Number Theory – level L2

Instructor: Dušan Djukić *Date: 4.6.2022.*

1. Find all primes p for which p^2-2, p^2+6 and p^2+10 are also primes.
2. By $[x]$ we denote the integer part of x. Find all positive integers for which $[\frac{n^2}{5}]$ is a prime number.
3. Determine all prime numbers p such that both $\frac{p+1}{2}$ and $\frac{p^2+1}{2}$ are perfect squares.
4. How many pairs of positive integers (x, y) are there for which $[x, y] = 20!$?
5. If n and $x < n^2$ are positive integers and x is coprime to $n^2(n^2+1)(n^2+2)\cdots(n^2+n-2)$, prove that x is prime.
6. Find all primes p, q such that $p^{q+1} + q^{p+1}$ is a perfect square.
7. Find all pairs of primes p, q for which $p^3 + 3q^3 - 32$ is also a prime.
8. Find the largest positive integer whose digits are all nonzero and distinct, and that is divisible by all its digits.
9. Find all positive integers n for which $n \cdot 2^n + 4$ is a perfect square.

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10. Find all positive integers n for which (a) $n(n-10)$; (b) $n^3 - n$ is a perfect square.
11. Determine all positive integers n for which $1! + 2! + \cdots + n!$ is a perfect square.
12. Find all positive integers n such that the sum of digits of $n!$ is equal to 9.
13. If $3n$ can be written in the form $x^2 + 2y^2$ for some integers x, y, prove that n can also be written in this form.
14. If primes p and q satisfy $p \mid q - 1$ and $q \mid p^3 - 1$, prove that $q = p^2 + p + 1$.
15. If $n^2 + 1$ has a divisor $d > n$, where $n > 1$, prove that in fact $d > n + \sqrt{n}$.
16. Find all pairs (n, d) of positive integers such that d is a divisor of n and $n^2 + d^2$ is divisible by $nd + 1$.
17. Can all numbers greater than 10^{100} be written as the sum of a prime and a perfect square?
18. Can every positive integer greater than 100^{100} be written as a sum of 15 fourth powers (some of which may be zero)?
19. Suppose each of the positive integers a, b, c, d is divisible by $ad - bc$. How much is $|ad - bc|$?

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20. If n is a positive integer, prove that at least one of the numbers $n, n+1, \dots, n+5$ is coprime with each of the remaining five numbers.
21. If a and b are positive integers and a^2+b^2 is divisible by ab, prove that $a=b$.
22. Prove that the largest power of 2 dividing $\frac{(2n)!}{n!}$ is 2^n.
23. If $n>1$, prove that $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$ is never an integer.
24. Positive integers a, b, c, d satisfy $ab=cd$. Prove that there exist positive integers x, y, u, v such that $a=xu$, $b=yv$, $c=xv$, $d=yu$.
25. Suppose x, y, z are rational numbers with $xyz=1$ such that both $x+y+z$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ are integers. Prove that $|x|=|y|=|z|=1$.
26. If a, b, c and $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$ are integers, prove that abc is a perfect cube.
27. Find all pairs of positive rational numbers (a, b) such that $a^b=b^a$.

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28. Find all positive integers n that cannot be written in the form $n = [a, b] + [b, c] + [c, a]$. (By $[x, y]$ we denote the LCM *(least common multiple)* of x and y.)

29. Suppose that positive integers x, y are such that $\frac{x^2+y^2+x}{xy}$ is an integer. Prove that x is a perfect square.

- *Chinese Remainder Theorem.* Let $n_1, n_2, \dots, n_k$ be pairwise coprime positive integers and let $a_1, a_2, \dots, a_k$ be any integers. Then the system of congruences
$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \quad \cdots\cdots \\ x \equiv a_k \pmod{n_k} \end{cases}$$
has a unique solution modulo $n_1 n_2 \cdots n_k$.

30. Find all x such that $x \equiv 1 \pmod{2019}$, $x \equiv 2 \pmod{2021}$ and $x \equiv 3 \pmod{2023}$.

Solutions: Number Theory – level L2

Instructor: Dušan Djukić

1. When divided by 7, a square always gives one of the remainders $0, 1, 2, 4$.
   – If $p^2 \equiv 1 \pmod 7$, then $7 \mid p^2 + 6$ is composite.
   – If $p^2 \equiv 2 \pmod 7$, then $7 \mid p^2 - 2$ is composite, unless $p = 3$ when it is 7.
   – If $p^2 \equiv 4 \pmod 7$, then $7 \mid p^2 + 10$ is composite.
   The only remaining cases are $p = 3$ and $p = 7$, and neither works.

2. When divided by 5, n^2 gives one of the residues $0, 1, 4$. Thus $\lfloor \frac{n^2}{5} \rfloor$ equals $\frac{n^2}{5}$, $\frac{n^2-1}{5} = \frac{(n-1)(n+1)}{5}$, or $\frac{n^2-4}{5} = \frac{(n-2)(n+2)}{5}$. Each of these is composite for $n > 7$, assuming it is an integer. The smaller cases give us $n \in \{4, 5, 6\}$ as the only solutions.

3. Let $p + 1 = 2a^2$ and $p^2 + 1 = 2b^2$. Then $p \mid p^2 - p = 2(b^2 - a^2) = 2(b - a)(b + a)$, but $0 < a, b < p$, so we must have $a + b = p$; hence $b - a = \frac{p-1}{2}$, so $a = \frac{p+1}{4}$. Solving the equation in p yields $p = 7$.

4. We have $20! = 2^{18}3^8 5^4 7^2 11{\cdot}13{\cdot}17{\cdot}19$, so let $x = 2^{r_1}3^{r_2}\cdots 19^{r_8}$ and $y = 2^{s_1}3^{s_2}\cdots 19^{s_8}$. Then $[x, y] = 2^{\max\{r_1,s_1\}} \cdots 19^{\max\{r_8,s_8\}}$. Therefore $r_1, s_1, \dots, r_8, s_8$ must be chosen so that $\max\{r_1, s_1\} = 18$, $\max\{r_1, s_1\} = 18$, $\max\{r_2, s_2\} = 8$, $\max\{r_3, s_3\} = 4$, $\max\{r_4, s_4\} = 2$, …, $\max\{r_8, s_8\} = 1$. The ordered pair (r_1, s_1) can be chosen in 37 ways, the ordered pair (r_2, s_2) in 17 ways, etc. The answer is $37{\cdot}17{\cdot}9{\cdot}5{\cdot}3^4$.

5. Every prime less than n divides the product $n^2 \cdots (n^2 + n - 2)$, but x is coprime to it, so it has no prime divisors less than n, i.e. it must be prime.

6. It can be $p = q = 2$. Assume p is odd and $p^{q+1} + q^{p+1} = x^2$. Then $p^{q+1} = (x - q^{\frac{p+1}{2}})(x + q^{\frac{p+1}{2}})$. If both factors $x \pm q^{\frac{p+1}{2}}$ are divisible by p, then $p \mid 2q^{\frac{p+1}{2}}$, so $p = q$, but then $x^2 = 2p^{p+1}$ is not a square. Thus we must have $x - q^{\frac{p+1}{2}} = 1$ and $2q^{\frac{p+1}{2}} + 1 = x + q^{\frac{p+1}{2}} = p^{q+1}$. This is also impossible for q odd, as then $p^{q+1} \equiv 1$ and $2q^{\frac{p+1}{2}} + 1 \equiv 3 \pmod 4$. Therefore $q = 2$, so $2^{\frac{p+3}{2}} = p^3 - 1 = (p - 1)(p^2 + p + 1)$, which is impossible because $p^2 + p + 1 > 1$ is odd.

7. If p, q were both odd, then the alleged prime would be even. So either $p = 2$ or $q = 2$. But in these cases $p^3 + 3q^3 - 32$ reduces to $3(q^3 - 8)$ and $p^3 - 8$ respectively, and both are factorizable. Only $(p, q) = (3, 2)$ works.

8. The required number cannot have both an even digit and a digit 5 (else it would be divisible by 10). If it has digit 5, it can have at most 5 digits. Assume it has no digit 5. Not all other digits are present, as then the sum of digits would be 40, not divisible by 3. So one more digit is missing, and if it is not 9, it must be 4. Hence our number consists of digits $1, 2, 3, 6, 7, 8, 9$.

Our number cannot start with 987, as neither of the possible four-digit endings divisible by 8, namely 1632, 2136, 3216, 6312, produces a multiple of 7. But the largest number starting with 986 works, and that is 9867312.

9. The given number is even, so let $n \cdot 2^n + 4 = (2x)^2$. Then $n \geqslant 3$ and $x = 2y + 1$ is obviously odd. Now $y(y+1) = 2^{n-4}n$. One of $y, y+1$ is even and one is odd, so one is divisible by 2^{n-4} and the other divides n. Therefore $2^{n-4} - 1 \leqslant y \leqslant n$, which is only possible for $n \leqslant 7$. The only solution is now $n = 7$.

10. (a) If $n(n-10) = m^2$, then $m^2 - (n-5)^2 = 25$. Since $25 = 5 \cdot 5 = 25 \cdot 1$, number 25 can be written as a difference of two squares in two ways: $25 = 5^2 - 0^2 = 13^2 - 12^2$. These yields $n = 10$ and $n = 18$.
(b) Since n and $n^2 - 1$ are coprime, both must be squares, which is only for $n = 1$.

11. The sum $1! + 2! + \cdots + n!$ ends in digit 3 whenever $n \geqslant 5$. Only the small cases remain and $n = 1, 3$ are the only solutions.

12. Hint: Can a number whose sum of digits is 9 be divisible by 11?

13. We first observe that $(a^2+2b^2)(x^2+2y^2) = (ax \pm 2by)^2 + 2(ay \mp bx)^2$. Taking $a = b = 1$ and $3n = x^2 + 2y^2$ we find that $n = (\frac{x \pm 2y}{3})^2 + 2(\frac{y \mp x}{3})^2$. But since $3 \mid y^2 - x^2$, either $\frac{y-x}{3}$ or $\frac{y+x}{3}$ is an integer, giving a valid representation for n.

14. Since $p \mid q - 1$, we have $p < q$ and $q \equiv 1 \pmod{p}$. Next, since $q \nmid p - 1$ and q is prime, we know that $q \mid p^2 + p + 1$. Also, $\frac{p^2+p+1}{q} \equiv \frac{1}{1} = 1 \pmod{p}$, but $\frac{p^2+p+1}{q}$ is less than $p + 1$, so it must be equal to 1.

15. Let $d = n + k$. Then $n + k \mid n^2 - k^2$ by default, so $n + k \mid k^2 + 1$ as well. Hence $n + k \leqslant k^2 + 1$, i.e. $n \leqslant k^2 - k + 1 < k^2$, unless $k = 1$ (which would imply $n = 1$).

16. Let $n = de$. Then $nd + 1 = d^2e + 1 \mid n^2 + d^2 = d^2(e^2 + 1)$. Since $(d^2, d^2e + 1) = 1$, we have $d^2e + 1 \mid e^2 + 1$, so $d^2e + 1 \mid e^2 - d^2e = e(e - d^2)$. Since $(e, d^2e + 1) = 1$, we get $d^2e + 1 \mid e - d^2$, but $|e - d^2| < \max\{e, d^2\} < d^2e + 1$, so we must have $e = d^2$. Hence $(n, d) = (d^3, d)$.

17. Can n^2 with $n > 10^{50}$ always be written in that form? If $n^2 = a^2 + p$, then $p = (n + a)(n - a)$, so $a = n - 1$ and $p = 2n - 1$, so it is possible only when $2n - 1$ is a prime, which is not always the case.

18. If x is odd, then $2 \mid x^2 + 1$ and $8 \mid x^2 - 1$. so $16 \mid x^4 - 1$. Also, if x is even, then $16 \mid x^4$. Thus every fourth power gives the remainder 0 or 1 modulo 16. In particular, if $16n$ is a sum of 15 fourth powers, then all these fourth powers are even, so n is also a sum of 15 fourth powers. But for instance, number 31 cannot be written as a sum of 15 fourth powers. Thus neither can the numbers $31 \cdot 16$, $31 \cdot 16^2$, etc.

19. The products $a \cdot d$ and $b \cdot c$ are divisible by $|ad - bc|^2$, so $|ad - bc|^2$ divides $ad - bc$. Therefore $ad - bc = \pm 1$.

20. Three of the six given numbers are even. Of the odd numbers, at most one is divisible by 3 and at most one is divisible by 5. Therefore at least one number is not divisible by 2, 3 or 5, and that one is coprime to the others.

21. Let $d = \gcd(a, b)$, $a = dx$, $b = dy$. Then $\frac{a^2+b^2}{ab} = \frac{x^2+y^2}{x}y$, so both x and y divide $x^2 + y^2$. But both are coprime to $x^2 + y^2$, so they must be 1, i.e. $a = b$.

22. Note that $\frac{(2n)!}{n!} = \frac{2\cdot4\cdot6\cdots(2n)\cdot1\cdot3\cdots(2n-1)}{1\cdot2\cdot3\cdots n} = 2^n \cdot 1 \cdot 3 \cdots (2n-1)$.

23. Consider k such that $2^k \leqslant n < 2^{k+1}$. Then in the sum $H = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ only one summand has a denominator divisible by 2^k (i.e. $v_2 \leqslant -k$, so H itself has a denominator divisible by 2^k in lowest terms and hence cannot be an integer.

24. Set $x = \gcd(a, c)$; then $u = \frac{a}{x}$, $v = \frac{c}{x}$ are integers as well. Moreover, $y = \frac{b}{v} = \frac{bx}{c}$. But $c \mid ba$ and $c \mid bc$ imply that $c \mid b\gcd(a, c)$, so y is also an integer. Thus this quadruple (x, y, u, v) works.

25. Suppose that e.g. $v_p(x) = -k < 0$. Since $v_p(x + y + z) \geqslant 0$, one of $v_p(y), v_p(z)$ is $-k$, so the other one (since $xyz = 1$) is $2k$. Thus two of $v_p(x), v_p(y), v_p(z)$ are negative and one is positive. But then among $v_p(\frac{1}{x}), v_p(\frac{1}{y}), v_p(\frac{1}{z})$ only one is negative, so $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ cannot be an integer.

26. If $v_p(\frac{a}{b}) = v_p(\frac{b}{c}) = v_p(\frac{c}{a}) = 0$, then $v_p(a) = v_p(b) = v_p(c)$ and $3 \mid v_p(abc)$. Else one of $v_p(\frac{a}{b}), v_p(\frac{b}{c}), v_p(\frac{c}{a})$ is negative and one is positive, say $v_p(\frac{a}{b}) = -k < 0 < v_p(\frac{c}{a})$. If $v_p(\frac{b}{c})$ is not $-k$, then $v_p(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}) = \min\{v_p(\frac{a}{b}, \frac{b}{c})\} < 0$, which is impossible because $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is an integer. Therefore $v_p(\frac{b}{c}) = -k$, so $v_p(b) = v_p(a) + k$, $v_p(c) = v_p(b) + k$ and again $3 \mid v_p(abc) = 3v_p(b)$. So in every case $v_p(abc)$ is a multiple of 3, which means that abc is a cube.

27. Let $b = qa$. Then $(qa)^a = a^{qa}$ reduces to $a = q^{\frac{1}{q-1}}$. Let $\frac{1}{q-1} = \frac{m}{n}$, where m and n are coprime positive integers. Then $q = \frac{m+n}{m}$ and $a = (\frac{m+n}{m})^{m/n}$ is rational, implying that both m and $m + n$ (being coprime) must be n-th powers. Thus $m = x^n$, $m + n = y^n$ and $n = y^n - x^n \geqslant (x + 1)^n - x^n = nx^{n-1} + \cdots$ which is always greater than n unless $n = 1$. It follows that $q = \frac{m+1}{m}$ and $(a, b) = \big((\frac{m+1}{m})^m, (\frac{m+1}{m})^{m+1}\big)$, where $m \in \mathbb{N}$.

28. Setting $b = c = 1$ we get $n = 2a + 1$. Moreover, setting $b = c = 2^k$ and $a = 2^k t$, we get $n = 2^k(2t + 1)$, which covers all positive integers except powers of two.

    Now suppose that $2^k = [a, b] + [b, c] + [c, a]$, where k is smallest possible. Clearly, $k \geqslant 2$. At least two of a, b, c must be even (otherwise $[a, b] + [b, c] + [c, a]$ will be odd), say $a = 2a'$ and $b = 2b'$, whereas c is odd due to minimality of k. But then $2^k = 2[a', b'] + 2[b', c] + 2[a', c]$, yielding a solution for 2^{k-1}, a contradiction.

29. We shall prove that $v_p(a) = k$ is even for any prime p. Assume $k > 0$ and $v_p(b) = \ell$. Then $v_p(ab) = k + \ell$, $v_p(a^2) = 2k$ and $v_p(b^2) = 2\ell$, but if $2\ell \neq k$, we have $v_p(b^2 + a) = \min\{k, 2\ell\} < v_p(a^2)$, so $v_p(a^2 + b^2 + a) = \min\{k, 2\ell\} < k + \ell$, contradicting the divisibility. Therefore $k = 2\ell$ is even, as desired.

30. Observe that $2x \equiv 2 \equiv -2017 \pmod{2019}$, $2x \equiv 4 \equiv -2017 \pmod{2021}$ and $2x \equiv 6 \equiv -2017 \pmod{2023}$. Hence $2x + 2017$ is divisible by $2019 \cdot 2021 \cdot 2023$, that is, $x \equiv \frac{2019\cdot2021\cdot2023-2017}{2} \pmod{2019 \cdot 2021 \cdot 2023}$.