Email training, N4

October 2-8

Problem 4.1. Solve the system of equations

$$\begin{cases}(x-1)(y-1)(z-1) = xyz-1\\(x-2)(y-2)(z-2) = xyz-2.\end{cases}$$

Problem 4.2. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.

Problem 4.3. Let $1 = d_0 < d_1 < \ldots < d_m = 4k$ be all positive divisors of $4k$, where k is a positive integer. Prove that there exists $i, 1 \leq i \leq m$ such that $d_i - d_{i-1} = 2$.

Problem 4.4. Let k be a positive integer such that $p = 8k+5$ is a prime number. The integers $r_1, r_2, \ldots, r_{2k+1}$ are chosen so that the numbers $0, r_1^4, r_2^4, \ldots, r_{2k+1}^4$ give pairwise different remainders modulo p. Prove that the product

$$\prod_{1\leq i<j\leq 2k+1} (r_i^4 + r_j^4)$$

is congruent to $(-1)^{k(k+1)/2}$ modulo p.

Problem 4.5. Given n, find a finite set S consisting of natural numbers larger than n, with the property that, for any $k \geq n$, the $k \times k$ square can be tiled by a family of $s_i \times s_i$ squares, where $s_i \in S$.

Problem 4.6. Ann and Max play a game on a 100×100 board. First, Ann writes an integer from 1 to 10000 in each square of the board so that each number is used exactly once. Then Max chooses a square in the leftmost column and places a token on this square. He makes a number of moves in order to reach the rightmost column. In each move the token is moved to a square adjacent by side or by vertex. For each visited square (including the starting one) Max pays Ann the number of coins equal to the number written in that square. Max wants to pay as little as possible, whereas Ann wants to write the numbers in such a way to maximise the amount she will receive. How much money will Max pay Ann if both players follow their best strategies?

Problem 4.7. A circle with center O is inscribed in an angle. Let A be the reflection of O across one side of the angle. Tangents to the circle from A intersect the other side of the angle at points B and C. Prove that the circumcenter of triangle ABC lies on the bisector of the original angle.

Problem 4.8. AB intersects two equal circles, is parallel to the line joining their centres, and all the points of intersection of the segment and the circles lie between A and B. From the point A tangents to the circle nearest to A are drawn, and from the point B tangents to the circle nearest to B are also drawn. It turns out that the quadrilateral formed by the four tangents extended contains both circles. Prove that a circle can be drawn so that it touches all four sides of the quadrilateral.

Solution submission deadline October 8, 2022