Email training, N13

February 2-12, 2020

Problem 13.1. Let $0 \leq x \leq 2$, $0 \leq y \leq 3$ and $x + y + z = 11$. Find the maximal possible value for expression xyz.

Problem 13.2. Determine all pairs of distinct real numbers (x, y) such that both of the following are true:

$$x^{100} - y^{100} = 2^{99}(x - y)$$
$$x^{200} - y^{200} = 2^{199}(x - y)$$

Problem 13.3. The tire can be driven 42.000 km when we put it in back side and 58.000 when we put it in front side. Find the maximal distance one may drive when he starts with a completely new set of 5 tires (one of them is spare tire).

Problem 13.4. Solve the system of equations

$$\begin{cases} x^{10} + x^2 = y^5 + y \\ y^6 + y^2 = 8x^3 + 2x \end{cases}.$$

Problem 13.5. Let a, b, c, d be real numbers with $0 \leqslant a, b, c, d \leqslant 1$. Prove that

$$ab(a - b) + bc(b - c) + cd(c - d) + da(d - a) \leqslant \frac{8}{27}.$$

When equality holds?

Problem 13.6. Integers $a_1, a_2, ..., a_{2020}$ satisfy $|a_k| = 1$. Prove that

$$\sum_{k=1}^{n} a_k a_{k+1} a_{k+2} a_{k+3} \neq 2,$$

where $a_{2020+j} = a_j$.

Problem 13.7. Determine the minimal value of

$$\left(x + \frac{1}{y}\right)\left(x + \frac{1}{y} - 2020\right) + \left(y + \frac{1}{x}\right)\left(y + \frac{1}{x} - 2020\right),$$

where x and y vary over the positive reals.

Solution submission deadline February 12, 2020