Problem 1. Let ABC be an acute triangle. Let H be the foot on AB of the altitude through C. Suppose that $AH = 3BH$. Let M and N be the midpoints of the segments AB and AC respectively. Let P be a point such that $NP = NC$ and $CP = CB$ and such that B and P lie on the opposite sides of the line AC.

Prove that $\angle APM = \angle PBA$.

Problem 2. Prove that every positive integer can be expressed as a sum of powers of 3, 4 and 7 in such a way that the representation doesn't contain two powers with the same base and the same exponent.

For example, $2 = 7^0 + 7^0$ and $22 = 3^2 + 3^2 + 4^1$ are not valid sums, but $2 = 3^0 + 7^0$ and $22 = 3^2 + 3^0 + 4^1 + 4^0 + 7^1$ are valid.

Problem 3. A sequence $a_1, a_2, a_3, \ldots$ of positive integers satisfies $a_{n+1} = a_n + 2d(n)$ for every $n \geqslant 1$, where $d(n)$ is the number of different positive divisors of n.

Is it possible that two consecutive terms of this sequence are perfect squares?

Problem 4. Let n be a positive integer, and consider a square of dimensions $2^n \times 2^n$. We cover this square by a number of (at least 2) rectangles, without overlaps, and in such a way that every rectangle has integer dimension and a power of two as area.

Show that two of the rectangles used must have the same dimensions. (Two rectangles are said to have the same dimensions if they have the same height and the same width, without rotating them.)

(Time: 2:00 pm – 6:40 pm)

1. Since N is the midpoint of AC, we have $NC = NA$. By assumption $NP = NC$, hence N is the circumcenter of the triangle ACP and $\angle APC = 90°$. Since $AH = 3BH$ and M is the midpoint of AB, it follows that $MH = BH$. Therefore the right triangles CHB and CHM are equal, which implies $CM = CB$. From $CP = CB$ we deduce that C is the circumcenter of the triangle PMB. As $\angle APC = 90°$, the line AP is tangent to this circle. By the tangent-chord theorem we have $\angle APM = \angle PBM = \angle PBA$.

2. Arrange all powers of 3, 4 and 7 with nonnegative integer exponents ascending: $x_1^{a_1} \leqslant x_2^{a_2} \leqslant x_3^{a_3} \leqslant \ldots$ (for example $x_5^{a_5} = 4$ and $x_9^{a_9} = 27$). Call the representations from the problem condition *proper representations*. We prove the stronger statement by induction on n: each integer between 1 and $x_1^{a_1} + x_2^{a_2} + \ldots + x_n^{a_n}$ has proper representation containing summands only from $x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}$. For n equals to 1, 2 and 3 it's clear. Assume that this is true for $n \geqslant 3$. Let the powers $x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}$ be $3^0, 3^1, \ldots, 3^a$, $4^0, 4^1, \ldots, 4^b$ and $7^0, 7^1, \ldots, 7^c$ then

$$x_1^{a_1} + x_2^{a_2} + \ldots + x_n^{a_n} \leqslant 3^0 + 3^1 + \ldots + 3^a + 4^0 + 4^1 + \ldots + 4^b + 7^0 + 7^1 + \ldots + 7^c =$$
$$= \frac{3^a - 1}{2} + \frac{4^b - 1}{3} + \frac{7^c - 1}{6} \leqslant \frac{x_{n+1}^{a_{n+1}} - 1}{2} + \frac{x_{n+1}^{a_{n+1}} - 1}{3} + \frac{x_{n+1}^{a_{n+1}} - 1}{6} = x_{n+1}^{a_{n+1}} - 1.$$

Hence by the induction assumption each integer between 1 and $x_{n+1}^{a_{n+1}} - 1$ has proper representation containing summands only from $x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}$. For any integer m such that $x_{n+1}^{a_{n+1}} \leqslant m \leqslant x_1^{a_1} + x_2^{a_2} + \ldots + x_{n+1}^{a_{n+1}}$ we write $m = x_{n+1}^{a_{n+1}} + (m - x_{n+1}^{a_{n+1}})$ and if $m \neq 0$, use the proper representation of $0 \leqslant m - x_{n+1}^{a_{n+1}} \leqslant x_1^{a_1} + x_2^{a_2} + \ldots + x_n^{a_n}$.

3. Answer: no.

Suppose the contrary: $a_n = x^2$ and $a_{n+1} = y^2$ for some positive integers x and y. Note that $a_{n+1} > a_n$ and they have the same parity, hence $a_n \geqslant n$ and $y \geqslant x + 2$. Therefore $2d(n) = a_{n+1} - a_n = y^2 - x^2 \geqslant (x+2)^2 - x^2 = 4x + 4$. This implies $d(n) \geqslant 2x + 2 = 2\sqrt{n} + 1$ which is impossible since $d(n) \leqslant 2\sqrt{n}$ for any integer $n \geqslant 1$.

4. Note that if in the covering of $2^n \times 2^n$ square all sides of the rectangles are even, we can divide all sides by 2 and obtain the covering of $2^{n-1} \times 2^{n-1}$ square. Hence without loss of generality assume that there is a rectangle with sidelength (let's say width) 1. Among all rectangles of width 1 choose the rectangle R with the largest height, denote it's height by 2^k. Consider all rows, containing the cells of R: each row must contain at least one cell of another rectangle of width 1. If we suppose that all such rectangles have different heights, then they contain at most $1 + 2 + \ldots + 2^{k-1} = 2^k - 1$ cells which is less than the number of rows. Hence there must be two rectangles of width 1 having the same heights.