Level 2 E-training, week 7

Due to 23:59, Friday, 23 October 2020

Problem 1. Prove that the longest median of any triangle bisects the shortest side.

Problem 2. Let p be an odd prime number and x, a, b be integers coprime with p such that both $x-1$ and $a-b$ are nonzero multiples of p. Prove that

(a) $p || \frac{x^p-1}{x-1}$

(b) $p || \frac{a^p-b^p}{a-b}$

$Note: p^k || m \ \ means \ p^k | m \ \ and \ \ p^{k+1} \not| m$

Problem 3. Find all reals x satisfying the equation

$$\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = 28$$

Problem 4. Each grid point of a cartesian plane is colored with one of three colors, whereby all three colors are used. Show that one can always find a right-angled triangle, whose three vertices have the same color.

Problem 5. For any circle γ and any point Y on the plane, we define $\mathcal{P}_\gamma(Y)$ as the power of the point Y with respect to γ. Now let Ω, Γ be two circles on the plane meeting at exactly two different points A, B and let ω be another circle on the plane passing through A and B. Show that $\frac{\mathcal{P}_\Omega(X)}{\mathcal{P}_\Gamma(X)}$ is constant over $X \in \omega$ (Naturally, we exclude $X \equiv A$ and $X \equiv B$).

Problem 6. Let $P = \{p^{2^k} | \, p \in \mathbb{P}, k \in \mathbb{N}_0\} = \{p_1 < p_2 < p_3 < \ldots\}$. Show that for any $n \in \mathbb{N}$ we have $\tau(p_1 p_2 \cdots p_n) = 2^n$.

Problem 7. We say that a finite set $\mathcal{S}$ of points in the plane is *balanced* if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$, and we say that C is an *equalizer* of A, B. Suppose that $\mathcal{S}$ is a balanced set such that for any 3 different points $X, Y, Z \in \mathcal{S}$ the circumcenter of XYZ is not a point of $\mathcal{S}$. Show that every two points of $\mathcal{S}$ have a unique equalizer.

Problem 8. Let $x, y > 0$. Prove that

$$1 + (x+y)^3 > 6xy\sqrt{x+y}$$