Hw 1) $\sum_{i=1}^{n} \frac{1}{x_i} = n$

find min of

$S := \sum_{i=1}^{n} \frac{x_i^{\,i}}{i}$

Sol) $x_1 + \frac{1}{x_1} \geq 2$

$x_2^2 + \frac{1}{x_2} + \frac{1}{x_2} \geq 3$

$\frac{x_2^2}{2} + \frac{1}{x_2} \geq \frac{3}{2}$

$\frac{x_3^3}{3} + \frac{1}{x_3} \geq \frac{4}{3}$

Similarly,

$$\frac{x_k^k}{k} + \frac{1}{x_k} \geq \frac{k+1}{k}$$

$$\left(\because \; x_k^k + \frac{1}{x_k} + \cdots + \frac{1}{x_k} \geq k+1 \right)$$

$\Longrightarrow$ by summing over $1 \leq k \leq n$ we get

$$S + n \geq \sum_{k=1}^{n} \frac{k+1}{k} = n + H_n$$

$$\Longleftrightarrow \; S \geq H_n \quad (\text{"="} \; x_k = 1)$$

H/W 2

$a^2 \leq 1,\ a^2+b^2 \leq 5,\ a^2+b^2+c^2 \leq 14$

$a^2+b^2+c^2+d^2 \leq 30$

Prove: $a+b+c+d \leq 10$

Sol

$(x_1^2+x_2^2+x_3^2+x_4^2)$

$(a+b+c+d)^2 \leq (\varepsilon_1 a^2+\varepsilon_2 b^2+\varepsilon_3 c^2+\varepsilon_4 d^2)$

$\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}+\frac{1}{\varepsilon_4}\right)$

$(y_1^2+y_2^2+y_3^2+y_4^2)$

$(\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \varepsilon_4)$

$(a\varepsilon_1 = b\varepsilon_2 = c\varepsilon_3 = d\varepsilon_4) \rightarrow \left(\frac{x_i}{y_i} = \text{con}\right)$

$\varepsilon_1 = 2\varepsilon_2 = 3\varepsilon_3 = 4\varepsilon_4$

$(\varepsilon_1 = 12,\ \varepsilon_2 = 6,\ \varepsilon_3 = 4,\ \varepsilon_4 = 3)$

Now, by Cauchy:

$$(a+b+c+d)^2 \leq (12a^2+6b^2+4c^2+3d^2)\left(\frac{1}{12}+\frac{1}{6}+\frac{1}{4}+\frac{1}{3}\right)$$

$$= \frac{5}{6}\left(\underbrace{12a^2+6b^2+4c^2+3d^2}_{T}\right)$$

But, from the given:

$$T = 3(a^2+b^2+c^2+d^2)+(a^2+b^2+c^2)$$

$$+2(a^2+b^2)+6a^2$$

$$\leq 90+14+10+6 = 120$$

Thus, $(a+b+c+d)^2 \leq \frac{5}{6}\cdot 120 = 10^2$

$$\Rightarrow a+b+c+d \leq 10$$

1] Find all nonzero x, y, z with $x = \frac{z}{y} - \frac{2y}{z}$,

$y = \frac{x}{z} - \frac{2z}{x}$, $z = \frac{y}{x} - \frac{2x}{y}$

2] For $x \in \mathbb{R}$. Find the max of

$$D = \left| \sqrt{x^2+8x+17} - \sqrt{x^2+4x+8} \right|$$

3] Let $a, b > 0$ and $ab \geqslant 1$

Prove:

$$\left(a+2b+\frac{2}{a+1}\right)\left(2a+b+\frac{2}{b+1}\right) \geqslant 16$$

(HW2: 5d2)

- $2a \le a^2 + 1$
- $2b \le \frac{b^2}{2} + 2$
- $2c \le \frac{c^2}{3} + 3$
- $2d \le \frac{d^2}{4} + 4$
- $$\underbrace{a^2 + \frac{b^2}{2} + \frac{c^2}{3} + \frac{d^2}{4}}_{= \frac{a^2+b^2+c^2+d^2}{4} + \frac{a^2+b^2+c^2}{12} + \frac{a^2+b^2}{6} + \frac{a^2}{2}} \le \frac{30}{4} + \frac{14}{12} + \frac{5}{6} + \frac{1}{2}$$

$= 10$

$\Longrightarrow 2\sum a \le 20 \Leftrightarrow \sum a \le 10$

1] $x = \frac{z}{y} - \frac{2y}{z}$

$\Longrightarrow xyz = z^2 - 2y^2$

$= y^2 - 2x^2$

$= x^2 - 2z^2$

$x^2 = y^2 = z^2$

$\rightarrow xyz = -x^2 \rightarrow yz = -x$

$xy = -z$

$xz = -y$

$xyz = -1 = -x^2 \rightarrow (x, y, z) = (\pm 1, \pm 1, -1)$

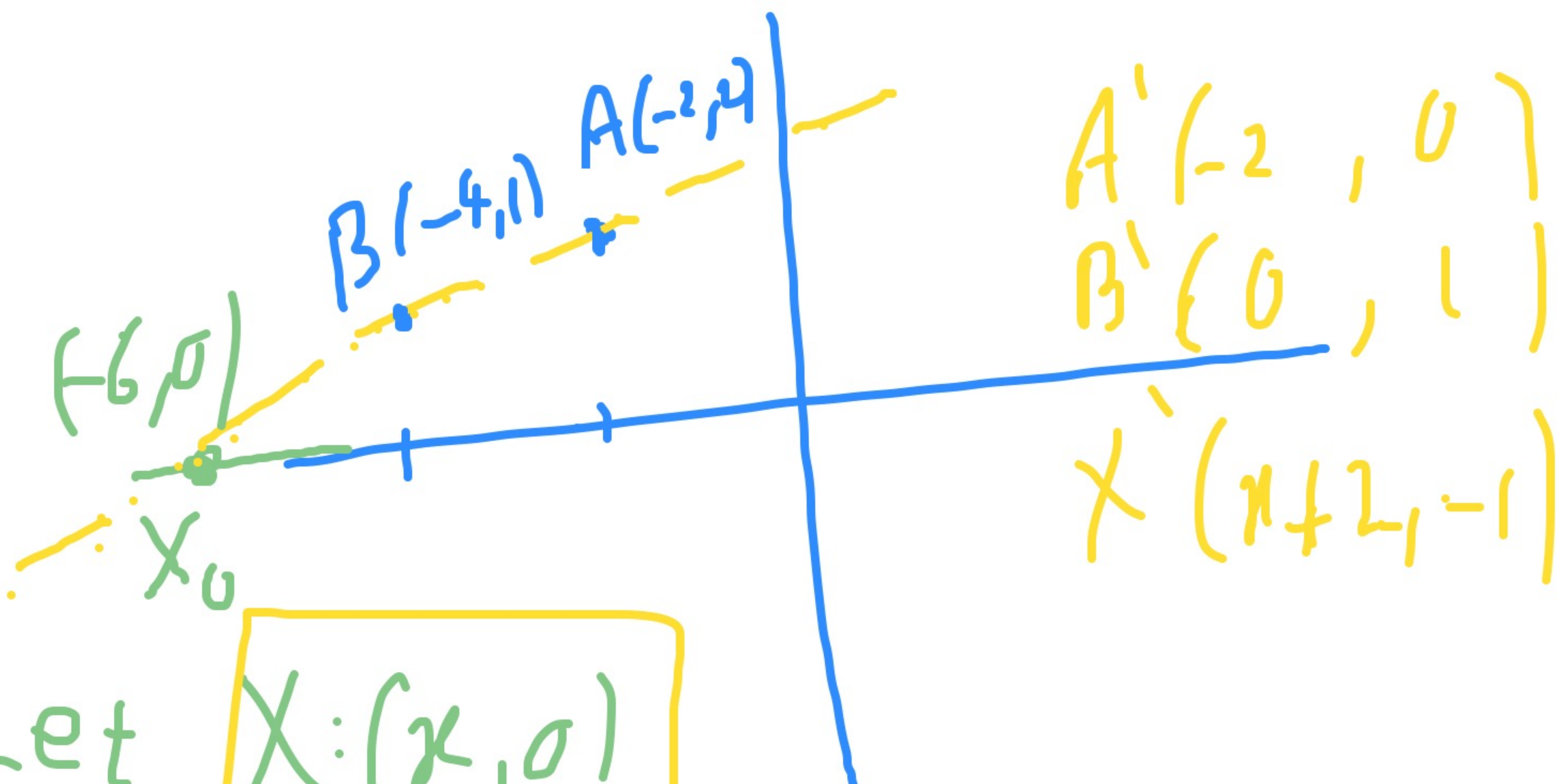
$, (\pm 1, \mp 1, 1)$

2]

$$\sqrt{x^2+8x+17}$$

$$=\sqrt{(x+4)^2+1^2} = XB$$

$$\sqrt{x^2+4x+8} = \sqrt{(x+2)^2+2^2} = XA$$



Let $\boxed{X:(x,0)}$

$$D = |XA - XB| \leq AB = \sqrt{5}$$

$\boxed{3}$

$(a+b \geq 2\sqrt{ab} \geq 2)$

$$a+2b+\frac{2}{a+1} \geq 2+b+\frac{2}{a+1}$$

$$= \frac{b+1}{2}+\frac{b+1}{2}+1+\frac{2}{a+1}$$

$$b+2a+\frac{2}{b+1} \geq \frac{a+1}{2}+\frac{2}{b+1}+1+\frac{a+1}{2}$$

$$\Rightarrow LHS \geq \left(\frac{\sqrt{(a+1)(b+1)}}{2}+3\right)^2$$

$(a+1)(b+1) = ab+a+b+1 \geq 4$

$$\geq (1+3)^2 = 16$$

$\blacksquare$

4) If $0 < a, b, c < 1$

show that

$a(1-b) \leq \frac{1}{4}$ or $b(1-c) \leq \frac{1}{4}$

or $c(1-a) \leq \frac{1}{4}$

Sol | $a(1-a) \leq \frac{1}{4}$

$b(1-b) \leq \frac{1}{4}$ (AM-GM)

$c(1-c) \leq \frac{1}{4}$

$\Longrightarrow \big(a(1-b)\big) \cdot \big(b(1-c)\big) \cdot \big(c(1-a)\big) \leq \frac{1}{4^3}$

the result immediately follows.

$\boxed{\text{Sol 2}}$ ~~(w.l.o.g $a \geq b \geq c$) $a \leq b \leq c$~~

w.l.o.g $a = \max(a, b, c)$

$\Rightarrow a \geq c$

$\Rightarrow c(1-a) \leq a(1-a) \leq \frac{1}{4}$.

$\boxed{\text{HW}}$ Let $a, b, c \in \mathbb{R}$ s.t

$$a+b+c \leq 4 \leq ab+bc+ca$$

Prove that at least two of the 3 inequalities below are true

$|a-b| \leq 2$, $|b-c| \leq 2$, $|c-a| \leq 2$