May Camp - 2021
NT L3 Primitive roots

Problems

1. Let p be a prime number. Is it always possible to arrange numbers $1, 2, \ldots, p-1$ along a circle so that that any three consecutive numbers a, b, c chosen in this order satisfy the congruence $b^2 \equiv ac \pmod{p}$?

2. Let
$$A_k = 1^k + 2^k + \ldots + (p-1)^k.$$
Prove that p^2 divides A_{pk}, $1 \leqslant k < p-1$.

3. Let a, b, c, d be elements of the set $\{0, 1, \ldots, 99\}$ such that $a + b \equiv c + d \pmod{100}$ and $2^a + 2^b \equiv 2^c + 2^d \pmod{101}$. Prove that $\{a, b\} = \{c, d\}$.

4. Is it true that if n is sufficiently large and $a_1, a_2, \ldots, a_n$ is an arbitrary permutation of $(1, 2, \ldots, n)$, then we can find integers i, d such that $1 \leqslant i < i + d < i + 2d \leqslant n$ and a_i, a_{i+d}, a_{i+2d} form an arithmetic progression?

5. Is there a positive integer k such that $p = 6k + 1$ is a prime and $\binom{3k}{k} \equiv 1 \pmod{p}$?

6. (*) Is there a positive integer n such that every nonzero digit in base 10 appears the same number of times in the decimal representation of each of the following numbers $n, 2n, 3n, \ldots, 2021n$.

Homework

1. Let p be a prime. Based on the existence of primitive roots modulo p, show that
$$A_k = 1^k + 2^k + \ldots + (p-1)^k, \ 1 \leqslant k \leqslant p-2$$
is divisible by p.

2. It is given that $p > 3$ is a prime number and $k \leqslant p-2$ is a positive integer. Let us find all k-element subsets of $\{1, 2, \ldots, p-1\}$, multiply their elements, raise the results to the power of p, and add them all together. Prove that the sum is divisible by p^2.

3. Let p be an odd prime and $1 < k < p$ an integer. Number a is called a k-th power residue modulo p if there exists a number $x \in \{1, 2, \ldots, p-1\}$ such that $x^k \equiv a \pmod{p}$. Find the number of k-th power residues.