

Email training, N3
September 8-14

Problem 3.1. Find all triples (a, b, c) such that $a = (b + c)^2$, $b = (c + a)^2$ and $c = (a + b)^2$.

Solution 3.1. Since a , b and c are squares of real numbers it means all of them are non negative. Assume $a \neq b$. Then

$$a - b = (b + c)^2 - (c + a)^2 = (b - a)(b + 2c + a).$$

Since $a - b \neq 0$, so $b + 2c + a = -1$, which is impossible since $b + 3c + a \geq 0$. So $a = b$ and in the same way $a = b = c$. Putting in the equation we get $a = 4a^2$, so $a = 0$ either $a = \frac{1}{4}$.

Answer: $a = b = c = 0$ and $a = b = c = \frac{1}{4}$.

Problem 3.2. Let a , b and c are pairwise different numbers. Solve the system of equations

$$\begin{cases} z + ay + a^2x + a^3 = 0 \\ z + by + b^2x + b^3 = 0 \\ z + cy + c^2x + c^3 = 0. \end{cases}$$

Solution 3.2. Consider the polynomial $t^3 + xt^2 + yt + z$. The numbers a, b, c are the roots of the polynomial. According to Viet's theorem one has

$$\begin{cases} a + b + c = -x \\ ab + bc + ca = y \\ abc = -z \end{cases}$$

Answer: $x = -a - b - c$, $y = ab + bc + ca$, $z = -abc$.

Problem 3.3. Solve equation in integers

$$x! + 13 = y^2.$$

Solution 3.3. Note that for the numbers bigger than 4 their factorial is divisible by 10, so their last digit is 0. Therefore for $x \geq 5$ the last digit of the left side of the equation is 3 so it can't be perfect square. So $x \leq 4$. By verifying each case $x = 1, 2, 3, 4$ we conclude that the equation has no solution in integers.

Problem 3.4. Let numbers x_1, x_2, \dots, x_n are given and each of them is equal either $+1$ or -1 . Prove that if

$$x_1x_2 + x_2x_3 + \dots + x_nx_1 = 0$$

then n is divisible by 4.

Solution 3.4. There are n terms in the sum, and half of them are $+1$ and half of them are equal -1 , so n is even. Now let's prove that there are even number of -1 's. From this we will conclude the statement of the problem. To obtain -1 we need consecutive written numbers $+1$ and -1 . Let's write numbers x_1, x_2, \dots, x_n around the circle and by starting from x_1 go around the circle. Note that we move from $+1$ to -1 as many times as we move from -1 to $+1$. So there are even number of terms -1 in the sum $x_1x_2 + x_2x_3 + \dots + x_nx_1$.

Problem 3.5. Chess king has started from some cell and by passing over each cell exactly once came back to original position. Prove that the king has done even number of diagonal moves.

Solution 3.5. The king has made 64 moves. Note that when king makes diagonal move then it doesn't change its cell color, otherwise he changes it. Since he came back to the same cell, it means that he made even number of non-diagonal moves. Since he made in total 64 moves, therefore the number of diagonal moves is even as well.

Problem 3.6. Let k is given and numbers from 1 to 100 are written on the board. Ali erases from the board arbitrary k numbers. Is it true that Bob may choose k numbers written on the board, which sum is equal to 100. Consider cases when a) $k = 8$, b) $k = 9$.

Solution 3.6. a) Let's group numbers from 1 to 24 to the pairs such that their sum is equal 25.

$$(1; 24), (2; 23), (3; 22), \dots, (11; 14), (12; 13) :$$

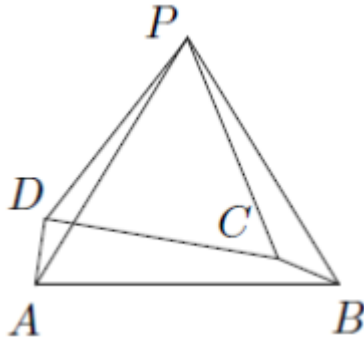
In total there are 12 pairs. If we remove any 8 numbers then there will be at least $12 - 8 = 4$ pairs where no any number is removed. If we take all numbers from these 4 groups then we get 8 numbers which sum is equal $4 \cdot 25 = 100$.

b) Let's remove numbers from 1 to 9. If we choose any 9 numbers from the remaining numbers then their sum will be at least $10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 = 126 > 100$. So we can't choose 9 numbers which sum is equal 100.

Answer: a) possible, b) not possible.

Problem 3.7. Let $ABCD$ be a quadrilateral with $AD = BC$ and $\angle A + \angle B = 120^\circ$ and let P be a point exterior to the $ABCD$ such that P and A lie at opposite sides of the line DC and the DPC is isosceles triangle. Prove that the APB is also isosceles triangle.

Solution 3.7. -



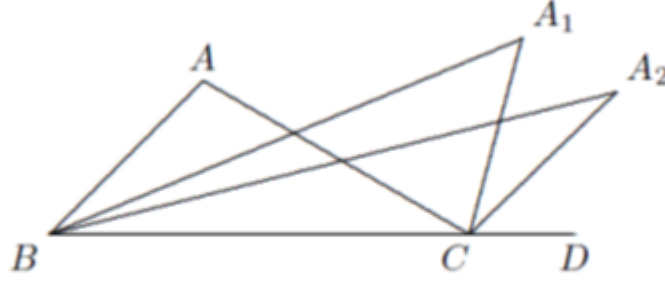
Note that $\angle ADC + \angle CDP + \angle BCD + \angle DCP = 360^\circ$.

Thus $\angle ADP = 360^\circ - \angle BCD - \angle DCP = \angle BCP$. As we have $DP = CP$ and $AD = BC$, the triangles ADP and BCP are congruent and $AP = BP$. Moreover, $\angle APB = 60^\circ$ since $\angle DPC = 60^\circ$ and $\angle DPA = \angle CPB$.

Problem 3.8. In the triangle ABC one has $\angle A = 96^\circ$. The segment BC is extended to an arbitrary point D . The angle bisectors of angles ABC and ACD intersect at A_1 , and the angle bisectors of A_1BC and A_1CD intersect at A_2 and so on... the angle bisectors of A_4BC and A_4CD intersect at A_5 . Find the size of $\angle BA_5C$ in degrees.

Solution 3.8. -

Since A_1B and A_1C bisect $\angle ABC$ and $\angle ACD$ respectively, $\angle A = \angle ACD - \angle ABC = 2(\angle A_1CD - \angle A_1BC) = 2\angle A_1$, therefore $\angle A_1 = \frac{1}{2}\angle A$.



Similarly, we have $\angle A_{k+1} = \frac{1}{2}\angle A_k$ for $k = 1, 2, 3, 4$. Hence

$$\angle A_5 = \frac{1}{2}\angle A_4 = \frac{1}{4}\angle A_3 = \frac{1}{2^3}\angle A_2 = \frac{1}{2^4}\angle A_1 = \frac{1}{2^5}\angle A = \frac{96^\circ}{32} = 3^\circ.$$