Email training, N7-8

October 8 - 19, 2019

Problem 7.1. Find all positive integers n and k for which

$$n^2 + 7 = 2 \cdot 3^k.$$

Problem 7.2. There are two irreducible rational numbers with denominators 600 and 700. Find the minimal possible value of the denominator of their sum.

Problem 7.3. Solve in integers

$$x^2 + y^2 + z^2 = 2xyz.$$

Problem 7.4. Prove that $40^{1963} + 1963^{40}$ is composite number.

Problem 7.5. Let n and q are positive integers, such that all prime divisors of q are greater than n. Show that

$$(q-1)(q^2-1)\dots(q^{n-1}-1) \equiv 0[n!].$$

Problem 7.6. Find all pairs of integers (m, n) such that

$$\binom{n}{m} = 1984.$$

Problem 7.7. Prove that for any positive integer n the following identity holds

$$\frac{2n-1}{2} - \frac{2n-2}{3} + \dots - \frac{2}{2n-1} + \frac{1}{2n} = \frac{1}{n+1} + \frac{3}{n+2} + \dots + \frac{2n-1}{2n}.$$

Problem 7.8. Let a, b, c be a positive real numbers such that $abc = 8$. Prove that

$$\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2} \geq 6.$$

Problem 7.9. Let a sequence of positive integers $a_1, a_2, \dots$ is given with $a_1 = 1$ and

$$a_{n+1} \leq 1 + a_1 + a_2 + \dots + a_n.$$

Prove that any positive integer N can be written as a sum of distinct terms of the sequence $\{a_n\}$.

Problem 7.10. In triangle ABC let $\angle C = 90^o$ and let D is the midpoint of AB. Let E and F are two points on AC and BC respectively, such that DE and DF are perpendicular. Prove that $EF^2 = AE^2 + BF^2$.

Problem 7.11. Let AB and CD are segments lying on the two sides of an angle whose vertex is O, such that A is between O and B, as well as C is between O and D. The line connecting the midpoints of the segments AD and BC intersects AB at M and CD at N. Prove that $\frac{OM}{ON} = \frac{AB}{CD}$.

Problem 7.12. -

(1) لدينا ABC مثلث متطابق الأضلاع النقطتان E, D تقعان على الضلعين AC, BC على الترتيب بحيث $AE = DC$. النقطة Q تقع على القطعة المستقيمة AD وبحيث $\angle AQB = 100^\circ$. أوجد قياس $\angle QBE$.

Problem 7.13. -

(2) لدينا $ABCD$ شكل رباعي مرسوم داخل دائرة قطرها AC وطول نصف قطرها 10، وكذلك $BA = BD$. إذا كان $BC = 6$ فأوجد طول CD.

Solution submission deadline October 19, 2019