HW2

$$f : A \longrightarrow B$$

$$g : B \longrightarrow C$$

$$h = g \circ f : A \longrightarrow C$$

f, g (1) injective (2) surjective $\overset{?}{\Longrightarrow}$ h injective / surjective

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(1) suppose $h(a) = h(a')$

$a, a' \in A \Rightarrow g(f(a)) = g(f(a'))$

$\Longrightarrow f(a) = f(a') \Rightarrow a = a'$

(2) $(\exists! \; \forall c \in C \; \exists a \in A$

$\text{s.t } h(a) = c)$

Let $c \in C$

g surjective $\Longrightarrow$

$\exists \, b \in B \text{ s.t } g(b) = c$

f surjective $\Longrightarrow$

$\exists \, a \in A \text{ s.t } f(a) = b$

$\Rightarrow h(a) = g(f(a)) = g(b) = c$ ✓

Problem 9

$f : \mathbb{N} \longrightarrow \mathbb{N}$

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

$\forall n \in \mathbb{N}$.

Solution

if $f(a) = f(b) \implies$

$$3a = f_3(a) + f_2(a) + f(a)$$
$$= f_3(b) + f_2(b) + f(b) = 3b$$

$\iff a = b \implies f$ is injective

Now: $f(f(f(1))), f(f(1)), f(1) \geq 1$

but $f_1(1) + f_2(1) + f(1) = 3$

$\implies f(1) = 1$

Now we prove by induction that $f(n) = n \quad \forall\, n \in \mathbb{N}$.

$n = 1$ proved ✓

Suppose

$f(1) = 1, f(2) = 2, \ldots, f(n-1) = n-1$

for some $n \geq 2$

by injectivity:

$$\left[\, f(m) \geq n \quad \forall\, m \geq n \quad (*) \,\right]$$

$$\left(\because f(m) \in \mathbb{N} \;;\; f(m) \neq f(k) = k \quad \forall\, 1 \leq k < n \right)$$

$$\overset{(*)}{\Longrightarrow} f(\overbrace{n}^{m}) \geq n$$

$$\overset{(*)}{\Longrightarrow} f(\underbrace{f(n)}_{m}) \geq n$$

$$\overset{(*)}{\Longrightarrow} f(f(\overset{m}{f(n)})) \geq n$$

$$\Longrightarrow 3n = f(n) + f(f(n)) + f(f(f(n))) \geq 3n$$

$\to$ equality case holds

$\to f(n) = n \quad \forall n \in \mathbb{N}$

which is indeed a solution

Problem 11) $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(f(x)) = (x-1)f(x) + 2$$

??

$\therefore \Longrightarrow f$ not surjective

Solution suppose f surjec

if $f(a) = f(b) \neq 0$

$\Rightarrow (a-1)f(a) = (b-1)f(b)$

$\Rightarrow a = b$ $(*)$

Pick t st $f(t) = 0$

$\Rightarrow f(0) = 2$

plug $x = 1 \Rightarrow f(f(1)) = 2 \neq 0$

$= f(0)$

$\Rightarrow f(1) = 0$ $(\because (*))$

now pick s s.t $f(s)=1$

$\Rightarrow 0 = (s-1)+2$

$\Rightarrow s = -1 \quad (f(-1)=1)$

now pick r s.t $f(r)=-1$

$\Rightarrow 1 = -(r-1)+2 \iff r=2$

$(f(2)=-1)$

now plug $x=0 \Rightarrow$

$-1 = (-1)2+2 = 0$ $\lightning$ $\blacksquare$

Problem 10 $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

$$(x+y) f(yf(x)) = x^2 (f(x) + f(y))$$

$\forall x, y > 0$

Solution First we prove that f is injective.

if $f(a) = f(b)$ now

$P(a, y), P(b, y) \Longrightarrow$

$$f(yf(a)) = \frac{a^2}{a+y}(f(a)+f(y))$$

$$\|$$

$$f(yf(b)) = \frac{b^2}{b+y}(f(b)+f(y))$$

$$\Longrightarrow \frac{a+y}{a^2} = \frac{b+y}{b^2} \quad \forall y>0$$

$$\frac{1}{a} + \frac{1}{a^2}y = \frac{1}{b} + \frac{1}{b^2}y$$

$$\Longrightarrow a=b \Longrightarrow \boxed{f \text{ injective}}$$

$P(1,1) \Longrightarrow$

$$2f(f(1)) = 2f(1)$$

$\xRightarrow{\text{injective}} f(1) = 1$

$P(1,x) \Longrightarrow$

$$(1+x)f(x) = 1 + f(x)$$

$$\Longleftrightarrow f(x) = \frac{1}{x} \quad \forall x > 0$$

Check: $LHS = (x+y)f(yf(x)) = (x+y)\dfrac{x}{y}$

$RHS = x^2\left(\dfrac{1}{x} + \dfrac{1}{y}\right) =$ $\blacksquare$

Problem 12 | $f : \mathbb{R} \to \mathbb{R}$

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

$\forall\ x, y$ (IMO 2010)

Solution

$P(0, x) \Rightarrow f(0) = f(0) \lfloor f(x) \rfloor$

Case 1: $f(0) \neq 0 \Rightarrow \lfloor f(x) \rfloor = 1$

$\forall x \in \mathbb{R} \cdot P(x, y)$

$f(\lfloor x \rfloor y) = f(x)$, $\forall x, y$

$P(1, x) \Rightarrow f(x) = f(1)\ \forall x$

$$\Rightarrow \boxed{\begin{array}{l} f(x) = c \quad \forall x \qquad \boxed{\text{sol } 1} \\ \text{where} \qquad \lfloor c \rfloor = 1 \end{array}}$$

Case 2: $f(0) = 0$

$P(t, x)$ s.t $0 < t < 1$

$$\Rightarrow 0 = f(t) \lfloor f(x) \rfloor$$

$$\Rightarrow 0 = f(t) \lfloor f(t) \rfloor$$

$\forall 0 < t < 1$

$$\Rightarrow \boxed{(*) \ \lfloor f(t) \rfloor = 0 \quad \forall 0 \leq t < 1}$$

Now $P(x,t)$, $0<t<1$

(*)

$\Rightarrow f(\lfloor x \rfloor t) = 0$

$\Leftrightarrow \boxed{f(nt) = 0 \quad (\square)}$

$\forall n \in \mathbb{Z},\ 0 \leq t < 1$

Let $x \in \mathbb{R}$; $\exists n \in \mathbb{Z}$

such that $0 \leq \frac{x}{n} < 1$

$\Rightarrow f(x) = f(n \cdot \frac{x}{n}) = 0 \ \forall x$

Therefore, we have one solution:

$$f(x) = c \quad \forall\, x \in \mathbb{R},$$

where $c = 0$ or $\lfloor c \rfloor = 1$ is constant $\blacksquare$