TST1 – sols

Problem 1 :

$$n = 5\tau(n)$$

. . . .

Solution :

Lemma : $\tau(n) < 2\sqrt{n}$

$\forall n \in \mathbb{N}$

Proof : if n is perfect square then $\tau(n) = 2k-1$

$$n = d_1 d_{2k-1} = \cdots = d_{k-1} d_{k+1} = d_k^2$$

$$d_1 < d_2 < \cdots < d_{k-1} < d_k = \sqrt{n}$$

$$\Rightarrow \quad k \leqslant \sqrt{n}$$

$$\Rightarrow \quad \tau(n) = 2k-1 < 2\sqrt{n}$$

<u>Case 2</u>: $\sqrt{n} \notin \mathbb{Z}$. $\quad \tau(n) = 2k$

$$n = d_1 d_{2k} = \cdots = d_k d_{k+1}$$

$$d_1 < d_2 < \cdots < d_k < \sqrt{n} < d_{k+1} < \cdots$$

$$\Rightarrow \quad k < \sqrt{n} \Rightarrow \tau(n) = 2k < 2\sqrt{n}$$

$\blacksquare$

Now back to the problem:

$$n = 5\tau(n) < 10\sqrt{n}$$

$$\Leftrightarrow \quad n < 100$$

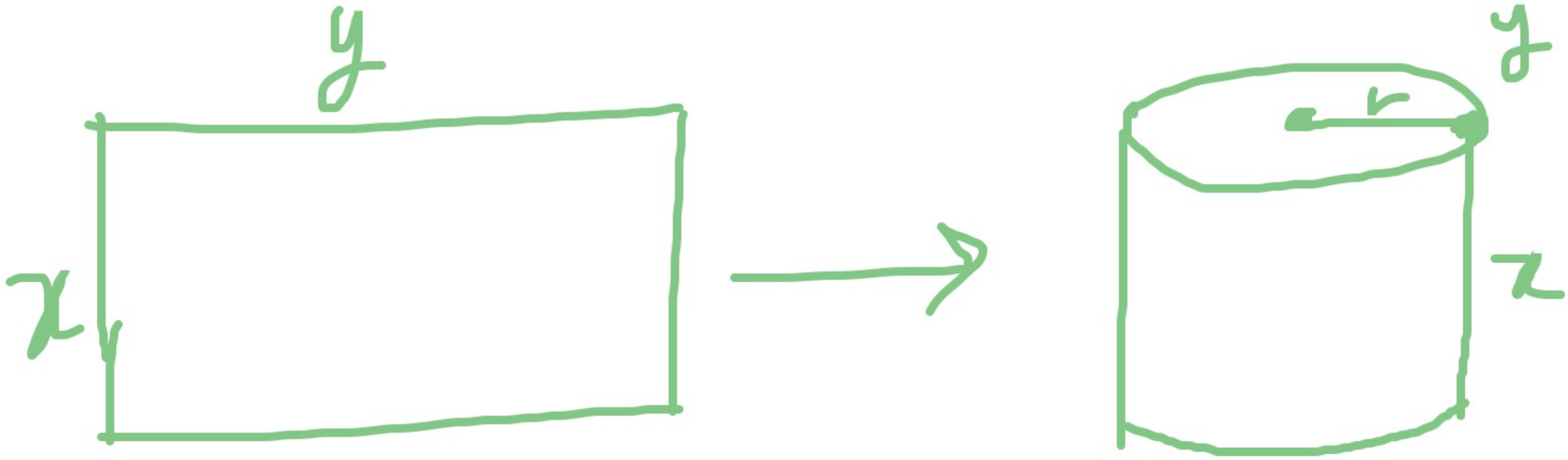
$$5 \mid n \Rightarrow n \in \{5, 10, \ldots, 95\}$$

$$\Rightarrow \boxed{n = 40, 60} \quad \blacksquare$$

$$\tau(40) = 8 = \frac{40}{5}$$

$$\tau(60) = 12 = \frac{60}{5}$$

Problem 2



$$x+y=3\pi \quad ; \quad y=2\pi r.$$

$$V=\pi r^2 x=\pi\left(\frac{y}{2\pi}\right)^2 x=\frac{xy^2}{4\pi}$$

$$x\cdot\frac{y}{2}\cdot\frac{y}{2}\leq\frac{(x+y)^3}{27} \quad \text{(AM-GM)}$$

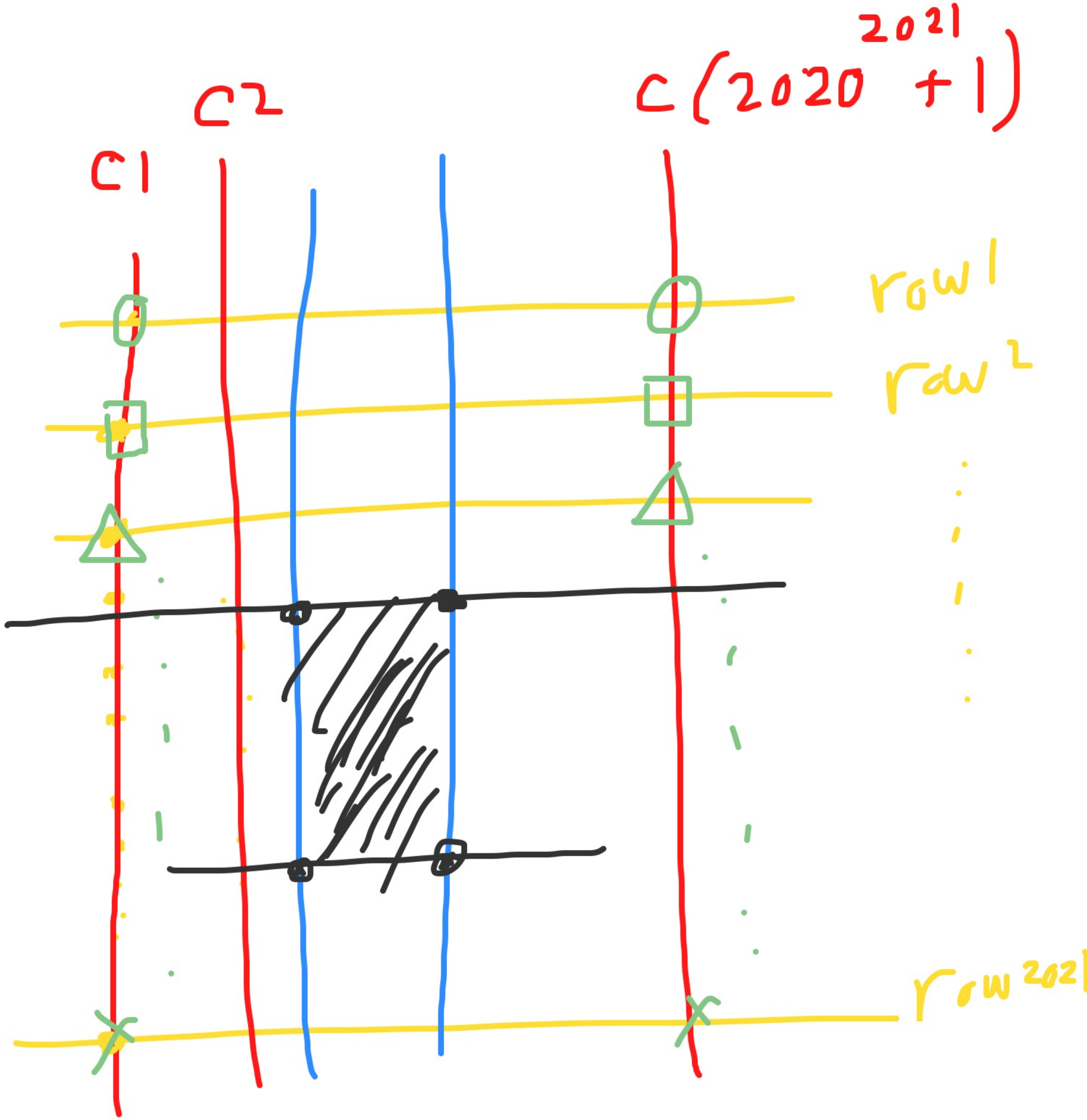
$$V = \frac{1}{\pi} \cdot \frac{xy^2}{4} \leq \frac{1}{\pi} \frac{(x+y)^3}{27}$$

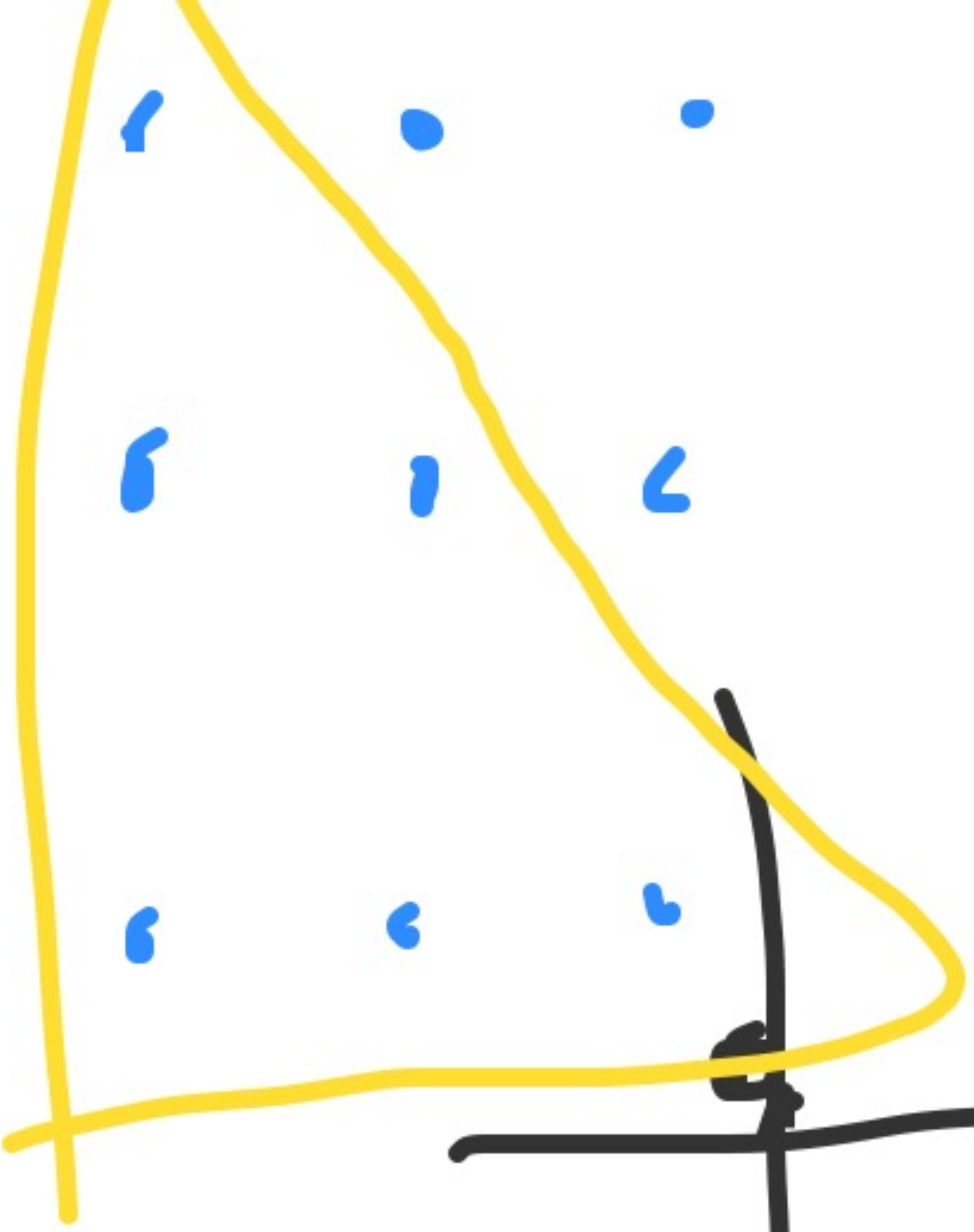
$$\boxed{= \pi^2}$$

"$=$" $\quad x = \frac{y}{2} = \pi$

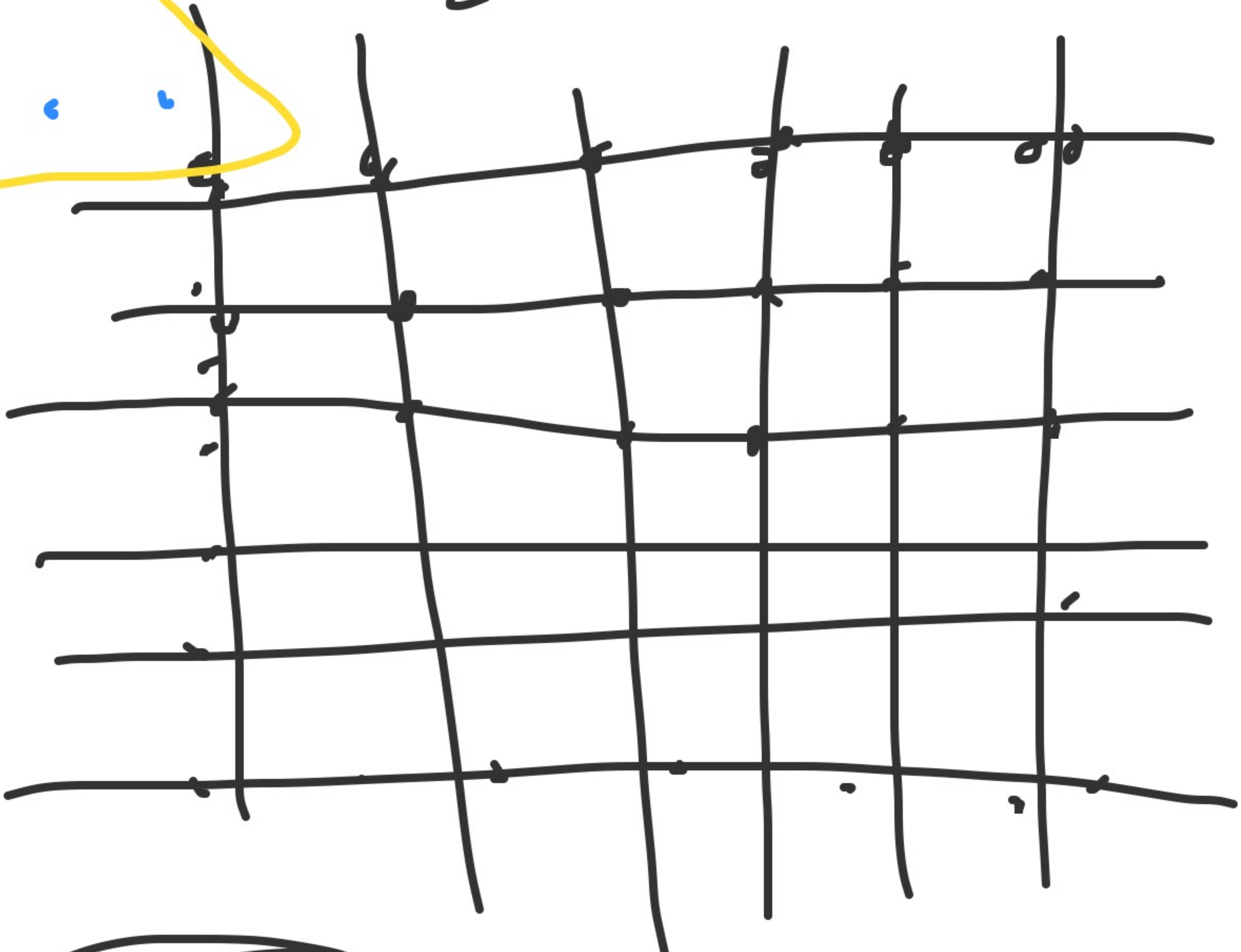
$\blacksquare$

Problem 3





2020 X 2020



rows

2020·2019 +1

S

S

2020 X 2019 +1

M

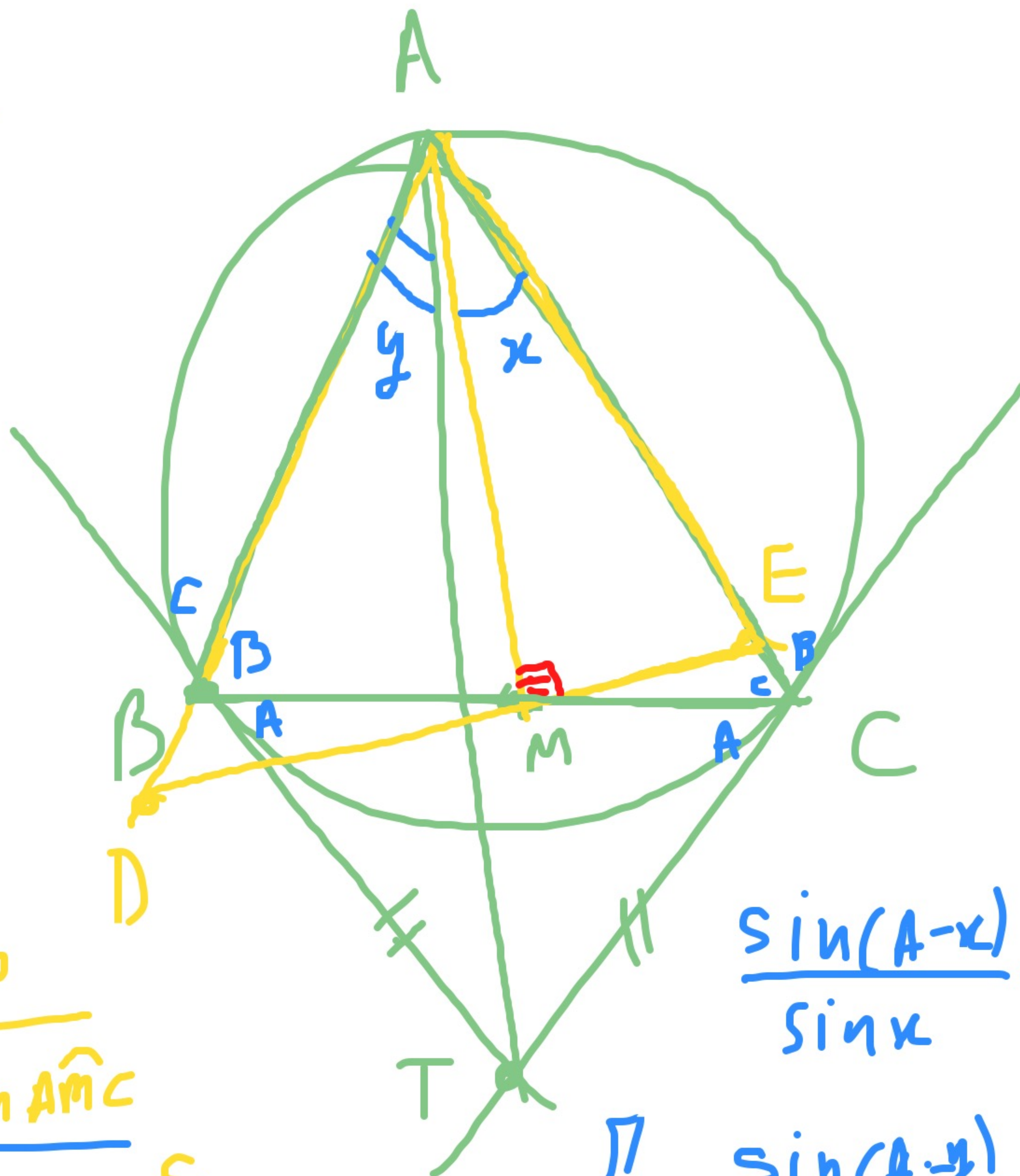
Problem 4

Lemma:

$\angle BAT = \angle CAM$

$AC = b$

$BC = c$



$$\frac{CM}{\sin x} = \frac{b}{\sin \widehat{AMC}}$$

$$\frac{BM}{\sin(A-x)} = \frac{c}{\sin \widehat{AMB}}$$

$$\frac{\sin(A-x)}{\sin x} = \frac{b}{c}$$

$$\frac{\sin(A-y)}{\sin y} = \frac{b}{c}$$

$$\frac{\sin(A-x)}{\sin x} = \frac{\sin(A-y)}{\sin y}$$

$$\frac{\sin A\cos x - \cos A\sin x}{\sin x} = \frac{\sin A\cos y - \cos A\sin y}{\sin y}$$

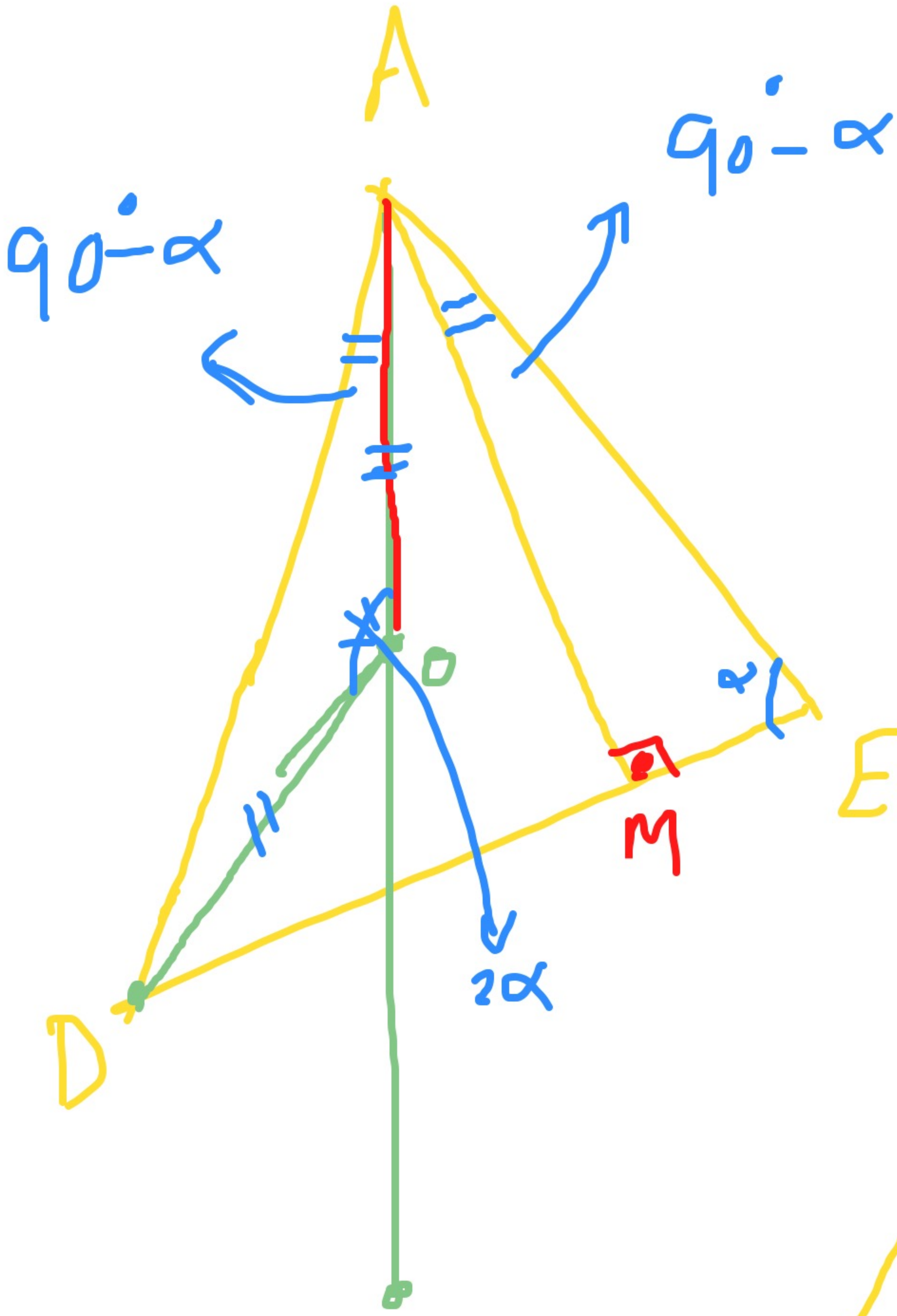
$$\sin A\frac{\cos x}{\sin x} - \cancel{\cos A} = \sin A\frac{\cos y}{\sin y} - \cancel{\cos A}$$

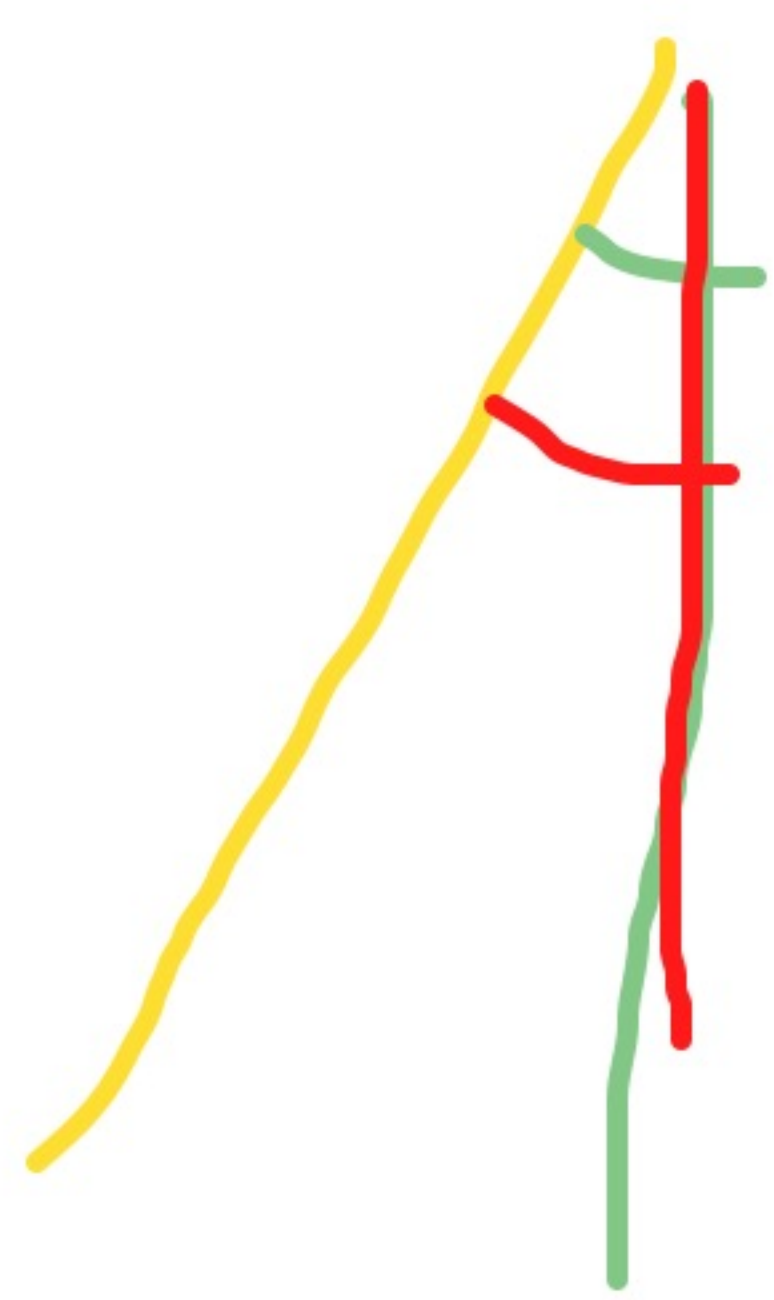
$$\frac{\cos x}{\sin x} = \frac{\cos y}{\sin y} \Longleftrightarrow \sin x\cos y - \sin y\cos x = 0$$

$$\|$$

$$\sin(x-y)$$

$$\Longrightarrow \boxed{x=y}$$





$$\Rightarrow \widehat{DAO} = \widehat{EAM} = \widehat{DAT} \quad \blacksquare$$