Problems Set

23 June 2020

Introductory Problems:

1. Let $1, 4, \ldots$ and $9, 16, \ldots$ be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

2. Given a sequence of six strictly increasing positive integers such that each number (besides the first) is a multiple of the one before it and the sum of all six numbers is 79, what is the largest number in the sequence?

3. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

4. Those irreducible fractions!

   (1) Let n be an integer greater than 2. Prove that among the fractions

   $$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n},$$

   an even number are irreducible.

   (2) Show that the fraction

   $$\frac{12n+1}{30n+2}$$

   is irreducible for all positive integers n.

5. A positive integer is written on each face of a cube. Each vertex is then assigned the product of the numbers written on the three faces intersecting the vertex. The sum of the numbers assigned to all the vertices is equal to 1001. Find the sum of the numbers written on the faces of the cube.

6. Call a number *prime looking* if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

Problems Set

27 June 2020

Introductory Problems:

7. A positive integer k greater than 1 is given. Prove that there exist a prime p and a strictly increasing sequence of positive integers $a_1, a_2, \ldots, a_n, \ldots$ such that the terms of the sequence

$$p + ka_1,\ p + ka_2,\ \ldots,\ p + ka_n,\ \ldots$$

are all primes.

8. Given a positive integer n, let $p(n)$ be the product of the nonzero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of S?

9. Let m and n be positive integers such that

$$\operatorname{lcm}(m, n) + \gcd(m, n) = m + n.$$

Prove that one of the two numbers is divisible by the other.

10. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

11. Show that for any positive integers a and b, the number

$$(36a + b)(a + 36b)$$

cannot be a power of 2.

12. Compute the sum of the greatest odd divisor of each of the numbers 2006, 2007, ..., 4012.

13. Compute the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 27000.

Problems Set

23/27 June 2020

Introductory Problems:

14. L.C.M of three numbers.

    (1) Find the number of ordered triples (a, b, c) of positive integers for which $\operatorname{lcm}(a, b) = 1000$, $\operatorname{lcm}(b, c) = 2000$, and $\operatorname{lcm}(c, a) = 2000$.

    (2) Let a, b, and c be integers. Prove that

    $$\frac{\operatorname{lcm}(a, b, c)^2}{\operatorname{lcm}(a, b)\operatorname{lcm}(b, c)\operatorname{lcm}(c, a)} = \frac{\gcd(a, b, c)^2}{\gcd(a, b)\gcd(b, c)\gcd(c, a)}.$$

15. Let x, y, z be positive integers such that

    $$\frac{1}{x} - \frac{1}{y} = \frac{1}{z}.$$

    Let h be the greatest common divisor of x, y, z. Prove that $hxyz$ and $h(y - x)$ are perfect squares.

16. Let p be a prime of the form $3k + 2$ that divides $a^2 + ab + b^2$ for some integers a and b. Prove that a and b are both divisible by p.

17. The number 27000001 has exactly four prime factors. Find their sum.

18. Find all positive integers n for which $n! + 5$ is a perfect cube.

19. Find all primes p such that the number $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

20. Call a positive integer N a *7-10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

21. If $a \equiv b \pmod{n}$, show that $a^n \equiv b^n \pmod{n^2}$. Is the converse true?

22. Let p be a prime, and let $1 \le k \le p - 1$ be an integer. Prove that

    $$\binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

Problems Set

28 June, 2020

Introductory Problems:

23. Let p be a prime. Show that there are infinitely many positive integers n such that p divides $2^n - n$.

24. Let n be an integer greater than three. Prove that $1! + 2! + \cdots + n!$ cannot be a perfect power.

25. Let k be an odd positive integer. Prove that

$$(1 + 2 + \cdots + n) \mid (1^k + 2^k + \cdots + n^k)$$

for all positive integers n.

26. Let p be a prime greater than 5. Prove that $p - 4$ cannot be the fourth power of an integer.

27. For a positive integer n, prove that

$$\sigma(1) + \sigma(2) + \cdots + \sigma(n) \leq n^2.$$

28. Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i + j}{\gcd(i, j)}$$

is an element of S for all i and j (not necessarily distinct) in S.

29. Knowing that 2^{29} is a nine-digit number all of whose digits are distinct, without computing the actual number determine which of the ten digits is missing. Justify your answer.

30. Prove that for any integer n greater than 1, the number $n^5 + n^4 + 1$ is composite.

Problems Set

29 June, 2020

Introductory Problems:

31. The product of a few primes is ten times as much as the sum of the primes. What are these (not necessarily distinct) primes?

32. A 10-digit number is said to be *interesting* if its digits are all distinct and it is a multiple of 11111. How many interesting integers are there?

Advanced Problems:

1. (a) Prove that the sum of the squares of 3, 4, 5, or 6 consecutive integers is not a perfect square.

   (b) Give an example of 11 consecutive positive integers the sum of whose squares is a perfect square.

2. Let $S(x)$ be the sum of the digits of the positive integer x in its decimal representation.

   (a) Prove that for every positive integer x, $\frac{S(x)}{S(2x)} \leq 5$. Can this bound be improved?

   (b) Prove that $\frac{S(x)}{S(3x)}$ is not bounded.

Problems Set

4 July, 2020

Introductory Problems:

33. Do there exist 19 distinct positive integers that add up to 1999 and have the same sum of digits?

34. Fractions in modular arithmetic.

    (1) Let a be the integer such that

    $$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{23}=\frac{a}{23!}.$$

    Compute the remainder when a is divided by 13.

    (2) Let $p>3$ be a prime, and let m and n be relatively prime integers such that

    $$\frac{m}{n}=\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(p-1)^2}.$$

    Prove that m is divisible by p.

    (3) Let $p>3$ be a prime. Prove that Let $p>3$ be a prime. Prove that

    $$p^2 \mid (p-1)!\left(1+\frac{1}{2}+\cdots+\frac{1}{p-1}\right).$$

Advanced Problems:

3. Most positive integers can be expressed as a sum of two or more consecutive positive integers. For example, $24=7+8+9$ and $51=25+26$. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore *interesting*. What are all the interesting integers?

4. Set $S=\{105,106,\ldots,210\}$. Determine the minimum value of n such that any n-element subset T of S contains at least two non-relatively prime elements.

Problems Set

5 July, 2020

Introductory Problems:

35. Let $p \geq 3$ be a prime. Determine whether there exists a permutation

$$(a_1, a_2, \ldots, a_{p-1})$$

of $(1, 2, \ldots, p-1)$ such that the sequence $\{ia_i\}_{i=1}^{p-1}$ contains $p-2$ distinct congruence classes modulo p.

36. Let $n > 1$ be an odd integer. Prove that n does not divide $3^n + 1$.

37. Let a and b be positive integers. Prove that the number of solutions (x, y, z) in nonnegative integers to the equation $ax + by + z = ab$ is

$$\frac{1}{2}[(a+1)(b+1) + \gcd(a, b) + 1].$$

Advanced Problems:

5. Let m and n be integers greater than 1 such that $\gcd(m, n-1) = \gcd(m, n) = 1$. Prove that the first $m-1$ terms of the sequence $n_1, n_2, \ldots,$ where $n_1 = mn + 1$ and $n_{k+1} = n \cdot n_k + 1$, $k \geq 1$, cannot all be primes.

6. For a positive integer k, let $p(k)$ denote the greatest odd divisor of k. Prove that for every positive integer n,

$$\frac{2n}{3} < \frac{p(1)}{1} + \frac{p(2)}{2} + \cdots + \frac{p(n)}{n} < \frac{2(n+1)}{3}.$$