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Muath Alghamdi

Sums of two squares 1 (L3)

1. (Warm up) Determine whether there exist rational numbers r and q such that $r^2 + q^2 = 15$.
2. Prove that there are infinitely many primes of the form $4m + 3$.
3. Prove that if $a^2 + b^2$ is divisible by a prime $p = 4m + 3$, then p divides both a and b.
4. The product of two numbers, each of which is a sum of two squares, is itself a sum of two squares.
5. Prove that if n is a sum of two squares, then $5n$ is also a sum of two squares.
6. Prove that if number n is divisible by 5 and n is a sum of two squares, then $n/5$ is also a sum of two squares.
7. Can 13×29 be expressed as a sum of two squares?

Homework

1. Prove that there are infinitely many integers of the form $4m + 1$ that are not a sum of two perfect squares.
2. Prove that if n is a sum of two squares, then $2n$ is also a sum of two squares.
3. Prove that if n is an even number and a sum of two squares, then $n/2$ is also a sum of two squares.
4. Determine whether there exist numbers $n \geqslant 15$ and $m \geqslant 15$ such that $n^2 + m^2 = 29 \cdot 41$.

Sums of two squares 2 (Level 3)

1. (Revision) Prove that for any integer m there exists a multiple of m that is not a sum of two squares.
2. (Revision) Determine whether number n^2+1 can have a factor of the form $4m-1$.
3. (Revision) Prove that if number n is divisible by 13 and n is a sum of two squares, then $n/13$ is also a sum of two squares.
4. Can 19×29 be expressed as a sum of two squares?
5. (Generalisation) If a number which is a sum of two squares is divisible by a prime which is a sum of two squares, then the quotient is a sum of two squares.
6. If a number which can be written as a sum of two squares is divisible by a number which is not a sum of two squares, then the quotient has a factor which is not a sum of two squares.
7. (Wilson's theorem) A natural number $n > 1$ is a prime number if and only if $(n-1)!+1 \equiv 0 \pmod{n}$. Prove it.
8. Prove that for any prime $p = 4n+1$ there exists m such that m^2+1 is divisible by p.
9. Number d is a factor of n^2+1. Prove that there exist infinitely many numbers m such that d is a factor of m^2+1.
10. How many positive integers $n < 1\,000$ satisfy the following condition: n^2+1 is divisible by 65?
11. Find a number that can be expressed as the sum of two squares in at least four different ways.
12. Prove that if number can be expressed as the sum of two squares in two different ways, then that number is composite.
13. Prove that there are infinitely many primes of the form $4m+1$.
14. Prove that if m^2+1 is divisible by a prime p, then p is a sum of two squares.

Homework

1. Find all integers a, b, c, d such that $a^2+b^2+c^2 = 8d-1$.
2. Can number $4^m(8n+7)$ be expressed as a sum of two squares?
3. Determine whether there exist integers m and $n > 1$ such that $\frac{m^2+1}{n^2-1}$ is an integer.
4. Factorise $1000009 = 235^2 + 972^2$.
5. Let x, y, z be integers and $4xy - x - y = z^2$. Prove that $x \leqslant 0$ and $y \leqslant 0$.

Sum of two squares (final part)

1. (Revision) Find all integers x, y, z such that $x^2y^2 = x^2 + y^2 + z^2$.

2. (Revision) Prove that number $N = p_1(p_2p_3)^4$, where $p_1 = 4m_1 + 1$, $p_2 = 4m_2 + 1$, and $p_3 = 4m_3 + 3$ are primes, can be expressed as sum of two squares in (at least) 5 different ways.

3. Prove that if $m^2 + 1$ is divisible by a prime p, then p is a sum of two squares.

4. Prove that there are infinitely many primes of the form $4m + 1$.