# Intensive Training

## Geometry

Day 3

27 March 2021

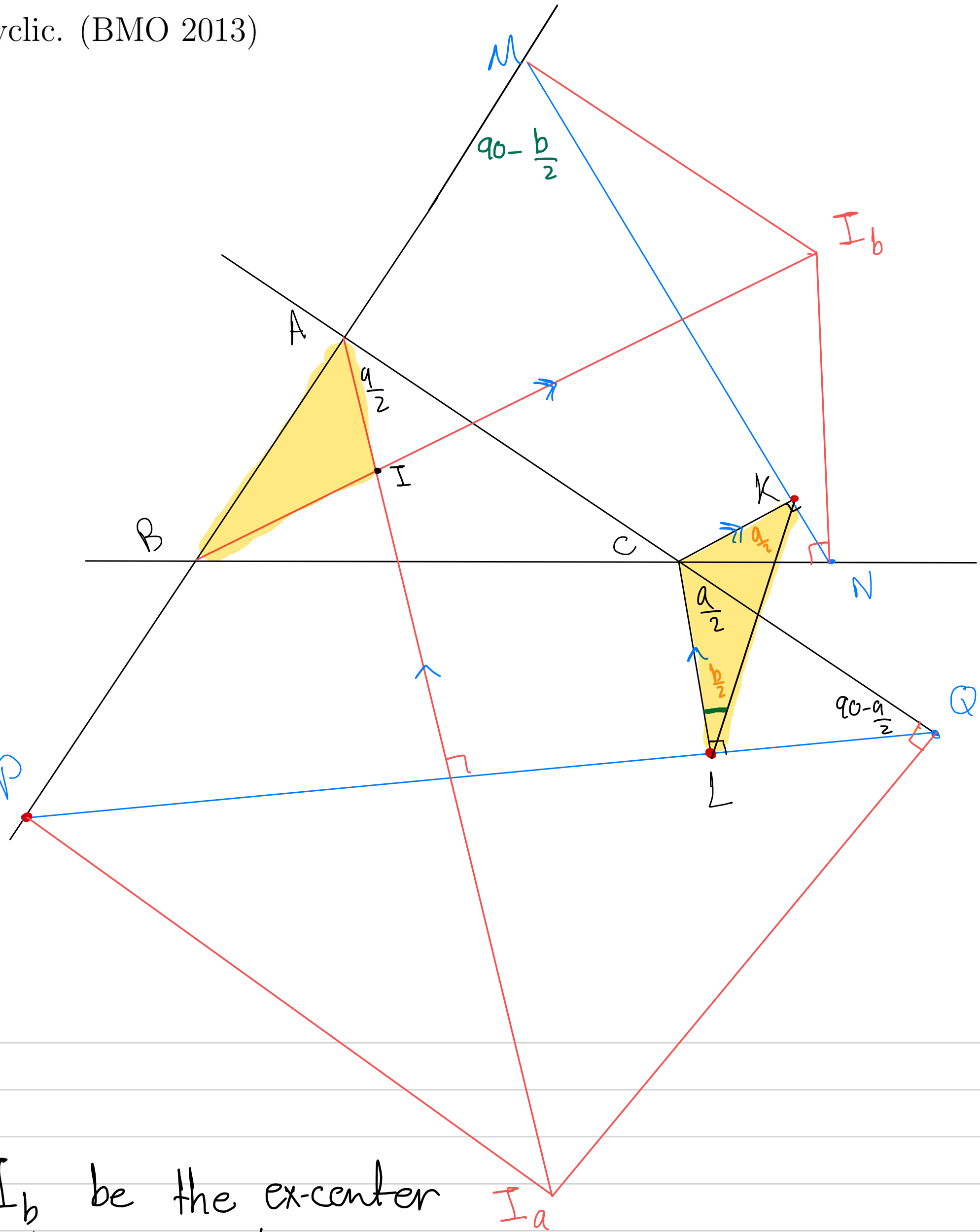
Includes solutions for:

BMO 2013 P1

EGMO 2018 P5

JBMO 2015 P3

9. In a triangle $ABC$, the excircle $\omega_a$ opposite $A$ touches $AB$ at $P$ and $AC$ at $Q$, while the excircle $\omega_b$ opposite $B$ touches $BA$ at $M$ and $BC$ at $N$. Let $K$ be the projection of $C$ onto $MN$ and let $L$ be the projection of $C$ onto $PQ$. Show that the quadrilateral $MKLP$ is cyclic. (BMO 2013)




Let $I_a$, $I_b$ be the ex-center with respect to $A$ and $B$. $\Rightarrow$ $API_aQ$ is a cyclic

$\angle AQP = \angle AI_aP = 90 - \angle I_aAP = 90 - \frac{a}{2}$

$\Rightarrow \angle LCQ = \frac{a}{2} \Rightarrow AI \parallel CL$ Similarly $BI \parallel CK$

We want to show that $\angle KLP = 180 - \angle NMP$

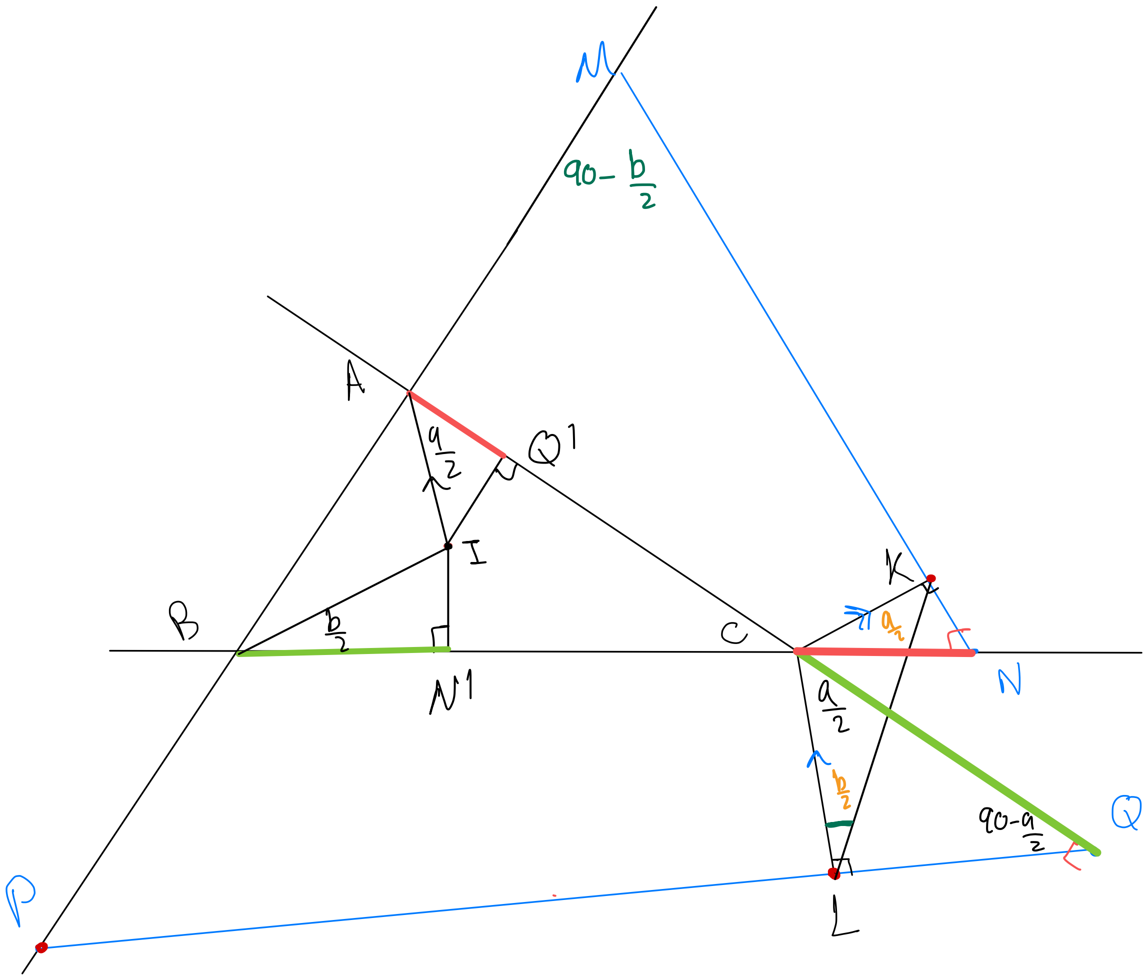
$\Leftrightarrow \angle KLP = 180 - (90 - \frac{b}{2})$

$\angle KLC = \frac{b}{2}$

It's sufficient to show that $\Delta AIB \sim \Delta KCL$

$\angle KCL = \angle I_aII_b = \angle AIB$

$$\frac{CL}{CK} = \frac{CQ \cos\frac{a}{2}}{CN \cos\frac{b}{2}} \begin{matrix} \to (\Delta CLQ) \\ \to (\Delta CKN) \end{matrix} \quad *$$




$$CQ = BN' \Rightarrow CQ = BI \cos\frac{b}{2}$$

Similarly $CN = AI \cos\frac{a}{2}$

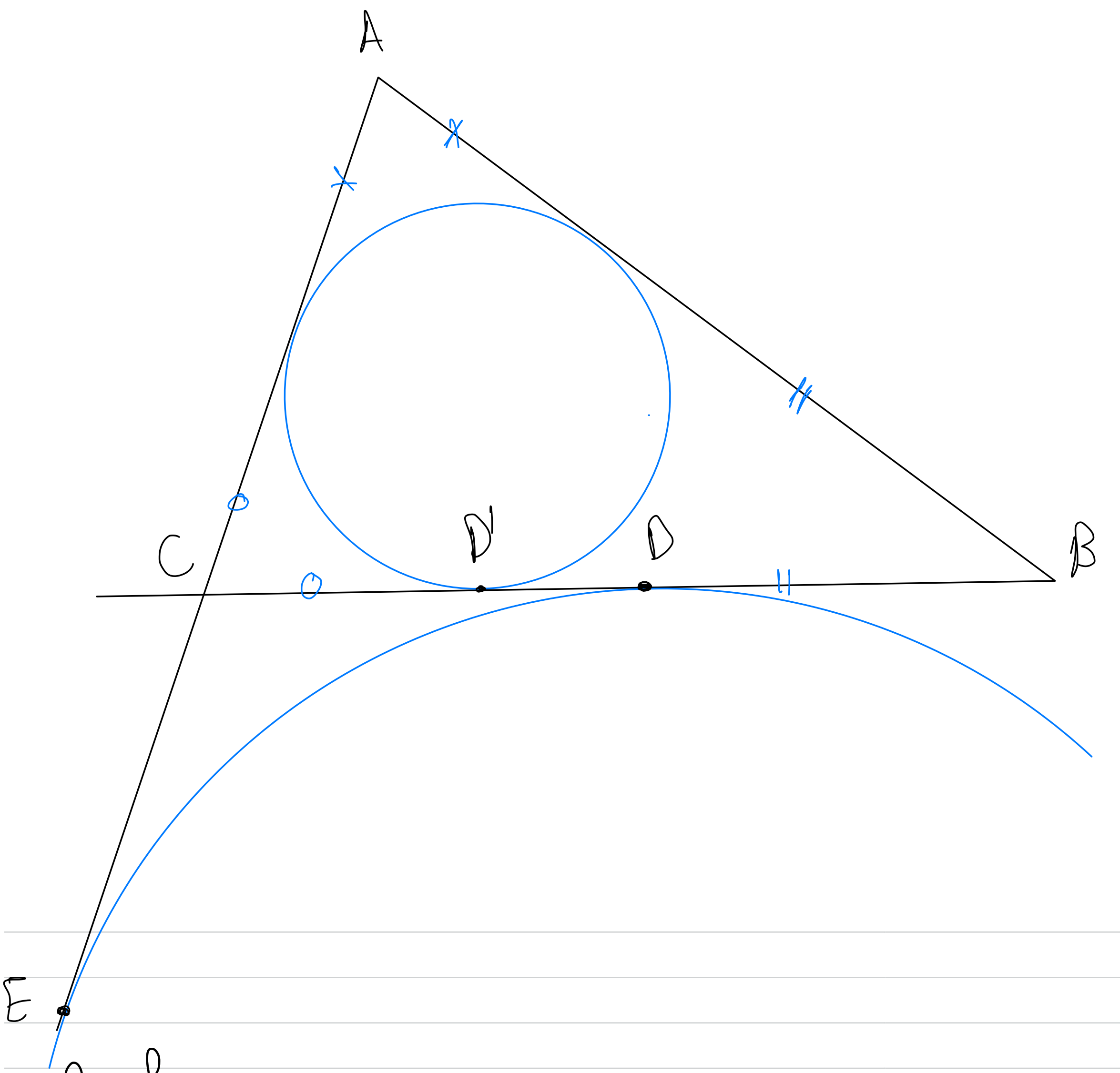
$$\Rightarrow \frac{CQ}{CN} = \frac{BI}{AI} \frac{\cos\frac{b}{2}}{\cos\frac{a}{2}}$$

In $*$:

$$\frac{CL}{CK} = \frac{BI}{AI} \frac{\cos\frac{b}{2}}{\cos\frac{a}{2}} \cdot \frac{\cos\frac{a}{2}}{\cos\frac{b}{2}} = \frac{BI}{AI}$$

Therefore, $\triangle KCL \sim \triangle AIB \Rightarrow KLPM$ cyclic $\square$

Lemma: in the following diagram, $BD = CD'$



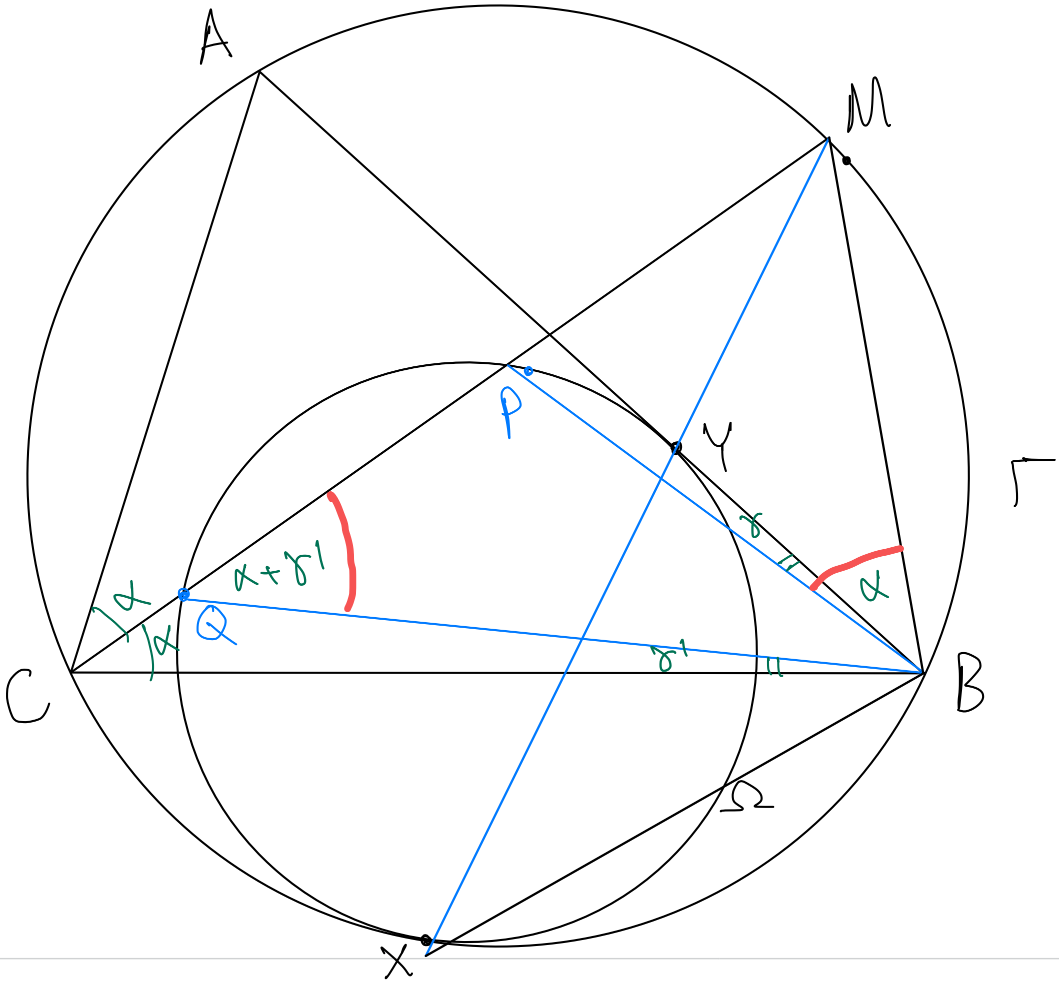

Proof:

$$\begin{cases} BD' = s - b \\ CD = s - b \end{cases}$$

left as a homework or already known.

$$\Longrightarrow CD = BD'$$

10. Let $\Gamma$ be the circumcircle of triangle $ABC$. A circle $\Omega$ is tangent to the line segment $AB$ and is tangent to $\Gamma$ at a point lying on the same side of the line $AB$ as $C$. The angle bisector of $\angle BCA$ intersects $\Omega$ at two different points $P$ and $Q$. Prove that $\angle ABP = \angle QBC$. (EGMO 2018 P5)




Let $M$ be the intersection of the angle bisector of $C$ with $\Gamma$. Assume that $\Omega$ is tangent to $\Gamma$ and $AB$ at $X, Y$

By the lemma, we knew that $X, Y, M$ are collinear

$MY \cdot MX = MP \cdot MQ$ (Power of $M$ w.r.t. $\Omega$)

We need to show that $\angle MBP = \angle MQB$

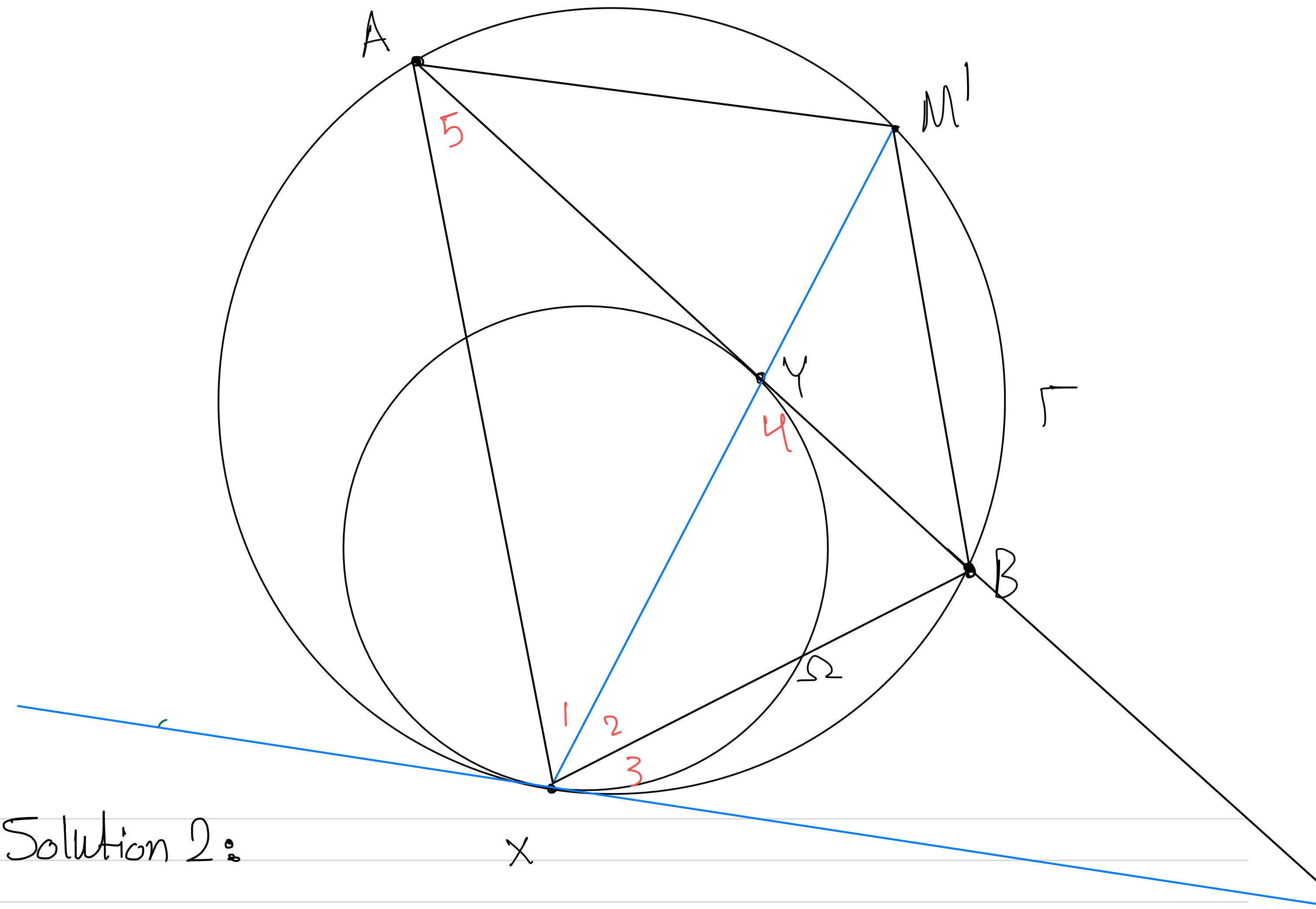
$$MB^2 = MP \cdot MQ$$

$\Leftrightarrow$ $MB^2 = MY \cdot MX \Leftrightarrow MB$ tangent to $(XYB)$

$\Leftrightarrow \angle BXY = \angle MBY$ ✓

Therefore $\angle MBP - \angle MBA = \angle MQB - \angle BCM \Leftrightarrow \angle ABP = \angle QBC$ $\square$

<u>Lemma</u>: $X, Y, M$ are collinear




Solution 2:

Let $M' = XY \cap \Gamma$ and $K = AB \cap$ tangent at $X$

$KY = KX \Rightarrow \angle 2 + \angle 3 = \angle 4$

$KX$ is tangent to $\Gamma \Rightarrow \angle 3 = \angle 5$ $\Big\}$ $\angle 2 + \angle 5 = \angle 4$ (1)

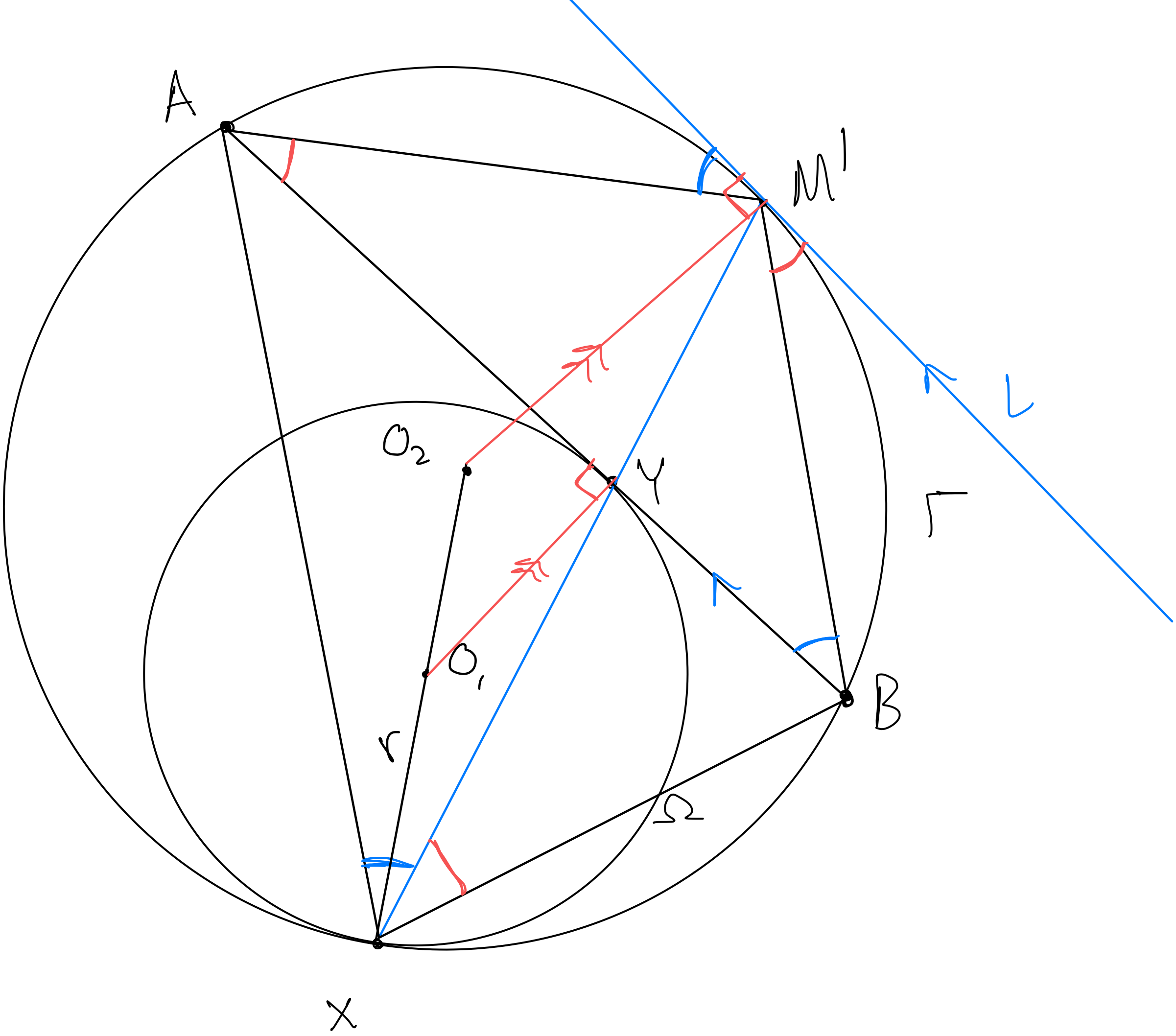
In $\triangle YXA$, $\angle 4 = \angle 1 + \angle 5$ (2)

From (1), (2)

$\angle 1 = \angle 2 \Rightarrow M'$ midpoint $\widehat{AB}$

$\Rightarrow M' = M$

Lemma: $X, Y, M$ are collinear




Solution 1:

Let $M' = XY \cap \Gamma$

By homothety from $\Omega$ to $\Gamma$, $L \parallel AB$. Another way.

let $O_1, O_2$ be the centers of $\Omega$ and $\Gamma$

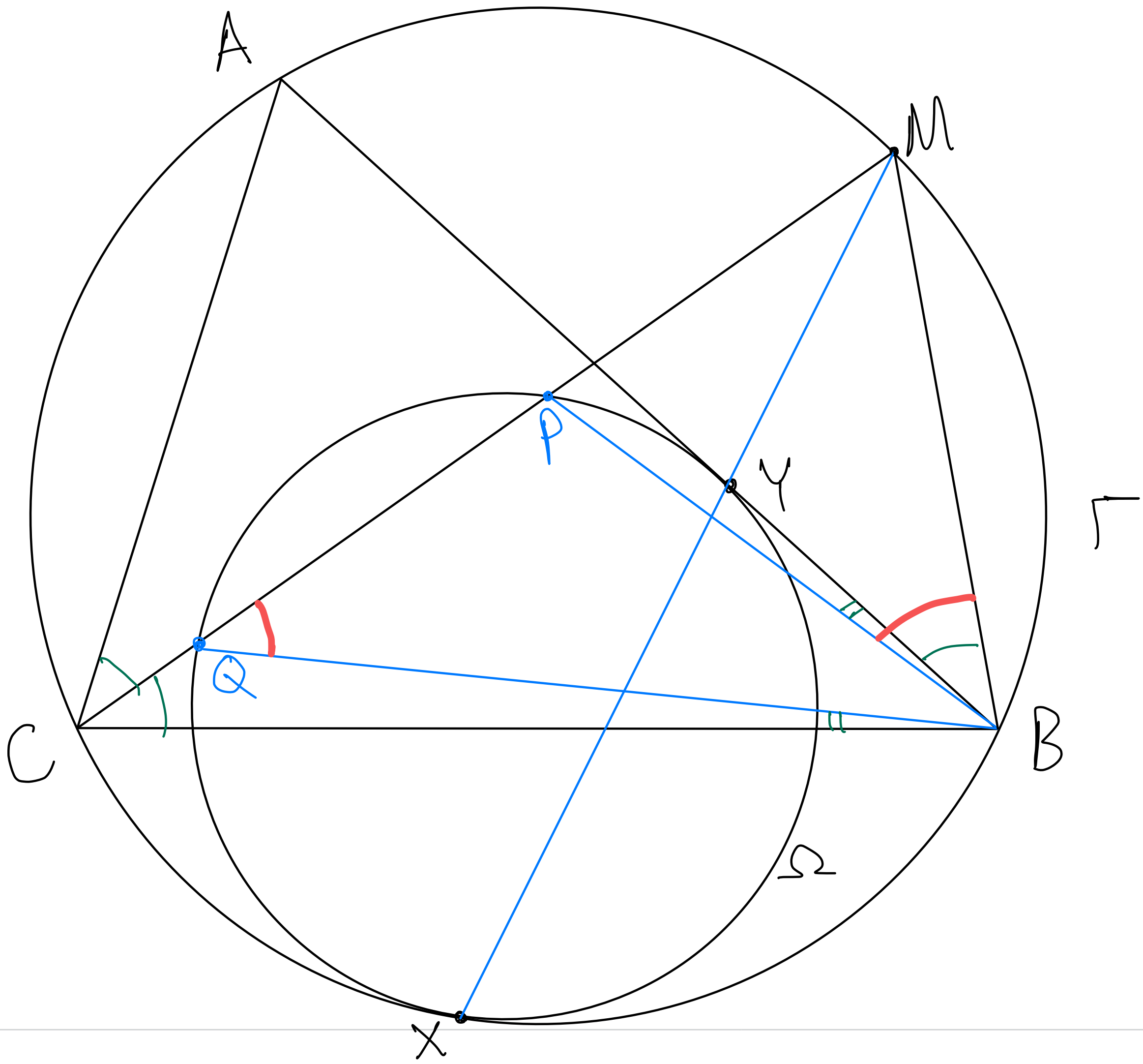
so $O_1, O_2, X$ are collinear

$$\frac{O_1X}{O_2X} = \frac{O_1Y}{O_2Y} \qquad \left(\text{because } \begin{array}{l} O_1X = O_1Y \\ O_2X = O_2Y \end{array}\right)$$

$\Rightarrow \triangle XO_1Y \sim \triangle XO_2M'$

$\Rightarrow O_1Y \parallel O_2M'$. but $O_1Y \perp AB$, $O_2M' \perp L$ $\Rightarrow AB \parallel L$

$\Rightarrow M' = M$ $\square$



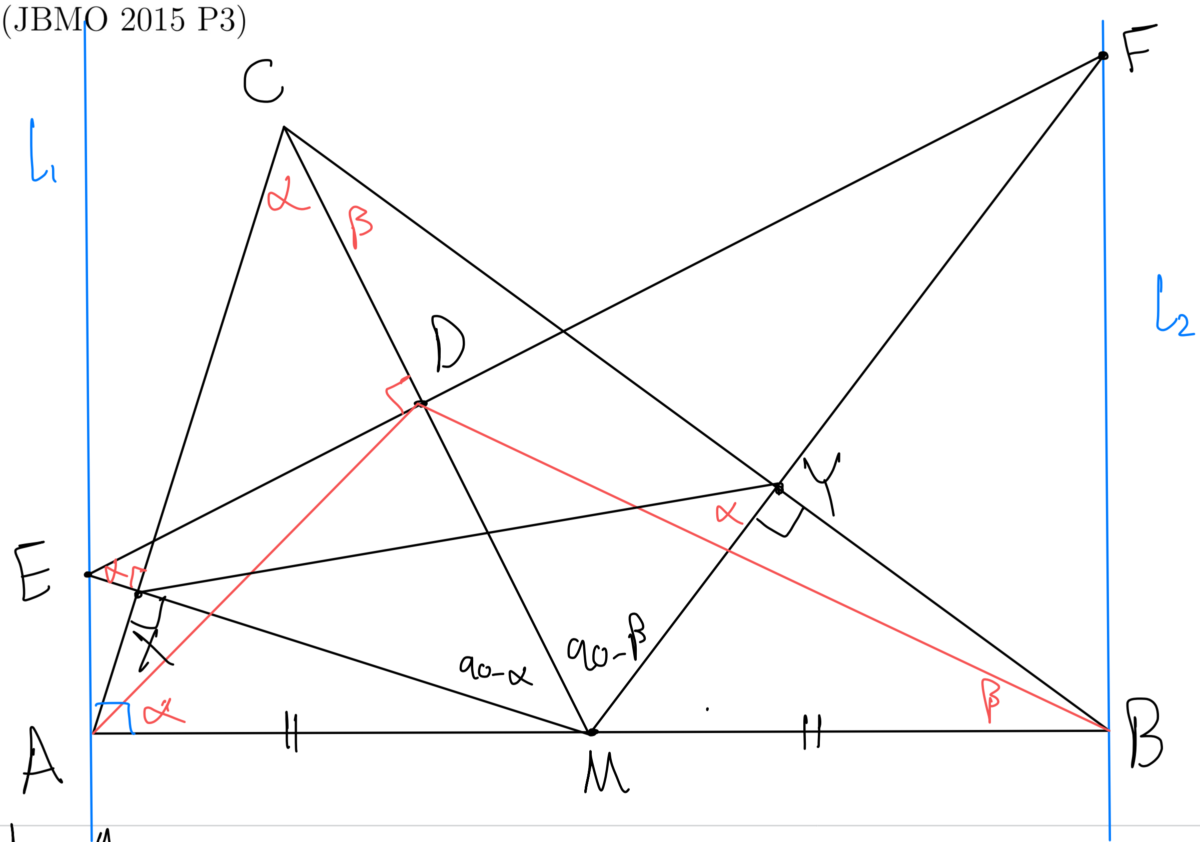

$$MB^2 = MP \cdot MQ$$

$$\Longleftrightarrow \quad MB^2 = MY \cdot MX \qquad \text{which's true}$$

11. Let $ABC$ be an acute triangle. The lines $l_1$ and $l_2$ are perpendicular to $AB$ at the points $A$ and $B$, respectively. The perpendicular lines from the midpoint $M$ of $AB$ to the lines $AC$ and $BC$ intersect $l_1$ and $l_2$ at the points $E$ and $F$, respectively. If $D$ is the intersection point of the lines $EF$ and $MC$, prove that

$$\angle ADB = \angle EMF.$$

(JBMO 2015 P3)




Solution 1:

$$\begin{cases} MA^2 = MX.ME \\ MB^2 = MY.MF \end{cases} \Rightarrow MX.ME = MY.MF \Rightarrow XYFE \text{ cyclic}$$

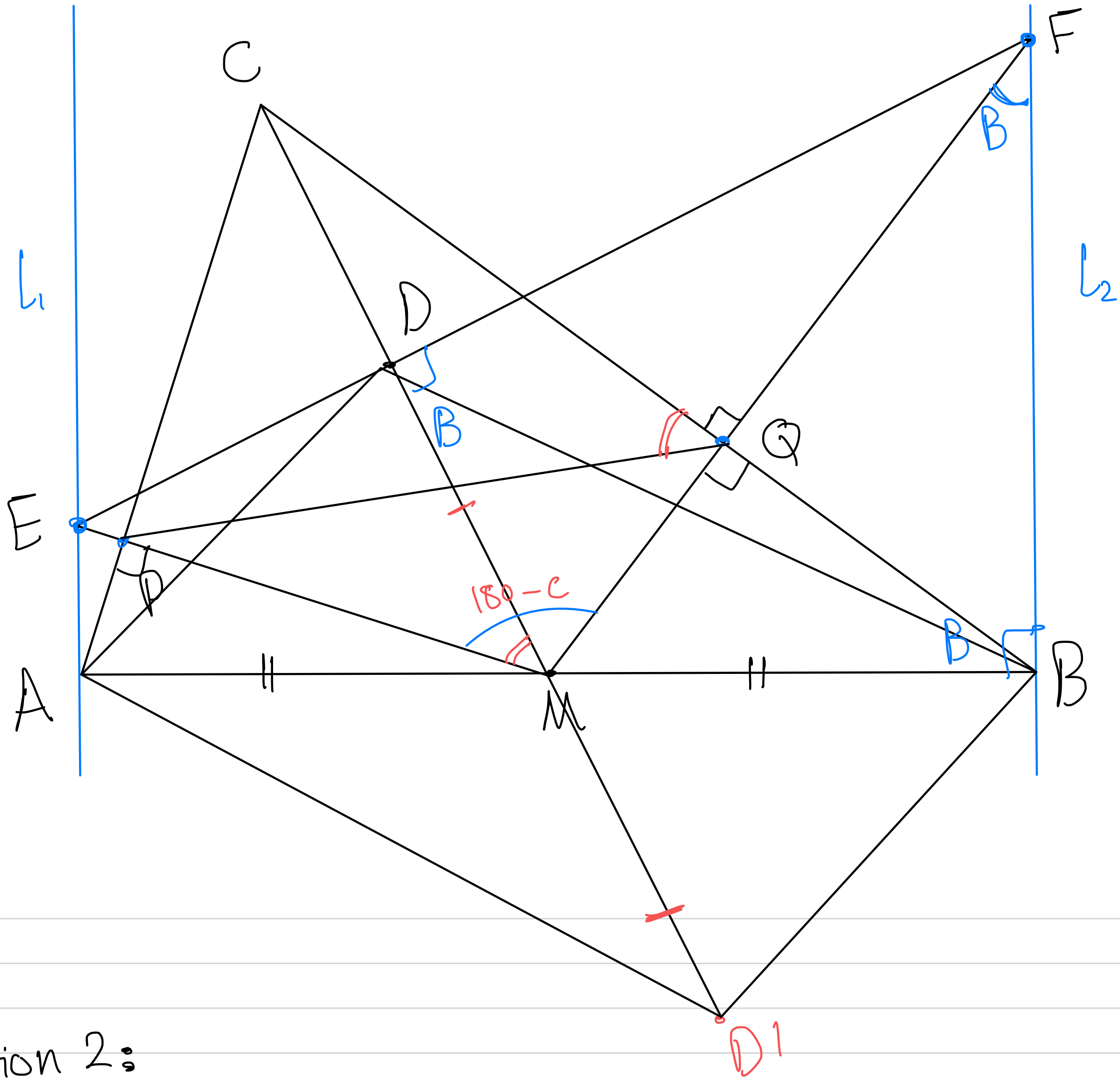
So $\angle FEX = \angle XYM$ (1)

However, $CXMY$ is a cyclic, so $\angle XYM = \angle XCM$ (2)

From (1), (2) $\angle FEX = \angle XCM$ (3) $\Rightarrow FE \perp CM$

$\Rightarrow EDMA$ cyclic $\Rightarrow \angle DAM = \angle FEX$
$= \angle XCM$
$= 90 - \angle XMC$

$\Rightarrow \angle DAM + \angle DBM = 180 - \angle EMF \Rightarrow \angle EMF = \angle BDA$ $\square$




Solution 2:

$CPMQ$ is a cyclic $\Rightarrow \angle PMQ = 180 - \angle C$

let $D'$ be on $CM$ such that $MD = MD'$

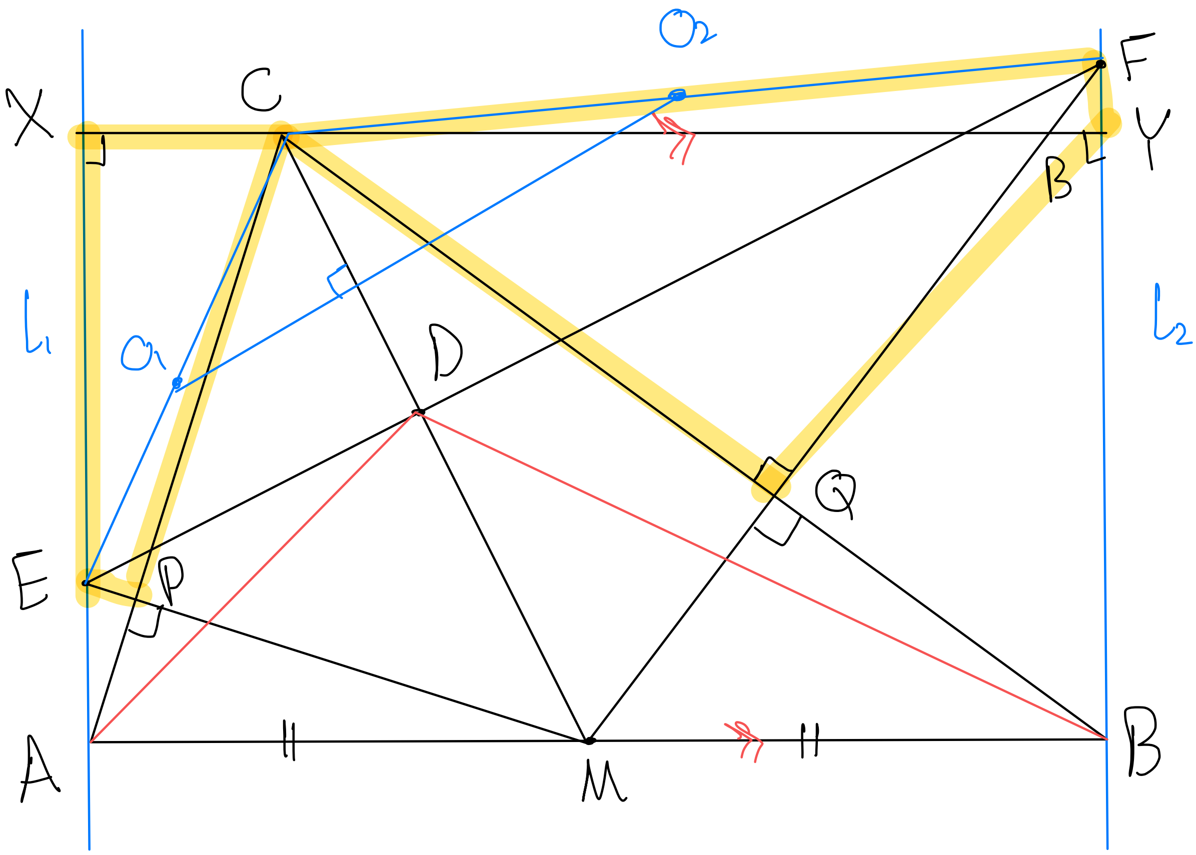
$AD'BC$ is a cyclic $\Leftrightarrow \angle CBA = \angle CD'A$

$\Leftrightarrow \angle CBA = \angle BDM$

$\Leftrightarrow \angle BFM = \angle BDM$

$\Leftrightarrow BFDM$ cyclic $\Leftrightarrow \angle FDM = 90$

$\Leftrightarrow EF \perp CM$




Proof 1 for $EF \perp CM$:

XCPE is a cyclic and YFCQ

$MP.ME = MA^2 = MB^2 = MQ.MY$

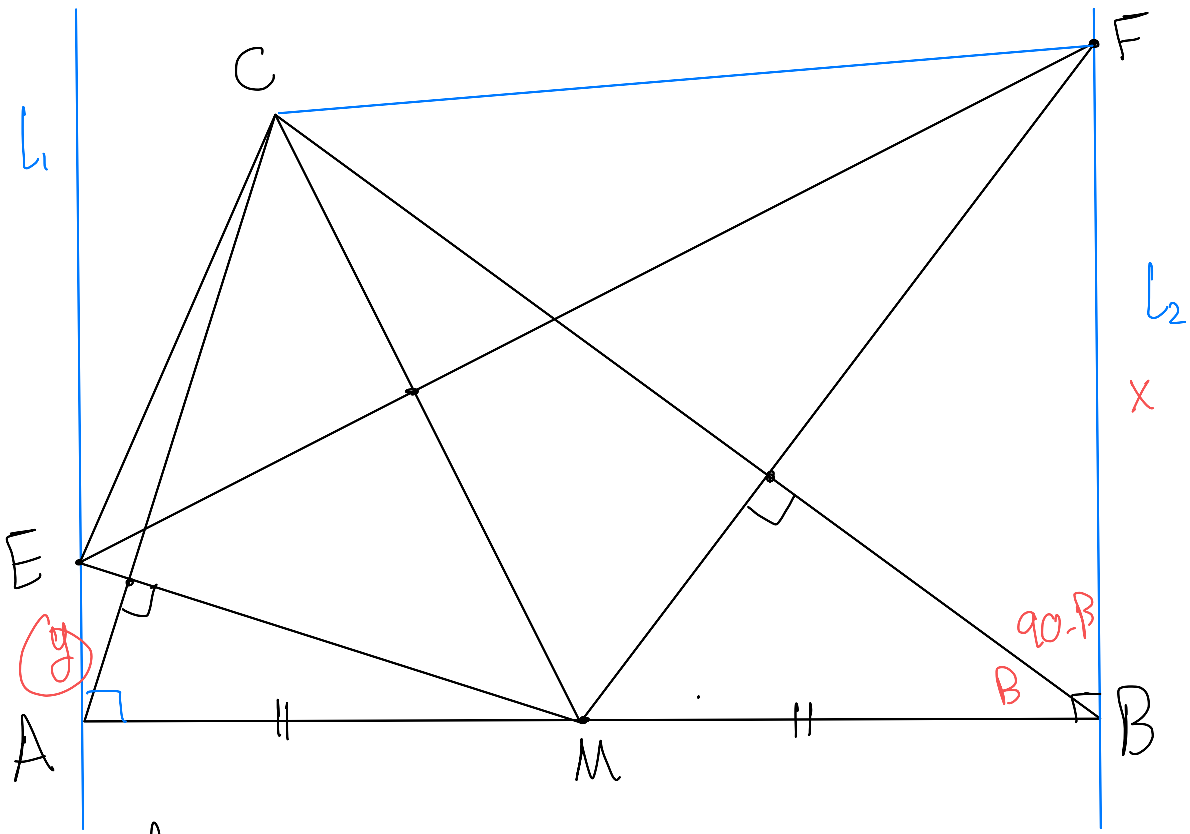
M is on the radical axis of (XCPE) and (YFCQ)

However, C is on the radical axis (intersection point)

MC is the radical axis of (XCPE) and (YFCQ)

$MC \perp O_1O_2$, $O_1$ midpoint of CE, $O_2$ mid point of CF $\Rightarrow$ $O_1O_2 \parallel EF$

Therefore $CM \perp EF$ $\square$




Proof 2 for $CM \perp EF$:

$CM \perp EF \iff CE^2 - CF^2 \overset{?}{=} ME^2 - MF^2$ (Carnot's Theorem)

① $ME^2 - MF^2 = (FB^2 + MB^2) - (AE^2 + AM^2)$

$= \boxed{FB^2 - EA^2}$

② $CE^2 - CF^2$

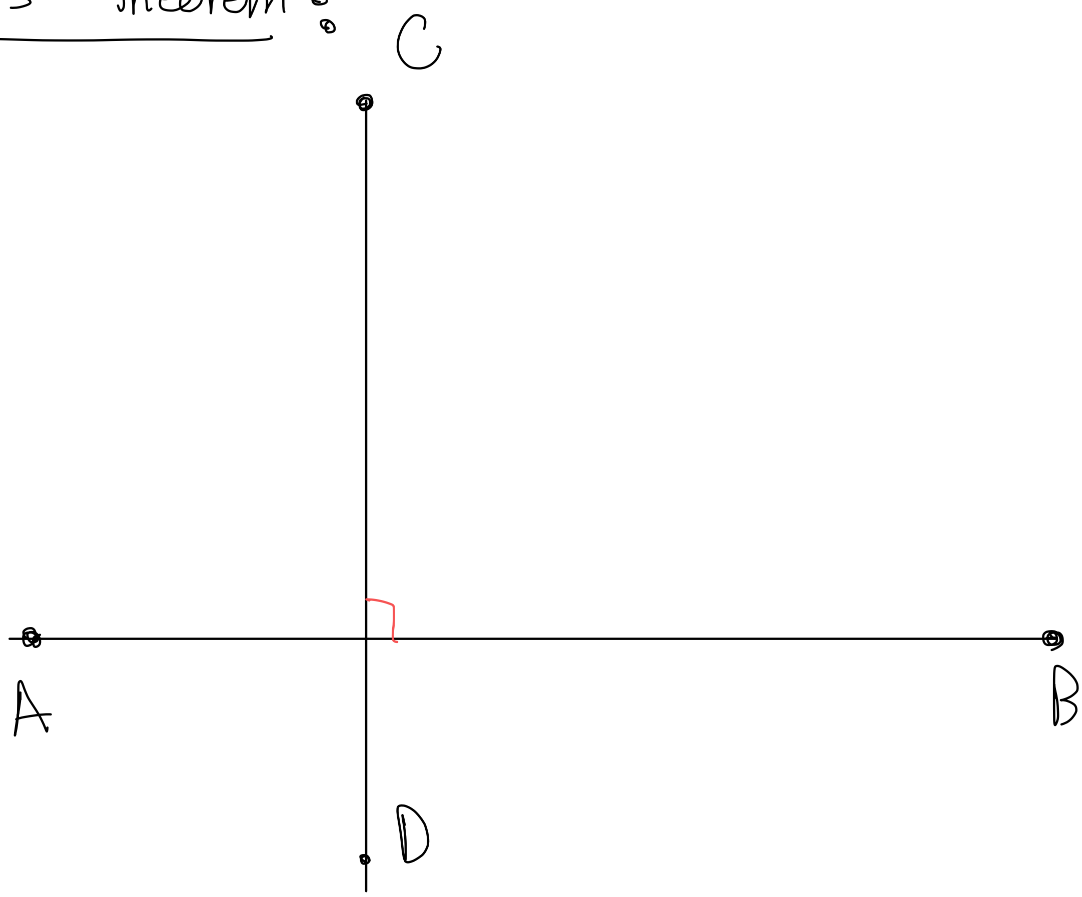
$ME \perp AC \Rightarrow CE^2 - EA^2 = CM^2 - MA^2$

$MF \perp CB \Rightarrow CF^2 - FB^2 = CM^2 - MB^2$

$\Rightarrow CE^2 - EA^2 = CF^2 - FB^2 \Rightarrow CE^2 - CF^2 = \boxed{EA^2 - FB^2}$

Therefore, $ME^2 - MF^2 = CE^2 - CF^2$ so $CM \perp EF$.

# Carnot's Theorem :

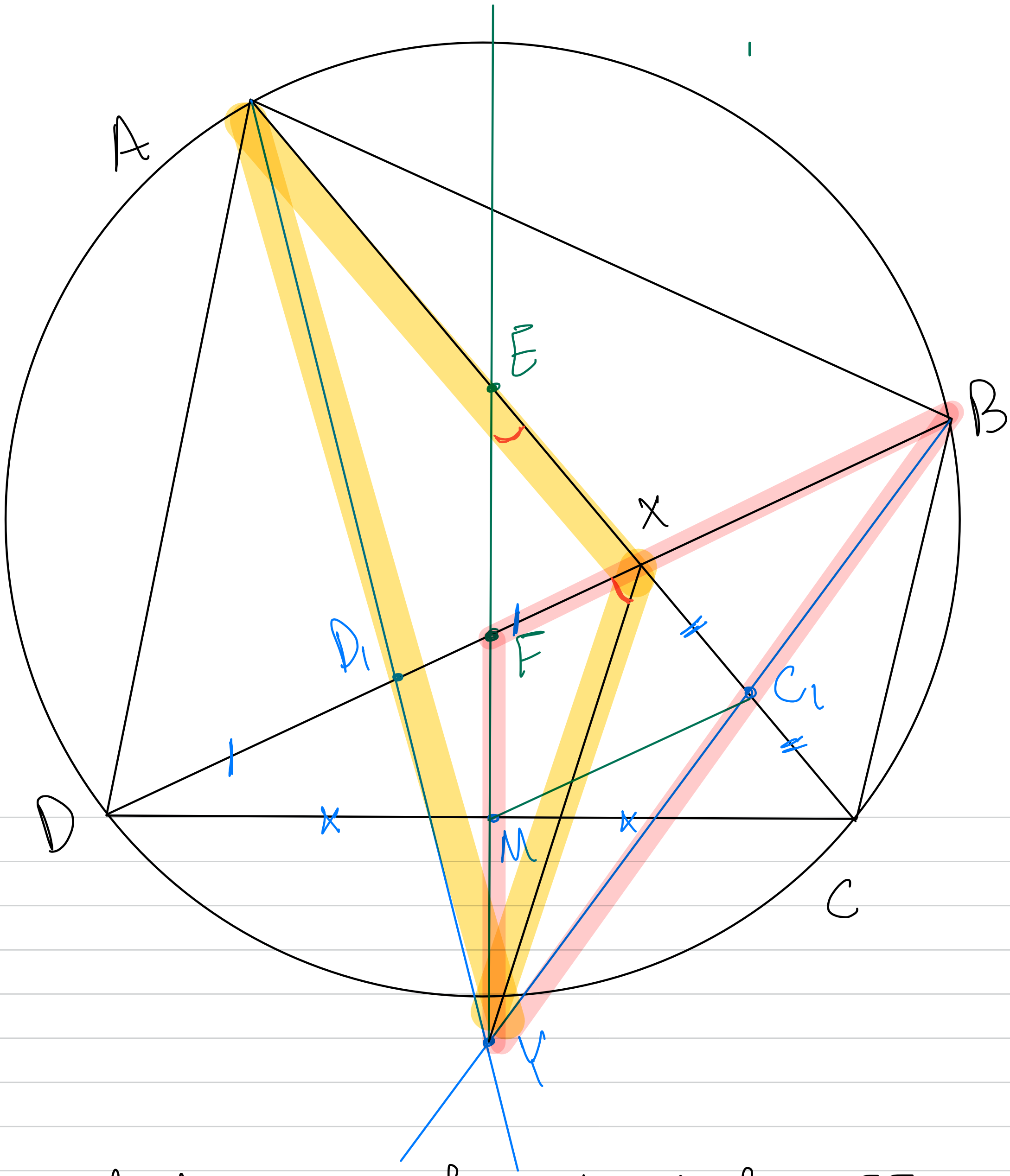



$CD \perp AB$ if and only if $AC^2 - BC^2 = AD^2 - BD^2$

proof : by Pythagorean theorem

left as an exercise

12. Let $ABCD$ be a cyclic quadrilateral, and let diagonals $AC$ and $BD$ intersect at $X$.Let $C_1, D_1$ and $M$ be the midpoints of segments $CX, DX$ and $CD$, respecctively. Lines $AD_1$ and $BC_1$ intersect at $Y$, and line $MY$ intersects diagonals $AC$ and $BD$ at different points $E$ and $F$, respectively. Prove that line $XY$ is tangent to the circle through $E, F$ and $X$. (EGMO 2016 P2)




$XY$ is tangent to $(XEF)$ if and only if $\angle XEF = \angle YXF$

Hint 1: $AD_1C_1B$ is a cyclic

Hint 2: $\Delta YXA \sim \Delta YFB$