— Combinatorics for L4 —

— January 2020 — Counting —

Problem 16.

Consider a simple graph with $n \geq 2$ vertices and $m \geq 1$ edges. Define *trianglity* of an edge to be the number of triangles to which this edge belongs. Denote by $\triangle$ the average edge trianglity in the given graph. Prove that

$$\frac{4m - n^2}{n} \leq \triangle < \sqrt{2m}.$$

Problem 17.

Given are positive integers n and k. During a party n guests met, some of whom are friends. It is known that no one has more than $2k$ friends, but every pair of guests who are not friends have at least k common friends. Prove that $n \leq 6k$.

Problem 18.

There are n people at a party. Prove that it is possible to choose two of them in such a way that among the remaining $n-2$ people, there are at least $\lfloor \frac{n}{2} \rfloor - 1$ who are either all friends or all strangers to the chosen two.

Problem 19.

In a tournament took part $2n+1$ participants. Every two of them played exactly one match and only one of them won (there were no tie games). Suppose everyone won exactly n times. Call three participants a *cyclic triangle* if each of them won with one of the remaining two and lost with the other one. Prove that the number of cyclic triangles in this tournament equals $1^2 + 2^2 + \ldots + n^2$.

Problem 20.

Let n and k be positive integers and let S be a set of n points in the plane such that:

(1) no three points in S are collinear, and

(2) for every P in S, there are at least k points in S equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.