Example 1. *Prove that the equation*

$$(x+1)^2 + (x+2)^2 + \cdots + (x+2001)^2 = y^2$$

is not solvable. (mod 3)

Example 2. *Find all pairs (p, q) of prime numbers such that*

$$p^3 - q^5 = (p+q)^2.$$

(Russian Mathematical Olympiad)

Example 3. *Prove that the equation $x^5 - y^2 = 4$ has no solutions in integers.*

(Balkan Mathematical Olympiad)

8. *Prove that the equation*

$$4xy - x - y = z^2$$

has no solution in positive integers.

(Euler)

Lemma: If $x^2 \equiv -1 \pmod{p}$ for some $x \in \mathbb{Z}$ and p - prime then $p = 4k+1$ for some $k \in \mathbb{Z}$.

Example 2. *Find all pairs (p, q) of prime numbers such that*

$$p^3 - q^5 = (p+q)^2.$$

(Russian Mathematical Olympiad)

$p^3 = q^5 + (p+q)^2 \quad (\text{mod } 3)$

$p^3 \equiv p \ (\text{mod } 3) \longrightarrow$ فيرما

$q^5 \equiv q \ (\text{mod } 3) \longrightarrow$ فيرما

$q^5 \equiv q^3 \cdot q^2 \equiv q \cdot q^2 \equiv q^3 \equiv q \quad (\text{mod } 3)$

$p \equiv q + (p+q)^2 \quad (\text{mod } 3)$

Case 1: $(p+q) \equiv 0 \quad (\text{mod } 3)$

$\Rightarrow \quad p \equiv q \ (\text{mod } 3) \Rightarrow p \equiv q \equiv 0 \ (\text{mod } 3)$

$\Rightarrow \quad p = q = 3$

Case 2: $(p+q) \equiv 1 \quad (\text{mod } 3)$

$p \equiv q + 1 \quad (\text{mod } 3)$

$\Rightarrow \quad p \equiv 1, \quad q \equiv 0 \quad \Rightarrow \quad q = 3$

$p^3 - 3^5 = (p+3)^2$

$p^3 - 243 = p^2 + 6p + 9$

$p^3 - p^2 - 6p = 252$

$p^3 < 255$ او $p \mid 252$

$p < 7$ $\quad p = 2, 3, 7$

$\Rightarrow \quad \boxed{p = 7}$

$p \equiv q + (p+q)^2 \pmod 3$

Case 3: $\underline{p+q \equiv -1} \pmod 3$

$$p \equiv q+1 \pmod 3$$

$$\left.\begin{array}{l} p+q \equiv -1 \\ p-q \equiv 1 \end{array}\right\} \begin{array}{l} 2p \equiv 0 \Rightarrow p \equiv 0 \pmod 3 \\ 2q \equiv -2 \equiv 1 \Rightarrow -q \equiv 1 \pmod 3 \\ \qquad q \equiv -1 \pmod 3 \end{array}$$

$$\Rightarrow \boxed{p=3}$$

$$p^3 - q^5 = 3^3 - q^5 = 27 - q^5 = \underline{(3+q)^2}$$

$$\Rightarrow 3+q < \sqrt{27}$$

$$3+q \leq 5$$

$$\longrightarrow q \leq 2 \Rightarrow q = 2 \quad X$$

Example 3. *Prove that the equation* $\boxed{x^5 - y^2 = 4}$ *has no solutions in integers.*

(Balkan Mathematical Olympiad)

$x^5 \equiv 0, +1, -1 \pmod{11}$

$x^3 \equiv 0, 1, -1 \pmod 7 \longleftarrow$

$x^2 \equiv 0, 1, -1 \pmod 5$

$x^{10} \equiv 1 \pmod{11}$

$11 \mid x^{10} - 1$

$11 \mid (x^5-1)(x^5+1)$

$\Rightarrow 11 \mid x^5 - 1$; $11 \mid x^5 + 1$

$\Rightarrow x^5 \equiv 0, 1, -1 \pmod{11}$

$x^6 \equiv 0, 1 \pmod 7$

$\xrightarrow{x \equiv 0\ [7]} x^3 \equiv 0 \pmod 7$

$\rightarrow x^6 \equiv 1 \pmod 7$

$0 \equiv x^6 - 1 \equiv \underline{(x^3-1)}\,\underline{(x^3+1)} \pmod 7$

$\Rightarrow \quad x^3 \equiv 1, -1 \pmod 7$

If p is a prime:

$x^{\frac{p-1}{2}} \equiv 0, 1, -1 \pmod p$

$x^{p-1} \equiv \underline{0, 1} \pmod p$

$x^5 \equiv 0, 1, -1 \pmod{11}$

$y^2 \equiv 0, 1, 5, 4, 3, -2 \pmod{11}$

$x^5 - y^2 \not\equiv 4 \pmod{11}$

$$5a^2+9b^2=13c^2$$

mod 5: $4b^2 \equiv 3c^2 \pmod 5$

$$-b^2 \equiv 3c^2 \pmod 5$$

$$b^2 \equiv -3c^2 \pmod 5$$

$$b^2 \equiv 2c^2 \pmod 5$$

$$0,1,-1 \equiv 0,2,-2 \pmod 5$$

$$\Rightarrow b \equiv c \equiv 0 \pmod 5 \Rightarrow \begin{array}{l} b=5b_1 \\ c=5c_1 \end{array}$$

$$\Rightarrow 5a^2+9(5b_1)^2 = 13(5c_1)^2$$

$$a^2+\underline{5(9b_1^2)} = \underline{5(13c_1^2)}$$

$$\Rightarrow 5 \mid a^2 \Rightarrow 5 \mid a \rightarrow a=5a_1$$

$$\Rightarrow (5a_1)^2+5(9b_1^2) = 5(13c_1^2)$$

$$5a_1^2+9b_1^2 = 13c_1^2$$

$$\Rightarrow 5 \mid a_1, \quad 5 \mid b_1, \quad 5 \mid c_1$$

$$\Rightarrow 5^2 \mid a, \quad 5^2 \mid b, \quad 5^2 \mid c \Rightarrow$$

$$5^n \mid a,b,c \quad \forall n \in \mathbb{Z}$$

ن.ع.

$$\Rightarrow a,b,c=0 \Rightarrow (0,0,0)$$

Lemma: If $x^2 \equiv -1 \pmod p$ for some $x \in \mathbb{Z}$ and p - prime $p \geqslant 3$ then $p = 4k+1$ for some $k \in \mathbb{Z}$.

$$x^2 \equiv -1 \pmod p \quad , \quad x^{p-1} \equiv 1 \pmod p$$

$$\Longrightarrow \quad (x^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \pmod p$$

$$\Longrightarrow \quad 1 \equiv x^{p-1} \equiv (-1)^{\frac{p-1}{2}} \pmod p$$

$$\Longrightarrow \quad 1 \equiv (-1)^{\frac{p-1}{2}} \pmod p \quad \Longrightarrow \quad \frac{p-1}{2} \text{ is even}$$

$$\Longrightarrow \quad 4 \mid p-1$$

$$\Longrightarrow \quad p = 4k+1$$

8. *Prove that the equation*

$$4xy - x - y = z^2$$

has no solution in positive integers.

(Euler)

$16xy - 4x - 4y = (2z)^2$

$16xy - 4x - 4y + 1 = (2z)^2 + 1$

$(4x-1)(4y-1) = (2z)^2 + 1$

lemma: $p = 4k+3$, $p \mid a^2 + b^2 \Rightarrow p \mid a, b$

$\rightarrow -a^2 \equiv b^2 \, [p]$

خيرما

$a^{p-1} \equiv b^{p-1} \, [p]$

$(a^2)^{\frac{p-1}{2}} \equiv (b^2)^{\frac{p-1}{2}} \, [p]$

$(a^2)^{\frac{p-1}{2}} \equiv (-a^2)^{\frac{p-1}{2}} \, [p]$

Aomp > Aops

$2(a^2)^{\frac{p-1}{2}} \equiv 0 \, [p]$

$2a^{p-1} \equiv 0 \, [p]$

$\Rightarrow a \equiv 0 \, [p] \Rightarrow b \equiv 0 \, [p] \Leftrightarrow p \mid a, b$

take p s.t. $p \mid 4x-1$ and $p \equiv 3 \, [4]$

$p \equiv 3 \, [4]$, $p \mid (2z)^2 + 1 \Rightarrow p \mid 1 \Rightarrow p = 1$ ↯

Example 2. *Let p and q be two primes. Solve in positive integers the equation*

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{pq}. \Rightarrow \frac{1}{x}<\frac{1}{pq} \Rightarrow pq<x$$

$$pqy+pqx = xy$$

$$pqy+pqx-xy=0$$

$$pq-y$$

$$\underline{pqy} + x(pq\underline{-y})=0$$

$$-p^2q^2+pqy+x(pq-y) = -p^2q^2$$

$$-pq(pq-y)+x(pq-y)=-p^2q^2$$

$$(pq-y)(x-pq)=\underline{-p^2q^2}$$

$$(y-pq)(x-pq)=\underline{p^2q^2}$$

$1, \ p^2q^2 \longleftrightarrow p^2q^2, 1$

$-p, \ pq^2 \longleftrightarrow q, qp^2$

$p^2, \ q^2 \longleftrightarrow q^2, p^2$

$p^2q, \ q \longleftrightarrow q^2p, p$

pq, pq

$(x,y) = (pq+1, pq+p^2q^2)$

$, (p+pq, pq^2+pq)$

$\vdots$

Example 3. *Determine all nonnegative integral pairs* (x, y) *for which*

$$(xy - 7)^2 = x^2 + y^2.$$

(Indian Mathematical Olympiad)

Example 4. *Solve the following equation in integers* x, y*:*

$$x^2(y - 1) + y^2(x - 1) = 1.$$

(Polish Mathematical Olympiad)

Example 5. *Find all integers* n *for which the equation*

$$x^3 + y^3 + z^3 - 3xyz = n$$

is solvable in positive integers.

(Titu Andreescu)

Example 6. *Find all triples of positive integers* (x, y, z) *such that*

$$x^3 + y^3 + z^3 - 3xyz = p,$$

where p *is a prime greater than* 3.

(Titu Andreescu, Dorin Andrica)

Example 8. *Find all primes* p *for which the equation* $x^4 + 4 = py^4$ *is solvable in integers.*

(Ion Cucurezeanu)

Example 3. *Determine all nonnegative integral pairs* (x, y) *for which*

$$(xy - 7)^2 = x^2 + y^2.$$

(Indian Mathematical Olympiad)

$$x^2y^2 - 14xy + 49 = x^2 + y^2$$

$x + y = s$, $xy = p$

$$\Rightarrow p^2 - 14p + 49 - s^2 + 2p = 0$$

$$\Rightarrow p^2 - 12p - s^2 + 49 = 0$$

$$\Rightarrow (p^2 - 12p + 36) - s^2 = -13$$

$$\Rightarrow (p - 6)^2 - s^2 = -13$$

$$\Rightarrow (s + p - 6)(s - p + 6) = 13$$

$$\begin{cases} s+p = 19 \\ s-p = -5 \end{cases} \quad \begin{cases} s+p = 7 \\ s-p = 7 \end{cases} \quad \begin{cases} s+p = -7 \\ s-p = -7 \end{cases} \quad \begin{cases} s+p = \\ s-p = \end{cases}$$

Example 4. *Solve the following equation in integers x, y:*

$$x^2(y-1) + y^2(x-1) = 1.$$

(Polish Mathematical Olympiad)

$x = a+1 \qquad y = b+1$

$$(a+1)^2 b + (b+1)^2 a = 1$$

$$(a^2+2a+1)b + (b^2+2b+1)a = 1$$

$$a^2b + b^2a + 4ab + a+b = 1$$

$$ab(a+b+4) + a+b+4 = 5$$

$$(a+b+4)(ab+1) = 5$$

1	5
5	1
-1	-5
-5	-1

Example 4. *Solve the following equation in integers x, y:*

$$x^2(y-1)+y^2(x-1)=1.$$

(Polish Mathematical Olympiad)

$$x^2y - x^2 + y^2x - y^2 = 1$$

$$xy(x+y) - (x^2+y^2) = 1$$

$$xy = a \qquad x+y = b$$

$$ab - (b^2 - 2a) = 1$$

$$ab - b^2 + 2a = 1$$

$$a(b+2) - b^2 = 1$$

$$a(b+2) = 1 + b^2$$

$$a = \frac{b^2+1}{b+2} = \frac{b^2-4}{b+2} + \frac{5}{b+2}$$

$$= b-2+\frac{5}{b+2}$$

$$b+2 \mid 5$$

$b+2=5 \qquad b+2=1 \qquad b+2=-1 \qquad b+2=-5$

$$\frac{b^2+1}{b+2} \in \mathbb{Z} \Rightarrow \quad b^2+1 \equiv 0 \pmod{b+2}$$

$$b+2 \equiv 0 \pmod{b+2}$$

$$\Rightarrow b \equiv -2 \pmod{b+2}$$

$$(-2)^2+1 \equiv 0 \pmod{b+2}$$

$$5 \equiv 0 \pmod{b+2}$$

$$\Rightarrow b+2 \mid 5$$

Example 5. *Find all integers n for which the equation*

$$x^3 + y^3 + z^3 - 3xyz = n$$

is solvable in positive integers.

(Titu Andreescu)

Case 1 $3 \mid n$ but $9 \nmid n$

$n = x^3+y^3+z^3-3xyz \equiv x+y+z \pmod 3$

$\Rightarrow x+y+z \equiv 0 \pmod 3$

$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-xz)$

$(x+y+z)^2 \equiv 0 \pmod 3$

$x^2+y^2+z^2+2xy+2yz+2xz \equiv 0 \pmod 3$

$x^2+y^2+z^2-xy-yz-xz \equiv 0 \pmod 3$

$\Rightarrow 3 \mid x^2+y^2+z^2-xy-yz-xz \Rightarrow 9 \mid n \rightarrow\leftarrow$

Case 2 $9 \mid n \longrightarrow$ there is solution

$n = 9k,\ k \in \mathbb{Z}^+$

$x^3+y^3+z^3-3xyz = 9k$

	x, y, z
$n=9$	$0,1,2$
$n=18$	$1,2,3$
$n=27$	$2,3,4$
$9k$	$(k-1, k, k+1)$

$$\underbrace{(x+y+z)}_{\substack{n \\ 3k}}\underbrace{\frac{1}{2}(\overbrace{(x-y)^2}^{1}+\overbrace{(y-z)^2}^{1}+\overbrace{(z-x)^2}^{4})}_{6} = 9k \longrightarrow (k-1, k, k+1)$$

$(x,\ y, z)$

Example 5. *Find all integers n for which the equation*

$$x^3 + y^3 + z^3 - 3xyz = n$$

is solvable in positive integers.

(Titu Andreescu)

Case 3: $n = 3k+1 \longrightarrow$ possible

$$\underbrace{(x+y+z)}_{3k+1}\frac{1}{2}\underbrace{\left((x-y)^2+(y-z)^2+(z-x)^2\right)}_{2} = 3k+1$$

$x, y, z = k, k, k+1$

Case 3: $n = 3k+2 \longrightarrow$ possible

$$\underbrace{(x+y+z)}_{3k+2}\frac{1}{2}\underbrace{\left((x-y)^2+(y-z)^2+(z-x)^2\right)}_{2} = 3k+1$$

$a, (a+1), a+1$

$a + (a+1) + (a+1) = 3k+2$

$3a+2 = 3k+2$

$\Rightarrow$

Example 6. *Find all triples of positive integers (x, y, z) such that*

$$x^3 + y^3 + z^3 - 3xyz = p,$$

where p is a prime greater than 3.

(Titu Andreescu, Dorin Andrica)

$(x+y+z)(x^2+y^2+z^2-xy-yz-xz)=p$

If $x+y+z=1 \Rightarrow \Leftarrow$

If $(x-y)^2+(y-z)^2+(z-x)^2=2$

$x+y+z=p$

$(x-y)^2=0, (y-z)^2=(z-x)^2=1$

$x=y \qquad y=z\pm 1, z=x\pm 1$

$x=z\pm 1, z=x\pm 1$

case 1: $x=z+1$

$x+y+z=3x-1=p \Rightarrow x=\frac{p+1}{3}$

$(x,y,z)=\left(\frac{p+1}{3}, \frac{p+1}{3}, \frac{p-2}{3}\right)$ cyc

case 2: $x=z-1$

$x+y+z=3x+1=p \Rightarrow x=\frac{p-1}{3}$

$(x,y,z)=\left(\frac{p-1}{3}, \frac{p-1}{3}, \frac{p+2}{3}\right)$ cyc

Example 8. *Find all primes p for which the equation $x^4+4=py^4$ is solvable in integers.*

(Ion Cucurezeanu)

$$x^4+4 = x^4+4x^2+4-4x^2$$
$$= (x^2+2)^2-(2x)^2$$
$$= (x^2-2x+2)(x^2+2x+2)$$

$$\gcd(\underbrace{x^2+2}_{a}\underbrace{-2x}_{b}, \underbrace{x^2+2}_{a}+\underbrace{2x}_{b}) =$$
$$\gcd(x^2+2, 2x), 2\gcd(x^2+2, 2x)$$

if $d \mid x^2+2, 2x$

$\hookrightarrow d \mid 2x^2+4 \rightarrow d \mid 2x^2+4-(2x)x$

$\rightarrow d \mid 4 \rightarrow d = 4, 1, 2$

However, if $d=4$ $\quad 4 \mid 2x, \; 4 \mid x^2+2$

$x=2k \Rightarrow 4 \mid 4k^2+2 \rightarrow \times$

Case 1: x is even

$\gcd(x^2+2, 2x)=2, \quad x=2a$

$$(x^2-2x+2)(x^2+2x+2) = py^4$$
$$(4a^2-4a+2)(4a^2+4a+2) = p$$

Lemma. $\gcd(a+b, a-b) = \begin{cases} \gcd(a,b) \\ 2\gcd(a,b) \end{cases}$

Assume that $\gcd(a,b) = d \Rightarrow \begin{array}{l} a = dx \\ b = dy \end{array}$, $\gcd(x,y) = 1$

$\Rightarrow \quad a+b, a-b = d(x+y), d(x-y)$

$\gcd(a+b, a-b) = d \gcd(x+y, x-y)$

if $s \mid x+y, x-y \Rightarrow \begin{cases} s \mid 2x \\ s \mid 2y \end{cases} \Rightarrow s \mid 2\gcd(x,y)$

$\Rightarrow s \mid 2 \Rightarrow s = 1, 2$

$\Rightarrow \quad \gcd(x+y, x-y) = 1, 2$

$\Rightarrow \quad \gcd(a+b, a-b) = d, 2d$ ✓

$p_1, p_2, \dots, p_n$ are different primes

$$P(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

$$a_i = p_1 p_2 \dots p_i$$

$$r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$$

$$\Rightarrow p_1 \mid r^n \Rightarrow p_1 \mid r$$

$$p_2 \mid r^{n-1} (r + p_1)$$

$$p_1^n r^n + (p_1^{n-1} a_1) r^{n-1} + \dots + a_{n-1} p' r + a_n = 0$$

if $n \geq 2$

$$\Rightarrow p_1^2 \mid a_n \Rightarrow \Leftarrow !!$$

$$\Rightarrow n = 1$$

$$x + p = 0$$

Problem 4.1. Different positive a, b, c are such that $a^{239} = ac - 1$ and $b^{239} = bc - 1$.Prove that $238^2(ab)^{239} < 1$.

$$\begin{cases} ac - 1 = a^{239} \\ bc - 1 = b^{239} \end{cases}$$

$$\cancel{(a-b)}c = a^{239} - b^{239} = \cancel{(a-b)}(\underbrace{a^{238} + a^{237}b + \cdots + b^{238}}_{S})$$

$$c = S \implies ac - 1 = a^{239}$$

$$\implies \cancel{a^{239}} + a^{238}b + \cdots + ab^{238} - 1 = \cancel{a^{239}}$$

$$\implies 1 = a^{238}b + \cdots + ab^{238}$$

$$\underset{\text{AM-GM}}{\geq} 238\sqrt[238]{(ab)^{1+2+\cdots+238}}$$

$$= 238\sqrt[\cancel{238}]{(ab)^{\frac{\cancel{238}\cdot 239}{2}}}$$

$$= 238\sqrt{(ab)^{239}}$$

$$\implies 1 \geq 238^2(ab)^{239}$$

$$\text{"="} \quad \cancel{a}^{\cancel{238}\,237}b = \cancel{a}b^{\cancel{238}\,237} \implies \boxed{a = b} \quad ↯$$

$$\implies 1 > 238^2(ab)^{239}$$

■

Problem: Prove that $x^2 = y^3 + z^5$ has infinetly meny solutions in positive integres.

$x = 3$ $\quad$ $1 + 8 = 9$ ✓

$y = 2$ $\quad$ $m^{30} y^3 + z^5 \cdot m^{30} = m^{30} \cdot x^2$

$z = 1$ $\quad$ $(m^{10} \cdot y)^3 + (z \cdot m^6)^5 = (x \cdot m^{15})^2$

$\Rightarrow (x, y, z) = (3m^{15}, 2m^{10}, m^6)$

Example 3. *Find all triples (x, y, z) of positive integers such that*

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}.$$

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=\frac{1}{z} \Rightarrow \quad z=\frac{xy}{x+y}$$

- gcd(ab, a+b) = 1 if a, b are relatively prime

assume that $\gcd(x,y)=d \Rightarrow \begin{matrix} x=da \\ y=db \end{matrix}$ $\quad \gcd(a,b)=1$

$$\Rightarrow \quad z=\frac{d^2ab}{da+db}=\frac{d\,ab}{a+b} \Rightarrow a+b \mid d$$

$$\Rightarrow d=(a+b)k$$

$k \in \mathbb{Z}$

$$\Rightarrow \quad z=\frac{d}{a+b}ab=k\,ab$$

$$\begin{cases} x=da=k(a+b)a \\ y=db=k(a+b)b \\ z=k\,ab \end{cases} \qquad \forall \; a,b,k \in \mathbb{Z}$$

(2) If a, b, c are positive integers satisfying

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c},$$

then $a^2+b^2+c^2$ is a square. Indeed,

(a,b,c) are solutions

$$a = m(m+n)k$$
$$b = n(m+n)k$$
$$c = kmn$$

$$a^2+b^2+c^2 = m^2(m+n)^2k^2 + n^2(m+n)^2k^2 + k^2m^2n^2$$

$$= k^2\left(m^2(m+n)^2 + n^2(m+n)^2 + m^2n^2\right)$$

$$= k^2\left((m+n)^2\underbrace{(m^2+n^2)}_{=(m+n)^2-2mn} + m^2n^2\right)$$

$$= k^2\left[(m+n)^2(m+n)^2 - (m+n)^2(2mn) + (mn)^2\right]$$

$$= k^2\left[(m+n)^4 - 2mn(m+n)^2 + (mn)^2\right]$$

$$= k^2\left[(m+n)^2 - mn\right]^2$$

2. *Show that the equation*

$$x^2 + y^2 = z^5 + z$$

has infinitely many relatively prime integral solutions.

(United Kingdom Mathematical Olympiad)

• حاصل ضرب مجموع مربعين = مجموع مربعين

$$(a^2+b^2)(c^2+d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$= (a^2c^2 + 2abcd + b^2d^2) + (a^2d^2 - 2abcd + b^2c^2)$$

$$= (ac+bd)^2 + (ad-bc)^2 \qquad \square$$

• إذا كان p عدد أولي على الصيغة $4k+1$ ، فإنه يمكن كتابة كحاصل جمع مربعين .

$$x^2 \equiv -1 \pmod p$$

$$x = \left(\frac{p-1}{2}\right)!$$

$$(p-1)! \equiv -1 \pmod p$$

$$1 \cdot 2 \cdot 3 \cdots (p-1)$$

$$(1\,(p-1))\,(2\,(p-2))\,(3\,(p-3)) \cdots \left(\left(\frac{p-1}{2}\right)\frac{p+1}{2}\right)$$

$$(-1)^{\frac{p-1}{2}}\; 1^2\; 2^2\; 3^2 \cdots \left(\frac{p-1}{2}\right)^2$$

$$\underbrace{(-1)^{\frac{p-1}{2}}}_{1} \underbrace{\left(\left(\frac{p-1}{2}\right)!\right)^2}_{x} \equiv -1 \pmod p$$

$$x^2 \equiv -1 \pmod p \rightarrow p \mid x^2+1$$

$$\underbrace{x^2+y^2}=\underbrace{z(z^4+1)}$$

$z=p \quad , \quad p=4k+1$

$p=\underbrace{a^2+b^2} \quad , \quad \underbrace{(z^4+1)}$

$$x^2+y^2=p^5+p$$

$$\gcd(x,y,p)=\begin{cases}1\\p\end{cases}$$

if $\gcd(x,y,p)=p \Rightarrow p \mid x \; , \; p \mid y$

$\Rightarrow p^2 \mid x^2+y^2$

$\Rightarrow p^2 \mid p^5+p$

$\Rightarrow p^2 \mid p \Rightarrow p \mid 1$

Example 1. *Prove that there are infinitely many triples* (x, y, z) *of integers such that*

$$x^3 + y^3 + z^3 = x^2 + y^2 + z^2.$$

(Tournament of Towns)

Example 2. *(a) Let* m *and* n *be distinct positive integers. Prove that there exist infinitely many triples* (x, y, z) *of positive integers such that*

$$x^2 + y^2 = (m^2 + n^2)^z,$$

with

(i) z *odd;* *(ii)* z *even.*

(b) Prove that the equation

$(4+9)^z$

$$x^2 + y^2 = 13^z$$

has infinitely many solutions in positive integers x, y, z.

Example 1. *Prove that there are infinitely many triples* (x, y, z) *of integers such that*

$$x^3 + y^3 + z^3 = x^2 + y^2 + z^2.$$

(Tournament of Towns)

Let $z = -y$

$$x^3 + y^3 + (-y)^3 = x^2 + y^2 + (-y^2)$$

$$\Rightarrow x^3 = x^2 + 2y^2 \quad \Rightarrow x^2 \mid 2y^2$$

Let $y = tx$

$$\Rightarrow x^3 = x^2 + 2t^2x^2 \quad \Rightarrow x = 1 + 2t^2$$

$$\Rightarrow y = t + 2t^3, \quad z = -t - 2t^3$$

$$(x, y, z) = (1 + 2t^2,\ t + 2t^3,\ -t^2 - 2t^3)$$

Example 2. *(a) Let m and n be distinct positive integers. Prove that there exist infinitely many triples (x, y, z) of positive integers such that*

$$x^2 + y^2 = (m^2 + n^2)^z,$$

with

(i) z odd; *(ii) z even.*

ai) $$(m^2+n^2)^{2k+1} = \underbrace{(m^2+n^2)^{2k}}_{\text{مربع كامل}} \; \underbrace{(m^2+n^2)}_{\text{مجموع مربعين}}$$

$$c^2(m^2+n^2) = (cm)^2 + (cn)^2$$

$$x = (m^2+n^2)^k \, m \qquad z = 2k+1$$
$$y = (m^2+n^2)^k \, n \quad ,$$

aii) $$(m^2+n^2)^{2k} = \underbrace{(m^2+n^2)^{2k-2}}_{\text{مربع كامل}} \; \underbrace{(m^2+n^2)^2}_{\left((m^2-n^2)^2 + (2mn)^2\right)}$$

$$= c^2\left[(m^2-n^2)^2 + (2mn)^2\right]$$

$$\Longrightarrow \begin{cases} x = c(m^2-n^2) = (m^2+n^2)^{k-1}(m^2-n^2) \\ y = (m^2+n^2)^{k-1}(2mn) \\ z = 2k \end{cases}$$

2. *Find all pairs of positive integers* (x, y) *for which*

$$x^2 - y! = 2001.$$

(Titu Andreescu)

$$x^2 = y! + 2001$$

$$x^2 = y! + 3 \times 29 \times 23$$

if $y \geq 6$, $9 \mid y! \Rightarrow 3 \mid y! + 2001$

$9 \nmid y! + 2001$

$\Rightarrow 3 \mid x^2$ but $9 \nmid x^2$

$\Longrightarrow \Longleftarrow$

$y < 6 \Rightarrow y = 1, 2, 3, 4, 5$

$$x^3+y^4=7$$

$(\text{mod } 13)$

$$\Longrightarrow \quad x^3 \equiv 0, 1, 8, 12 \pmod{13}$$

$$y^4 \equiv 0, 1, 3, 9 \pmod{13}$$

$$x^3 \equiv 7-y^4 \pmod{13}$$

$$0, 1, 8, 12 \equiv 7, 6, 4, 11 \pmod{13}$$

$$3^x - 2^y = 7$$

(mod 8) :

$$3^x \equiv 1, 3, \pmod 8$$

$$2^y \equiv 0 \pmod 8 \text{ if } y \geq 3$$

$$7 \equiv -1 \pmod 8$$

$\Longrightarrow$ $y = 1, 2$

$y=1$: $3^x = 2+7 = 9 \Longrightarrow x = 2$

$y=2$: $3^x = 4+7 = 11 \Longrightarrow\Longleftarrow$

2

$$3^x - 2^y = 7$$

- $y \geq 2$;

mod 3 $\Longrightarrow -(-1)^y \equiv 1 [3]$ $\qquad x = 2m+1$

$\Longrightarrow (-1)^y \equiv -1 [3]$ $\qquad y = 2n+1$

$\Longrightarrow$ فردي y

mod 4 $\Longrightarrow (-1)^x \equiv -1 [4]$

$\Longrightarrow$ فردي x

mod 5 $\Longrightarrow (-2)^{2m+1} - 2^{2n+1} \equiv -2\cdot 4^m - 2\cdot 4^n$

$\equiv -2(-1)^m - 2(-1)^n \equiv 2 [5]$

$\Longrightarrow (-1)^m + (-1)^n \equiv -1 [5]$ ↯

5. Determine all nonnegative integral solutions $(x_1, x_2, \ldots, x_{14})$ if any, apart from permutations, to the Diophantine equation

$$x_1^4 + x_2^4 + \cdots + x_{14}^4 = 15999.$$

(8th USA Mathematical Olympiad)

6. Find all pairs (x, y) of integers such that

$$x^3 - 4xy + y^3 = -1.$$

(G.M. Bucharest)

7. Find all triples (x, y, z) of nonnegative integers such that

$$5^x 7^y + 4 = 3^z.$$

(Bulgarian Mathematical Olympiad)

5. Determine all nonnegative integral solutions $(x_1, x_2, \ldots, x_{14})$ if any, apart from permutations, to the Diophantine equation

$$x_1^4 + x_2^4 + \cdots + x_{14}^4 = \underbrace{15999.}_{-1 \,\equiv\, 15} = 16000 - 1$$

(8th USA Mathematical Olympiad)

(mod 16)

if $n = 2k \Rightarrow n^4 = 16k$

$\Rightarrow n^4 \equiv 0 \pmod{16}$

if $n = 2k+1$

$n^4 - 1 = \underbrace{(n-1)(n+1)}_{8}\underbrace{(n^2+1)}_{2}$

$\Rightarrow n^4 \equiv 1 \pmod{16}$

$$\Rightarrow \underset{\substack{0\\1}}{x_1^4} + \underset{\substack{0\\1}}{x_2^4} + \cdots + \underset{\substack{0\\1}}{x_{14}^4} \equiv 0, 1, 2, \ldots, 14$$

$\not\equiv 15 \pmod{16}$