— Combinatorics for L3 —

— April 13, 2019 — Graph theory (1) —

Definition: One of the most common and formal definition of the term **graph** is the following: graph is a pair of 2 sets V and E, where V refers to the set of **vertices** and E refers to the set of **unordered pairs of vertices**, which are called **edges**.
Degree of a vertex is the number of vertices that are incident to this vertex.

1. There are 50 towns and 100 roads in the country. How many roads are coming out from each town, if it is known that this number is equal for all of them?

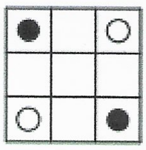
2. Does there exist a graph in which the set of degrees of the vertices is

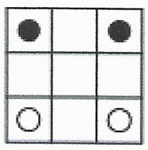
a) $\{9, 8, 8, 7, 6, 6, 3, 2, 1\}$

b) $\{8, 7, 6, 5, 4, 4, 3, 2, 1\}$

c) $\{8, 8, 7, 6, 6, 5, 4, 3, 2\}$?

3. Two black and two white knights are placed on a 4×4 board as it is shown on the right picture. Several moves were done. Is it possible that now the knights are placed on the 4×4 board as it is shown on the left picture?





Two fairly well-known facts:

- In a group of people each person has two friends (as usual friendship is mutual). Prove that if each person takes hands of both of his friends then the whole group will be split into the "round dances".
- Prove that the sum of degrees of all vertices of an arbitrary graph is an even number.

4. Does there exist 7 english words such that each of them has one common letter with exactly three other words?

5. Twenty teams take part in the football competition. After all teams have played two games, the organizers will decide to split them into three different groups in such a way, that there will not exist two teams from the same group which have already played with each other. Prove that it will always be possible to do so.

6. Number 1, 2, ..., 15 are written on both sides of 15 different cards (only one number on each side of the card). Each number is used exactly two times. Prove that it is possible to put these cards on the table in such a way, that each number occurs exactly one time.

7. **(Homework)** Prove that in every company of people there exist two distinct people who know the same number of others in this company.

8. **(Homework)** A set of 20 problems was given to 20 children. It happened that each child has solved exactly two problems and each problem has been solved by exactly 2 children. Prove that it is possible to arrange parsing of the problems in such a way that each person will parse exactly one problem and each problem will be parsed only once.

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— April 14, 2019 — Graph theory (2) —

Definition: A **cycle** of length k is a set C of $k \geq 3$ distinct edges in E, such that there is a sequence of not necessarily distinct vertices $v_1, v_2, \ldots, v_k$ such that for each $1 \leq i \leq k$ $\{v_i, v_{i+1}\}$ is in C, and $\{v_k, v_1\}$ is in C. If do not write that "... and $\{v_k, v_1\}$ is in C" in the definition of a cycle, you get the definition of a **path** of length k. To get the definitions of a **simple cycle** and a **simple path** you need to specify that $v_1, v_2, \ldots, v_k$ are pairwise distinct.
Graph is **connected** iff there is a path between any two vertices. And finally, a **tree** is a connected graph without cycles.

9. Prove that in a tree there is only one simple path between any two vertices.

10. Prove that if in a graph G there is only one simple path between any two vertices, then G is a tree. From the previous problem and this problem conclude that "a graph is a tree $\Leftrightarrow$ there is only one path between any two its vertices."

11. a) Prove that in any tree there is at least one vertex of degree 1 (this vertex is called **a leaf**).
b) Prove that in any tree with at least one edge there are at least two leaves.

12. Prove that if in a tree an arbitrary edge was removed, the remaining graph will not be connected.

13. Prove that in a tree with n vertices there are exactly $n-1$ edges.

14. An arbitrary connected graph is given. Prove that it is possible to delete some edges of this graph in a way that the remaining graph will be a tree.

A **spanning tree** of a graph G is a tree which includes all of the vertices of G. There can be several spanning trees for a graph G. The previous problems shows that for any connected graph there is a spanning tree (for not connected graph it is obvious that there is no spanning tree :)).

15. Prove that if a connected graph has exactly n vertices and n -1 edges than this graph is a tree.

From the previous problem it follows that if there are n vertices and we want to draw as less as possible edges to get a connected graph, then we will draw a tree.

16. A net looks like a 5×10 rectangle. Find the biggest number of strings (=sides of the cells) that can be cut out from the net such that the net will not divide into several pieces.

17. Given connected graph G. Prove that it is possible to delete one of the vertices and all edges which come out from it such that the remaining graph will be still connected.

18. There are 100 towns in a country with some routes between them. It is known that there is a path between any two towns in this country. Prove that there is a path which contains all of the towns and in which you need to go on less or equal than **a)** 198 **b)** 196 routes.