Intensive Training

Geometry

Day 1

20 March 2021

Includes solutions for:

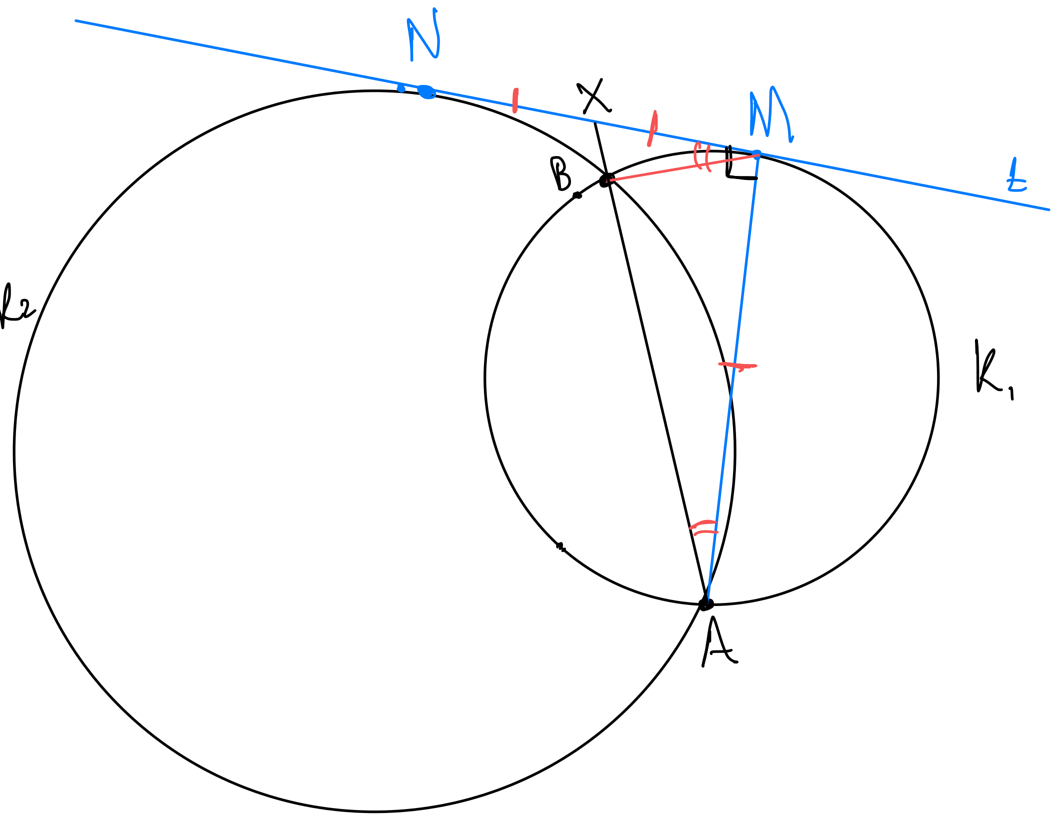
JBMO 2012 P2

JBMO 2016 P1

EGMO 2012 P1

EGMO 2013 P1

1. Let the circles k_1 and k_2 intersect at two points A and B, and let t be a common tangent of k_1 and k_2 that touches k_1 and k_2 at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate the angle NMB. (JBMO 2012)



$MN = 2AM$, $t \perp AM$

Since AM is orthogonal to the tangent, AM is a diameter in k_1.

Let $AB \cap MN = X$, then $\begin{cases} XM^2 = XB \cdot XA \\ XN^2 = XB \cdot XA \end{cases} \Rightarrow XM = XN$

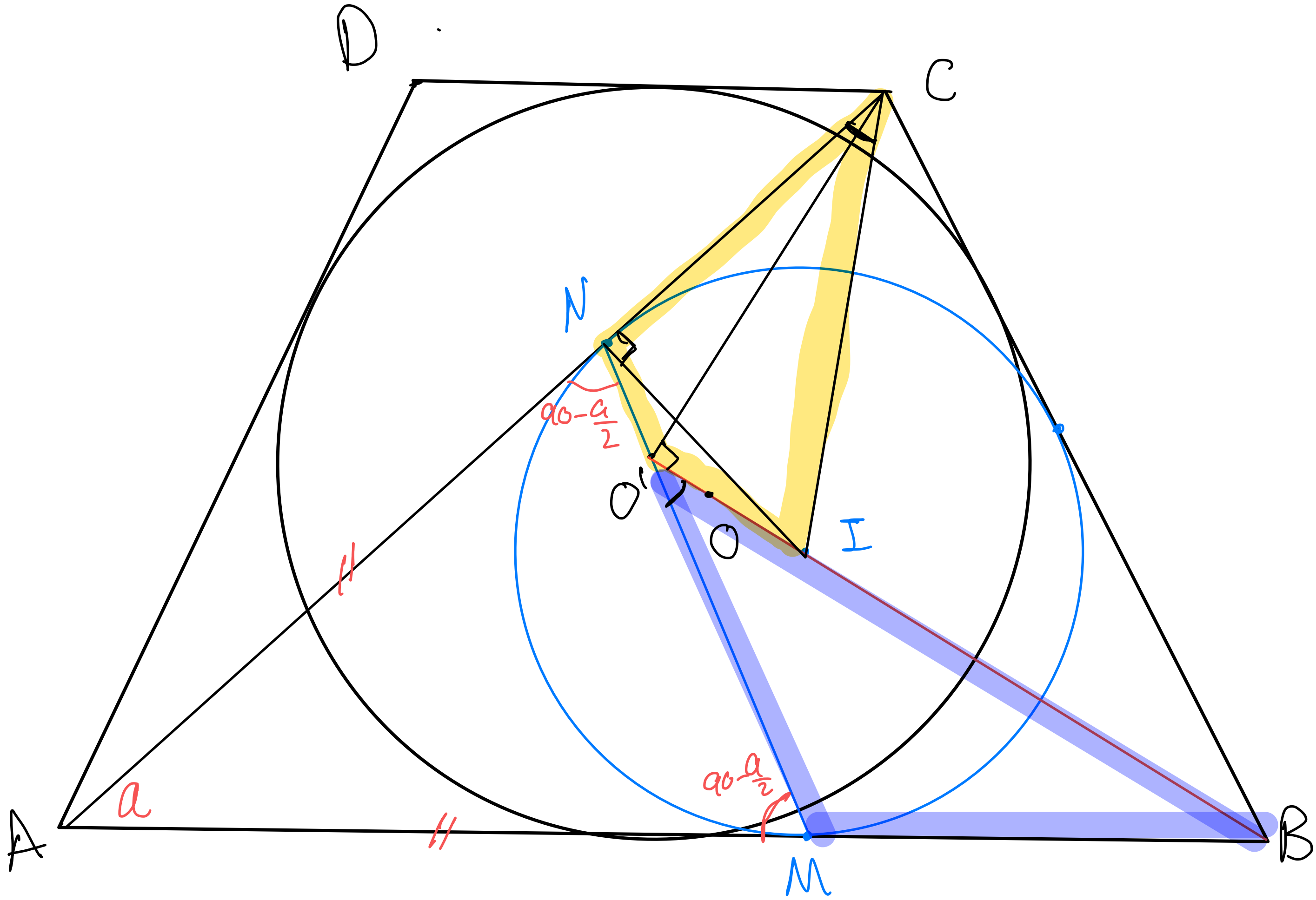
However $MN = 2AM \Rightarrow \boxed{XM = AM}$

In AMX: $\angle AMX = 90$, $AM = MX \Rightarrow \angle MAX = 45^\circ$

but $\angle XMB = \angle MAX$ (XM tangent)

$\angle XMB = 45^\circ \Rightarrow \angle NMB = 45^\circ$

2. A trapezoid $ABCD$ ($AB||CD, AB > CD$) is circumscribed. The incircle of the triangle ABC touches the lines AB and AC at the points M and N, respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN. (JBMO 2016)



B, I, O are on the same line because OB and IB are angle bisectors for $\angle ABC$. (1)

We also know that $\angle BOC = 180 - \frac{\angle CBA}{2} - \frac{\angle BCD}{2} = 90$ (2)

Let $O' = BI \cap MN$. we want to show that $\angle CO'B = 90$

It's sufficient to show that $ICNO'$ is cyclic.

$$\angle ICN = \angle ICA = \frac{1}{2}\angle BCA = \frac{c}{2} \quad (3)$$

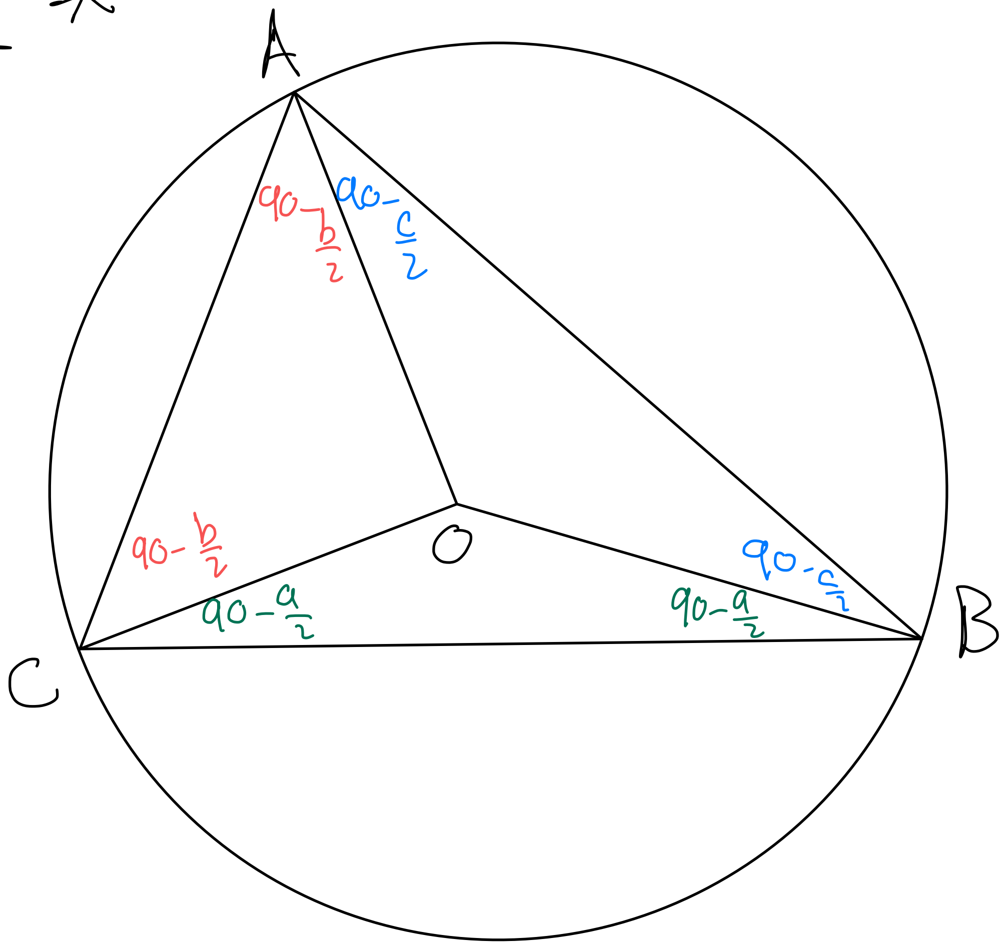
$$\angle BO'M = \angle O'MA - \angle MBO' = \left(90 - \frac{a}{2}\right) - \left(\frac{b}{2}\right) = \frac{c}{2} \quad (4)$$

From 3, 4: $ICNO'$ is a cyclic

$\Rightarrow \angle IO'C = 90$

but B, I, O' are collinear and $\angle COB = 90° \Rightarrow O' = O$ $\square$

Review: *



What is $\angle OAB$?

Since O is the center of (ABC), $\angle AOB = 2\angle ACB = 2C$

In $\triangle AOB$:

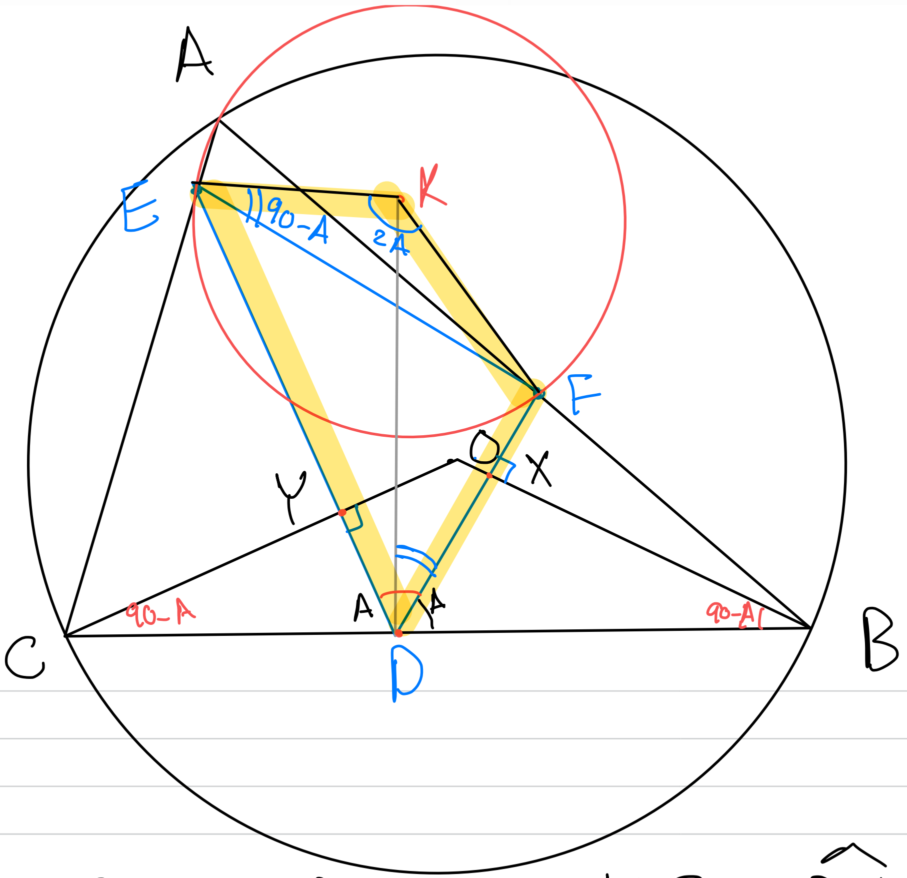
$$AO = OB \Rightarrow 2\angle BAO = 180 - C$$

$$\Rightarrow \angle BAO = 90 - \frac{C}{2}$$

3. Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.) Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular. (EGMO 2012)

السؤال الأول. ليكن ABC مثلثاً و O مركز الدائرة المارة بروؤسه. النقاط D ، E ، و F تقع في داخل الأضلاع BC ، CA ، و AB على الترتيب ، بحيث أن DE عمودي على CO و DF عمودي على BO. (على سبيل المثال النقطة D تقع على المستقيم BC و D تقع بين B و C على هذا المستقيم وهكذا).

ليكن K مركز الدائرة المارة بروؤس المثلث AFE. أثبتي أن المستقيمين DK و BC متعامدان.



Let $DF \cap OB = X$, $DE \cap OC = Y$. Since $\widehat{DYO} = \widehat{DXO} = 90$,

$DXOY$ is a cyclic $\Rightarrow \angle FDE = \angle XDY = 180 - 2\angle XOY = 180 - 2\hat{A}$

$\Rightarrow \angle FDE = 180 - 2A$ (1)

K is the center of $(AEF) \Rightarrow \angle EKF = 2\angle EAF = 2A$ (2)

From (1), (2), $EKFD$ cyclic. $\Rightarrow \angle FDK = \angle KEF = 90 - A$

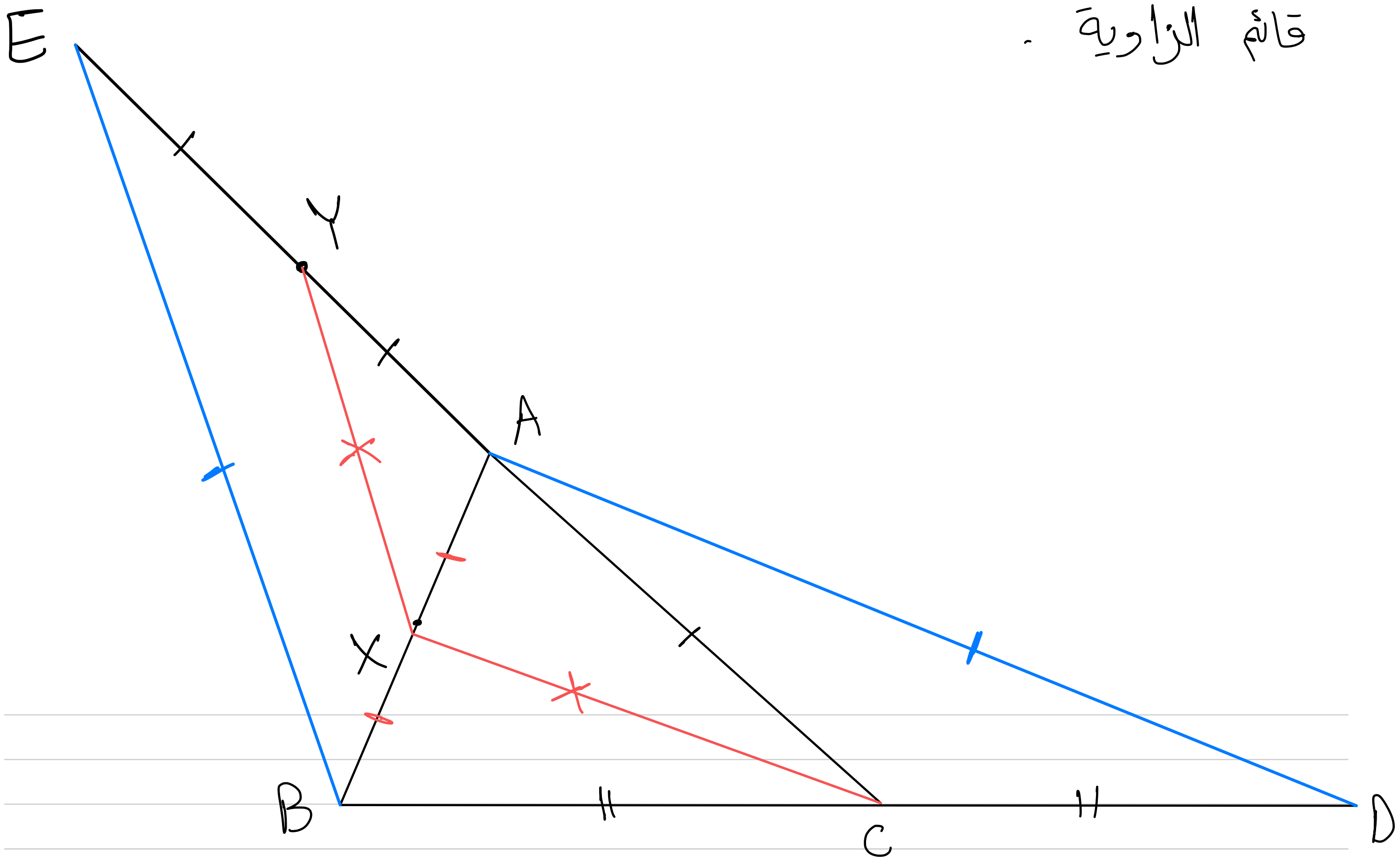
$\Rightarrow \angle KDB = \angle KDF + \angle FDB = 90 - A + \angle FDB$

but $\angle FDB = 90 - \angle DBO = 90 - (90 - A)$ from $*$

$\Rightarrow \angle KDB = 90 \Rightarrow KD \perp BC$

4. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled. (EGMO 2013)

4) تم مد الضلع BC في المثلث ABC من جهة C إلى D حيث أن $CD=BC$ وتم مد الضلع CA من جهة A إلى E حيث أن $AE=2CA$. أثبت أنه إذا كانت $AD=BE$ ، فإن المثلث ABC قائم الزاوية .



Solution 1:

Let X, Y be the midpoints of AB, AE

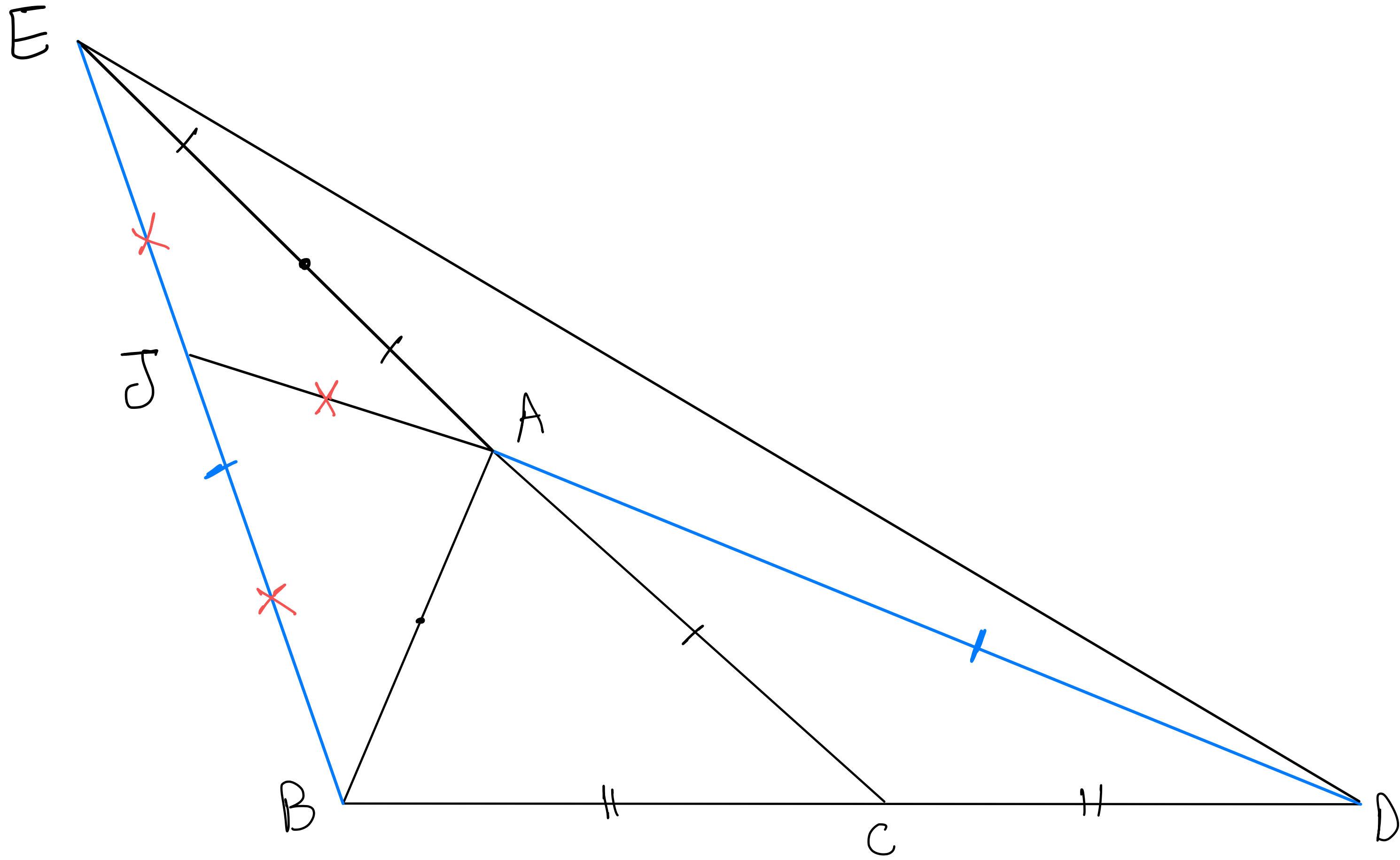
In $\triangle AEB$: $XY = \frac{1}{2} BE$

In $\triangle ABD$: $XC = \frac{1}{2} AD$

but $AD = BE$, so $XY = XC$

In $\triangle CXY$, $CX = XY$ and A is the midpoint of YC

$\Rightarrow XA \perp YC \Rightarrow \angle CAB = 90$ $\square$



Solution 2:

Let $J = AD \cap BE$

EC is a median in ΔEDB, $\dfrac{EA}{AC} = 2$

$\Rightarrow$ A is the centriod of ΔEBD

$\Rightarrow$ DJ is a median in ΔEBD

$\Rightarrow$ $AJ = \frac{1}{2} AD = \frac{1}{2} BE$

$\Rightarrow$ $JE = JA = JB \Rightarrow \angle EAB = 90$

$\Rightarrow$ $\angle BAC = 90$

$\square$