January Camp 2022
Problems

Geometry – L2

Inversion

Problem 1. Let ω be a circle internally tangent to circle Ω at S. Let SA and SB be diameters of ω, Ω, respectively. Let o be the circle tangent to AB at C and tangent to ω and Ω. Prove that

$$\frac{2}{SC} = \frac{1}{SA} + \frac{1}{SB}.$$

Problem 2. Let ω be a circle with center A. Let B be a point on ω. Consider circle Ω tangent to perpendicular bisector of AB, ω, and line AB at D. Prove that $AB = BD$.

Problem 3. Let ABC na a right triangle with $\sphericalangle BAC = 90°$. Let o_1 be circle with diameter AC. Let o_2 be the circle tangent to BC at D, to segment AB and externally to o_1. Prove that $AC = DC$.

Problem 4. Let o_1, o_2 with radii r_1 and r_2 be externally tangent at A. Let o_3, o_4 with radii r_3 and r_4 be externally tangent at A, but the are not tangent to circles o_1, o_2. Prove that there exists circle tangent to o_1, o_2, o_3, o_4 iff

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}.$$

Problem 5. Let A, B, C, D be arbitrary points on a plane. Prove that

$$AC \cdot BC \leq AD \cdot BC + AB \cdot CD.$$

Problem 6. Let P be a point inside a triangle ABC such that

$$\sphericalangle APB - \sphericalangle ACB = \sphericalangle APC - \sphericalangle ABC.$$

Let D, E be the incenters of triangles APB andAPC, respectively. Show that the lines AP, BD, CE meet at a point.

Problem 7. Circles k_1, k_2, k_3, k_4 are such that k_2 and k_4 each touch k_1 and k_3. Show that the tangency points are collinear or concyclic.

Problem 8. Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\sphericalangle PAB$.

Incenter and excenter

Problem 9. Let $ABCD$ be a cyclic quadrilateral such that incircles of triangles ABD and ABC are congruent. Decide whether incircles of triangles CDB and CDA are also congruent.

Problem 10. A trapezoid $ABCD$ in which $AB \parallel CD$ and $AB > CD$, is circumscribed. The incircle of the triangle ABC touches the lines AB and AC at the points M and N, respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

Problem 11. Points A, B lie on circle ω. Points C and D are moved on the arc AB, such that CD has constant length. Points I_1, I_2 are incenters of ABC and ABD, respectively. Prove that line I_1I_2 is tangent to some fixed circle.

Problem 12. Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that $KL \parallel AC$. Show that the incenters of triangles ABK and CBL are equidistant from the midpoint of the arc AC, containing point B, of the circumcircle of triangle ABC.

Problem 13. Let ABC be a triangle with $\sphericalangle BAC = 60°$. Let D and E be the feet of the perpendiculars from A to the external angle bisectors of ABC and ACB, respectively. Let O be the circumcenter of the triangle ABC. Prove that the circumcircles of the triangles ADE and BOC are tangent to each other.

Problem 14. The circle Γ has centre O, and BC is a diameter of Γ. Let A be a point which lies on Γ such that $\sphericalangle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets Γ at E and at F. Prove that I is the incentre of the triangle CEF.

Problem 15*. Let AD be altitude in acute-angled triangle ABC. Points M and N are projections of point D onto AB and AC. Lines MN and AD intersect circumcirlce ω of triangle ABC respectively at points P, Q and A, R. Prove that D is incenter of PQR.

Problem 16*. Let I be the incenter of $\triangle ABC$. Denote by D, $S \neq A$ intersections of AI with BC and circumcircle ω of ABC, respectively. Let K, L be incenters of triangles DSB and DCS. Let P be a reflection of I with respect to KL. Prove that $BPC = 90°$.

Midarc

Problem 17. Let ABC be a triangle and M be the midpoint of arc BAC. Let X and Y lie on AB and AC such that $BX = CY$. Prove that $AXYM$ are concyclic.

Problem 18. In triangle ABC point I is its incenter. The circle passing through A and I intersects AB and AC at X and Y, respectively. Prove that $BX + CY = BC$.

Problem 19. Let ω be circumcircle of an acute triangle ABC. Point X lies inside ABC, such that $\sphericalangle BAX = 2\sphericalangle XBA$ and $\sphericalangle XAC = 2\sphericalangle ACX$. M is midpoint of arc BC of ω, which contains point A. Show that $XM = XA$.

Problem 20. Let I be incenter of triangle ABC. Let M and N be midarc point of arc BAC and midpoint of BC. Prove that $\sphericalangle AMI = \sphericalangle INB$.

Problem 21$^\bigstar$. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcenter of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

Problem 22. Let M be the midpoint of side BC of triangle ABC. Let I and J be incenters of triangles ABM and AMC. Prove that circumcircle of triangle AIJ passes through midarc BAC.

Problem 23. Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Menelaus

Problem 24. Let the external angle bisector of $\sphericalangle BAC$ intersect BC at A'. Define B', C' analogously. Prove that A', B', C' are collinear.

Problem 25. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let X be a point on segment AB. Put $P = BC \cap AD$, $Y = CD \cap PX$, $R = AY \cap BD$ and $T = PR \cap AB$. Prove that

$$\frac{1}{AT} = \frac{1}{AX} + \frac{1}{AB}.$$

Problem 26. In triangle ABC let D be the point on the segment BC, and E on the segment CE, for which $BD = CE = AB$. Let ℓ be the line through D that is parallel to AB. If $M = \ell \cap BE$ and $F = CM \cap AB$ prove that

$$AE \cdot BF \cdot CD = (AB)^3.$$

Problem 27. In triangle ABC internal angle bisectors t_a, t_b, t_c meet BC, CA, AB at U, V, W, respectively; and medians m_a, m_b, m_c intersect BC, CA, AB at L, M, N, respectively. Let $m_a \cap t_b = P$, $m_b \cap t_c = Q$, $m_c \cap t_a = R$. Prove that

$$\frac{AR}{RU} \cdot \frac{BP}{PV} \cdot \frac{CQ}{QW} \geq 8.$$

Problem 28. Let D and E be points on sides AB and AC of a triangle ABC such that $DE \parallel BC$. Let P be an interior point of triangle ADE. Lines PB and PC intersect DE at F, G, respectively. Prove that AP is a radical axis of circumcircles of triangles PDG and PFE.

Problem 29. Let $ABCD$ be a parallelogram. Points K and L lie on the sides AB and AD, respectively. Line segments DK and BL intersect at P. Point Q is chosen such that $AKQL$ is a parallelogram. Prove that P, Q, C are collinear.

Problem 30. Let $ABCD$ be a convex quadrilateral. A line k intersects DA, AB, BC and CD at X, Y, Z and T, respectively. Prove that

$$\frac{DX}{XA} \cdot \frac{AY}{YB} \cdot \frac{BZ}{ZC} \cdot \frac{CT}{TD} = 1.$$

Ceva

Problem 31. Let ABC be a triangle with $\sphericalangle A = 100^\circ$, $\sphericalangle B = 60^\circ$, and let $M \in BC$ and $N \in AC$ be points for which $\sphericalangle BAM = 30^\circ$ and $\sphericalangle ABN = 20^\circ$. Prove that the lines AM, BN and the bisector of $\sphericalangle ACB$ are concurrent.

Problem 32. Let ABC be a right triangle with right angle at C. On sides BC and CA build squares $BEFC$ and $CGHA$, respectively. Let D be the feet of altitude from C to AB. Prove that AE, BH and CD concur.

Problem 33. A circle meets the sides BC, CA, and AB of triangle ABC at points A_1; A_2, B_1; B_2, and C_1; C_2. Prove that the lines AA_1, BB_1, and CC_1 are concurrent if and only if the lines AA_2, BB_2, and CC_2 are concurrent

Problem 34. Let ABC be a triangle. Prove that lines joining midpoints of the sides with midpoints of the corresponding altitudes pass through a single point.

Problem 35. Let $ABCDEF$ be a hexagon inscribed in a circle ω. Show that the diagonals AD, BE, CF are concurrent if and only if

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Problem 36. Prove that in triangle ABC interior bisector of angle A, median of triangle from B and altitude from C concur iff

$$\tan A = \frac{\sin C}{\cos B}.$$

Problem 37. In an acute triangle ABC a semicircle ω centered on the side BC is tangent to the sides AB and AC at points F and E, respectively. If X is the intersection of BE and CF, show that $AX \perp BC$.

Problem 38. Prove that in regular 30-gon diagonals A_1A_{19}, A_3A_{24} and A_8A_{28} concur.

Problem 39. Let P be a point inside equilateral triangle ABC. Let AP, BP, CP meet sides BC, CA, AB are A_1, B_1, C_1. Prove

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$$

When does equality hold?