Email training, N5

September 22-28

Problem 5.1. Prove the inequality

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\ldots-\frac{1}{99}+\frac{1}{100}>\frac{1}{5}.$$

Solution 5.1.

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\ldots-\frac{1}{99}+\frac{1}{100}=\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}\right)+$$
$$\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\ldots+\left(\frac{1}{98}-\frac{1}{99}\right)+\frac{1}{100}>$$
$$\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}\right)=\frac{13}{60}>\frac{1}{5}.$$

Problem 5.2. Show that for all $n \geq 1$ one has

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}<2.$$

Solution 5.2. It's not possible to prove the statement by induction, since the left side is increasing and the right side is staying constant. However, by induction it's possible to prove stronger result, from which will follow the statement of the problem. Let's prove that

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\leq 2-\frac{1}{n}.$$

The statement is trivial for $n=1$, ie $1 \leq 2-1$. Assume that the statement holds for n and let's prove it for $n+1$.

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}++\frac{1}{(n+1)^2}\leq 2-\frac{1}{n}+\frac{1}{(n+1)^2}=$$
$$2-\frac{n^2+n+1}{n(n+1)^2}=2-\frac{n^2+n+1}{(n^2+n)(n+1)}<2-\frac{1}{n+1}.$$

Problem 5.3. Prove that for any numbers a, b, $c>0$ the following inequality holds

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\geq\frac{2}{a}+\frac{2}{b}-\frac{2}{c}.$$

Solution 5.3. After bringing to the common denominator and eliminating abc we get the following equivalent inequality

$$a^2+b^2+c^2\geq 2bc+2ac-2ab$$

which is equivalent to

$$(a+b)^2+c^2\geq 2c(a+b).$$

The last one is the known inequality $x^2+y^2\geq 2xy$.

Problem 5.4. How many integer solutions has the following inequality

$$\left(x-\frac{1}{2}\right)^1\cdot\left(x-\frac{3}{2}\right)^3\cdot\ldots\cdot\left(x-\frac{2017}{2}\right)^{2017}<0.$$

Solution 5.4. First notice, that we may erase the powers over breckets, since all powers are odd. After that we get a polynomial of degree 1009. Since 1009 is odd, then it tends to $-\infty$ when $x \to -\infty$, so the inequality has infinitely many solutions.

Answer: Infinitely many.

Problem 5.5. Find the maximum value of expression $\sqrt{x^2+y^2}$ if it's known that

$$\{-4 \leq y - 2x \leq 2, \quad 1 \leq y - x \leq 2\}.$$

Solution 5.5. Let's notice, that the given region is a quadrilateral with it's internal region. In fact we need to find the most far point of the quadrilateral from the point $(0,0)$. It's obvious, that we are looking for the one of the vertices of the quadrilateral. By simple calculation we get the following vertices $A(6,8)$, $(0,2)$, $C(-1,0)$ and $D(5,6)$. From these points the point A is the most far and has the distance 10.

Answer: 10.

Problem 5.6. Quadrilateral $ABCD$ is given such that

$$\angle DAC = \angle CAB = 60^o,$$

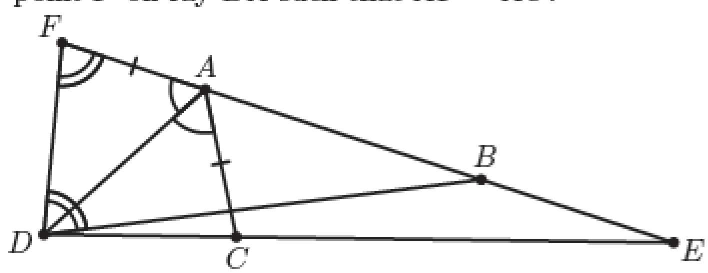
and

$$AB = BD - AC.$$

Lines AB and CD intersect each other at point E. Prove that $\angle ADB = 2\angle BEC$.

Solution 5.6. -

Consider point F on ray BA such that $AF = AC$.



Knowing that $AB = BD - AC$, it is implied that $BF = BD$. Therefore

$$\left.\begin{array}{l} AF = AC \\ AD = AD \\ \angle FAD = \angle CAD = 60^\circ \end{array}\right\} \implies \triangle FAD \cong \triangle CAD. \qquad (1)$$

Note that

$$\angle BEC = \angle FAD - \angle ADC \overset{(1)}{=} 60^\circ - \angle ADF. \qquad (2)$$

On the other hand

$$\begin{aligned} \angle ADB = \angle FDB - \angle ADF &= \angle AFD - \angle ADF \\ &= (120^\circ - \angle ADF) - \angle ADF \\ &= 120^\circ - 2\angle ADF \\ &\overset{(2)}{=} 2\angle BEC. \end{aligned}$$

So the claim of the problem is proved.

Problem 5.7. There are $n > 2$ lines on the plane in general position; Meaning any two of them meet, but no three are concurrent. All their intersection points are marked, and then all the lines are removed, but the marked points are remained. It is not known which marked point belongs to which two lines. Is it possible to know which line belongs where, and restore them all?

Solution 5.7. Draw the lines which each of them contains $n-1$ marked points, at least. All the original lines are among these lines. Conversely, let some line ℓ contains some $n-1$ marked points. They are points of meet of some pairs of the original lines $(\ell_1; \ell_2)$, $(\ell_3; \ell_4)$, $\ldots$, $(\ell_{2n-3}; \ell_{2n-2})$. Since $n > 2$, we have $2n-2 > n$, so ℓ_i coincides with ℓ_j for some $1 \leq i < j \leq 2n-2$. Then these lines belong to distinct pairs in the above list, and the two corresponding marked points belong to $\ell_i = \ell_j$. But then also $\ell = \ell_i$, and we are done.

Answer: Yes, it is.

Problem 5.8. Find all quadrilaterals $ABCD$ such that all four triangles DAB, CDA, BCD and ABC are similar to one-another.

Solution 5.8. -

First assume that $ABCD$ is a concave quadrilateral. Without loss of generality one can assume $\angle D > 180°$, in other words D lies inside of triangle ABC. Again without loss of generality one can assume that $\angle ABC$ is the maximum angle in triangle ABC. Therefore

$$\angle ADC = \angle ABC + \angle BAD + \angle BCD > \angle ABC.$$

Thus $\angle ADC$ is greater than all the angles of triangle ABC, so triangles ABC and ADC cannot be similar. So it is concluded that $ABCD$ must be convex.

Now let $ABCD$ be a convex quadrilateral. Without loss of generality one can assume that the $\angle B$ is the maximum angle in the quadrilateral. It can be written that

$$\angle ABC > \angle DBC, \quad \angle ABC \geq \angle ADC \geq \angle BCD.$$

Since triangles ABC and BCD are similar, it is implied that $\angle ABC = \angle BCD$ and similarly, all the angles of $ABCD$ are equal; Meaning $ABCD$ must be a rectangle. It is easy to see that indeed, all rectangles satisfy the conditions of the problem.

Answer. All rectangles.