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Day 4

Problem 1. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying $\frac{a_{n-1}+a_{n+1}}{2} \geq a_n$ for all positive integers n. Show that

$$\frac{a_0 + a_{n+1}}{2} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}$$

holds for all positive integers n.

Problem 2. Let $a_0, a_1, \ldots, a_N$ be real numbers satisfying $a_0 = a_N = 0$ and

$$a_{i+1} - 2a_i + a_{i-1} = a_i^2$$

for $i = 1, 2, \ldots, N-1$. Prove that $a_i \leq 0$ for $i = 1, 2, \ldots, N-1$.

Problem 3. Given $x_1, x_2, \ldots, x_n$ real numbers, prove that there exists a real number y such that

$$\{y - x_1\} + \{y - x_2\} + \ldots + \{y - x_n\} \leq \frac{n-1}{2}.$$

where $\{x\} = x - \lfloor x \rfloor$.

Problem 4. Determine all integers $n > 1$ for which the inequality

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq (x_1 + x_2 + \ldots + x_{n-1})x_n$$

holds for all reals $x_1, x_2, \ldots, x_n$.

Problem 1. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying $\underbrace{\frac{a_{n-1}+a_{n+1}}{2} \geq a_n}$ for all positive integers n. Show that

$$\underbrace{\frac{a_0 + a_{n+1}}{2} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}}$$

holds for all positive integers n.

$n=1$: $\frac{a_0+a_2}{2} \geqslant a_1$ ✓

$n=2$: $a_0 + a_3 \geqslant a_1 + a_2$

- $a_0 + a_2 \geqslant 2a_1$
- $a_1 + a_3 \geqslant 2a_2$

$\rightarrow a_0 + a_1 + a_2 + a_3 \geqslant 2a_1 + 2a_2$

$a_0 + a_3 \geqslant a_1 + a_2$ ✓

$n=3$: $\frac{a_0+a_4}{2} \geqslant \frac{a_1+a_2+a_3}{3}$

$3a_0 + 3a_4 \geqslant 2a_1 + 2a_2 + 2a_3$

$a_0 + a_2 \geqslant 2a_1$

$a_4 + a_2 \geqslant 2a_3$

$\rightarrow a_0 + a_4 \geqslant 2a_1 + 2a_3 - 2a_2$

- $2(a_0+a_4) \overset{?}{\geqslant} 4a_2$

$\underline{a_0+a_4} \overset{?}{\geqslant} 2a_2 \leftarrow$

$a_0 + a_4 \geqslant \underline{2a_1 + 2a_3 - 2a_2} \geqslant 4a_2 - 2a_2 = 2a_2$

$\hookrightarrow a_0 + a_4 \geqslant a_1 + a_3$

$\frac{a_0+a_n}{2} \geqslant \frac{a_1 + \ldots + a_{n-1}}{n-1}$: خطوة الاستقراء

$a_{n+1} + a_{n-1} \geqslant 2a_n$

$\downarrow$

$(n-1)(a_0+a_n) \geqslant 2(a_1 + \ldots + a_{n-1})$

$\frac{a_0+a_{n+1}}{2} \overset{?}{\geqslant} \frac{a_1 + \ldots + a_n}{n}$

$n(a_0 + a_{n+1}) \overset{?}{\geqslant} 2(a_1 + \ldots + a_n)$

$n a_{n+1} + n a_{n-1} \geqslant 2n a_n$

$n(a_{n+1} + a_0) - na_0 + na_{n-1} \geqslant 2na_n$

$n(a_{n+1} + a_0) \geqslant 2na_n + na_0 - na_{n-1} \overset{?}{\geqslant} 2(a_1 + \ldots + a_n)$

$2(n-1)a_n + na_0 - na_{n-1} \overset{?}{\geqslant} 2(a_1 + \ldots + a_{n-1})$

$\underbrace{2a_n + a_0 - a_{n-1} + \frac{a_0 - a_{n-1}}{n-1}}_{\overset{?}{\geqslant}\, a_0 + a_n} \overset{?}{\geqslant} \frac{2}{n-1}(a_1 + \ldots + a_{n-1})$

$\Leftrightarrow \quad a_n - a_{n-1} + \frac{a_0 - a_{n-1}}{n-1} \overset{?}{\geqslant} 0$

$i<j$

$$\boxed{a_i + a_j \geq a_{i+1} + a_{j-1}}$$

هل

$$\geq a_{i+2} + a_{j-2} \geq \dots \geq a_{\lfloor \frac{i+j}{2} \rfloor} + a_{\lceil \frac{i+j}{2} \rceil}$$

$$a_i + a_{i+2} \geq 2a_{i+1}$$

$$a_j + a_{j-2} \geq 2a_{j-1}$$

$$\rightarrow a_i + a_j \geq a_{i+1} + a_{j-1} + \underbrace{(a_{i+1} + a_{j-1} - a_{i+2} - a_{j-2})}_{\geq 0}$$

بالاستقراء

$$\geq a_{i+1} + a_{j-1}$$

$$n(a_0 + a_{n+1}) \geq 2(a_1 + a_2 + \dots + a_n)$$

n مرة

$$\begin{cases} a_0 + a_{n+1} \geq a_1 + a_n \\ a_0 + a_{n+1} \geq a_2 + a_{n-1} \\ \vdots \\ a_0 + a_{n+1} \geq a_n + a_1 \end{cases}$$

$$\rightarrow n(a_0 + a_{n+1}) \geq 2(a_1 + a_2 + \dots + a_n)$$

Problem 2. Let $a_0, a_1, ..., a_N$ be real numbers satisfying $a_0 = a_N = 0$ and

$$a_{i+1} - 2a_i + a_{i-1} = a_i^2 \qquad \underline{a_{i+1} + a_{i-1} \geq 2a_i}$$

for $i = 1, 2, ..., N-1$. Prove that $a_i \leq 0$ for $i = 1, 2, ..., N-1$.

$$\underbrace{\sum_{i=1}^{N-1} a_i^2}_{\geq 0} = \sum_{i=1}^{N-1} a_{i+1} - 2a_i + a_{i-1} = a_N - a_{N-1} - a_1 + a_0$$

$$= -(a_{N-1} + a_1)$$

$$\Rightarrow \quad a_1 + a_{N-1} \leq 0 \quad \to (1)$$

$$\underbrace{\sum_{i=1}^{k} a_i^2}_{\geq 0} = \sum_{i=1}^{k} a_{i+1} - 2a_i + a_{i-1} = a_{k+1} - a_k - a_1$$

$$\Rightarrow \quad a_1 + a_k \gtrsim a_{k+1} \quad \to (2)$$

$$(*) \quad \boxed{\underbrace{\sum_{i=k+1}^{N-1} a_i^2}_{\geq 0}} = \cancelto{0}{a_N} - a_{N-1} - a_{k+1} + a_k$$

$$\Rightarrow \quad a_{N-1} + a_{k+1} \geq a_k \quad \to (3)$$

$(1),(2),(3) \to a_1 + a_{N-1} = 0$

$(3) \to a_{k+1} \geq a_k + a_1$, $(2) \Rightarrow \underline{a_{k+1} = a_k + a_1}$

$(*) \Rightarrow a_i = 0 \quad \forall\, i = 0, 1, \ldots, N$

حالة المساواة

فرض $M = a_k = \max\{a_i\}$

ازای $k = 0, N \Leftarrow a_i \le 0$ کل i.

ازای $1 \le k \le N-1$:

$$a_k^2 = a_{k+1} - 2a_k + a_{k-1}$$
$$= a_{k+1} + a_{k-1} - 2M \le M + M - 2M = 0$$

$$\rightarrow a_k^2 \le 0$$
$$M^2 \le 0 \quad \Rightarrow \quad a_i = 0 \quad \forall i.$$

Problem 3. Given $x_1, x_2, ..., x_n$ real numbers, prove that there exists a real number y such that

$$\{y - x_1\} + \{y - x_2\} + ... + \{y - x_n\} \le \frac{n-1}{2}.$$

where $\{x\} = x - \lfloor x \rfloor$.

نفرض التناقض

$y = x_i$: $\{x_i - x_1\} + \{x_i - x_2\} + \dots + \{x_i - x_n\} > \frac{n-1}{2}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \{x_i - x_j\} > \frac{n(n-1)}{2}$$

$$\underbrace{\sum_{\substack{i \neq j \\ \in \{1,2,\dots,n\}}} \left(\{x_i - x_j\} + \{x_j - x_i\} \right)}_{\le \frac{n(n-1)}{2}} > \frac{n(n-1)}{2}$$

$\{a\} + \{-a\} \in \{0, 1\}$

$\{x\} = x - \lfloor x \rfloor$

$0 \le \{z\} < 1$

$\{a\} + \{-a\} \le 1$

$\le \frac{n(n-1)}{2}$

تناقض

Problem 4. Determine all integers $n > 1$ for which the inequality

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holds for all reals $x_1, x_2, ..., x_n$.

$$\underbrace{x_1^2 - x_1x_n}_{+\frac{1}{4}x_n^2} + \underbrace{x_2^2 - x_2x_n} + \dots + \underbrace{x_{n-1}^2 - x_{n-1}x_n} + x_n^2 \geq 0$$

$$\Leftrightarrow \left(x_1 - \frac{x_n}{2}\right)^2 + \left(x_2 - \frac{x_n}{2}\right)^2 + \dots + \left(x_{n-1} - \frac{x_n}{2}\right)^2 - \frac{n-1}{4}x_n^2 + x_n^2 \geq 0$$

$$\underbrace{\sum_{i=1}^{n-1} \left(x_i - \frac{x_n}{2}\right)^2}_{\geq 0} \geq \frac{n-5}{4} x_n^2 \qquad \forall x_i \in \mathbb{R}$$

$$\to n \leq 5$$

$$\boxed{n \in \{2,3,4,5\}}$$

دائماً المتباينة صحيحة

$$x_i = \frac{1}{2} \quad i = 1, 2, \to n-1, \quad x_n = 1$$

$$0 \geq \frac{n-5}{4} \quad \text{if } n > 5 \quad \longrightarrow$$ تناقض