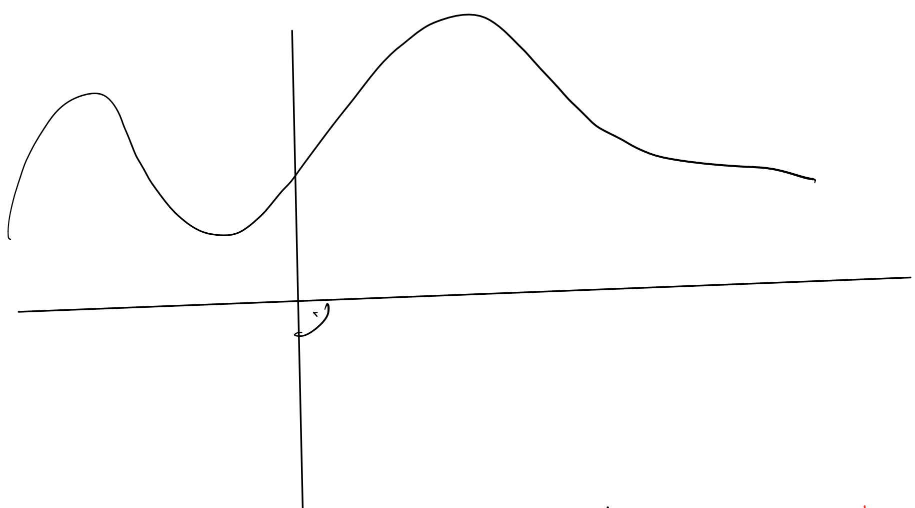
$\boxed{P8}$

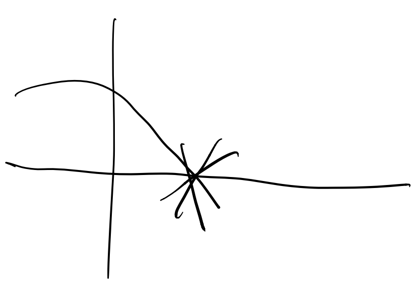
$$f_1, f_2, f_3, f_4 \in \underset{\text{real polynomials}}{\underset{\|}{\mathbb{R}[X]}}$$

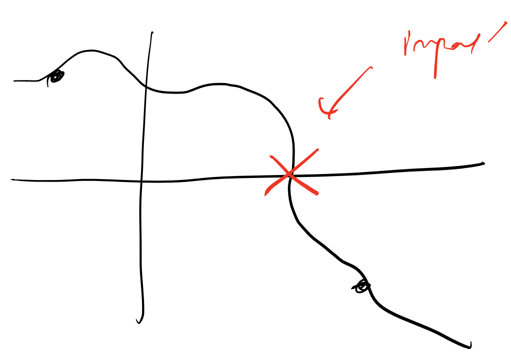
$\forall_{i,j}$ $f_i + f_j$ is even / pos or negative,

$f$ is $\in \mathbb{R}[X]$ then:

$f$ has no real root iff has constant sign.







$f_1 + f_2 + f_3 + f_4$ has real root

$\Downarrow$

$\exists\, i$ : $f_i$ has no any real root

---

Suppose $\forall_i$ $f_i$ is either positive or negative

$\vdots$

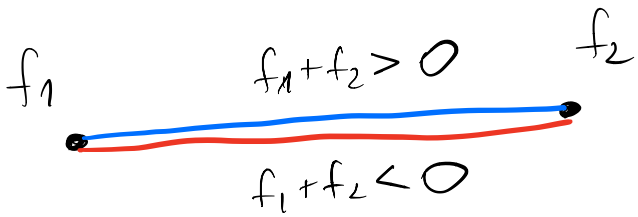
f polynomial then f has no real root iff

$\forall_x$ $f(x) > 0$ or

$\forall_x$ $f(x) < 0$

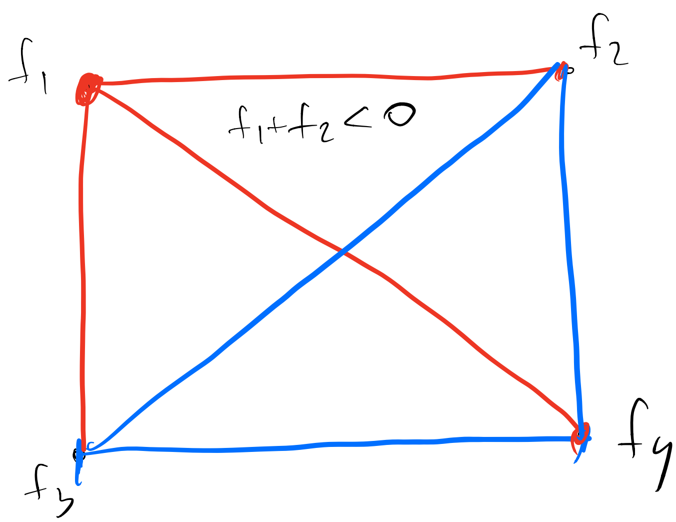
---

## $\boxed{\text{Male graph}}$





$\underline{f_i > 0 \text{ or } f_i < 0}$ .




Clai: $\exists$ vertex

$$f_1+f_2<0$$

$$f_1+f_3<0$$

$$f_1+f_4<0$$

$$\downarrow$$

$$3f_1+f_2+f_4+f_4<0$$

$$+\begin{cases} f_2+f_3>0 \\ f_2+f_4>0 \\ f_3+f_4>0 \end{cases}$$

$$2(f_2+f_3+f_4)>0$$

$f_1<0$ so $f_1$ ha no $\boxed{\text{real}}$ root $\square$

AM-GM

$$a_1^2 + 2a_2^3 + \dots + n a_n^{n+1} < 1$$

$$\Rightarrow \quad 2a_1 + 3a_2^2 + \dots + (n+1) a_n^n < 3$$

$$a_1^2 + \frac{1}{2^2} \geq 2\sqrt{a_1^2 \cdot \frac{1}{2^2}} = a_1$$

so

$$2a_1 \leq 2a_1^2 + \frac{1}{2}$$

$$a_2^3 + a_2^3 + \frac{1}{2^3} \geq 3\sqrt[3]{a_2^3 a_2^3 \cdot \frac{1}{2^3}} =$$

$$= 3a_2^2 \cdot \frac{1}{2}$$

$$3a_2^2 \leq 4a_2^3 + \frac{1}{2^2} \quad \dots$$

$$\underbrace{\underbrace{a_k^{k+1} + \dots a_k^{k+1}}_{k \text{ copies}} + \frac{1}{2^{k+1}}} \geq$$

$$\geq (k+1)\sqrt[k+1]{a_k^{k+1} \cdots a_k^{k+1} \cdot \frac{1}{2^{k+1}}}$$

$$= (k+1) \cdot a_k^k \cdot \frac{1}{2}$$

$$(k+1) a_k^k \leq 2k \cdot a_k^{k+1} + \frac{1}{2^k}$$

so the sum

$$\sum^{1}_{1} (k+1) a_k^k \leq 2 \cdot \sum^{1}_{1} k a_k^{k+1} + \sum^{1}_{1} \frac{1}{2^k} <$$

$$< 2 \cdot 1 + \sum^{1}_{1} \frac{1}{2^k} <$$

$$< \boxed{2 \cdot 1 + 1 = 3} \quad \square$$

P9 All monotoni $f:\mathbb{R}\to\mathbb{R}$

$$f(f(x)-y)+f(x+y)=0$$

$$-(-x-y)-(x+y)=0$$

$$\boxed{1^\circ \quad \text{Comput } f(0)\ ?}$$

$x=y=0$

$$f(f(0))+f(0)=0$$

$x=f(0)$, $y:=f(f(0))$

$$f(\underbrace{f(f(0))-f(f(0))}_{=0})+f(\underbrace{f(0)+f(f(0))}_{=0})=0$$

$$f(0)+f(0)=0$$

$\Downarrow$

$$\boxed{f(0)=0}$$

Is there any other $t \neq 0$

s.t $f(t) = 0$?

If yes put $y := f(x)$

$$f(0) + f(f(x) + x) = 0$$

$$\underset{0}{\shortparallel} \qquad f(x) + x = 0$$

$$\boxed{\cancel{f(x) = -x}}$$

- Why $f$ is odd?

$x := 0$

$$f(-y) + f(y) = 0$$

✓

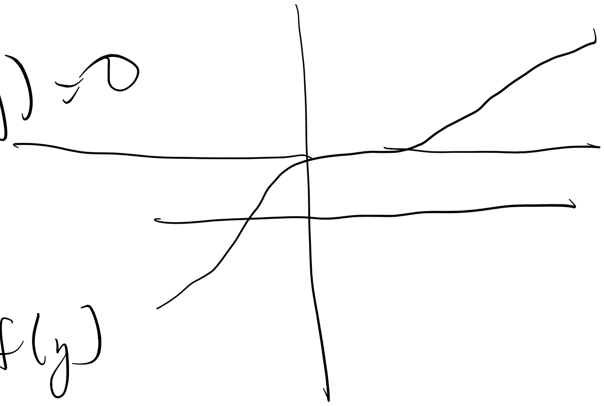
$t \neq 0$ s.t $f(t) = 0$

$\boxed{x = t}$

$f(-y) + f(t+y) = 0$

$\Downarrow$

$\forall y \quad f(y+t) = f(y)$



f is periodic

f is monotone $\Rightarrow$ constant

$\Downarrow$

f - constant

$f(0) = 0 \Rightarrow f(x) \equiv 0$

13 (IMO) $\sim$ 9