Combinatorial geometry

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Problems – June 1

1. Given 100 points inside a unit circle, prove that some two are on a distance less than $\frac{2}{9}$.

2. *Silvester's problem.* Given a finite set S of points in the plane, not are collinear, prove that there is a line containing exactly two of these points.

3. Given a convex polygon $\mathcal{P}$ of area 1, prove that there is a triangle of area $\frac{1}{4}$ that is fully contained in $\mathcal{P}$.

   Now prove there is also such a triangle of area $\frac{3}{8}$.

4. (a) Prove that among any four points inside a unit circle there are two on a distance less than $\sqrt{2}$.

   (b) Prove that among any five points inside a unit sphere there are two on a distance less than $\sqrt{2}$.

5. A plane figure of area greater than 51 is placed within a 1×101 rectangle. Prove that this figure must contain two points on mutual distance 1.

6. We are given n blue and n red points such that no three are collinear. Prove that we can draw n pairwise disjoint segments, each connecting a red point and a blue point.

7. A finite set of points in the plane is given. The distance between any two points is at most 1. Prove that it is always possible to remove one point and divide the rest into two parts, such that the distance between any two points in the same part is strictly less than 1.

8. Given a positive integer n, let f be a map from the plane to the real numbers with the property that $f(A_1) + f(A_2) + \cdots + f(A_n) = 0$ whenever $A_1A_2\ldots A_n$ is a regular n-gon. Prove that f is constantly zero.

9. An even number of unit segments in the plane form a closed (non-degenerate) polygonal line. Prove that some two of its vertices are on a distance greater than 1.

10. A set S consists of n points in the plane, no three of which are collinear. Prove that there is a set P of $2n-5$ points such that every triangle with the vertices in S contains at least one point from P in its interior.

11. A *corner* is a polygonal line in the (coordinate) plane consisting of one vertical and one horizontal segment.

    We are given n red and n blue points in the plane whose all projections onto the coordinate axes are distinct. Prove that we can draw n pairwise disjoint corners, each connecting a red point to a blue point.

12. A set with $n > 2$ vectors is given in the plane. Call a vector *lengthy* if its length is not less than that of the sum of all other vectors. If each vector in the set is lengthy, prove that the sum of all these vectors is zero.

13. Prove that, among any 111 unit vectors in the plane with the sum zero, one can find 55 vectors whose sum has a length not exceeding 1.

14. We are given n^2 points inside or on the edges of a unit square. Prove that we can always connect them all by a polygonal line of length less than $2n + 1$.

15. We are given four congruent right-angled triangles made of paper. In each step we can cut one triangle along its altitude. Can we ever obtain a set of pairwise incongruent triangles? (And how many cuts do we need at least?)

16. Finitely many unit equilateral triangles are given in the plane, all of which are translates of each other. Every two triangles have a common point. Prove that there are three points such that each triangle contains at least one of them.

17. By *clipping* a convex polygon we mean choosing two adjacent sides, say AB and BC, taking their midpoints M and N and removing the triangle BMN. If we start with a regular hexagon of area 1 and clip it finitely many times, prove that the remaining area will always be greater than $\frac{2}{3}$.

18. A *strip* is a part of the plane lying between two parallel lines. Suppose that S is a set of points in the plane such that any three points can be covered with a strip of width 1. Prove that the entire set S can be covered with a strip of width 2.

19. A regular n-gon with n odd is dissected into triangles by nonintersecting diagonals. Prove that exactly one of these triangles is acute.

20. For which n is it possible to dissect a regular n-gon into isosceles triangles by nonintersecting diagonals?

21. A regular n-gon is dissected into triangles by nonintersecting diagonals. At most how many of these triangles can be pairwise incongruent?