Lecture 2

Training Camp

15 November 2020

Level 2

Lifting The Exponent Lemma (LTE)

Theorem 3.3.1. *For p being an odd prime relatively prime to integers a and b with $p \mid a - b$ then*

$$v_p\left(a^n - b^n\right) = v_p\left(a - b\right) + v_p\left(n\right).$$

Corollary 3.3.1. *For p being an odd prime relatively prime to a and b with $p \mid a - b$ and n is a **odd** positive integer than*

$$v_p\left(a^n + b^n\right) = v_p\left(a + b\right) + v_p\left(n\right)$$

Theorem 3.3.2. *If $p = 2$ and n is even, and*

- $4 \mid x - y$ *then* $v_2(x^n - y^n) = v_2(x - y) + v_2(n)$
- $4 \mid x + y$ *then* $v_2(x^n - y^n) = v_2(x + y) + v_2(n)$

$v_2(x^n - y^n) =$

$v_2(x - y) + v_2(x+y) + v_2(n) - 1$

4. (ISL 1991) Find the largest k such that $1991^k \mid 1990^{1991^{1992}} + 1992^{1991^{1990}}$.

Hint: $1990^{1991^{1992}} = \left(1990^{1991^2}\right)^{1991^{1990}}$

$$S = \left(1990^{1991^2}\right)^{1991^{1990}} + (1992)^{1991^{1990}}$$

$$1990^{1991^2} + 1992 \equiv (-1)^{1991^2} + 1 \equiv 0 \pmod{1991}$$

$\boxed{1991 = 11 \times 181}$ and 11, 181 are primes

from LTE

$$v_{11}(S) = v_{11}\big(\underbrace{1990^{1991^2} + 1992}_{\geqslant 1}\big) + \underbrace{v_{11}\left(1991^{1990}\right)}_{= 1990}$$

$$v_{181}(S) = v_{181}\big(\underbrace{1990^{1991^2} + 1992}_{\geqslant 1}\big) + \underbrace{v_{181}\left(1991^{1990}\right)}_{= 1990}$$

$$k = \min\left(v_{11}(S), v_{181}(S)\right)$$

$$k \geqslant 1991$$

4. (ISL 1991) Find the largest k such that $1991^k \mid 1990^{1991^{1992}} + 1992^{1991^{1990}}$.

$$11 \mid 1990^{1991^2} + 1992$$

$$11^2 \overset{?}{\mid} 1990^{1991^2} + 1992$$

$$1990^{1991^2} + 1992 \equiv 54^{1991^2} + 56 \pmod{121}$$

$1991^2 \equiv 11 \pmod{\phi(121)}$

$$\equiv 54^{11} + 56 \pmod{121}$$

$$\not\equiv 0 \pmod{121}$$

$$v_{11}\left(1990^{1991^2} + 1992\right) = 1 \implies v_{11}(S) = 1991$$

$$\implies \boxed{k = 1991}$$

4. (ISL 1991) Find the largest k such that $1991^k \mid 1990^{1991^{1992}} + 1992^{1991^{1990}}$.

$$11^2 \nmid 1990^{1991^2} + 1992$$

طريقة أخرى لاثبات.

LTE:

$$11 \mid 1990 + 1992$$

$$v_{11}\left(1990^{1991^2} + 1992^{1991^2}\right) = v_{11}(1990+1992) + v_{11}(1991^2)$$
$$= 3$$

$$\Rightarrow 11^2 \mid 1990^{1991^2} + 1992^{1991^2} \quad (1)$$

$$11^2 \mid 1990^{1991^2} + 1992 \iff 11^2 \mid 1992^{1991^2} - 1992$$
$$\iff 11^2 \mid 1992\left(1992^{1991^2-1} - 1\right)$$
$$\iff 11^2 \mid 1992^{1991^2-1} - 1^{1991^2-1}$$

LTE:

$$11 \mid 1992 - 1$$

$$v_{11}\left(1992^{1991^2-1} - 1\right) = v_{11}(1992-1) + v_{11}(1991^2 - 1)$$
$$= 1 + 0 = 1$$

$$11^2 \nmid 1990^{1991^2} + 1992$$

اذن!

$$\Rightarrow v_{11}(S) = 1991 \Rightarrow \boxed{k = 1991}$$

5. (AIME 2018) Find the smallest n such that 3^n ends with 01 when written in base 143.

3^n ends with 01 base 143, $3^n = \overline{a_k a_{k-1} \cdots a_1 a_0}_{(143)}$, $a_1 = 0$, $a_0 = 1$

$$\Rightarrow \quad 3^n = 143^k a_k + 143^{k-1} a_{k-1} + \cdots + 143 a_1 + a_0$$

$a_0 = 1, a_1 = 0 \quad \Leftrightarrow \quad 3^n \equiv 143 a_1 + a_0 \pmod{143^2}$

$$\Leftrightarrow \quad 3^n \equiv 0 + 1 = 1 \pmod{143^2}$$

$3^n \equiv 1 \pmod{11}$, $3^n \equiv 1 \pmod{13}$ 1

$\mathrm{ord}_{11} 3 = 5$, $\mathrm{ord}_{13} 3 = 3$

$\Rightarrow \quad 5 \mid n, \; 3 \mid n \Rightarrow 15 \mid n \Rightarrow n = 15 n_1$

$$v_{11}(3^n - 1) = v_{11}(3^{15n_1} - 1) = v_{11}((3^5)^{3n_1} - 1)$$ 2

$$= v_{11}(3^5 - 1) + v_{11}(3n_1) = 2 + v_{11}(3n_1) \geq 2$$

5. (AIME 2018) Find the smallest n such that 3^n ends with 01 when written in base 143.

$$\nu_{13}(3^n-1) = \nu_{13}(3^{15n_1}-1) = \nu_{13}\left((3^3)^{5n_1}-1\right)$$

$$= \nu_{13}(3^3-1) + \nu_{13}(5n_1)$$

$$= 1 + \nu_{13}(n_1)$$

لما ان $\nu_{13}(3^n-1) \geq 2$ فان $\nu_{13}(n_1) \geq 1$

$13 \mid n_1$

$\nu_{13}(3^n-1) \geq 2 \iff 13 \mid n_1$ (3)

(1), (2), (3)

$143^2 \mid 3^n-1 \iff 3\cdot 5\cdot 13 \mid n \iff 195 \mid n$

اصغر n ممكنة هي $\boxed{195}$

Example 3.3.1 (AoPS). *Let $p > 2013$ be a prime. Also, let a and b be positive integers such that $p|(a+b)$ but $p^2 \nmid (a+b)$. If $p^2|(a^{2013}+b^{2013})$ then find the number of positive integer $n \leq 2013$ such that $p^n|(a^{2013}+b^{2013})$*

$p \mid a+b, \; p^2 \nmid a+b \implies v_p(a+b) = 1$

$p^2 \mid a^{2013}+b^{2013} \implies v_p(a^{2013}+b^{2013}) \geq 2$

Case 1: $\gcd(a,p) = 1$

$\implies \gcd(b,p) = 1$ and by LTE

$$v_p(a^{2013}+b^{2013}) = v_p(a+b) + \underbrace{v_p(2013)}_{\text{but } p>2013}$$
$$= 1 + 0$$

$v_p(a^{2013}+b^{2013}) = 1 \longrightarrow \longleftarrow$

Case 2: $p \mid a$

$\implies p \mid b \implies p^{2013} \mid a^{2013}, \; p^{2013} \mid b^{2013}$

$\implies p^{2013} \mid a^{2013}+b^{2013}$ إذن كل $2013 \geq n$ تحقق أن $p^n \mid a^{2013}+b^{2013}$

إذن يوجد 2013 قيمة ممكنة

Example 3.3.2 (AMM). *Let a, b, c be positive integers such that $c \mid a^c - b^c$. Prove that $c \mid \frac{a^c-b^c}{a-b}$.*

Assume that p is a prime divisor of c and

$x = \nu_p(c)$ $\quad (p^x \parallel c)$

$c \mid \dfrac{a^c-b^c}{a-b} \iff p^{\nu_p(c)} \mid \dfrac{a^c-b^c}{a-b}$ for all prime divisors p

$c \mid a^c - b^c \Rightarrow p^x \mid a^c - b^c$

Case 0: $p \mid a-b$, $p \mid a$, $p \mid b$

$a = px$, $b = py \Rightarrow \cdots$

$$\frac{a^c-b^c}{a-b} = \frac{p^c x^c - p^c y^c}{p(x-y)} = p^{c-1}\left(\frac{x^c-y^c}{x-y}\right)$$

$p^{c-1} \geq c \geq p^x \Rightarrow p^x \mid \dfrac{a^c-b^c}{a-b}$

Case 1: $p \nmid a-b$

$\Rightarrow p^x \mid \dfrac{a^c-b^c}{a-b} \in \mathbb{Z}$

Example 3.3.2 (AMM). *Let a, b, c be positive integers such that $c \mid a^c - b^c$. Prove that $c \mid \frac{a^c-b^c}{a-b}$.*

<u>Case 2</u> : $p \mid a - b$

بما أن $p \mid a-b$ فيمكننا استعمال LTE :

<u>if $p \neq 2$:</u>

$$v_p(a^c - b^c) = v_p(a-b) + v_p(c)$$

$$v_p(a^c - b^c) = v_p(a-b) + x$$

$$v_p\left(\frac{a^c - b^c}{a-b}\right) = x \implies p^x \mid \frac{a^c - b^c}{a-b}$$

<u>if $p=2$:</u>

c is even

$$v_2(a^c - b^c) = \boxed{v_2(a-b)} + \underbrace{v_2(a+b)}_{\geq 1} + \boxed{v_2(c)} - 1$$

$$\geq v_2(a-b) + v_2(c)$$

$$\implies v_2\left(\frac{a^c - b^c}{a-b}\right) \geq v_2(c) = x \implies 2^x \mid \frac{a^c - b^c}{a-b}$$

إذن من case 1 , case 2 نستنتج أن

$$c \mid \frac{a^c - b^c}{a-}$$

Example 3.3.3 (IMO 1999). *Find all pairs of positive integers* (x, p) *such that* p *is prime,* $x \leq 2p$*, and* $x^{p-1} \mid (p-1)^x + 1$.