Saudi Arabia 2022 – Math Camp

Level 4

Geometry – Projective Geometry

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Harmonic Division 2

Theorem 4: Let A, B, C be three points lying on a circle ω. Let the tangents at A and C to ω intersect at a point P and let the line PB intersect ω again at D. Then $ABCD$ is harmonic.

Theorem 5: Let A, B, C, D be four points lying in this order on a line d. If X is a point not lying on this line, then if two of the following three propositions are true, then the third is also true:

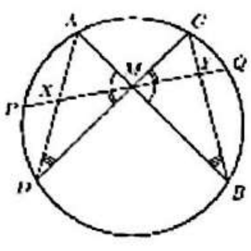
(a) $(A, C; B, D) = -1$.
(b) XB is the internal angle bisector of $\angle AXC$.
(c) $XB \perp XD$.

Theorem 6: Consider a circle ω and a point P outside the circle ω. The points C and D are on ω such that PC and PD are the tangents from P to ω. A line l through P intersects ω at A and B (with A between P and B). The segments AB and CD meet at Q. Then $(P, Q; A, B)$ is harmonic.

Theorem 7: Points $A; C; B; D$ lie on a line in this order, and M is the midpoint of CD. Then $(A; B; C; D)$ is harmonic if, and only if, $AC \cdot AD = AB \cdot AM$.

Example

Butterfly Theorem: For any chord PQ of a circle, let M be the midpoint of PQ. If we draw two other chords, AB and CD, through M, draw the line segments AD and CB, and let X and Y the intersection points of AD and CB with PQ, then M is the midpoint of XY.



Problems

5. The tangents to the circumcircle of ΔABC at B and C intersect at D. Prove that AD is the symmedian of ΔABC.

6. (China TST/2002) Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let O be the foot of the perpendicular from P to EF. Show that $\angle BOC = \angle AOD$.

7. (Brazil/2007) Let $ABCD$ be a convex quadrilateral, P the intersection of lines AB and CD, Q the intersection of lines AD and BC and O the intersection of diagonals AC and BD. Show that if $\angle POQ = 90°$ then PO is the bisector of $\angle AOD$ and OQ is the bisector of $\angle AOB$.

8. (Vietnam/2009) Let A, B be two fixed points and C is a variable point such that $\angle ACB = \alpha$, a constant in the range $[0°;\ 180°]$. The incircle of ΔABC with incenter I touches sides AB, BC, CA at points D, E, F, respectively. AI, BI intersect EF at M, N respectively. Prove that the length of MN is constant and the circumcircle of ΔDMN passes through a fixed point.