Email training, N3

Level 4, September 27-October 3

Problem 3.1. Prove there exist infinitely many positive integers divisible by 2021 and each of them containing the same number of digits 0, 1, ..., 9.

Solution 3.1. Since $gcd(10, 2021) = 1$, then, according to Euler's theorem $10^{\phi(2021)} - 1$ is divisible by 2021. So

$$10^{\phi(2021)} - 1 = \underbrace{99......99}_{\phi(2021)} = 9 \cdot \underbrace{11\dots11}_{\phi(2021)}$$

is divisible by 2021. Since numbers 9 and 2021 are coprime, then $\underbrace{11\dots11}_{\phi(2021)}$ is divisible by 2021.

So

$$N = \underbrace{11\dots11}_{\phi(2021)}\underbrace{22\dots22}_{\phi(2021)}\dots\underbrace{99\dots99}_{\phi(2021)}\underbrace{00\dots00}_{\phi(2021)}$$

is multiple of $\underbrace{11\dots11}_{\phi(2021)}$, which is divisible by 2021. It means, that N is divisible by 2021. By "gluing" numbers N together one gets infinitely many integers satisfying to the conditions of the problem.

Problem 3.2. Do there exist positive integers m and n such that the decimal representation of 5^m starts with 2^n and the decimal representation of 2^m starts with 5^n.

Solution 3.2. Lets assume such an m and n exists. Then

$$2^m = 5^n \times 10^x + A$$

where $0 < A < 10^x$. From this follows, that

$$0 < 2^m - 5^n \cdot 10^x < 10^x$$

which means

$$\frac{2^m}{5^n + 1} < 10^x < \frac{2^m}{5^n}. \tag{3.2.1}$$

In the same way one gets

$$5^m = 2^n \cdot 10^y + B,$$

where $0 < B < 10^y$, therefore

$$\frac{5^m}{2^n + 1} < 10^y < \frac{5^m}{2^n}. \tag{3.2.2}$$

By multiplying (3.2.1) and (3.2.2) we get

$$\frac{10^m}{10^n + 2^n + 5^n + 1} < 10^{x+y} < 10^{m-n}.$$

Its easy to conclude, that

$$\frac{10^m}{10^{n+1}} < \frac{10^n}{4 \cdot 10^n} < \frac{10^m}{10^n + 2^n + 5^n + 1},$$

therefor

$$\frac{10^m}{10^{n+1}} < 10^{x+y} < 10^{m-n}$$

which is impossible.

Answer: Don't exist.

Problem 3.3. Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate the sum
$$\sum_{k=0}^{2021} f\left(\frac{k}{2021}\right).$$

Solution 3.3. Lets prove, that $f(x) + f(1-x) = 1$. Indeed
$$f(x) + f(1-x) = \frac{9^x}{9^x+3} + \frac{9^{1-x}}{9^{1-x}+3} =$$
$$\frac{9^x}{9^x+3} + \frac{9}{9+3\cdot 9^x} = \frac{9^x}{9^x+3} + \frac{3}{3+\cdot 9^x} = 1:$$
So
$$\sum_{k=0}^{2021} f\left(\frac{k}{2021}\right) = \sum_{k=0}^{1010}\left(f\left(\frac{k}{2021}\right) + f\left(\frac{2021-k}{2021}\right)\right) = 1011.$$
Answer: 1011.

Problem 3.4. Let the sequence a_i is defined in the following way: $a_1 = m \in Z_+$ and inductively $a_{i+1} = a_i + \left[\sqrt{a_i}\right]$. Prove that the sequence a_i contains infinitely many perfect square.

Solution 3.4. Let $m = n^2 + r$ where $0 \le r < 2n+1$. In the case of $r = 0$ the statement of the problem is satisfied. Consider two cases:

Case 1: Let $1 \le r \le n$. Then
$$a_1 = n^2 + n + r$$
and
$$a_2 = n^2 + 2n + r = (n+1)^2 + (r-1).$$
So after two interations we get a number which is 'closer' to perfect square. Therefore we can state, that a_{2r} is a perfect square.

Case 2: Let $n+1 \le r \le 2n$. Then
$$a_1 = n^2 + n + r = (n+1)^2 + (r-n-1)$$
where $0 \le r-n \le n-1$. If $r-n = 0$ then we are done, otherwise we come to the case 1, and conclude, that $a_{1+2(r-n-1)}$ is a perfect square.

Problem 3.5. In the cells of the grid 10×10 are written positive integers, all of them less than 11. It is known that the sum of 2 numbers written in the cells having common vertex is a prime number. Prove that there are 17 cells containing the same number.

Solution 3.5. Let's do contradiction, which means no number is written more than 16 times. Lets split the board to 25 squares of size 2×2.

Let A is the number of squares that don't contain number 1. Then $A \ge 25 - 16 = 9$. Since we have at most 16 numbers 1. then $25 - A \le 16$, so $A \ge 9$. Let split integers from 2 to 10 to the groups $\{3, 6, 9\}$, $\{5\}$, $\{7\}$, $\{2, 4, 8, 10\}$. From the condition of the problem follows, that in each square can be at most one number from each group. Since the square has 4 cells, it means that in each square that don't contain 4, has numbers from all groups. So in the mentioned above A squares there are both 5 and 7. In other squares can be at most $16 - A$ times number 5 and $16 - A$ times number 7. So, in other squares number 5 and 7 may appear at most $16 - A + 16 - A = 32 - 2A$ times, but in each square should be at least one of them. So there are $32 - 2A \ge 25 - A$ which means $A \le 7$. But we had $A \ge 9$, which is contradiction.

Problem 3.6. In each cell of a chessboard (sizes 8×8) is put a rock. At each step one can remove from the board one rock which beats an odd number of other rocks (for example in initial configuration top-left rock beats 2 rocks). Find the maximal possible number of rocks one can remove from the board.

Solution 3.6. First, lets prove that one can't remove 4 a rock from the corner cell. Assume at some moment one of them is removed. It means that rock beats exactly 1 rock, which is not possible, since on its row and on its column there are other corner rocks.

Now, lets prove that also one more rock will remain on the board. Assume one removes a rock and it remain only 4 corner rocks. Consider that rock and its position on the board. If it is on 1st or 8th row or column, then it beats just 2 corner rock, otherwise it beats no any rock, so it can't be removed. So at least 5 rock will remain on the board.

Below is an example how to remove rock to keep only 5 rock on the board (numbers are the order how to remove rocks).

	58	56	54	52	50	48	
	59	57	55	53	51	49	47
1	5	9	13	17	21	46	25
2	6	10	14	18	22	45	26
3	7	11	15	19	23	44	27
4	8	12	16	20	24	43	28
29	30	32	34	36	38	42	40
	31	33	35	37	39	41	

Answer: 59.

Problem 3.7. Given ΔABC where $AB < AC$, M is the midpoint of BC. The circle O passes through A and is tangent to BC at B, intersecting the lines AM, AC at D, E respectively. Let $CF \parallel BE$, intersecting BD extended at F. Let the lines BC and EF intersect at G. Show that $AG = DG$.