HELLO

SQUARES

GOOD?

$(a^{2n+1}+b^{2n+1}) = (a+b)(a^n - \dots)$

1

$\boxed{\text{Problem 39}}$ Solve in integers equation

$$x^2+4=y^5.$$

Used $\boxed{\text{mod } 11}$ $= 2\cdot 5+1$

$x^2+6^2 = y^5+2^5$

$x^2+36 = y^5+32$

$y+2 \mid y^5+2^5$ so

$y+2 \mid x^2+6^2$

$y+2 \equiv 3 \mod 4$, so $p=4l+3 \mid x^2+6 \Rightarrow p \mid x,\ p \mid 6 \Rightarrow p=2,3$ ↯

$y=1$

Proof if x is even $\Rightarrow y$ is even, so

$$\left.\begin{array}{l} y^5 \equiv 0 \pmod{16} \\ x^2+4 \equiv 4, 8 \pmod{16} \end{array}\right\} \text{contradiction}$$

so x is odd. then

$y^5 = x^2+4 \equiv 1 \mod 4$, so

$\boxed{y \equiv 1 \mod 4}$

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$x^3+7=y^2$

if x is even $\Rightarrow$ y is ~~even~~ odd but $y^2 \equiv 1 \mod 8$, so

$x^3+7=y^2 \equiv 1 \mod 8$, so

$\underbrace{x^3 \equiv -6 \equiv 2}_{\text{impossible}} \mod 8$

so x is odd, y is even. $y^2 \equiv 0 \pmod 4 \Rightarrow$

$x^3 = y^2-7 \equiv -7 \equiv 1 \pmod 4$, so $x \equiv 1 \mod 4$

$\underset{\boxed{3 \bmod 4}}{(x+2)} \mid \underset{2^3}{x^3+8} = y^2+1$, so y^2+1 has divisor of the form $4k+3$

$\mod 13$

$\lightning$

$3^k - 1 = m^2 + n^2$ has infinitely many positive integer solutions

(3)

HINT: $k = 2^s$ s integer

k - power of 2.

$$3^{2^s} - 1 = (3^2 - 1)(3^2 + 1)(3^4 + 1) \cdots (3^{2^{s-1}} + 1) = A^2 + B^2$$

$(3^4 - 1)$

$3^8 - 1$

$8 = 2^2 + 2^2$

sum of square

also

also

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$x^4 - 4 = y^2 + z^2$ has no integer solution

(4)

odd exp.

<u>HINT</u> Prove that some prime of the form $4k+3$ divides $x^4 - 4$

Need to find $p \equiv 4k+3$ with odd exponent in decomposition of $x^4 - 4$

1) x is odd

2) x is even :) $x = 2k \Rightarrow 4(4k^4 - 1) = y^2 + z^2$

so

$x^4 - 4 = y^2 + z^2$ (5)

if $\boxed{x \text{ is odd}}$

$x^4 - 4 = (x^2-2)(x^2+2)$ then $x^2+2 \equiv 3 \pmod 4$, so

there exists prime $p = 4l+3$ with odd exponent in x^2+2.

But $p \nmid x^2-2$, otherwise $p \mid x^2-2$, $p \mid x^2+2 \Rightarrow p \mid 4$ ↯.

Odd exponent $x^2+2 = p^{\boxed{2S+1}}$ ← exponent

$\equiv 3 \bmod 4$ $\Longrightarrow$ p has odd exponent

$(4l+3)(4l_2+3) \equiv (4l_3+1)$

$(4l+1)(4l_2+1) = \underline{4l_3+1}$.

and $p \mid y^2+z^2$

↯.

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Case x is <u>odd</u>

$x^4-4=(x^2-2)(x^2+2)$, if x is odd then

$$\boxed{x^2+2 \equiv 3 \pmod 4}$$

A

$\parallel$

Fact If the number is of the form $4k+3$ then there exists prime $p \equiv 3 \pmod 4$ which divides A and has odd exponent.

$A = p^{2s+1} \cdot A'$, $p \nmid A'$.

But if $p \mid x^2+2 \Rightarrow p \nmid x^2-2$. Otherwise

$p \mid x^2+2-(x^2-2)=4 \Rightarrow p=2 \not\equiv 3 \bmod 4$

So p has odd exponent in $x^4-4=y^2+z^2$

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Case $x = 2k$ even. Then

$$4(4k^4-1) = y^2+z^2 \Rightarrow 4(2k^2-1)(2k^2+1) = y^2+z^2 \quad (7)$$

- If k is even, then $2k^2-1 \equiv 3 \pmod 4$ so by the last fact $\exists\ p \equiv 3 \pmod 4$ s.t. $p \mid 2k^2-1$ and p has odd exponent in $2k^2-1$. But $p \nmid 2k^2+1$ (otherwise $p \mid 2$) so p has odd exponent in $4(2k^2-1)(2k^2+1)$. $\lightning$
- If is odd then $2k^2+1 \equiv 3 \pmod 4$, the same argument implies contradiction. $\lightning$

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Next

Prove $x^8+1=n!$ has finitely many solutions. (8)

$x^8+1=(x^4)^2+1$ = Sum of two squares —

→ cannot have prime divisor of the form $\boxed{4k-3}$

In particular. $\boxed{n \leq 3}$ because $n \geq 3$ then

$3 \mid n! \rightarrow 3 \equiv 3 \bmod 4$

so $n = 0, 1, 2 \Rightarrow$ finitely many solution.

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Find all n-tuples $(a_1, a_2, \dots, a_n)$ positive integers (a)

Such that

$$\underbrace{(a_1!-1)(a_2!-1)\cdots(a_n!-1)-16}_{= x^2} \text{ is square.}$$

$$\boxed{(a_1!-1)(a_2!-1)\cdots(a_n!-1) = x^2+4^2} \quad ??$$

if $a_i > 3$ then $\boxed{a_i!-1 \equiv 3 \pmod 4}$ so

$p \equiv 3 \pmod 4$ s.t $p \mid a_i!-1$. So $p \mid x^2+4^2 \Rightarrow p \mid 4$ ✗

So $\forall i \in \{1, \dots, n\}$, $\boxed{a_i \leq 3}$.

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$a_i \leq 3 \Rightarrow a_i = 1, 2, 3$, $a_i = 1$? , $a_i - 1 = 0$

$0 = x^2 + 4^2 \rightarrow$ absurd.

$\boxed{a_i = 2, 3}$ Put $a_i = 2$ the $a_i! - 1 = 1$

Let m denots number of 3's.

$$\underbrace{(3!-1)(3!-1)\cdots(3!-1)}_{m} = x^2 + 4^2$$

$$\boxed{5^m = x^2 + 4^2} \Rightarrow x \text{ is } \underline{\text{odd}}$$

so modulo 8 $\Rightarrow$ m is even $m = 2s \Rightarrow$

$$(5^s - x)(5^s + x) = 4^2 \Rightarrow s = 1, \; m = 2$$

$$m^2-1 \mid 3^m + (n!-2)^m.$$ Assume $\underline{n \geq 2}$

(1)

1) m is even $\Leftarrow$ prove it.

if m is odd $\Rightarrow$ $8 \mid m^2-1$, so

$$8 \mid \underbrace{3^m + (n!-2)^m}_{\boxed{\text{odd}}} \quad ↯$$

so m is even. then $m^2-1 \equiv 3 \pmod 4$

and so $\exists$ $p \equiv 3 \pmod 4$ s.t $p \mid \underbrace{3^m + (n!-2)^m}_{\text{sum of 2 squares}}$ ↯

$p \mid 3$, $p \mid n!-2$

$p = 3$ $\quad 3 \mid n!-2$ $\Rightarrow$ $n \leq 2$

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FIND ALL positive integers (m, n) such that

$$m^2 - 1 \mid 3^m + (n! - 2)^m$$ ← Next

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Solve in integers

$$x^2 = y^7 + 7$$

$$\Downarrow$$

$$\underbrace{x^2 + 11^2}_{121} = \underbrace{y^7 + 2^7}_{128}$$

$y + 2$

:)

~~29~~

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$\underline{m=1,\ n=2}$ $\quad n>2$

$n=1 \Rightarrow m^2-1 \mid 3^m+(-1)^m$ so

Either i° $m^2-1 \mid 3^m+1$ and m is even or

ii° $m^2-1 \mid 3^m-1$ and m is odd

1° $m^2-1 \equiv 3 \pmod 4 \mid 3^{m}+1^2$ (every), the same argument holds. ↯

2° $8 \mid m^2-1 \mid 3^m-1$, m is odd — impossible

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so $\underline{m=2}$, $m^2-1 \mid 3^m + (2!-2)^m = 3^m$

$(m-1)(m+1) \mid 3^m$

$\parallel$

$(m-1)(m+1) = 3^k$ for some $k \leq m$.

$\underset{3^s}{\underset{\parallel}{(m-1)}}\underset{3^t}{\underset{\parallel}{(m+1)}}$ $\quad 3^t - 3^s = 2$ only possible if

$\boxed{m-1=1} \Rightarrow m=2$

$\boxed{n=m=2}$

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PROBLEM $\boxed{x^2 = y^7 + 7}$

q in A $\square$ A', $q \nmid A'$

$A = q \cdot v_q$

Case $y < -1$ – no solutions, so $y+2 > 0$. Take this equation modulo 4. $\Rightarrow$ $y \equiv 1 \bmod 4$ NOW

Key identity $\boxed{x^2 + 11^2 = y^7 + 2^7} =$

$$= (y+2)\left(y^6 - 2y^5 + 4y^4 - 8y^3 + 16y^2 - 32y + 64\right)$$

Since $y \equiv 1 \pmod 4 \Rightarrow y+2 = 3 \pmod 4$, so $\exists\, p \equiv 3 \bmod 4$ such that p has odd exponent in $\boxed{y+2}$.

Key observation $p \nmid y^6 - 2y^5 + 4y^4 - 8y^3 + 16y^2 - 32y + 64$ Why?

$p \mid y+2 \Rightarrow y \equiv -2 \pmod p$ so

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$$0 \equiv y^6 - 2y^5 + 4y^4 - 8y^3 + 16y^2 - 32y + 64 \equiv$$

$$(-2)^6 - 2(-2)^5 + 4\cdot(-2)^4 - 8(-2)^3 + 16(-2)^2 - 32(-2) + 64 =$$

$$= 64 + 64 + 64 + 64 + 64 + 64 + 64 = 7\cdot 64$$

$$\begin{cases} p \mid 7\cdot 64 \text{ and moreover} \\ p \mid x^2 + 11^2 \end{cases}$$

$$\Downarrow \quad p = 7, 2 \quad p \mid x, \; p \mid 11 \quad \Rightarrow \; ↯$$

so $p \mid (y+2)$ and $p \nmid y^6 - 2y^5 + 4y^4 - 8y^3 + 16y^2 - 32y + 64$

odd exp.

p has odd exponent in $y^7 + 2^7$

$$\overset{\shortparallel}{x^2 + 11^2}$$

$$v_p(y+2) = 2l+1, \; v_p(____) = 0 \quad ↯$$

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$$\boxed{A^2+B^2 \rightarrow p=4k+3 \text{ has alway even exponent}}$$

Key point is to find identity

$$\{\text{Something}\} = A^2+B^2$$

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$$0 \equiv y^6 - 2y^5 + 4y^4 - 8y^3 + 16y^2 - 32y + 64 \equiv$$

$$(-2)^6 - 2(-2)^5 + 4\cdot(-2)^4 - 8(-2)^3 + 16(-2)^2 - 32(-2) + 64 =$$

$$= 64 + 64 + 64 + 64 + 64 + 64 + 64 = 7\cdot 64$$

$$\left\{\begin{array}{l} p \mid 7\cdot 64 \text{ and moreover} \\ p \mid x^2 + 11^2 \end{array}\right. \Rightarrow \lightning$$

$$\Downarrow$$

$$p = 7, 2$$

$$p \mid x, \; p \mid 11$$

so $p \mid (y+2)$ and $p \nmid y^6 - 2y^5 + 4y^4 - 8y^3 + 16y^2 - 32y + 64$

odd exp.

p has odd exponent in $y^7 + 2^7$

$$\Downarrow$$

$$x^2 + 11^2$$

$$v_p(y+2) = 2l+1, \; v_p(_\;_\;_\;_) = 0$$

$\lightning$

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