Saudi Arabia 2022 – Math Camp

Level 4

Geometry – Projective Geometry

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Poles and Polars

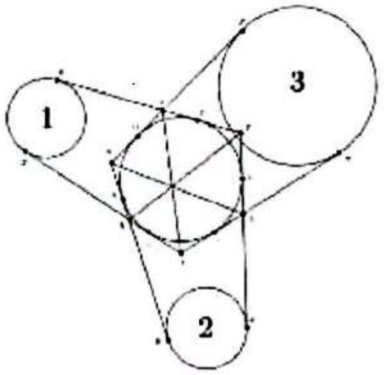
Given a circle ω with center O and radius r and any point $A \neq O$. Let A' on ray OA such that $OA \cdot OA' = r^2$. The line a through A' perpendicular to OA is called *polar of A with respect ω* and A is called the *pole of a with respect to ω.*

Theorem 1 (La Hire's Theorem): If a point A lies on the polar b of point B with respect to ω, then B lies on the polar a of A with respect to ω.

Theorem 2: Consider a circle ω and a point P outside the circle ω. The points C and D are on ω such that PC and PD are the tangents from P to ω. Then CD is the polar of P with respect to ω.

Theorem 3 (Brokard's Theorem): The quadrilateral $ABCD$ is inscribed in the circle k with center O. Let $E = AB \cap CD$, $F = AD \cap BC$ and $G = AC \cap BD$. Then O is the orthocenter of the triangle EFG.

This triangle EFG is self-polar with respect k. In other words, $\overleftrightarrow{EF} = g$, $\overleftrightarrow{EG} = f$ and $\overleftrightarrow{FG} = e$ are the polars of G, F and E, respectively.



Example

Brianchon's Theorem: Let $ABCDEF$ be a hexagon with an inscribed circle ω. Then lines AD, BE and CF concur.

Problems

9. (AIME II/2018) The incircle ω of triangle ABC is tangent to BC at X. Let $Y \neq X$ be the other intersection of AX with ω. Points P and Q lie on AB and AC, respectively, so that PQ is tangent to ω at Y. Assume that $AP = 3$, $PB = 4$, $AC = 8$, and $AQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. (ELMO Shortlist/2012) ABC is a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD = \angle DPC$.

11. (Romania TST/2008) Let $ABCD$ be a convex quadrilateral and let $O \in AC \cap BD$, $P \in AB \cap CD$, $Q \in BC \cap DA$. If R is the projection of O on theline PQ prove that the orthogonal projections of R on the sidelines of $ABCD$ are concyclic.

12. (IMO Shortlist/2004) In a cyclic quadrilateral $ABCD$, let E be the intersection of AD and BC (so that C is between B and E), and F be the intersection of AC and BD. Let M be the midpoint of side CD, and $N \neq M$ be a point on the circumcircle of ΔABM such that $B/MA = NB/NA$. Show that E, F, N are collinear.