Inequalities 2

1. For any $a > 0, b > 0$ and $c > 0$ prove that

$$(ax + by + cz)(bx + cy + az)(cx + ay + bz) \geq (a + b + c)^3 xyz.$$

2. For any $a \geq 0, b \geq 0$ prove that

$$\frac{a^4 + a^2b^2 + b^4}{3} \geq \frac{a^3b + b^3a}{2}.$$

3. Let $a_1, a_2, \ldots, a_{2020}$ be positive numbers such that $a_1 + a_2 + \ldots + a_{2020} = 1$. Find the minimum value of the following expression

$$\frac{2021 + a_2}{a_1 + a_2 + a_3} \cdot \frac{2021 + a_3}{a_2 + a_3 + a_4} \cdot \ldots \cdot \frac{2021 + a_1}{a_{2020} + a_1 + a_2}.$$

4. Let $a, b, c > 0$. Prove that

$$3(a + b + c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}}.$$

5. Let a, b, c be positive numbers. Prove that

$$\frac{a^2}{a\sqrt{a^2 + 8bc}} + \frac{b^2}{b\sqrt{b^2 + 8ca}} + \frac{c^2}{c\sqrt{c^2 + 8ab}} \geq 1.$$

6. Let a_i be positive numbers and $\sum\limits_{i=1}^{n} a_i^2 = 1$. Prove that

$$\sum_{i=1}^{n} \left(\frac{1}{a_i} - a_i \right) \geq (n - 1)\sqrt{n}.$$

7. Let a_i, b_i be positive numbers. Prove that

$$\sqrt{\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2} \leq \sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2}.$$

8. Let a, b, c be positive numbers such that $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

9. Let $a_1, \ldots, a_n$ be positive numbers. Prove that

$$\left(1+\frac{a_1^2}{a_2}\right)\cdot\left(1+\frac{a_2^2}{a_3}\right)\cdot\ldots\cdot\left(1+\frac{a_n^2}{a_1}\right) \geq (1+a_1)\cdot(1+a_2)\cdot\ldots\cdot(1+a_n).$$

10. Real numbers $a_1, \ldots, a_n$, $n>1$, satisfy the following inequalities $a_1 < a_2 < \ldots < a_n$. Prove that

$$a_1a_2^4 + a_2a_3^4 + \ldots + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \ldots + a_1a_n^4.$$

Homework

1. Prove that if $a_1 < a_2 < \ldots < a_n$, then

$$a_1^{a_2}\cdot a_2^{a_3}\cdot\ldots\cdot a_n^{a_1} \geq a_2^{a_1}\cdot a_3^{a_2}\cdot\ldots\cdot a_1^{a_n}.$$