Training Session in Mathematics February-March 2022

Level 3

Nikola Petrović

Lesson 1

Bijective Proofs

Problems:

1. Which subsets of $\{1, 2, \dots, n\}$ are more numerous: subsets in which no two elements are consecutive numbers or subsets which can be broken up into disjoint pairs of consecutive numbers?

2. Let p be a prime number and k a positive integer less than p. Show that the number of subsets of $\{1, 2, \dots, p\}$ whose sum is divisible by p is by 2 larger than the number of subsets whose sum has a remainder k when divided by p. Use this to find the number of subsets of $\{1, 2, \dots, p\}$ whose sum is divisible by p.

3. Let A_i be the number of subsets of $\{1, 2, \dots, 3n\}$ whose sum has a remainder i when divided by 3. Show that $A_0 - 2^n = A_1 = A_2$ and use this to find A_0, A_1 and A_2.

4. A partition is a decomposition of a positive integer n into a sum of (not necessarily distinct) positive integers called parts or terms. For example, $9 = 4 + 3 + 1 + 1$ is a partition of number 9 into 4 parts. Two partitions that only differ in the order of terms are considered equal. Show that the number of partitions of n into k parts is equal to the number of partitions of $n + \binom{k}{2}$ into k distinct parts for $n \geq k$.

5. Let n be a positive integer. Prove that the number of partitions of n equals the number of partitions of $2n$ with n parts.

6. Let n and k be a positive integers. Prove that the number of partitions with at most k parts is equal to the number of partitions of n of parts which are not larger than k.

7. Let n be a positive integer. Show that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts. b - se 2

8. Let n be a positive integer. Show that the number of partitions of n into numbers not divisible by 3 is equal to the number of partitions of n where no 3 parts are equal. b - se 3

9. A partition is called good if, having sorted the parts into a non-increasing sequence, for each k the k-th part is equal to the number of parts at least as large as k. Show that the number of good partitions of n is equal to the number of partitions of n into distinct odd numbers.

10. Let $\{a_n\}_{n \geq 0}$ be a non-decreasing, unbounded sequence of non-negative integers with $a_0 = 0$. Let the number of members of the sequence not exceeding n be b_n. Prove that

$$(a_0 + a_1 + \cdots + a_m)\overset{+}{}(b_0 + b_1 + \cdots + b_n) \geq (m + 1)(n + 1).$$

11. A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid. $3\left(\binom{n+1}{3} + \binom{n+1}{4}\right) = 3\binom{n+2}{4}$

HW

12. An n-term sequence $(x_1, x_2, \dots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

13. Let ~~N~~ n be an integer greater than 1 and let T_n be the number of non empty subsets S of $\{1, 2, \ldots\ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

14. Let n be a positive integer. A proper latice path is a path connecting two points on the lattice that takes only unit up and right steps. A good proper path is a proper path which does not pass above the line connecting the two points (though it may contain points on the line itself).

    (a) Show that the number of proper paths from $(0, 0)$ to (n, n) which are *not* good is equal to the number of proper paths from $(-1, 1)$ to (n, n).

    (b) Use the result in the previous section to find the number of good proper paths from $(0, 0)$ to (n, n).

15. Show that the number of steps determined in the previous problem is equal to:

    (a) The number of binary trees with $n + 1$ leaves. (A binary tree, inductively defined, consists of either a lone root point, called a leaf, or a root point connected to two roots of two, smaller, binary trees.)

    (b) The number of ordered trees with $n + 1$ vertices. (An ordered tree, inductively defined, consists of either a lone root point, or a root point connected to several, i.e. one or more, root points of ordered trees which are arranged in a specific order. For example, if there are three branches from the root, and one further branch, the cases where the extra branch is on the left, middle and right branch of the root will be regarded as three different ordered trees.)

    (c) The number of triangulations of a $(n + 2)$-gon.

    (d) The number of ways to tile a stair-step shape of size n ($n, n-1, n-2, \ldots, 1$ fields respectively in each row starting from the same vertical line) with n rectangles.

    ~~(e) The number of ways to pair $2n$ points on the circle into n chords so that no two chords~~ intersect.

    (f) The number of ways to decompose the set of vertices of a regular n-gon into subsets whose convex hulls do not intersect each other.

    (g) The number of permutations without an increasing subsequence of length 3. (The three terms do not have to be consecutive).

    (h) The number of ways of labeling a $2 \times n$ board with numbers from 1 to $2n$ such that each row and each column is increasing.

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Lesson 2

Generating functions

Problems:

1. Use generating functions to prove that the number of ways of representing n as a sum of distinct natural numbers is equal to the number of ways of representing n as a sum of odd numbers.

2. Let p be an odd prime number. Use generating functions to find the number of subsets of $\{1, 2, \dots, p\}$ whose sum of elements is divisible to p.

3. Catalan numbers are numbers c_n satisfying $c_0 = 1$ and for $n > 0$, $c_n = \sum_{i=1}^{n-1} c_i c_{n-1-i}$. Find the generating function for the Catalan numbers and use it to find the formula for them as a function of n.

4. Show that each of the following numbers satisfies the recurrence relation for Catalan numbers $c_n = \sum_{i=1}^{n-1} c_i c_{n-1-i}$:

   (a) The number of binary trees with $n+1$ leaves. (A binary tree, inductively defined, consists of either a lone root point, called a leaf, or a root point connected to two roots of two, smaller, binary trees.)

   (b) The number of ordered trees with $n+1$ vertices. (An ordered tree, inductively defined, consists of either a lone root point, or a root point connected to several, i.e. one or more, root points of ordered trees which are arranged in a specific order. For example, if there are three branches from the root, and one further branch, the cases where the extra branch is on the left, middle and right branch of the root will be regarded as three different ordered trees.)

   (c) The number of triangulations of an $(n+2)$-gon.

   (d) The number of ways to tile a stair-step shape of size n ($n, n-1, n-2, \dots, 1$ fields respectively in each row starting from the same vertical line) with n rectangles.

   (e) The number of ways to pair $2n$ points on the circle into n chords so that no two chords intersect.

   (f) The number of ways to decompose the set of vertices of a regular n-gon into subsets whose convex hulls do not intersect each other.

   (g) The number of permutations without an increasing subsequence of length 3. (The three terms do not have to be consecutive).

   (h) The number of ways of labeling a $2 \times n$ board with numbers from 1 to $2n$ such that each row and each column is increasing.

5. Let n be positive integer and fix $2n$ distinct points on a circle. Determine the number of ways to connect the points with n arrows (oriented line segments) such that all of the following conditions hold: each of the $2n$ points is a startpoint or endpoint of an arrow; no two arrows intersect; and there are no two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that A, B, C and D appear in clockwise order around the circle (not necessarily consecutively).

6. Find the generating function of the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and use it to find the closed formula (not depending on other terms) for F_n using the roots of $x^2 + x - 1 = 0$.

7. For any positive n prove that the number of natural numbers whose digits are from the set $\{1,3,4\}$ and whose sum of digits is $2n$ is a perfect square. For example for $n = 3$ there are $9 = 3^2$ such numbers: 111111, 1113, 1131, 1311, 3111, 33, 114, 141, 411. (Hint: $x^4+x^3+x-1 = (x^2+1)(x^2+x-1)$.

8. A sequence is defined by $a_1 = 1$, and $a_{n+1} = 2a_n + n$. Using a generating function, find the formula for a_n.

9. Find the generating function for the following sequences:

   (a) $a_n = \lfloor \frac{n+2}{2} \rfloor$

   (b) $a_n = 3^n + n$

   (c) $a_n = \binom{n}{k}$ as a function of k

10. Let $f_1 = 1$, $f_{2n} = f_n$ and $f_{2n+1} = f_n + f_{n+1}$ for all positive integer n. Prove that the generating function $F(x) = \sum_{k=1}^{\infty} f_k x^{k-1}$ satisfies: $F(x) = (1+x+x^2)F(x^2)$.

11. Let $G(x) = \frac{1}{1-2x-x^2}$ be the generating function of a sequence $a_n, n \geq 0$. Prove that for each n: $a_n^2 + a_{n+1}^2 = a_{2n+2}$.

12. Find the sum of the number of non-negative integer solutions of each of the following equations: $x + 2y = n$, $2x + 3y = n-1, \ldots, nx + (n+1)y = 0$.

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Lesson 3

Combinatorial geometry

Problems:

1. The faces of a polyhedron are colored in either black or white so that there are more black faces than white faces and no two black faces share an edge. Prove that it is impossible to inscribe a sphere inside the polyhedron.

2. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

4. A 100 circles of radius one are given in the plane. A triangle formed by the centres of any three given circles has area at most 2017. Prove that there is a line intersecting at least three of the circles.

5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and $60°$ rotational symmetry (that is, there is a point O such that a $60°$ rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2 + 3n + 1$ citizens at $3n^2 + 3n + 1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

6. Consider a rectangle R partitioned into 2016 smaller rectangles such that the sides of each smaller rectangle is parallel to one of the sides of the original rectangle. Call the corners of each rectangle a vertex. For any segment joining two vertices, call it basic if no other vertex lie on it. (The segments must be part of the partitioning.) Find the maximum/minimum possible number of basic segments over all possible partitions of R.

7. Does there exist a set S of 100 points in a plane such that the center of mass of any 10 points in S is also a point in S?

8. Cover the following sets with mutually disjoint line segments (Open sets are denoted with '(' and ')' and closed sets are denoted with '[' and ']'):

   (a) Closed triangular area without one point in the interior: $[ABC] \setminus E,\ E \in (ABC)$.

   (b) Closed triangle: $[ABC]$.

   (c) Closed n-gon: $[A_1A_2 \dots A_n]$.

   (d) Closed circle: $[k]$.

   (e) Open triangular area: (ABC).

   (f) Open n-gon: $(A_1A_2 \dots A_n)$.

(g) Open circular area: (k).

9. Finitely many unit circles are given in a plane such that the area of their union is S. Prove that it is possible to select a subset of these circles such that no two circles in the subset intersect with each other and the area of these circles is at least $2S/9$.

10. There are 2019 points given in the plane. A child wants to draw k (closed) discs in such a manner, that for any two distinct points there exists a disc that contains exactly one of these two points. What is the minimal k, such that for any initial configuration of points it is possible to draw k discs with the above property?

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Lesson 4

Double Counting and Game Theory

Problems:

1. At a contest there are m candidates and $n \geq 3$ judges where n is an odd integer. Each candidate is evaluated by a judge as either passing or failing. Suppose that each pair of judges agrees on at most k candidates. Prove that: $\dfrac{k}{m} \geq \dfrac{n-1}{2n}$.

2. Given are n pairwise intersecting convex k-gons on the plane. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

3. An table with m rows and n columns is given where n is even and where every number inside each cell is either 0 or 1. Every two rows are different in at least $1 + n/2$ cells. Prove that $m \leq 1 + n/2$ and that the equality cannot hold when n is divisible by 4.

4. At a competition of n students, for each group of 3 students there is a student they are conspiring to get rid of. Any student who conspiring to get rid of student A in some group of three is an *enemy* of student A. Prove that there is a student with at least $\sqrt[3]{(n-1)(n-2)}$ enemies.

5. At a social gathering of students, every boy knows at least one girl. Prove that there exists a set S of at least half the students at the gathering such that every boy in S knows an odd number of girls in S.

6. Two players, A and B take turns in the following game with A starting first. On a table there are initially two piles with m and n coins, respectively. In each move, a player takes from one pile the number of coins which divides the number of coins on the other pile. The player who takes the last coin(s) on the table is the winner. For which values of (m, n) does player A have a winning strategy and for which values of (m, n) does player B have a winning strategy?

7. You are cheating at a trivia contest. For each question, you can peek at each of the $n > 1$ other contestants' guesses before writing down your own. For each question, after all guesses are submitted, the host announces the correct answer. A correct guess is worth 0 points. An incorrect guess is worth -2 points for other contestants, but only -1 point for you, since you hacked the scoring system. After announcing the correct answer, the host proceeds to read the next question. Show that if you are leading by 2^{n-1} points at any time, then you can surely win first place.

8. A snake is a set of fields on a board such that they can be arranged in a sequence such that each field of the snake is to the right or above the previous field. Find the number of ways to tile an $m \times n$ board with snakes.

9. Some squares of a $n \times n$ board ($n > 2$) are black, the rest are white. In every white square we write the number of all the black squares having at least one common vertex with it. Find the maximum possible sum of all these numbers.

10. Let n be an odd integer. Consider a square lattice of size $n \times n$, consisting of n^2 unit squares and $2n(n+1)$ edges. All edges are painted in red or blue so that the number of red edges does not exceed n^2. Prove that there is a cell that has at least three blue edges.

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Lesson 5

How to write mathematical proofs

Sample problem and solution:

There are n coins in one pile and two players A and B take turns with the following game. In each move a player whose turn it is may either take a coin from a pile and discard it, or divide a pile into (at least 2) equal piles. Player A starts first. When a player takes the last coin from a pile it gone. The winner is the player who takes the last coin off the table. For which n does player A have a winning strategy and for which n does player B have a winning strategy.

Solution. We will prove that player A has a winning strategy for all integers n except when n is an odd prime.

We first prove that a game must end. Assign to each coin the number of coins in its group as its value. Clearly, the total value on the table is finite and in each move it decreases by at least 1, thus it must eventually reach 0 since it can't ever be negative.

Now we proceed to analyze the individual cases for the value of n. The case $n = 1$ is trivial. For even numbers $n = 2k$, player A splits the coins into two equal piles and proceeds to play symmetrically, this always guaranteing him a move and thus a win.

In the case of $n = p$, an odd prime number, A has the choice of removing a coin, in which case B can apply the strategy for even numbers, or A splits the odd number of coins into groups of one, in which case the remaining moves are predetermined and B takes the last coin. Thus, A loses if n is an odd prime.

Finally, in the case of an odd composite number, we have $n = pq$ for some prime p and $q > 1$, In this case, player A divides the coins into piles of p and plays the prime number strategy for each individual pile as second player, since it is player B's turn.

Thus, we have proven that A has a winning strategy for all initial numbers of coins except for odd primes.

Tips on how to write solutions:

1. Organize your proof into discrete steps. Each step of the proof, unless one sentence short, should ideally get its own paragraph.

2. In the first paragraph, optionally write the topic sentence for the entire proof, for example: 'We will probe the problem using mathematical induction on n' or 'Let us assume the contrary' if you're going to prove the problem by contradiction.

3. Before starting the main body of the proof it is a good idea to define all terms that you will be using. Good definitions of terms can often make writing solutions much easier. For example, instead of repeatedly listing several properties your objects have to hold, you may use the adjective 'good' or any other suitable adjective to describe this class of objects.

4. Start each paragraph with a topic sentence. This sentence describes what you will prove in the paragraph.

5. Use adverbs and adverb phrases at the beginnings of sentences to help the reader navigate through the proof:

   - : A follows B: 'Thus,' 'Hence,' 'Therefore,' 'It follows that', 'From A, it follows that,' 'As a result,'...

- : B contrasts A: 'However,' 'On the other hand,' 'In contrast,'...
- : Listing of cases: 'First,' 'Second,' ..., 'Finally,' (for the last case).
- : Use 'Without loss of generality,' when it is convenient to make an arbitrary choice, where the proof for the other choice(s) follows analogously.
- Use 'It suffices to prove' when you want to prove a stronger claim or an alternate equivalent claim.
- Use 'We have thus shown that' when you want to summarize your result in order to prepare the reader for the subsequent arguments.
- Use 'Note that' when a result emerges from your proof that's not immediately important, but will be important later in the proof.
- Use 'This completes the proof.' if the proof has come to an end, but it's not immediately obvious. (For example, if the final argument proven is far from the statement of the problem.)
- Use 'Let A be...' when you want to define A.
- Use 'We assume that...' when making an assumption for a subset of the proof. Use 'We now assume that...' when the assumptions change and you're now making a new set of assumptions for the next case.

6. Pictures allowed, but must clearly be stated and described. For example: 'We will color the squares of the chessboard as in the picture.'

7. All theorems that have a name are useable. However, in many cases, especially if the theorem substantially helps to solve the problem, you will be required not only to refer to the theorem, but to refer to the system to which it is applied and how it is applied. (e.g. 'We apply Turan's theorem for the set of n mathematicians and $k = 4$ to conclude that if no 4 mathematicians are friends with each other there are at most $\frac{n^2}{2}(1-\frac{1}{k-1}) = \frac{n^2}{2}(1-\frac{1}{3}) = \frac{n^2}{3}$ friendships among them.')

8. Always write using legible handwriting.

9. If in doubt, include something even if it is obvious. If it disrupts the flow of the proof, you can choose to quickly write the proof of the statement in brackets. (e.g. 'We have that $(n-2)(n-4)(n-6)$ is divisible by 3 (since each of the three terms gives a different remainder modulo 3).')

10. If necessary, it is good to summarize the proof with a final statement.