Email training, N10

November 17 - 26, 2019

Problem 10.1. Positive integers 1 and 2 are written on the board in laboratory. Every morning professor Ali erases the written numbers from the board and writes their arithmetical and harmonic means. It occurs that

a) At some moment $\frac{941664}{665857}$ was written on the board. Determine either it is written as arithmetical or harmonic mean and determine the other number written on the board.

b Determine if some moment the number $\frac{35}{24}$ can be written on the board.

Solution 10.1. a) Assume at some moment we have number a and b on the board. Then next day we will have numbers

$$\frac{a+b}{2} \quad \text{and} \quad \frac{2}{1/a+1/b} = \frac{2ab}{a+b}.$$

Note that

$$\frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab,$$

which means that the product of numbers is invariant, so the product is equal $1 \cdot 2 = 2$. If at some moment one number is equal to $\frac{941664}{665857}$ then the another number is $2 : \frac{941664}{665857} = \frac{665857}{470832}$.

b) We have

$$MIN < HARMONIC\ MEAN < ARITHMETICALMEAN < MAXIMUM,$$

so the biggest number on the board decreases and the minimal number increases. Note, that after second day we have arithmetical means equal to $\frac{17}{12}$ which is less than $\frac{35}{24}$. So after second day all number written on the board will be always less than $\frac{35}{24}$. Not possible.

Problem 10.2. Let non-regular polygon is inscribed to the circle. Each step one may choose a vertex A which divides the arc between two neighbour to the A vertices into non equal parts and move the point A to the midpoint of that arc. Is it possible that after 100 moves one gets a polygon which is equal to the original one?

Solution 10.2. Consider the sum of the arc length squares. Since for $a \neq b$ one has

$$a^2 + b^2 > \left(\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2,$$

then the sum of squares is strictly decreasing, so after 10 steps it can't be equal to the original value.

Problem 10.3. Let 10 students stay around the circle. During the step is allowed that one student changes his position and stays somewhere else around the circle. Find the minimal number of moves that needed that students stay in increasing order by height in clockwise (from shortest to tallest) independent from the original positions.

Solution 10.3. Let's students are staying in increasing order by height in counter-clockwise order. If we make less than 8 moves, then at least 3 students will not change their position, so they will not satisfy to the condition.

Enumerate students $1, 2, 3, \ldots, 10$ in decreasing order. Let 1 and 2 don't change their position. Then 3 goes and stays next to 3 in clockwise order, then 4 goes and stays next to 3 and so on.

Problem 10.4. Let the cake has a form of regular pentagon (5-gon). There is a nuts in the center of it. At each step one is allowed to cut a piece of the cake which has a form of triangle and the line which cuts the triangle doesn't contain vertex of the polygon (originally it was pentagon). Is it possible to achieve to the nuts?

Solution 10.4. Consider general convex polygon and enumerate it's vertices $A_1A_2\dots A_n$. Obviously in first step we may achieve a nut if it's in the triangle $A_iA_{i+1}A_{i+2}$. We state, that if you can't get it after one step, then you will never get it. Indeed, after each step the total area of triangles of type $A_iA_{i+1}A_{i+2}$ decreases, so new points that are achievable will not appear. Note, that for regular pentagon it's center isn't achievable.

Problem 10.5. Let several pairwise different positive number are written on the board. It is known that the sum of any two of them is power of 2. Find the maximal possible number of numbers written on the board.

Solution 10.5. Consider the biggest number a. Assume $2^{n-1} < a \le 2^n$. Then for any other integer b from the group we have

$$2^{n-1} < a + b < 2^{n+1},$$

so $a + b = 2^n$, which means b is uniquely defined. So at most two number might be written on the board. Example 5 and 3.

Problem 10.6. Let 100 apples are put in a row such that the difference of weights of neighbour apples is less than 1 gram. Prove that one may combine them in 50 groups, each group having 2 apples such, and put them in a row such, that the difference of weights of neighbor groups is less than 1 gram.

Solution 10.6. Order apples by weight and denote their weights by

$$a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_{99} \ge a_{100}.$$

Note that we have $a_i \le a_{i+1}+1$.Group them in the following way: put i apple with $101-i$'th one. Then, for $i \le 50$ we have

$$\big(a_{i+1} + a_{100-i}\big) - \big(a_i + a_{101-i}\big) = \big(a_{i+1} - a_i\big) + \big(a_{100-i} - a_{101-i}\big) \le a_{100-i} - a_{101-i} \le 1,$$

and

$$\big(a_{i+1} + a_{100-i}\big) - \big(a_i + a_{101-i}\big) = \big(a_{i+1} - a_i\big) + \big(a_{100-i} - a_{101-i}\big) \ge a_{i+1} - a_i \ge -1.$$

Problem 10.7. Let the triangle is divided into 3 similar triangles. Prove that the original triangle can be divided into 2 similar triangles.

Solution 10.7. Let C is the biggest angle of triangle ABC is divided to 3 triangles with segments AX, BX, CX where X is inside ABC. Since $\angle AXB > \angle ACB$, then AXB can be equal to AXC and BXC in other triangles. It means $\angle AXB = \angle AXC = \angle BXC = 120^o$. Then $AX = BX = CX$ and we have equilateral triangle.

Now assume that we split the triangle into two triangle by a line passing through a vertex of triangle and then one of them is divided into two triangles in the same way. By considering the biggest angle of last two triangles we conclude that they are right-angle triangles. By considering options we conclude that the original triangle is either isoscale triangle either right angle triangle.

So, anyway our triangle is equilateral, isoscale or right-angle. In all cases we can't divide them into 2 similar triangles by appropriate altitude.