Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = xf(x) + y$$

for every $x, y \in \mathbb{R}$.

Solution: Let $P(x, y)$ be the assertion in the given equation.

$$P(0, x) \Rightarrow f(f(x)) = x \;\forall\, x \in \mathbb{R} \qquad (*)$$

Now if $f(a) = f(b)$ then $a = f(f(a)) = f(f(b)) = b$, and hence f is injective. Now, by $(*)$:

$$P(f(x), y) \Rightarrow f(f(x)^2 + f(y)) = f(x)f(f(x)) + y = xf(x) + y = f(x^2 + f(y))$$

and so by injectivity, $f(x)^2 + f(y) = x^2 + f(y)$. So, $f(x) = x$ or $-x$ for every $x \in \mathbb{R}$, in particular, $f(0) = 0$. Now we are only left to check whether there is an overlap, that is, assume there are $u, v \in \mathbb{R}$ such that $f(u) = u$ and $f(v) = -v$, then

$$P(u, v) \Rightarrow u^2 + v = f(u^2 - v) \in \{u^2 - v, -u^2 + v\}$$

but this means $u = 0$ or $v = 0$, and so there is no overlap, and we get only two solutions (can be easily checked):

$$(1) \quad f(x) = x \;\; \forall\, x$$

$$(2) \quad f(x) = -x \;\; \forall\, x$$

□

Problem 2 (IMO 2019). Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(f(a + b)) = f(2a) + 2f(b)$$

for every $a, b \in \mathbb{Z}$.

Solution: Let $P(a, b)$ be the assertion in the given equation.

$$P(0, n + 1) \Rightarrow f(f(n + 1)) = f(0) + 2f(n + 1)$$

$$P(1, n) \Rightarrow f(f(n + 1)) = f(2) + 2f(n)$$

hence, $f(n + 1) - f(n) = \frac{f(2)-f(0)}{2}$, a common constant difference. Therefore,

$$..., f(-1), f(0), f(1), f(2), ...$$

is an arithmatic progression, and so $f(n) = cn + d \;\forall\; n$, and some constants c, d. Now, to finish the problem, we only need to determine the possible values for the constants c, d. To do that, we simply substitute in the given equation:

$$LHS = f(f(a + b)) = cf(a + b) + d = c^2(a + b) + cd + d$$

$$RHS = f(2a) + 2f(b) = 2c(a + b) + 3d$$

and so $c^2 = 2c$ and $(c + 1)d = 3d$. Hence, $(c, d) = (0, 0)$ or $(2, d)$ and we get the only two solutions (can be easily checked):

$$(1) \quad f(n) = 0 \;\; \forall\, n \in \mathbb{Z}$$

$$(2) \quad f(n) = 2n + d \;\forall\, n \in \mathbb{Z} \text{ and some constant } d \in \mathbb{Z}$$

□

Problem 3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(m) + f(n)) = m + n$$

for every $m, n \in \mathbb{N}$.

1st Solution: Let $P(m, n)$ be the assertion in the given equation. If $f(a) = f(b)$ for some $a, b \in \mathbb{N}$ then $P(a, n), P(b, n)$ give:

$$a + n = f(f(a) + f(n)) = f(f(b) + f(n)) = b + n \Rightarrow a = b$$

so f is injective. Now let $n \in \mathbb{N}$. notice that

$$P(n, 2), P(n + 1, 1) \Rightarrow f(f(n) + f(2)) = n + 2 = (n + 1) + 1 = f(f(n + 1) + f(1))$$

and so $f(n + 1) - f(n) = f(2) - f(1)$, a common constant difference. So $f(1), f(2), f(3), ...$ is an arithmatic sequence, in particular, $f(n) = cn + d$ for every $n \in \mathbb{N}$ and some constants c, d. Substiting in the given equation we get:

$$m + n = f(c(m + n) + 2d) = c^2(m + n) + d(2c + 1)$$

so $c^2 = 1$ and $d(2c + 1) = 0$, which gives $d = 0$ and $c = 1$ (as $c = -1$ gives negative values). Thus, the only solution is $f(n) = n$ for all $n \in \mathbb{N}$. □

2nd Solution: First, we prove f is injective as in *Solution*1. Now we claim that $f(n) \leq n$ for every $n \in \mathbb{N}$. Indeed, suppose $f(a) > a$ for some $a \in \mathbb{N}$, then

$$P(f(a) - a, a) \Rightarrow f(f(f(a) - a) + f(a)) = f(a)$$

and so by injectivity we get $f(a) < f(f(a) - a) + f(a) = a < f(a)$, a contradiction! Hence, $f(n) \leq n$ for every $n \in \mathbb{N}$. Now we can see that

$$m + n = f(f(m) + f(n)) \leq f(m) + f(n) \leq m + n$$

which gives an equality case. So $f(n) = n$ for every $n \in \mathbb{N}$. □

Remark : the 2nd solution works even when we replace $\mathbb{N}$ by $\mathbb{R}^+$.

Problem (HW), Cauchy Equation. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + y) = f(x) + f(y)$$

for every $x, y \in \mathbb{Q}$.

Solution: Let $P(x, y)$ be the assertion in the given equation.

$$P(0, 0) \Rightarrow f(0) = 0, \text{ and } P(x, -x) \Rightarrow f(-x) = -f(x) \ \forall\, x \in \mathbb{Q}$$

Also notice that the given condition can be generalised to

$$f(x_1 + x_2 + ... + x_n) = f(x_1) + f(x_2) + ... + f(x_n)$$

Now let $n \in \mathbb{N}$. We have

$$f(n) = f(1 + 1 + ... + 1) = nf(1)$$

then $f(-n) = -f(n) = -nf(1)$. So $f(m) = mf(1)$ for every $m \in \mathbb{Z}$. Now fix an $r \in \mathbb{Q}$ and let $n \in \mathbb{N}$ such that $nr \in \mathbb{Z}$. Then

$$nrf(1) = f(nr) = f(r + r + ... + r) = nf(r) \Rightarrow f(r) = rf(1) \ \ \forall\, r \in \mathbb{Q}$$

Which is indeed a solution. $\square$

Problem 4. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies that $x + f(x) = f(f(x))$ for every $x \in \mathbb{R}$. Find all solutions of the equation $f(f(x)) = 0$.

Solution: First, if $f(a) = f(b)$ then $a = f(f(a)) - f(a) = f(f(b)) - f(b) = b$, hence f is injective. Now we set $x = 0$ in the given equation, we get $f(0) = f(f(0))$, and so by injectivity we see that $f(0) = 0 \Rightarrow f(f(0)) = 0$. Hence, 0 is a solution, and again, by injectivity, there are no other solutions! $\square$

Problem 5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(y) + y) = f(x)f(y) + y$ for all reals x and y.

Solution: Let $P(x, y)$ be the assertion in the given equation. $P(0, x)$ gives

$$f(x) = f(0)f(x) + x \ \ \forall\, x$$

then $f(0) \neq 1$ as for otherwise $x = f(x)(1-f(0)) = 0\, \forall\, x$ which is absurd. This implies that $f(x) = \frac{x}{1-f(0)}\ \forall\, x$, which gives $f(0) = 0$ and so $f(x) = x\ \forall x$, which is indeed a solution. $\square$

Problem 6 (ISL 2002). Find all functions f from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

holds for all real x and y.

Solution: Let $P(x, y)$ be the assertion in the given equation.

$$P(-f(x), x) \Rightarrow f(f(-f(x)) - x) = f(0) - 2x$$

this immediately implies that f is surjective (the LHS is an image of f and the RHS can take any real value when x vary), and so there is an $r \in \mathbb{R}$ such that $f(r) = 0$. Now

$$P(r, y) \Rightarrow f(y) = 2r + f(f(y) - r) \ \ \forall\, y \qquad (*)$$

then again, by surjectivity of f, the number $f(y)$ can take any real number, in particular, let's write $f(y) = x + r$ for some $x \in \mathbb{R}$. Then $(*)$ can be rewritten as $x + r = 2r + f(x)$, or

$$f(x) = x - r \ \ \forall\, x \in \mathbb{R}$$

which is indeed a solution, regardless of the value of the constant r. (can be easily checked) $\square$

Problem 7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = f(x)^2 + y$$

for every $x, y \in \mathbb{R}$.

Solution: Let $P(x, y)$ be the assertion in the given equation.

$$P(0, x) \Rightarrow f(f(x)) = f(0)^2 + x \qquad (*)$$

this immediately gives that f is surjective, moreover, if $f(a) = f(b)$ then $(*)$ gives $a = f(f(a)) - f(0)^2 = f(f(b)) - f(0)^2 = b$, and so f is injective as well. Now, $\exists\, r \in \mathbb{R}$ such that $f(r) = 0$. Now, we use $(*)$ and the following substitution

$$P(r, x) \Rightarrow x + f(0)^2 = f(f(x)) = x \Rightarrow f(0) = 0$$

so $(*)$ becomes $f(f(x)) = x$ for every $x \in \mathbb{R}$. And we'll use this fact in the following substitution:

$$P(f(x), y) \Rightarrow x^2 + y = f(f(x))^2 + y = f(f(x)f(f(x)) + f(y)) = f(xf(x) + f(y)) = f(x)^2 + y$$

and this is just $f(x)^2 = x^2\ \forall\, x$, and we complete as in Problem 1 (proving that there is no overlap), to get the only two solutions (can be easily checked):

$$(1) \quad f(x) = x\ \ \forall\, x$$

$$(2) \quad f(x) = -x\ \ \forall\, x$$

□

Problem 8. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n)$ is a perfect square for any $n \in \mathbb{N}$, and

$$f(m + n) = f(m) + f(n) + 2mn \quad \forall\, m, n \in \mathbb{N}.$$

Solution: Setting $m = 1$ we get $f(n + 1) = f(n) + 2n + f(1)$ for every $n \in \mathbb{N}$ (so every value can be calculated from the previous one). Then an easy induction gives

$$f(n) = n^2 + n(f(1) - 1)\ \ \forall\, n \in \mathbb{N}$$

Now if $f(1) = 1$ then $f(n) = n^2$ for every natural n, which is clearly a solution. While if $f(1) > 1$, we can write $f(1) \in \{2k, 2k + 1\}$ (based on its parity) for some $k \in \mathbb{N}$. Then $f(k^2) \in \{k^4 + 2k^3 - k^2, k^4 + 2k^3\}$, and then we can easily verify that

$$(k^2 + k - 1)^2 < k^4 + 2k^3 - k^2 < k^4 + 2k^3 < (k^2 + k)^2$$

a contradiction to the first condition! Thus, the only solution is $f(n) = n^2\ \ \forall\, n \in \mathbb{N}$. □

Problem (HW2). Prove that the composition of injective (surjective) functions gives an injective (surjective) function.

Solution: We start by arbitrary $f : A \to B$ and $g : B \to C$, then their composition is $h = g \circ f : A \to C$.
Now suppose that both f and g are injective. If $h(a) = h(a')$ for some $a, a' \in A$ then $g(f(a)) = g(f(a'))$. By the injectivity of g we see that $f(a) = f(a')$, then by the injectivity of f we see that $a = a'$. Thus, h is injective as well.
Now suppose both f and g are surjective. Let $c \in C$ be arbitrary. By surjectivity of g, there is $b \in B$ such that $c = g(b)$, and by surjectivity of f, there is $a \in A$ such that $b = f(a)$. Thus, $c = g(b) = g(f(a)) = h(a)$, which means that h is surjective. $\square$

Problem 9. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer n we have

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

Solution: If $f(m) = f(n)$ then

$$3m = f(f(f(m))) + f(f(m)) + f(m) = f(f(f(n))) + f(f(n)) + f(n) = 3n$$

and hence f is injective. Next we prove by induction that $f(n) = n \ \forall\, n \in \mathbb{N}$, for $n = 1$ we have $f(1), f(f(1)), f(f(f(1))) \geq 1$ (because the range is $\mathbb{N}$), but

$$f(f(f(1))) + f(f(1)) + f(1) = 3$$

so we must have an equality case, that is, $f(1) = 1$. Now suppose that $f(k) = k$ for every $k < n$, where $n \geq 2$ is an integer. Then by injectivity we see that $f(m) \geq n$ whenever $m \geq n$ (because $f(m)$ is a positive integer which doesn't belong to the set $\{f(1), f(2), ..., f(n-1)\} = \{1, 2, 3, ..., n-1\}$). Then $f(n) \geq n$, which also means $f(f(n)) \geq n$ and $f(f(f(n))) \geq n$, then again, due to the given condition we see that we must have an equality case, in particular $f(n) = n$, and the induction statement is proved. Thus, $f(n) = n \ \forall\, n \in \mathbb{N}$. $\square$

Problem 10. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(x + y)f(yf(x)) = x^2(f(x) + f(y))$$

for every $x, y > 0$.

Solution: Let $P(x, y)$ be the assertion in the given equation. If $f(a) = f(b)$ for some $a, b > 0$ then $P(a, y), P(b, y)$ implies that

$$\frac{a + y}{a^2} = \frac{f(a) + f(y)}{f(yf(a))} = \frac{f(b) + f(y)}{f(yf(b))} = \frac{b + y}{b^2}$$

for every $y > 0$. This immediately gives $a = b$, hence f is injective. Now

$$P(1, 1) \Rightarrow 2f(f(1)) = 2f(1) \Leftrightarrow f(f(1)) = f(1)$$

and so by injectivity we see that $f(1) = 1$. Next,

$$P(1, x) \Rightarrow (1 + x)f(x) = x^2(1 + f(x))$$

which is just $f(x) = \frac{1}{x}$ for every $x > 0$, and it is indeed a solution (we can easily check). $\square$

Problem 11. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = (x-1)f(x) + 2 \ \ \forall \ x$$

Show that f is not surjective.

Solution: Suppose on contrary that f is surjective. Now if $f(a) = f(b) \neq 0$ then plugging $x = a$ and $x = b$ gives $a = b$, let's call this property $(*)$.
Now, by surjectivity, we can pick an element $s \in \mathbb{R}$ for which $f(s) = 0$. Plug $x = s$ and $x = 1$ we get $f(0) = f(f(1)) = 2$, then by $(*)$ we see that $f(1) = 0$.
Now pick an element $t \in \mathbb{R}$ such that $f(t) = 1$. Plug $x = t$ we get $t = -1$ or $f(-1) = 1$.
Now pick an element $u \in \mathbb{R}$ such that $f(u) = -1$. Plug $x = u$ we get $u = 2$ or $f(2) = -1$.
Finally, plug $x = 0$ we get $-1 = 0$, a contradiction!! So we are done. $\square$

Problem 12 (IMO 2010). Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

Solution: Let $P(x, y)$ be the assertion in the given equation.
$P(0, x)$ gives $f(0)(\lfloor f(x) \rfloor - 1) = 0$ for all x, then we need to consider two separate cases:
$Case1:$ $f(0) \neq 0$, then $\lfloor f(x) \rfloor = 1$ for every $x \in \mathbb{R}$, and so $P(1, x)$ gives $f(x) = f(1)$ for every x. Therefore, we get the 1st solution, a constant in the interval $[1, 2)$.
$Case2:$ $f(0) = 0$, then $P(t, t)$, where $0 \leq t < 1$ gives $0 = f(t)\lfloor f(t) \rfloor$ or $\lfloor f(t) \rfloor = 0$ for every $0 \leq t < 1$. Finally, let $x \in \mathbb{R}$, there is a nonzero integer n such that $0 \leq \frac{x}{n} < 1$ (we find such an n by letting it having sufficiently large absolute value with the same sign as x). $P(n, \frac{x}{n})$ gives $f(x) = 0$ and showing the other solution, the zero function.
Thus, we can write the solution as (can be easily checked):

$$f(x) = C \quad \forall \ \ x \in \mathbb{R}$$

where $C \in \{0\} \cup [1, 2)$ is a constant. $\square$

Problem 13 (ISL 2018). Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$.

Solution: Let $P(x, y)$ be the assertion in the given equation.
The idea here is to make one side symmetric between x, y and the other side not symmetric, so if we plug $P(f(x), y)$ we get

$$f(f(x)^2 f(y)^2) = f(f(x))^2 f(y) \ \ \forall \ x, y \in \mathbb{Q}^+$$

and so by symmetry we get $f(f(x))^2 f(y) = f(f(y))^2 f(x)$ or, by rearranging the last equation, $\frac{f(f(x))^2}{f(x)} = \frac{f(f(y))^2}{f(y)}$ for any $x, y \in \mathbb{Q}^+$. This immediately gives that the expression $\frac{f(f(x))^2}{f(x)}$ is constant over $x \in \mathbb{Q}^+$, call this constant k, then $f(f(x))^2 = kf(x)$ for any $x \in \mathbb{Q}^+$. Rearrange the last equation to get $\frac{f(x)}{k} = (\frac{f(f(x))}{k})^2$. Now the last equation can be generalised in the following manner:

$$\frac{f(x)}{k} = (\frac{f(f(x))}{k})^2 = (\frac{f(f(f(x)))}{k})^4 = ... = (\frac{f(f(...(f(x))...))}{x})^{2^{n-1}}$$

where f occurs in the RHS n times. This means that the positive rational number $\frac{f(x)}{k}$ is a perfect 2^nth power for every $n \in \mathbb{N}$. But the only such number is 1, so $\frac{f(x)}{k} = 1$ or just $f(x) = k \ \ \forall\ x \in \mathbb{Q}^+$ (a constant function). Now we only need to determine the possible values of this constant, by plugging in the original equation:

$$k = k^3 \Rightarrow k = 1$$

Thus, the only solution is $f(x) = 1 \ \ \forall x \in \mathbb{Q}^+$. $\square$

Problem Test3-P3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$n^2 - 1 \leq f(n)f(f(n)) \leq n^2 + n$$

for every natural number n.

Solution: First, we prove that f is injective. Indeed, suppose on contrary that it is not, so there are positive integers $a < b$ such that $f(a) = f(b)$. But then

$$b^2 - 1 \leq f(b)f(f(b)) = f(a)f(f(a)) \leq a^2 + a < (a+1)^2 - 1 \leq b^2 - 1$$

a contradiction!! Hence, f is injective. Now plug $n = 1$ we see that $f(1)f(f(1)) = 1$ or 2. So $f(1) = 1$ or 2, but if $f(1) = 2$ then we must have $f(f(1)) = 1$ and so $f(2) = 1$. But now if we plug $n = 2$ we'll get

$$3 \leq f(2)f(f(2)) = 1 \cdot f(1) = 2$$

a contradiction!! Hence, $f(1) = 1$. Next, we prove by induction that $f(n) = n \ \ \forall\ n \in \mathbb{N}$. Indeed, the base case $n = 1$ is proved, suppose that $f(n) = n$ for every $n < k$, where $k \geq 2$. Then by injectivity we see that

$$f(m) \geq k \ \ \forall\ m \geq k \qquad (*)$$

So from $(*)$ we see that $f(k) \geq k$, then again by $(*)$, we see that $f(f(k)) \geq k$ as well. But given that

$$f(k)f(f(k)) \leq k^2 + k = k(k+1)$$

and so $f(k), f(f(k)) \leq k + 1$. Now if $f(k) \neq k$ then $f(k) = k + 1$ and we immediately get $f(k+1) = f(f(k)) = k$. But it is given that

$$k^2 + 2k \leq f(k+1)f(f(k+1)) = kf(k) = k^2 + k$$

a contradiction!! Therefore, $f(k) = k$, finishing the induction. Thus, we get the only solution $f(n) = n \ \ \forall\ n \in \mathbb{N}$. $\square$

Problem 14. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(2y) = f(x + y)$$

for every two real numbers x, y.

1*st Solution:* Let $P(x, y)$ be the assertion in the given equation. $P(2x, y)$ gives

$$f(2x)f(2y) = f(2x + y) \ \ \forall\, x, y \in \mathbb{R}$$

then by symmetry we see that $f(2x+y) = f(x+2y)$ for any two reals x, y $(*)$. But clearly, $(*)$ means f is constant, to see this, fix $t \in \mathbb{R}$ and plug $(x, y) = (-\frac{t}{3}, \frac{2t}{3})$ in $(*)$, we obtain $f(t) = f(0) := C$ for any $t \in \mathbb{R}$. Now we just check the possible values of this constant by plugging in the given equation, we see that $C^2 = C$, so $C = 0$ or 1. Therefore, we get two constant solutions to the given equation:

$$(1) \ \ f(x) = 0 \ \ \forall\, x \quad \text{and} \ \ (2) \ \ f(x) = 1 \ \ \forall\, x.$$

□

2nd Solution: $P(x, 0)$ gives $f(x)f(0) = f(x)$ for every $x \in \mathbb{R}$. So let's consider two cases,
$Case1:$ $f(0) \neq 1$. Then we get the 1st solution: $f(x) = 0 \ \ \forall\, x \in \mathbb{R}$.
$Case2:$ $f(0) = 1$. Then $P(x, -x) \Rightarrow f(x)f(-2x) = 1$, which means $f(x) \neq 0 \ \forall\, x \in \mathbb{R}$. Now $P(0, x) \Rightarrow f(2x) = f(x)$, and so $P(x, x)$ implies

$$f(x)^2 = f(x)f(2x) = f(2x) = f(x)$$

and this gives the other solution $f(x) = 1 \ \forall\, x \in \mathbb{R}$. □

Problem 15. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the inequality

$$f(f(f(x))) \geq f(x + y) + y$$

holds for any $x, y \in \mathbb{R}$.

Solution: Let $P(x, y)$ be the assertion in the given inequality. $P(x, f(f(x)) - x)$ gives $x \geq f(f(x))$ for any $x \in \mathbb{R}$ $(*)$. Now if we replace x by $f(x)$ in $(*)$ we'll get

$$f(x) \geq f(f(f(x))) \geq f(x + y) + y \ \ \forall\, x, y$$

Then replace y by $y - x$ in the last inequality we get

$$f(x) \geq f(y) + y - x \Leftrightarrow f(x) + x \geq f(y) + y \ \ \forall\, x, y$$

but this asymmetry immediately gives that the expression $f(x)+x$ is constant over $x \in \mathbb{R}$, call this constant C. Therefore, we'll get one solution to our inequality (can be easily checked):

$$f(x) = C - x \ \ \forall\, x \in \mathbb{R}$$

where C is a real constant. □

Problem 16. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n) = 2f(f(n))$ for every integer n.

Solution: We simply generalise the given equation into:

$$f(n) = 2f(f(n)) = 4f(f(f(n))) = ... = 2^{k-1}f(f(...f(n)...))$$

where f occurs k times on the rightmost side. Hence, $2^m | f(n)$ for every $m \in \mathbb{N}$, which immediately gives the only possible solution: $f(n) = 0 \;\; \forall \, n \in \mathbb{Z}$. □

Problem 17. Show that the function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x + f(x) + y) = x + f(x) + f(y)$$

for all $x, y \in \mathbb{R}$ is bijective. (a bijective function is both injective and surjective)

Solution: Just plug $y = -f(x)$, we get $f(x) = x + f(x) + f(-f(x))$, or simply $f(-f(x)) = -x \;\; \forall \, x \in \mathbb{R}$. And from the last equation we can obviously see that f is bijective (injective surjective). □

Problem 18. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(x) + 2y) = x + f(f(x)) + 2f(y)$$

for every $x, y \in \mathbb{R}$.

Solution: Let $P(x, y)$ be the assertion in the given equation. $P(-2x, x)$ gives

$$f(-2x + f(-2x) + 2x) = -2x + f(f(-2x)) + 2f(x) \Leftrightarrow f(x) = x \;\; \forall x \in \mathbb{R}$$

which is indeed a solution. □