Problem

APMO 2015 ← FIRST TASK

$$6(6a^2+3b^2+c^2) = 5n^2$$

$$\Downarrow$$

$$a=b=c=n=0$$

1° We can assume $\gcd(a,b,c,n) = \cancel{1} = d$

$a = da_1,\ b = db_1,\ c = dc_1,\ n = dn_1$

$$6(6\cancel{d^2}a_1^2 + 3\cancel{d^2}b_1^2 + \cancel{d^2}c_1^2) = 5n_1^2\cancel{d^2}$$

$$6(6a_1^2+3b_1^2+c_1^2) = 5n_1^2$$

$6 \mid 5n^2 \Rightarrow 6 \mid n$, $n = 6m$.
$6 = 2\cdot 3$

$$\cancel{6}(6a^2+3b^2+c^2) = 5\cdot \cancel{36}^{6} m^2 \quad |:3$$

$$\boxed{2a^2+b^2+\frac{c^2}{3} = 10m^2}$$

$3 \mid c$.

$$6(6a^2+3b^2+c^2)=5n^2 \qquad n=6m$$

$$6(6a^2+3b^2+c^2)=\cancel{30m^2}\ 180m^2$$

$$36a^2+18\cdot 4r^2+6\cdot 4s^2=180m^2$$

$$36a^2+72r^2+24s^2=180m^2$$

$$8 \quad 8 \quad 8 \quad 8$$

$\Downarrow$

a is even.

$2|a,\ 2|b,\ 2|c,\ 6|n = 2|n$.

but I assumed $\gcd(a,b,c,n)=1$

↯

We can write $c = 3\cdot d$.

$$\boxed{2a^2 + b^2 + 3d^2 = 10 m^2}$$

OK?

$(x^2 \equiv 0, 1, 4 \text{ , modulo } 8)$

$0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$

$0, 1, 4, 1, 0, 1, 4, 1$

$$\Rightarrow + \left| \begin{array}{l} 2a^2 \equiv 0, 2 \pmod 8 \\ b^2 \equiv 0, 1, 4 \pmod 8 \\ 3d^2 \equiv 0, 3, 4 \pmod 8 \end{array} \right.$$

$2a^2 + b^2 + 3d^2 = 10m^2 \equiv 0, 2 \mod 8$

$\Downarrow$

$\boxed{b^2, 3d^2 \text{ are even}}$

$\Downarrow$

$\boxed{b, d \text{ are even}}$

$3\cdot d$

$\shortparallel$

$\boxed{c - \text{even}}$

$\boxed{\begin{array}{l} b = 2r \\ c = 2s \end{array}}$

$\cancel{6(6 \cdot 4a^2 + 3 \cdot 4r^2 + 4s^2) = 5}$

$\boxed{25}$

$ab \mid a^2+b^2+1 \implies \frac{a^2+b^2+1}{ab} = 3.$

$$\frac{a^2+b^2+1}{ab} = k$$

$a \geq b.$ $\quad a = b? \implies \frac{2a^2+1}{a^2} \in \mathbb{Z}.\quad a^2 \mid 1 \implies \boxed{a=1}$

$(1,1),\quad k = \frac{1^2+1^2+1}{1 \cdot 1} = \boxed{3}$

Assume $k \neq 3 \implies a \neq b \implies$ WLOG $\boxed{a > b}$.

Quadratic eq.

$$a^2+b^2+1 = k \cdot ab$$

$$a^2 - a(kb) + b^2+1 = 0$$

$\mathbb{Z}$. $\quad$ $\mathbb{Z}$.

Vieta's $\begin{cases} a + x_2 = kb \\ a \cdot x_2 = b^2+1 \end{cases} \implies$ $x_2 = kb - a$

$x_2 = \frac{b^2+1}{a} > 0$

$x_2 > 0$ integer.

TAKE (a,b) with smallest possible sum.

$$x_2 + \cancel{b} \geq a + \cancel{b}$$

$$x_2 \geq a$$

$$\| $$

$$\frac{b^2+1}{a} \geq a \Rightarrow b^2+1 \geq a^2 \geq (b+1)^2 = b^2+2b+1 > b^2+1 \; ⚡$$

BUT $a > b \Rightarrow a \geq b+1$.

$$\Downarrow$$

$$\boxed{k=3}$$

$$xy \mid x^2+y^2+1 \Longrightarrow x^2+y^2+1 = 3xy$$

Exercise Find all integer (x,y) (positive)

$x \mid y^2+1$, $\quad y \mid x^2+1$. $\Rightarrow \gcd(x,y)=1$

if $\gcd(x,y)=d$

$x \mid x^2+y^2+1$ and $y \mid x^2+y^2+1$

$xy \mid x^2+y^2+1$!!!

$\Downarrow$ Previous problem

$\boxed{x^2+y^2+1=3xy}$ $\quad (x,y)$ $(1,1)$

HINT there is ∞ solutions

$(a,b) \to (b, 3b-a) \to \to \to \dots \; b = 1$

$b > 3b-a$

if $b=1$ then $a^2+2=3a \Rightarrow a=1$, or 2.

$(a,b) \to (b, 3b-a) \to (3b-a, 3(3b-a)-b)$

$(a_1, b_1) = (1,1)$

$(a_{n+1}, b_{n+1}) = (b_n, 3b_n - a_n)$

$a_n = ?$ $\quad b=1$.

$$\begin{cases} a_{n+1} = b_n \\ b_{n+1} = 3b_n - a_n \end{cases} //$$

$a_{n+2} = 3a_{n+1} - a_n$

$a_n = F_{2n+1}$

$\begin{matrix} 1 & 2 & & \\ a_1, & a_2 & \dots & a_n \end{matrix}$

$$x^2+y^2+1 = 3xy$$

Let (a,b) be a solution with $a>b$.

$$a^2+b^2+1 = 3ab$$

Quadratic eq. in $\boxed{\cancel{b}}$ a

$$\cancel{b^2 - b(3a) + a^2+1=0}$$

$$a^2 - a\cdot 3b + b^2+1=0$$

$\boxed{x_2 = 3b-a}$

$$\underbrace{(3b-a)^2-3(3b-a)\cdot b + b^2+1=0}$$

$(b, 3b-a)$ is also solution.

$\boxed{\text{TRICK}}$

$a>b$

$(a,b) \longrightarrow (b, 3b-a)$ but

$$\underbrace{(a-b)}_{>0}(a-2b) = a^2-3ab+2b^2 = b^2-1>0$$

$\boxed{a>2b}$ so $3b-a<b$

Markov equatio

$$x^2+y^2+z^2=3xyz$$

Previou problem is Markov eq. for $\boxed{z=1}$

$\boxed{k=2}$

$$\frac{a^2+b^2+1}{ab+a+b} = 2 \Rightarrow \boxed{a^2+b^2+1 = 2ab+2a+2b}$$

$$\cancel{b+a} \quad \boxed{(b-a-1)^2 = b^2+a^2+1-2ab+2a-2b} = 4a$$

$$4a = (b-a-1)^2$$

$$a = d^2 \Rightarrow \quad 4d^2 = (b-a-1)^2$$

$$b-a-1 = \pm 2d$$

$$b = d^2 \pm 2d + 1 = (d \pm 1)^2$$

$\boxed{(d^2, (d \pm 1)^2)}$ for any d. $(4, 9)$

$(1,1)$, $\boxed{(d^2, (d\pm1)^2)}$.

$\boxed{k \geq 3}$

$$a^2+b^2+1 = k(ab+a+b)$$

$a \geq b$

$$a^2 - a(kb+1) + b^2+1 - kb = 0$$

$$x_2 = kb+1-a \in \mathbb{Z}.$$

$$x_2 = \frac{b^2+1-kb}{a} \neq 0$$

$$x_2 \leq -1$$

$$b^2+1-kb \leq -a$$

$$k \geq \frac{a+b^2+1}{b}$$

$$a^2+b^2+1 \geq \left(\frac{a+b^2+1}{b}\right)(ab+a+b)$$

$$ba^2+b^3+b \geq (a+b^2+1)(ab+a+b) \quad \text{∢}$$

$\boxed{k \geq 3}$

$\boxed{x_2 \geq a}$

$$\| \quad \frac{b^2+1-kb}{a} \geq a$$

$b^2+1-kb \geq a^2$

$3b \leq kb \leq b^2+1-a^2$

$3b \leq b^2+1-a^2$

$\leq \boxed{b + a^2 \leq (b-1)^2 \cancel{b}}$

$\cancel{b^2-2b+1 \geq b+b^2}$

$3b \leq$

$a > b$

$\boxed{3b \leq kb \leq b^2+1-a^2}$

$\cancel{2b \leq b^2+1-a^2}$

$\boxed{(b-1)^2 - a^2 \geq 0}$

$\boxed{a \geq b}$

$a^2+b > (b-1)^2$

$a > b$

$b^2+b > b^2-2b+1$

Scanned with CamScanner

(x_2, b) (a, b) minimal.

$$b + x_2 \geq a + b$$

$$\boxed{x_2 \geq a}$$

$$\cancel{b^2 + 1 - k}$$

$$kb + 1 - a \geq a$$

$$\boxed{k \geq \frac{2a-1}{b}}$$

$$\frac{b^2+1-kb}{a} \geq a$$

$$b^2 + 1 - kb \geq a^2$$

$$\leq kb \leq b^2 - a^2 + 1$$

$$k = \frac{a^2+b^2+1}{ab+a+b} = k \geq \frac{(2a-1)}{b}$$

$$\cancel{a^2b} + b^3 + b \geq \cancel{2a^2b} + 2a^2 + 2ab - \cancel{ab} - a - b$$

$$a^2b + 2a^2 + ab - a - b$$

$$b^3 + 2b + a \geq a^2b + 2a^2 + ab > b^3$$

Scanned with CamScanner

$ab-1 \mid a^2+b^2 \Rightarrow \dfrac{a^2+b^2}{ab-1} = 5$

5 is reali $1,2$

$a \geq b$ $\qquad \underline{a=b}$ $\qquad \dfrac{2a^2}{a^2-1}$ so $a^2-1 \mid 2a^2$

$a^2-1 \mid 2a^2-2$, $\quad a^2-1 \mid 2a^2 \rightarrow a^2-1 \mid 2$.

$a^2-1 = \cancel{1}, \cancel{-1}, \cancel{2}, \cancel{-2}$.

$\boxed{\text{WLOG} \quad a > b}$ $\qquad$ If $b=1$ $\qquad \dfrac{a^2+1}{a-1}$ $\qquad (2,1)$ $(3,1)$

$a-1 \mid a^2+1$ but $a-1 \mid a^2-1$ so

$a-1 \mid 2 \Rightarrow a-1 = \cancel{-1}, 1, 2, \cancel{-2}$

$a=2$

$a=3$

$\underline{a > b}$

$x_2 + b \geq a + b$ From Vieta

$$x_2 \geq a$$

$$\frac{b^2+1-kb}{a} \geq a$$

$$b^2+1-kb \geq a^2$$

$$3b \leq kb \leq b^2+1-a^2$$

BUT $k \geq 3$

$$b^2 \leq a^2 \leq b^2+1-3b$$

$$\cancel{b^2} \leq \cancel{b^2}+1-3b$$

$$3b \leq 1$$

↯.

$x_2 \geq a$

$\frac{b^2+k}{a} \geq a$

$\boxed{k \geq a^2-b^2}$

$kb-a \geq a$

$kb \geq 2a$

$\frac{a^2+b^2}{ab-1} =$

But

$\frac{a^2+b^2}{ab-1} = k \geq a^2-b^2$

$a \geq$

$a^2+\cancel{b^2} \geq (a^2-b^2)(ab-1) = ab(a^2-b^2)-a^2+\cancel{b^2}$

$2a^2 \geq ab(a^2-b^2) \quad | : a$

$\boxed{2a \geq b(a^2-b^2)}$ $\cancel{\geq b(2a+1)}$

$(2,1)$ $\quad\quad$ $(3,1)$

$\frac{2^2+1^2}{2\cdot 1-1} = 5$ $\quad\quad$ $\frac{3^2+1^2}{3\cdot 1-1} = 5$.

So we can assume $\boxed{a > b > 1}$.

$$\frac{a^2+b^2}{ab-1} = k$$

$$a^2 - a(kb) + b^2 + k = 0$$

$$\left.\begin{array}{l} x_2 = kb - a \in \mathbb{Z}. \\ x_2 = \frac{b^2+k}{a} > 0 \end{array}\right\}$$ x_2 is positive integer.

(x_2, b) Take smallest (a,b)

$$x_2 + \cancel{b} \geq a + \cancel{b}$$

$a > b > 1$

$2a \geq b(a^2 - b^2)$?

$a > b > 1 \Rightarrow b^2 \leq (a-1)^2$

$2a \geq b(a^2 - b^2) \geq b(a^2 - (a-1)^2) =$

$= b(a^2 - (a^2 - 2a + 1)) = b(2a-1)$

$2a \geq b(2a-1) \geq 2(2a-1)$

$b \geq 2$

↯. Ok?

$$(mn-1) \mid (m+n-1)^2$$ in positive integer

By Vieta

$$\frac{(m+n-1)^2}{mn-1} = k > 0$$

Claim: $k = 3$ OR 4. :)

$$m^2 - m(kn+2-2n) + (n-1)^2 + k = 0 \quad ?$$

$m \geq n$

$$x_2 = kn+2-2n-m \in \mathbb{Z}.$$

$$x_2 = \frac{(n-1)^2+k}{m} > 0$$

$$(mn-1) \mid (n^2-n+1)^2.$$

$$\Downarrow$$

$\boxed{1^\circ}$ Prove that $mn-1 \mid (m+n-1)^2$:) ?

$$0 \equiv (n^2-n+1)^2 \equiv (n^2-n+1+mn-1)^2 = n^2(n+m-1)^2 \mod (mn-1)$$

$$= (n^2-n+mn)^2$$

$$= (n(n+m-1))^2$$

$$= n^2(n+m-1)^2 \quad \text{but } (mn-1, n) = 1$$

Ok!

$$\boxed{mn-1 \mid (n+m-1)^2}$$

(m,n) such that sum $m+n$ is <u>smallest</u>.

$$x_2 + \cancel{n} \geq m + \cancel{n}$$

$$x_2 \geq m$$

$$\frac{(n-1)^2+k}{m} \geq m$$

$$\boxed{k \geq m^2 - (n-1)^2 = (m-n+1)(m+n-1)}$$

$\boxed{1^\circ}$ $m > n$. then

$$k \geq \underbrace{(m-n+1)}_{\geq 1}(m+n-1) \geq 2(m+n-1)$$

It means that $\frac{(m+n-1)^2}{mn-1} = k \geq 2(m+n-1) \quad \Big| : (m+n-1)$

$\boxed{m \geq n \geq 0}$ so $m > 1$

$$(2n-1)(m+1) = 2mn + 2n - m - 1 \leq$$

$$\leq \cancel{m} + n + \cancel{1} + 2n - \cancel{m} - \cancel{1} = 3n$$

$$4n - 2 < (2n-1)(\underbrace{m+1}_{>2}) \leq 3n$$

$4n - 2 < 3n$ so $\boxed{n = 1}$ so

$\boxed{n < 2}$ $m - 1 = 1$

$mn - 1 \mid (m+n-1)^2 \Rightarrow \boxed{m-1} \mid m^2 \Rightarrow m = 2$

$\gcd(m-1, m) = 1$. $(2, 1)$

$$\frac{2m+n-1}{mn-1} \geq 2 \Rightarrow (m+n-1) \geq 2mn-2$$

$$\boxed{m+n+1 \geq 2mn}$$

$m > n$.

$$\cancel{(m+1)(n+1)} \geq 3mn.$$

$$(2n-1)(m+1) = 2mn+2n-m-1 \leq \cancel{m}+n+\cancel{1}+2n-\cancel{m}-\cancel{1} = 3n$$

$$\boxed{(2n-1)(m+1) \leq 3n}$$

But $m \geq n$. Suppose: $\boxed{m \geq 1}$ then:

$$3n \geq (2n-1)(m+1) \geq (2n-1)\cdot 2 = 4n-2$$

$$\boxed{n \leq 2}$$

$m \geq 1$.

$n \leq 2$

$n = 1, 2$

$(2,1)$ and so

$$\frac{(m+n-1)^2}{mn-1} = \frac{(2+1-1)^2}{2\cdot 1-1} = \boxed{4}.$$

$\boxed{2^\circ}$ $\underline{m=n}$

$$\frac{(2m-1)^2}{m^2-1} \in \mathbb{Z}.$$

$\boxed{m \geq 1}$

$$m^2-1 \mid 4m^2-4m+1 = 4m^2-4-4m+5 \quad \text{so}$$

$$m^2-1 \mid -4m+5$$

$$m^2-1 \mid |4m-5|$$

$$4m-5 \geq m^2-1 \nearrow$$

$$m^2-4m+4 \leq 0$$

$$(m-2)^2 \leq 0$$

$$\boxed{m=2}$$

$(2,2)$

$$\frac{(2+2-1)^2}{2\cdot 2-1} = \boxed{3}$$

$n=m=2$

$$\frac{(m+n-1)^2}{mn-1} = 3 \quad \text{or} \quad 4$$

$$\boxed{\frac{x^2+y^2+1}{xy} = 3}$$

$$(m+n-1)^2 = 3mn-3$$

$$2mn + m^2+n^2-2m-2n+1 = 3mn-3$$

$$m^2+n^2-2m-2n+4 = mn \quad | \cdot 2$$

$$(m-n)^2+(m-2)^2+(n-2)^2 = 0$$

$$\boxed{m = n = 2}$$

$$\frac{(m+n-1)^2}{mn-1} = 4$$

$$\left\{ \begin{array}{c} (t^2+1,\ t^2-2t+2) \\ (2,2) \end{array} \right\}$$

$$(m+n-1)^2 = 4mn-4$$

$$(m-n)^2 - 2(m+n) + 5 = 0$$

$m-n \geq 1$ - odd $\qquad m-n = 2t-1$

$$4t^2-4t+1 - 2(m+n)+5 = 0$$

$$4t^2-4t+6 = 2(m+n)$$

$$\begin{cases} 2t^2-2t+3 = m+n \\ 2t-1 = m-n \end{cases}$$

$$2t^2+2 = 2m \Rightarrow \boxed{m = t^2+1}$$

$$n = m-2t+1 = t^2+1-2t+1 = \boxed{t^2-2t+2}$$

Plan

Revision (or not) When number is a sum of 2 squares.

$$n = a^2+b^2 \iff n = 2^r \cdot \underbrace{p_1^{\alpha_1} \cdots p_k^{\alpha_k}}_{p_i = 4k+1} \; \underbrace{q_1^{2\beta_1} \cdots q_l^{2\beta_l}}_{4k+3}$$

Next plan

Chinese Remainder Theorem (Revision)

Quadratic Residues (Introduction)

Diophantine problems / Divisibility / many details

LITTLE FERMAT ?

Euler theorem ? $a^{\varphi(n)} \equiv 1 \pmod{n}$

Willson ?

CRT.

Order and generator ?