Preparation for Saudi Arabia Team 2021

May/June Session: Level 3

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Lesson 4

Muirhead's Inequality

Problems:

1. Let a, b and c be positive real numbers. Show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 3+2\frac{a^3+b^3+c^3}{abc}.$$

2. Let a, b and c be positive real numbers. Show that

$$\frac{a+b}{a^2+b^2}+\frac{a+c}{a^2+c^2}+\frac{b+c}{b^2+c^2} \geq \frac{3(a^3+b^3+c^3)}{a^4+b^4+c^4}.$$

3. Let a, b and c be positive real numbers. Show that

$$\frac{a^2+bc}{b^2+bc+c^2}+\frac{b^2+ca}{c^2+ca+a^2}+\frac{c^2+ab}{a^2+ab+b^2} \geq 2.$$

4. Let a, b, c be positive real numbers such that $abc=1$. Show that

$$\frac{a^3}{(1+b)(1+c)}+\frac{b^3}{(1+a)(1+c)}+\frac{c^3}{(1+a)(1+b)} \geq \frac{3}{4}.$$

5. Let a, b, c be positive real numbers such that $abc=1$. Show that

$$\frac{1}{a^5\left(b^2+c^2\right)}+\frac{1}{b^5\left(c^2+a^2\right)}+\frac{1}{c^5\left(a^2+b^2\right)} \geq \frac{3}{2}.$$

6. Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5} \geq 0.$$