Email training, N3

September 25-October 1

Problem 3.1. Find an example of a sequence of natural numbers $1 \leq a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ with the property that every positive integer m can be uniquely written as $m = a_i - a_j$, with $i > j \geq 1$.

Solution 3.1. We consider the sequence

$$a_1 = 1, a_2 = 2,$$

$$a_{2n+1} = 2a_{2n},$$

$$a_{2n+2} = a_{2n+1} + r_n,$$

where r_n is the smallest natural number that cannot be written in the form $a_i - a_j$, with $i, j =\leq 2n+1$. It satisfies to the conditions of the problem

Problem 3.2. Prove the identity

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \frac{\binom{n}{0}}{x} - \frac{\binom{n}{1}}{x+1} + \frac{\binom{n}{2}}{x+2} - \ldots + (-1)^n \frac{\binom{n}{n}}{x+n}.$$

Solution 3.2. By applying the identity

$$\frac{1}{(x+a)(x+b)} = \frac{1}{a-b}\Big(\frac{1}{x+b} - \frac{1}{x+a}\Big)$$

multiple times one may get the following relation

$$\frac{n!}{x(x+1)(x+2)\ldots(x+n)} = \sum_{k=0}^{n} \frac{A_k}{x+k}.$$

By multiplying both sides by $x(x+1)(x+2)\ldots(x+n)$ and by putting $n = -k$ one gets

$$n! = A_k \cdot (-k) \cdot (-k+1) \cdot (-k+2) \cdot \ldots \cdot (-1) \cdot 1 \cdot 2 \cdot \ldots \cdot (n-k)$$

so

$$A_k = \frac{(-1)^k A_k}{k!(n-k)!} = (-1)^k \binom{n}{k}.$$

Problem 3.3. Prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Solution 3.3. Let's apply Bernoulli inequality.

$$\left(1+\frac{5}{n}\right)^n > 1+n\cdot\frac{5}{n}=6,$$

therefore

$$1+\frac{5}{n} > 6^{1/n}.$$

Also

$$\left(1+\frac{-5}{6n}\right)^n > 1+n\cdot\frac{-5}{6n}=\frac{1}{6},$$

$$\left(\frac{6n-5}{6n}\right)^n > \frac{1}{6},$$

$$6 > \left(\frac{6n}{6n-5}\right)^n,$$

$$6^{1/n} > \frac{6n}{6n-5} = 1+\frac{5}{6n-5}.$$

Problem 3.4. Let $x, y, z \geq 0$ and $x+y+z=3$. Prove that

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+yz+zx.$$

Solution 3.4. One has

$$3(x+y+z)=(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx).$$

Hence it follows that

$$xy+yz+zx=\frac{1}{2}(3x-x^2+3y-y^2+3z-z^2).$$

Then

$$\sqrt{x}+\sqrt{y}+\sqrt{z}-(xy+yz+zx)=$$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}+\frac{1}{2}(x^2-3x+y^2-3y+z^2-3z)$$

$$=\frac{1}{2}\sum_{cyc}(x^2-3x+2\sqrt{x})=\frac{1}{2}\sum_{cyc}\sqrt{x}(\sqrt{x}-1)^2(\sqrt{x}+2)\geq 0.$$

Problem 3.5. Let $a, b, c > 0$. Prove that

$$\frac{a+b}{a^2+b^2}+\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution 3.5. By applying the AM-GM for the denominator one gets

$$\frac{a+b}{a^2+b^2} \leq \frac{a+b}{2ab} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right).$$

By applying the same estimation for 2 other expressions of the left side and by taking the sum we get the desired statement.

Problem 3.6. Let $n > 3$, $x_1, x_2, \ldots, x_n > 0$ and $x_1 x_2 \ldots x_n = 1$. Prove that

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \ldots + \frac{1}{1+x_n+x_nx_1} > 1.$$

Solution 3.6.

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \ldots + \frac{1}{1+x_n+x_nx_1} >$$

$$\frac{1}{1+x_1+x_1x_2+x_1x_2x_3+\ldots+x_1x_2\ldots x_{n-1}} +$$

$$\frac{1}{1+x_2+x_2x_3+x_2x_3x_4+\ldots+x_2x_3\ldots x_n} + \ldots +$$

$$\frac{1}{1+x_n+x_nx_1+x_nx_1x_2+\ldots+x_nx_1\ldots x_{n-2}}.$$

Denote $S = 1 + x_1 + x_1x_2 + \ldots + x_1x_2\ldots x_{n-1}$. By multiplying the nominator and denominator of second term by x_1, of the third term by x_1x_2 and son on in n-th term by $x_1x_2\ldots x_{n-1}$ and by taking into account that $x_1x_2\ldots x_n = 1$ one gets

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \ldots + \frac{1}{1+x_n+x_nx_1} >$$

$$\frac{1}{S} + \frac{x_1}{S} + \frac{x_1x_2}{S} + \ldots + \frac{x_1x_2\ldots x_{n-1}}{S} = 1.$$

Problem 3.7. -

Let M be the midpoint of the side AC of a triangle ABC and let H be the foot point of the altitude from B. Let P and Q be the orthogonal projections of A and C on the bisector of angle B. Prove that the four points M, H, P and Q lie on the same circle.

Solution 3.7. -

Solution. If $|AB| = |BC|$, the points M, H, P and Q coincide and the circle degenerates to a point. We will assume that $|AB| < |BC|$, so that P lies inside the triangle ABC, and Q lies outside of it.

Let the line AP intersect BC at P_1, and let CQ intersect AB at Q_1. Then $|AP| = |PP_1|$ (since $\triangle APB \cong \triangle P_1PB$), and therefore $MP \parallel BC$. Similarly, $MQ \parallel AB$. Therefore $\angle AMQ = \angle BAC$. We have two cases:

(*i*) $\angle BAC \le 90°$. Then A, H, P and B lie on a circle in this order. Hence $\angle HPQ = 180° - \angle HPB = \angle BAC = \angle HMQ$. Therefore H, P, M and Q lie on a circle.

(*ii*) $\angle BAC > 90°$. Then A, H, B and P lie on a circle in this order. Hence $\angle HPQ = 180° - \angle HPB = 180° - \angle HAB = \angle BAC = \angle HMQ$, and therefore H, P, M and Q lie on a circle.

Problem 3.8. -

ABCD is a trapezium , $AD \parallel BC$. P is the point on the line AB such that $\angle CPD$ is maximal. Q is the point on the line CD such that $\angle BQA$ is maximal. Given that P lies on the segment AB, prove that $\angle CPD = \angle BQA$.

Solution 3.8. -

Solution. The property that $\angle CPD$ is maximal is equivalent to the property that the circle CPD touches the line AB (at P). Let O be the intersection point of the lines AB and CD, and let ℓ be the bisector of $\angle AOD$. Let A', B' and Q' be the points symmetrical to A, B and Q, respectively, relative to the line ℓ. Then the circle AQB is symmetrical to the circle $A'Q'B'$ that touches the line AB at Q'. We have

$$\frac{|OD|}{|OA'|} = \frac{|OD|}{|OA|} = \frac{|OC|}{|OB|} = \frac{|OC|}{|OB'|}.$$

Hence the homothety with centre O and coefficient $|OD|/|OA|$ takes A' to D, B' to C, and Q' to a point Q'' such that the circle $CQ''D$ touches the line AB, and thus Q'' coincides with P. Therefore $\angle AQB = \angle A'Q'B' = \angle CQ''D = \angle CPD$ as required.