Competition Preparation for Saudi Arabia Team: Level 4+

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Lesson 6

Invariants

Problems:

1. Numbers 1 to 20 are written on a blackboard. In each move you're allowed to erase two numbers from the blackboard, a and b, and instead of them write the number $ab + a + b$. Prove that the final number on the board will not depend on the sequence of moves and find the final number.

2. On an infinite board each square in an $n \times n$ group of squares has a coin placed over it. In each move, a coin is allowed to jump over an adjacent coin if the opposite square is empty and then the coin that was jumped over is removed from the board. For which n is it possible for just one coin to remain on the board after a sequence of moves.

3. A group of n children arranged in a circle play a sharing game. In the beginning one child has m candies while the remaining children have none and in each move a child with at least two candies gives one candy to each of his two neighbors. The process continues until no more moves are possible.

   (a) If $m = n$, for which values of n will the game end?

   (b) If $m < n$ show that the game ends regardless of the sequence of moves.

4. On a single pile there are 1001 stones. In a single move one is allowed to remove one stone from a pile and split the remaining stones on the pile into two non-zero piles of arbitrary amount. Is it possible to end up only with piles containing three stones?

5. A $(2n + 1) \times (2n + 1)$ board is colored in the chessboard fashion. One is in one move allowed to select a rectangle on the board and reverse all colors. Find the minimum number of moves needed to make the entire board be of the same color.

6. On a blackboard n non-negative integer numbers are written. One is allowed to select two numbers from the board x and y such that $x \geq y$ and replace them with $x - y$ and $2y$. Determine for which n-tuples of initial numbers is it possible to increase the number of zeros on the blackboard to $n - 1$.

7. We have n coins placed on one square of an infinite one-dimensional grid. In each move we select two consecutive squares such that one on the left has at least 2 coins more and move a coin from the left square to the right square. The process continues until no more moves can be made. Show that the final configuration of coins is independent of the sequence of moves made.