Saudi Arabia 2022
Level 3
Geometry – Homothety 3
Regis

Homothety of Circles: Consider two circles ω_1 and ω_2 with centers O_1 and O_2 and radii r_1 and r_2, respectively. Then the *exsimilicenter* of ω_1 and ω_2 os the point E on the line O_1O_2 **not** on segment O_1O_2 such that

$$\frac{EO_1}{EO_2}=\frac{r_1}{r_2}$$

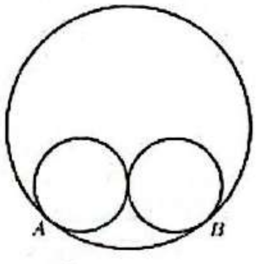
and the *insimilicenter* of ω_1 and ω_2 is the point I on the segment O_1O_2 such that

$$\frac{IO_1}{IO_2}=\frac{r_1}{r_2}$$

Monge-D'Alembert (Circle) Theorem: The pairwise exsmilicenter of three distinct circles are collinear. In particular, for three circles where one is not inside other the pairwise intersections of their common external tangents are collinear.

Problems

13. (AMC 10A/2018) Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



(A) 21 **(B)** 29 **(C)** 58 **(D)** 69 **(E)** 93

14. (AMC 12B/2011) Triangle ABC has $AB = 13$, $BC = 14$, and $AC = 15$. The points D, E and F are the midpoints of AB, BC and AC, respectively. Let $X \neq E$ be the intersection of circumcircles of ΔBDE and ΔCEF. What is $XA + XB + XC$?

(A) 24 **(B)** $14\sqrt{3}$ **(C)** $\frac{195}{8}$ **(D)** $\frac{129\sqrt{7}}{14}$ **(E)** $\frac{69\sqrt{2}}{4}$

15. (AMC 12B/2018) In ΔABC with side lengths $AB = 13$, $AC = 12$, and $BC = 5$, let O and I denote the circumcenter and incenter, respectively. A circle with center M is tangent to the legs AC and BC and to the circumcircle of ΔABC. What is the area of ΔMOI?

(A) $\frac{5}{2}$ **(B)** $\frac{11}{4}$ **(C)** 3 **(D)** $\frac{13}{4}$ **(E)** $\frac{7}{2}$

16. (AIME II/2007) Four circles ω, ω_A, ω_B and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B and ω_C. If the sides of triangle ABC are 13, 14 and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

17. (BMO/1990) Let ABC be an acute triangle and let A_1, B_1, C_1 be the feet of its altitudes. The incircle of the triangle $A_1B_1C_1$ touches its sides at the points A_2, B_2, C_2. Prove that the Euler lines of triangles ABC and $A_2B_2C_2$ coincide.

18. (Brazil Junior/2021) Let ABC be an acute-angled triangle. Let A_1 be the midpoint of the arc BC which contain the point A. Let A_2 and A_3 be the foot(s) of the perpendicular(s) of the point A_1 to the lines AB and AC, respectively. Define B_2, B_3, C_2, C_3 analogously.
a) Prove that the line A_2A_3 cuts BC in the midpoint.
b) Prove that the lines A_2A_3, B_2B_3 and C_2C_3 are concurrent.

19. Let Ω be the circumcircle of the triangle ABC and let ω_a be the circle tangent do the segment CA, segment AB and Ω. Define ω_b and ω_c similarly. Let ω_a, ω_b and ω_c touch Ω at points A', B' and C', respectively. Prove that the lines AA', BB' and CC' concur on the line OI, where O and I are the circumcenter and the incenter of ABC, respectively.

20. (IMO Shortlist/2006) Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concyclic.

21. (Brazil/2017) In triangle ABC, let r_A be the line that passes through the midpoint of BC and is perpendicular to the internal bisector of $\angle BAC$. Define r_B and r_C similarly. Let H and I be the orthocenter and incenter of ABC, respectively. Suppose that the three lines r_A, r_B, r_C define a triangle. Prove that the circumcenter of this triangle is the midpoint of HI.

22. (EGMO/2018) Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q. Prove that $\angle ABP = \angle QBC$.

23. (Romania TST/2007) Given a triangle ABC, let Γ_A be a circle tangent to AB and AC, let Γ_B be a circle tangent to BC and BA and let Γ_C be a circle tangent to CA and CB. Suppose that Γ_A, Γ_B and Γ_C are tangent to one another. Let D be the tangency point of Γ_B and Γ_C, E be the tangency point of Γ_A and Γ_C and F be the tangency point of Γ_A and Γ_B. Prove that the lines AD, BE and CF are concurrent.