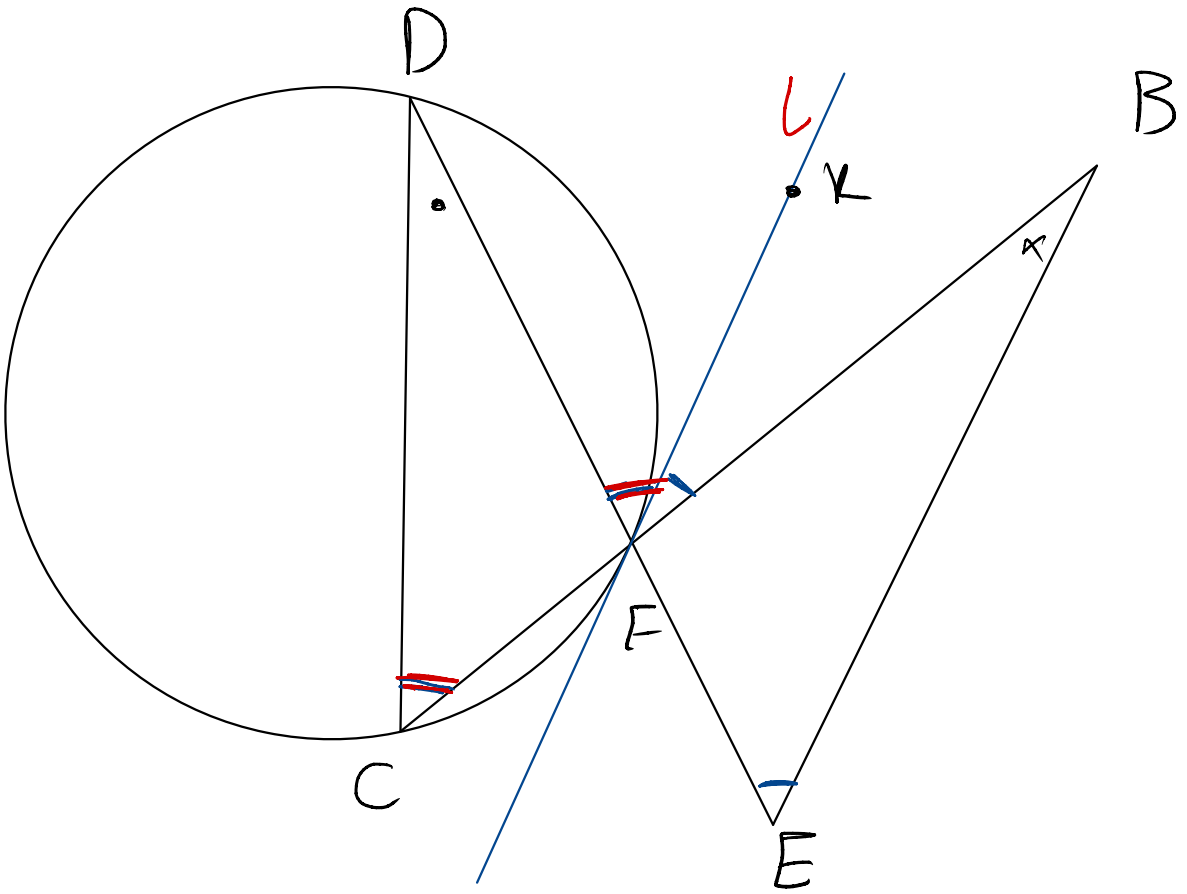
Intensive Training
Geometry

Day 7
10 April 2021

Includes solutions for:

BMO 2012 – P1
JBMO 2019 Shortlist

Lemma: (BFE) is tangent to (DFC) if and only if $\angle BFD = \angle DCF + \angle BEF$



Let L be tangent to (DFC) at F

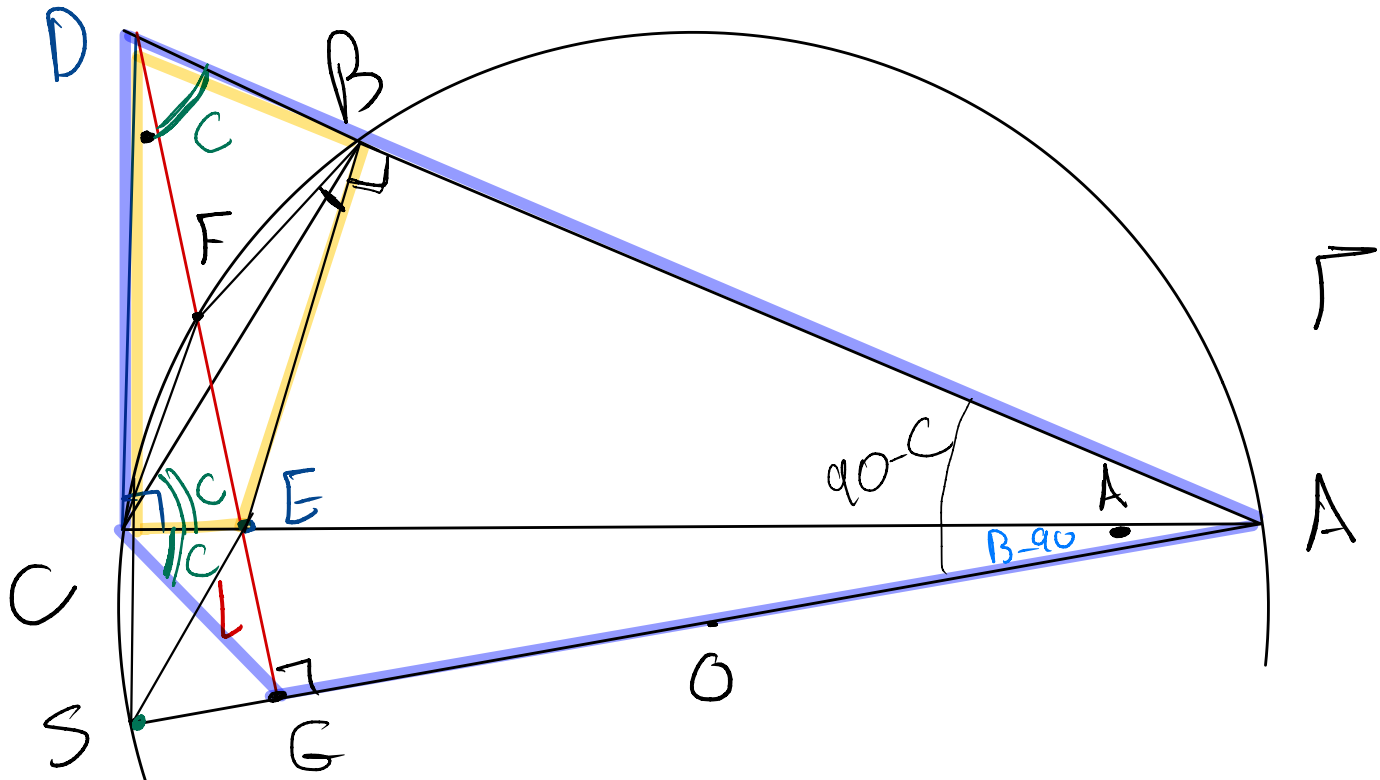
$\Rightarrow \angle DCF = \angle DFK$

L is tangent to $(BFE) \iff \angle BFK = \angle BEF$

$\iff \angle BFD - \angle DFK = \angle BEF$

$\iff \angle BFD = \angle DCF + \angle BEF$

20. Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let l be the line through D which is perpendicular to AO. Let E be the point of intersection of l with the line AC, and let F be the point of intersection of Γ with l that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F. (BMO 2012 P1)



Let $\angle B = \angle ABC$, $G = L \cap AO$, $S = AO \cap DC$

$\Rightarrow$ ADCG cyclic

$\Rightarrow \angle CDE = \angle CDG = \angle CAO$ (1)

In $\triangle OAC$:

$2\angle CAO + \angle AOC = 180$

$\Rightarrow 2\angle CAO = 180 - (360 - 2B)$

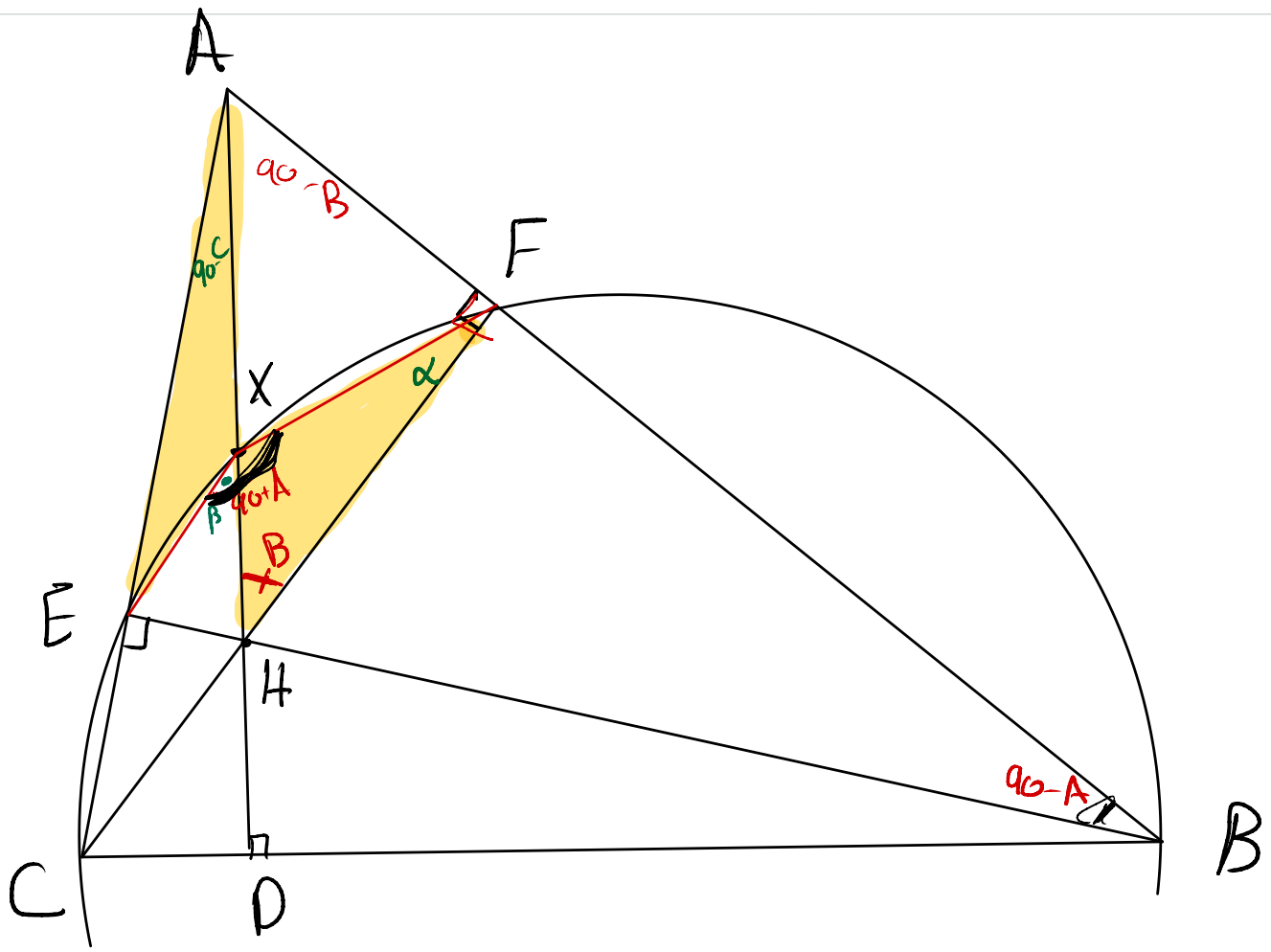
$\Rightarrow \angle CAO = B - 90$

$\angle OAB = \angle OAC + \angle CAB = (B-90) + A = 90 - C$

$\Rightarrow \angle ADG = C \Rightarrow \angle BDE = \angle BCE = C$

$\Rightarrow$ BDEC cyclic $\Rightarrow \angle EBA = 90$

$\Rightarrow$ AG and AC are altitudes $\triangle ADS$



$\Rightarrow$ E is the orthocenter.

The problem now is equivalent to ΔABC with H orthocenter and D, E, F feets of altitudes.

Let $X = AD \cap (BC)$. We want to show that

$\angle EAX + \angle XFH = \angle EXH$. $*$ $\leftarrow$ by lemma

$\angle EAX = \angle CAD = 90 - C$

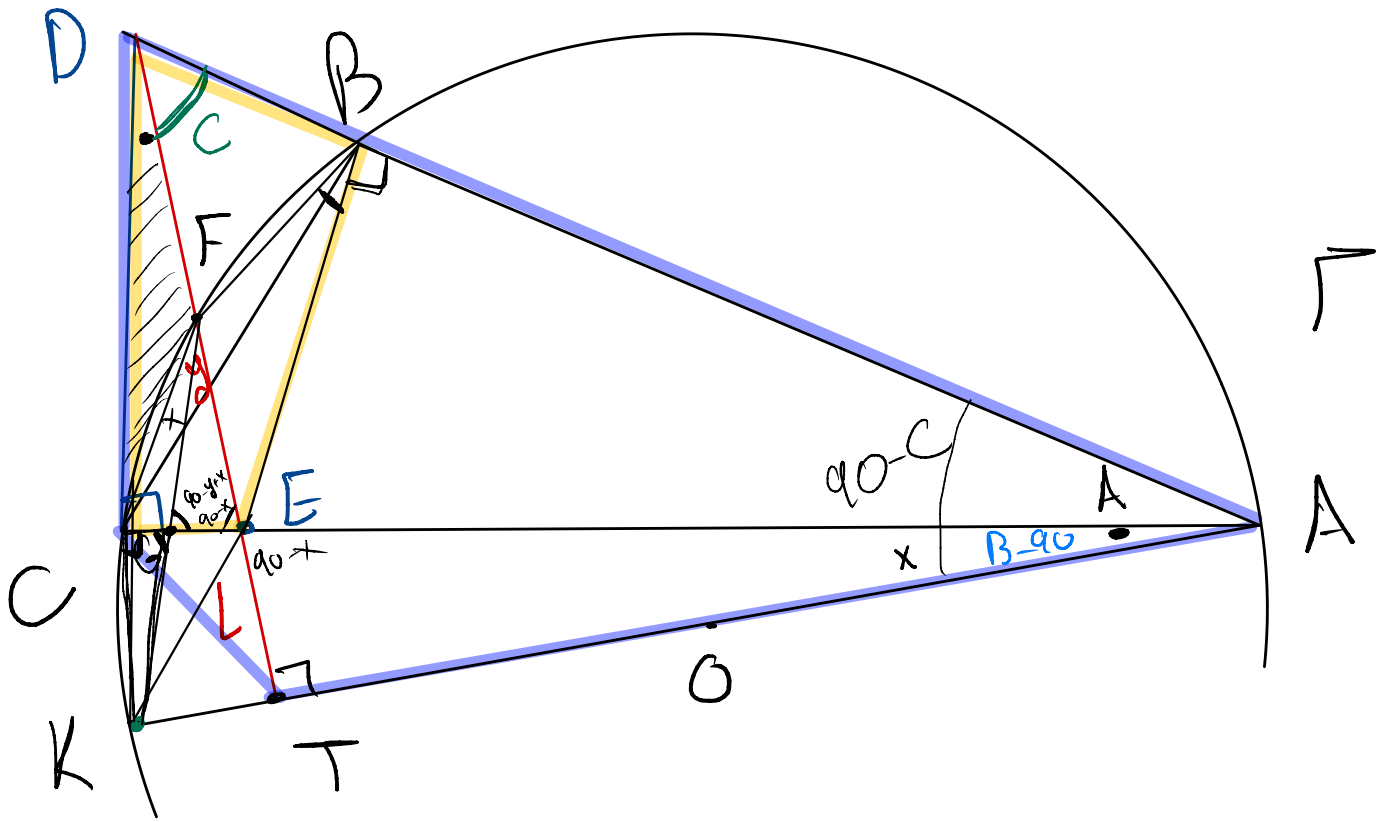
Also, $\angle EXF = 180 - \angle FBE = 180 - (90 - A) = 90 + A$

$\Rightarrow \angle EXH = 90 + A - \angle FXH$

$* \Leftrightarrow 90 - C + \angle XFH = 90 + A - \angle FXH$

$\Leftrightarrow \angle XFH + \angle XFH = A + C = 180 - B = 180 - \angle FHX$

which is true



Solution 2: Let $L \cap AO = T$, $AO \cap DC = K$, $FK \cap AC = G$

We showed that E is the orthocenter of ADK

Let $x = \angle KFC$, $y = \angle KFE$

$$\left.\begin{aligned} x = \angle KFC &= \angle KAC \quad (\Gamma) \\ &= \angle TDC \quad (ATCD) \end{aligned}\right\} \Rightarrow KF \text{ tangent to } \Delta DFC$$

KF tangent to $\Delta BFE \Leftrightarrow \angle KFE = \angle FBE$

$\Leftrightarrow \angle FBE = y \Leftrightarrow \widehat{FC} + \widehat{CK} = y$

$\Leftrightarrow \widehat{FC} + x = y \Leftrightarrow \widehat{FC} = y - x$

21. Let ABC be a triangle with circumcircle . Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B || l_C$. The second intersections of l_B and l_C with are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B || OP || l_C$.

① $\triangle AGD \sim \triangle AEF \sim \triangle ABC$

$\angle ADG = \angle ACB$ (from w)

$\angle DGA = \angle AEF$ ($l_B \parallel l_C$)

$= \angle ABC$ (from w

$\Rightarrow \triangle AGD \sim \triangle ABC$)

② O_1, A, O_2 are collinear

$\angle O_2AE = 90 - \angle EFA = 90 - C$

$\angle O_1AG = 90 - \angle ADG = 90 - C$

③ $\triangle OO_1O_2 \sim \triangle ABC$

$O_1O \perp AD$, $O_2O \perp AE$

$\Rightarrow \begin{cases} \angle AO_1O = \angle AGD = B \\ \angle AO_2O = \angle AFE = C \end{cases}$

$\Rightarrow \triangle OO_1O_2 \sim \triangle ABC$

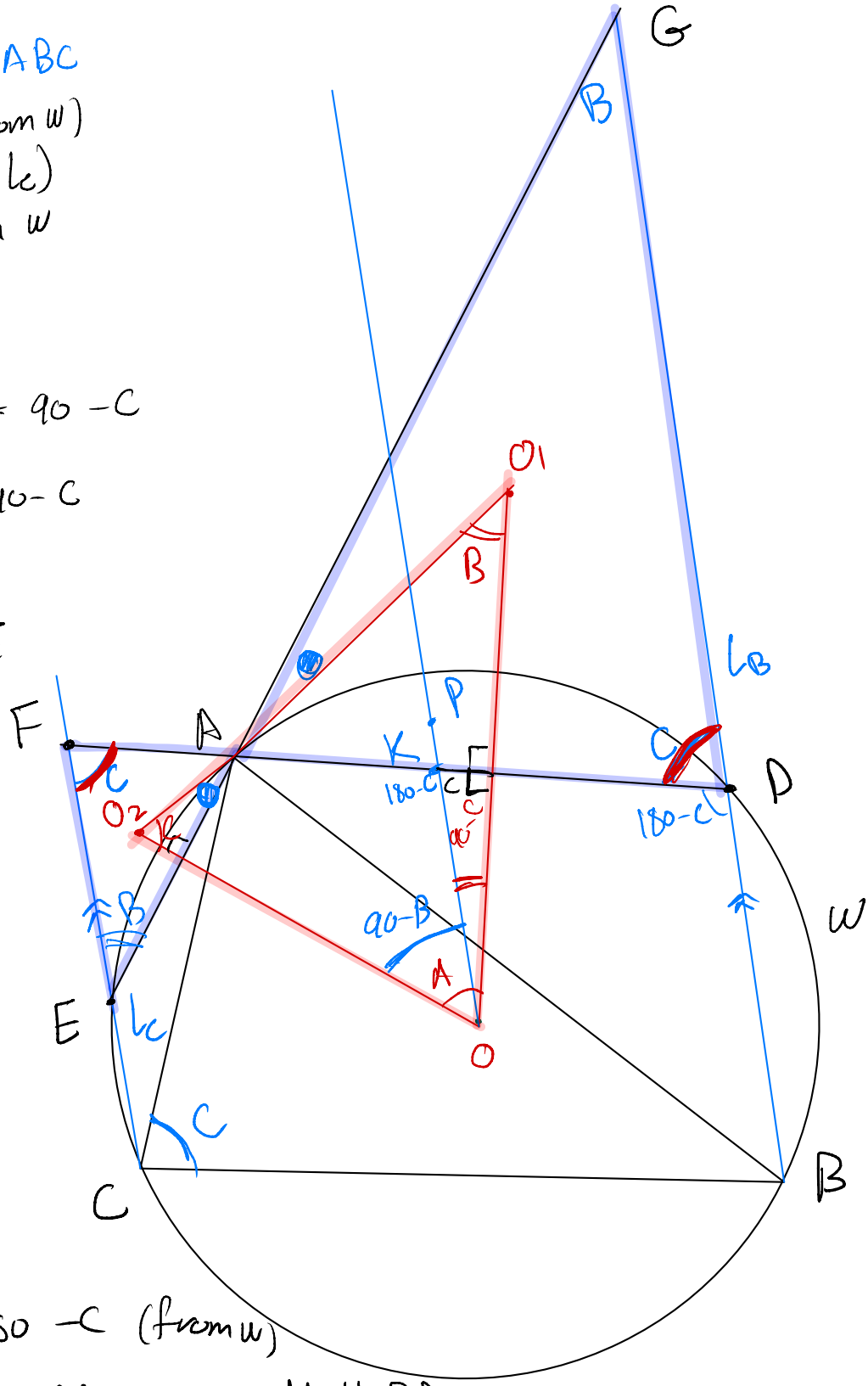
④ $\angle POO_1 = 90 - C$

⑤ $OP \parallel l_B$

$K = OP \cap AD$

$\angle OKA = 180 - C$

However, $\angle BDA = 180 - C$ (from w)

$\Rightarrow \angle OKA = \angle ODA \Rightarrow OK \parallel BD$

$\Rightarrow OP \parallel l_B$