Level 2 E-training, week 2

Due to 23:59, Friday, 18 September 2020

Problem 1. Let $x, y, z > 0$. Show that

$$\frac{4x+1}{3y} + \frac{4y+1}{z} + \frac{3z+1}{2x} > 6$$

Can we replace 6 with any bigger constant?

Problem 2. On a board there are 6 nails, each two connected by a rope. Each rope is colored in one of 6 given distinct colors. Is it possible that, for each three distinct colors, there will be three nails connected with ropes of these three colors?

Problem 3. Let ABC be a triangle and M is the midpoint of BC. Let N be the midpoint of AM. Points D, E, lie on segments AB, AC, respectively. It is known that

$$\frac{AB}{AD} + \frac{AC}{AE} = 4$$

Show that D, N, E are collinear.

Problem 4. Find all $n \in \mathbb{N}$ such that $8n^6 - 4n^3 + 1$ is prime.

Problem 5. Let $n \in \mathbb{N}$. Show that

$$\tau(n) < 2\sqrt{n}$$

Problem 6. Let the circles k_1 and k_2 intersect at two points A and B, and let t be a common tangent of k_1 and k_2 that touches k_1 and k_2 at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate $\angle NMB$ in degrees.

Problem 7. Let $\mathcal{R}$ be the set of all right triangles of integer sidelengths. Let

$$\mathcal{A} = \{[ABC]\ |\Delta ABC \in \mathcal{R}\}$$

Find, with proof, the greatest common divisor of all elements of $\mathcal{A}$.