Saudi Arabia 2022 – Math Camp

Level 3

Geometry – Power of a Point and Radical Axis

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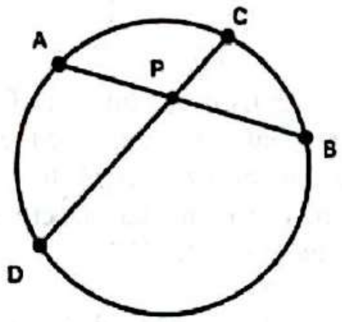
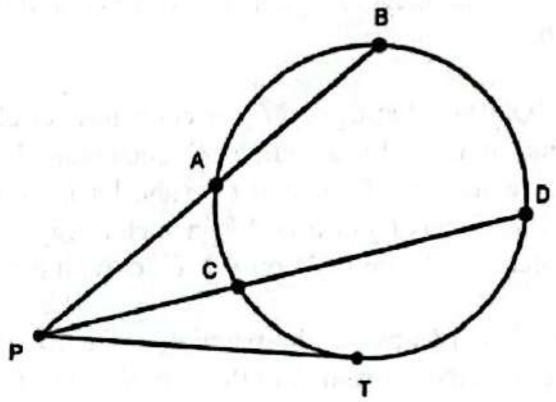
Theorem 1 (Chord Theorem – "Power of a Point"). Let Γ be a circle, and P a point. Let a line through P meet Γ at points A and B, and let another line through P meet Γ at points C and D. Then

$$PA \cdot PB = PC \cdot PD$$

If P lies outside Γ and we draw PT tangent to Γ at T, then

$$PA \cdot PB = PC \cdot PD = PT^2$$





Proof.
For the first case, we have $\angle PDA = \angle PBC$ (arc AC) and $\angle DPA = \angle BPC$ (opposite at P). So $\Delta PDA \sim \Delta PBC(AA)$ and

$$\frac{PA}{PC} = \frac{PD}{PB} \Leftrightarrow PA \cdot PB = PC \cdot PD$$

In the second case, we also have $\Delta PDA \sim \Delta PBC(AA)$ and $PA \cdot PB = PC \cdot PD$.

For the other part we have $\angle PDT = \angle PTC$ (arc CT), $\angle DPT = \angle TPC$ and $\Delta PDT \sim \Delta PTC$ (AA). The ratio of the sides gives us

$$\frac{PT}{PC} = \frac{PD}{PT} \Leftrightarrow PT^2 = PC \cdot PD$$

Theorem 2 (Converse to power of a point). Let A, B, C, D be four distinct points. Let lines AB and CD intersect at P. Assume that either
(1) P lies on both line segments AB and CD, or
(2) P lies on neither line segments.
Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$

Proof.
Suppose that P lies on both line segments AB and CD. We have $\angle DPA = \angle BPC$ (opposite at P) and

$$PA \cdot PB = PC \cdot PD \Leftrightarrow \frac{PA}{PC} = \frac{PD}{PB} \Leftrightarrow \Delta PDA \sim \Delta PBC \Leftrightarrow \angle PDA = \angle PBC$$

This occur iff A, B, C, D are concyclic.
Case (2) is analogous.

Problems

1. (AMC/2020-12B) In unit square $ABCD$ the inscribed ω intersects CD at M and AM intersects ω at a point P different from M. What is AP?

(A) $\frac{\sqrt{5}}{12}$ (B) $\frac{\sqrt{5}}{10}$ (C) $\frac{\sqrt{5}}{9}$ (D) $\frac{\sqrt{5}}{8}$ (E) $\frac{2\sqrt{5}}{15}$

2. (AIME I/2019) In convex quadrilateral $KLMN$ side MN is perpendicular to diagonal KM, side KL is perpendicular to diagonal LN, $MN = 65$, and $KL = 28$. The line through L perpendicular to side KN intersects diagonal KM at O with $KO = 8$. Find MO.

3. (Brazil/2013) Let Γ be a circle and A a point outside Γ. The tangent lines to Γ through A touch Γ at B and C. Let M be the midpoint of AB. The segment MC meets Γ again at D and the line AD meets Γ again at E. Given that $AB = a$, $BC = b$, compute CE in terms of a and b.

4. (USAMO/1998) Let C_1 and C_2 be concentric circles, with C_2 in the interior of C_1. Let A be a point on C_1 and B a point on C_2 such that AB is tangent to C_2. Let C be the second point of intersection of AB and C_1, and let D be the midpoint of AB. A line passing through A intersects C_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

5. (Russia/2012) Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the foot of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C and D lie on the same circle.

6. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.