Intensive Training 2021
Level 3 Geometry Problems

1. Let the circles k_1 and k_2 intersect at two points A and B, and let t be a common tangent of k_1 and k_2 that touches k_1 and k_2 at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate the angle NMB. (JBMO 2012)

2. A trapezoid $ABCD$ ($AB||CF$,$AB > CD$) is circumscribed.The incircle of the triangle ABC touches the lines AB and AC at the points M and N,respectively.Prove that the incenter of the trapezoid $ABCD$ lies on the line MN. (JBMO 2016)

3. Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.) Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular. (EGMO 2012)

4. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled. (EGMO 2013)

5. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BCD = 90°$ and $\angle ABC > \angle CDA$. Let Q and R be points on segments BC and CD, respectively, such that line QR intersects lines AB and AD at points P and S, respectively. It is given that $PQ = RS$.Let the midpoint of BD be M and the midpoint of QR be N.Prove that the points M, N, A and C lie on a circle. (EGMO 2017)

6. Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120°$, and let M be the midpoint of AB. Let P be a variable point of the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N. Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P. (EGMO 2018)

7. Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear. (JBMO 2013)

8. Consider an acute triangle ABC of area S. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocentres of the triangles MNC, respectively MND. Find the area of the quadrilateral AH_1BH_2 in terms of S. (JBMO 2014)

9. In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic. (BMO 2013)

10. A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$,and AB is not paralel to CD.Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k. (BMO 2018)