Email training, N7

Level 3, October 25-31

Problem 7.1. Find all positive integers n such that
$$3^{n-1} + 5^{n-1} | 3^n + 5^n.$$

Problem 7.2. The numbers in the sequence $101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

Problem 7.3. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ divides $abc - 1$.

Problem 7.4. Prove that every integer can be written as a sum of 5 perfect cubes (may be negative).

Problem 7.5. Let $\phi_n(m) = \phi(\phi_{n-1}(m))$, where $\phi_1(m) = \phi(m)$ is the Euler totient function, and set $\omega(m)$ the smallest number n such that $\phi_n(m) = 1$. If $m < 2^\alpha$, then prove that $\omega(m) \leq \alpha$.

Problem 7.6. Consider the lattice in the plane, from which we may cut rectangles, but only by making cuts along the lines of the lattice. Prove that for any integer $m > 12$ one may cut a rectangle o area grater than m such, that from that rectangle one can't cut a rectangle of area m.

Problem 7.7. Let ABC be a triangle with midpoints K, M, N of sides BC , CA , AB respectively. Let AD, BE, CF are the altitudes of the triangle ABC and let U, V, W are the midpoints of FD, DE , EF respectively. Prove that KW, MV, NU intersect at one point.

Solution submission deadline October 31, 2021

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