HOMEWORK

Enough to find solution for $n=1$, $n=2$

For $n=1$: $(1,2)$ For $n=2$ $(3,5)$

Problem 18

$x^2+xy+y^2=7^n$ is solvable for any $n \geq 1$

$9+15+25=\underline{49}$

Idea (x,y) is a solution for n, then $(7x,7y)$ is solution for $n+2$

Indeed $(7x)^2+7x\cdot(7y)+(7y)^2 = 7^2(x^2+xy+y^2) = 7^2\cdot 7^n = 7^{\boxed{n+2}}$

$n=$ 1 3 5 7 9 11 ...

2 4 6 8 10

Problem 19

$$\frac{1}{x_1^2}+\frac{1}{x_2^2}+\dots+\frac{1}{x_n^2}=1$$ is solvable for $n \geq 1$ in integers

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(x,y) - solution for n then $(2x-y,\ x+3y)$ is solution for $\boxed{n+1}$ $\quad n \longrightarrow n+1$.

$$(2x-y)^2 + (2x-y)(x+3y) + (x+3y)^2 =$$

$$= 4x^2 - 4xy + y^2 + 2x^2 + 6xy - xy - 3y^2 + x^2 + 6xy + 9y^2 =$$

$$= 7x^2 + 7xy + 7y^2 = 7\underbrace{(x^2+y^2+xy)}_{7^n} = \boxed{7^{n+1}}$$

$(n \longrightarrow n+1$

$\boxed{n=1}$ $\quad (1,2) \longrightarrow (2\cdot 1-2,\ 1+3\cdot 2) \longrightarrow \ldots$

for any $n \geq 1$ you get solution :)

$(ax+by,\ cx+dy)$

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Problem 6 Prove that for any $n \geq 6$

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \dots + \frac{1}{x_n^2} = 1$$ is solvable

in integers.

HINT:

$$\frac{1}{a^2} = \frac{1}{4a^2} + \frac{1}{4a^2} + \frac{1}{4a^2} + \frac{1}{4a^2} = \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2}$$

If we have solution for n

$$\left.\begin{aligned} &\frac{1}{x_1^2} + \frac{1}{x_2^2} + \dots + \frac{1}{x_n^2} = 1 \\ &\qquad\qquad\qquad\qquad \| \\ &\qquad\qquad \frac{1}{(2x_n)^2} + \frac{1}{(2x_n)^2} + \frac{1}{(2x_n)^2} + \frac{1}{(2x_n)^2} \end{aligned}\right\} \text{solution for } n+3$$

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We have jump $n \longrightarrow n+3$, $\boxed{n \geq 6}$ so enough to find solutions for $n = 6, 7, 8$.

$$\boxed{n \rightsquigarrow n+3}$$

$\boxed{n=6}$

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \frac{1}{36} = 1.$$

$\boxed{n=7}$

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = 1$$

$\boxed{n=8}$

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{16} + \frac{1}{144} + \frac{1}{144} = 1$$

You <u>said</u> $\frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{36} + \frac{1}{36} = 1.$

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Decide whether

$$x^4 + y^3 = z! + 7 \quad \text{has} \quad \infty \text{ or } < \infty$$

Solutions in $\boxed{\text{positive}}$

$$\underline{3 \cdot 4 + 1 = 13}$$

Consider $z \geq 13$, then for sure $13 \mid z!$ so

$$x^4 + y^3 \equiv 7 \pmod{13}$$

$\downarrow$

$\boxed{x^4 \equiv y^3 + 7}$ No possible $\rightsquigarrow$

So $z < 13 \Rightarrow z = 0, 1, 2, \ldots, 12$.

$x^4 + y^3 =$ Some number

IN TOTAL ONLY FINITELY MANY SOL.

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17

$$x^2 - y^2 = 2xyz$$

$$\cancel{(d_1, d_1, 0)}$$
$$\cancel{(x_1, y_1, z)}$$

$d = \gcd(x, y) > 1$

$x = d x_1$

$y = d y_1$

$\Rightarrow$

$$\cancel{d^2} x_1^2 - \cancel{d^2} y_1^2 = 2 \cdot \cancel{d^2} x_1 y_1 z$$

$$\boxed{x_1^2 - y_1^2 = 2 x_1 y_1 z} \quad \times 1$$

$z = 0$

$\gcd \boxed{(x_1, y_1) = 1}$

Start with solution (x, y) with minimal sum, and you see that $\left(\frac{x}{d}, \frac{y}{d}\right)$ is also solution with smaller sum.

Alternative way

$x_1 \mid y_1^2 \rightarrow$ ↯

$\downarrow$

or $x_1 = 1$ $\quad 1 - y_1^2 = 2 y_1 z \Rightarrow y_1 (y_1 + 2z) = 1$

$y_1 \mid x_1^2 \rightarrow y_1 = 1$ $\quad z = 0$

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$4ab-1 \mid (a-b)^2 \Rightarrow \underline{a=b}$ in positive integers

<u>Vieta Jumping</u>

Small hint

$$\frac{\overset{>0}{(a-b)^2}}{\underset{>0}{4ab-1}} = k > 0$$

$\boxed{a > b}$

$a+b$ $\boxed{\text{minimal}}$

$$(a-b)^2 = k(4ab-1)$$

$$a^2-2ab+b^2 = 4kab-k$$

$$\boxed{a^2-a(2b+4kb)+b^2+k=0} \quad (a)$$

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$$\boxed{X^2 - X(2b+4kb) + b^2 + k = 0}$$

One solution is $X_1 = a$.

Consider X_2.

From Vieta

$$\underset{\overset{\parallel}{a}}{X_1} + X_2 = 2b + 4kb \Rightarrow X_2 = \overbrace{2b}^{\in\mathbb{Z}} + \overbrace{4kb}^{\in\mathbb{Z}} - \overbrace{a}^{\in\mathbb{Z}} \in \mathbb{Z}.$$

$$\underset{\overset{\parallel}{a}}{X_1} X_2 = b^2 + k \Rightarrow X_2 = \frac{b^2+k}{a} > 0$$

so X_2 is positive integer (X_2, b) is also a solution,

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So

$$x_2 + \cancel{b} \geq a + \cancel{b}$$

$$\left.\begin{array}{c} x_2 \geq a \\ \| \\ \dfrac{b^2+k}{a} \end{array}\right\} \Rightarrow \frac{b^2+k}{a} \geq a \Rightarrow k \geq a^2 - b^2$$

$a = b$

$$\frac{(a-b)^2}{4ab-1} = k \geq a^2 - b^2 \Rightarrow \cancel{(a-b)^2} \geq (a^2-b^2)(4ab-1)$$

$$\| $$

$$\cancel{(a-b)}(a+b)(4ab-1)$$

$a > b$

$$a-b \geq \underbrace{(4ab-1)}_{\geq 1}(a+b) \geq a+b$$

impossible

impossible integer.

$\lightning$

so $a = b$

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a, b odd positive integers such that

$$2ab+1 \mid a^2+b^2+1$$

Prove: $\underline{a=b}$. $\qquad a>b$

$$\frac{a^2+b^2+1}{2ab+1}=k$$

$$a^2+b^2+1=2abk+k$$

$$a^2-a\cdot 2bk+b^2+1-k=0$$

$$\boxed{X^2-X\cdot 2bk+b^2+1-k=0}$$

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Viet formulas:

$$\overset{a}{\overset{\shortparallel}{X_1+X_2}} = 2bk \Rightarrow \boxed{X_2 = \underbrace{2bk}_{\text{even}} - \underbrace{a}_{\text{odd}}} \in \mathbb{Z}.$$

$$\underset{a}{\underset{\shortparallel}{X_1 X_2}} = b^2+1-k \Rightarrow X_2 = \frac{b^2+1-k}{a}$$

(a,b) is solution so WE WANT (x_2, b) to be solution.

so what WE NEED to check:

$\boxed{1^\circ}$ x_2 is odd ✓ positive integer ✓

- Why x_2 is nonzero?: Otherwise

$x_2 \neq 0$

$x_2 = 0$

$2bk - a = 0 \Rightarrow a = \overbrace{2bk}^{\text{even}}$ ↯

Since $x_2 \in \mathbb{Z}$, not positive $x_2 \leq -1$.

$$\frac{b^2+1-k}{a} \leq -1 \Rightarrow b^2+1-k \leq -a$$

$$\Downarrow$$

$$\boxed{k \geq b^2+1+a}$$

so

$$\frac{a^2+b^2+1}{2ab+1} = k \geq b^2+1+a \rightsquigarrow a^2+\cancel{b^2}+\cancel{1} \geq (2ab+1)(b^2+1+a)$$

$$= 2ab(b^2+1+a)+\cancel{b^2}+\cancel{1}+a$$

so $a^2 \geq 2ab(b^2+1+a)+a \quad | : a$

$$\boxed{a \geq 2 \cdot b \cdot (b^2+1+a)+1} \quad \lightning \Rightarrow x_2 > 0$$

$$\in \mathbb{Z}.$$

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(x_2, b) solution as well (a, b),

take $a+b$ minimal. Then

$$x_2 + \cancel{b} \geq a + \cancel{b}$$

$$x_2 \geq a$$

$$\parallel$$

$$\frac{b^2+1-k}{a} \geq a \quad \Longrightarrow \quad b^2+1-k \geq a^2$$

$$k \leq b^2+1-a^2$$

but we assumed $a > b \Rightarrow a \geq b+1$, so

$$0 < k \leq b^2+1-a^2 \leq b^2+1-(b+1)^2 = b^2+1-b^2-1-2b = -2b < 0$$

$\lightning$

OK?

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$$x^2+y^2+x+y+1=k\cdot xy$$

We need to prove that $k=5$. & WLOG $\underline{x\geq y}$. ~~But~~ Assume $k\neq 5$. Prove (firstly) that $\underline{x>y}$

$\underline{x\neq y}$

Suppose $\underline{x=y}$ then $2x^2+2x+1$ is divisible by x^2 so

$x^2 \mid 2x+1 \Rightarrow x^2 \leq 2x+1 \Rightarrow$

$\Rightarrow (x-1)^2 \leq 2 \Rightarrow x=1, \cancel{2}$

$x=1$ $y=1$ so $\boxed{\frac{1^2+1^2+1+1+1}{1\cdot 1}=5}$

Assume $\boxed{x>y}$

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x, y positive integers.

x. $x^2+y^2+x+y+1$ is divisible by xy

Prove

$$\frac{x^2+y^2+x+y+1}{xy} = 5$$

$x, y \in \mathbb{Z}_+$ s.t. $xy \mid x^2+y^2+x+y+1$. Prove that

$$\frac{x^2+y^2+x+y+1}{xy} = 5$$

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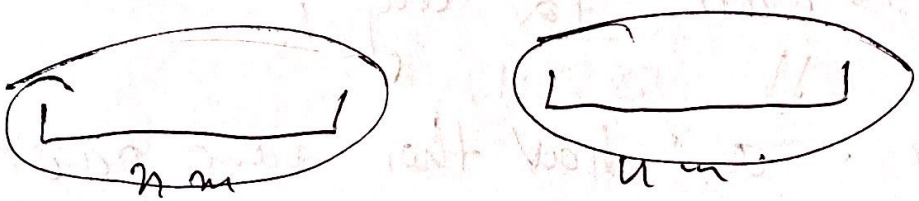
Problem Suppose you have $2n+1$ integers such that if you remove any one number then rest $2n$ numbers may be decompose into two groups each containing n numbers, with equal sum. Prove all numbers are equal.

$a_1, a_2, \ldots, a_{2n+1}$

$2n$ numbers.



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Hint: all numbers have the same parity

$a_1, a_2, a_3, \dots, a_{2n+1}$

$S = a_1 + a_2 + \dots + a_{2n+1}$

$\forall_{i \in \{1,\dots,2n+1\}}$ $S - a_i =$ sum of $2n$ number $=$ even

$(___)$ $(___)$

n num n numbers

wrt equal sum

fixed

$\boxed{S} - a_i$ is even for any i!

$\Downarrow$

numbers a_i have the same parity

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if $a_1, a_2, \ldots, a_{2n+1}$ are even the.

$$\boxed{\frac{a_1}{2}, \frac{a_2}{2}, \ldots, \frac{a_{2n+1}}{2}}$$ have the same property as $a_1 \ldots a_{2n+1}$

$2 \mid a_1,\ 2 \mid a_2, \ldots,\ 2 \mid a_{2n+1}$

$$\frac{a_1}{2}, \frac{a_2}{2}, \ldots, \frac{a_{2n+1}}{2}$$

$\in \mathbb{Z}$ $\in \mathbb{Z}$ $\in \mathbb{Z}$

$$\frac{a_1+1}{2}, \frac{a_2+1}{2}, \ldots, \frac{a_{2n+1}+1}{2}$$

$\in \mathbb{Z}$ $\in \mathbb{Z}$ $\in \mathbb{Z}$

a_1 - smaller

$a_1 + C,\ a_2 + C, \ldots, a_{2n+1} + C$. $\frac{a_1+1}{2} < a_1$ ↯

$a_1, a_2, \dots, a_{2n+1}$ have the same parity?

1) $2 \mid a_1,\ 2 \mid a_2, \dots, 2 \mid a_{2n+1}+1$

~~take~~

$a_1 + C,\ a_2 + C, \dots, a_{2n+1} + C$

$a_1, a_2, \dots, a_{2n+1} \geq 0$

WLOG a_1 - minimal number.

$$\boxed{\frac{a_1}{2}, \frac{a_2}{2}, \dots, \frac{a_{2n+1}}{2}}$$

$x_1 \quad x_2 \quad \dots \quad x_{2n+1}$

$\frac{a_1}{2} < a_1$ ↯

Take $a_1 \leq a_2, \dots \leq a_{2n+1}$ be the set satisfying problem's condition with the smallest possible a_1.

$\frac{a_1+1}{2}, \dots, \frac{a_{2n+1}+1}{2}$

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Remark (Difficult) the statement holds for $\boxed{\text{rational numbers}}$

Tomorrow 10:00, Last Homework problem + New Homework.

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