Intensive Training 2021
Level 3 Geometry Problems

Day 1 (20 March):

1. Let the circles k_1 and k_2 intersect at two points A and B, and let t be a common tangent of k_1 and k_2 that touches k_1 and k_2 at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate the angle NMB. (JBMO 2012)

2. A trapezoid $ABCD$ ($AB||CF$,$AB > CD$) is circumscribed.The incircle of the triangle ABC touches the lines AB and AC at the points M and N,respectively.Prove that the incenter of the trapezoid $ABCD$ lies on the line MN. (JBMO 2016)

3. Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.) Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular. (EGMO 2012)

4. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled. (EGMO 2013)

Day 2 (25 March):

5. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90°$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$.Let the midpoint of BD be M and the midpoint of QR be N.Prove that the points M, N, A and C lie on a circle. (EGMO 2017)

6. Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point of the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N. Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P. (EGMO 2018)

7. Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear. (JBMO 2013)

8. Consider an acute triangle ABC of area S. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocentres of the triangles MNC, respectively MND. Find the area of the quadrilateral AH_1BH_2 in terms of S. (JBMO 2014)

More problems:

9. In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic. (BMO 2013 P1)

10. Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$. (EGMO 2018 P5)

11. Let ABC be an acute triangle.The lines l_1 and l_2 are perpendicular to AB at the points A and B, respectively.The perpendicular lines from the midpoint M of AB to the lines AC and BC intersect l_1 and l_2 at the points E and F, respectively.If D is the intersection point of the lines EF and MC, prove that

$$\angle ADB = \angle EMF.$$

(JBMO 2015 P3)

12. Let $ABCD$ be a cyclic quadrilateral, and let diagonals AC and BD intersect at X.Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respecctively. Lines AD_1 and BC_1 intersect at Y, and line MY intersects diagonals AC and BD at different points E and F, respectively. Prove that line XY is tangent to the circle through E, F and X. (EGMO 2016 P2)

13. A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$,and AB is not paralel to CD.Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k. (BMO 2018)

14. Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F. (BMO 2012)

15. Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear. (BMO 2015)

16. Let $ABCD$ be a cyclic quadrilateral with $AB < CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD. (BMO 2016)

17. Let $ABCD$ be a trapezium inscribed in a circle Γ with diameter AB. Let E be the intersection point of the diagonals AC and BD . The circle with center B and radius BE meets Γ at the points K and L (where K is on the same side of AB as C). The line perpendicular to BD at E intersects CD at M. Prove that KM is perpendicular to DL. (BMO 2014)