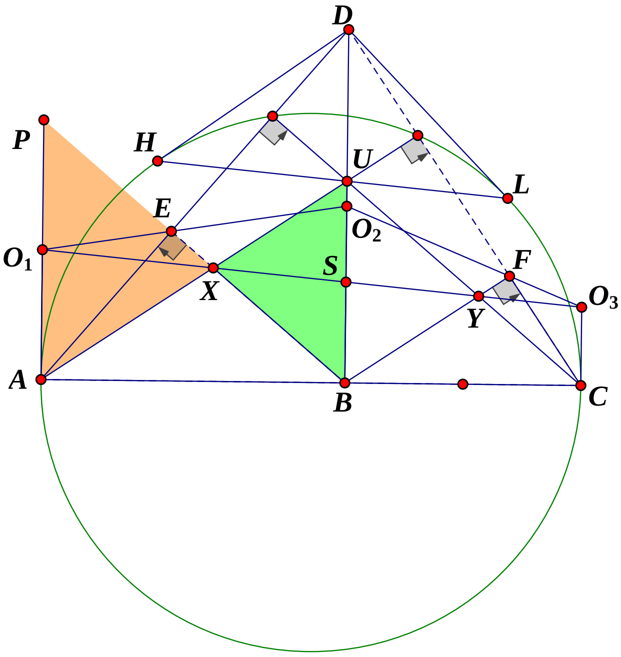


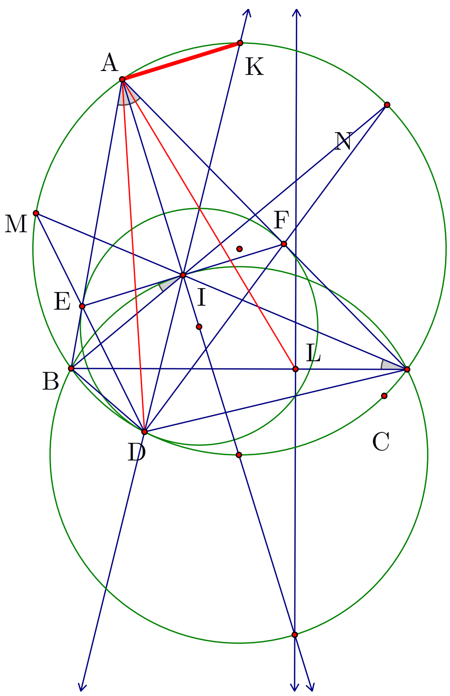
We know that: HL || O1O3

BG ⊥ O1O3.

We need to prove B, G symmetric w.r.t O1O3 iff O1O3 bisects BG iff O1O3 bisects BG' with G' is some point on HL.

HL passes through which special point of triangle DAC? --> U: orthocenter of triangle DAC. Use Brocard or pole-polar.
---> U is the point G'.
Rephrase the original problem:
Take O1 on O2E such that AO1 $\perp$ AC.
Take O3 on O2F such that CO2 $\perp$ AC.
--> we need to prove O1O3 bisects BU.
BE cuts AO1 at P --> O1 is the midpoint of AP.
AU cuts BE = X, BF cuts CU = Y.
XUB, XAP are homothety --> O1, X, S are collinear. Similarly: O3, S, Y are collinear.
BXUY is the parallelogram --> X, S, Y are collinear --> O1, S, O3 are collinear.





circle (T) that tangent to AB, AC and (ABC) --> Mixtilinear. take E, F are tangent points of (T) on AB, AC. Denote I as the incenter.

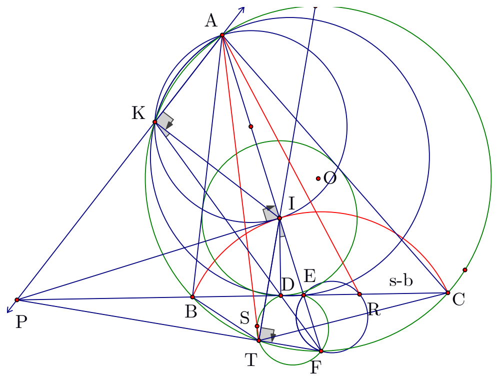
(1) *We need to proof I is midpoint of EF.*

Proof of (1): AE = AF and AI is the angle bisector of $\angle$EAF, so we just need to prove I, E, F are collinear. We know that (problem 5): DE bisects arc AB and DF bisects arc AC. Also M, I, C and N, I, B are collinear. By Pascal --> E, I, F are collinear.

(2) *DI bisects the arc BAC.*

$\angle$BDE = $\angle$BDM = 1/2. $\angle$ACB = $\angle$BIE --> DBEI is cyclic --> $\angle$BDI = $\angle$AEI. Similarly: $\angle$CDI = $\angle$AFI so $\angle$AEI = $\angle$AFI --> $\angle$BDI = $\angle$CDI --> done.

(3) *AD, AL are isogonal (L is the tangent point of A-excircle with BC).*



Since FD = FR (D, E symmetric w.r.t midpoint of BC)

--> DF / sin∠DEF = ∠FR/sin∠FER

--> (TEF)≅(REF) and symmetric w.r.t EF.

Take S as the symmertric of R over AI --> S lies on AT, (TEF), (J) --> done.

Problem 7*. Let ABC be a triangle with circumcenter O and incenter I, ex-center in angle A is J. Denote D as the tangent point of là (I) on BC and the angle bisector of angle A cuts $BC,(O)$ respectively at E,F. The circle (DEF) meeds (O) again at T. Prove that AT passes through an intersection of (J) and (DEF).

DF cuts (O) again at K --> FI^2 = FD.FK = FE.FA --> A,K,D,E are concyclic. Angle chasing: ∠AKI = 90.

So AK, DE, TF are concurrent --> P.

Lemma: AK cuts BC at P --> ∠PIA = 90: take radical axis of (BIC), (AIK) and (ABC) --> BC, AK and the line through I, ⊥ AI are concurrent.

And FT.FP = FI^2 --> IT ⊥ PF --> IT passes through the midpoint L of arc BAC and T is the tangent point of Mixtilinear circle with (O).

We need to prove that: **AT, AR are isogonal conjugate.**

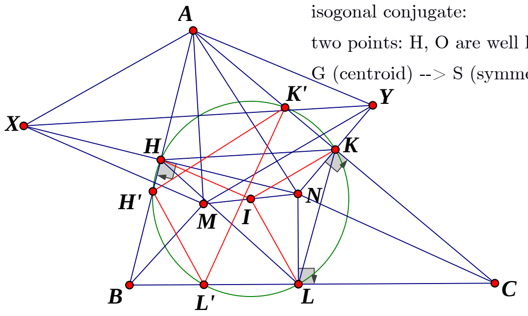
1st way: use inversion --> center A, power AB.AC and reflect through AI.

B <--> C, BC <--> (O), X <--> X' with X is any point on BC so X' lies on (O) and AX, AX' are isogonal conjugate.

Mixtilinear circle <--> excircle, so T <--> R.

2nd way: calculation: sin∠TAB/sin∠TAC = sin∠RAC/sin∠RAB.

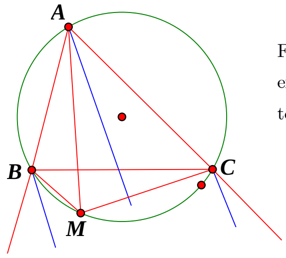
Problem set 4. Isogonal conjugate



isogonal conjugate:
two points: H, O are well known.
G (centroid) --> S (symmedian).

Take projections of N onto AB, AC --> H, K
Circumcenter of AHK --> midpoint of AN --> orthocenter of AHK lies on AM --> AM perpendicular to HK.
Denote I as the midpoint of MN --> IH = IK. AX = AY = AN and AM $\perp$ HK, HK || XY --> AM is perpendicular bisector of XY.
--> MX = MY --> IH = XM / 2, IK = YM / 2 so IH = IK.
Take L as projection of N onto BC. Similarly: IH = IK = IL --> I is the circumcenter of (HKL). And I is also the center of (H'K'L').
When (M, N) = (H, O) --> (I) is nine point circle.

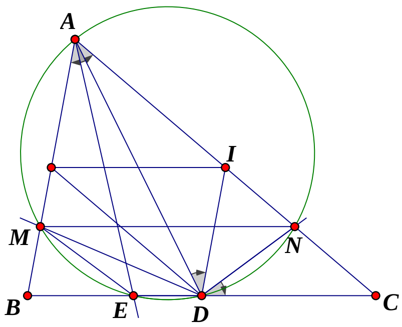
If we take M lies on circle (ABC) and H, K, L are projections of M on BC, CA, AB --> Simson line. So if **N** is the isogonal conjugate of M, and I is the midpoint of MN --> IH = IK = IL. (???)



For all other cases --> there exists exactly one N isogonal conjugate to M in triangle ABC.

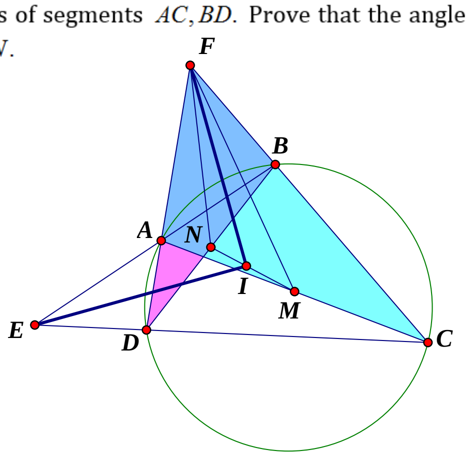
Problem 1. Let ABC with median AD. Denote DM, DN, AE as the symmedian of triangles ABD, ACD, ABC. Prove that $MNED$ is the isosceles trapezoid.

MB/MA = (DB/DA)^2
NC/NA = (DC/DA)^2
DB = DC --> MB/MA = NC / NA
--> MN || BC
Take I as midpoint of AC
--> $\angle$CDN = $\angle$ADI = $\angle$DAB = $\angle$IAE --> AEDN cyclic.
Similarly: AMED cyclic so MNDE cyclic --> done.



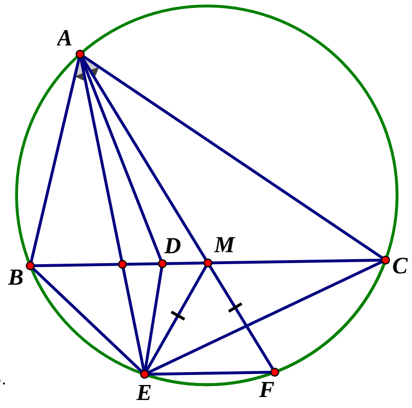
Problem 2. Let $ABCD$ be a cyclic quadrilateral with $AB \cap CD = E$, $AD \cap BC = F$. Denote M, N are midpoints of segments AC, BD. Prove that the angle bisectors of angle E, F meet on the line MN.

Since FAC ~ FBD --> FM, FN are isogonal conjugate in angle F.
Take I = intersection of MN with angle bisector of F --> FI is also the angle bisector of MFN.
IN / IM = FN / FM = BD / AC.
Similarly: EN / EM = BD /AC so IN/IM = EN/EM --> angle bisector of E also passes through I.



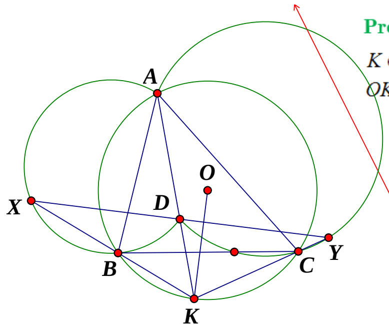
Problem 3. Let ABC be a non-isosceles triangle with $AB^2 + AC^2 = 2BC^2$. Denote AD, AM as the internal angle bisector and the median of triangle. Take E such that D is the incenter of triangle AME. Prove that $MA = 3ME$.

AE is the symmedian.
Suppose that AE intersects (ABC) at E'
--> ABE'C is the harmonic quadrilateral
--> AB / AC = E'B / E'C so angle bisector of ∠BAC and ∠BE'C and BC are concurrent.
So E'A is also the symmedian of triangle E'BC, E'M is the median of triangle E'BC
--> E'D is the angle bisector of ∠BE'C and the same for ∠AE'M --> D is the incenter of triangle AME' --> E = E'.



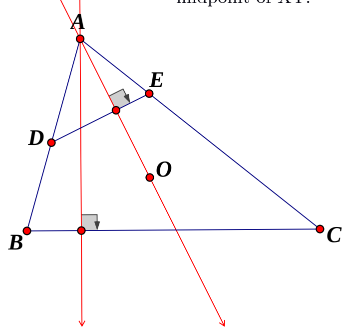
MA^2 = (AB^2+AC^2)/2 - BC^2/4 = 3/4. BC^2 -->
Extend AM cuts (ABC) at F.
MA.MF = MB.MC = BC^2 / 4 --> MF = BC^2 / (4MA).
MA/ME = MA / MF = MA^2 / (MA.MF) = 4MA^2 / BC^2 = 3.

Problem 4. Let ABC be a triangle inscribed in circle (O) and the symmedian AK with $K \in (O)$. Suppose that KB cuts (ADB) at X and KC cuts (ADC) at Y. Prove that $OK \perp XY$ and D is the midpoint of XY. D is any point on AK (Saudi TST 2016)



1st way, XBCY is cyclic. ∠KCB = ∠KXY = 90 - ∠OKB
--> OK ⊥ XY ; XY is the antiparallel of KBC
and KA is the symmedian of KBC --> KA is the median of KXY.
∠XDA = ∠XBA = ∠ACK = ∠KDY --> **X, D, Y** collinear --> D is the midpoint of XY.

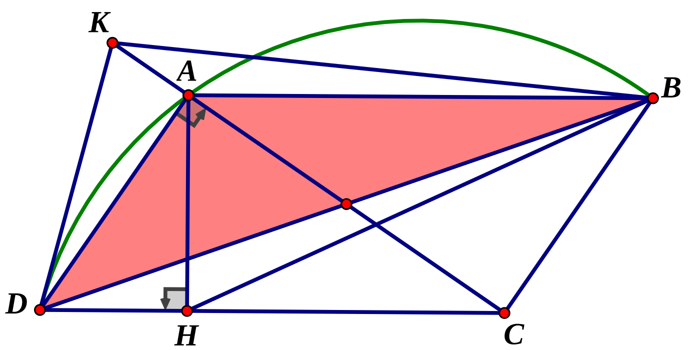
2nd way,
XY is the antiparallel of KBC and KO passes through the center of KBC --> by lemma, we get KO ⊥ XY.
Another way to prove X, D, Y are collinear:
KX.KB=KD.KA=KC.KY
and (ABC) passes through K --> inversion of center K --> X, D, Y are collinear



Lemma: DE is the antiparallel of ABC
Take two corresponding lines in triangle ADE and ABC (median-median, altitude-altitude, ...)
--> they are isogonal conjugate

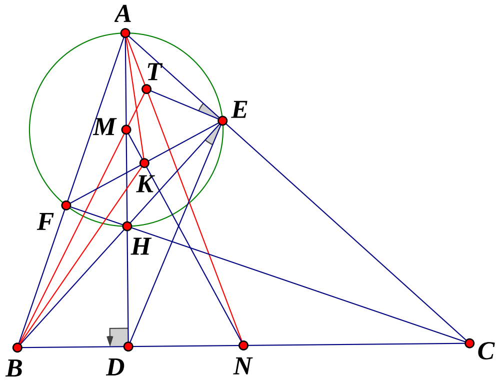
Problem 5. Let $ABCD$ be a parallelogram with $\angle CAD = 90°$ and H is the projection of A on CD. Suppose that the tangent line of circle (ABD) at D cuts AC at K. Prove that $\angle KBA = \angle HBD$.

two points are isogonal conjugate in a triangle



∠DAH = ∠ACD = ∠CAB
∠KDA = ∠ABD = ∠BDC
DK, DH are isogonal
AH, AK are isogonal
--> H, K are isogonal in triangle ADB --> BH, BK are isogonal --> done.

Problem 6. Let ABC be a triangle with altitudes AD, BE intersect at H. Denote M, N as the midpoints of segments AH, BC. BM cuts AN at T. Prove that $\angle TED = 90°$.



CF is the altitude and K is the midpoint of EF.

M, K, N are collinear.

BM and BK are isogonal in ∠ABE.

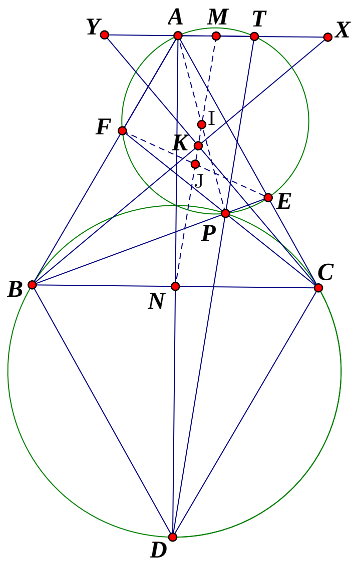
AK and AT are isogonal in ∠BAE.

--> K, T are isogonal points in triangle ABE.

--> ET and EK are isogonal.

--> ∠AET = ∠KEB = ∠HED.

--> ∠TED = 90.



Problem 7. Let $ABDC$ be a rhombus and P lies inside triangle ABC such that $\angle BPC = 180° - \angle A$. Suppose that $BP \cap AC = E$, $CP \cap AB = F$ and X, Y are excenters respect to angles B, C in triangles ABE, ACF. Circles (AEF) cuts XY at T. Prove that TP passes through D and the center of (AEF) is equidistance to X, Y.

A, E, F, P are cyclic and B, C, P, D are cyclic.

∠DPC = ∠DBC = ∠ABC = 180 - ∠FAT

= 180 - (180 - ∠FPT) = ∠FPT.

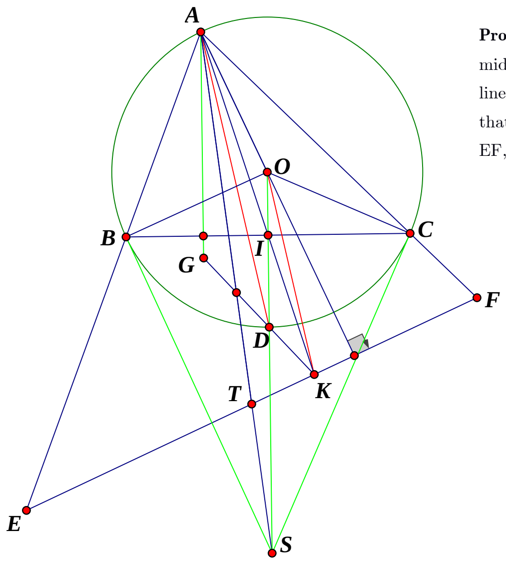
--> D, P, T collinear.

We just need to prove: AY = TX.

M is the midpoint of XY and BCXY is the trapazoid --> M, K, N are collinear.

Denote I, J as midpoints of AP, EF --> I, J, N are collinear by Gauss line.

And I, K, J are collinear by problem 2 --> and NI || DT --> NM || DT --> M is the midpoint of AT.



Problem 8. Let ABC be an acute, non-isosceles triangle inscribed in circle (O) and I is the midpoint of BC, D is the midpoint of minor arc BC. Take K on AI such that OK || AD. The line through K, perpendicular to AO cuts AB, AC at E, F respectively. Take G on KD such that AG perpendicular to BC. Prove that A lies on the line joining midpoints of segments EF, DG.

AO, AG are isogonal.

EF ⊥ AO --> EF is the antiparallel of BC.

T: midpoint of EF --> AT, AI are isogonal.

AT is the median of AEF so AT is the symmedian of ABC -->

AT passes through S = intersection of two tangent lines from B, C of (O).

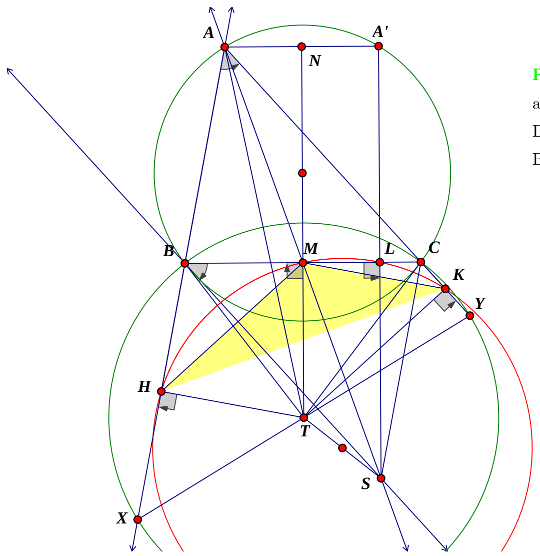
We need to prove: AS bisects DG iff AGSD is parallelogram

iff GS || AD || OK or AG = DS.

OD / DS = KD / DG = KI / AI = OI / ID.

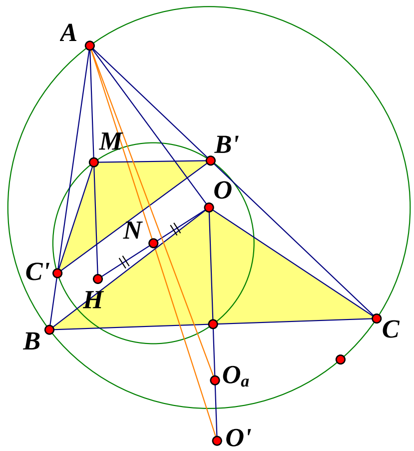
--> OI / ID = OD / DS

iff OI / OD = OD/OS iff OI.OS = OD^2 = R^2.



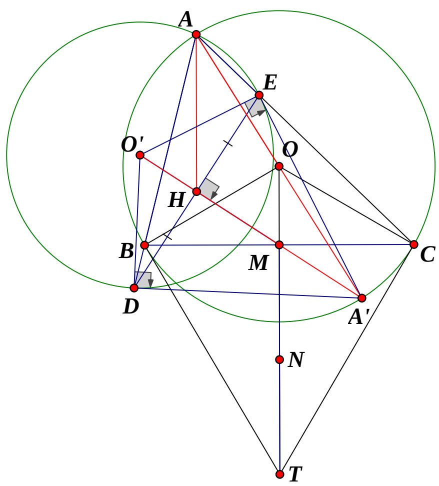
Problem 9. Let ABC be an acute, non-isosceles triangle inscribed in circle (O) and M is the midpoint of BC. Tangent lines of (O) from B, C intersect at T. Denote H, K as projections of T on AB, AC. Take A' on (O) such that AA' || BC and L is the projection of A' on BC. Prove that H, K, L, M are concyclic.

The antiparalel through T cuts AB, AC at X, Y
--> H, K is the midpoint of BX, CY.
HM || XC, XC ⊥ AC, TK ⊥ AC --> HM || TK.
Similarly: KM || TH so HMKT is parallelogram.
We also write: MHK is the Pedal triangle of T in triangle ABC.
What is the isogonal conjugate S point of T?
$\angle SBH = \angle TBC = \angle BAC$ --> BS || AC. Similarly: CS || AB --> ABSC is the parallelogram.
So MN || A'S (with N is the midpoint of AA') --> A'S ⊥BC --> A', L, S are collinear. So by the theorem about isogonal conjugate points T, S --> 4 points H, M, L, K lie on a same circle.

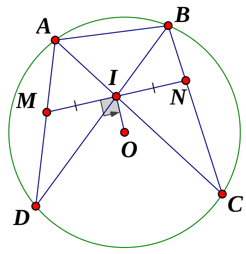


We know that: (H, O) are isogonal and (G, S) are also (S: symmedian).
I, Ia, Ib, Ic: are isogonal with itselft.
(T, D) with T: intersection of two tangent lines from B, C and ABDC is parallelogram.
Question here: what is the isogonal conjugate of nine-point center N?

AB'C' ~ ABC.
M is the circumcenter of AB'C'.
O is the circumcenter of ABC.
Role of N in AB'C' --> N is the circumcenter of (MB'C').
Take Oa is the circumcenter of (OBC).
--> AN and AOa are corresponding so they isogonal conjugate in angle A. Take Ob, Oc are center of (OCA), (OAB) then AOa, BOb, COc are concurrent at K --> isogonal conjugate of N.
K: Kosnita point of triangle ABC.



Problem 10.
Let ABC be a triangle inscribed in a fix circle (O) with BC is fix and A vary on (O). Denote H as the orthocenter of triangle ABC and take D, E on AB, AC respectively such that H is the midpoint of DE. Prove that when A moves on (O), the center of (ADE) belongs a fixed circle.



Take M as midpoint of BC --> apply Butterfly theorem --> HM ⊥ DE.
1st idea: TB, TC are tangent lines of (O).
2nd idea: Prove: MH / MO' is fixed.
AH, AO are isogonal conjugate and AH is the median of triangle ADE --> AO is the symmedian.
Take A' = AO cuts HM --> A' lies on (O) and A'D, A'E are tangent lines of (ADE).

BOCT ~ DO'EA', take N as the midpoint of MT so MA' / MO' = NT / NO = k fixed since O, M, T are fixed points --> O' is the image of A' from the homothety of center M and ratio -k.

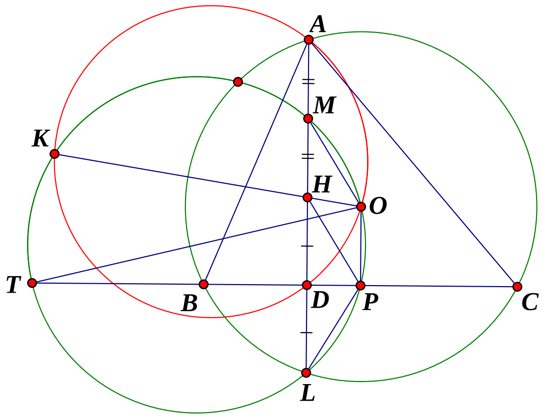
Problem in BMO TST 1.

O1, M, O2 are collinear; the angle bisector of MHO1 || AC; the angle bisector of MHO2 || AB --> ARHS is parallelogram.

XR/XH = MR/MH; YS/YH = MS/MH.

MS = MR --> XR/XH = YS/YH --> XY || O1O2.

Extra question: Let O1O2 meets BC at T. Denote O as circumcenter of ABC and circle (MOT) meets OH againt at K. Prove that the center of (AOK) belongs to the midline of triangle ABC.



First, denote P as midpoint of BC --> PH is the radical axis of (O1), (O2) --> PH $\perp$ O1O2, PH || MO -->

OM $\perp$ O1O2 --> we can remove O1, O2 to make the problem simpler. We need to prove: A, K, D, O concyclic.

AD meets (O) again at L --> H, L symmetric w.r.t BC --> PL = PH = MO so MOPL is an isosceles trapezoid --> T, M, O, P, L concyclic.

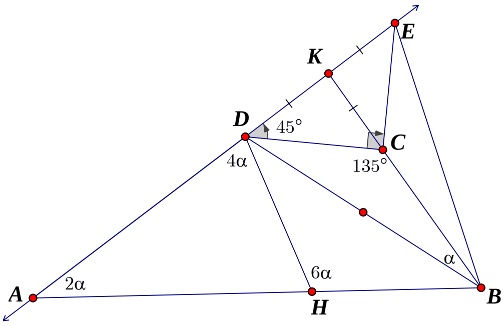
HD.HA = HM.HL = HK.HO --> A, K, O, D concyclic.

Problem set 5. Advanced angle chasing

Problem 1. Let $ABCD$ be a quadrilateral such that

$$\begin{cases} \angle ADC = 135^\circ \\ \angle ADB - \angle ABD = 2.\angle DAB = 4.\angle CBD \\ BC = \sqrt{2}.CD \end{cases}.$$

Prove that $AB = BC + AD$.



Take E on AD such that ∠DCE = 90 --> DCE is a isosceles right triangle --> DE = CD.$\sqrt{2}$ = CB.

To prove AB = BC + AD, we need to prove: AE = AB (?)

Take H on AB such that: HD = HB --> ∠HDB = ∠HBD

--> ∠ADH = ∠ADB - ∠ABD = 4α.

BC cuts AD = K --> ∠DBA = 90 - 3α

--> ∠ABK = 90 - 2α --> AKB = 90.

∠DCB = 180 - ∠DCK = 135.

BC is the perpendicular bisector of DE

--> BD = BE --> ∠EBD = 2α.

∠ADB = 90 + α

∠AEB = ∠ADB - ∠DBE = 90 -α.

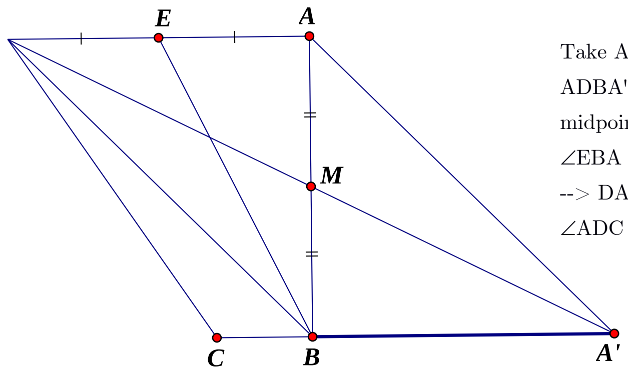
Angle chasing on triangle ABE -->

∠ABE = 90 -α = ∠AEB --> AB=AE.

Problem 2. Let $ABCD$ be a quadrilteral with $\angle A = \angle B = 90^\circ$, $AB = AD$. Denote E as the midpoint of AD and suppose that

$$CD = BC + AD, AD > BC.$$

Prove that $\angle ADC = 2.\angle ABE$.



Take A' on BC such that BA' = AD so CD = CA'.

ADBA' is the parallelogram --> A'D cuts AB at midpoint M.

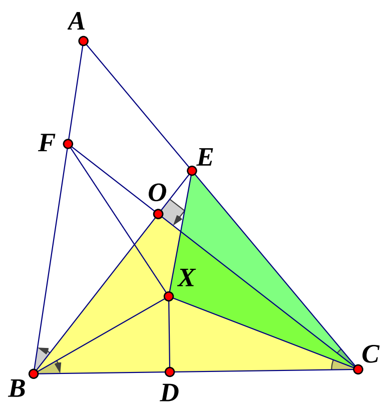
∠EBA = ∠ADA' = ∠DA'B = ∠CDA'.

--> DA' is angle bisector of ∠ADC so

∠ADC = 2∠ADA' = 2∠ABE.

Problem 3. Let ABC be a triangle. Take points E, F on CA, AB respectively such that $BE \perp CF$. Denote O as the intersection of BE, CF and X lies inside triangle ABC such that $\angle ACO = \angle XCB, \angle ABO = \angle XBC$. Prove that $\angle EXF + \angle BAC = 90^\circ$.

(ABC is an acute triangle)



X, O are isogonal conjugate in ABC.

If we can prove O, X, C, E are concyclic ?

Let D as the projection of X onto BC.

BOF ~ BDX and COE ~ CDX.

BDO ~ BXF (s.a.s) and

CDO ~ CXE (s.a.s).

--> ∠AFX + ∠AEX = (180 - ∠XFB) +

(180 - ∠XEC) = 360 - (∠DOB+∠DOC)

= 360 - ∠BOC = 270.

--> ∠A+∠EXF = 360 - 270 = 90.

Quadriteral ADEB and C

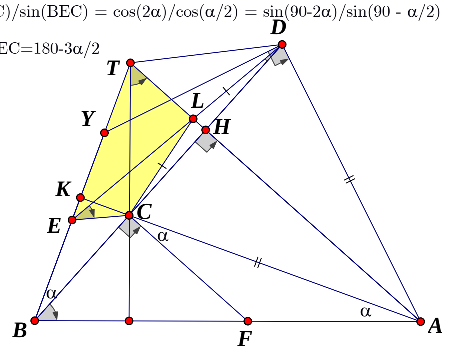
--> Ceva

sinα/sin(2α).cos α/sin(α/2).sin(DEC)/sin(BEC).sinα/cos(2α)=1.

--> sin(DEC)/sin(BEC).cos(α/2)/cos(2α)=1

--> sin(DEC)/sin(BEC) = cos(2α)/cos(α/2) = sin(90-2α)/sin(90 - α/2)

∠DEC+∠BEC=180-3α/2



Problem 4. Let ABC be a triangle with $\angle C = \angle A + 90°$. Take D on the ~~opposite~~ ray of ray BC such that $AC = AD$. Take E such that E, A lie on different side respect to BC $\angle EBC = \angle A; \angle EDC = \frac{1}{2}\angle A$. Prove that $\angle CED = \angle ABC$.

Take F on AB such that ∠BCF = 90 --> FC = FA and ∠FCA = ∠FAC = α; ∠EBF = ∠ACD = ∠ADC = 90 - α.

BE cuts AC at K --> ∠BKA = 90; ∠ABC = 90 - 2α.

Let Dx as the line ⊥ DA --> DE is the angle bisector of ∠BDx and Dx cuts BK at Y.

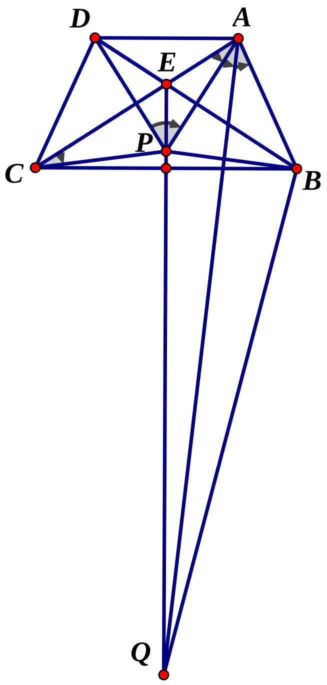
∠YBD = α = ∠YDB --> YB=YD. AKYD is cyclic.

∠ECB = 90 - 3α / 2 (we need it), but ∠YED = 3α/2.

H is the midpoint of CD and AH cuts BE at T --> C is the orthocenter of triangle TAB --> D is the reflection of C over AT --> D on (TAB).

DE cuts AT at L, we need to prove TECL is cyclic (then we finish because ∠CEL = ∠CTL = ∠ABC).

∠BTC = ∠BAC = α and ∠LDC = α/2, LD=LC --> ∠ELC = α.



Problem 5. Let ABC be a triangle with $\angle B = 2.\angle C$. Take P, Q on the perpendicular bisector of BC such that

$$\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}.$$

Prove that Q is the circumcenter of triangle CPB.

BE is angle bisector of ∠ABC --> BE = CE --> P, Q, E are collinear.

We have: QB=QC, so we need to prove: QP = QC, QB = QP.

QB = QP iff APQB is kite iff (AB = AP | AQ ⊥ PB).

We know that AE.AC = AB^2 so if we can prove AE.AC = AP^2 --> done.

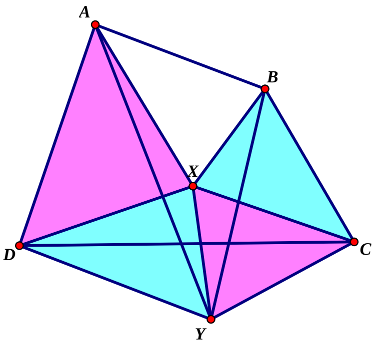
Take the reflection of A over the PQ --> DC = AB, ABCD is isosceles trapezoid. We have PA = PD and

∠DAP = ∠DAC + ∠CAP =∠C+∠A/3

=1/3.(3∠C+∠A)=1/3.(∠C+∠B+∠A)=60.

--> APD is the equilateral --> AD = AP = BP.

B, E, D are collinear --> ADB is isosceles --> AB = AD = AP.



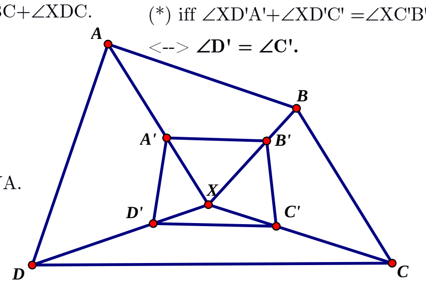
Problem 6. Let ABC be convex quadrilateral and X lying inside it such that

$$XA \cdot XC^2 = XB \cdot XD^2 \text{ and } \angle AXD + \angle BXC = \angle CXD.$$

Prove that $\angle XAD + \angle XCD = \angle XBC + \angle XDC$. (*)

Similarly, YBC ~ YAD.
--> ∠YBC = ∠YAD.
--> ∠XBC - ∠XBY = ∠XAD - ∠XAY
--> ∠XBC - ∠XCD = ∠XAD - ∠XDC.
--> ∠XAD+∠XCD = ∠XBC+∠XDC.

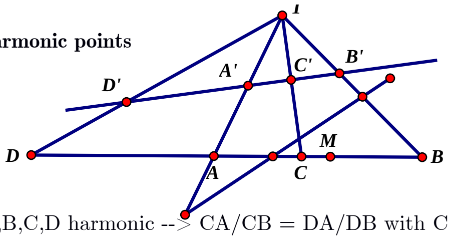
1st idea: tangent line from X of (XDC)
2nd idea: inversion of center X, power k > 0.
A, B, C, D --> A',B',C',D'.
XA.XA'=XB.XB'=XC.XC'=XD.XD' = k.
--> XA'.XC'^2 = XB'.XD'^2 and
(*) iff ∠XD'A'+∠XD'C' =∠XC'B'+∠XC'D'
<--> **∠D' = ∠C'.**



Construct Y such that: XAD ~ XYC: **XA/XY=AD/YC=XD/XC** (1) and
∠AXD=∠CXY --> ∠BXC=∠DXY. We need to prove: **XB/XC=XY/XD** (2)
From (1) --> XY = XA.XC/XD=XB.XD/XC --> (2) --> XBC ~ XYD.
∠BXY=∠CXD=∠AXY and XA/XY=XD/XC=XY/XB --> XCD ~ XBY ~ XYA.
∠XCD = ∠XBY = ∠XYA, ∠XDC = ∠XYB = ∠XAY.

Problem set 6. Harmonic points, bundle, quadrilateral

harmonic points



A,B,C,D harmonic --> CA/CB = DA/DB with C, D different side w.r.t AB.

(AB,CD)=-1 (double ratio).

If M is the midpoint of AB

--> MA^2 = MB^2 = MC.MD (Newton identity)

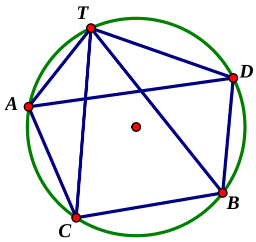
--> DC.DM = DA.DB and CD.CM=CA.CB (Maclaurin identity)

T(AB,CD)=-1 is harmonic quartet/bundle.

line (d) cuts TA, TB, TC, TD at A', B', C', D'

--> (A'B', C'D')=-1.

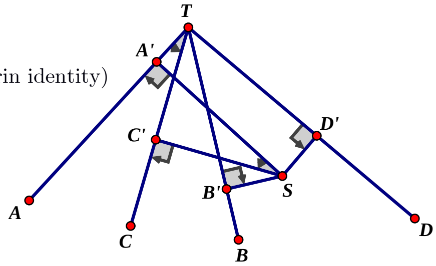
if (d) || TD --> C' is the midpoint of A'B'



T(AB,CD)=-1

--> ACBD is harmominc quadrilateral

--> AB is the symmedian of triangle ACD, BCD; similar with CD.

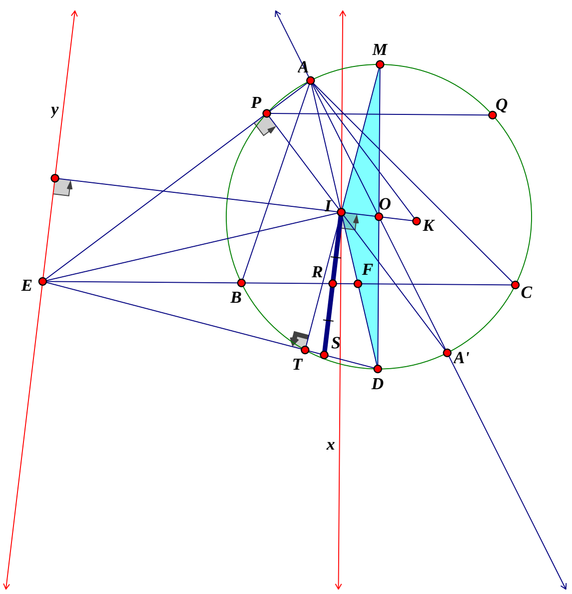


T(AB,CD)=-1

take any point S

draw 4 lines perpendicular to TA, TB, TC, TD

--> **S(A'B',C'D')=-1.**



Problem in test 2.

Original --> shooting lemma 2 times + angle chasing / power point to circle.

Extra questions:

1) Take K as the reflection of I over O --> Prove that $\angle EAK = 90$.

2) The line passes through I, and $\perp$ OI cuts BC, DE at R, S --> R is the midpoint of IS.

Take the diameter AA' --> P, I, A' are collinear since $\angle API = 90$. And AIA'K is the parallelogram --> AK || IA' --> AK $\perp$ AE.

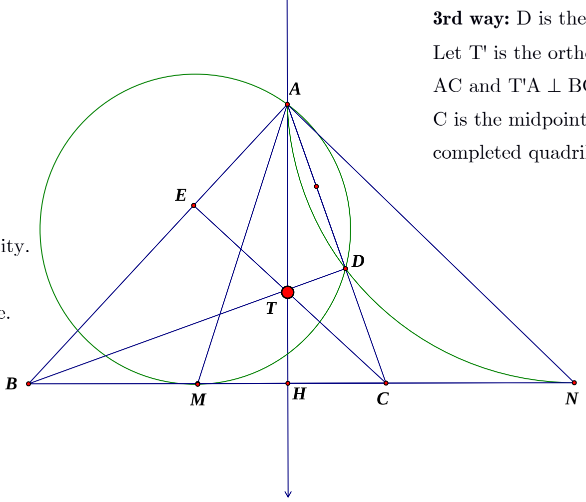
(?) Prove OR || SK (but maybe it's not easy to prove).

* IO is the median of triangle IMD so if the draw Ix || DM --> **I(MD, Ox)=-1**: harmonic quartet, take E ED $\perp$ IM, EI $\perp$ ID, EF $\perp$ Ix, Ey $\perp$ IO --> **E(DI, Fy)=-1**, the big line || Ey and intersects EI at I, EF at R and ED at S --> done.

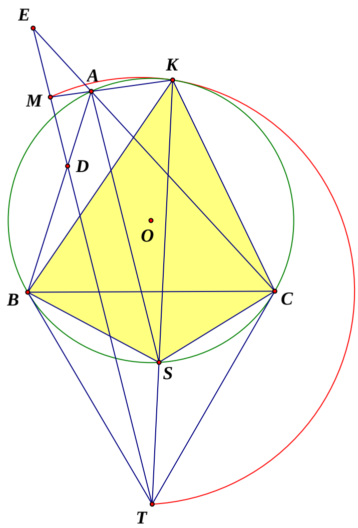
1st idea: H is projection of A on BC
--> CH.CB=CD.CA=CM^2.
--> BM.BN = (BC-CM).(BC+CN)
=BC^2 - CM^2
=BC^2 - CH.CB=
=BC.BH=BE.BA --> done.

2nd idea: Harmonic idea:
C is the midpoint of MN.
CM^2 = CN^2 = CD.CA=CH.CB
--> (MN,HB) = -1 by Newton's identity.
Let apply Maclaurin:
BM.BN = BH.BC = BE.BA --> done.

Problem 1. Let ABC be an acute, non-isosceles triangle with BD, CE are altitudes. Take points $M, N \in BC$ such that $(ADM), (ADN)$ are tangent to BC. Prove that M, N, A, E are concyclic.



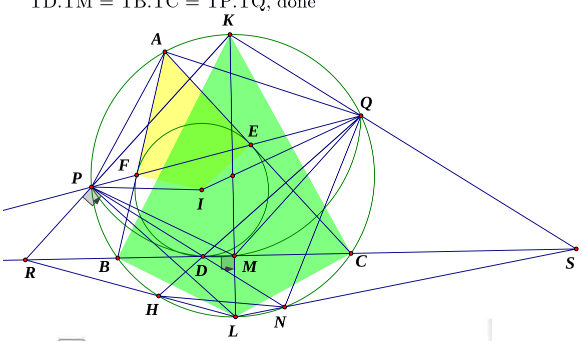
3rd way: D is the Humpty point of AMN
Let T' is the orthocenter of AMN --> T'D ⊥ AC and T'A ⊥ BC --> T=T'.
C is the midpoint --> Miquel point of completed quadrilateral --> E is on (AMN).



Problem 2. Let ABC be an acute, non-isosceles triangle with circumcenter O. Tangent lines to (O) at B, C meet at T. A line passes through T cuts segments AB at D and cuts ray CA at E. Take M as midpoint of DE and suppose that MA cuts (O) again at K. Prove that (MKT) is tangent to (O).

TK cuts (O) againt at S --> BKCS is harmonic
--> A(KS,BC)=-1.
And we have AK bisects DE --> AS || DE.
Take the homothety center K --> (O)≡(KAS)
and (KMT) are tangent, done!

PQ cuts BC at T
AD, BE, CF are concurrent --> By Ceva, Menelaus --> (TD, BC)=-1.
TD.TM = TB.TC = TP.TQ, done



G6. Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

Problem 3. Let ABC be an acute, non-isosceles triangle with circumcenter O, incenter I and (I) tangent to BC, CA, AB at D, E, F respectively. Suppose that EF cuts (O) at P, Q. Prove that (PQD) bisects segment BC. (IMO SL 2019)

Angle chasing (?)
1st part. PD cuts (O) at N --> CNBQ is harmonic quadrilateral --> QN is the symmedian of BQC and QM is median --> ∠NQB = ∠MQC and ∠NQC=∠MQB
∠BQP=1/2.m(BP), ∠BDP=1/2.(m(BP)+m(NC))
--> ∠PQM=∠PDB --> P, Q, M, D cyclic.
2nd part: ∠DPA+∠DQA = ∠QIP.
* ∠DPA+∠DQA=∠DPK+∠DQK=**∠MPK+∠MQK (1)**.
K,L is the midpoint of the arc BC of (O). RPML is cyclic --> ∠MPK = ∠MLR. Similarly, ∠MQK = ∠MLS --> RHS(1) = ∠RLS = ∠PIQ (?) Two kites AEIF, KCLB are similar --> there is a spiral similarity maps AEIF --> KCLB.

Problem 4. Let ABC be a triangle with $AB < AC$ and incircle (I) tangent to BC at D. Take K on AD such that $CD = CK$. Suppose that AD cuts (I) at G and BG cuts CK at L. Prove that K is the midpoint of CL. (Chinese TST 2008)

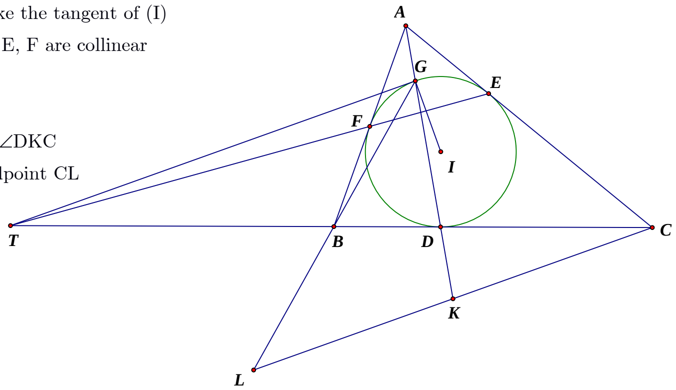
GEDF is harmonic --> take the tangent of (I) at G cuts BC at T --> T, E, F are collinear

--> (TD, BC)=-1.

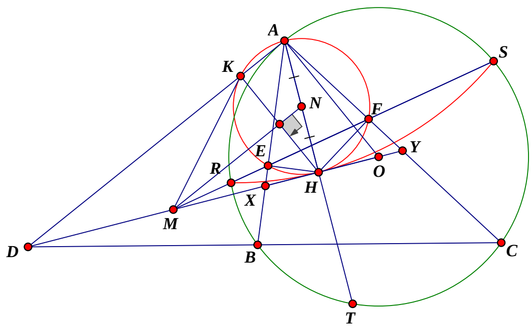
--> **G(TD,BC)=-1**

∠TGD=∠TDG=∠KDC=∠DKC

--> **CL || TG** --> K is midpoint CL



Problem 5. Let ABC be a triangle with $AB < AC$ inscribed in (O). Tangent line at A of (O) cuts BC at D. Take H as the projection of A on OD and E, F as projections of H on AB, AC. Suppose that EF cuts (O) at R, S. Prove that (HRS) is tangent to OD.



AH cuts (O) at T --> DT is tangent to (O) --> ABTC is harmonic --> AT is symmedian. A(DH,BC)=-1 --> A(DH,XY)=-1.

(HRS) tangent to OD iff MR.MS = MH^2.

If we call M' is the midpoint of DH so by Newton's identity: M'H^2 = M'X.M'Y. Circle (AH) of center N, cuts AD again at K --> AD, AH, AX, AY intersect (AH) at K, H, E, F --> KEHF is harmonic quadrilateral.

M = dianogal EF cuts tangent line of (N) at H --> M is the intersection of tangent at H and K of (N) --> MN ⊥ HK, but HK ⊥ AD --> M is the midpoint of DH.

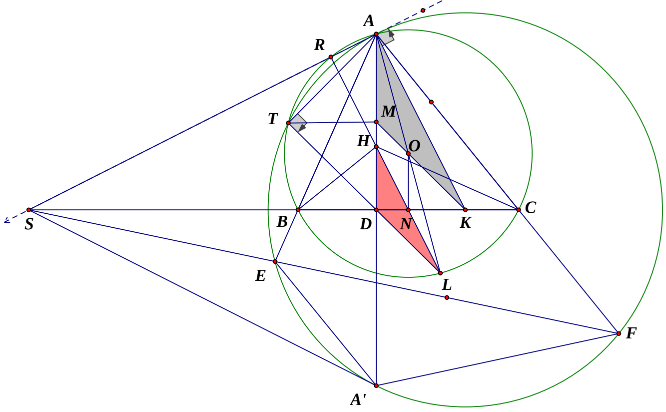
MH^2 = MD^2 = MX.MY = ME.MF (since XEFY is cyclic).

We need to prove: MR.MS = MX.MY.

But MR.MS = power from M to (O) = MO^2 - AO^2.

AO^2 = OH.OD=(MO-MH).(MO+MH)=MO^2 - MH^2.

Problem 6. Let ABC be an acute, non-isosceles triangle with altitude AD $(D \in BC)$, M is the midpoint of AD and O is the circumcenter. Line ~~AO~~ MO meets BC at K and circle of center K, radius KA cuts AB, AC at E, F respectively. Prove that AO bisects EF.



AD, AO are isogonal.

AO is the median of AEF iff AD is the symmedian iff **AEA'F** is the harmonic quadrilateral.

(K), (O) meet again at T --> AT ⊥ OK, and we also have AT ⊥ TD --> TD || OK.

Take S on BC s.t AS tangent to (K), we need to prove A(AA',EF)=-1 iff A(SD,BC)=-1.

We need to HN || AK.

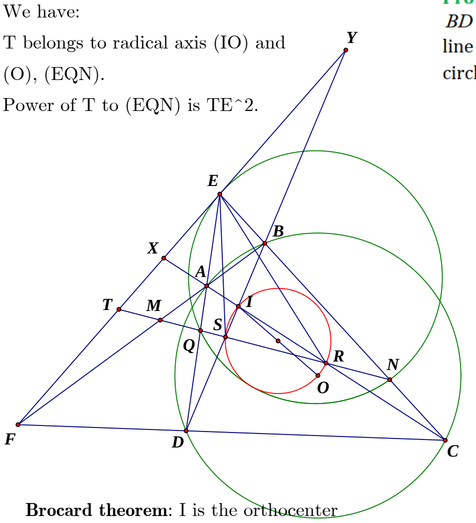
LD⊥AT, MO⊥AT --> AMK and HDL.

HN || AK iff DK/NK = **DA/AH**, but we know DA=2DM, AH=2ON --> **DA/AH** = DM/ON and we need: DK/NK = DM/ON which is true since ON || DM --> R is on AS --> (SD,BC)=-1.

We have:

T belongs to radical axis (IO) and (O), (EQN).

Power of T to (EQN) is TE^2.



Brocard theorem: I is the orthocenter of triangle EOF

Problem 7. Let $ABCD$ be a cyclic quadrilateral with O is circumcenter and AC meets BD at I. Suppose that rays DA, CD meet at E and rays BA, CD meet at F. The Gauss line of $ABCD$ meets AB, BC, CD, DA at points M, N, P, Q respectively. Prove that the circle of diameter OI is tangent to two circles $(ENQ), (FMP)$.

E(AB,IF) = -1 by Ceva, Menelaus; AC, BD cuts EF at X, Y --> (XY,EF)=-1, T is midpoint of EF. TE^2 = TF^2 = TX.TY (Newton). Note that: (AC,XI) = (BD,YI)=-1. RA^2=RC^2=RI.RX, <u>XA.XC=XI.XR</u> (Maclaurin).

IA.IC = IX.IR and IB.ID = IY.IS, but IA.IC=IB.ID --> IX.IR=IY.IS implies that XYRS is cyclic. **S, R are on (IO) --> X belongs to the radical axis of (O) and (IO), same with Y --> XY: radical axis.**

Power of T to (O) is TE^2.

TX.TY=TR.TS --> XYRS is cyclic. We need to prove: TQ.TN = TE^2.

We know that: ER, ES are isogonal (since EAC, EBD are similar)

∠TQE = ∠TEN iff ∠TES = ∠TRE (TE is tangent to (ERS)).

--> TE is tangent to (EQN). TU is the tangent line of (EQN)

--> TU^2 = TE^2 = TS.TR = TQ.TN.

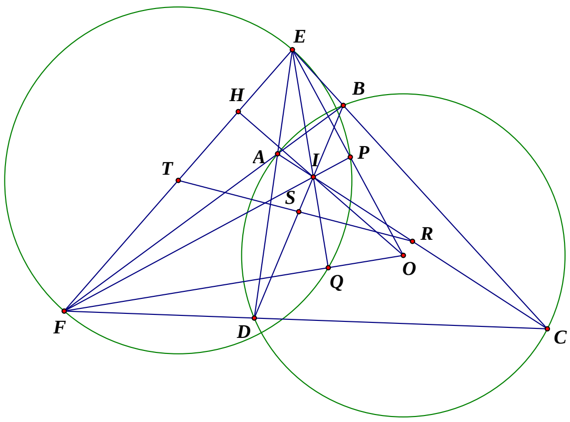
By harmonic bundles, E, I, U are collinear. Since TE = TU = TF --> EI ⊥ UF, but by Brocard, EI ⊥ OF --> F, U, O are collinear --> IUO = 90 --> U belongs to (IO) --> TU is common tangent line of (IO) and (EQN)

Remark about Brocard theorem:

OE, OF cuts (EF) at P, Q

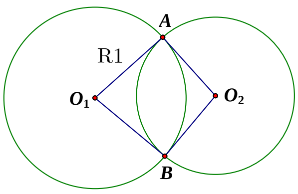
--> EQ⊥OF, but by Brocard's theorem: I is the orthocenter of triangle EOF.

E, I, Q are collinear; similar to F, I, P.



ABCD.EF is the completed quadrilateral.

Prove (EF) and (O) are orthogonal.



two circles:

- tangent;
- orthogonal.

~

two line:

- parallel.
- perpendicular

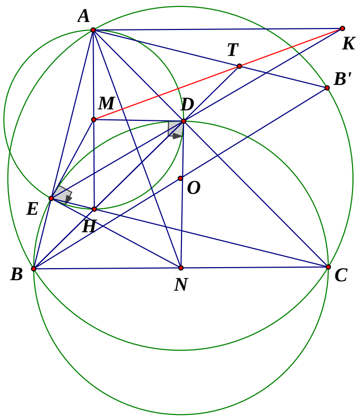
(O1), (O2) are orthogonal

iff ∠O1AO2 = ∠O1BO2 = 90.

iff O1A, O1B are tangent line of (O2).

iff power from O1 to (O2) is R1^2.

Problem set 7. Point circle



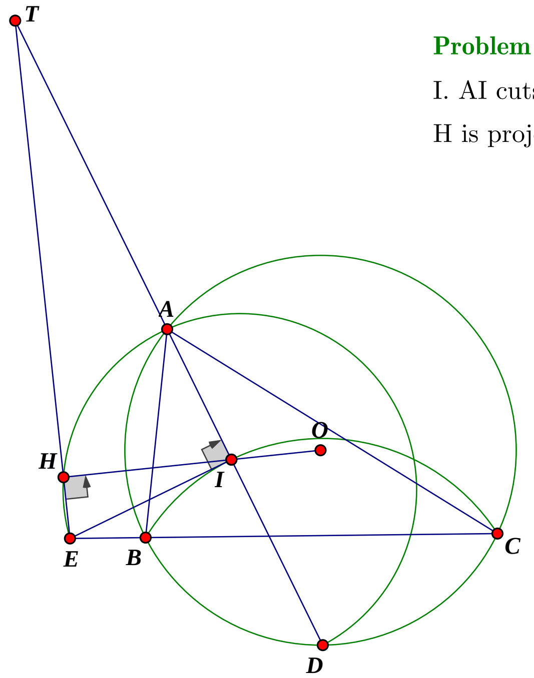
Problem 1: Let ABC be a triangle with BB' is the diameter of (O), AB' cuts BD at T. MT cuts DE at K. Prove that: AK || BC.

* "point circle" or "degenerate circle" ~ radius = 0.

M is the midpoint AH --> MD, ME are tangent lines of (BC) so MA^2 = MD^2 = ME^2 = power from M to (BC). --> The line through M, perpendicular to AN is the radical axis (d).

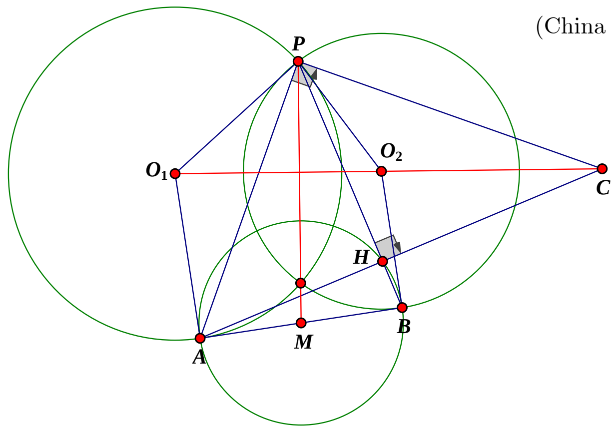
Note that ∠B'AC = ∠B'BC = ∠ABD, thus TA^2 = TB.TD implies that T lies on (d).

So MT = (d) and K lies on (d) --> KA^2 = KD.KE --> ∠KAD = ∠AED = ∠ACB --> AK || BC.



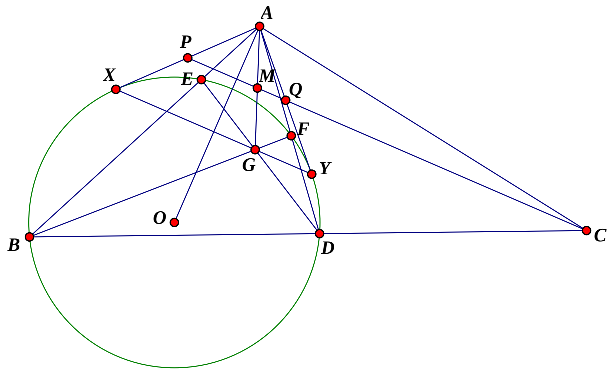
Problem 2: Let ABC be a triangle with circumcenter O, incenter I. AI cuts (O) againt at D. Take E on BC such that $\angle AIE = 90$, H is projection of E on IO. Prove that A, H, E, D are concyclic.

EI^2 = EB.EC --> E belongs to radical axis of (I,0) and (O).
EH $\perp$ OI --> EH is the radical axis.
EH cuts AD at T --> TI^2 = TA.TD,
but TI^2 = TH.TE --> TA.TD = TH.TE --> done.



Problem 3: Circle (O1), (O2) meet at P. AB is the common tangent line of these circles with A on (O1), B on (O2). Take C on O1O2 such that AC $\perp$ BP. Prove that AP $\perp$ PC. (China TST 2013)

$\angle$CPA = 90 iff CP^2 = CH.CA
CP^2 = power from C to (P;0).
CH.CA = power from C to some circle ~ (AB).
But we know that PM $\perp$ O1O2 and we need to prove C belongs to radical axist of (P),(M).
We just need to prove O1O2 is radical axis, and it follows by:
O1P^2 = O1A^2, O2P^2 = O2B^2.



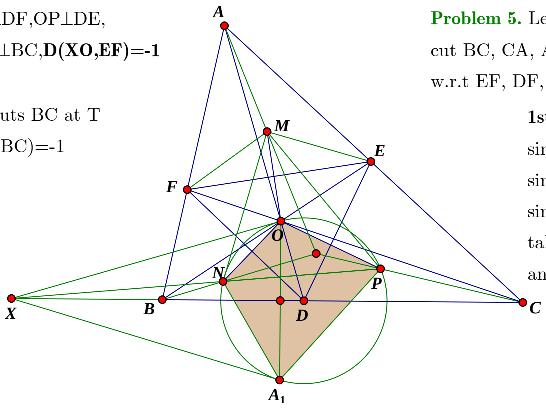
Problem 4: Let ABC be a triangle with $\angle B < \angle A$. Take D on BC such that $\angle CAD = \angle B$. A random circle (O) passes through B, D cuts AB, AD at E, F respectively. BF cuts DE at G, M is the midpoint of AG. Prove that CM $\perp$ AO.

By angle chasing --> CA is tangent to (ABD) --> CA^2 = CB.CD --> consider (A;0), (O) then C belongs to radical axis of them.
CM $\perp$ AO iff <u>M belongs to radical axis</u>.
Take the tangent AX, AY of (O) --> XY is the polar of A in (O) and X, G, Y are collinear.
Take P, Q are the midpoint of AX, AY --> P, M, Q are collinear and we know that PQ is the radical axis of (O) and (A;0) --> done.

ON$\perp$DF, OP$\perp$DE, OA1$\perp$BC, **D(XO,EF)=-1 (*)**

EF cuts BC at T

(TD,BC)=-1



Problem 5. Let ABC be an acute triangle with O lies inside it. AO, BO, CO cut BC, CA, AB at D, E, F respectively. M, N, P are the reflections of O w.r.t EF, DF, DE. Prove that AM, BN, CP are concurrent.

1st way: trigonometrical Ceva

$\sin\angle FAM$ / sin FAM = FM/EM. $\sin\angle MFA/\sin\angle MEA$.

$\sin\angle FAM$ / $\sin\angle MAE$. $\sin\angle AEM/\sin\angle MEF$. $\sin\angle MFE$/sinMFA=1.

$\sin\angle FAM/\sin\angle MAE = \sin\angle AFM/\sin\angle MEA.AF/AE.ME/MF$.

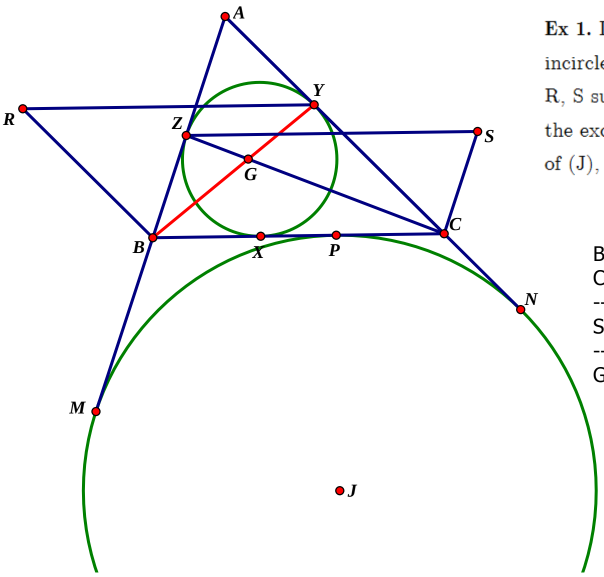
take the product of these three ratios --> done since $\prod(AF/AE)=1$ and $\prod(ME/MF)=\prod(OE/OF)=1$.

2nd way: Desargues theorem --> X = NP cuts BC, Y = PM cuts CA, Z = MN cuts AB; AM, BN, CP are concurrent iff X, Y, Z are collinear. A1 is the reflection of O w.r.t BC --> DO=DP=DN=DA1 so PONA1 is cyclic.

(*) --> O(OA1, NP)=-1:

OP cuts (D) at P, ON cuts (D) at N, OA1 cuts (D) at A1 and OO cuts (D) at O --> PONA1 is harmonic.

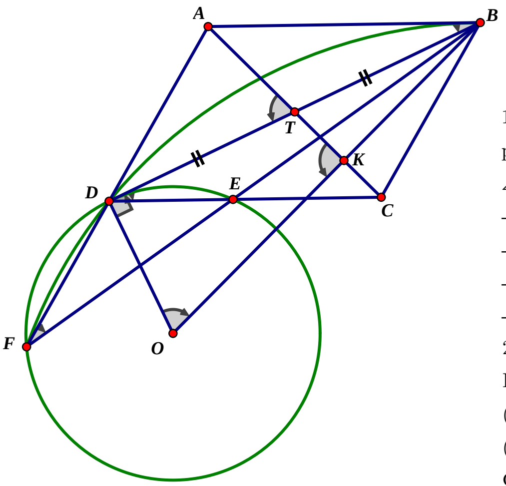
If we can prove **PONA1 is harmonic,** so XO, XA1 is the tangent lines of (D); BC is perpendicular bisector of OA1, BC cuts NP at X so XO, XA1 are tangent lines of (D) --> $XO^2 = XN.XP$ --> X belongs to the radical axis (O;0) and (MNP).



Ex 1. Let ABC be a triangle with X, Y, Z are tangent points of incircle with BC, CA, AB. Denote G as intersection of BY, CZ. Take R, S such that **BCYR** and **BCSZ** are parallelograms and denote (J) as the excircle of ABC w.r.t angle A. Prove that BY is the radical axis of (J), point circle R and then, prove that GR = GS.

(IMO Shortlist 2009)

BR=CY=CX=BP=BM = tangent from B to (J)
CY=s-b, CN=s-c --> YN = a = BC = YR
--> BY is the radical axis of (J) and (R).
Similarly, CZ is the radical axis of (J) and (S).
--> G = BY cuts CZ --> G is the radical center -->
GR = GS.



Ex 2. Let ABCD be a parallelogram and AC cuts BD at T. A circle of center O, radius OD is tangent to BD at D and cuts segment CD at E, cuts ray AD at F. Suppose that B, E, F are collinear. Prove that $\angle ATD = \angle DOB$.

1st way: DB is tangent line of (O) --> $TB^2 = TD^2 =$ power from T to (O).

$\angle ABD = \angle BDE = \angle DFE = \angle DFB$ (since B, E, F collinear)

--> AB is tangent to (BDF).

--> $AB^2 = AD.AF$ = power from A to (O).

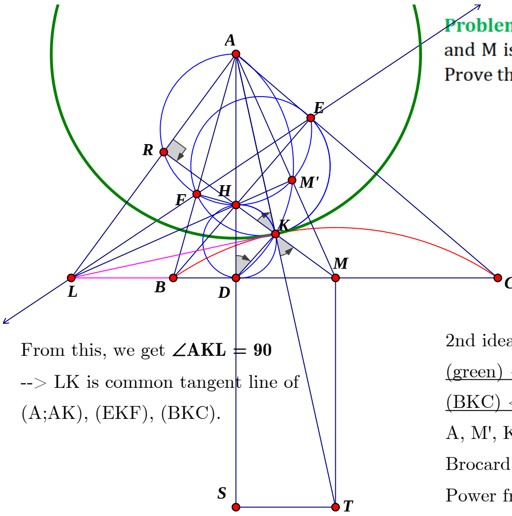
--> AC is the radical axis of (O) and (B).

--> DTKO is cyclic --> done.

2nd way: need to prove $\angle TKO = 90$ by 4 points lemma $BC^2 - BT^2 = OC^2 - OT^2$

($OT^2 = OD^2 + DT^2 = OD^2 + BT^2$),

($OC^2 = OD^2 + CE.CD$: power from C to (O)) iff CD.CE = CB^2 iff CB is tangent to (BDE)

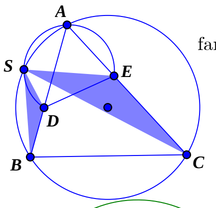


Problem 1. Let ABC be an acute, non isosceles triangle with orthocenter H, altitude AD and M is the midpoint of BC. On the ray HM, take K such that AK is tangent to (HK~~M~~). D
Prove that the circle of center A, radius AK is tangent to (KBC).

1st idea: AK cuts perpendicular bisector of BC at T --> need to prove T is the center of (BKC) (*) Take power from D: DB.DC = AD.DH =AD.(AD-AH)=AD^2 - AD.AH = AD^2 - AK^2
(*) iff DB.DC = TK^2 - DT^2 iff **AD^2** - AK^2 = TK^2 - DT^2.
AT^2 = AS^2+**ST**^2=AD^2+TM^2+2AD.MT+**DM**^2.
TM^2+DM^2 = TD^2, AT^2 = (AK+KT)^2 ...
--> AK.TK = AD.MT, true since △ADK~△TKM (a.a).

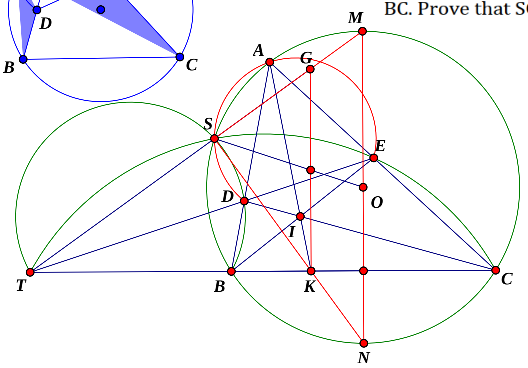
From this, we get **∠AKL = 90**
--> LK is common tangent line of (A;AK), (EKF), (BKC).

2nd idea: take the inversion of center A, power AK^2 then:
(green) <-> (green), H <-> D, K <->K, B <-> F, C <-> E
(BKC) <-> (EKF). So need to prove: (EKF) tangent to (green)
A, M', K, D are concyclic.
Brocard theorem for BCEF.AL --> H is the orthocenter of AML.
Power from L to (KEF): LE.LF=LB.LC=**LR.LA**=LD.LM
Power from L to (A;AK): LA^2-AK^2=AL^2-AH.AD=AL^2-AL.AR=**AL.LR**



familiar figure

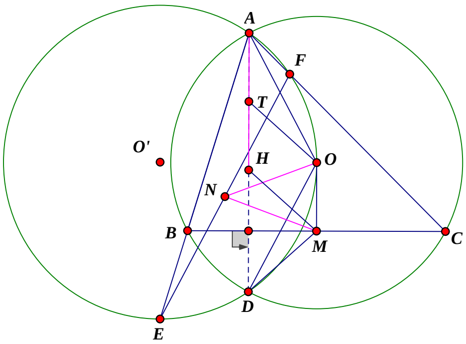
Problem 2. Let ABC be a triangle inscribed in circle (O). On the segments AB, AC, take D, E such that AD=AE. Suppose that BE cuts CD at I, AI cuts BC at K and EF cuts BC at T. Circles (TBD), (TCE) meet again at S. On the ray TS, take G such that GK perpendicular to BC. Prove that SO bisects GK.



By Ceva --> (TK, BC)=-1.
Miquel's point for BDEC.AT --> S is on (ABC), (ADE).
AD = AE --> **BD/CE = BK/CK** (Ceva).
S(TK,BC)=-1 project onto (O) --> BMCN is harmonic.
We need to prove M, N are midpoints of arc BC of (O) iff ST, SK are the angle bisectors of ∠BSC.
1st way: angle chasing.
2nd way: S is the center of spiral similarity --> SDB similar SEC --> SB/SC = BD/EC = BK/CK --> SK is the internal angle bisector and ST is the external angle bisector.

Problem 3. Let ABC be a triangle inscribed in circle (O), M is the midpoint of BC and point D on (O) such that AD perpendicular to BC. (AOD) cuts AB, AC again at E, F respectively. Prove that the midpoint of EF is equidistance to O, M. **NO = NM**

DBC, DEF are similar
(O, O'), (M, N) are corresponding points.

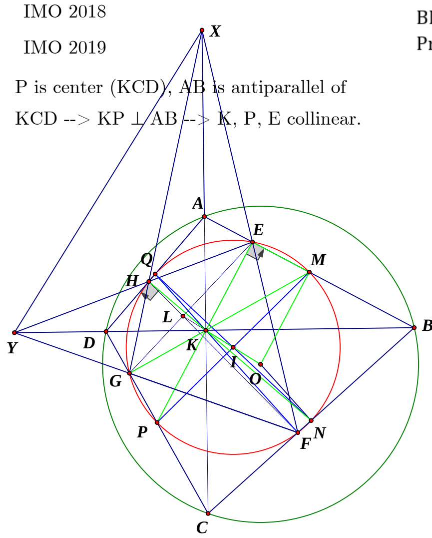


We have: OD || EF since AH, AO are isogonal
--> EFDO is isosceles trapezoid --> NO=ND. (1)
Take H is the orthocenter and T is midpoint of AH
--> TH=OM --> TO=HM=DM --> TOMD is also isosceles trapezoid.
D is the center of spiral similarity:
B --> E, C --> F: DBC --> DEF.
O is center of (DBC), O' is the center of (DEF)
O --> O', M is midpoint BC --> N is midpoint EF.
△DMN ~ △DOO', but O'O=O'D --> NM=ND. (2)
(1) & (2) --> done.

IMO 2018

IMO 2019

P is center (KCD), AB is antiparallel of KCD --> KP ⊥ AB --> K, P, E collinear.



Problem 4. Let ABCD be a quadrilateral inscribed in circle (O) with AC perpendicular to BD at K. Denote E, F, G, H as projections of K onto AB, BC, CD, DA and EG cuts FH at L. Prove that O, K, L are collinear.

Angle chasing --> KE is the angle bisector of ∠HEF. Similar with KF,KG,KH --> K is incenter of EFGH; EFGH is cyclic --> bicentric quadrilateral.
M, N, P, Q are midpoints of sides --> QK ⊥ BC (by angle chasing / isogonal) --> QK || ON, similar to KN || OQ --> **ONKQ is parallelogram** --> QN bisects OK; the same with MP. **MNPQ is rectangle** --> midpoint of QN, PM is the center of (MNPQ) --> K, I, O are collinear.

<u>2nd way</u>: KEMO is right trapezoid --> take I is the midpoint of KO --> IE = IM; similar: IF=IN, IP=IG, IQ = IH --> 8 points belongs to circle of center I.

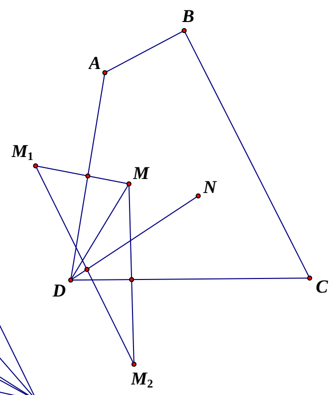
<u>3rd way:</u> K, O are isogonal conjugate in ABCD --> 8 points are concycylic.

Angle chasing: DHEB is cyclic (w1) since ∠AEH = ∠AKH = ∠ADK --> HE is the radical axis of (w1) and (I); GF is radical axis of (w2)=(DGFB) and (I) --> HE, GF, BD are concurrent at Y; EF, AC, GH concurrent at X.

By Brocard theorem: EFGH.XY --> IL ⊥ XY; XY is the radical axis of (O), (I) --> XY ⊥ OI --> L is on IO --> I, O, K, L are collinear.

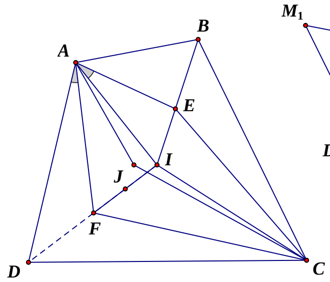
Remark about isogonal conjugate in quadrilateral :

In triangle ABC, if point M is not on (ABC), BC, CA, AB --> there exist unique N s.t (M, N) are isogonal conjugate
But in quadrilateral ABCD, take any point M --> not sure there exist N (aops) --> there exist M iff ∠AMB+∠CMD=180 iff ∠BMC+∠DMA=180. (*)



Example. ABCD with incenter I. Take E on IB, F on ID s.t ∠EAF = 1/2. ∠DAB. Prove that ∠ECF = 1/2. ∠BCD.

We have: I, F are isogonal conjugate in AECD since I satisfy the condition (*), I, F are on angle bisector of ∠CDA, ∠IAE = ∠FAD --> CI, CF are isogonal --> done.



We need to check the statement: if M, N are isogonal conjugate in ABCD then 8 projections of M, N onto sides of ABCD are concyclic --> true

Problem 5. Let ABC be a triangle inscribed in circle (O) with M is the midpoint of BC. Take D on BC, differs from M, such that (ABD) cuts segment AC at E, circle (ACD) cuts segment AB at F.

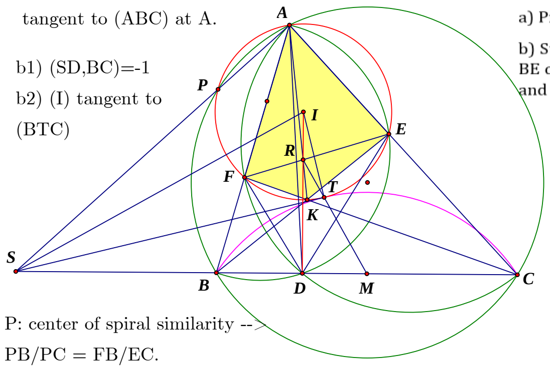
a) Prove that if AD is the symmedian then EF || BC and DE = DF.

b) Suppose that AD is not the symmedian, (AEF) cuts (O) again at P and AP cuts BC at S. BE cuts CF at K and AK cuts EF at R. Segment RM cuts (AEF) at T. Prove that (SD,BC)=-1 and (AEF) is tangent to (BTC).

AD is symmedian iff EF || BC iff (AEF) is tangent to (ABC) at A.

b1) (SD,BC)=-1

b2) (I) tangent to (BTC)



a) **1st way:** ∠EBD = ∠DAE = ∠MAB; ∠FCD = ∠FAD = ∠MAC --> Ceva sin --> BE, CF, AM are concurrent --> EF || BC.
Angle chasing --> DE=DF.

2nd way: CD.CB = CE.CA, BD.BC = BF.BA, we know BD/CD = (AB/AC)^2 --> CE/BF = AC/AB --> EF || BC.

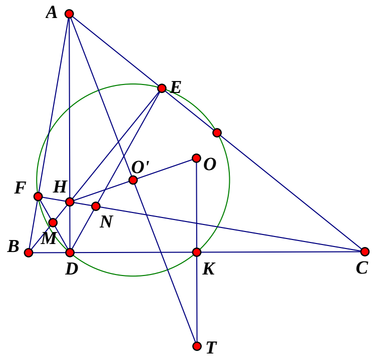
△SPB ~ △SCA: SB/SA=PB/CA=SP/SC; △SPC ~ △SBA: SC/SA=PC/AB=SP/SB; --> divide them: SB/SC = PB/PC.AB/AC=FB/EC.AB/AC=DB/DC.
--> (SD,BC)=-1. We have another way by inversion center A, any power

P: center of spiral similarity --> PB/PC = FB/EC.

SP.SA=SB.SC=SD.SM ?=? ST^2.

AEKF.BC, Brocard theorem --> IR ⊥ BC, point D is Miquel point --> I, R, D are collinear --> DR.DI = DB.DC (since R is the orthocenter of IBC) = DM.DS --> R is orthocenter of triangle ISM --> RM ⊥ IS --> TM ⊥ IS; polar (d) ⊥ IS --> d passes through T --> ST tangent to (I) --> ST is the common tangent of (I) & (BTC).

BC is the polar of R in (I), and S is on BC --> polar (d) of S will passes through R.



Problem 1. Let ABC be a non-isosceles triangle with altitudes AD, BE, CF with orthocenter H. Suppose that $DF \cap HB = M, DE \cap HC = N$ and T is the circumcenter of triangle HBC. Prove that $AT \perp MN$.

MD.MF = MB.MH, similar with N

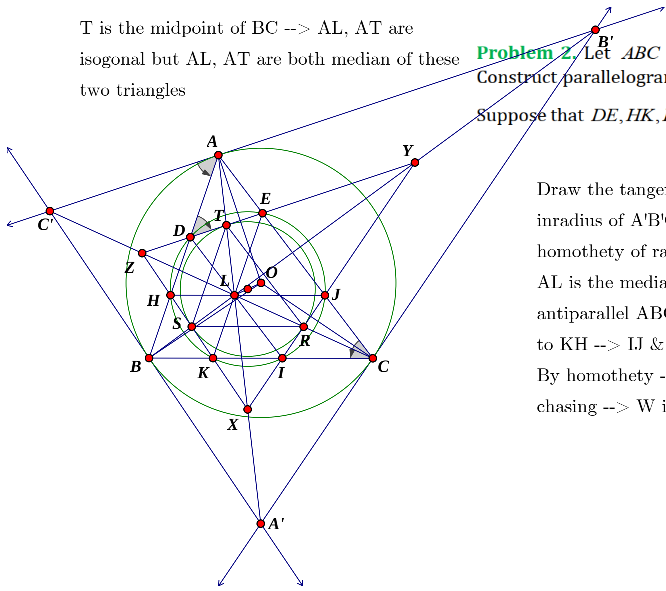
--> MN is the radical axis of (DEF) & (HBC)

center of (DEF) is O', midpoint of HO.

MN ⊥ O'T, AHTO is parallelogram --> O',

A, T are collinear --> AT ⊥ MN.

T is the midpoint of BC --> AL, AT are isogonal but AL, AT are both median of these two triangles

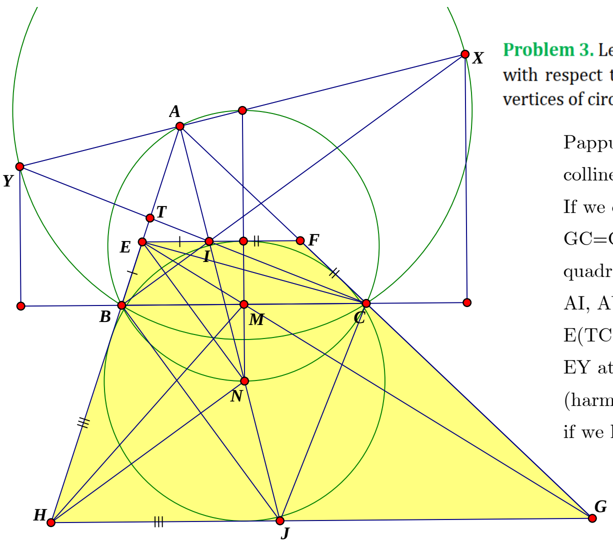


Problem 2. Let ABC be triangle with the symmedian point L and circumradius R. Construct parallelograms $ADLE, BHLK, CILJ$ such that $D,H \in AB; K,I \in BC; J,E \in CA$. Suppose that DE, HK, IJ pairwise intersect at X,Y,Z. Prove that inradius of XYZ is $\frac{R}{2}$.

Draw the tangent of (O) at A, B, C and they intersect at A', B', C' --> inradius of A'B'C' is R --> need to prove A'B'C' and XYZ are homothety of ratio 2, center L.

AL is the median of ADE, also the symmedian of ABC --> DE is antiparallel ABC. IJ bisects LC & IJ || A'B' --> IJ bisects LA'; similar to KH --> IJ & KH both bisects LA' --> X is the midpoint of LA'.

By homothety --> midpoint W of LO is the incenter of XYZ. By angle chasing --> W is center of DEJIKH.



Problem 3. Let ABC be triangle with M is the midpoint of BC and X,Y are excenters with respect to angle B,C. Prove that MX, MY intersect AB, AC at four points are vertices of circumscribe quadrilateral.

Pappus's theorem --> (XAY) & (BMC) --> E, I, F are collinear. We will prove EF || BC || HG (*)

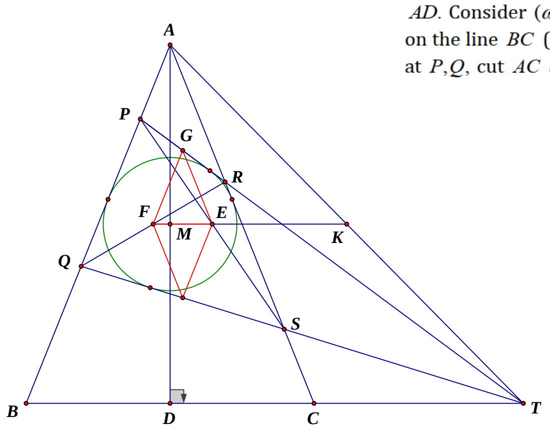
If we can prove (*), then angle chasing --> EI = EB, FI=FC, GC=GJ, HJ = HB --> EF+GH=EH+FG --> circumscribe quadrilateral.

AI, AY are angle bisectors --> (TC, IY)=-1

E(TC,IY)=-1, the line BC cuts ET at B, cuts EC at C, cuts EY at M and M is the midpoint of BC --> EI || BC.

(harmonic) & (parallel) & (midpoint)

if we have two then we can think the other



Problem 4. Let ABC be a triangle with $AB = AC$ and M is the midpoint of the altitude AD. Consider (ω) as the circle of center M and tangent to AB, AC. From some point T on the line BC (outside triangle ABC), construct two tangents of (ω) and they cut AB at P, Q, cut AC at R, S. Prove that $PQ = RS$.

Consider completed quadrilateral PQSR.AT.

Newton's theorem in circumscribe quadrilteral PQSR --> center M, midpoints E, F of PS, QR are collinear.

But E, F and midpoint K of AT are collinear (by Gauss) --> M, E, F, K are collinear --> MK ⊥ AD.

Take G is the midpoint of PR --> GE = RS/2, GF = PQ/2

GE || AC, GF || AB, EF || BC --> GEF is isosceles, done.

<u>**We also have: AP = SC, AR = BQ**</u> --> AR.SC = AP.BQ

Menelaus for AQS with P, R, T collinear:

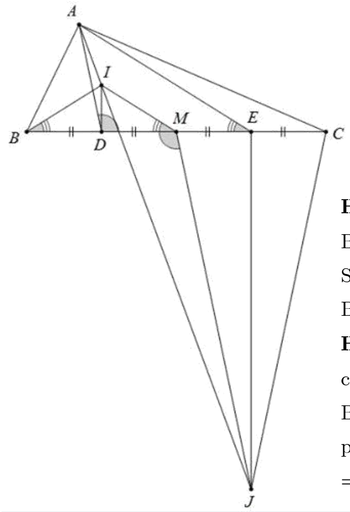
PA/PQ.RS/AR.TQ/TS=1.

TSQ and line TBC --> BQ/BA.CA/CS.TS/TQ = 1

AR/AP = BQ/CS --> AR/BQ = AP/CS.

PA+BQ = AR+SC --> (AR-BQ)/BQ=(AP-CS)/CS

but BQ > CS --> AR=BQ & AP=CS.



Problem 5. Let ABC be a triangle with D, E are tangent points of incircle and excircle respect to vertex A with the segment BC. Suppose that $BC = 2|AB - AC|$, prove that $\angle BAC = 2\angle DAE$.

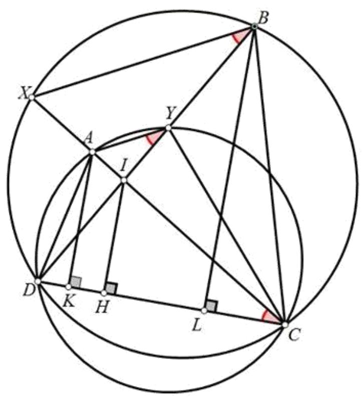
Problem 6. Let $ABCD$ be a convex quadrilateral and non-cyclic. Denote X_a as the product of $\mathcal{P}_{A/(BCD)}$ with S_{BCD}. Defind X_b, X_c, X_d similarly. Prove that

$$|X_a| = |X_b| = |X_c| = |X_d|.$$

Hint 5. Take J as ex-center of ABC and M is midpoint of BC. Then we know that DB=DM, IM || AE, JM || AD. Suppose that AB < AC, then 2BC=AC-AB, so BD=DM=ME=EC. So by angle chasing, we get conclusion.

Hint 6. Take the intersection points as picture. By angle chasing, we have AY || XB, thus

BY/AX = IY/IA = IC/ID. Denote H, K, L as the projections then: S[BCD]/S[CDA]=BL/AK=BL/IH.IH/AK = DB/DI.CI/CA=BD/AC.IC/ID = BD.BY/AC.AX = pow(B,(CDA))/pow(A,(BCD)) --> |Xa|=|Xb|.



Problems for self-training

A. Harmonic points.

Problem 1. Let $ABCD$ be a quadrilateral inscribed in a circle (O) and E,F,K be the intersections of AC and BD, AD and BC, AB and CD. Denote M as the midpoint of CD and suppose EF intersects the circumcircle of MAB, the line AB,CD at N,J,I. Prove that I belongs to the circumcirlce of ABM and $\frac{MA}{MB}=\frac{NA}{NB}$.

Problem 2. Let ABC be a non isosceles triangle with incircle (I). This circle is tangent to BC,CA,AB at D,E,F respectively. Suppose that AD intersect (I) at the second point P with $\angle BPC=90°$. Denote $\{K\}=BC\cap EF,\{P,Q\}=PC\cap(I)$. Prove that $QP=QD$ and $EA+AP=PD.$

Problem 3. Let ABC be a triangle with M lies inside this triangle. Suppose that AM,BM,CM intersect BC,CA,AB at D,E,F respectively. Take a point X on BC such that $\angle AMX=90°$ and Y,Z are symmetric point of X respect to line DE,DF. Denote T is the symmetric point of M respect to BC.

a) Prove that XM,XT are tangent lines of (YZT).

b) Prove that X,Y,Z are collinear.

Problem 4. Let ABC be a triangle with AH is the altitude and K is the midpoint of segment AH. The incircle (I) of ABC is tangent to BC at D. Suppose that DK intersects (I) at the second point T. Prove that (TBC) is tangent to (I).

B. Pole and anti-pole.

Problem 1. Let ABC be a triangle with I lies inside triangle. Suppose that the rays IA,IB,IC intersect BC,CA,AB at D,E,F respectively. A circle (O) passes through D,E,F intersect AD,BE,CF at the second points M,N,P. Suppose that tangent lines of (O) at D,M intersect each other at X, tangent lines of (O) at E,N intersect each other at Y, tangent lines of (O) at F,N intersect each other at Z. Prove that X,Y,Z collinear and the line passes through these point perpendicular to IO.

Problem 2. Let ABC be an acute triangle with AD,BE,CF are altitudes and H is the orthocenter. Suppose that DE,CF intersect each other at M and DF,BE intersect each other at N. Prove that the line passes through A and perpendicular to MN passes through the circumcircle of triangle BCH.

Problem 3. Let ABC be a triangle with incircle (I). Suppose that $(I)$is tangent to BC,CA,AB at D,E,F. On the segment BC, take a point M such that $IM \parallel EF$. On the segment AC, take a point N such that $IN \parallel DF$. Denote P, Q as the projections of D on EF and projection of E on DF respectively.

a) Prove that $IP \perp AM, IQ \perp BN$.

b) Prove that AM, BN, IF are concurrent.

Problem 4. Let ABC be a triangle with (I) is the incircle. Denote D, E, F as the tangent points of (I) on BC, CA, AB respectively. Suppose that DE intersects AB at P. A line passes through C and intersects AB, EF at M, N respectively. Two lines PN, AC intersect each other at Q. Prove that IM is perpendicular to FQ.

Problem 5. Let $ABCD$ be a quadrilateral and suppose that (O) is a circle which tangent to AB, BC, CD, DA at M, N, P, Q respectively. Denote d as a line passes through C and perpendicular to OC and E, F as the intersections of NQ, MP with d.

a) Suppose that AD, BC, MP are concurrent, prove that EB, FD, OC are concurrent.

b) Suppose that EB, FD, OC are also concurrent, are AD, BC, MP also concurrent?

C. Inversion.

Problem 1. Let $ABCD$ be a quadrilateral inscribed a circle (O). The tangent line of (O) at T intersects AB, AC at M, N. Prove that $\frac{1}{DM} + \frac{1}{DN} = \frac{BC}{DB \cdot DC}$.

Problem 2. Let s be the semiperimeter of a triangle ABC. Points E, F are taken on line AB such that $CE = CF = p$. Prove that the circumcircle of EFC is tangent to the excircle of triangle ABC corresponding to AB.

Problem 3. Let (O) be a fixed circle with a fixed diameter AB, take a fixed point I on the segment AB differs from A, B. An arbitrary line d (differs from AB) passes through I intersects (O) at P, Q. The line m is the tangent line of (O) at B and AP, AQ intersects m at M, N.Prove that the center of (AMN) belongs to a fixed line.

Problem 4. Let ABC be a triangle with circumcircle (O) and incircle (I). Let A_0, B_0, C_0 be the tangent point of BC, CA, AB and $(O_a), (O_b), (O_c)$ be the circumcircle of triangle AB_0C_0, BC_0A_0 and CA_0B_0. Suppose that A_1, B_1, C_1 are the second intersections of $(O_a), (O_b), (O_c)$ and (O), respectively. Prove that A_0A_1, B_0B_1, C_0C_1 are concurrent at a point N which belongs to the Euler line of triangle $A_0B_0C_0$.

D. Mixtilinear circle.

Let ABC be a triangle with O is the circumcenter, I is the incenter. Denote $(M_a),(M_b),(M_c)$ as the internal Mixtilinear circle respect to vertex A,B,C.

Problem 1. Denote X as the tangent point of (M_a) and (O). Prove that AO is the tangent line of circle (AIX).

Problem 2. Denote $\angle xAy$ be an angle and a fixed circle (I) that tangent to Ax, Ay. An arbitrary tangent line of (I) intersects the rays Ax, Ay at B, C. Prove that the circle (ABC) always tangent to a fixed circle.

Problem 3. Let A_1 be the tangent point of (O) and (M_a) and A_1A_2 is the diameter of (M_a). Define B_2, C_2 similarly. Prove that AA_2, BB_2, CC_2 are concurrent.

Problem 4. Let X, Y, Z be the tangent point of (M_a) with $(O), AB, AC$. Suppose that AX intersects YZ at N and IX intersects BC at M. Prove that $MN \parallel AI$ and the circle (XNI) is tangent to (O).

Problem 5. Let T be the tangent point of $(O),(M_a)$. Denote x, y, z as the length of tangent segments of (M_a) that pass through A, B, C respectively. Prove that

$$\frac{x}{TA}=\frac{y}{TB}=\frac{z}{TC}.$$

Problem 6. Let X be the tangent point of (O) and (M_a). Prove that XM_b, XM_c are symmetric respect to the bisector of angle BXC.

Problem 7. Let ABC be a fixed triangle with $\angle A < \angle B < \angle C$ and (O) is a circumcircle. On the minor arc BC of (O), take an arbitrary point D. Suppose that CD intersects AB at E and BD intersects AC at F. Denote (O_1) as the center of circle lies inside triangle EBD, tangent to EB, ED and also tangent to (O). Denote (O_2) as the center of circle lies inside triangle FCD, tangent to FC, FD and also tangent to (O).

a) Let M be the tangent point of (O_1) with BE and N be the tangent point of (O_2) with CF. Prove that the circle with diameter MN always passes through a certain fixed point.

b) The line passes through M and parallel to CE intersects AC at P, the line passes through N and parallel to BF intersect AB at Q. Prove that the circumcircles of triangle AMP, ANQ both tangent to a fixed circle.