**Problem 3.** Determine all the pairs $(p, n)$ of a prime number $p$ and a positive integer $n$ for which $\frac{n^p+1}{p^n+1}$ is an integer.

**N5.** Find all triples $(p, x, y)$ consisting of a prime number $p$ and two positive integers $x$ and $y$ such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of $p$.

*(Belgium)*

**Example 3.4** (Math Prize Olympiad 2010)

Prove that for every positive integer $n$, there exists integers $a$ and $b$ such that $4a^2 + 9b^2 - 1$ is divisible by $n$.

**Problem 4.1** (IMO 1989). Prove that for every positive integer $n$, there exists $n$ consecutive positive integers such that none of them is a power of a prime.

**Problem 4.2.** Let $n$ be a positive integer. Determine, in terms of $n$, the number of $x \in \{1, 2, \ldots, n\}$ for which $x^2 \equiv x \pmod{n}$.

Let $P \in \mathbb{Z}[x]$ and for any integer $n$ either $P(n) = 5$ or $P(n) = 2022$. Prove that either all $P(n) = 5$ or $P(n) = 2022$.

$a - b \mid P(a) - P(b) \Rightarrow$

$P(x) \equiv P(y) \pmod{n}$ if $x \equiv y \pmod{n}$

(AoPS). *Show that for $c \in \mathbb{Z}$ and a prime $p$, the congruence $x^x \equiv c \pmod{p}$ has a solution.*

(ISL 2005 N6). *Let $a, b$ be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers $n$. Prove that $a = b$.*

**Example 3.1** (USAMO 2008/1)

Prove that for each positive integer $n$, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \ldots k_n - 1$ is the product of two consecutive integers.

**Problem 4.3** (IMO 2009/1). Let $n$ be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ (here $k \geq 2$) be distinct integers in the set $\{1, 2, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

**Problem 4.4** (APMO 2009/4). Prove that for any positive integer $k$, there exists an arithmetic progression

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \ldots, \quad \frac{a_k}{b_k}$$

of rational numbers, where $a_i$, $b_i$ are relatively prime positive integers for each $i = 1, 2, \ldots, k$, and moreover the $2k$ numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ are pairwise distinct.

**Problem 4.5** (ELMO Shortlist, Evan Chen). Find all triples $(a, b, c)$ of positive integers such that if $n$ is a positive integer not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

**Problem 4.1** (IMO 1989). Prove that for every positive integer $n$, there exists $n$ consecutive positive integers such that none of them is a power of a prime.

**Problem 4.2.** Let $n$ be a positive integer. Determine, in terms of $n$, the number of $x \in \{1, 2, \ldots, n\}$ for which $x^2 \equiv x \pmod{n}$.

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(AoPS). *Show that for $c \in \mathbb{Z}$ and a prime $p$, the congruence $x^x \equiv c \pmod{p}$ has a solution.*

(ISL 2005 N6). *Let $a, b$ be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers $n$. Prove that $a = b$.*

**Problem 4.3** (IMO 2009/1). Let $n$ be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ (here $k \geq 2$) be distinct integers in the set $\{1, 2, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

1. (Poland 2003) A polynomial $f$ with integer coefficients has the property that $\gcd(f(a), f(b)) = 1$ for some integers $a \neq b$. Prove that there is an infinite set of integers $S$ such that $\gcd(f(m), f(n)) = 1$ whenever $m, n$ are distinct elements of $S$.

2. Prove that for all positive integers $k$ and $n$ there exists a set $S$ of $n$ consecutive positive integers such that each $x \in S$ has at least $k$ distinct prime divisors that do not divide any other element of $S$.

3. A lattice point is called visible if its coordinates are relatively prime integers. Prove that for any positive integer $k$ there is a lattice point whose distance from each visible lattice point is greater than $k$.

4. a) Prove that for all $n \geq 1$ there is a positive integer $a$ such that $a, 2a, \ldots, na$ are all perfect powers.

   b) (Balkan 2000) Prove that for all $n \geq 1$ there is a set $A$ of $n$ positive integers such that for all $1 \leq k \leq n$ and all $x_1, x_2, \ldots, x_k \in A$ the number $\frac{x_1 + x_2 + \ldots + x_k}{k}$ is a perfect power.

**Example 3.4** (Math Prize Olympiad 2010)

Prove that for every positive integer $n$, there exists integers $a$ and $b$ such that $4a^2 + 9b^2 - 1$ is divisible by $n$.

① if $5 \nmid n \Rightarrow$ $a \equiv 2 \cdot 5^{-1} \pmod{n}$

$b \equiv 1 \cdot 5^{-1} \pmod{n}$

$\Rightarrow 4a^2 + 9b^2 - 1 \equiv \frac{16}{25} + \frac{9}{25} - 1 \equiv 0 \pmod{n}$

② if $n = 5^k \Rightarrow$ $a \equiv 2^{-1} \pmod{n}$ $\quad \frac{n+1}{2}$

$b \equiv 0 \pmod{n}$

$\Rightarrow 4a^2 + 9b^2 - 1 \equiv \frac{4}{4} + 0 - 1 \equiv 0 \pmod{n}$

If $m = 5^k n$, $5 \nmid n$

$\exists\, a_1, b_1$ from ① $\quad 4a_1^2 + 9b_1^2 - 1 \equiv 0 \pmod{n}$

$\exists\, a_2, b_2$ from ② $\quad 4a_2^2 + 9b_2^2 - 1 \equiv 0 \pmod{5^k}$

From CRT: $\quad a \equiv a_1 \pmod{n} \quad b \equiv b_1 \pmod{n}$

$a \equiv a_2 \pmod{5^k} \quad b \equiv b_2 \pmod{5^k}$

$\Rightarrow 4a^2 + 9b^2 - 1 \equiv 4a_1^2 + 9b_1^2 - 1 \equiv 0 \pmod{n}$

$4a^2 + 9b^2 - 1 \equiv 4a_2^2 + 9b_2^2 - 1 \equiv 0 \pmod{5^k}$

$\Rightarrow \underbrace{n\, 5^k}_{m} \mid 4a^2 + 9b^2 - 1$

$m = n\, 3^\beta$

$a \equiv 0 \,[n]$ $\quad a \equiv 2^{-1} \,[3^\beta]$

$b \equiv 3^{-1} \,[n]$ $\quad b \equiv 0 \,[3^\beta]$

$9b^2 \equiv 1 \,[n]$ $\quad \Rightarrow 4a^2 \equiv 1 \,[3^\beta]$

$n \mid m^{p(n-1)} - 1$ prove that $\gcd(m^{n-1}-1, n) > 1$

Assume that $\gcd(__, __) = 1$

$$m^{p(n-1)} - 1 = \underbrace{(m^{n-1}-1)}\,\underbrace{(_____)}_{n}$$

Let $q$ be a prime that divides $n$.

$\Rightarrow \quad m^{p(n-1)} \equiv 1 \pmod q \quad \Rightarrow \quad d_q \mid p(n-1)$

$m^{n-1} \not\equiv 1 \pmod q \qquad d_q \nmid n-1$

$\operatorname{ord}_q m = d_q$

Let $v_p(n-1) = \alpha$. $\Rightarrow \quad \boxed{v_p(d_q) = v_p(n-1) + 1}$

$d_q \mid q-1 \Rightarrow q \equiv 1 \pmod{p^{\alpha+1}} \quad \forall\, q \mid n$

$\Rightarrow n \equiv 1 \pmod{p^{\alpha+1}} \Rightarrow p^{\alpha+1} \mid n-1$

$\Rightarrow v_p(n-1) \geq \alpha+1 \quad \longrightarrow \longleftarrow !!$

$\alpha \quad \geq \alpha+1 \quad \times$

**N5.** Find all triples $(p, x, y)$ consisting of a prime number $p$ and two positive integers $x$ and $y$ such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of $p$.

*(Belgium)*

Solution:

If $p=2 \Rightarrow x+y=2^k \Rightarrow (x,y)=(n, 2^k-n)$ s.t. $1 \le n < 2^k$

Assume that $p>2$ $\begin{cases} x^{p-1}+y = p^\alpha \\ x+y^{p-1} = p^\beta \end{cases}$

$v_p(y) = v_p(x^{p-1}) = (p-1)\, v_p(x)$ and similarly $v_p(x) = (p-1) v_p(y)$

$\Rightarrow v_p(x) = v_p(y) = 0 \Rightarrow p \nmid x, y$

W.L.O.G. assume that $\alpha < \beta$ $\quad x+y^{p-1} \equiv 0 \pmod{p^\alpha}$

but $y \equiv -x^{p-1} \pmod{p^\alpha} \Rightarrow x + x^{(p-1)^2} \equiv 0 \pmod{p^\alpha}$

$p \nmid x$ $\quad \Rightarrow p^\alpha \mid 1 + x^{p(p-2)}$

$\Rightarrow p \mid x^{\gcd(p(p-2),\,(p-1))} - 1$ $\quad x^{2p(p-1)} \equiv 1 \pmod p$

$\Rightarrow p \mid x^2 - 1$, but $p \nmid x-1$

$\Rightarrow p \mid x+1$

$\Rightarrow v_p(x^{p(p-2)}+1) = v_p(x+1) + v_p(p(p-2))$

$\Rightarrow \alpha \le v_p(x^{p(p-2)}+1) = v_p(x+1)+1$

$\Rightarrow \alpha - 1 \le v_p(x+1) \Rightarrow p^{\alpha-1} \mid x+1$ $\quad p \le x+1$

However, $x^{p-1} < p^\alpha \Rightarrow x^{p-1} < p(x+1)$ $\quad x < x^{p-2} < 2p$

$x \to p-1$

$\boxed{\text{if } \alpha = 1 \Rightarrow p=2.}$

$\Rightarrow x^{p-2} < p + \frac{p}{x} < 2p$

$\Rightarrow x = p-1,\ p<5,\ \alpha = 2$

$\Rightarrow x = 2,\ p = 3 \Rightarrow y = 5$ $\quad (2,5,3)$ $(5,2,3)$

**Problem 4.1** (IMO 1989). Prove that for every positive integer $n$, there exists $n$ consecutive positive integers such that none of them is a power of a prime.

We want to find $a$ such that each term in $a, a+1, \ldots, a+(n-1)$ is divisible by at least two distinct primes.

$$\begin{cases} a \equiv 0 \pmod{p_1 p_2} \\ a+1 \equiv 0 \pmod{p_3 p_4} \\ \vdots \\ a+(n-1) \equiv 0 \pmod{p_{2n-1}\, p_{2n}} \end{cases} \Rightarrow \begin{array}{l} a \equiv 0 \ [p_1 p_2] \\ a \equiv -1 \ [p_3 p_4] \\ \vdots \\ a \equiv -(n-1) \ [p_{2n-1}\, p_{2n}] \end{array}$$

$\exists\ a$ s.t. $\quad a \equiv r \ [p_1 p_2 \cdots p_{2n}]$

**Problem 4.2.** Let $n$ be a positive integer. Determine, in terms of $n$, the number of $x \in \{1, 2, \ldots, n\}$ for which $x^2 \equiv x \pmod{n}$.

Assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$\Rightarrow \quad x^2 \equiv x \pmod{p_i^{\alpha_i}}$

$\Rightarrow \quad p_i^{\alpha_i} \mid x(x-1)$ and $\gcd(x, x-1) = 1$

$\Rightarrow \quad p_i^{\alpha_i} \mid x$ or $p_i^{\alpha_i} \mid x-1$

$$\Rightarrow \quad k \begin{cases} x \equiv 0, 1 \pmod{p_i^{\alpha_i}} \text{ for each } i \\ x \equiv 0, 1 \quad (\ p_j^{\alpha_j}) \\ \vdots \end{cases}$$

each set of choices of 0, 1 forms one residue mod $n$ by CRT $\Longrightarrow$ # residues $= 2^k$

Let $P \in \mathbb{Z}[x]$ and for any integer $n$ either $P(n)=5$ or $P(n)=2022$. Prove that either all $P(n)=5$ or $P(n)=2022$.

$a-b \mid P(a)-P(b) \Rightarrow$

$P(x) \equiv P(y) \pmod{n}$ if $x \equiv y \pmod{n}$

Assume that $\exists\, a,b \in \mathbb{Z}$ s.t. $P(a)=5$, $P(b)=2022$

if $n \equiv a \pmod{5} \Rightarrow P(n) \equiv P(a) \pmod{5}$

$\Rightarrow P(n) \equiv 0 \pmod{5}$

$\Rightarrow P(n)=5$

Similarly if $n \equiv b \pmod{2022} \Rightarrow P(n)=2022$

However, $\exists\, n \in \mathbb{Z}^+$ s.t.

$$\begin{cases} n \equiv a \pmod{5} \\ n \equiv b \pmod{2022} \end{cases}$$ from CRT.

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$A, B$

if $n \equiv a \pmod{A} \Rightarrow P(n) \equiv P(a) = 5 \,[A]$

$n \equiv b \pmod{B} \Rightarrow P(n) = P(b) = 2022 \,[B]$

**Problem 4.3** (IMO 2009/1). Let $n$ be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ (here $k \geq 2$) be distinct integers in the set $\{1, 2, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

Assume that $n \mid a_k(a_1 - 1)$ and let $p$ be a prime $p^r \mid n$

$\Rightarrow a_i \equiv 0 \pmod{p^r}$ or $a_{i+1} \equiv 1 \pmod{p^r}$

$\Rightarrow$ either all $a_i$'s $\equiv 0 \pmod{p^r}$

or all $a_i$'s $\equiv 1 \pmod{p^r}$

(AoPS). *Show that for $c \in \mathbb{Z}$ and a prime $p$, the congruence $x^x \equiv c \pmod{p}$ has a solution.*

$\exists \quad x \quad$ s.t.

$$\underbrace{x^0, x^1, \ldots, x^{p-2}}_{p-1}$$

$$\begin{cases} x \equiv 1 \pmod{p-1} \\ x \equiv c \pmod{p} \end{cases}$$

$$\Longrightarrow \quad x^x \equiv c^x \pmod{p}$$

$$\equiv c^1 \pmod{p}$$

$p$

(ISL 2005 N6). *Let $a, b$ be positive integers such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers $n$. Prove that $a = b$.*

$$a^n + n \mid b^n + n \quad \forall\, n \in \mathbb{Z}$$

$$\Longleftrightarrow \quad a^n + n \mid b^n - a^n$$

$$\Rightarrow \quad a^n \equiv b^n \pmod{a^n + n}$$

Take $q \mid a^n + n$

$$\Rightarrow \quad \underset{\substack{\equiv \\ a}}{a^n} \equiv \underset{\substack{\equiv \\ b}}{b^n} \pmod{q}$$

$$n \equiv 1 \pmod{q-1}$$

$$a \equiv b \pmod{q} \longrightarrow$$

$$a^n \equiv -n \pmod{q}$$

$$a \equiv -n \pmod{q}$$

$$\Rightarrow \quad \exists\, n \in \mathbb{Z}$$

$$\Rightarrow \quad q \mid a - b$$

**Problem 4.3** (IMO 2009/1). Let $n$ be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ (here $k \geq 2$) be distinct integers in the set $\{1, 2, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k-1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

Assume that $p \mid n$, and $p \mid a_k(a_1 - 1)$

if $p \mid a_1(a_2 - 1) \Rightarrow \cdots \Rightarrow a_1 \equiv a_2 \equiv \cdots \equiv a_k \equiv \underbrace{0, 1}_{r} \pmod{n}$

Case $r = 0$

$\Rightarrow p^\alpha \mid a_1(a_2 - 1)$

$\gcd(p, a_2 - 1) = 1$

$\Rightarrow p^\alpha \mid a_1 \Rightarrow p^\alpha \mid a_2, a_3, \ldots, a_k$

Similarly if $r = 1$

$a_1 \equiv a_2 \equiv \cdots \equiv a_k \pmod{p^\alpha}$

$\forall\ p^\alpha \mid n$

$\Rightarrow a_1 \equiv a_2 \equiv \cdots \equiv a_k \pmod{n}$

$a \mid x - y \leftarrow x \equiv y \pmod{a}$

$b \mid x - y \leftarrow x \equiv y \pmod{b}$ $\Rightarrow$ $\underline{ab \mid x - y}$

23. (China 2007) Let $n > 1$ be an integer. Prove that $2n-1$ is a prime number if and only if for any $n$ pairwise distinct positive integers



$a_1, a_2, \ldots, a_n$ there exist $i, j \in \{1, 2, \ldots, n\}$ such that

$$\frac{a_i + a_j}{\gcd(a_i, a_j)} \geq 2n - 1$$

Assume that $2n-1 = p$ is a prime and there exist $a_1, a_2, \ldots, a_n$ pairwise distinct such that for all $i, j$

$$\frac{a_i + a_j}{\gcd(a_i, a_j)} < p$$

We can assume that $\gcd(a_1, a_2, \ldots, a_n) = 1$ otherwise we can divide by the gcd.

<u>Case 1:</u> $\exists\, a_i$ such that $p \mid a_i$

$p \mid a_i$ & $p \nmid a_j$

$$\frac{a_i + a_j}{\gcd(a_i, a_j)} \geq \frac{a_i}{\gcd(a_i, a_j)} \geq p$$

$$a_i \not\equiv a_j \,[p]$$

$$2n-2\,[p] \qquad n$$

$$\underbrace{(1, p-1), \ldots, \left(\frac{p-1}{2}, \frac{p+1}{2}\right)}_{n-1}$$

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$$p \mid \frac{a_i + a_j}{\gcd(a_i, a_j)} \geq p$$

$$a_i \equiv a_j \,[p] \qquad a_i = k_1 p + r \Rightarrow a_j = k_2 p + r$$

$$k_1 > k_2$$

$$(a_i, a_j) = (a_i, a_i - a_j) = (a_i, p(k_1 - k_2))$$

$$= (a_i, k_1 - k_2) \leq k_1 - k_2$$

$$\frac{a_i + a_j}{(a_i, a_j)} \geq \frac{a_i}{(a_i, k_1 - k_2)} \geq \frac{k_1 p + r}{k_1 - k_2} \geq p$$

if $a_i \equiv a_j \ [P]$

$$\frac{a_i - a_j}{\gcd(a_i, a_j)} \geq P$$

$$\Rightarrow \quad \frac{a_i + a_j}{\gcd(a_i, a_j)} \geq P$$

23. (China 2007) Let $n > 1$ be an integer. Prove that $2n-1$ is a prime number if and only if for any $n$ pairwise distinct positive integers



$a_1, a_2, \ldots, a_n$ there exist $i, j \in \{1, 2, \ldots, n\}$ such that

$$\frac{a_i + a_j}{\gcd(a_i, a_j)} \geq 2n - 1$$

Assume that $2n-1 = xy$ where $x, y > 1$

$$\underbrace{1, 2, 3, \ldots, x}_{A}, \quad \underbrace{x+1, x+3, \ldots, xy-x}_{\substack{n-x \text{ even numbers} \\ B}}$$

<u>Case 1</u> $a_i, a_j \in A$

$$\Rightarrow \frac{a_i + a_j}{\gcd(a_i, a_j)} \leq a_i + a_j \leq \underbrace{x + x - 1}_{2x-1 < 2x < xy}$$

<u>Case 2:</u> $a_i, a_j \in B$

$$\Rightarrow \frac{a_i + a_j}{\gcd(a_i, a_j)} \leq \frac{a_i + a_j}{2} < \frac{(xy-x)+(xy-x)}{2} = xy - x < xy$$

<u>Case 3:</u> $a_i \in A$, $a_j \in B$

$$\Rightarrow \frac{a_i + a_j}{\gcd(a_i, a_j)} \leq a_i + a_j < \underbrace{x + xy - x}_{xy}$$

"=" $\Leftrightarrow a_i = x$, $a_j = xy - x$ but $\frac{a_i + a_j}{\gcd(-,-)} = \frac{xy-y}{x} \in \mathbb{Z}$

$\rightarrow\leftarrow$

1. (Poland 2003) A polynomial $f$ with integer coefficients has the property that $\gcd(f(a), f(b)) = 1$ for some integers $a \neq b$. Prove that there is an infinite set of integers $S$ such that $\gcd(f(m), f(n)) = 1$ whenever $m, n$ are distinct elements of $S$.

2. Prove that for all positive integers $k$ and $n$ there exists a set $S$ of $n$ consecutive positive integers such that each $x \in S$ has at least $k$ distinct prime divisors that do not divide any other element of $S$.

3. A lattice point is called visible if its coordinates are relatively prime integers. Prove that for any positive integer $k$ there is a lattice point whose distance from each visible lattice point is greater than $k$.

4. a) Prove that for all $n \geq 1$ there is a positive integer $a$ such that $a, 2a, ..., na$ are all perfect powers.

   b) (Balkan 2000) Prove that for all $n \geq 1$ there is a set $A$ of $n$ positive integers such that for all $1 \leq k \leq n$ and all $x_1, x_2, ..., x_k \in A$ the number $\frac{x_1+x_2+...+x_k}{k}$ is a perfect power.

6. (AMM) Prove that there are arbitrarily long sequences of consecutive integers, none of which can be written as the sum of two perfect squares.

7. Let $f$ be a nonconstant polynomial with integer coefficients and let $n$ and $k$ be positive integers. Prove that there is a positive integer $a$ such that each of the numbers $f(a), f(a+1), \ldots, f(a+n-1)$ has at least $k$ distinct prime divisors.

9. (IMO 1999 Shortlist) Find all positive integers $n$ for which there is an integer $m$ such that $2^n - 1 \mid m^2 + 9$.

10. (Bulgaria 2003) A finite set $C$ of positive integers is called good if for any $k \in \mathbf{Z}$ there exist $a \neq b \in C$ such that $\gcd(a+k, b+k) > 1$. Prove that if the sum of the elements of a good set $C$ equals 2003, then there exists $c \in C$ such that the set $C - \{c\}$ is good.

$$\varphi(ab) = \varphi(a)\,\varphi(b) \quad \text{if} \quad \gcd(a,b)=1$$

$$\left.\begin{array}{ll} r_1, r_2, \ldots, r_{\varphi(a)} & \pmod{a} \\ g_1, g_2, \ldots, g_{\varphi(b)} & \pmod{b} \end{array}\right\} \varphi(a)\,\varphi(b)$$

$$s_1, s_2, \ldots, s_{\varphi(ab)} \quad \pmod{ab} \quad \varphi(ab)$$

1. (Poland 2003) A polynomial $f$ with integer coefficients has the property that $\gcd(f(a), f(b)) = 1$ for some integers $a \neq b$. Prove that there is an infinite set of integers $S$ such that $\gcd(f(m), f(n)) = 1$ whenever $m, n$ are distinct elements of $S$.

Assume that we have $a_1, a_2, \ldots, a_k$ such that $f(a_i)$ and $f(a_j)$ are relatively prime for all $i, j$

$$p \nmid f(a_{k+1}) \quad \forall \quad p \mid f(a_1) \ldots f(a_k)$$

if $p \mid f(a_j) \Rightarrow \underline{a_{k+1} \equiv a_{j-1}} \pmod{f(a_j)}$

$a_0 = a_k \Rightarrow f(a_{k+1}) \equiv \underbrace{f(a_{j-1})}_{\not\equiv 0 \ [p]} \pmod{p}$

$\Rightarrow p \nmid f(a_{k+1})$

Note that if $p \mid f(a_j)$ then $p \nmid f(a_i)$ $\forall \ i \neq j$

from CRT $\exists$ such $a_{k+1}$ $\blacksquare$

2. Prove that for all positive integers $k$ and $n$ there exists a set $S$ of $n$ consecutive positive integers such that each $x \in S$ has at least $k$ distinct prime divisors that do not divide any other element of $S$.

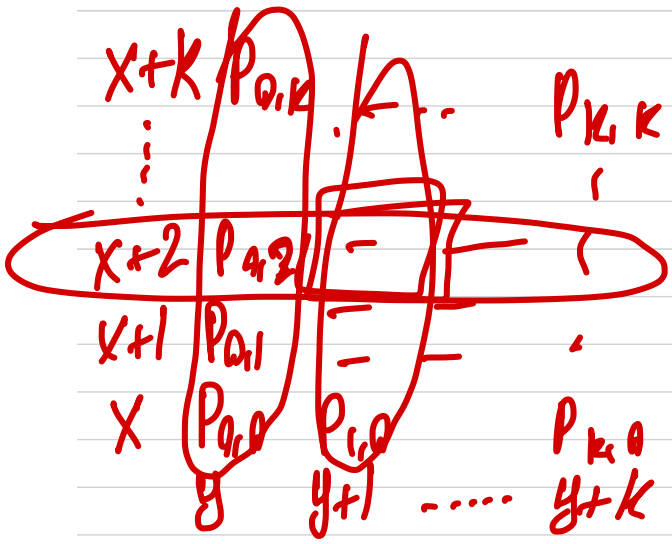
$a \equiv 0 \; [p_1 p_2 \cdots p_k]$

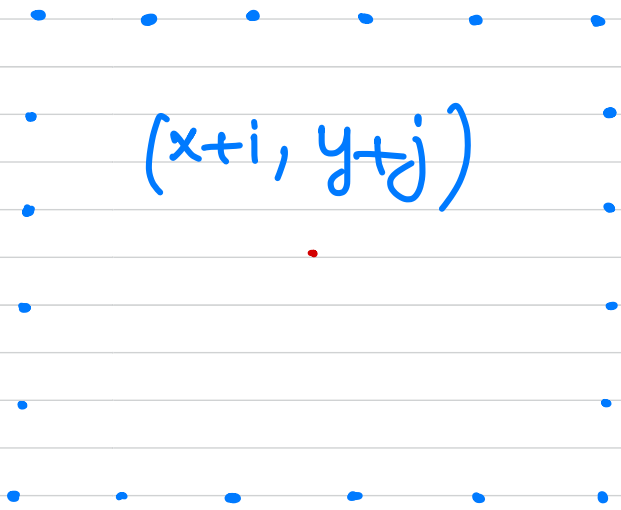
$a \equiv -1 \; [p_{k+1} p_{k+2} \cdots p_{2k}]$

$\vdots$

$a \equiv -n+1 \; [p \cdots]$

3. A lattice point is called visible if its coordinates are relatively prime integers. Prove that for any positive integer $k$ there is a lattice point whose distance from each visible lattice point is greater than $k$.




$x \equiv -i \pmod{p_{i,j}}$

$y \equiv -j \pmod{p_{i,j}}$

4. a) Prove that for all $n \geq 1$ there is a positive integer $a$ such that $a, 2a, ..., na$ are all perfect powers.

b) (Balkan 2000) Prove that for all $n \geq 1$ there is a set $A$ of $n$ positive integers such that for all $1 \leq k \leq n$ and all $x_1, x_2, ..., x_k \in A$ the number $\frac{x_1+x_2+...+x_k}{k}$ is a perfect power.

a) $a = 2^{\alpha_1} 3^{\alpha_2} 4^{\alpha_3} \cdots n^{\alpha_{n-1}}$, let $p_1, p_2, \cdots, p_n$ be $n$ distinct primes

let $\alpha_1 + 1 \equiv \alpha_2 \equiv \alpha_3 \equiv \cdots \equiv \alpha_{n-1} \equiv 0 \, [p_1]$

$\alpha_1 \equiv \alpha_2 + 1 \equiv \cdots \cdots \equiv \alpha_{n-1} \equiv 0 \, [p_2]$

$\vdots$

$\alpha_1 \equiv \cdots \cdots \equiv \alpha_{n-1} \equiv 0 \, [p_n]$

CRT done

$i a_i = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdots i^{\alpha_i + 1} (i+1)^{\alpha_{i+1}} \cdots n^{\alpha_{n-1}}$

$p_i \mid \alpha_1, \alpha_2, \cdots, \alpha_i + 1, \alpha_{i+1}, \cdots, \alpha_n$

$\Rightarrow i a_i = x^{p_i}$ for some $x \in \mathbb{N}$

b) (Balkan 2000) Prove that for all $n \geq 1$ there is a set $A$ of $n$ positive integers such that for all $1 \leq k \leq n$ and all $x_1, x_2, ..., x_k \in A$ the number $\frac{x_1+x_2+...+x_k}{k}$ is a perfect power.

$x_1 = n!a$, $x_2 = 2 \cdot n!a$ . . . $x_n = n! \cdot n \cdot a$

$a \cdot M_k$, $M_k < N$

6. (AMM) Prove that there are arbitrarily long sequences of consecutive integers, none of which can be written as the sum of two perfect squares.

Let $P_1 \equiv P_2 \equiv \cdots \equiv P_n \equiv 3\,[4]$ be distinct prime numbers

we choose $A$ s.t

$$A+i \equiv P_i \;[P_i^2] \quad \forall\; i \in \{1,2,\cdots,n\}$$

$$P_i \mid A+i, \quad P_i^2 \nmid A+i$$

if $A+i = a^2+b^2$, $P_i \mid a^2+b^2$, $P_i \equiv 3\,[4]$

$\Rightarrow P_i \mid a,\ P_i \nmid b \Rightarrow P_i^2 \nmid a^2+b^2 \Rightarrow P_i^2 \mid A+i$

$\Rightarrow\Leftarrow$

7. Let $f$ be a nonconstant polynomial with integer coefficients and let $n$ and $k$ be positive integers. Prove that there is a positive integer $a$ such that each of the numbers $f(a), f(a+1), \ldots, f(a+n-1)$ has at least $k$ distinct prime divisors.

<u>Lemma:</u> if $P \in \mathbb{Z}[x]$ then $P(n)$ is divisible by $\infty$-ly many primes.

Suppose that there are finite number of prime numbers dividing $P(x)$

$\boxed{1}$ $p \mid P(n) \quad \forall n \in \mathbb{Z}$ $\qquad Q(x) = \frac{P(x)}{p}$

$\boxed{2}$ $p \nmid P(a), \quad a \in \mathbb{Z}$

$p_i \nmid P(a_i)$

$n \equiv a_1 \pmod{p_1}$

$\vdots$

$n \equiv a_k \pmod{p_k}$

$P(a) \equiv P(pk+a) \; [p]$

$n \equiv a \; [p]$

$p_1 p_2 \ldots p_k \nmid P(n)$

$p(n) = 1$

$p_i \mid f(a_i) \quad \forall i = 0, 1, \ldots, N$

$a \equiv a_i \; [p_i] \quad \forall 0 \le i < k$

$a+1 \equiv a_i \; [p_i] \quad \forall k \le i < 2k$

$\vdots$

$a+n \equiv a_i \; [p_i] \quad \forall (n-1)k \le i < nk$

from CRT $\exists a$