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Geometry – L3

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Contents

Problems

Similarity

Problem 1. In triangle ABC there is point P lying inside such that

$$\sphericalangle BAP = \sphericalangle ACP \quad \text{and} \quad \sphericalangle PBA = \sphericalangle PAC.$$

Let X and Y be midpoints of AB and AC respectively. Prove that A, Y, P and X are concyclic.

Problem 2. In triangle ABC in which $\sphericalangle BAC = 60^\circ$ tehre is point T lying inside such that

$$\sphericalangle ATB = \sphericalangle CTA = 120^\circ.$$

Let X and Y be midpoints of AB and AC, respectively. Prove that A, Y, T and X are concyclic.

Problem 3. In triangle ABC there is point P lying inside such that

$$\sphericalangle BAP = \sphericalangle ACP \quad \text{oraz} \quad \sphericalangle PBA = \sphericalangle PAC.$$

Let O be a circumcenter of triangle ABC. Prove that $\sphericalangle APO = 90^\circ$.

Problem 4. Circle ω_1 passes through vertices A and B of triangle ABC and intersects BC at D. Circle ω_2 passes through vertices B and C and intersects AB at E and circle ω_1 again at F. Assume that A, E, D, C lie on circle ω_3 with center O. Prove that $\sphericalangle BFO = 90^\circ$.

Problem 5. In cyclic convex quadrilateral $ABCD$ lines AB and CD intersects at P, while BC and AD intersects at Q. Let K and N be midpoints of AC and BD. Prove that $\sphericalangle PKQ + \sphericalangle PNQ = 180^\circ$.

Problem 6. In cyclic convex quadrilateral $ABCD$ lines AB and CD intersects at P, while BC and AD intersects at Q. Let K and N be midpoints of AC and BD. Prove that bisectors of angles APD and BQA intersect on KN.

Problem 7. Let M be a midpoint of side BC of triangle ABC. Denote by P and Q projections of M on AB and AC, respectively. Let N be a midpoint of side PQ. Prove that $AO \parallel MN$, where O is the circumcenter of triangle ABC.

Problem 8. Let ABC be a triangle. Let K be a midpoint of BC and M be a point on the segment AB. Let $L = KM \cap AC$ and C lies on the segment AC between A and L. Let N be a midpoint of ML. Assume line AN intersects circumcircle of triangle ABC at $S \neq N$. Prove that circumcircle of triangle KSN is tangent to BC.

Problem 9. In convex quadrilateral $ABCD$ diagonals AC and BD intersect at P. Circumcircles of triangles ADP and PCB intersect at $S \neq P$. Let M and N be midpoints of diagonals AC and BD, respectively. Prove that M, P, N and S are concyclic.

Butterfly Theorem

Problem 10. Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\sphericalangle FHP = \sphericalangle BAC$.

Pascal's Theorem

Problem 11. The incircle of a convex quadrilateral $ABCD$ touches sides AB, BC, CD, DA at K, L, M, N, respectively. Prove that points $X = AB \cap CD$, $Y = BC \cap DA$, $Z = NK \cap LM$ are collinear.

Problem 12. Let ABC be a triangle. Let AXC and BYC be triangles constructed outside of triangle ABC such that

$$\sphericalangle CAX + \sphericalangle CBY = 180^\circ \quad \text{and} \quad \sphericalangle ACX = \sphericalangle BCY = 15^\circ.$$

Prove that all lines XY, corresponding to different positions of points X and Y, have a common point.

Problem 13. Point F lies on side DE of convex pentagon $ABCDE$ and satisfies

$$\sphericalangle FAC = \sphericalangle DBC \quad \text{and} \quad \sphericalangle FCA = \sphericalangle EBA.$$

Prove that $\sphericalangle BAE + \sphericalangle BCD = 180^\circ$.

Problem 14. Let $ABCD$ be a convex quadrilateral with circumcenter O and incenter I. Diagonals AC and BD meet at E. Prove that points O, I, and E are collinear

Problem 15. From a point P in the interior of triangle ABC, perpendiculars PQ and PR are dropped to the sides BC and AC. Also, from vertex C, perpendiculars CS and CT are drawn to the extensions of AP and BP. Prove that the point of intersection of SQ and TR always lies on AB.

Problem 16. Let ABC be a triangle such that $AC = BC$. Let P and Q lie inside triangle such that

$$\sphericalangle PAC = \sphericalangle ABQ \quad \text{and} \quad \sphericalangle PBC = \sphericalangle BAQ.$$

Prove that C, P, Q are collinear.

Problem 17. A convex pentagon $ABCDE$ with $BC = CD$ and $DE = EA$ is inscribed in a circle ω. Set $K = AD \cap CE$ and $L = BD \cap CE$. Point M is symmetric to D with respect to K and point N is symmetric to D with respect to L. Prove that lines EM and CN meet at the circle ω.

Problem 18. Let $ABCDE$ be a convex pentagon with $\sphericalangle BCD = \sphericalangle DEA = 90°$. Lines AC and BE meet at P. Assume points A, B, C, E lie on a circle with center O. Prove that points D, O, P are collinear.

Problem 19. Let ABC be an acute-angled triangle. Let D be the foot of the perpendicular from C to AB, and let E and F be the feet of the perpendiculars from D to BC and CA, respectively. Let K and L be the midpoints of BC and CA, respectively. Set $P = FK \cap EL$. Prove that the line PD passes through the circumcenter O of triangle ABC.

Problem 20*. Let $ABCD$ be a cyclic quadrilateral with circumcircle Ω. Its diagonals intersect at the point X. The incenter of triangle XBC is J, and the center of the circle (external) that touches the lines AB and CD and the segment BC is J. Prove that if M is the midpoint of the arc CB of Ω, then the points I, M and J are collinear.

Problem 21*. Let ABC be an acute-angled triangle with circumcircle Ω and it's center O. Let D be a point different than A and C lying on the arc AC of Ω which does not contain B. Let E be a point on side AB that $\sphericalangle ADE = \sphericalangle OBC$. Let F be a point on side BC that $\sphericalangle CDF = \sphericalangle OBA$. Prove that $\sphericalangle DEF = \sphericalangle DOC$ and $\sphericalangle DFE = \sphericalangle DOA$.

Problem 22*. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O, and E is the intersection of diagonals AC and BD. Let P be an interior point of $ABCD$ such that

$$\sphericalangle PAB + \sphericalangle PCB = \sphericalangle PBC + \sphericalangle PDC = 90°.$$

Prove that O, P and E are collinear.

Problem 23. Let O be a circumcenter of triangle ABC. Let K lies on segment AB, and let L lies on segment AC such that K, O and L are collinear. Let R be a midpoint of BL, and let S be a midpoint of CK. Prove that $\sphericalangle BAC = \sphericalangle SOR$.

Solutions

Similarity

Problem 1. In triangle ABC there is point P lying inside such that

$$\sphericalangle BAP = \sphericalangle ACP \quad \text{and} \quad \sphericalangle PBA = \sphericalangle PAC.$$

Let X and Y be midpoints of AB and AC respectively. Prove that A, Y, P and X are concyclic.

Solution. Note that triangles ABP and CBP are similar, so $\sphericalangle PXB = \sphericalangle PYC$, so A, Y, P and X are concyclic. $\square$

Problem 2. In triangle ABC in which $\sphericalangle BAC = 60^\circ$ tehre is point T lying inside such that

$$\sphericalangle ATB = \sphericalangle CTA = 120^\circ.$$

Let X and Y be midpoints of AB and AC, respectively. Prove that A, Y, T and X are concyclic.

Solution. Note that triangles ABT and CBT are similar, so $\sphericalangle TXB = \sphericalangle TYC$, so A, Y, T and X are concyclic. $\square$

Problem 3. In triangle ABC there is point P lying inside such that

$$\sphericalangle BAP = \sphericalangle ACP \quad \text{oraz} \quad \sphericalangle PBA = \sphericalangle PAC.$$

Let O be a circumcenter of triangle ABC. Prove that $\sphericalangle APO = 90^\circ$.

Solution. Note that triangles ABP and CBP are similar, so $\sphericalangle PXB = \sphericalangle PYC$, so A, Y, P and X are concyclic. Also X and Y are projections of O onto AB and AC, so A, Y, P, X and O lie on circle with diameter AO. $\square$

Problem 4. Circle ω_1 passes through vertices A and B of triangle ABC and intersects BC at D. Circle ω_2 passes through vertices B and C and intersects AB at E and circle ω_1 again at F. Assume that A, E, D, C lie on circle ω_3 with center O. Prove that $\sphericalangle BFO = 90^\circ$.

Solution. The quadrilateral $ABDF$ is inscribed in the circle ω, therefore

$$\sphericalangle FDC = \sphericalangle BAF = \sphericalangle EAF.$$

Similarly $\sphericalangle FEA = \sphericalangle DCF$. Hence, triangle AEF is similar to triangle DCF. Let K and L be the midpoints of the segments AE and CD, respectively. Then FK and FL are the medians of similar triangles AEF and DCF drawn from the vertices of the corresponding angles. Hence points B, K, L and F lie on the same circle.

The perpendiculars to the segments AE and CD meet at the center O of circle ω. Hence, points K and L lie on a circle with diameter BO. Thus, point F lies on a circle with diameter BO, therefore $FO \perp BO$. □

Problem 5. In cyclic convex quadrilateral $ABCD$ lines AB and CD intersects at P, while BC and AD intersects at Q. Let K and N be midpoints of AC and BD. Prove that $\sphericalangle PKQ + \sphericalangle PNQ = 180°$.

Solution. Note that triangles PAC and PDB are similar, so $\sphericalangle PKC = \sphericalangle PNB$. Similarly $\sphericalangle QKN = \sphericalangle DNQ$. Merging both equalities we get statement. □

Problem 6. In cyclic convex quadrilateral $ABCD$ lines AB and CD intersects at P, while BC and AD intersects at Q. Let K and N be midpoints of AC and BD. Prove that bisectors of angles APD and BQA intersect on KN.

Solution. Note that triangles PAC and PDB are similar, so $\sphericalangle APK = \sphericalangle NPD$, so bisectors of angles APD and KPN coincide. Similarly bisectors of angles BQA and NQK coincide. Let bisector of angle KPN intersects KN at X. Then from angle bisector theorem and similarity of triangles PAC and PDB we get

$$\frac{KX}{XN} = \frac{KP}{NP} = \frac{AC}{BD}.$$

But similarity of triangles ACQ and DBQ gives

$$\frac{AC}{BD} = \frac{KQ}{QN} \Longrightarrow \frac{KX}{XN} = \frac{KQ}{QN}.$$

By the reverse of angle bisector theorem we are done. □

Problem 7. Let M be a midpoint of side BC of triangle ABC. Denote by P and Q projections of M on AB and AC, respectively. Let N be a midpoint of side PQ. Prove that $AO \parallel MN$, where O is the circumcenter of triangle ABC.

Solution. Let AM intersects circumcircle of triangle ABC at M'. Then triangles PMQ and $BM'C$ are similar, so $\sphericalangle NMP = \sphericalangle MM'B$. Therefore perpendicular bisector of BC is a symmedian of triangle PMQ, which easily finish the proof. □

Problem 8. Let ABC be a triangle. Let K be a midpoint of BC and M be a point on the segment AB. Let $L = KM \cap AC$ and C lies on the segment AC between A and L. Let N be a midpoint of ML. Assume line AN intersects circumcircle of triangle ABC at $S \neq N$. Prove that circumcircle of triangle KSN is tangent to BC.

Solution. Let A' be the reflection of A wrt point N. Then triangles BSC and AMA' are similar. Therefore $\measuredangle ANM = \measuredangle CKS$, which easily implies statement. $\square$

Problem 9. In convex quadrilateral $ABCD$ diagonals AC and BD intersect at P. Circumcircles of triangles ADP and PCB intersect at $S \neq P$. Let M and N be midpoints of diagonals AC and BD, respectively. Prove that M, P, N and S are concyclic.

Solution. It follows immediately from similarity of triangles ASC and BSD. $\square$

Butterfly Theorem

Problem 10. Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\measuredangle FHP = \measuredangle BAC$.

Solution. Let S be the reflection of point H with respect to side AB. It is well-known that S lies on the circumcircle ω of triangle ABC. Let PF intersects BS at X. From butterfly theorem we may conclude that $PF = FX$, so triangles PFH and SFX are congruent. But $\measuredangle BAC = \measuredangle BSC$, so we are done. $\square$

Pascal's Theorem

Problem 11. The incircle of a convex quadrilateral $ABCD$ touches sides AB, BC, CD, DA at K, L, M, N, respectively. Prove that points $X = AB \cap CD$, $Y = BC \cap DA$, $Z = NK \cap LM$ are collinear.

Solution. Apply Pascal's theorem for hexagon $KKMMLN$ and $NNLLKM$. $\square$

Problem 12. Let ABC be a triangle. Let AXC and BYC be triangles constructed outside of triangle ABC such that

$$\sphericalangle CAX + \sphericalangle CBY = 180^\circ \quad \text{and} \quad \sphericalangle ACX = \sphericalangle BCY = 15^\circ.$$

Prove that all lines XY, corresponding to different positions of points X and Y, have a common point.

Solution. Let Z be an intersection of XB and YC. Then $ABZC$ is inscribed in some circle Ω. Now let P and Q be intersections of Ω with AX and AY, respectively. Then from Pascal's theorem applied for hexagon $APBZCQ$ it follows that XY passes through intersection of PC and BQ which is a fixed point by angle condition.

□

Problem 13. Point F lies on side DE of convex pentagon $ABCDE$ and satisfies

$$\sphericalangle FAC = \sphericalangle DBC \quad \text{and} \quad \sphericalangle FCA = \sphericalangle EBA.$$

Prove that $\sphericalangle BAE + \sphericalangle BCD = 180^\circ$.

Solution. Let $X = BE \cap CF$ and $Y = BD \cap AF$. From the given condition $ABCXY$ lie on one circle. Now it is enough to apply converse of the Pascal's theorem. □

Problem 14. Let $ABCD$ be a convex quadrilateral with circumcenter O and incenter I. Diagonals AC and BD meet at E. Prove that points O, I, and E are collinear

Solution. Let AI, BI, CI, DI intersect circumcircle of $ABCD$ at E, F, G, H. Lines EG and FH are diameters $ABCD$ so their intersection is O. Let $X = EB \cap CH$. Applying Pascal's for $ACHDBE$ we see that P, X, I are collinear. From Pascal's theorem for hexagon $GCHFBE$ we see that O, X, I are collinear. □

Problem 15. From a point P in the interior of triangle ABC, perpendiculars PQ and PR are dropped to the sides BC and AC. Also, from vertex C, perpendiculars CS and CT are drawn to the extensions of AP and BP. Prove that the point of intersection of SQ and TR always lies on AB.

Solution. Apply Pascal's theorem for hexagon $PQRSTC$. □

Problem 16. Let ABC be a triangle such that $AC = BC$. Let P and Q lie inside triangle such that

$$\sphericalangle PAC = \sphericalangle ABQ \quad \text{and} \quad \sphericalangle PBC = \sphericalangle BAQ.$$

Prove that C, P, Q are collinear.

Solution. Let $X = BP \cap AQ$ and $Y = AP \cap QB$. Then $AXYB$ lie on one circle Ω and AC, BC are tangent to Ω. Apply Pascal's theorem for hexagon $AAXYBB$. □

Problem 17. A convex pentagon $ABCDE$ with $BC = CD$ and $DE = EA$ is inscribed in a circle ω. Set $K = AD \cap CE$ and $L = BD \cap CE$. Point M is symmetric to D with respect to K and point N is symmetric to D with respect to L. Prove that lines EM and CN meet at the circle ω.

Solution. Since AC and BE intersects at I – incenter of triangle ABC, by Pascal's theorem it is enough to prove that M, I and N are collinear. From the well known lemma $IEDC$ is a kite, so KL is a midline of triangle DMN and we are done. □

Problem 18. Let $ABCDE$ be a convex pentagon with $\sphericalangle BCD = \sphericalangle DEA = 90°$. Lines AC and BE meet at P. Assume points A, B, C, E lie on a circle with center O. Prove that points D, O, P are collinear.

Solution. Let PD and PC intersects circumcircle at X and Y, respectively. Then YB and XA are diameters so intersect at O. Apply Pascal's theorem for hexagon $ADYXCB$. □

Problem 19. Let ABC be an acute-angled triangle. Let D be the foot of the perpendicular from C to AB, and let E and F be the feet of the perpendiculars from D to BC and CA, respectively. Let K and L be the midpoints of BC and CA, respectively. Set $P = FK \cap EL$. Prove that the line PD passes through the circumcenter O of triangle ABC.

Solution. Easy to see that $PQMN$ lie on circle with center at midpoint of OD. Let DP and DQ intersect this circle at X, Y. Apply Pascal's theorem for hexagon $MNXYPQ$. □

Problem 20★. Let $ABCD$ be a cyclic quadrilateral with circumcircle Ω. Its diagonals intersect at the point X. The incenter of triangle XBC is J, and the center of the circle (external) that touches the lines AB and CD and the segment BC is J. Prove that if M is the midpoint of the arc CB of Ω, then the points I, M and J are collinear.

Solution. Let I_D, I_A be incenters of triangles DBC and ABC. Similarly let J_D, J_A be excenters of triangles DBC and ABC. Then points I_D, I_A, J_D, J_A, B, C are concyclic, so it is enough to apply Pascal's theorem for hexagon $I_D I_A J_D J_A BC$. □

Problem 21★. Let ABC be an acute-angled triangle with circumcircle Ω and it's center O. Let D be a point different than A and C lying on the arc AC of Ω which does not contain B. Let E be a point on side AB that $\sphericalangle ADE = \sphericalangle OBC$. Let F be a point on side BC that $\sphericalangle CDF = \sphericalangle OBA$. Prove that $\sphericalangle DEF = \sphericalangle DOC$ and $\sphericalangle DFE = \sphericalangle DOA$.

Solution. Let DE and DF intersect circumcircle at K and L, respectively. Then AL and AO are isogonal, so $AL \perp BC$. Similarly $CK \perp AB$. Therefore from Pascal's theorem for hexagon $ABCKLD$ it follows that orthocenter H lies on line EF. Now since the reflection of H with respect to sides lies on circumcircle of triangle ABC we are done by easy angle chase. □

Problem 22★. A convex quadrilateral $ABCD$ with $AC \neq BD$ is inscribed in a circle with center O, and E is the intersection of diagonals AC and BD. Let P be an interior point of $ABCD$ such that

$$\sphericalangle PAB + \sphericalangle PCB = \sphericalangle PBC + \sphericalangle PDC = 90^\circ.$$

Prove that O, P and E are collinear.

Solution. X □

Problem 23. Let O be a circumcenter of triangle ABC. Let K lies on segment AB, and let L lies on segment AC such that K, O and L are collinear. Let R be a midpoint of BL, and let S be a midpoint of CK. Prove that $\sphericalangle BAC = \sphericalangle SOR$.

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