Saudi Arabia 2022
Level 3
Geometry – Homothety 2
Regis

Example 1: Let AB be a chord of a circle Ω. Let ω be a circle tangent to chord AB at K and internally tangent to ω at T. Then ray TK passes through the midpoint M of the arc AB of Ω not containing T.
Moreover, $MA^2 = MB^2$ is the power of M with respect to ω.

Example 2: (Pascal's Theorem) Given 6 points (which can be coincident) on the circumference of a circle labelled A, C, E, B, F, and D in that order around the circle, the intersections of AB and DE, AF and CD, and BC and EF are collinear.

Problems

7. (IMO/1978) In ΔABC, $AB = AC$. A circle is tangent internally to the circumcircle of ABC and also to the sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of ΔABC.

8. (IMO/1992) In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exist two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of ΔPQR.

9. Let I, G and N be the incenter, the centroid and the Nagel Point of triangle ABC. Prove that I, G and N are collinear and that $\frac{\overline{GN}}{\overline{GI}} = -2$. *The excircles touch sides BC, CA and AB at points X, Y and Z and the Nagel Point is the concurrent point of AX, BY and CZ.*

10. (IMO/1983) Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

11. (IMO/1999) Two circles Γ_1 and Γ_2 are inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the center of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively. Prove that CD is tangent to Γ_2.

12. (APMO/2000) Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA respectively and O the point in which the perpendicular at P to BA meets AN produced. Prove that QO is perpendicular to BC.