Saudi Arabia 2022 – Math Camp

Day 4 (Part 1) - Level 4+

Geometry - Miscellaneous problems

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0. (IMO/2018) A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point $X$ lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \text{ and } \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

1. (Brazil/2011) Let $ABC$ be a triangle and $H$ its orthocenter. The lines $BH$ and $CH$ intersect $AC$ and $AB$ at points $D$ and $E$, respectively. The circumcircle of $ADE$ intersects the circumcircle of $ABC$ at $F \neq A$. Prove that the bisectors of $\angle BFC$ and $\angle BHC$ concur at a point on $BC$.

2. (Russia/2012) The points $A_1$, $B_1$ and $C_1$ lie on the sides $BC$, $CA$ and $AB$ of the triangle $ABC$, respectively. Suppose that $AB_1 - AC_1 = CA_1 - CB_1 = BC_1 - BA_1$. Let $O_A$, $O_B$ and $O_C$ be the circumcenters of triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$ respectively. Prove that the incenter of triangle $O_AO_BO_C$ is the incenter of triangle $ABC$.

3. (Bulgaria TST/2004) The points $P$ and $Q$ lie on the diagonals $AC$ and $BD$, respectively, of a quadrilateral $ABCD$ such that $\frac{AP}{AC} + \frac{BQ}{BD} = 1$. The line $PQ$ meets the sides $AD$ and $BC$ at points $M$ and $N$. Prove that the circumcircles of the triangles $AMP$, $BNQ$, $DMQ$ and $CNP$ are concurrent.

4. (Canada/2016) Let $\Delta ABC$ be an acute-angled triangle with altitudes $AD$ and $BE$ meeting at $H$. Let $M$ be the midpoint of segment $AB$, and suppose that the circumcircles of $\Delta DEM$ and $\Delta ABH$ meet at points $P$ and $Q$ with $P$ on the same side of $CH$ as $A$. Prove that the lines $ED$, $PH$, and $MQ$ all pass through a single point on the circumcircle of $\Delta ABC$.