Email training, N4

September 15-21

Problem 4.1. Prove that for all $n \geq 4$ the following inequalities hold $n! > 2^n$ and $2^n \geq n^2$.

Solution 4.1. For $n > 3$ one has

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot n > 1 \cdot 2 \cdot 2 \cdot 2^2 \cdot 2 \cdot \ldots \cdot 2 = 2^n.$$

For $n = 4$ one has $2^4 = 16 = 4^2$. By induction, for $n \geq 4$ one has

$$2^{n+1} \geq 2n^2 = (n+1)^2 + (n–1)^2 - 2 > (n+1)^2.$$

Problem 4.2. It is known that $a < 1$, $b < 1$ and $a+b \geq 0.5$. Prove that $(1-a)(1-b) \leq \frac{9}{16}$.

Solution 4.2. From the conditions of the problem follows that $1 - a \geq 0$ and $1 - b \geq 0$. By using the AM-GM inequality one gets

$$\sqrt{(1-a)(1-b)} \leq \frac{(1-a)+(1-b)}{2} = 1 - \frac{a+b}{2} \leq \frac{3}{4}.$$

By taking the square one gets the desired inequality.

Problem 4.3. Read the proof of Bernouli inequality. Conclude that $8^{91} > 7^{92}$.
(https://www.youtube.com/watch?v=7BZWeWZoVcY).

Solution 4.3.

$$\frac{8^{91}}{7^{91}} = \left(1 + \frac{1}{7}\right)^{91} > 1 + \frac{91}{7} > 7,$$

therefore

$$8^{91} > 7 \cdot 7^{91} = 7^{92}.$$

Problem 4.4. By using Bernouli inequality prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Solution 4.4.

$$\left(1 + \frac{5}{n}\right)^n > 1 + n \cdot \frac{5}{n} = 6,$$

therefore

$$1 + \frac{5}{n} > 6^{1/n}.$$

Also

$$\left(1 + \frac{-5}{6n}\right)^n > 1 + n \cdot \frac{-5}{6n} = \frac{1}{6},$$

$$\left(\frac{6n-5}{6n}\right)^n > \frac{1}{6},$$

$$6 > \left(\frac{6n}{6n-5}\right)^n,$$
$$6^{1/n} > \frac{6n}{6n-5} = 1 + \frac{5}{6n-5}.$$

Problem 4.5. Let
$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$
Prove that for $n \geq 3$ one has $a_{n+1} \geq a_n$ and based on this conclude that $a_{2019} > \frac{3}{5}$.

Solution 4.5. For $n = 3$ one has
$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} > \frac{3}{5}.$$

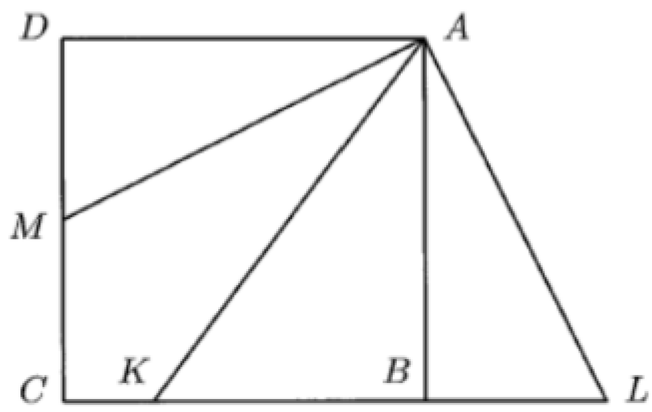
Note, that
$$a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0,$$
so $a_{2019} > a_{2018} > \ldots > a_4 > a_3 > \frac{3}{5}$.

Problem 4.6. Let a, b, c are positive and less than 1. Prove that
$$1 - (1-a)(1-b)(1-c) > k,$$
where $k = max(a, b, c)$.

Solution 4.6. Since $0 < 1-a, 1-b, 1-c < 1$ therefore one may state that
$$1 - k > (1-a)(1-b)(1-c),$$
since in right side one multiplier is equal to $1-k$ and two others are positive and less than one. From that inequality immediately follows that
$$1 - (1-a)(1-b)(1-c) > k.$$

Problem 4.7. In the square $ABCD$ let K is a point on the side BC and the bisector of $\angle KAD$ meets the side CD at point M. Prove that $AK = DM + BK$.

Solution 4.7. -
Extend KB to L such that $BL = DM$.



Since $AB = AD$ and $\angle ABL = 90° = \angle ADM$, triangles ABL and ADM are congruent. Hence $\angle BAL = \angle DAM$ and $\angle ALK = \angle AMD$. Now

$$\begin{aligned}
\angle KAL &= \angle BAL + \angle KAB \\
&= \angle MAD + \angle KAB \\
&= \angle MAK + \angle KAB \\
&= \angle MAB \\
&= \angle AMD
\end{aligned}$$

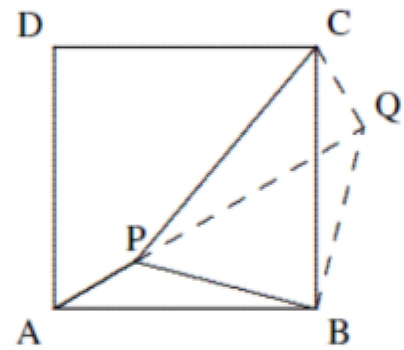
since AB and DC are parallel. It follows that $\angle KAL = \angle ALK$, and therefore $AK = KL = KB + BL = KB + DM$.

Problem 4.8. Let $ABCD$ is a square, P is an inner point such that $PA : PB : PC = 1 : 2 : 3$. Find $\angle APB$ in degrees.

Solution 4.8. -

Without loss of generality, we assume that $PA = 1, PB = 2, PC = 3$.

Rotate the $\triangle APB$ around B by $90°$ in clockwise direction, such that $P \to Q, A \to C$, then $\triangle BPQ$ is an isosceles right triangle, therefore



$$PQ^2 = 2PB^2 = 8, CQ^2 = PA^2 = 1,$$

therefore, by Pythagoras' Theorem,

$$PC^2 = 9 = CQ^2 + PQ^2, \ \ \angle CQP = 90°.$$

Hence $\angle APB = \angle CQB = 90° + 45° = 135°$.