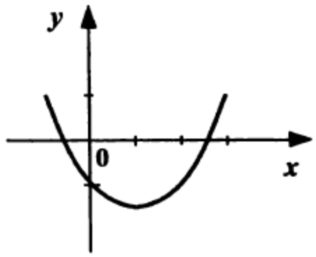
Email training, N1

September 11-16, 2022

Problem 1.1. Below is the graph of function $y = ax^2 + bx + c$. Define the signs of a, b and c.



Solution 1.1. It's obvious that $a > 0$. Since the x coordinate of parabol vertex is positive, therefore $\frac{-b}{2a} > 0$ and $b < 0$. Since $f(0) = c$ and $f(0) < 0$, so $c < 0$.

Answer: $a > 0, b < 0, c < 0$.

Problem 1.2. Solve the system of equations

$$\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \\ \frac{1}{x^3} + \frac{1}{y^3} = \frac{35}{216} \end{cases}$$

Solution 1.2. Denote $\frac{1}{x} = a, \frac{1}{y} = b$. Then

$$\begin{cases} a + b = \frac{5}{6} \\ a^3 + b^3 = \frac{35}{216}. \end{cases}$$

From $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$ follows that

$$ab = \frac{(a+b)^3 - (a^3+b^3)}{3(a+b)} = \frac{1}{6}.$$

So $a + b = \frac{5}{6}$ and $ab = \frac{1}{6}$. It means a and b are solutions of equation

$$z^2 - \frac{5z}{6} + \frac{1}{6} = 0.$$

One gets $a = \frac{1}{2}$, $b = \frac{1}{3}$ or $a = \frac{1}{3}$, $b = \frac{1}{2}$. From this one easily gets x and y.

Answer: $x = 2, y = 3$ and $x = 3, y = 2$.

Problem 1.3. Prove that $1^9 + 2^9 + 3^9 + \ldots + 16^9$ is divisible by 17.

Solution 1.3. It is known fact that $a^n + b^n$ is divisible by $a + b$ for odd n. Therefore

$$1^9 + 16^9 \vdots 17, 2^9 + 15^9 \vdots 17, \ldots, 8^9 + 9^9 \vdots 17.$$

So is their sum.

Problem 1.4. Find all three digit integers $\overline{abc}$ that satisfy to the equation

$$\overline{abc} = 11(a^2 + b^2 + c^2).$$

Solution 1.4. $\overline{abc} \vdots 11$, so either $b = a + c$, or $b = a + c - 11$.

In first case one gets

$$100a + 10(a + c) + c = 11(2a^2 + 2ac + 2^2) \Leftrightarrow 2a^2 + 2a(c - 5) + 2c^2 - c = 0,$$

from this follows that $c \vdots 2$ and $c \leq 5$. One easily gets $c = 0, a = 5, b = 5$.

In second case one gets

$$2a^2 + 2a(c - 16) + 2c^2 - c + 131 = 0,$$

so c is odd and since $D = c(14 - 3c) \geq 6$ follows that $c \leq 3$. One easily gets that $a = 8, b = 0, c = 3$.

Answer: $550, 803$.

Problem 1.5. There are 85 cubes in the shop, each of them painted in one color. Prove that either there exist 10 cubes all of them having the same color, either there exist 10 cubes all of them having different colors.

Solution 1.5. If there are cubes of 10 different colors then we may take one cube of each color and we are done. So assume that there are cubes of at most 9 colors. Then, by Pigeonhole principle there are $\frac{85}{9} = 9\frac{4}{9} > 9$ cubes of some color.

Problem 1.6. Is it possible to write 7 integers around the table with total sum equal to 1000, such that the difference of two neighbor numbers by absolute value is equal to

a) 1,
b) 2.

Solution 1.6. a) Within 2 neighbor numbers one is odd and one is even. Chose an odd number and calculate the parity of others O-E-O-E-O-E-O. Since the first and last numbers are also neighbors and both are odd, then their difference cannot be equal to 1.

b) Consider the residue mode 4. If we consider 2 neighbor numbers then either they give residues 0 and 2, either they give residues 1 and 3. In both cases we have a sequences $0 - 2 - 0 - 2 - 0 - 2 - 0$ or $1 - 3 - 1 - 3 - 1 - 3 - 1$, where the fist one and the last one are equal, so their difference is divisible by 4. But according to the condition of the problem their difference must be 2. We got contradiction.

Answer: a) not possible, b) not possible.