Day 3

Problem 1. Positive reals $a_1, a_2, ...$ satisfy:

- $a_{n+1} = a_1^2 \cdot a_2^2 \cdot ... \cdot a_n^2 - 3 \quad \forall n$ positive integers
- $\frac{1}{2}(a_1 + \sqrt{a_2 - 1})$ is a positive integer.

Prove that $\frac{1}{2}(a_1 \cdot a_2 \cdot ... \cdot a_n + \sqrt{a_{n+1} - 1})$ is a positive integer.

Problem 2. Let $a_0 > 0$ be a real number, and let

$$a_n = \frac{a_{n-1}}{\sqrt{1 + 2020 \cdot a_{n-1}^2}}, \quad n = 1, 2, ..., 2020$$

Show that $a_{2020} < \frac{1}{2020}$

Problem 3. The real numbers $x_1, x_2, ..., x_{2011}$ satisfy

$$x_1 + x_2 = 2x_1', \quad x_2 + x_3 = 2x_2', \ ..., \quad x_{2011} + x_1 = 2x_{2011}'$$

where $x_1', x_2', ..., x_{2011}'$ is a permutation of $x_1, x_2, ..., x_{2011}$. Prove that $x_1 = x_2 = ... = x_{2011}$.

Problem 4. Let $x_1, x_2, ..., x_n$ $(n \geq 2)$ be real numbers greater than 1. Suppose that $|x_i - x_{i+1}| < 1$ for $i = 1, 2, ..., n-1$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + ... + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} < 2n - 1$$

Problem 5. Let $a_0, a_1, a_2, ...$ be an infinite sequence of real numbers satisfying $\frac{a_{n-1} + a_{n+1}}{2} \geq a_n$ for all positive integers n. Show that

$$\frac{a_0 + a_{n+1}}{2} \geq \frac{a_1 + a_2 + ... + a_n}{n}$$

holds for all positive integers n.

Problem 6. Let $a_0, a_1, ..., a_N$ be real numbers satisfying $a_0 = a_N = 0$ and

$$a_{i+1} - 2a_i + a_{i-1} = a_i^2$$

for $i = 1, 2, ..., N-1$. Prove that $a_i \leq 0$ for $i = 1, 2, ..., N-1$.

Problem 7. Given $x_1, x_2, ..., x_n$ real numbers, prove that there exists a real number y such that

$$\{y - x_1\} + \{y - x_2\} + ... + \{y - x_n\} \leq \frac{n-1}{2}.$$

where $\{x\} = x - \lfloor x \rfloor$.

Problem 1. Positive reals $a_1, a_2, \ldots$ satisfy:

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Prove that $\frac{1}{2}(a_1 \cdot a_2 \cdot \ldots \cdot a_n + \sqrt{a_{n+1} - 1})$ is a positive integer.

$P_n = a_1 a_2 \cdots a_n \longrightarrow \boxed{a_{n+1} = P_n^2 - 3}$

المطلوب : $\frac{1}{2}(P_n + \sqrt{P_n^2 - 4})$ صحيح موجب

$b_n = \frac{1}{2}(P_n + \sqrt{P_n^2 - 4}) \in \mathbb{Z}^+$ مطلوب

$\Leftrightarrow 2b_n - P_n = \sqrt{P_n^2 - 4}$

$\rightarrow 4b_n^2 - 4b_n P_n + \cancel{P_n^2} = \cancel{P_n^2} - 4$

$\rightarrow b_n^2 - b_n P_n + 1 = 0 \longrightarrow \boxed{P_n = b_n + \frac{1}{b_n}}$

$x^2 - P_n x + 1 = 0$

$\underline{b_n \text{ و } \frac{1}{b_n}}$ حلولها

$P_n = b_n + \frac{1}{b_n}$ ← من فيتا

$\underline{P_n = P_{n-1} \cdot a_n} = \underline{P_{n-1}(P_{n-1}^2 - 3)} = P_{n-1}^3 - 3P_{n-1}$

نفترض $P_{n-1} = b_{n-1} + \frac{1}{b_{n-1}} \Leftarrow$

$P_n = P_{n-1}^3 - 3P_{n-1}$

$= \left(b_{n-1} + \frac{1}{b_{n-1}}\right)^3 - 3\left(b_{n-1} + \frac{1}{b_{n-1}}\right)$

$= b_{n-1}^3 + \cancel{3b_{n-1}} + \cancel{3 \cdot \frac{1}{b_{n-1}}} + \frac{1}{b_{n-1}^3} - \cancel{3b_{n-1}} - \cancel{3 \cdot \frac{1}{b_{n-1}}}$

$= b_{n-1}^3 + \frac{1}{b_{n-1}^3}$

$$\boxed{b_n} = \frac{1}{2}\left(P_n + \sqrt{P_n^2 - 4}\right) = \frac{1}{2}\left(b_{n-1}^3 + \frac{1}{b_{n-1}^3} + \sqrt{b_{n-1}^6 - 2 + \frac{1}{b_{n-1}^6}}\right)$$

$$= \frac{1}{2}\left(b_{n-1}^3 + \frac{1}{b_{n-1}^3} + \left(b_{n-1}^3 - \frac{1}{b_{n-1}^3}\right)\right)$$

$$= \boxed{b_{n-1}^3}$$

بالاستقراء إذا كان b_{n-1} صحيح موجب $\Leftarrow$ $b_n = b_{n-1}^3$ صحيح موجب

و b_1 صحيح موجب من المعطى

Problem 2. Let $a_0 > 0$ be a real number, and let

$$a_n = \frac{a_{n-1}}{\sqrt{1 + 2020 \cdot a_{n-1}^2}}, \quad n = 1, 2, ..., 2020$$

Show that $a_{2020} < \frac{1}{2020}$

$$a_n \overset{?}{<} \frac{1}{n}$$

$$a_1 = \frac{a_0}{\sqrt{1+2020a_0^2}} \overset{?}{<} 1$$

$$a_0^2 \overset{?}{<} 1 + 2020a_0^2$$

$$2019a_0^2 + 1 \overset{?}{>} 0 \checkmark$$

$$a_{n-1} < \frac{1}{n-1} \xrightarrow{?} a_n < \frac{1}{n}$$

$$a_n = \frac{a_{n-1}}{\sqrt{1+2020a_{n-1}^2}} \overset{?}{<} \frac{1}{n}$$

$$\Leftrightarrow n^2 a_{n-1}^2 \overset{?}{<} 1 + 2020\, a_{n-1}^2$$

$$(n^2-2020)\, a_{n-1}^2 \overset{?}{<} 1$$

$$a_{n-1}^2 < \frac{1}{(n-1)^2} \overset{?}{<} \frac{1}{n^2-2020}$$

$$\Leftrightarrow n^2 - 2020 \overset{?}{<} n^2 - 2n + 1$$

$$2n+1 \overset{?}{<} 2020$$

$$\underline{b_n = n a_n}, \quad b_n = n a_n = n \cdot \frac{(n-1)a_{n-1}}{\sqrt{(n-1)^2 + 2020(n-1)^2 a_{n-1}^2}}$$

$n = 1, 2, ---$

$$\boxed{b_n = \frac{n\, b_{n-1}}{\sqrt{(n-1)^2 + 2020\, b_{n-1}^2}}}, \quad b_{2020} \overset{?}{<} 1$$

$$b_n \overset{?}{<} 1 \Leftrightarrow n^2 b_{n-1}^2 \overset{?}{<} (n-1)^2 + 2020\, b_{n-1}^2$$

$$(n^2-2020)\, b_{n-1}^2 \overset{?}{<} (n-1)^2$$

$$2020 \times 2019 \times b_{2019}^2 \overset{?}{<} 2019^2$$

$$b_{2019}^2 \overset{?}{<} \frac{2019}{2020}$$

$$\frac{2019^2\, b_{2018}^2}{2018^2 + 2020\, b_{2018}^2} \overset{?}{<} \frac{2019}{2020}$$

$$2020 \times 2019\, b_{2018}^2 \overset{?}{<} 2018^2 + 2020\, b_{2018}^2$$

$$2020 \times 2018\, b_{2018}^2 \overset{?}{<} 2018^2$$

$$b_{2018}^2 \overset{?}{<} \frac{2018}{2020}$$

$$b_n^2 \overset{?}{<} \frac{n}{2020} \quad \Leftrightarrow \quad n^2 a_n^2 < \frac{n}{2020}$$

$$a_n^2 < \frac{1}{2020\, n}$$

$$\frac{n^2 b_{n-1}^2}{(n-1)^2 + 2020\, b_{n-1}^2} \overset{?}{<} \frac{n}{2020}$$

$$\Leftrightarrow \quad 2020\, n\, b_{n-1}^2 \overset{?}{<} (n-1)^2 + 2020\, b_{n-1}^2$$

$$\Leftrightarrow \quad \underbrace{b_{n-1}^2 \overset{?}{<} \frac{n-1}{2020}}$$

$$b_1^2 < \frac{1}{2020}$$

$$b_1 = a_1 = \frac{a_0}{\sqrt{1 + 2020 a_0^2}} \overset{?}{<} \sqrt{\frac{1}{2020}}$$

$$\Leftrightarrow \quad 2020\, a_0^2 \overset{?}{<} 1 + 2020\, a_0^2$$

$$\Leftrightarrow \quad 0 \overset{?}{<} 1 \quad \checkmark$$

نثبت بالاستقراء $a_n < \frac{1}{\sqrt{2020 n}}$ $\quad n = 1, 2, \ldots$

$n=1$: $a_1 < \frac{1}{\sqrt{2020}}$ — — —

$n=k \Rightarrow n=k+1$:

$$\Rightarrow \quad a_{2020} < \frac{1}{\sqrt{2020^2}} = \frac{1}{2020}$$

الحل الثاني

$$a_n^2 = \frac{a_{n-1}^2}{1+2020a_{n-1}^2}$$

$$\frac{1}{a_n^2} = \frac{1+2020a_{n-1}^2}{a_{n-1}^2} = \frac{1}{a_{n-1}^2} + 2020$$

$$\Rightarrow \frac{1}{a_{2020}^2} = \frac{1}{a_0^2} + 2020 \times 2020 > 2020^2$$

$$\rightarrow a_{2020} < \frac{1}{2020}.$$

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$M = \max\{x_i\}$ $\quad \nexists j : x_j > M$

يوجد n من المتتابعة قيمتهم M .

افرض $2011 > n$

$x_i + x_{i+1} = 2M$

n معادلة أو n i

$x_{i-1} + x_i = 2M$

$\rightarrow x_{i-1} = x_i = M$

$2M = x_i + x_{i+1} \leq M + M = 2M$

$\Rightarrow x_i = x_{i+1} = M$

حل ثان

$(x_i + x_{i+1})^2 = (2x_i')^2$

$x_i^2 + 2x_i x_{i+1} + x_{i+1}^2 = 4x_i'^2$

$\rightarrow 2\sum x_i^2 + 2\sum x_i x_{i+1} = 4\sum x_i^2$

$2\sum x_i x_{i+1} = 2\sum x_i^2$

AM-GM: $x_i^2 + x_{i+1}^2 \geq 2x_i x_{i+1}$

$\rightarrow 2\sum x_i^2 \geq 2\sum x_i x_{i+1} \rightarrow x_i = x_{i+1}$

أو

$\sum (x_{i+1} - x_i)^2 = 0 \rightarrow x_i = x_{i+1}$

Problem 4. Let $x_1, x_2, ..., x_n$ $(n \geq 2)$ be real numbers $\underbrace{\text{greater than 1}}_{A}$. Suppose that $\underbrace{|x_i - x_{i+1}| < 1}_{L}$ for $i = 1, 2, ..., n-1$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + ... + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} < 2n - 1$$

الحل بالاستقراء : $n = 2$:

$$x_1^2 + x_2^2 \overset{?}{<} 3x_1x_2 \iff \frac{x_1}{x_2} + \frac{x_2}{x_1} \overset{?}{<} 3$$

$$\iff \underbrace{(x_1 - x_2)^2}_{<1} \overset{?}{<} \underbrace{x_1x_2}_{1<} \quad \checkmark$$

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + ... + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_1} \overset{?}{<} 2n+1 \quad : n+1$$

نريد •

$$\frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_1} - \frac{x_n}{x_1} \leq 2$$

$$\frac{x_{n+1}^2 - x_{n+1}x_n + x_1x_n}{x_{n+1}x_1} \overset{?}{\leq} 2$$

$$\iff \frac{x_{n+1}(x_{n+1} - x_n) + x_1(x_n - x_{n+1})}{x_{n+1}x_1} \overset{?}{\leq} 1$$

$$\iff \frac{(x_{n+1} - x_n)(x_{n+1} - x_1)}{x_{n+1}x_1} \overset{?}{\leq} 1$$

$$\frac{|x_{n+1} - x_n| \cdot |x_{n+1} - x_1|}{x_{n+1}x_1} < \left| \frac{x_{n+1} - x_1}{x_{n+1}x_1} \right| \overset{?}{\leq} 1$$

$$= \left| \frac{1}{x_1} - \frac{1}{x_{n+1}} \right|$$

$$< \max\left\{ \frac{1}{x_1}, \frac{1}{x_{n+1}} \right\} < 1$$

حل ثان

$$\frac{x_i}{x_{i+1}} \qquad x_i\,,\ x_{i+1} > 1\ ,\ |x_{i+1} - x_i| < 1$$

if $x_i \le x_{i+1}$: $0 < \frac{x_i}{x_{i+1}} \le 1$

if $x_i \ge x_{i+1}$: $1 \le \frac{x_i}{x_{i+1}} < 2$

$$\left(x_i - x_{i+1} < 1 < x_{i+1}\right)$$

ليكن k هو عدد i بحيث $x_i \le x_{i+1}$

$$\frac{x_1}{x_2} + \dots + \frac{x_{n-1}}{x_n} < k \cdot 1 + (n-1-k)\cdot 2$$
$$= 2n - 2 - k$$

$$\cdot\ \frac{x_n}{x_1} \overset{?}{\le} k+1$$

$$x_n < x_1 + k < x_1 + kx_1 = (k+1)x_1$$

$$\rightarrow \frac{x_n}{x_1} < k+1$$

$$\Rightarrow \sum \frac{x_i}{x_{i+1}} + \frac{x_n}{x_1} < 2n-1$$