— Algebra for L3 —

— February Camp, 2022 — Problems —

Warm-up.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $[0,1]$, $(0,\infty)$, ...
- $f\colon D\to C$ means that with each $x\in D$ we associate exactly one number $f(x)\in C$; sets D, C are called the *domain*, *codomain* of f, respectively
- function $f\colon D\to C$ is *increasing* (resp. *nondecreasing*) if for every $x,y\in D$ such that $x<y$ we have $f(x)<f(y)$ (resp. $f(x)\le f(y)$); *decreasing* and *nonincreasing* functions are defined similarly; all these functions are said to be *monotonic* (this includes *constant* functions i.e. both nonincreasing and nondecreasing)
- function $f\colon D\to C$ with domain symmetric with respect to zero is called *even* if $f(-x)=-f(x)$ for every $x\in D$, and *odd* if $f(-x)=-f(x)$ for every $x\in D$
- function $f\colon D\to C$ is called *periodic* if there exists $0<t\in D$ such that for every $x\in D$: $f(x+t)=f(x)$
- Cauchy equation: $f\colon D\to D$, $\forall x,y\in D$ $f(x+y)=f(x)+f(y)$ has the only solutions $f(x)=cx$ for $c\in D=\mathbb{N},\mathbb{Z},\mathbb{Q}$; for $D=\mathbb{R}$ to ensure these are the only solutions we need an extra assumption (e.g. monotonicity, continuity), otherwise there exist other (so called *pathological*) solutions

H1. Find all functions $f\colon\mathbb{Q}_+\to\mathbb{Q}_+$ such that for every $x\in\mathbb{Q}_+$

$$f(x+1)=f(x)+1 \quad\text{and}\quad f(x^3)=(f(x))^3.$$

Solution. By easy induction we get that if $x\in\mathbb{Q}_+$ and $m\in\mathbb{Z}_+$, then $f(x+m)=f(x)+m$. Now take $x=p/q$ and $m=q^2$. We have

$$(f(x+m))^3=f(x^3+3p^2+3pq^3+q^6)=f(x^3)+3p^2+3pq^3+q^6$$

because $3p^2+3pq^3+q^6\in\mathbb{Z}$. On the other hand,

$$(f(x)+m)^3=(f(x))^3+3q^2(f(x))^2+3q^4f(x)+q^6.$$

The left-hand sides of the two above are equal, so so are the right-hand sides, i.e.

$$(f(x))^2+q^2f(x)=x^2+q^2x \iff (f(x)-x)(f(x)+x+n^2)=0.$$

The second factor is positive, so $f(x)=x$ for every x.

2. Prove that every function $f\colon\mathbb{R}\to\mathbb{R}$ can be uniquely represented as a sum of an even function and an odd function.

Solution. Suppose that for every $x\in\mathbb{R}$ we have $f(x)=e(x)+o(x)$, where e is even and o is odd. Then plugging $-x$ instead of x we get $f(-x)=e(-x)+o(-x)=e(x)-o(x)$, so solving this system of equations for e and o, we obtain

$$e(x)=\frac{f(x)+f(-x)}{2} \quad\text{and}\quad o(x)=\frac{f(x)-f(-x)}{2}.$$

On the other hand, it's easy to check that e and o defined as above are even and odd, respectively. Therefore they are the unique pair of such functions.

3. Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that for every $x,y\in\mathbb{R}$

$$f(x+y)-f(x-y)=f(x)f(y).$$

SOLUTION. Plugging $x=y=0$ gives $f(0)=0$. Plugging $x=0$ gives $f(y)=f(-y)$ for every y. Plugging $y=x$, using $f(x)=f(-x)$, and plugging $y=-x$ gives

$$f(2x)=f(x)^2=f(x)f(-x)=-f(2x)$$

for every x, i.e. $f\equiv 0$.

4. Find all functions $f\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}$ such that for every $x\neq 0$

$$f\left(\frac{1}{x}\right)+\frac{f(-x)}{x}=x.$$

SOLUTION. Putting $-\frac{1}{x}$ instead of x, we get

$$f(-x)-x\left(\frac{1}{x}\right)=-\frac{1}{x},$$

which combined with the given equation gives

$$f(x)=\frac{x^3+1}{2x}.$$

It's easy to check that this function satisfies the given equation.

5. Suppose that $f\colon \mathbb{R}\to\mathbb{R}$ satisfies for every $x\in\mathbb{R}$

$$f(x)=f(2x)=f(1-x).$$

Prove that f is periodic.

SOLUTION. We have for every x

$$f(x)=f(1-x)=f(2-2x)=f(2x-1)=f(x-\tfrac{1}{2}),$$

do f is $\frac{1}{2}$-periodic.

One can even show that it's $\frac{1}{2^n}$-periodic for every $n\in\mathbb{N}$, but this does not mean it's constant! E.g. the Dirichlet function defined as $f(x)=1$ if $x\in\mathbb{Q}$ and $f(x)=0$ otherwise, is non-constant and t-periodic for every $t\in\mathbb{Q}_+$. Another example is $f(x)=m$ if $x=p/q\in\mathbb{Q}$ is in the reduced form and m is the largest odd divisor of q, and $f(x)=0$ if $x\notin\mathbb{Q}$.

6. Find all functions $f\colon \mathbb{Q}\to\mathbb{Q}$ such that $f(1)=2$ and for every $x,y\in\mathbb{Q}$

$$f(xy)+f(x+y)=f(x)f(y)+1.$$

SOLUTION. We easily prove that $f(x)=x+1$ for all $x\in\mathbb{Z}$ and that $f(x+1)=f(x)+1$ for $x\in\mathbb{Q}$, which can be extended to $f(x+m)=f(x)+m$ for every $x\in\mathbb{Q}$ and $m\in\mathbb{Z}$. Therefore

$$f(\tfrac{p}{q})+p+q+1=f(p)+f(\tfrac{p}{q}+q)=f(\tfrac{p}{q})f(q)+1=(q+1)f(\tfrac{p}{q})+1,$$

so $f(\frac{p}{q})=\frac{p}{q}+1$ for an arbitrary $\frac{p}{q}\in\mathbb{Q}$. It's easy to check that $f(x)=x+1$ indeed satisfies the initial equation.

WARM-UP.

- Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that $f(x+y)=f(x)+y$ for all $x,y\in\mathbb{R}$.
- Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that $f(x)f(y)=f(xy)+x+y$ for all $x,y\in\mathbb{R}$.

H2. Find all functions $f\colon \mathbb{Z}\to\mathbb{Z}$ such that

$$f(a)^2+f(b)^2+f(c)^2=2f(a)f(b)+2f(b)f(c)+2f(c)f(a)$$

for all integers a, b, c satisfying $a+b+c=0$.

SOLUTION. Cf. P4/IMO2012 (or A1/IMOSL2012) e.g. on AoPS.

8. Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that for every $x\in\mathbb{R}$:

$$f(-x)=-f(x),\quad f(x+1)=f(x)+1,\quad f\left(\frac{1}{x}\right)=\frac{f(x)}{x^2}\ (\text{if } x\neq 0).$$

SOLUTION. The idea behind the proof is that our assumptions can be used to express $f(x)$ in terms of $f(-x)$ or $f(1+x)$ or $f(\frac{1}{x})$ only. In particular, we can make a cycle

$$x\rightsquigarrow x+1\rightsquigarrow \frac{1}{x+1}\rightsquigarrow -\frac{1}{x+1}\rightsquigarrow \frac{x}{x+1}\rightsquigarrow 1+\frac{1}{x}\rightsquigarrow \frac{1}{x}\rightsquigarrow x.$$

Performing the actual manipulations gives

$$\begin{aligned}
f(x)&=f(x+1)-1=(x+1)^2 f\left(\frac{1}{x+1}\right)-1=-(x+1)^2 f\left(-\frac{1}{x+1}\right)-1\\
&=-(x+1)^2\left(f\left(\frac{x}{x+1}\right)-1\right)-1=-(x+1)^2\left(f\left(\frac{x+1}{x}\right)\cdot\frac{x^2}{(x+1)^2}-1\right)-1\\
&=-x^2 f\left(\frac{x+1}{x}\right)+x^2+2x=-x^2\left(1+f\left(\frac{1}{x}\right)\right)+x^2+2x=-x^2 f\left(\frac{1}{x}\right)+2x\\
&=-f(x)+2x,
\end{aligned}$$

so $f(x)=x$ for $x\notin\{0,-1\}$. Moreover $f(0)=0$ and $f(-1)=-1$ follow directly from $f(0)=-f(0)$ and $f(-1)=f(0)-1$. Finally we check that $f(x)=x$ satisfies all given conditions.

9. Find all functions $f\colon \mathbb{N}\to\mathbb{N}$ such that for every $x,y\in\mathbb{N}$

$$xf(y)+yf(x)=(x+y)f(x^2+y^2).$$

SOLUTION. All constant functions satisfy the given equation. Suppose that for some $x\neq y$ we have $f(x)<f(y)$. Then

$$(x+y)f(x)=xf(x)+yf(x)<xf(y)+yf(x)<xf(y)+yf(y)=(x+y)f(y),$$

so $f(x)<f(x^2+y^2)<f(y)$. This means that between any two distinct values of f we can find another value of f which is impossible as f takes only integer values.

10. Functions $f,g\colon \mathbb{N}\to\mathbb{N}$ satisfy for every $n\in\mathbb{N}$

$$g(f(n))=g(n)-n.$$

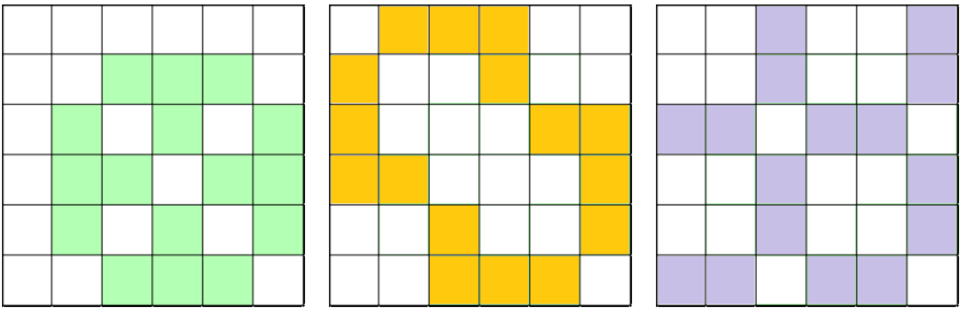
Find all possible values of $f(0)$.

SOLUTION. Let m be such that $g(m)$ is the smallest value attained by g. The given equation gives that $g(f(m))=g(m)-m\leq g(m)$, which implies $m=0$ (otherwise $g(f(m))$ would be smaller than the smallest possible value of g). Moreover, this implies $f(m)=0$ as 0 is the only argument for which g attains its minimum. So if such functions f, g exist, then $f(0)=0$. It remains to prove that there exist such functions — take e.g. $f\equiv 0$ and $g(n)=n$.

11. Let $S=\{0,1,2,3,4,5\}$ be the set of residues modulo 6 and let $\oplus$ be the addition modulo 6 (e.g. $4\oplus 3=1$). Find at least one two-argument function $f\colon S\times S\to\{0,1\}$ which is not constantly zero with the following properties:

- $f(x,y)=f(y,x)$ for every two $x,y\in S$;
- for every $x,y\in S$ both numbers
$$\sum_{i=0}^{2}\sum_{j=0}^{2} f(x\oplus i, y\oplus j) \quad \text{and} \quad \sum_{i=0}^{3}\sum_{j=0}^{3} f(x\oplus i, y\oplus j)$$
are even.

Solution. We easily observe that there are 0's on the diagonal (i.e. $f(x,x)=0$) and that if we fix nine values in the first two rows ($x\in\{0,1\}$), then everything else will be uniquely determined by our conditions. It is enough to fix $f(0,1)$, $f(0,2)$, $f(1,2)$ (almost) arbitrarily and solve a systems of (many) linear equations mod 2 (with variables being the missing 6 values in the first two columns). The only solutions (up to a toroidal translation along the diagonal) are:



where the white cells stand for zeroes, and the colorful — for ones.

12. Given a function $f\colon\mathbb{R}\to\mathbb{R}$, if for all $x,y\in\mathbb{R}$ the equality
$$f(xy+x+y)=f(xy)+f(x)+f(y)$$
holds, prove that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$.

Solution. Cf. IMOSL 1979, e.g. on AoPS.

13. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that for every $x,y,\in\mathbb{R}$
$$f(x+f(y))-f(x)=(x+f(y))^4-x^4.$$

Solution. If $f\equiv 0$, then f is a solution. Suppose that for some $a\in\mathbb{R}$ we have $f(a)=b\neq 0$. Plugging $y=a$ into the initial equation, we get
$$f(x+b)-f(x)=(x+b)^4-x^4=4x^3b+6x^2b^2+4xb^3+b^4.$$

As $b \neq 0$, $g(x) = f(x+b) - f(x)$ is a non-degenerate cubic polynomial, i.e. it attains all real values. In other words, for every $x \in \mathbb{R}$ there exists $w \in \mathbb{R}$ such that $g(w) = x$, i.e. $f(w+b) - f(w) = f(x)$. Fix any $x \in \mathbb{R}$ and let w be a number as above. Then plugging $(-f(w), w+b)$ in place of (x,y) in the initial equation, we get

$$f(x) - f(-f(w)) = x^4 - (-f(w))^4.$$

On the other hand, plugging $(-f(w), w)$ to the initial equation, we obtain

$$f(0) - f(-f(w)) = -(-f(w))^4.$$

Joining the two last results, we get

$$f(x) = x^4 + f(0),$$

and we easily check that all such functions satisfy the given equation.

14. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$

$$f(x + f(x+y)) = f(x-y) + f(x)^2.$$

Solution. Take $x = a + f(a)$, $y = -f(a)$ to get

$$f(a + f(a)) = 0$$

for every $a \in \mathbb{R}$. But for $x = a$, $y = 0$ we have

$$f(a + f(a)) = f(a) + f(a)^2,$$

so joining the two relations we have $f(a) \in \{0, -1\}$ for every $a \in \mathbb{R}$. But if $f(x) = -1$ for some x, then

$$f(x + f(x+y)) - f(x-y) = 1$$

for all $y \in \mathbb{R}$, hence $f(x + f(x+y)) = 0$ and $f(x-y) = -1$ (as that's the only combination of two possible values of f resulting in 1). Hence $f \equiv -1$ but this function fails to satisfy the given equation. We conclude that the only possibility is $f \equiv 0$ and check that it indeed is a solution.

15. Find all pairs of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$:

$$f(x)f(y) = g(x)g(y) + g(x) + g(y).$$

Solution. Let $h(x) = g(x) + 1$, then

$$f(x)f(y) + 1 = h(x)h(y),$$

so in particular $h(x)^2 = f(x)^2 + 1$. Hence

$$(f(x)^2 + 1)(f(y)^2 + 1) = h(x)^2 h(y)^2 = (f(x)f(y) + 1)^2 \iff (f(x) - f(y))^2 = 0,$$

i.e. f is constant, so so is h, so so is g. Thus the solutions are all pairs $f \equiv c$, $g \equiv d$ s.t. $c^2 + 1 = (d+1)^2$.

16. Functions $f, g\colon (0,2) \to (0,2)$ satisfy for every $x \in (0,2)$

$$f(g(x)) = g(f(x)) = x \quad \text{and} \quad f(x) + g(x) = 2x.$$

Prove that $f(1)=g(1)$.

Solution. Suppose that $f(1)=1+c$ for $c\in(0,1)$. Then

$$f(1+c)=2+2c-g(1+c)=2+2c-g(f(1))=2+2c-1=1+2c$$

and inductively $f(1+nc)=1+(n+1)c$ for each integer $n\geq 1$. But this means $(n+1)c<1$ for every n, which is impossible.

Therefore $f(1)\leq 1$ and fully analogously (problem conditions are symmetric) $g(1)\leq 1$, so by $f(1)+g(1)=2$ we have $f(1)=g(1)=1$.

17. Given is a function $f\colon\mathbb{R}\to\mathbb{R}$ and $\alpha\neq 0$. Suppose that for every $x,y\in\mathbb{R}$

$$f\left(\frac{x+y}{\alpha}\right)=\frac{f(x)+f(y)}{\alpha}.$$

Prove that for every $x,y\in\mathbb{R}$

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}.$$

Solution. We have

$$\begin{aligned}\frac{f(x)+f(y)}{2}&=\frac{\alpha}{2}\cdot\frac{f(x)+f(y)}{\alpha}\\&=\frac{\alpha}{2}\cdot f\left(\frac{x+y}{\alpha}\right)\\&=\frac{\alpha}{2}\cdot f\left(\frac{\frac{x+y}{2}+\frac{x+y}{2}}{\alpha}\right)\\&=\frac{\alpha}{2}\cdot\frac{2f\left(\frac{x+y}{2}\right)}{\alpha}\\&=f\left(\frac{x+y}{2}\right).\end{aligned}$$

18. Function $f\colon\mathbb{R}\to\mathbb{R}$ satisfies $f(x+1)-f(x)=2x+1$ for every $x\in\mathbb{R}$ and $|f(x)|\leq 1$ for $x\in[0,1]$. Prove that $|f(x)|\leq x^2+2$.

Solution. Take $f(x)=x^2+g(x)$. Then g is 1-periodic, so if $x=\lfloor x\rfloor+a$, then

$$|f(x)|\leq x^2+|g(x)|=x^2+|g(a)|\leq x^2+|f(a)|+a^2\leq x^2+2.$$

Extra1. We are given a wooden cube with its surface painted green. There are 33 different planes, each located between some two opposite faces of the cube and parallel to them, which dissect the cube into small cuboidal blocks. Given that the number of blocks with at least one green face equals the number of blocks with no green faces, determine the total number of blocks into which the cube is dissected.

Solution. It is easy to see that there have to be at least four planes in each of three possible directions (if in one of the directions there are less than five layers of blocks, then the number of at-least-one-green-side blocks is greater than the number of inside blocks).

Denote the numbers of planes in different directions by $a+3$, $b+3$, $c+3$, where a, b, c are positive integers. It follows that $(a+3)+(b+3)+(c+3)=33$, so $a+b+c=24$. The problem condition can be rewritten as

$$(a+4)(b+4)(c+4)=2(a+2)(b+2)(c+2)$$

which yields $abc=240=2^4\cdot 3\cdot 5$. Since $a+b+c$ is even, either (1) exactly one or (2) all three of numbers a, b, c are even. In the case (1), one of the numbers a, b, c (w.l.o.g. a) is divisible by 16 and since $a+b+c=24$, we have $a=16$. It follows that $b+c=8$ and $bc=15$, so $\{b,c\}=\{3,5\}$. We can now calculate the total number of blocks:

$$(a+4)(b+4)(c+4)=20\cdot 7\cdot 9=1260.$$

In the case (2) we have w.l.o.g. $a=4x$, $b=2y, c=2z$ where $xyz=15$ and $2x+y+z=12$. The only possibility is $x=3$, $\{y,z\}=\{1,5\}$, which gives

$$(a+4)(b+4)(c+4)=16\cdot 6\cdot 14=1344.$$