Number Theory – group L4

Instructor: Dušan Djukić *Date: 26.11.2021.*

1. Given n positive integers, denote by d_k the greatest common divisor of all product of k of these integers. Prove that $d_k^2 \mid d_{k-1}d_{k+1}$ for $2 \leqslant k \leqslant n-1$.
2. Find all triples of positive integers (x, y, z) such that each of the numbers x^2-1, y^2-2, z^2-4 is divisible by $x+y+z$.
3. If $n \in \mathbb{N}$, prove that $\sum_{i=1}^{n}[\frac{n}{i}]^2 = \sum_{i=1}^{n}(2i-1)[\frac{n}{i}]$.
4. Let n and a be given positive integers. Suppose that for every $m \in \mathbb{N}$ there is an integer x such that $x^n \equiv a \pmod{m}$. Prove that $a = y^n$ for some integer y.
5. Prove that there exist infinitely many positive integers k for which the equation $\frac{x}{\tau(x)} = k$ has no solutions in $\mathbb{N}$.
6. We are given two integers a and b of different parities. Prove that there is an integer c such that $c+a$, $c+b$ and $c+ab$ are all perfect squares.
7. Find all pairs of positive integers a and b such that $\operatorname{lcm}(a+1, b+1) = a^2 - b^2$.
8. Find all triples of positive integers a, b, c such that $a^2+b^2 = c^2$ and $a^3+b^3 = (c-1)^3-1$. (HW)
9. Prove that for every $n \in \mathbb{N}$ there exist n pairwise disjoint integers whose sum of squares equals their sum of cubes. (HW)
10. Let a_i, b_i $(1 \leqslant i \leqslant k)$ be real numbers. Define $x_n = [a_1 n + b_1] + \cdots + [a_k n + b_k]$. If $x_1, x_2, \ldots$ is an arithmetic sequence, prove that $a_1 + a_2 + \cdots + a_k$ is an integer. (HW)

Number Theory – group L4

Instructor: Dušan Djukić *Date: 27.11.2021.*

8. Find all triples of positive integers a, b, c such that $a^2+b^2 = c^2$ and $a^3+b^3 = (c-1)^3-1$.
9. Prove that for every $n \in \mathbb{N}$ there exist n distinct integers whose sum of squares equals their sum of cubes.
10. Let a_i, b_i $(1 \leqslant i \leqslant k)$ be real numbers. Define $x_n = [a_1 n + b_1] + \cdots + [a_k n + b_k]$. If $x_1, x_2, \ldots$ is an arithmetic sequence, prove that $a_1 + a_2 + \cdots + a_k$ is an integer.
11. Suppose that all divisors of n have been divided into pairs so that the sum in each pair is a prime. Prove that all these sums are distinct.
12. Find all positive integers x for which $3x^4 + 10x^2 + 3$ is a square.
13. Find all positive integers n for which $n^4 + 1$ has a divisor d satisfying $n^2 \leqslant d \leqslant n^2 + 3n + 7$.
14. Denote by $\omega(n)$ the number of distinct prime divisors of n. Given any three positive integers a, b, c, prove that there exists a positive integer n for which $\omega(an + c) \geqslant \omega(bn + c)$.
15. Find all $n \in \mathbb{N}$ for which the sum of digits of $n!$ is equal to 9. (HW)
16. Given a positive integer n, we write down the fraction $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, each in lowest terms. Define $f(n)$ to be the sum of the numerators of these fractions. Find all n for which $f(n)$ and $f(999n)$ have opposite parities. (HW)
17. Prove that there are infinitely many integers $n > 0$ for which $[\sqrt[3]{n^2}] + [\sqrt[3]{n}]$ divides n. (HW)

Number Theory – group L4

Instructor: Dušan Djukić *Date: 28.11.2021.*

15. Find all $n \in \mathbb{N}$ for which the sum of digits of $n!$ is equal to 9.

16. Given a positive integer n, we write down the fraction $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}$, each in lowest terms. Define $f(n)$ to be the sum of the numerators of these fractions. Find all n for which $f(n)$ and $f(999n)$ have opposite parities.

17. Prove that there are infinitely many integers $n > 0$ for which $[\sqrt[3]{n^2}] + [\sqrt[3]{n}]$ divides n.

18. Let $1 = d_1 < d_2 < d_3 < \cdots < d_k = 4n$ be all divisors of $4n$, where $n \in \mathbb{N}$. Prove that there is an index i for which $d_{i+1} - d_i = 2$.

19. Denote by $d_k(n)$ the number of divisors of n that are not less than k. Evaluate $d_1(2021) + d_2(2022) + \cdots + d_{2020}(4040)$.

20. If $n > 1$ is an integer, prove that $4^n + 2^n + 1$ cannot be divisible by $3^n - 2^n$.

21. Given a prime p, find all triples of positive integers a, b, c such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{p}$ and $a + b + c < 2p^{3/2}$.

22. Let p be a prime. Consider all pairs (a, b) of positive integers with $a < b < \frac{p}{2}$ for which $p \mid a^2 + b^2$. Prove that the sum $\sum a$ over all such pairs (a, b) equals $\frac{p^2-1}{24}$. (HW)

23. Find all pairs of positive integers (a, b) for which a is odd, b is a power of 2, and $a^2 - ab + b^2$ is a perfect square. (HW)

24. Is there a positive integer n such that both $n - 2015$ and $\frac{n}{2015}$ are positive integers having exactly 2015 divisors? (HW)

Number Theory – group L4

Instructor: Dušan Djukić *Date: 29.11.2021.*

22. Let p be a prime. Consider all pairs (a, b) of positive integers with $a < b < \frac{p}{2}$ for which $p \mid a^2 + b^2$. Prove that the sum $\sum a$ over all such pairs (a, b) equals $\frac{p^2-1}{24}$.
23. Find all pairs of positive integers (a, b) for which a is odd, b is a power of 2, and $a^2 - ab + b^2$ is a perfect square.
24. Is there a positive integer n such that both $n - 2015$ and $\frac{n}{2015}$ are positive integers having exactly 2015 divisors?
25. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant a(b-1) + b(c-1) + c(a-1) + 1$. Show that equality occurs for infinitely many triples (a, b, c).
26. Prove that there exist infinitely many positive integers that are less than the sum of their proper divisors (so, excluding 1 and itself), but cannot be written as a sum of several (distinct) divisors.
27. If a and b are positive integers such that $\mathrm{lcm}[a, b] + \mathrm{lcm}[a+2, b+2] = 2 \cdot \mathrm{lcm}[a+1, b+1]$, prove that $a \mid b$ or $b \mid a$. (HW)
28. We are given $n \geqslant 3$ consecutive odd three-digit numbers. Prove that these n numbers can be ordered in a sequence $b_1, b_2, \dots, b_n$ so that the number $\overline{b_1 b_2 \dots b_n}$, obtained by writing these numbers one after another in the decimal system, be composite. (HW)
29. Denote by $P(n)$ the largest prime divisor of $n \in \mathbb{N}$. Prove that there are infinitely many positive integers n such that $P(n) < P(n+1) < P(n+2)$. (HW)
30. Suppose that positive integers $a_1, a_2, \dots, a_n$ have the property that each of the quotients $k_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ for $i = 1, 2, \dots, n$ is an integer (here $a_0 = a_n$ and $a_{n+1} = a_1$). Prove that $2n \leqslant k_1 + k_2 + \dots + k_n \leqslant 3n$. (HW)
31. Find all prime numbers p such that $\frac{p^2-p-2}{2}$ is a perfect cube. (HW)

Number Theory – group L4

Instructor: Dušan Djukić *Date: 1.12.2021.*

27. If a and b are positive integers such that $\operatorname{lcm}[a,b]+\operatorname{lcm}[a+2,b+2]=2\cdot\operatorname{lcm}[a+1,b+1]$, prove that $a \mid b$ or $b \mid a$.

28. We are given $n \geqslant 3$ consecutive odd three-digit numbers. Prove that these n numbers can be ordered in a sequence $b_1, b_2, \dots, b_n$ so that the number $\overline{b_1 b_2 \dots b_n}$, obtained by writing these numbers one after another in the decimal system, be composite.

29. Denote by $P(n)$ the largest prime divisor of $n \in \mathbb{N}$. Prove that there are infinitely many positive integers n such that $P(n) < P(n+1) < P(n+2)$.

30. Suppose that positive integers $a_1, a_2, \dots, a_n$ have the property that each of the quotients $k_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ for $i = 1, 2, \dots, n$ is an integer (here $a_0 = a_n$ and $a_{n+1} = a_1$). Prove that $2n \leqslant k_1 + k_2 + \dots + k_n \leqslant 3n$.

31. Find all prime numbers p such that $\frac{p^2-p-2}{2}$ is a perfect cube.

32. Find all positive integers n with the following property: Whenever $n \mid xy+1$ for some integers x, y, it also holds that $n \mid x+y$.

33. Prove that for every positive integer n there exists a positive integer x such that $\varphi(x) = n!$.

34. Call an integer an *almost-square* if it is a product of two consecutive integers. Prove that every almost-square can be written as a quotient of two other almost-squares.

35. Find all pairs of positive integers a, b for which $(a^3+b)(b^3+a)$ is a power of 2.

36. Denote by b_n the number of binary unit digits of a positive integer n. We call n *lively* if $b_n \mid n$. Prove that (a) there are no 5 consecutive positive integers that are all lively; (b) there are infinitely many instances of 4 consecutive positive integers that are all lively.

Number Theory – group L4

Instructor: Dušan Djukić *Date: 3.12.2021.*

33. Prove that for every positive integer n there exists a positive integer x such that $\varphi(x) = n!$.

34. Call an integer an *almost-square* if it is a product of two consecutive integers. Prove that every almost-square can be written as a quotient of two other almost-squares.

35. Find all pairs of positive integers a, b for which $(a^3 + b)(b^3 + a)$ is a power of 2.

36. Denote by b_n the number of binary unit digits of a positive integer n. We call n *lively* if $b_n \mid n$. Prove that (a) there are no 5 consecutive positive integers that are all lively; (b) there are infinitely many instances of 4 consecutive positive integers that are all lively.

37. Find all pairs of positive integers x, y such that $[\sqrt{x}] = [\sqrt{y}]$, $x \mid y^4 + 1$ and $y \mid x^4 + 1$.

38. Find all positive integers n for which $2^n - 1$ has exactly n divisors.

39. There are $n > 2$ integers on the board with the GCD equal to 1. In each step we are allowed to increase or decrease one of the numbers by a multiple of another number. Find the smallest k for which it is always possible to obtain number 1 by a sequence of k such steps.

40. Find all values of n for which there are positive integers a, b, c, d for which $a+b+c+d = n$ and $abc + abd + acd + bcd$ is divisible by n.

41. Denote by $S(x)$ the sum of decimal digits of a positive integer x. Prove that there exist 50 distinct positive integers $n_1, n_2, \dots, n_{50}$ such that $n_1 + S(n_1) = n_2 + S(n_2) = \dots = n_{50} + S(n_{50})$.

42. We perform a sequence of operations of the following types: if the number is even, we divide it by 2, and if it is odd, we multiply it by some power of 3 (we choose one) and add 1. Prove that, starting from any number, we can reach number 1 in finitely many such operations.

Number Theory – group L4

Instructor: Dušan Djukić *Date: 4.12.2021.*

37. Find all pairs of positive integers x, y such that $[\sqrt{x}] = [\sqrt{y}]$, $x \mid y^4 + 1$ and $y \mid x^4 + 1$.

38. Find all positive integers n for which $2^n - 1$ has exactly n divisors.

39. There are $n > 2$ integers on the board with the GCD equal to 1. In each step we are allowed to increase or decrease one of the numbers by a multiple of another number. Find the smallest k for which it is always possible to obtain number 1 by a sequence of k such steps.

40. Find all values of n for which there are positive integers a, b, c, d for which $a+b+c+d = n$ and $abc + abd + acd + bcd$ is divisible by n.

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42. We perform a sequence of operations of the following types: if the number is even, we divide it by 2, and if it is odd, we multiply it by some power of 3 (we choose one) and add 1. Prove that, starting from any number, we can reach number 1 in finitely many such operations.

43. Does there exist a surjective function $f : \mathbb{Q}^+ \to \mathbb{Q}$ such that $f(xy) = f(x) + f(y)$ for all rational $x, y > 0$?

44. A sequence of positive integers $a_1, a_2, \dots$ is such that $n \leqslant a_n \leqslant n + 2021$ for all n and $(a_m, a_n) = 1$ whenever $(m, n) = 1$. If a prime p divides a_n, prove that also $p \mid n$.

45. Positive integers x and $y < x$ are such that $x^2 + y^2 - 2$ is divisible by $x^2 - y^2$. Prove that $x^2 + y^2 - 2$ and $x^2 - y^2$ have the same sets of prime divisors.

Solutions – group L4

Instructor: Dušan Djukić *Nov.26–Dec.4, 2021*

1. We will prove that $v_p(d_{k-1}d_{k+1}) \geqslant v_p(d_k^2)$ for every prime p. Let the exponents at p in the given numbers be $r_1 \leqslant r_2 \leqslant \dots \leqslant r_n$. Then $v_p(d_i) = r_1 + \dots + r_i$, so $v_p(d_{k-1}d_{k+1}) = 2(r_1 + \dots + r_{k-1}) + r_k + r_{k+1}$ and $v_p(d_k^2) = 2(r_1 + \dots + r_k) \leqslant v_p(d_{k-1}d_{k+1})$.

2. Denote $s = a + b + c$. Modulo s we have $x^2 \equiv 1$, $y^2 \equiv 2$ and $(x+y)^2 \equiv z^2 \equiv 4$. It follows that $2xy = (x+y)^2 - x^2 - y^2 \equiv 1 \pmod{s}$, so $4 \cdot 1 \cdot 2 \equiv 4x^2y^2 = (2xy)^2 \equiv 1 \pmod{s}$. Therefore $s \mid 7$, and for $s = 7$ we have a unique solution $(x, y, z) = (1, 4, 2)$.

3. Use induction on n (base $n = 1$). For the inductive step, when $n - 1$ increases to n, only the summands corresponding to $i \mid n$ change, as then $[\frac{n}{i}] = [\frac{n-1}{i}] + 1$. Verify that both sides of the equality get exactly the same increment.

4. Take $m = a^2$. There is $x \in \mathbb{Z}$ such that $a^2 \mid x^n - a$. So if p is any prime divisor of a and $v_p(a) = k$, then $v_p(x^n - a) \geqslant 2k$, which implies $v_p(x^n) = k$. Thus $n \mid k$, and since this holds for all p, number a must be an n-th power.

5. We will prove that if $k = p^{p-1}$, where $p \geqslant 5$ is a prime, the given equation has no solutions. Suppose that $x = p^{p-1}\tau(x)$ for some $x \in \mathbb{N}$. Clearly, $r = v_p(x) \geqslant p - 1$. Moreover, $p^{p-1} = \frac{x}{\tau(x)} = \frac{p^r}{\tau(p^r)} \cdot \frac{x/p^r}{\tau(x/p^r)}$, which implies $\frac{x/p^r}{\tau(x/p^r)} = \frac{r+1}{p^{r+1-p}}$. If $r = p - 1$, then $\frac{x/p^r}{\tau(x/p^r)} = p$, which is impossible because $p \nmid x/p^r$. On the other hand, if $r \geqslant p+1$, then $\frac{x/p^r}{\tau(x/p^r)} \leqslant \frac{p+2}{p^2} < 1$, which is also impossible. Finally, we have a contradiction for $r = p$ as well: $\frac{x/p^r}{\tau(x/p^r)} = \frac{p+1}{p}$ - indeed, note that $\tau(y) \leqslant \frac{y}{2} + 1 < \frac{p}{p+1}y$ for $y \geqslant p + 1$.

6. We will set c so that $c + a = k^2$ and $c + b = (k+1)^2$ are two consecutive squares. Then $b - a = 2k + 1$, so the corresponding c will be $c = k^2 - a = \frac{a^2+b^2+1-2ab-2a-2b}{4}$. Luckily, then $c + ab = (\frac{a+b-1}{2})^2$.

7. Let $a + 1 = dx$ and $b + 1 = dy$, where d, x, y are positive integers with $x > y$ and $\gcd(x, y) = 1$. Then $\operatorname{lcm}(a+1, b+1) = dxy = (dx-1)^2 - (dy-1)^2 = d(x-y)(dx+dy-2)$ and hence $dx + dy - 2 = \frac{xy}{x-y}$. Since $x - y$ is coprime to both x and y, we must have $x - y = 1$. The previous equality becomes $d(2x-1) - 2 = x(x-1)$, so $2x - 1 \mid x^2 - x + 2 \mid (2x-1)^2 + 7$ and consequently $2x - 1 \mid 7$, i.e. $x = 1$ or $x = 4$. The first option fails and the second one yields $d = 2$ and $(a, b) = (7, 5)$.

8. Fix c and assume $a < b$. If $a+b = x$, then $a^3+b^3 = x(3c^2-x^2)$ is increasing in x, and x is itself increasing in b. Since $b \leqslant c-2$, we have $(c-1)^3 - 1 = a^3 + b^3 \leqslant \sqrt{4c-4}^3 + (c-2)^3$. Solving this inequality will give us $c \leqslant 10$. Now it is easy to manually check the small cases. The only solution is $(a, b, c) = (6, 8, 10)$.

9. For $n = 3$ we can find infinitely many examples (x, y, z) by setting $y = -z$: setting $\frac{y}{x} = k$ we easily find $(x, y, z) = (2k^2 + 1, k(2k^2 + 1), -k(2k^2 + 1))$.

   An example for $n = 3r$ can be constructed as a union of r such triples. For $n = 3r + 1$ or $3r + 2$ it suffices to append this $3r$-tuple by a 0 and/or 1.

10. Let $x_n = an + b$ for some constants $a, b \in \mathbb{Z}$. Denote $A = a_1 + \cdots + a_k$ and $B = b_1 + \cdots + b_k$. Summing the inequalities $a_i n + b_i - 1 \leqslant [a_i n + b_i] \leqslant a_i n + b_i$ over $1 \leqslant i \leqslant k$ yields $An + B - k \leqslant x_n \leqslant An + B$, i.e. $B - b - k \leqslant (a - A)n \leqslant B - b$ for all n, and this is only possible if $A = a$.

11. Hint: prove that if (a, b) is one such pair, then ab cannot exceed n, Deduce that in fact $ab = n$ in all such pairs. Now all pars look like $(a, \frac{n}{a})$, but these sums are distinct for all pairs.

12. We have $3x^4 + 10x^2 + 3 = (3x^2 + 1)(x^2 + 3)$. The GCD of $3x^2 + 1$ and $x^2 + 3$ divides $3(x^2 + 3) - (3x^2 + 1) = 8$, so it is 1, 2, 4 or 8. Therefore $3x^2 + 1$ and $x^2 + 3$ are either squares, or squares multiplied by 2. However, $x^2 + 3 = 2a^2$ is impossible modulo 3, so both factors must be squares. Then $x^2 + 3 = a^2$, which is only possible for $x = 1$. This is a solution indeed.

13. Denoting $d = n^2 + a$, we see that d also divides $a^2 + 1$, so let $a^2 + 1 = kd$. Then $a = \frac{k}{2} + \sqrt{kn^2 + \frac{k^2}{4} - 1}$; if $k \geqslant 13$, then automatically $a > 3n + 7$. On the other hand, for $k < 13$, $k \mid a^2 + 1$ is possible only for $k = 1, 2, 5, 10$. The discriminant $kn^2 + \frac{k^2}{4} - 1$ cannot be a square if $k = 1, 2, 5$ (modulo 2 or 3). Only $k = 10$ remains, and then $a \leqslant 3n + 7$ is equivalent to $n \leqslant 10$. And indeed, $n = 2$ and $n = 10$ are solutions, with the corresponding divisors $d = 17$ and $d = 137$ (note that $10001 = 73 \cdot 137$).

14. Suppose, to the contrary, that $\omega(an + c) < \omega(bn + c)$ for every n. Then for any $k \in \mathbb{N}$, substituting the numbers $a^{k-1}, a^{k-2}b, \ldots, b^{k-1}$ for n, we have $\omega(a^k + c) < \omega(a^{k-1}n + c) < \omega(a^{k-2}b^2 + c) < \cdots < \omega(b^k + c)$, and hence $\omega(b^k + c) \geqslant k + 1$. However, the product of first $k + 1$ primes is bigger than $k!$, so $k! \leqslant b^k + 1 < (b + 1)^k$, which is false for k big enough.

15. Hint: Can a number whose sum of digits is 9 be divisible by 11?

16. Let $n = 2^k m$ with m odd. The numerator in $\frac{a}{n}$ in lowest terms is even if and only if $2^{k+1} \mid a$, and there are $[\frac{n}{2^{k+1}}] = \frac{m-1}{2}$ such values of a. In the remaining $n - \frac{m-1}{2}$ fractions the numerator is odd, so $f(n) \equiv n - \frac{m-1}{2} \pmod 2$. Then $f(999n) \equiv 999n - \frac{999m-1}{2} \pmod 2$, so $f(999n) - f(n) \equiv 499m \equiv 1 \pmod 2$. Hence $f(n)$ and $F(999n)$ have opposite parities for every n.

17. Denote $f(n) = [\sqrt[3]{n^2}] + [\sqrt[3]{n}]$. We will prove that for every positive integer k there exists $n \in \mathbb{N}$ such that $n = k \cdot f(n)$. Fix k and consider $g(n) = n - kf(n)$. Increase n one by one. In each step, $f(n)$ increases by 0, 1 or 2, so $g(n)$ either increases by 1 or decreases. However, $g(1) < 0$ and $g(n)$ grows to infinity as $n \to \infty$, so it cannot skip zero.

18. Suppose the statement is false. Consider the largest pair of even divisors $(a, a+2)$ of $4n$ (there is one such pair: $(2,4)$). Then $a+1$ must also be a divisor and it is odd, so $2a+2$ is a divisor. Moreover, either $2a$ or $2a+4$ is not divisible by 8 and hence is a divisor of $4n$. Thus we find a larger pair, namely $(2a, 2a+2)$ or $(2a+2, 2a+4)$.

19. The summand $d_i(2020+i)$ counts the divisors not less than i, so a possible divisor $k \leqslant 2020$ would be counted only through the summands $d_1(2021), \dots, d_k(2020+k)$. Since k divides exactly one of the numbers $2021, 2022, \dots, 2020+k$, it follows that it has been counted exactly once. Thus the divisors from 1 to 2020 have been counted 2020 times in total.

    Additionally, the divisors from 2021 to 4040 have been counted once each. This gives the sum of 4040.

20. If n is even, then $3^n - 2^n$ is divisible by 5, but $4^n + 2^n + 1$ is not. Now assume n is odd. The number $9^n + 3^n + 1 = 4^n + 2^n + 1 + (3^n - 2^n) + (9^n - 4^n)$ is also divisible by $3^n - 2^n$. However, $9^n + 3^n + 1 = (3^n - 1)^2 + (3^{\frac{n+1}{2}})^2$ is a sum of two squares, and $3^n - 2^n \equiv 3 \pmod 4$ for $n > 1$, so this is impossible. This leaves $n = 1$ as the only possibility.

21. Some of a, b, c are multiples of p. If it is only one, say $p \mid a = pu$, then $\frac{u-1}{pu} = \frac{1}{p} - \frac{1}{pu} = \frac{1}{b} + \frac{1}{c}$, so $p \mid u - 1$, which makes $a \geqslant p(p+1)$ too big.

    If exactly two are multiples of p, say $a = pu$, $b = pv$, then $\frac{u+v}{uv} = \frac{c-p}{c}$, so $p \mid uv - u - v$, but then $u + v \geqslant 2\sqrt{p}$ and $a + b$ is again too big.

    Therefore $a = pu$, $b = pv$, $c = pw$ are all multiples of p. Then $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 1$, which has the only solutions $(3,3,3)$, $(2,4,4)$, $(2,3,6)$.

22. If (a, b) is one of the pairs, then either $(b-a, b+a)$ or $(b-a, p-b-a)$ is also one of the pairs. In particular, $b - a$ is the smaller element in some pair as well.

    Moreover, if two pairs (a, b) and (c, d) give the same difference $d - c = b - a$, then also $c + d \equiv \pm(a+b)$, so the pairs are the same.

    It follows that the differences $b - a$ are precisely a permutation of elements a. Therefore the sum of a over all pairs is half the sum of b and equals $\frac{1}{3}(1 + 2 + \cdots + \frac{p-1}{2})$.

23. Let $b = 2^n$ and $a^2 - ab + b^2 = c^2$. We can rewrite this as $3 \cdot 2^{2n-2} = \frac{3}{4}b^2 = c^2 - (a - \frac{b}{2})^2 = (c + a - 2^{n-1})(c - a + 2^{n-1})$.

    We easily check the cases $n \leqslant 2$ and find no solutions. Assume that $n \geqslant 3$. Then the factors $c + a - 2^{n-1}$ and $c - a + 2^{n-1}$ are even and not both multiples of 4, so one of them equals ± 2 or ± 6. Assuming c is positive, both factors are positive as well. Checking all four possibilities we find only two possibilities for $n \geqslant 3$: $a = 2^{2n-4} + 2^{n-1} - 3$ or $a = 3 \cdot 2^{2n-4} + 2^{n-1} - 1$, and in addition, $a = 3$ for $n = 3$.

24. Both $n - 2015 = (2015a)^2$ and $\frac{n}{2015} = b^2$ are squares, so $(b+1)(b-1) = 2015a^2$. Since $2015 = 5 \cdot 13 \cdot 31$, none of these squares can have more than three prime divisors, so each in fat consists only of the primes $5, 13, 31$. So do $b+1$ and $b-1$ and their difference is 2, but modulo 13 this is easily checked to be impossible.

25. The given product of GCD's divides both abc and $(b-1)(c-1)(a-1)$, so it divides the difference, which is $a(b-1)+b(c-1)+c(a-1)+1$.

    We find an equality case by setting $(a,b,c)=(n,n=1,n+2)$, with $3\mid n-1$.

26. We can take e.g. $n=2^kp$, where p is a prime with $2^k<p<2^{k+1}-1$. The sum of its proper divisors is $n+2^{k+1}-p-2$. However, if some of them add up to n, then the remaining ones add up to $p-2$, and this is not possible, because all proper divisors less than p are even.

27. Case $a=b$ is trivial. Assume w.l.o.g. that $a>b$ and let $[a,b]=ka$, $[a+1,b+1]=\ell(a+1)$ and $[a+2,b+2]=m(a+2)$. Reducing the equality $ka+m(a+2)=2\ell(a+1)$ modulo $a+1$ we get $a+1\mid m-k$, but $k,m\leqslant b+2\leqslant a+1$, so we must have $k=m$. Then $\ell=k=m$, and since $k\mid b$ and $\ell\mid b+1$, we obtain $k=1$, that is, $b\mid a$.

28. If $n=5$, one of $b_1,\dots,b_n$ is divisible by 5, so put it on the last place. If $n=4$, order them as b_1,b_2,b_4,b_3 to get a number divisible by 11. And if $n=3$, they will always yield a multiple of 3.

29. Given an arbitrary odd prime p, we will try to find n of the form $n=p^k-1$. We want to find k for which $P(p^k-1)<P(p^k)=p<P(p^k+1)$.

    Suppose there is no such k. Since $p^2-1=(p-1)(p+1)$, we have $P(p^2-1)<p$, so we must have $P(p^2+1)<p$ as well. Now, since $p^4-1=(p^2-1)(p^2+1)$, we have $P(p^4-1)<p$, which forces $P(p^4+1)<p$ as well. Next, $P(p^8-1)<p$, forcing $P(p^8+1)<p$ as well, etc. Inductively, $P(f_i)<p$ for each $i\in\mathbb{N}_0$, where $f_i=p^{2^i}+1$. However, since $f_i=(p-1)f_0f_1\cdots f_{i-1}+2$, the numbers $P(f_i)$ are pairwise distinct, and this is a finishing contradiction.

30. The LHS is easy: $k_1+k_2+\dots+k_n=\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)+\left(\frac{a_2}{a_3}+\frac{a_3}{a_2}\right)+\dots+\left(\frac{a_n}{a_1}+\frac{a_1}{a_n}\right)\geqslant 2n$.

    For the right-hand side we induct on n. The base case $n=1$ is trivial, so assume $n>1$ and $a_{n-1}<a_n=\max\{a_1,a_2,\dots,a_n\}$. Then we must have $k_n=1$ and $a_n=a_{n-1}+a_1$. It follows that $\frac{a_{n-2}+a_1}{a_{n-1}}=k_{n-1}-1$ and $\frac{a_{n-1}+a_2}{a_1}=k_1-1$ are integers as well, so by the inductive hypothesis for $n-1$ numbers $a_1,a_2,\dots,a_{n-1}$ we have $(k_1-1)+k_2+\dots+k_{n-2}+(k_{n-1}-1)<3(n-1)$, which immediately implies $k_1+\dots+k_n<3n$.

31. One solution is $p=2$, but there is also an unexpected solution.

32. Denote by x^{-1} the multiplicative inverse of x modulo n. We are given that $n\mid x-x^{-1}$ (mod n), i.e. $n\mid x^2-1$ whenever $(x,n)=1$.

    If n has a prime factor $p\geqslant 5$, we can take x so that $x\equiv 2 \pmod p$ and $x\equiv 1$ modulo all other prime divisors of n. Then $(x,n)=1$, but $p\nmid x^2-1$, contradicting the condition. Hence $n=2^a3^b$. But if $n\nmid 24$, the condition fails for $x=5$. It follows that $n\mid 24$, and these n actually work because $24\mid x^2-1$ whenever $(x,6)=1$.

33. Take $x=n\prod_p\frac{p}{p-1}$, where the product is over the primes $p\leqslant n+1$. Prove that it works.

34. Note that $(x-1)x=\frac{(x^2-1)x^2}{x(x+1)}$.

35. If $a = b$, the only solution is $(1, 1)$. Also, if a and b are even and $v_2(a) < v_2(b)$, then $v_2(b^3 + a) = v_2(a)$, so $b^3 + a$ cannot be a power of 2.

    Assume a and b are odd and $a < b$. Then $a^3 + b < b^3 + a$ and both are powers of 2, so $a^3 + b \mid b^3 + a$. Hence $a^3 + b \mid (a^9 + b^3) - (b^3 + a) = a(a^8 - 1)$. However, by the LTE lemma, $v_2(a^3 + b) \leqslant v_2(a^8 - 1) = v_2(a^2 - 1) + 2$, so $a^3 + b$ also divides $4(a^2 - 1)$. It follows that $a^3 + b \leqslant 4(a^2 - 1)$, so $a \leqslant 3$, and the only solution is $(a, b) = (3, 5)$.

36. (a) The numbers $4k + 1$ and $4k + 2$ have the same number of binary unit digits, so they cannot both be lively. However, in every five consecutive numbers one can find $4k + 1$ and $4k + 2$.

    For (b), set n so that $b_n = 6$, $b_{n+1} = 7$, $b_{n+2} = 4$ and $b_{n+3} = 5$. We can take e.g. $n = 2^a + 2^b + 2^c + 14$ with $a > b > c > 4$. Then see how to make $n, n + 1, n + 2, n + 3$ all lively.

37. Let $x = n^2 + a$, $y = n^2 + b$ with $0 \leqslant a, b \leqslant 2n$. We have that $k = \frac{(x-y)^4+1}{xy} = \frac{(a-b)^4+1}{(n^2+a)(n^2+b)}$ is an integer, but clearly $k \leqslant 16$ and k divides a fourth power plus 1. Such k has no odd prime factors that are not 1 (mod 8), and also $4 \nmid k$, so we must have $k \in \{1, 2\}$.

    Finally, it is easy to see that $k = 2$ is impossible due to parity and $k = 1$ is impossible for $n > 1$ due to size. The only solution is $\{x, y\} = \{1, 2\}$.

38. Let $n = 2^k m$ with m odd. Then $n = \tau(2^n - 1) = \tau(2^m - 1)\tau(2^m + 1) \cdots \tau(2^{2^{k-1}m} + 1)$ is divisible by 2^{k+1} (each factor is even) unless some of the $k + 1$ factors is a square. But $2^x + 1$ is a square only for $x = 3$ and $2^x - 1$ is a square only for $x = 1$, so we must have $m = 1$ or 3.

    If $m = 1$, then $\tau(2^m + 1) = \cdots = \tau(2^{2^{k-1}m} + 1) = 2$, so all these factors are primes, but $2^{2^5} + 1$ is composite, so We must have $k \leqslant 5$,

    Similarly, if $m = 3$, we must have $k = 1$ for $2^6 + 1$ is composite. Hence the solutions are $n = 1, 2, 4, 6, 8, 16, 32$.

39. We start with a lemma:

    Lemma. If a, b, m are nonzero integers with $(a, b) = 1$, then there exists $k \in \mathbb{Z}$ such that $(a + kb, m) = 1$. $\square$

    We claim that n steps always suffice. If $(a_1, \ldots, a_{n-1}) = 1$, then for some integers x_i we have $x_1 a_1 + \cdots + x_{n-1} a_{n-1} = 1 - a_n$, so by adding the multiples $x_i a_i$ to a_n we obtain 1 in $n - 1$ steps. We proceed to the general case: $d = (a_{n-1}, a_n)$ and $e = (a_1, \ldots, a_{n-2})$. Clearly, $(d, e) = (\frac{a_{n-1}}{d}, \frac{a_n}{d}) = 1$, so by the Lemma there exists k such that $\frac{a_{n-1}+ka_n}{d}$ is coprime to e. Then we also have $(a_1, \ldots, a_{n-2}, a_{n-1} + ka_n) = 1$. As before, we need further $n - 1$ steps to replace the number a_n by 1.

    Let us prove that $n - 1$ may not be enough. Suppose $p_1, \ldots, p_n > 2$ are different primes. By the Chinese remainder theorem there exist integers $a_1, \ldots, a_n$ such that $a_i \equiv 0 \pmod{p_j}$ for $j \neq i$ and $a_i \equiv 2 \pmod{p_i}$. Suppose that we have applied $n - 1$ steps. Then there exists i such that no multiple of a_i was ever added. Thus the given numbers did not change modulo p_i, so none of them could become 1.

40. We have that $abc+(ab+ac+bc)(n-a-b-c) \equiv -(a+b)(b+c)(c+a)$ is divisible by n, so n cannot be prime. On the other hand, if $n = xy$ $(x, y > 1)$ is composite, then one can take $(a, b, c, d) = (1, x-1, y-1, (x-1)(y-1))$.

41. For an arbitrary $k \in \mathbb{N}$, consider the numbers

$$a_k = 1\underbrace{00\ldots0}_{10^k+k+1} \quad \text{and} \quad b_k = \underbrace{99\ldots9}_{10^k}1\underbrace{00\ldots0}_{k}.$$

Then $a_k + S(a_k) = b_k + S(b_k) = 10^{10^k+k+1} + 1$. Define the sequence $k_0 = 0$ and $k_{i+1} = 10^{k_i} + k_i + 2$. Now consider any of the numbers $n = x_0 + x_1 + \cdots + x_5$, where $x_i \in \{a_{k_i}, b_{k_i}\}$. The digits of $x_0, \ldots, x_5$ do not interfere mutually, so $S(n) = S(x_0)+\cdots+S(x_5)$. Hence $n+S(n) = (10^{10^{k_0}+k_0+1}+1)+\cdots+(10^{10^{k_5}+k_5+1}+1)$, which iS the same for each of the 64 possible choices of number n.

42. Don't use the Collatz conjecture.

43. There is a bijection between the primes and the rationals. Thus we can define f on the set P of primes so that $f(P)$ is the entire $\mathbb{Q}$. Finally, $f(P)$ determines f on all of $\mathbb{Q}$ as $f(\prod p_i^{n_i}) = \sum n_i f(p_i)$.