ABOUT SYMMETRIC 3-VARIABLE INEQUALITIES

- pqr technique & Schur's inequality.
- Standardize and finding the best parameter.

Problem 1. Let $x, y, z > 0$ and $xy + yz + zx = 3$. Prove that

$$(x^2+1)(y^2+1)(z^2+1) \geq 8.$$

Problem 2. Let $x, y, z > 0$ and $x + y + z = 3$. Prove that

$$3 + \frac{12}{xyz} \geq 5\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Problem 3. Let $a, b, c > 0$ and $abc = 1$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 3 \geq 2(a+b+c).$$

Problem 4. Let $a, b, c > 0$, prove that

a) $(a^3+b^3+c^3)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right) + 6(ab+bc+ca) \geq 9(a^2+b^2+c^2)$.

b) $2(a^2+b^2+c^2) + 3(abc)^{\frac{2}{3}} \geq (a+b+c)^2$.

Problem 5.

a) Find the maximum value of k such that the following inequality is true for all a, b, c are non-negative numbers:

$$2(a^3+b^3+c^3) - k(a^3+b^3+c^3-3abc) \geq ab(a+b) + bc(b+c) + ca(c+a).$$

b) Find the maximum value of k such that the following inequality is true for all a, b, c are non-negative numbers satisfying $a+b+c = ab+bc+ca > 0$:

$$a+b+c+k(abc-1) \geq 3.$$

c) Find the minimum value of k such that the following inequality is true for all a, b, c are non-negative numbers satisfying $ab+bc+ca = 3$:

$$(a+k)(b+k)(c+k) \geq (k+1)^3.$$

Problem 6. Let a, b, c be non-negative numbers such that

$$2(a^2+b^2+c^2) + 3(ab+bc+ca) = 5(a+b+c).$$

Prove that

$$4(a^2+b^2+c^2) + 2(ab+bc+ca) + 7abc \leq 25.$$

Problem 7*. (Saudi Arabia TST 2015) Let $a, b, c > 0$ such that $a^2+b^2+c^2 = 3$. Prove that

$$a+b+c \geq 3 \cdot \sqrt[5]{abc}.$$