— ALGEBRA FOR L2 —

— FEBRUARY CAMP, 2022 — BASIC INEQUALITIES —

WARM-UP.

- $x^2 \geq 0$ for each $x \in \mathbb{R}$
- $x^2 \leq x$ if and only if $0 \leq x \leq 1$
- if $a > 0$ and $x > y > 0$, then $\frac{a}{x} < \frac{a}{y}$ and $\frac{x}{a} > \frac{y}{a}$
- several useful identities (every $\pm$ should be replaced with the same sign):
$$(1 \pm x)(1 \pm y) = 1 \pm x \pm y + xy, \quad (x \pm y)^2 = x^2 \pm 2xy + y^2.$$

1. Prove that if real numbers $x, y \in \mathbb{R}$ satisfy $x^2 + x \leq y$, then $y^2 + y \geq x$.

SOLUTION. By $y^2 \geq 0$, the given assumption, and $x^2 \geq 0$, we have

$$y^2 + y \geq 0 + y = y \geq x^2 + x \geq 0 + x = x.$$

2. Prove that if a, b, c are positive real numbers, then

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} > \sqrt{a+b+c}.$$

SOLUTION. We have $\sqrt{a+b} < \sqrt{a+b+c}$ and similarly for $\sqrt{b+c}$ and $\sqrt{c+a}$. By increasing the denominator, we decrease the fraction:

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} > \frac{a}{\sqrt{a+b+c}} + \frac{b}{\sqrt{a+b+c}} + \frac{c}{\sqrt{a+b+c}} = \frac{a+b+c}{\sqrt{a+b+c}} = \sqrt{a+b+c}.$$

3. Prove that if a, b, c are positive real numbers, then

$$\frac{a}{a+1} + \frac{b}{(a+1)(b+1)} + \frac{c}{(a+1)(b+1)(c+1)} < 1.$$

SOLUTION. When we expand the left-hand side and put ecerything as a single fraction, we'll see that the numerator is precisely 1 less than the denominator (and both are positive). The other way to see this is to add $\frac{1}{(a+1)(b+1)(c+1)}$ to the left-hand side:

$$\begin{aligned}
\frac{a}{a+1} + \frac{b}{(a+1)(b+1)} + \frac{c}{(a+1)(b+1)(c+1)} &< \frac{a}{a+1} + \frac{b}{(a+1)(b+1)} + \frac{c+1}{(a+1)(b+1)(c+1)} \\
&= \frac{a}{a+1} + \frac{b}{(a+1)(b+1)} + \frac{1}{(a+1)(b+1)} \\
&= \frac{a}{a+1} + \frac{b+1}{(a+1)(b+1)} \\
&= \frac{a}{a+1} + \frac{1}{a+1} = 1.
\end{aligned}$$

4. Real numbers a, b, c, d satisfy $a+b=cd$ and $c+d=ab$. Prove that
$$(a+1)(b+1)(c+1)(d+1)\geq 0.$$

Solution. In the assumptions it's seen that the pairs of variables a, b and c, d come together. Let us try to group them like this in our hypothesis, too:

$$(a+1)(b+1)(c+1)(d+1)=(ab+a+b+1)(cd+c+d+1)=(a+b+c+d+1)^2.$$

We conclude the solution with the observation that squares are always non-negative.

5. Given are real numbers $a,b\in(0,1)$. Prove that
$$a\sqrt{b}+b\sqrt{a}+1>3ab.$$

Solution. Note that if $a\in(0,1)$, then $\sqrt{a}>a$ (because $a>a^2$) and similarly $\sqrt{b}>b$. Moreover, $1=1\cdot 1>a\cdot b$, so

$$a\sqrt{b}+b\sqrt{a}+1>a\cdot b+b\cdot a+a\cdot b=3ab.$$

6. Given are real numbers $x,y\in(0,1)$. Prove that
$$x(1-y)^2+y(1-x)^2<(1-xy)^2.$$

Solution. Expanding the squares on both sides, we can rewrite the desired inequality as follows:

$$\begin{aligned} x-2xy+xy^2+y-2xy+yx^2 &< 1-2xy+x^2y^2,\\ x+y+xy(x+y) &< 1+2xy+x^2y^2,\\ (1+xy)(x+y) &< (1+xy)^2. \end{aligned}$$

Because $1+xy>0$, the last inequality is equivalent to

$$x+y<1+xy \iff 0<1-x-y+xy \iff 0<(1-x)(1-y).$$

The last inequality is true because $1-x$ and $1-y$ are positive (by the problem's assumption).

7. Positive numbers a, b, c satisfy $a\leq 1$, $b\leq 2$, $c\leq 3$. Prove that $a+b+c\geq abc$.

8. Distinct positive numbers a, b satisfy $a+b=1$. Prove that
$$\left|\frac{a-b}{\sqrt{1-a^2}-\sqrt{1-b^2}}\right|<\sqrt{3}.$$