Day 2

Problem 1. Let n be an odd positive integer, and let $x_1, x_2, ..., x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\dots,n}(x_i^2 + x_{i+1}^2) \le \max_{j=1,\dots,n}(2x_jx_{j+1})$$

where $x_{n+1} = x_1$.

Problem 2. Three different non-zero real numbers a, b, c satisfy the equations $a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a} = p$, where p is a real number. Prove that $abc + 2p = 0$.

Problem 3. The sequence (a_n) is defined as: $a_1 = 1007$ and $a_{i+1} \ge a_i + 1$.
Prove the inequality:

$$\frac{1}{2016} > \sum_{i=1}^{2016} \frac{1}{a_{i+1}^2 + a_{i+2}^2}$$

$\mathbb{Z}^+$ مختلفة

Problem 4. Positive integers $a_1, a_2, ..., a_{2020}$ are pairwise distinct, and define $a_{2021} = a_1, a_{2022} = a_2$. Prove that there exists an integer $1 \le n \le 2020$ such that $a_n^2 + a_{n+1}^2 \ge a_{n+2}^2 + n^2 + 3$.

Problem 5. Positive reals $a_1, a_2, ...$ satisfy:

- $a_{n+1} = a_1^2 \cdot a_2^2 \cdot ... \cdot a_n^2 - 3 \quad \forall n$ positive integers
- $\frac{1}{2}(a_1 + \sqrt{a_2 - 1})$ is a positive integer.

Prove that $\frac{1}{2}(a_1 \cdot a_2 \cdot ... \cdot a_n + \sqrt{a_{n+1} - 1})$ is a positive integer.

Problem 6. Let $a_0 > 0$ be a real number, and let

$$a_n = \frac{a_{n-1}}{\sqrt{1 + 2020 \cdot a_{n-1}^2}}, \quad n = 1, 2, ..., 2020$$

Show that $a_{2020} < \frac{1}{2020}$

Problem 2. Three different non-zero real numbers a, b, c satisfy the equations $a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a} = p$, where p is a real number. Prove that $abc + 2p = 0$.

الحل الأول :

$$a - b + \frac{2}{b} - \frac{2}{c} = 0 \iff a - b = \frac{2}{c} - \frac{2}{b} = \frac{2(b-c)}{bc}$$

بنفس الطريقة :

$$c - a = \frac{2(a-b)}{ab}, \quad b - c = \frac{2(c-a)}{ca}$$

$$\rightarrow \prod(a-b) = \frac{8\prod(a-b)}{a^2b^2c^2} \iff a^2b^2c^2 = 8$$

$$abc = \pm 2\sqrt{2}$$

$$ab + 2 = bp, \quad bc + 2 = cp, \quad ca + 2 = ap$$

$$\rightarrow b(a-c) = (b-c)p, \quad c(b-a) = (c-a)p, \quad a(c-b) = (a-b)p$$

بالضرب

$$\rightarrow abc \prod_{cyc}(b-a) = \prod_{cyc}(a-b)\, p^3$$

$$\rightarrow p^3 = -abc = \mp 2\sqrt{2} = \mp(\sqrt{2})^3$$

$$\rightarrow p = \mp\sqrt{2}$$

$$\Rightarrow abc + 2p = 2\sqrt{2} - 2\sqrt{2} = 0$$

أو $= -2\sqrt{2} + 2\sqrt{2} = 0$

الحل الثاني :

$$p = a + \frac{2}{b} = a + \frac{2}{p - \frac{2}{c}} = a + \frac{2}{p - \frac{2}{p - \frac{2}{a}}}$$

$$= a + \frac{2}{p - \frac{2a}{pa-2}} = a + \frac{2(pa-2)}{p(pa-2) - 2a}$$

$$\rightarrow p(p(pa-2) - 2a) = a(p(pa-2) - 2a) + 2(pa-2)$$

$$\iff p^3a - 2p^2 - 2ap = p^2a^2 - \cancel{2pa} - 2a^2 + \cancel{2pa} - 4$$

$$\Leftrightarrow \quad \underline{p^3 a} - \underline{2p^2} - 2ap - \underline{p^2 a^2} + 2a^2 + 4 = 0$$

$$\Leftrightarrow \quad p^2(pa - 2 - a^2) + 2\underbrace{(-ap + a^2 + 2)} = 0$$

$$\Leftrightarrow \quad (p^2 - 2)(pa - 2 - a^2) = 0$$

- $pa - 2 - a^2 = 0$

  $p = c + \frac{2}{a} \rightarrow pa = ca + 2$ $\Big\} \rightarrow ca = a^2 \rightarrow c = a$

  $\Rightarrow\Leftarrow$

- $p^2 = 2$

$$\rightarrow p^2 = \cancel{2} = \left(a + \frac{2}{b}\right)\left(b + \frac{2}{c}\right) = ab + 2\frac{a}{c} + \cancel{2} + \frac{4}{bc}$$

$$\rightarrow abc + 2a + \frac{4}{b} = 0$$

$$\Leftrightarrow abc + 2\left(a + \frac{2}{b}\right) = 0$$

$$\Leftrightarrow abc + 2p = 0$$

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$$\min_{i=1,\dots,n} (x_i^2 + x_{i+1}^2) \leq \max_{j=1,\dots,n} (2x_j x_{j+1})$$

wlog $x_1 \geq x_2 \geq x_3$

$\max(2x_i x_{i+1}) = 2x_2 x_1$

$\min(x_i^2 + x_{i+1}^2) = x_3^2 + x_2^2$

$2x_2 x_1 \overset{?}{\geq} x_3^2 + x_2^2$

$x_2 x_1 \geq x_2^2$

$x_2 x_1 \geq x_3^2$

$\rightarrow 2x_2 x_1 \geq x_3^2 + x_2^2$

$x_i \geq x_{i+1}, x_{i-1}$

or $x_i \leq x_{i+1}, x_{i-1}$

من فردية n يوجد i

بحيث

$x_{i-1} \leq x_i \leq x_{i+1}$

أو $x_{i-1} \geq x_i \geq x_{i+1}$

Problem 3. The sequence (a_n) is defined as: $a_1 = 1007$ and $a_{i+1} \geq a_i + 1$.
Prove the inequality:

$$\frac{1}{2016} > \sum_{i=1}^{2016} \frac{1}{a_{i+1}^2 + a_{i+2}^2}$$

$a_i \geq 1006 + i$

$$a_i^2 + a_{i+1}^2 \geq 2a_i a_{i+1} \geq 2a_i(a_i+1)$$

$$\rightarrow \sum \frac{1}{a_i^2 + a_{i+1}^2} \leq \sum \frac{1}{2a_i(a_i+1)} = \frac{1}{2}\sum \frac{1}{a_i(a_i+1)} \leq \frac{1}{2}\left[\frac{1}{1008} + \frac{1}{1009} + \dots + \dots\right]$$

$$= \frac{1}{2}\left[\frac{1}{1008} + \frac{1}{a_{n}}\right] = \frac{1}{2016} - \frac{1}{2024} < \frac{1}{2016}$$

$$\frac{1}{a_i^2 + a_{i+1}^2} < \frac{1}{2a_i a_{i+1}} \leq \frac{a_{i+1} - a_i}{2a_i a_{i+1}} = \frac{1}{2}\left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right)$$

$$\sum_{i=1}^{2016} \frac{1}{a_{i+1}^2 + a_{i+2}^2} < \frac{1}{2}\left(\frac{1}{a_2} - \frac{1}{a_{2018}}\right) \leq \frac{1}{2016} - \frac{1}{2a_{2018}} < \frac{1}{2016}$$

Problem 4. Positive integers $a_1, a_2, ..., a_{2020}$ are pairwise distinct, and define $a_{2021} = a_1, a_{2022} = a_2$. Prove that there exists an integer $1 \le n \le 2020$ such that $a_n^2 + a_{n+1}^2 \ge a_{n+2}^2 + n^2 + 3$.

بفرض التناقض : $a_n^2 + a_{n+1}^2 < a_{n+2}^2 + n^2 + 3$ لكل n

$$a_n^2 + a_{n+1}^2 \le a_{n+2}^2 + n^2 + 2$$

$$\rightarrow \quad 2\sum a_i^2 \le \sum a_i^2 + \sum n^2 + 2 \times 2020$$

$$\Leftrightarrow \quad \sum_{i=1}^{2020} a_i^2 \le \underbrace{\sum_{n=1}^{2020} n^2 + 2 \times 2020}$$

$$\bullet \quad \sum_{i=1}^{2020} a_i^2 \ge \underbrace{1^2 + 2^2 + \dots + 2020^2}$$

إذا كان $2021 \le a_i$ $\leftarrow$ $a_i^2 \ge 2020^2 + 2 \times 2020 + 1$ $\leftarrow$ تناقض

$$\Rightarrow \{a_1, a_2, \dots, a_{2020}\} = \{1, 2, \dots, 2020\}$$

$$\sum_{n=1}^{2020} n^2 + 2 \times 2020 + 1 = 1^2 + \dots + 2018^2 + 2019^2 + 2021^2 =$$ ثاني أصغر مجموع

تناقض مع الطرف الأيمن من المتباينة

$\Leftarrow$ الأعداد يجب أن تكون من 1 إلى 2020

$$a_{n+2}^2 \ge a_{n+1}^2 + a_n^2 - n^2 - 2 \qquad n \in \{1, \dots, 2020\}$$

$$a_{n+2}^2 \ge \underbrace{a_{n+1}^2 - 2} \quad \leftarrow \quad \underline{a_n \ge n}$$ افرض

$$\Rightarrow \quad a_{n+2} \ge a_{n+1} + 1$$

$$\underline{a_{n+2} \ge n+2} \quad \leftarrow \quad \underline{a_{n+1} \ge n+1}$$ افرض

$\Leftarrow$ $\boxed{\text{لو كان } a_i \ge i, a_{i+1} \ge i+1 \Leftarrow a_{i+2} \ge i+2}$

$a_1 \geq 1$ لو كان $a_2 \geq 2$ $\Leftarrow$ $a_i \geq i$ لكل i

$\Leftarrow$ $a_1 = a_{2021} \geq 2021$ تناقض

• لو كان $a_2 = 1$:

$$a_3^2 \geq a_2^2 + a_1^2 - 1 - 2 = a_1^2 - 2$$

$$\Rightarrow a_3 \geq a_1 + 1 \geq 2 + 1 = 3$$

إذا كان $4 \leq a_4$:

$i \leq a_i$ لكل i $\Leftarrow$ $a_1 = a_{2021} \geq 2021$ تناقض

• لو كان $a_4 = 2$:

$$a_4^2 \geq a_3^2 + a_2^2 - 2^2 - 2$$

$$\Leftrightarrow 4 \geq a_3^2 + 1 - 4 - 2 = a_3^2 - 5$$

$$\Rightarrow 9 \geq a_3^2 \Rightarrow a_3 = 3$$

ولكن $3 = a_3 \geq a_1 + 1 \geq 3$ $\Leftarrow$ $a_1 = 2 = a_4$ تناقض

• لو كان $a_4 = 3$:

$$a_4^2 \geq a_3^2 + a_2^2 - 2^2 - 2$$

$$\Leftrightarrow 9 \geq a_3^2 + 1 - 4 - 2 = a_3^2 - 5$$

$$\Leftrightarrow 14 \geq a_3^2$$

ولكن $a_3 \geq 3$ $\Leftarrow$ $a_3 = 3 = a_4$ تناقض

بالتالي يوجد n بحيث $a_n^2 + a_{n+1}^2 \geq a_{n+2}^2 + n^2 + 3$