March Online Camp 2021

Algebra – Level L3

Problems

Problem 1. Solve in reals the following system of equations
$$\begin{cases}(x+y)^3 = 8z\\(y+z)^3 = 8x\\(z+x)^3 = 8y\end{cases}$$

Problem 2. Solve in reals the following equation
$$(x^4+3y^2)\sqrt{|x+2|+|y|} = 4|xy^2|.$$

Problem 3. Solve in positive reals the following system of equations
$$\begin{cases}a^3+b^3+c^3 = 3d^3\\b^4+c^4+d^4 = 3a^4\\c^5+d^5+a^5 = 3b^5\end{cases}$$

Problem 4. For any integer $n \geq 3$ find all real sequences $(x_1, x_2, \ldots, x_n)$, such that
$$\sum_{i=1}^{n} x_i = n \quad \text{and} \quad \sum_{i=1}^{n}(x_{i-1}-x_i+x_{i+1})^2 = n,$$
here we assume $x_0 = x_n$ i $x_{n+1} = x_1$.

Problem 5. Solve in positive reals the following system of equations
$$\begin{cases}2x^3 = 2y(x^2+1)-(z^2+1)\\2y^4 = 3z(y^2+1)-2(x^2+1)\\2z^5 = 4x(z^2+1)-3(y^2+1)\end{cases}$$

Problem 6. Solve in reals the following system of equations

$$\begin{cases} 2x + x^2y = y \\ 2y + y^2z = z \\ 2z + z^2x = x \end{cases}$$

Problem 7. Find all lists $(x_1, x_2, \ldots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

- $x_1 \le x_2 \le \ldots \le x_{2020}$;
- $x_{2020} \le x_1 + 1$;
- there is a permutation $(y_1, y_2, \ldots, y_{2020})$ of $(x_1, x_2, \ldots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i + 1)(y_i + 1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ *is a permutation of* $(1, 2, 2)$*, and they are both permutations of* $(2, 2, 1)$*. Note that any list is a permutation of itself.*

Problem 8. Let f_1, f_2, f_3, f_4 be real polynomials such that sum of any two does not have a real root. Prove that if polynomial $f_1 + f_2 + f_3 + f_4$ has a real root, then at least one polynomial among f_1, f_2, f_3, f_4 does not have any real root.

Problem 9. Let a_1, a_2, $\ldots$, a_n be positive real numbers such that

$$a_1^2 + 2a_2^3 + \cdots + na_n^{n+1} < 1.$$

Prove that

$$2a_1 + 3a_2^2 + \cdots + (n+1)a_n^n < 3.$$

Problem 10. Find all monotonic functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following holds

$$f(f(x) - y)) + f(x + y) = 0.$$

Problem 11. Let $\mathcal{A}$ be a set of real numbers such that for any $x \in \mathcal{A}$, if $x \neq 0$ and $x \neq 1$ then

$$\frac{x+1}{x} \in \mathcal{A} \quad \text{and} \quad \frac{2x-1}{x-1} \in \mathcal{A}.$$

Prove that if $2 \in \mathcal{A}$, then $\mathcal{A}$ contains all rational numbers greater then 1.

Problem 12. Let $\mathbb{N}_0$ be the set of all non-negative integers. Let $f, g\colon \mathbb{N}_0 \to \mathbb{N}_0$ be functions such that for all $n \in \mathbb{N}_0$ the following holds

$$g(f(n)) = g(n) - n.$$

Find all possible values of $f(0)$.

Problem 13. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \ldots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \ldots (1 + a_n)^n > n^n.$$

Problem 14. Let n be an odd positive integer, and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n} (x_i^2 + x_{i+1}^2) \leq \max_{j=1,\ldots,n} (2x_j x_{j+1})$$

where $x_{n+1} = x_1$.

Problem 15. Let a, b, c, d be positive real numbers such that $a, c > 1$ and $b, d < 1$. Prove that

$$\frac{a}{ab + c + 1} + \frac{b}{bc + d + 1} + \frac{c}{cd + a + 1} + \frac{d}{da + b + 1} > 1.$$

Problem 16. Suppose that F, G, H are polynomials of degree at most $2n+1$ with real coefficients such that:

i) For all real x we have

$$F(x) \leq G(x) \leq H(x).$$

ii) There exist distinct real numbers $x_1, x_2, \ldots, x_n$ such that

$$F(x_i) = H(x_i) \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

iii) There exists a real number x_0 different from $x_1, x_2, \ldots, x_n$ such that

$$F(x_0) + H(x_0) = 2G(x_0).$$

Prove that

$$F(x) + H(x) = 2G(x)$$

for all real numbers x.

Problem 17. Let $f, g\colon (0, 2) \to (0, 2)$ be such that for any $x \in (0, 2)$ the following equalities hold

$$f(g(x)) = g(f(x)) = x \quad \text{and} \quad f(x) + g(x) = 2x.$$

Prove that $f(1) = g(1)$.

Problem 18. Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be non-negative reals such that $x_1 + x_2 + \ldots + x_n = 1$. Prove that

$$\max\{x_1, x_2, \ldots, x_n\} \cdot \left(1 + 2 \sum_{1 \leq i < j \leq n} \min\{x_i, x_j\}\right) \geq 1.$$

Problem 19. Find the number of solutions of the following system of equations

$$\begin{cases} x_2 + x_1^2 = 4x_1 \\ x_3 + x_2^2 = 4x_2 \\ x_4 + x_3^2 = 4x_3 \\ \ldots \\ x_n + x_{n-1}^2 = 4x_{n-1} \\ x_1 + x_n^2 = 4x_n \end{cases}$$

in non-negative real numbers.

Problem 20. Find all functions $f \colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ such that for any $n > 0$ the following inequalities hold

$$(n-1)^2 < f(n)f(f(n)) < n^2 + n.$$

Problem 21. Prove that if the polynomials P and Q have a real root each and

$$P(1 + x + Q(x)^2) = Q(1 + x + P(x)^2),$$

then $P \equiv Q$.

Problem 22. The polynomial

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + 1$$

with nonnegative real coefficients has n real roots. Prove that

- $f(2) \geq 3^n$,
- $f(x) \geq (x+1)^n$ for all $x \geq 0$,
- $a_k \geq \binom{n}{k}$ for all $k = 1, 2, \ldots, n-1$.

Problem 23. Let $x_1, x_2, \ldots, x_n$ (where $n \geq 2$) be real numbers greater than 1. Suppose that $|x_i - x_{i+1}| < 1$ for $i = 1, 2, \ldots, n-1$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} < 2n - 1.$$

Problem 24. Let P be a polynomial with real coefficients such that $P(x) \geq 0$ for any real x. Prove that
$$P(x) = A(x)^2 + B(x)^2$$
for some real polynomials $A(x)$ and $B(x)$.

Pro☠lem 25. Let P be a polynomial with real coefficients such that $P(x) > 0$ for any $x \in [a, b]$ $(a < b)$. Prove that
$$P(x) = A(x)^2 + (x-a)(b-x)\sum_{i=1}^{m} B_i(x)^2,$$
for some real polynomials A, B_1, B_2, $\ldots$, B_m.

Problem 26. Let P be a polynomial with real coefficients such that
$$P(x) = U_1(x)^2 + U_2(x)^2 + \ldots + U_k(x)^2$$
for some real polynomials U_1, U_2, $\ldots$, U_k. Prove that
$$P(x)^2 = V_1(x)^4 + V_2(x)^4 + \ldots + V_m(x)^4$$
for some real polynomials V_1, V_2, $\ldots$, V_m.

Problem 27. Real numbers x_1, x_2, x_3, x_4 are roots of the fourth degree polynomial $W(x)$ with integer coefficients. Prove that if $x_1 + x_2$ is a rational number and $x_1 x_2$ is a irrational number, then $x_1 + x_2 = x_3 + x_4$.

Problem 28. For each positive integer n, determine the smallest possible value of the polynomial
$$W_n(x) = x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \ldots + (2n-1)x^2 + 2nx.$$

Problem 29. Let $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ be $n \geq 2$ reals. Prove the inequality
$$a_1 a_2 \ldots a_{n-1} + (2a_2 - a_1)(2a_3 - a_2)\ldots(2a_n - a_{n-1}) \geq 2a_2 a_3 \ldots a_n.$$

Problem 30. Let c be a real number such that the polynomial
$$P(x) = x^5 - 5x^3 + 4x - c$$
has five distinct real roots x_1, x_2, x_3, x_4, x_5. Determine, depending on c, the sum of the absolute values of the coefficients of the polynomial
$$Q(x) = (x - x_1^2)(x - x_2^2)(x - x_3^2)(x - x_4^2)(x - x_5^2).$$

Problem 31. Are there polynomials of the form

$$x^n \pm x^{n-1} \pm \ldots \pm x \pm 1$$

such that all their roots are real?

Problem 32. Given that the polynomial

$$P(x) = x^n - 2nx^{n-1} + 2n(n-1)x^{n-2} + \ldots + a_0$$

has only real roots, find all real roots.

Problem 33. The polynomial

$$x^3 + ax^2 + bx + c,$$

with $c \neq 0$, has three distinct real roots. Prove that the polynomial

$$x^3 - bx^2 + acx - c^2$$

also has three real roots.

Problem 34. Prove that for any integer $n \geq 2$ the following number

$$n^{2n} - n^{n+2} + n^n - 1$$

is divisible by $(n-1)^3$.

Problem 35. Find all non-zero polynomials $P(x)$ with real coefficients such that

$$P(x)^3 = P(x^3)$$

for any real x.

Problem 36. Let $P(x)$ and $Q(x)$ be monic polynomials with real coefficients and $\deg P(x) = \deg Q(x) = 10$. Prove that if the equation $P(x) = Q(x)$ has no real solutions, then equation $P(x+1) = Q(x-1)$ has a real solution.

Problem 37. Let $P(x)$ be a polynomial such that

$$P(8) + P(11) < 19 < P(12) + P(7).$$

Prove that there are real numbers a, b such that

$$a + b = P(a) + P(b) = 19.$$

Problem 38. Let
$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$$
has roots $x_1, x_2, \ldots, x_n$ and
$$Q(x) = x^{n+1} + b_n x^n + \ldots + b_0$$
has roots $y_1, y_2, \ldots, y_{n+1}$. Prove that
$$P(y_1) \cdot \ldots \cdot P(y_{n+1}) = Q(x_1) \cdot \ldots \cdot Q(x_n).$$

Problem 39. Let a, b and c be positive real numbers. Prove that
$$\frac{a^2 b(b-c)}{a+b} + \frac{b^2 c(c-a)}{b+c} + \frac{c^2 a(a-b)}{c+a} \geq 0.$$

Problem 40. Let a, b, c be positive real numbers with sum 1. Prove that
$$a^2 + b^2 + c^2 + 2\sqrt{3abc} \leq 1.$$

Problem 41. Prove that for any positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds
$$\frac{a_1^3}{a_1^2 + a_1 a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2 a_3 + a_3^2} + \ldots + \frac{a_n^3}{a_n^2 + a_n a_1 + a_1^2} \geq \frac{a_1 + a_2 + \ldots + a_n}{3}.$$

Problem 42. Let $a, b > 0$ and $n \geq 1$. Find the greatest value of
$$\frac{x_1 x_2 \ldots x_n}{(a + x_1)(x_1 + x_2) \ldots (x_{n-1} + x_n)(x_n + b)},$$
where $x_1, x_2, \ldots, x_n$ are positive reals.

Problem 43. Given real $c > -2$. Prove that for positive reals $x_1, \ldots, x_n$ satisfying:
$$\sum_{i=1}^{n} \sqrt{x_i^2 + c x_i x_{i+1} + x_{i+1}^2} = \sqrt{c+2} \cdot \left(\sum_{i=1}^{n} x_i \right).$$
holds $c = 2$ or $x_1 = \ldots = x_n$

Problem 44. Prove that for any real numbers $a, b, c \geq 1$ the following inequality holds:
$$\sqrt{a-1} + \sqrt{b-1} + \sqrt{c-1} \leq \sqrt{a(bc+1)}.$$

Problem 45. Let $a, b, c, d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be real numbers such that

$$\sin a + \sin b + \sin c + \sin d = 1$$

and

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}.$$

Prove that $a, b, c, d \in [0, \frac{\pi}{6}]$.

Problem 46. Let a, b, c be positive real numbers. Prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geq abc\left(a^3b^3 + b^3c^3 + c^3a^3\right) + a^2b^2c^2\left(a^3 + b^3 + c^3\right).$$

Problem 47. Let a, b, c be real numbers. Prove that

$$\sqrt{2(a^2+b^2)}+\sqrt{2(b^2+c^2)}+\sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2+3(b+c)^2+3(c+a)^2}.$$

Problem 48. Let p be a polynomial with positive real coefficients. Prove that if $p\left(\frac{1}{x}\right) \geq \frac{1}{p(x)}$ is true for $x = 1$, then it is true for all $x > 0$.

Problem 49. Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Problem 50. Prove that for any real numbers $1 < a, b, c < 2$ the following inequality holds

$$\frac{b\sqrt{a}}{4b\sqrt{c} - c\sqrt{a}} + \frac{c\sqrt{b}}{4c\sqrt{a} - a\sqrt{b}} + \frac{a\sqrt{c}}{4a\sqrt{b} - b\sqrt{c}} \geq 1.$$

Problem 51. Prove that for any real numbers $0 < a, b, c < 1$ the following inequality holds

$$\frac{a}{bc+1} + \frac{b}{ca+1} + \frac{c}{ab+1} \leq 2.$$

Problem 52. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ which satisfy conditions:

- $f(x) + f(y) \geq xy$ for all real x, y,
- for each real x exists real y, such that $f(x) + f(y) = xy$.

Problem 53. Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 54. Let k, m and n be three different **positive integers**. Prove that

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right) \leq km - 2$$

and

$$\left(k - \frac{1}{k}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) \leq kmn - (k + m + n).$$

Problem 55. Let $P(x)$, $Q(x)$ be two different polynomials with real coefficients such that $P(Q(x)) = Q(P(x))$. Prove that for all positive integers n the following polynomial:

$$\underbrace{P(P(\ldots P(P}_{n}(x))\ldots)) - \underbrace{Q(Q(\ldots Q(Q}_{n}(x))\ldots))$$

is divisible by $P(x) - Q(x)$.

Pro☠lem 56. Let $n \geq 3$ and x_1, x_2, $\ldots$, x_n be distinct real such that

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = 1.$$

Prove that there are distinct numbers a, b, c, d among $\{x_1, x_2, \ldots, x_n\}$ such that

$$a + b + c + nabc \leq \sum_{i=1}^{n} x_i^3 \leq a + b + d + nabd.$$