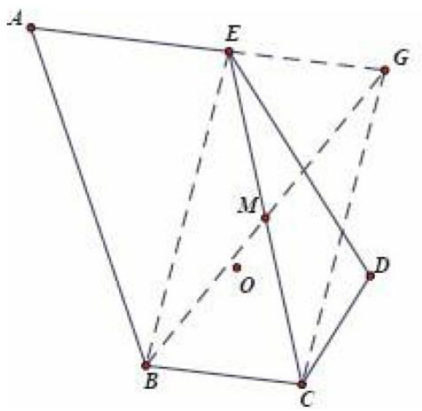
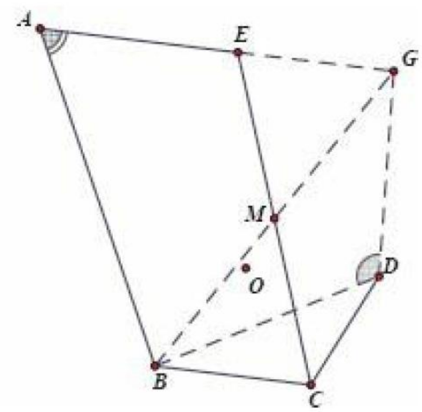
4.1. Refer to the diagram below. $ABCDE$ is a pentagon such that $BC \parallel AE$, $AB = BC + AE$ and $\angle B = \angle D$. Let M be the midpoint of CE and O be the circumcenter of ΔBCD. Show that if $OM \perp MD$ then $\angle CDE = 2\angle ADB$.

Sol: Refer to the left diagram below. Extend AE to G such that $BC = EG$. Since $AB = BC + AE$, we have $AB = AG$.



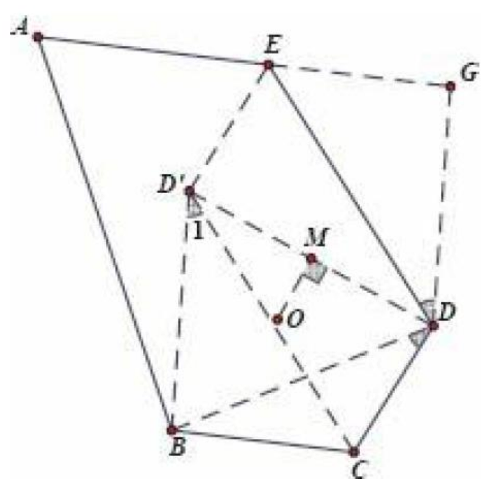


Now $\angle ABG = \angle AGB = \angle CBG$ (because $AE \parallel BC$), i.e., BG bisects $\angle ABC$. It is also easy to see that $BCGE$ is a parallelogram where M is the center.

We claim that A, B, D, G are coney clic. (1)

Notice that (1) would imply that $\angle ADB = \angle AGB$, which leads to the conclusion because $\angle AGB = \angle CBG = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE$.

Refer to the right diagram above. It suffices to show that $\angle BDG = 180° - \angle A$, where $180° - \angle A = \angle ABC = \angle CDE$. Hence, it suffices to show $\angle BDG = \angle CDE$, or $\angle BDC = \angle EDG$. (2)



Let D' be the reflection of D about OM. Refer to the diagram below. Since $OD = OD'$, D' must lie on $\odot O$ whose radius is OD. Notice that $\odot O$ is exactly the circumcircle of ΔBCD, i.e., B, C, D, D' are concyclic.

Now $\angle BDC = \angle 1$. (3)

On the other hand, one sees that $CDED'$ is a parallelogram because DD' and CE bisect each other at M

It follows that $CD' = DE'$ and $CD' \parallel DE$. Now it is easy to see that $\Delta BCD' \cong \Delta GED$ (S.A.S.). We conclude that $\angle EDG = \angle 1$. (4).

(3) and (4) imply (2), which completes the proof.

On the other hand