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Geometry – L4

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Contents

Problems

Problem 1. Let ABC be a triangle satysfying $2(AB - AC) = BC$. Let D be a point on segment BC satysfying $AB + BD = AC + CD$. Prove that $2\sphericalangle ADC = \sphericalangle ACB$.

Problem 2. Let ABC be a triangle and DEF its intouch triangle. Let I_A be an A-excircle of ABC and consider midpoint M of the segment DI_A. Prove that circumcircle of BCM is tangent to the incircle of the triangle ABC.

Problem 3. Let ABC be a triangle such that $AB < AC$. The circle ω is tangent to AB at B and to the segment AC at D. Let E be a projection of D on BC. The circle ω intersects circumcircle of ABC at B and P. Prove that $\sphericalangle CPE = 2\sphericalangle ACB$.

Problem 4*. Let I and O be incenter and circumcenter of triangle ABC. Let P and Q lie on segments AC and BC, respectively such that $PA = AB = BQ$. Prove that circumradius of triangle CPQ equals OI.

Problem 5. Consider a convex pentagon $ABCDE$ and a variable point X on its side CD. Suppose that points K, L lie on the segment AX such that $AB = BK$ and $AE = EL$ and that the circumcircles of triangles CXK and DXL intersect for the second time at Y. As X varies, prove that all such lines XY pass through a fixed point, or they are all parallel.

Problem 6. Two circles Γ_1 and Γ_2 meet at two distinct points A and B. A line passing through A meets Γ_1 and Γ_2 again at C and D respectively, such that A lies between C and D. The tangent at A to Γ_2 meets Γ_1 again at E. Let F be a point on Γ_2 such that F and A lie on different sides of BD, and $2\sphericalangle AFC = \sphericalangle ABC$. Prove that the tangent at F to Γ_2, and lines BD and CE are concurrent.

Problem 7. Let ABC be an acute triangle ($AB < AC$) in which AH is an altitude, AM is median, and O is circumcenter. Perpendicular bisectors of AB and AC intersect AH at P and Q. Let J be the circumcenter of triangle OPQ. Prove that $\sphericalangle CAJ = \sphericalangle BAM$.

Problem 8. Let M be a midpoint of side BC of triangle ABC. Denote by P and Q projections of M on AB and AC, respectively. Let N be a midpoint of side PQ. Prove that $AO \parallel MN$, where O is the circumcenter of triangle ABC.

Problem 9*. Hexagon $ABCDEF$ is circumscribed around circle with center O. Prove that if O is the circumcenter of ACE, then circumcircles of triangles OAD, OBE, OCF concur at point different than O.

Problem 10★. Let $ABCD$ be a quadrilateral inscribed in a circle with center O. On sides BC and AD we build externally triangles BCE and ADF such that

$$BE = CE, \quad AF = DF \quad \text{and} \quad \measuredangle BEC + \measuredangle AOD = \measuredangle AFD + \measuredangle BOC = 180^\circ.$$

Let M and N are midpoints of AB and CD, respectively. Prove that BN, CM and EF concur.

Problem 11★. Let ABC be an isosceles triangle ($AC = BC$) with circumcircle ω. Let M be the midpoint of segment AB. The line tangent to a circle with diameter CM and passing through B, different than BC, intersects ω again at D. Prove that there is circle tangent to DC, CA, AB and circle ω.

Problem 12. Let ABC be an acute scalene triangle with circumcircle ω and incenter I. Suppose the orthocenter H of BIC lies inside ω. Let M be the midpoint of the longer arc BC of ω. Let N be the midpoint of the shorter arc AM of ω. Prove that there exists a circle tangent to ω at N and tangent to the circumcircles of BHI and CHI.

Problem 13. Let $ABCD$ be a described quadrilateral. The segments AB, BC, CD and DA are the diameters of the circles ω_1, ω_2, ω_3 and ω_4, respectively. Prove that there exists a circle tangent to all of the circles ω_1, ω_2, ω_3 and ω_4.

Problem 14★. Convex quadrilaterals $ABCD$ and $PQRS$ have equal areas. Moreover

$$AB = PQ, \quad BC = QR, \quad CD = RS, \quad DA = SP.$$

Prove that there exist points P', Q', R', S' which lie on a plane of quadrilateral $ABCD$, such that

$$AP' = BQ' = CR' = DS'$$

and quadrilaterals $PQRS$ and $P'Q'R'S'$ are congruent.

Problem 15. In triangle ABC the incircle ω centred at I touches segment BC at D. Let AH be the altitude of triangle ABC. Point K is symmetric to H with respect to the point D. Moreover given is tangent KL to ω, where L lies on AC. Prove that ID bisects BL.

Problem 16. Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

Problem 17. Let ABC be a triangle with circumcircle Ω and mixtilinear circles ω_A, ω_B, ω_C. Assume that ω_A is tangent to Ω at T_A. Let incircle of triangle ABC with center I is tangent to BC, CA and AB are D, E, F, respectively. Prove that:

(1) Point I is the midpoint of the segment connecting tangent points of ω_A with AB and AC.

(2) $T_A A$ is symmedian of triangle $DT_A E$.
(3) The line passing through the tangent point of ω_A with Ω and the incenter I of ABC intersects Ω at midpoint M of the arc BAC.
(4) Quadrilaterals BT_AID and CT_AIE are cyclic.
(5) Quadrilaterals BT_AID and CT_AIE are harmonic.
(6) Point T_A is the center of spiral similarity mapping AI to ID.
(7) Denote by Q the tangent point of the A-excircle and BC. Then, $\sphericalangle BAT_A = \sphericalangle QAC$, i.e. AT_A and AQ are isogonal with respect to ABC.
(8) Lines MT_A, AQ intersect on ω.
(9) $T_A A$ and $T_A D$ are isogonal with respect to $BT_A C$.
(10) Let N be a midpoint of arc BC of Ω. Lines BC, $T_A N$, B'C' are concurrent, where B', C' are tangency points of ω_A with AB and AC.

Solutions

Problem 1. Let ABC be a triangle satysfying $2(AB - AC) = BC$. Let D be a point on segment BC satysfying $AB + BD = AC + CD$. Prove that $2\sphericalangle ADC = \sphericalangle ACB$.

Solution. Let $E,\ I,\ M$ be a tangency point of the incircle with BC, incenter of triangle ABC and midpoint of BC.

Let D' be a touching point of A-exircle of ABC with BC. Then

$$AB + BD' = AB + CE = AB + \frac{BC + AC - AB}{2} = \\ = \frac{AB + AC + BC}{2} = AC + CD',$$

so $D = D'$.

Moreover let P and S be points on AD, that $PM \perp BC$ and $SE \perp BC$. Homothety with center A sending incenter to excenter sends IE to line passing through D and perpendicular to BC. Thus S lies on incricle of triangle ABC, so SE is a diameter of that circle. Thus $PM = \frac{1}{2}SE = IE$.

Since $AB > AC$, points $B,\ D,\ E,\ C$ lie in that order on BC. We have

$$EM = \frac{1}{2}DE = \frac{1}{2}(CD - CE) = \frac{1}{2}(BE - CE) \\ = \frac{1}{4}((BC + AB - AC) - (BC + AC - AB)) = \frac{1}{4}\left(2(AB - AC)\right) = \frac{1}{4}BC.$$

Therefore $DM = EM = EC$, so triangles DMP and CEI are congruent, hence

$$\sphericalangle ADC = \sphericalangle ICE = \frac{1}{2}\sphericalangle ACB.$$

□

Problem 2. Let ABC be a triangle and DEF its intouch triangle. Let I_A be an A-excircle of ABC and consider midpoint M of the segment DI_A. Prove that circumcircle of BCM is tangent to the incircle of the triangle ABC.

Solution. Denote the incircle of triangle ABC by o. Then $DF \parallel I_AB$. Similarly, $DE \parallel I_AC$. Midline of trapezoid $DFBI_A$ passes through midpoints of BF and BD, so it is a radical axis of o and point B. Similarly midline of trapezoid $DECI_A$ is a radical axis of o and C. Therefore M is a radical center of o, B and C. Thus $MB^2 = MD \cdot MT = MC^2$, where T is a second intersection of MD with o. From $MB^2 = MD \cdot MT$ we get that triangles MBT and MDB are similar, so $\sphericalangle BTM = \sphericalangle MBD$. Similarly, $\sphericalangle MTC = \sphericalangle DCM$. Therefore

$$\begin{aligned} &\sphericalangle BTC + \sphericalangle CMB = \sphericalangle BTM + \sphericalangle MTC + \sphericalangle CMB = \\ &= \sphericalangle MBC + \sphericalangle BCM + \sphericalangle CMB = 180^\circ, \end{aligned}$$

so T lies on circumcircle ω of triangle BCM.

We are left with proving that o and ω have the same tangent at T. Let k be the tangent line to o at T. Then

$$\sphericalangle(k, TD) = \sphericalangle TDB = \sphericalangle TMB + \sphericalangle MBD = \sphericalangle TCB + \sphericalangle BCM = \sphericalangle TCM,$$

thus k is also tangent to ω. □

Problem 3. Let ABC be a triangle such that $AB < AC$. The circle ω is tangent to AB at B and to the segment AC at D. Let E be a projection of D on BC. The circle ω intersects circumcircle of ABC at B and P. Prove that $\sphericalangle CPE = 2\sphericalangle ACB$.

Solution. Let ω intersects BC at $X \neq B$. Notice that

$$\sphericalangle XPC = \sphericalangle BPC - \sphericalangle BPX = 180^\circ - \sphericalangle BAC - \sphericalangle ABC = \sphericalangle DCX,$$

so circumcircle of XPC is tangent to AC. Since PX is a radical axis of ω and circumcircle of triangle XPC, it mast passes through midpoint N of AC. Then $NC = NE = ND$ and

$$\sphericalangle NPC = \sphericalangle NCE = \sphericalangle NEC,$$

so N, E, C, P are concyclic. Therefore

$$\sphericalangle CPE = \sphericalangle DNE = 2\sphericalangle ACB.$$

□

Problem 4★. Let I and O be incenter and circumcenter of triangle ABC. Let P and Q lie on segments AC and BC, respectively such that $PA = AB = BQ$. Prove that circumradius of triangle CPQ equals OI.

Solution. X □

Problem 5. Consider a convex pentagon $ABCDE$ and a variable point X on its side CD. Suppose that points K, L lie on the segment AX such that $AB = BK$ and $AE = EL$ and that the circumcircles of triangles CXK and DXL intersect for the second time at Y. As X varies, prove that all such lines XY pass through a fixed point, or they are all parallel.

Solution. Let ω_B be the circle with center B and radius AB and ω_E the circle of center E and raidus AE. So $K = AX \cap \omega_B$ and $L = \omega_E \cap AX$. Let $F \in \omega_B$, $G \in \omega_E$ such that F, A, G are aligned and this line is parallel to CD. Also let P, Q be the second intersections of FC and GD with ω_B and ω_E respectively.

Let $Z = FC \cap GD$, and ℓ_Z the line trough Z parallel to CD and so also to FG, and let $XY \cap \ell_Z = W$. By Reim's theorem, $CXKP$ and $DXLQ$ are cyclic. By Reim's theorem, $YPZW$ is cyclic and so is $YQZW$. In other words $PYQZW$ is a cyclic pentagon. Therefore XY always passes through the second intersection of ℓ_Z and circumcircle of ZPQ. □

Problem 6. Two circles Γ_1 and Γ_2 meet at two distinct points A and B. A line passing through A meets Γ_1 and Γ_2 again at C and D respectively, such that A lies between C and D. The tangent at A to Γ_2 meets Γ_1 again at E. Let F be a point on Γ_2 such that F and A lie on different sides of BD, and $2\sphericalangle AFC = \sphericalangle ABC$. Prove that the tangent at F to Γ_2, and lines BD and CE are concurrent.

Solution. Let the bisector of angle ABC intersects Γ_2 at G and let $X = AF \cap CE$. Then $\sphericalangle XCF = \sphericalangle ECB = \sphericalangle EAB = \sphericalangle BFA = \sphericalangle BFX$ thus $BCXF$ is cyclic. Moreover $\sphericalangle CBG = \sphericalangle CFA = \sphericalangle CFX = \sphericalangle CBX$, so B, G, and X are collinear. The result follows from the Pascal Theorem for hexagon $FFGBDA$. □

Problem 7. Let ABC be an acute triangle $(AB < AC)$ in which AH is an altitude, AM is median, and O is circumcenter. Perpendicular bisectors of AB and AC intersect AH at P and Q. Let J be the circumcenter of triangle OPQ. Prove that $\sphericalangle CAJ = \sphericalangle BAM$.

Solution. Note that ABC and OPQ are similar. The scale of that similarity equals the ratio of AH and altitude from O in triangle OPQ which is equal to HM.

On the other hand, the similarity ratio is equal to the ratio of circumradii of these triangles. Therefore

$$\frac{AO}{OJ} = \frac{AH}{HM} \quad \text{oraz} \quad AO \perp OJ,$$

so $\triangle AOJ \sim \triangle AHM$, thus

$$\sphericalangle BAM = \sphericalangle BAH + \sphericalangle HAM = \sphericalangle CAO + \sphericalangle OAJ = \sphericalangle CAJ.$$

□

Problem 8. Let M be a midpoint of side BC of triangle ABC. Denote by P and Q projections of M on AB and AC, respectively. Let N be a midpoint of side PQ. Prove that $AO \parallel MN$, where O is the circumcenter of triangle ABC.

Solution. Let AM intersects circumcircle of triangle ABC at M'. Then triangles PMQ and $BM'C$ are similar, so $\sphericalangle NMP = \sphericalangle MM'B$. Therefore perpendicular bisector of BC is a symmedian of triangle PMQ, which easily finish the proof. □

Problem 9★. Hexagon $ABCDEF$ is circumscribed around circle with center O. Prove that if O is the circumcenter of ACE, then circumcircles of triangles OAD, OBE, OCF concur at point different than O.

Solution. Let K, L, M, N, P, Q be tangency point of ω with respectively sides AB, BC, CD, DE, EF, FA. Invert picture wrt incenter of $ABCDEF$. Then points K, L, M, N, P, Q are fixed. Points A, B, C, D, E, F are sending to midpoints A', B', C', D', E', F' of QK, KL, LM, MN, NP, PQ. Circumcircles of triangles OAD, OBE and OCF are sending to lines connecting opposite sides of hexagon $KLMNPQ$. Since $OA = OC = OE$, then $OA' = OC' = OE'$. From Pythagorean theorem easy to see that segments KQ, LM, NP are equal.

Let α be inscribed angle of ω based on arc KQ. Since $KQ = LM$, then $KLMQ$ is isosceles trapezoid. Therefore from parallelities $KM \parallel B'C'$, $LQ \parallel B'A'$, $C'A' \parallel MQ$ it follows

$$\sphericalangle A'C'B' = \sphericalangle QMK = \alpha = \sphericalangle MQL = \sphericalangle C'A'B'.$$

Similarly

$$\sphericalangle E'C'D' = \sphericalangle C'E'D' = \alpha \quad \text{oraz} \quad \sphericalangle A'E'F' = \sphericalangle E'A'F' = \alpha.$$

The statement follows from Jacobi Theorem. □

Problem 10★. Let $ABCD$ be a quadrilateral inscribed in a circle with center O. On sides BC and AD we build externally triangles BCE and ADF such that

$$BE = CE, \quad AF = DF \quad \text{and} \quad \sphericalangle BEC + \sphericalangle AOD = \sphericalangle AFD + \sphericalangle BOC = 180^\circ.$$

Let M and N are midpoints of AB and CD, respectively. Prove that BN, CM and EF concur.

Solution. Let P be a point of intersection of AC and BD. Then

$$\sphericalangle ACD = \sphericalangle ABD = \frac{1}{2}\sphericalangle AOD = 90^\circ - \frac{1}{2}\sphericalangle BEC = \sphericalangle BCE = \sphericalangle CBE.$$

Triangles ABP and DCP are similar, so BMP and CNP are similar, thus $\sphericalangle BPM = \sphericalangle CPN$.

From Jacobi theorem applied for triangle BPC we get that BN, CM and PE have the common point X. Similarly AN, DM and PF concur at Y. From Pappus theorem for hexagon $ANBDMC$ we get that Y, P, X are collinear, therefore E, X, P, Y, F are collinear. □

Problem 11★. Let ABC be an isosceles triangle ($AC = BC$) with circumcircle ω. Let M be the midpoint of segment AB. The line tangent to a circle with diameter CM and passing through B, different than BC, intersects ω again at D. Prove that there is circle tangent to DC, CA, AB and circle ω.

Solution. Let Ω be circle tangent to AC, AB and ω. Let D' be a second intersection of tangent Ω passing through C with ω. It is enough to prove that $D' = D$. We will show that $\sphericalangle ABD + \sphericalangle ACD' = 180°$.

Let J be the center of Ω, K and L are projections of J on AC and AB, and let N be midpoint of MC. Then $\sphericalangle ACD' = 2\sphericalangle KCJ = 180 - 2\sphericalangle CJK$ and $\sphericalangle ABD = 2\sphericalangle MBN$, so it is enough to prove that $\sphericalangle MBN = \sphericalangle CJK$ (or equivalently $\sphericalangle ACJ = \sphericalangle MNB$).

Denote the incenter of triangle ABC by I. From the well known fact I is a midpoint of KL. Also I lies on CM and AJ.

Let JK and CM intersects at P. Then IM and JL are parallel (both are perpendicular to AB), hence $\frac{KI}{IL} = \frac{KP}{PJ}$, since $KI = IL$, then $KP = PJ$.

Right-angled triangles CKP and CMB are similar, becouse $\sphericalangle KCP = \sphericalangle MCB$. Theorefore $\frac{CK}{KP} = \frac{CM}{MB}$. Since $KJ = 2KP$ and $NM = \frac{1}{2}CM$, we get $\frac{CK}{KJ} = \frac{NM}{MB}$. Therefore right-angled triangles CKJ and NMB are similar, so $\sphericalangle MBN = \sphericalangle CJK$.

$\square$

Problem 12. Let ABC be an acute scalene triangle with circumcircle ω and incenter I. Suppose the orthocenter H of BIC lies inside ω. Let M be the midpoint of the longer arc BC of ω. Let N be the midpoint of the shorter arc AM of ω. Prove that there exists a circle tangent to ω at N and tangent to the circumcircles of BHI and CHI.

Solution. Denote the circumcircles of BHI and CHI by ω_1 and ω_2 and their centers by O_1 and O_2, respectively. Let O be the center of ω. Let R be the radius of ω.

Since H is the orthocenter of triangle BIC it follows that I is the orthocenter of triangle BHC. Therefore

$$\begin{aligned}\sphericalangle HIB &= 180° - (\sphericalangle BHI + \sphericalangle IBH) = 180° - (90° - \sphericalangle CBH + 90° - \sphericalangle BHC) = \\ &= 180° - \sphericalangle HCB.\end{aligned}$$

Denote by r the radius of circle ω_1, then from sine law we get

$$\begin{aligned}2r &= \frac{HB}{\sin\sphericalangle HIB} = \frac{HB}{\sin(180° - \sphericalangle HIB)} = \frac{HB}{\sin\sphericalangle HCB} = \\ &= \text{diameter of circumcircle of the triangle } BHC.\end{aligned}$$

Using the same argument for triangles CIH i BIC we see that r is equal to radii of ω_1, ω_2, circumcircles of BIC and BHC.

From the following angle chase it follows that

$$\sphericalangle BHC = 180^\circ - \sphericalangle BIC = 180^\circ - \left(180^\circ - \frac{1}{2}\sphericalangle CBA - \frac{1}{2}\sphericalangle BCA\right) =$$
$$= \frac{1}{2}(\sphericalangle CBA + \sphericalangle BCA) = 90^\circ - \frac{1}{2}\sphericalangle BAC.$$

Since H lies inside ω and $\sphericalangle BAC$ is acute we conclude that

$$\sphericalangle BAC < \sphericalangle BHC = 90^\circ - \frac{1}{2}\sphericalangle BAC < 90^\circ$$

so

$$2r = \text{diameter of circumcircle of } BHC = \frac{BC}{\sin\sphericalangle BHC} < \frac{BC}{\sin\sphericalangle BAC} = 2R,$$

thus $r < R$.

Let $\sphericalangle BAC = \alpha$, $\sphericalangle CBA = \beta$, $\sphericalangle ACB = \gamma$. Then

$$\sphericalangle BO_1I = 2\sphericalangle BHI = 2(90^\circ - \sphericalangle CBH) = 2\sphericalangle ICB = \gamma,$$

so

$$\sphericalangle IBO_1 = 90^\circ - \frac{1}{2}\sphericalangle BO_1I = 90^\circ - \frac{\gamma}{2} = \frac{\alpha+\beta}{2},$$

and finally

$$\sphericalangle ABO_1 = \sphericalangle IBO_1 - \sphericalangle IBA = \frac{\alpha+\beta}{2} - \frac{\beta}{2} = \frac{\alpha}{2} = \sphericalangle BAI.$$

This shows that $BO_1 \parallel AI$, and moreover, rays $BO_1^{\rightarrow}$, $AI^{\rightarrow}$ determine opposite directions. Similarly, rays $CO_2^{\rightarrow}$, $AI^{\rightarrow}$ are parallel and determine opposite directions. Therefore rays are parallel and $BO_1^{\rightarrow}$, $CO_2^{\rightarrow}$ determine the same direction. Since $BO_1 = r = CO_2$, it follows that vectors $\overrightarrow{BO_1}$, $\overrightarrow{CO_2}$ are equal. Denote this vector by $\vec{v}$.

Note that $ON \perp AM$. Moreover

$$\sphericalangle IAM = \sphericalangle IAC + \sphericalangle CAM = \sphericalangle IAC + \sphericalangle CBM =$$
$$= \sphericalangle IAC + \frac{1}{2}(180^\circ - \sphericalangle BMC) = \sphericalangle IAC + \frac{1}{2}(180^\circ - \sphericalangle BAC) = 90^\circ,$$

so $AM \perp AI$, hence $ON \parallel AI \parallel BO_1 \parallel \vec{v}$. Let X be a point such that $\overrightarrow{OX} = \vec{v}$. Since $ON \parallel \vec{v}$, X lies on line ON. It actually lies on ray $ON^{\rightarrow}$ since rays $ON^{\rightarrow}$, $AI^{\rightarrow}$ determine opposite directions.

Note that translation by $\vec{v}$ maps triangle BCO to triangle O_1O_2X. Therefore $O_1X = BO = R$ and $O_2X = CO = R$.

Let ω' be the circle centered at X with radius $R - r > 0$.

Observe that $O_1X = R = r + (R-r)$, so ω' is tangent externally to ω_1. For similar reason it is tangent externally to ω_2. Moreover $OX = r = R-(R-r) = ON - XN$, so ω' is tangent to ω internally at point N. $\square$

Problem 13. Let $ABCD$ be a described quadrilateral. The segments AB, BC, CD and DA are the diameters of the circles ω_1, ω_2, ω_3 and ω_4, respectively. Prove that there exists a circle tangent to all of the circles ω_1, ω_2, ω_3 and ω_4.

Solution. X □

Problem 14★. Convex quadrilaterals $ABCD$ and $PQRS$ have equal areas. Moreover

$$AB = PQ, \quad BC = QR, \quad CD = RS, \quad DA = SP.$$

Prove that there exist points P', Q', R', S' which lie on a plane of quadrilateral $ABCD$, such that

$$AP' = BQ' = CR' = DS'$$

and quadrilaterals $PQRS$ and $P'Q'R'S'$ are congruent.

Solution. We will first prove the following lemma:

Lemma 1. There are at most two quadrilaterals (modulo congruent) with consecutive sides of length a, b, c, d and area F.

Proof. We will use the following formula for the area of quadrilateral with sides a, b, $c$and d :

$$F = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\phi},$$

where $p = a + b + c + d$, and ϕ is the arithmetic mean of the angles α and β between the sides a, b and c, d, respectively.

From this formula we see that

$$\cos^2\phi = \frac{(p-a)(p-b)(p-c)(p-d) - F^2}{abcd},$$

so ϕ can take at most two different values. It remains to show that side lengths and ϕ determine unique quadrilateral, but this is obvious since if we increase the diagonal e, then ϕ is also increased, so e is determined by ϕ but quadrilateral is determined by a, b, c, d and e. □

If $ABCD$ and $PQRS$ are congruent we can take $P'Q'R'S' = ABCD$. Suppose that $ABCD \not\equiv PQRS$. Let ϕ_1 and ϕ_2 denote angles from the lemma of $ABCD$ and $PQRS$, respectively.

If $ABCD$ (or $PQRS$) are cyclic quadrilaterals, then $\phi_1 = \phi_2 = 90°$, so $PQRS \equiv ABCD$, contrary to the assumption. Therefore, none of the quadrilateral $ABCD$, $PQRS$ is cyclic. Let X be the point of intersection of perpendicular bisectors of AC and BD. Then of course $AX = CX \neq BX = DX$. Let $x = AX$ and $y = BX$. Take P', Q', R' and S' on $\overrightarrow{XA}$, $\overrightarrow{XB}$, $\overrightarrow{XC}$ and $\overrightarrow{XD}$, respectively such that

$$XQ' = XS' = x \quad \text{and} \quad XP' = XR' = y.$$

Note that X is an intersection of perpendicular bisectors of $P'R'$ and $Q'S'$, too but $XA = x \neq y = XP'$.

Therefore $ABCD \not\equiv P'Q'R'S'$ but

$$AXB \equiv Q'XP', BXC \equiv R'XQ', CXD \equiv S'XR' \quad \text{and} \quad DXA \equiv P'XS'.$$

Hence $ABCD$ and $P'Q'R'S'$ have the same areas and $PQRS \not\equiv ABCD$, so by the lemma $PQRS \equiv P'Q'R'S'$. Moreover

$$AP' = BQ' = CR' = DS' = |x - y|.$$

□

Problem 15. In triangle ABC the incircle ω centred at I touches segment BC at D. Let AH be the altitude of triangle ABC. Point K is symmetric to H with respect to the point D. Moreover given is tangent KL to ω, where L lies on AC. Prove that ID bisects BL.

Solution. Let M and N be the midpoints of segments BL and AK, respectively. Points N, I, D lies on the midline of triangle AHK, because $AH \perp BC$ and $ID \perp BC$.

Moreover, since points M, N and I are collinear (according to Newton-Gauss line theorem), so must I, M, D. Hence we are done. □

Problem 16. Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

Solution. For simplicity, we consider only the case when X lies inside segment BA_0. All other cases are similar.

Let B_0 and C_0 be the midpoints of segments AC and AB, respectively. Let one tangent meet segment AC_0 at P and let the other tangent meet segment CB_0 at Q.

By the Gauss-Newton Theorem for circumscribed quadrilateral $APXQ$, the midpoints of segments AA_0, AX, and PQ are collinear. Therefore, the midpoint R of segment PQ lies on the midline of triangle ABC opposite to vertex A.

Let S be the reflection of point A about point R. Then S lies on line BC, and quadrilateral $APSQ$ is a parallelogram. Therefore,

$$\frac{C_0P}{A_0S} = \frac{B_0Q}{A_0S}$$

and so $C_0P = B_0Q$. □

Problem 17. Let ABC be a triangle with circumcircle Ω and mixtilinear circles ω_A, ω_B, ω_C. Assume that ω_A is tangent to Ω at T_A. Let incircle of triangle ABC with center I is tangent to BC, CA and AB are D, E, F, respectively. Prove that:

(1) Point I is the midpoint of the segment connecting tangent points of ω_A with AB and AC.
(2) $T_A A$ is symmedian of triangle DT_AE.
(3) The line passing through the tangent point of ω_A with Ω and the incenter I of ABC intersects Ω at midpoint M of the arc BAC.
(4) Quadrilaterals BT_AID and CT_AIE are cyclic.
(5) Quadrilaterals BT_AID and CT_AIE are harmonic.
(6) Point T_A is the center of spiral similarity mapping AI to ID.
(7) Denote by Q the tangent point of the A-excircle and BC. Then, $\sphericalangle BAT_A = \sphericalangle QAC$, i.e. AT_A and AQ are isogonal with respect to ABC.
(8) Lines MT_A, AQ intersect on ω.
(9) T_AA and T_AD are isogonal with respect to BT_AC.
(10) Let N be a midpoint of arc BC of Ω. Lines BC, T_AN, B'C' are concurrent, where B', C' are tangency points of ω_A with AB and AC.

Solution. See the following article for solutions $\square$

References

- Art of Problem Solving - `https://artofproblemsolving.com`
- Polish Mathematical Olympiad - `https://om.mimuw.edu.pl`
- Homepage of D. Burek - `http://dominik-burek.u.matinf.uj.edu.pl`