Problem 1A. Let a, b and c be positive integers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

is also a positive integer. Prove that abc is a perfect cube.

Problem 2A. Prove that

$$\frac{(m,n)}{n}\binom{n}{m}$$

is a positive integer for all positive integers $n \geqslant m \geqslant 1$.

Problem 3A. Let n be a positive integer. Prove that

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$$

is not an integer.

Problem 4A. Let k and n be positive integers. Can the number

$$S(n,k) = \frac{1}{n} + \frac{1}{n+13} + \ldots + \frac{1}{n+13k}$$

be a positive integer?

Problem 5A. Let $n > 1$ be a positive integer. Prove that the polynomial

$$\frac{x^n}{n\,!} + \frac{x^{n-1}}{(n-1)\,!} + \cdots + 1$$

does not have rational roots.

Problem 6A. Let n and d, $n > d$, be positive integers. Prove the inequality

$$\mathrm{LCM}(n, n-1, \ldots, n-d) \geqslant \frac{n(n-1)\cdots(n-d)}{d!}.$$

Problem 7A. Let p and q be coprime positive integers. Prove that for all positive integers $n > m$ the following inequality holds

$$\mathrm{LCM}(qm+p, q(m+1)+p, \ldots, qn+p) \geqslant m\binom{n}{m}.$$

Problem 8A. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Let us assume that there is a positive integer $N > 1$ such that for all $n \geqslant N$ the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \quad \text{is a positive integer.}$$

Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geqslant M$.

Problem 9A. Find all pairs of positive integers (a, b) such that $a^{b^2} = b^a$.

Problem 10A. Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined with

$$f(n) = n^{2021} - n!$$

is injective.

Problem 11A. Let $n \geqslant 2018$ be a positive integer and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct positive integers not greater than $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$$

forms an arithmetic progression. Prove that the terms of this sequence are equal.