May Online Camp 2021

Number Theory

Level L2

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(01-06-2021, 10:25)

Problems

Problem 1. A positive integer is called *nice* if it can be represented as a sum of two squares of non-negative integers. Prove that any positive integer is the difference of two nice numbers.

Problem 2. Let a, b be positive integers such that ab and $(a+1)(b+1)$ are squares. Prove that there is integer $n > 1$ such that $(a+n)(b+n)$ is also square.

Problem 3. Find all the natural numbers a, b, c such that:

- $a^2 + 1$ and $b^2 + 1$ are primes,
- $(a^2 + 1)(b^2 + 1) = (c^2 + 1)$.

Problem 4. Let a, b, c, d be a positive integers such that $ad = b^2 + bc + c^2$. Prove that $a^2 + b^2 + c^2 + d^2$ is composite.

Problem 5. Let a, b, c be positive integers. Prove that there is a positive integer n such that

$$(a^2 + n)(b^2 + n)(c^2 + n)$$

is a perfect square.

Problem 6. Let a, $b > 1$ be integers such that $a^2 + b$, and $a + b^2$ are primes. Prove $\gcd(ab + 1, a + b) = 1$.

Problem 7. Let n be a positive integer. Prove that there exists positive integers a and b, such that

$$a^2 + a + 1 = (n^2 + n + 1)(b^2 + b + 1).$$

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that

$$a = \frac{x+y}{z} \quad \text{and} \quad b = \frac{xy}{z}.$$

Problem 9. We say that a positive integer is an almost square, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.

Problem 10. Let a, b, z be positive integers such that $ab = z^2 + 1$. Prove that there are positive integers such x, y such that

$$\frac{a}{b} = \frac{x^2 + 1}{y^2 + 1}.$$

Problem 11. Prove that there are infinitely many pairwise distinct positive integers a, b, c and d such that $a^2 + 2cd + b^2$ and $c^2 + 2ab + d^2$ are squares.

Problem 12. Let a, b, c, n be positive integers such that the following conditions hold

(i) numbers a, b, c, $a + b + c$ are pairwise coprime,
(ii) number $(a + b)(b + c)(c + a)(a + b + c)(ab + bc + ca)$ is a perfect n-th power.

Prove, that the product abc can be expressed as a difference of two perfect n-th powers.

Pro☠lem 13. Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Problem 14. Prove that any rational number may be written as

$$\frac{a^2 + b^3}{c^5 + d^7},$$

where a, b, c, d are positive integers.

Problem 15. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a+b+c+d$ and s divides $abc+abd+acd+bcd$.

Solutions

Problem 1. A positive integer is called *nice* if it can be represented as a sum of two squares of non-negative integers. Prove that any positive integer is the difference of two nice numbers.

Solution. Note that

$$2a-1=a^2-(a-1)^2 \quad \text{and} \quad 2a=(a^2+1^2)-(a-1)^2.$$

We just need to make sure that all nice numbers must be positive which is fine as soon as $a \geq 2$.

But for $a=1$ we can certainly write $1=(1^2+1^2)-1^2$ and $2=2^2-(1^2+1^2)$. $\square$

Discussion.

Problem 2. Let a, b be positive integers such that ab and $(a+1)(b+1)$ are squares. Prove that there is integer $n>1$ such that $(a+n)(b+n)$ is also square.

Solution. Take $n=ab$. Then

$$(a+n)(b+n)=ab(a+1)(b+1)$$

is a product of squares and hence it is a square. $\square$

Discussion.

Problem 3. Find all the natural numbers a, b, c such that:
- a^2+1 and b^2+1 are primes,
- $(a^2+1)(b^2+1)=(c^2+1)$.

Solution. Assume that $a \geqslant b$, then $c^2+1 \leqslant (a^2+1)^2$ and so $c<a^2+1$. Now

$$a^2+1 \mid (c^2+1)-(a^2+1)=(c-a)(c+a).$$

Note that

$$0<c-a<c+a<a^2+a+1<2(a^2+1),$$

so $a^2-a+1=c$ and hence $b^2-1=a^2-2a+2$. But a^2+1 and b^2+1 are primes only for $a=1$ or 2. Since $b>0$, then it follows that $a=2$, $b=1$, $c=3$. $\square$

Discussion.

Problem 4. Let a, b, c, d be a positive integers such that $ad = b^2 + bc + c^2$. Prove that $a^2 + b^2 + c^2 + d^2$ is composite.

Solution. Given condition implies that $2ad = b^2 + c^2 + (b + c)^2$. Therefore

$$(a + d)^2 - (b + c)^2 = (a + d)^2 - 2ad + b^2 + c^2 = a^2 + b^2 + c^2 + d^2.$$

Thus

$$a^2 + b^2 + c^2 + d^2 = (a + d + b + c)(a + d - b - c).$$

The factor $a + b + c + d < a^2 + b^2 + c^2 + d^2$ is greater then 1, since otherwise $a = b = c = d = 1$. Hence $a + b + c + d$ is a proper divisor of $a^2 + b^2 + c^2 + d^2$. □

Discussion.

Problem 5. Let a, b, c be positive integers. Prove that there is a positive integer n such that

$$(a^2 + n)(b^2 + n)(c^2 + n)$$

is a perfect square.

Solution. Let $n = ab + bc + ca$, then

$$(a^2 + n)(b^2 + n)(c^2 + n) = (a^2 + ab + bc + c)(b^2 + ab + bc + ca)(c^2 + ab + bc + ca) =$$
$$= (a + b)^2(b + c)^2(c + a)^2.$$

□

Discussion.

Problem 6. Let a, $b > 1$ be integers such that $a^2 + b$, and $a + b^2$ are primes. Prove $\gcd(ab + 1, a + b) = 1$.

Solution. Assume $p \mid ab + 1, a + b$ for some prime p. Then, we have $(a + 1)(b + 1) = ab + 1 + a + b \equiv 0 \pmod{p}$. Thus, we have $a + 1 \equiv 0 \pmod{p}$ or $b + 1 \equiv 0 \pmod{p}$.

WLOG, we assume $a \equiv -1 \pmod{p}$. Since, $a + b \equiv 0 \pmod{p}$, we must have $b \equiv 1 \pmod{p}$. Then, $b^2 + a \equiv 0 \pmod{p}$. Therefore, $b^2 + a = p$ must be satisfied. Then, $p > b^2 \geq b \equiv 1$. Therefore, $b = 1$, a contradiction. □

Discussion.

Problem 7. Let n be a positive integer. Prove that there exists positive integers a and b, such that

$$a^2 + a + 1 = (n^2 + n + 1)(b^2 + b + 1).$$

Solution. Take $a = n^2$ and $b = n-1$, then
$$\frac{a^2+a+1}{b^2+b+1} = \frac{n^4+n^2+1}{(n-1)^2+(n-1)+1} = \frac{n^4+n^2+1}{n^2-n+1} = n^2+n+1.$$
□

Discussion.

Problem 8. Let a, b be positive integers such that $a \mid b+1$. Prove that there exists positive integers x, y, z such that
$$a = \frac{x+y}{z} \quad \text{and} \quad b = \frac{xy}{z}.$$

Solution. Take
$$x = \frac{b+1}{a}, \quad y = \frac{b(b+1)}{a}, \quad z = \frac{(b+1)^2}{a^2}.$$
□

Discussion.

Problem 9. We say that a positive integer is an almost square, if it is equal to the product of two consecutive positive integers. Prove that every almost square can be expressed as a quotient of two almost squares.

Solution. Note that
$$a(a-1) = \frac{(a^2-1)a^2}{(a-1)a}.$$
□

Discussion.

Problem 10. Let a, b, z be positive integers such that $ab = z^2+1$. Prove that there are positive integers such x, y such that
$$\frac{a}{b} = \frac{x^2+1}{y^2+1}.$$

Solution. Let $x = z+a$ and $y = z+b$. Then
$$\begin{aligned}\frac{x^2+1}{y^2+1} &= \frac{(z+a)^2+1}{(z+b)^2+1} = \frac{z^2+1+2za+a^2}{z^2+1+2zb+b^2} = \\ &= \frac{ab+2za+a^2}{ab+2zb+b^2} = \frac{a(a+b+2z)}{b(a+b+2z)} = \frac{a}{b}.\end{aligned}$$
□

Discussion.

Problem 11. Prove that there are infinitely many pairwise distinct positive integers a, b, c and d such that $a^2 + 2cd + b^2$ and $c^2 + 2ab + d^2$ are squares.

Solution. It is enough to take distinct a, b, c and d for which $ab = cd$. For example, $b := 6a$, $c := 2a$ and $d := 3a$. □

Discussion.

Problem 12. Let a, b, c, n be positive integers such that the following conditions hold

(i) numbers a, b, c, $a + b + c$ are pairwise coprime,
(ii) number $(a+b)(b+c)(c+a)(a+b+c)(ab+bc+ca)$ is a perfect n-th power.

Prove, that the product abc can be expressed as a difference of two perfect n-th powers.

Solution. Note that, $(a + b + c, a) = (a + b + c, b + c) = 1$. Moreover, we also have, $(a + b + c, a + b) = (a + b + c, c) = 1$ and $(a + b + c, a + c) = (a + b + c, b) = 1$. Therefore, $(a + b)(a + c)(b + c)$ and $a + b + c$ are coprime.

Next, let $p \mid a+b$ be a prime number. We shall prove that $p \nmid ab+bc+ca$. Assume the converse. Let $p \mid ab+bc+ca$. Then $p \mid a+b \Rightarrow p \mid ac+bc$, and thus, $p \mid ab$. Thus, either $p \mid a$, in which case $p \mid a + b$ yields $p \mid b$, contradicting with the coprimality of a and b. Similar holds for $p \mid b$. Thus, $\gcd((a + b)(a + c)(b + c), ab + bc + ca) = 1$.

Now, this yields that, $\gcd((a + b)(a + c)(b + c), (a + b + c)(ab + bc + ca)) = 1$. Since the product is a perfect power, it therefore holds that for some m, k integers, $(a + b)(a + c)(b + c) = m^n$ and $(a + b + c)(ab + bc + ca) = k^n$. Thus,

$$abc = (a + b + c)(ab + bc + ca) - (a + b)(a + c)(b + c) = k^n - m^n,$$

as claimed. □

Discussion.

Pro☠lem 13. Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Solution. Note that given condition is equivalent to

$$(a - b)(a + b + d) = (c + d)(d + a - c),$$

so by Four Numbers Theorem there exist $p, q, r, s \in \mathbb{N}$ such that

$$a - b = pq, \quad a + b + d = rs, \quad c + d = pr, \quad d + a - c = qs.$$

Therefore

$$3(ab + cd) = (r - q)(r + q)(s^2 - ps + p^2)$$

which is composite. □

Discussion.

Problem 14. Prove that any rational number may be written as
$$\frac{a^2+b^3}{c^5+d^7},$$
where a, b, c, d are positive integers.

Solution. For any positive integers p, q the following holds
$$\frac{p}{q} = \frac{p}{q}\cdot\frac{p^5q^4+p^{14}q^6}{p^5q^4+p^{14}q^6} = \frac{p^6q^4+p^{15}q^6}{p^5q^5+p^{14}q^7} = \frac{(p^3q^2)^2+(p^5q^2)^3}{(pq)^5+(p^2q)^7}.$$
□

Discussion.

Problem 15. Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a+b+c+d$ and s divides $abc+abd+acd+bcd$.

Solution. Observe that $a+b+c+d \mid abc+abd+acd+bcd$ is equivalent to
$$\begin{aligned} 0 &\equiv abc+(ab+bc+ca)d \\ &\equiv abc-(a+b+c)(ab+bc+ca) \\ &\equiv -(a+b)(b+c)(c+a) \pmod{a+b+c+d}. \end{aligned}$$
Note that $a+b$, $b+c$, $c+a$ are each less than $a+b+c+d$, so the condition cannot hold if $s=a+b+c+d$ is prime. Moreover, each non-prime $s=mn$ can be attained by taking $a=1$, $b=m-1$, $c=n-1$, and $d=(m-1)(n-1)$, so the answer follows. □

References

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- Homepage of Dominik Burek - http://dominik-burek.u.matinf.uj.edu.pl