Problem 1. The square 20×20 was cut along the grid lines into several smaller (not necessarily different) squares. For each square it's sidelength was written on the board.

Determine the maximal possible amount of different numbers on the board.

Problem 2. Let $n > 2$ be an integer such that n does not divide any element of the set $\{2^n - 1, 3^n - 1, \ldots, (n-1)^n - 1\}$.

(a) Prove that n is a square-free number

(b) Does it necessarily follow than n is a prime number?

Problem 3. Determine all pairs of polynomials (P, Q) with real coefficients satisfying

$$P(x + Q(y)) = Q(x + P(y))$$

for all real numbers x and y.

Problem 4. Point M lies on the side AB of circumscribed quadrilateral $ABCD$. Points I_1, I_2 and I_3 are incenters of $\triangle MBC$, $\triangle MCD$ and $\triangle MDA$.

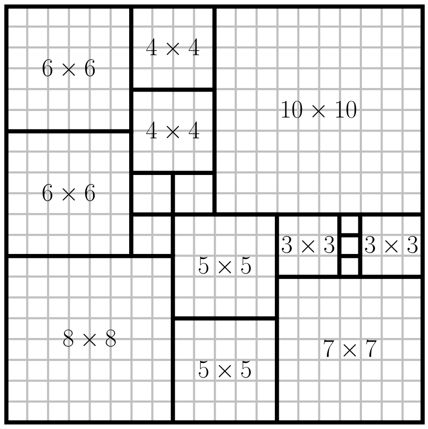
Prove that the points M, I_1, I_2 and I_3 lie on a circle.

(Time: 1:00 pm – 5:00 pm)

1. Answer : 9.

The minimal possible sum of ten distinct perfect squares is $1^2 + 2^2 + \ldots + 10^2 = 385$. The next such sum contains at least one perfect square not less than 11^2, hence it equals $385 - 10^2 + 11^2 = 406 > 400$. Therefore the maximal possible amount of different sizes is at most 10 and, moreover if it equals 10 then the sidelengths are $1, 2, \ldots, 10$.

Suppose there exists a cut with 10 different sidelengths. Mark the squares with sides 6, 7, 8, 9 and 10, one of each kind. Consider the sum of projections of all marked squares to the side AB of the initial 20×20 square. This sum equals $6+7+8+9+10 = 40 = 2\cdot 20$. Hence by the Dirichlet's principle either 1) each unit segment of AB is covered with exactly 2 projections; or 2) some unit segment of AB is covered with at least 3 projections. Clearly, second case is impossible since the sum of projections of the corresponding marked squares to the other side of 20×20 square exceed 20 and they overlap. Consider first case. Let the projections of $a \times a$ and $b \times b$ squares cover the leftmost unit segment of AB and the projections of $c \times c$ and $d \times d$ squares cover the rightmost unit segment of AB. Then the $(\max(a, b) + 1)$-th on the left unit segment of AB can be covered only by the projection of the remaining fifth marked square — a contradiction with the assumption of this case.



Hence there can be at most 9 different sidelengths. The example above shows that such cut is possible (all sidelenghts exceeding 2 are marked).

2. Answer : b) no.

a) Suppose n is not a square-free and let $n = pa$ where $p \mid a$. Then $a+1 \equiv 1 \pmod p$ whence $(a+1)^{n-1} + (a+1)^{n-2} + \ldots + 1 \equiv n \equiv 0 \pmod p$. So the number $(a+1)^n - 1 =$ $= a((a+1)^{n-1} + (a+1)^{n-2} + \ldots + 1)$ is divisible by $ap = n$, a contradiction.

b) Take for example $n = 15$. Consider any $a^n - 1$ from $\{2^{15}-1, 3^{15}-1, \ldots, 14^{15}-1\}$ and suppose $m \mid a^{15} - 1$. Clearly if $p \mid a$ then $p \nmid a^{15} - 1$ hence $\gcd(a, 15) = 1$. Since $\varphi(15) = 8$ we have $a^{15} \equiv 1 \pmod{15} \iff a^{16} \equiv a \pmod{15} \iff 1 \equiv a \pmod{15}$ which is impossible.

3. Answer : Either $P \equiv Q$ or $P(x) = x + a$ and $Q(x) = x + b$ for some real a and b.

If either P or Q is constant then clearly $P \equiv Q$. Suppose neither P nor Q is constant. Write $P(x) = ax^n + bx^{n-1} + R(x)$ and $Q(x) = cx^m + dx^{m-1} + S(x)$ with $n, m > 1$, $a \neq 0 \neq c$, $\deg R < n - 1$, $\deg S < m - 1$. The leading term of $P(x + Q(y))$ with respect to x is ax^n and the leading term of $Q(x + P(y))$ with respect to x is cx^m. Hence $m = n$ and $a = c$. The x^{n-1}-coefficient of the polynomial $P(x + Q(y))$ is $anQ(y) + b$ and the corresponding coefficient of $Q(x + P(y))$ is equal to $anP(y) + d$. Therefore $anQ(y) + b = anP(y) + d$ for every $y \in \mathbb{R}$. Hence $P(x) = Q(x) + t$ where $t = \frac{b-d}{an}$ is a constant. If $t = 0$ then $P \equiv Q$ and such polynomials satisfy required conditions. Suppose $t \neq 0$ and substitute $P(x) = Q(x) + t$ to $P(x + Q(y)) = Q(x + P(y))$ to obtain $Q(x + Q(y)) + t = Q(x + Q(y) + t)$. Since $x + Q(y)$ attains all real values, we can write $Q(x) + t = Q(x + t)$. This implies that $Q(kt) = Q(0) + kt$ for $k \in \mathbb{Z}$, therefore

$Q(x) = Q(0) + x$ and $P(x) = x + Q(0) + t$ for every real x. Such polynomials clearly satisfy required conditions.

Alternative solution. Substitute $x = -P(y)$ to obtain $P(Q(y) - P(y)) = Q(0)$. If $Q(y) - P(y)$ is not a constant, then $Q(y) - P(y)$ is a polynomial which takes infinitely many values, so $P(x) = Q(0)$ with infinitely many x, hence $P(x)$ is a constant. In this case P and Q are two equal constant polynomials which is a valid solution. Suppose $Q(y) - P(y) = c$ where c is a constant. If $c = 0$ then $P \equiv Q$ which is also a valid solution. If $c \neq 0$, we get $P(x + P(y) + c) = P(x + P(y)) + c$. Similarly to the first solution we can write x instead of $x + P(y)$, get $P(x + c) = P(x) + c$ and finish the solution.

4. We will use the following well-known fact:

Fact 1. *Let ω_1, ω_2 and ω_3 be three pairwise nonintersecting circles. Denote by t_{12} the length of common internal tangent of ω_1 and ω_2; by t_{23} the length of common internal tangent of ω_2 and ω_3; and by t_{13} the length of common external tangent of ω_1, ω_3. If $t_{13} = t_{12} + t_{23}$ then there exists common tangent of ω_1, ω_2 and ω_3 which separates ω_2 from ω_1 and ω_3.*

Denote by ω_1, ω_2 and ω_3 the incircles of the triangles MBC, MCD and MDA respectively. Keep the notations for t_{12}, t_{23} and t_{13} from fact 1. Calculation shows that:

$$t_{12} = \frac{1}{2}(MC + MD - CD - MB - MC + BC);$$
$$t_{23} = \frac{1}{2}(MC + MD - CD - MD - MA + DA);$$
$$t_{13} = \frac{1}{2}(MB + MC - BC + MD + MA - DA).$$

Since the quadrilateral $ABCD$ is circumscribed, $AB + CD = BC + DA$, hence we can apply the fact 1 for ω_1, ω_2 and ω_3. Denote by ℓ the common tangent of ω_1, ω_2 and ω_3. Consider the triangle MKL formed by the lines MC, MD and ℓ. Since I_1I_2 and I_2I_3 are external angle bisectors of this triangle, $\angle I_1I_2I_3 = 90^\circ - \frac{1}{2}\angle KML = 180^\circ - \angle I_1MI_3$. So the quadrilateral $MI_1I_2I_3$ is cyclic.

Alternative solution. We will use the following well-known fact:

Fact 2. *Let ABC be a non-isosceles triangle ($AB \neq BC$). Denote the incenter of triangle ABC by I. Points A_0 and C_0 are marked on the segments BC and BA. Prove that points B, I, C_0 and A_0 are concyclic iff $AC_0 + CA_0 = AC$.*

Let the circumcircle of the triangle MI_1I_3 intersect the lines AD, MB and MC at the points P, Q and R respectively. From the fact 2 we obtain $DP + CP = CD$ and $AP + BQ = AB$. Hence $BC = AB + CD - AD = DP + CP + AP + BQ - AD = BQ + CP$ and the fact 2 implies that I_2 lies on the the circumcircle of the triangle MI_1I_3.