$A \subseteq \mathbb{R}$, :

$x \in A \Longrightarrow \frac{x+1}{x} \in A$, $\frac{2x-1}{x-1} \in A$

$\neq 0, 1$

$2 \in A$, prove that $\mathbb{Q}_{>1} \subseteq A$.

Suppose $\frac{p}{q} \notin A$, $p > q$ (it means $\frac{p}{q} > 1$)

with minimal denominator q.

with smaller numerator.

Prove that there is a fraction

not in A

$\frac{p}{q} \neq 2 \rightsquigarrow p-2q \neq 0$.

$$\frac{\frac{q}{p-q}+1}{\frac{q}{p-q}} = \frac{p}{q}$$

$\frac{1}{q} \in A$ $\quad \frac{x+1}{x} = \frac{p}{q} \notin A$

$\Downarrow$

$x \notin A$

$\frac{q}{p-q} \notin A$, $\frac{2x-1}{x-1} = \frac{p}{q}$?

$\frac{p}{q} \notin A$, q minimal

$$\frac{x+1}{x} = \frac{p}{q} \implies x \notin A$$

$(x+1)q = px$

$x(p-q) = q$

$x = \frac{q}{p-q}$

$\frac{q}{p-q} \notin A$

$$\frac{2x-1}{x-1} = \frac{p}{q}$$

$2xq - q = px - p$

$x(2q-p) = q-p$

$x = \frac{p-q}{p-2q}$

$\frac{p-q}{p-2q} \notin A$

• $1 < \frac{p}{q} < 2 \overset{\text{Check}}{\implies} \frac{q}{p-q} > 1$ so

but $\boxed{q < p}$ so contradiction

$\frac{p}{q} > 2 \Rightarrow \frac{p-q}{\underbrace{p-2q}_{\neq 0}} > 1$ $\quad 2 \in A$

$p-q > \cancel{p}-2q$

$2q > q$

$p-q < p$.

taking minimal object

II way

$x \in A,\ x > 1 \Rightarrow \frac{x+1}{x} \in A$

$\frac{x+1}{x} > 1$

$A \ni \frac{2 \cdot \frac{x+1}{x} - 1}{\frac{x+1}{x} - 1} = \frac{2x+2-x}{\frac{x+1-x}{x}} = x+2$

So if $x > 1 \rightsquigarrow x+2 \in A$
$x \in A$

if we the $x > 0 \rightsquigarrow$
$x+1 \in A \Rightarrow x+2 \in A$ ✓

$\rightsquigarrow x+t \in A$ for any $t = 1, 2, \ldots$

Induction on n,

$\frac{k}{n} \in A$ for any $k > n$ $\quad (*)$

$n = 1$ $\quad k > 1 \in A$

$3 = \frac{2 \cdot 2 - 1}{2 - 1} \in A$

$n = 1$ ✓

Assume $(*)$ holds for all $n < n_0$

We will prove for n_0.

$\frac{n_0}{n}, \frac{n_0 + n}{n} \in A$ for $n < n_0$

$k = n_0 > n$, $k = n_0 + n > n$

$$\frac{\frac{n_0}{n} + 1}{\frac{n_0}{n}} = \frac{n}{n_0} + 1, \quad \frac{2 \cdot \frac{n_0 + n}{n} - 1}{\frac{n_0 + n}{n} - 1} = \frac{n}{n_0} + 2$$

$\cap$

A

$$\Downarrow \quad \frac{n}{n_0} + t = \frac{tn_0 + n}{n_0} \in \mathcal{L}$$

Any $k > n_0$ $\quad k = tn_0 + n$

P12

$$g(f(n)) = g(n) - n$$

Find all possible values of $f(0)$

$$f, g : \mathbb{N}_{\geq 0} \longrightarrow \mathbb{N}_{\geq 0}.$$

take minimal value of $g(m)$

$\boxed{m \neq 0}$

take $g(m)$ - minimal value of g

what if the minimal value is taken for some $m \neq 0$. ?

Let $m \neq 0$ be such that

$g(m)$ is minimal

$$g(f(m)) = g(m) - m < g(m)$$

if $m \neq 0$

$\lightning$

so $g(m)$ is minimal iff $m = 0$

$$g(f(u)) = g(u) - u$$

$$g(f(0)) = g(0)$$

minimal

iff

$\Downarrow$

$$\boxed{f(0) = 0}$$

$g(m)$ is minimal iff $m=0$

$$g(f(0)) = \underset{\text{minimal}}{\underset{\|}{g(0)}}$$

so $g(f(0))$ is minimal

$$\Updownarrow$$

$$\boxed{f(0)=0}$$

P13)

$n \geq 3$

$a_2, a_3, \ldots, a_n \in \mathbb{R}_+$

$a_2 a_3 \cdots a_n = 1$.

Prove;

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n$$

$$a_k+1 = \Big(a_k + \underbrace{\frac{1}{k-1} + \dots + \frac{1}{k-1}}_{k-1}\Big) \quad \text{"="} \quad a_k = \frac{1}{k-1}$$

AM-GM

$$\geq k \sqrt[k]{a_k \cdot \frac{1}{(k-1)^{k-1}}}$$

$$a_k + 1 \geq k \sqrt[k]{a_k \cdot \frac{1}{(k-1)^{k-1}}}$$

$$(a_k+1)^k \geq k^k \cdot a_k \cdot \frac{1}{(k-1)^{k-1}} =$$

$$= \frac{k^k}{(k-1)^{k-1}} \cdot a_k$$

$$\prod_{k=2}^{n} (a_k+1)^k \geq \prod_{k=2}^{n} \frac{k^k}{(k-1)^{k-1}} \cdot a_k =$$

$$= \frac{\cancel{2^2}}{1^1} \cdot \frac{\cancel{3^3}}{\cancel{2^2}} \cdot \frac{\cancel{4^4}}{\cancel{3^3}} \cdot \dots \cdot \frac{n^n}{\cancel{(n-1)^{n-1}}} \underbrace{a_2 \cdot \dots \cdot a_n}_{= 1}$$

$$= n^n$$

$$a_2 = 1, \quad a_3 = \frac{1}{2}, \quad a_4 = \frac{1}{3}, \ldots, \quad a_n = \frac{1}{n-1}$$

$$a_2 a_3 \cdots a_n = \frac{1}{(n-1)!} \neq 1.$$

$n \geq 3$

$2 \nmid n \qquad x_1, \ldots, x_n \geq 0$

$$\min_{i=1\ldots n} (x_i^2 + x_{i+1}^2) \leq \max_{j=1\ldots n} (2x_j x_{j+1})$$

$2 \nmid n$

$$\left.\begin{array}{l} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \\ \vdots \\ x_n - x_{n-1} \\ x_1 = x_{n+1} - x_n \end{array}\right\}$$

n numbers
odd number.

there are to consecutive of the same sign Why?

$$\begin{array}{lll} b-a & P & N \\ c-b & N & P \\ a-c & P & N \end{array}$$

$$\exists j : (x_{j+1} - x_j)(x_{j+2} - x_{j+1}) \geq 0$$

P N P N . . . . P $\qquad x_j \leq x_{j+1} \leq x_{j+2}$

P N P N P N P N

N P N P N P N P