Training Session in Mathematics, November-December 2021

Level 2

Nikola Petrović

Lesson 1

Pigeonhole principle

Problems:

- **Pigeonhole principle:** For any n elements distributed into k disjoint sets, there is a set with at least $\left\lceil \frac{n}{k} \right\rceil$ elements, where $\lceil x \rceil$ is the smallest integer number not smaller than x (for example, $\lceil 2 \rceil = 2$, $\lceil \pi \rceil = 4$, and so on).

Problems:

1. Show that among any $n+1$ different numbers from the set $\{1, 2, \ldots, 2n\}$ it is possible to find two numbers that differ by n. Show that the statement no longer holds if we replace '$n+1$' with n.
2. Assuming friendship is a reciprocal relation (i.e., if person A is friends with person B, then so is B friends with A), show that among n people there are two people with the same number of friends.
3. Show that among 5 points inside an equilateral whose side has length 1, there are two such that the distance between them is less than $1/2$.
4. Show that among 9 points inside a square of area 1, there are three points that form a triangle of area less than $\frac{1}{8}$.
5. On each of the 20 faces of an icosahedron a non-negative integer is written. The total sum of the numbers written is 39. Prove that there are at least two faces sharing a vertex that have the same number written on them.
6. On a 3×7 board each square is colored either black or white. Prove that there are four squares whose centers form a rectangle that are all colored the same color.
7. A collection of subsets of $\{1, 2, \ldots, n\}$ is given such that every 2 subsets in the collection have at least one element in common. Prove that the number of subsets in the collection is at most 2^{n-1}.
8. Show that for any 5 points on a sphere, one can find a closed hemisphere (including the corresponding great circle) that contains 4 out of 5 points.
9. Lattice points in three-dimensional Euclidean space are those points whose coordinates are of the form (a, b, c), where a, b and c are integers. Show that among 9 arbitrarily selected lattice points, there are two such that the line segment connecting them contains at least one lattice point in its interior (not necessarily one of the 9 selected points).
10. Show that for any n integer numbers $a_1, a_2, \ldots, a_n$ there exist indices i and j, $1 \leq i \leq j \leq n$, such that $a_i + a_{i+1} + \ldots + a_j$ is divisible by n
11. Show that there exists a number of the form $111\ldots111$ that is divisible by 2021.
12. Show that among any $n+1$ different numbers from the set $\{1, 2, \ldots, 2n\}$ it is possible to find two numbers such that one number divides the other. Show that the statement no longer holds if we replace '$n+1$' with n.

V 315 ≠ 351
$\overline{V}$ 111, 353, 992
C 315 = 351 → 135
$\overline{C}$ 121 = 112

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الاهتمام بالترتيب مع تواقيق = V

Lesson 2
Variations, Combinations and Permutations

شرح:

Problems:

- **Variation:** A variation is an ordered selection of k elements which belong to a set of n elements. The number of variations is $V_k^n = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$.
- **Variation with repetition:** A variation with repetition $\overline{V}_k^n$ is an ordered selection of k elements which belong to a set of n elements wherein the same element can be selected multiple times. The number of variations with repetition is $\overline{V}_k^n = n^k$.
- تواقيق **Combination:** A combination is an unordered selection of k elements which belong to a set of n elements. The number of combinations is $C_k^n = \frac{V_k^n}{k!} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$.
- مسموح التكرار **Combination with repetition:** A combination with repetition is an unordered selection of k elements which belong to a set of n elements wherein the same element can be selected multiple times. The number of combinations with repetition is $\overline{C}_k^n = \binom{n+k-1}{k}$.
- ترتيب الأشياء على صف **Permutation:** A permutation is an ordering of n elements. The number of permutations is $P_n = n!$.
- مسموح التكرار **Permutation with repetition:** A permutation with repetition is an ordering of n elements where certain groups of elements are deemed identical. If a permutation of n elements contains classes of identical elements with $\alpha_1, \alpha_2, \dots, \alpha_k$ elements each $(\alpha_1 + \dots + \alpha_k \leq n)$, then the number of permutations with repetition is $P_{n,\alpha_1,\dots,\alpha_k} = \frac{n!}{\alpha_1!\dots\alpha_k!}$.

Problems:

1. Give a combinatorial interpretation for why $P_{n,k,n-k} = C_k^n$.
2. Prove the formula for the number of combinations with repetition.
3. In how many ways can you place 7 different balls in 3 boxes such that at most 1 ball can be places in the first box, at most 2 balls can be placed in the second box and at most 6 balls can be placed in the third box.
4. Let n be a natural number. A positive integer d is said to be *ascending* if in its decimal representation: $d_m d_{m-1} \dots d_2 d_1$ we have $0 < d_m \leq d_{m-1} \leq \dots \leq d_2 \leq d_1$ For example, 1337 and 2455566799 are ascending integers. Find the number of ascending integers which are less than 10^n.
5. In a mathematical committee there are $2n$ mathematicians, $n \geq 3$ and each is in a quarrel with exactly 1 other mathematician. They are supposed to divide into three committees for the upcoming competition: one for composing the problems, one for grading the problems and one for organizing the event. In how many ways can they do this so that each committee has at least two members and no two mathematicians who are in a quarrel are on the same committee?
6. Find the number of ways to tile a 3×3 board in 4 colors such that no two fields in the same row or the same column have the same color.

7. Find the number of integer sequences such that the first number is 1, the last number is 25 and each number after the first is larger by 2 or 3 than the previous number.

8. There are 14 books arranged consecutively on a bookshelf. Find the number of ways of selecting 5 books out of these 14 such that no two adjacent books are selected.

9. Find the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ such that for each $i = 1, 2, \ldots, n$ the first i numbers of the permutation are consecutive integers.

10. In how many ways can you arrange 6 volumes of an encyclopedia on a shelf such that volume 1 is not in the first or last position, volumes 2 and 3 are next to each other and volumes 4 and 5 are not next to each other?

11. Let k, l and m be positive integers. Let S be the set of integer points (x, y, z) such that $1 \leq x \leq k$, $1 \leq y \leq l$ and $1 \leq z \leq m$. A flea starts from the point $(1, 1, 1)$ in S and wants to reach point (k, l, m) by only jumping along points in S. When in point (x, y, z) it can jump to one of the following points: $(x + 1, y, z)$, $(x, y + 1, z)$, $(x, y, z + 1)$, $(x, y, z - 1)$, but it cannot jump to a point it was already in. In how many different ways can the flea reach (k, l, m)?

12. A pawn is located on the second row and first column (the $a2$ square) of an 8×8 chessboard. In how many ways can the pawn reach the 8-th row (ignoring the moves of other pieces) if it does not move two squares on its first move? A pawn can move from a square (x, y) to squares $(x, y + 1)$, $(x - 1, y + 1)$ and $(x + 1, y + 1)$ (the latter two if it captures a piece).

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Lesson 3
Basic game theory

- A game is typically a large set of possible states where certain states are deemed to be *winning*, i.e. the player who reaches the state wins the game, or *losing*, i.e. a player who reaches the state loses the game.
- A winning strategy in a two-player game is a sequence of moves guarantees the victory to one player or a loss to the opposing player.
- A winning strategy typically entails dividing the set of all game configurations into sets W and L, where W represents all the states in which a player making the move will win the game, and L represents the states in which the player making the move will lose the game.
- In order for the game strategy to work you need four conditions:
  - The winning state should be located in W or the losing state in L.
  - From each state in W a move can be made to a state in L.
  - From each state in L every move leads to a state in W.
  - The rules of the game should guarantee that the final state is eventually reached (e.g. through a semi-invariant).

Problems:

1. On two piles there are initially m and n coins respectively. Two players, A and B take turns removing at least one coin from one of the piles. Player A starts first. For which values of m and n will player A have a winning strategy and for which m and n will player B have a winning strategy if:
   (a) the player who takes the last coin wins,
   (b) the player who takes the last coin loses?
2. Alice and Bob play the following game. They arrange and 2017 cookies in a row on a table and label them consecutively from 1 to 2017. Then, in the k-th turn a player eats k consecutive cookies still left on the board, wherein Alice takes odd-numbered turns and Bob takes even-numbered turns. The game ends when there is one cookie left on the table. If the cookie's number is odd, then Alice wins, otherwise Bob wins. Which player has a winning strategy and how should they play to win?
3. Will and Bob play the following game. Will places his 8 white coins on the first row of an 8×8 chessboard and Bob places his 8 black coins on the 8-th row of the same chessboard. They take turns moving their coins, with Will starting first. Each player on their turn moves one of his coins in either direction along the column the coin is in however many steps he wants (but at least one!) with the restriction that he cannot land on the space with their opponent's coin. The player who is first unable to make a move loses. Which player has a winning strategy and how should they play the game in order to win? A step is a movement of a coin to a field adjacent to the one the coin is currently on.

4. Alice and Bob play the following game, taking turns with Alice starting first. On a table there are 2020 coins. In the first move Alice takes some coins from the table, but not all. In each subsequent move a player can only take a number of coins from the table that divides the number of coins taken in the previous move (a natural number is divisible by itself). The winner is the player who takes the last coin. Which player has a winning strategy and how should they play to win the game?

5. Alice and Bob play the following game. There are two piles of coins on a table, the first pile with m coins and the second pile with n coins, such that $m \geq n$. In each turn, a player is allowed to take off an arbitrary number of coins from one of the piles or an equal number of coins from both piles with the restriction that the number of coins in the first pile can never become smaller than the number of coins in the second pile. The winner is the player who takes the last coin off the table. For which values of m and n does Alice have a winning strategy and for which values does Bob have a winning strategy?

6. On the lower-left corner of a $m \times n$ chessboard a king is placed. Players A and B take turns moving the king with the restriction that the king cannot move to the left, downward or diagonally down-left (It can move in the remaining 5 directions). The winner is the player who places the king in the upper-right corner. For which values of m and n does which player have a winning strategy?

7. For a positive integer n, two payers A and B play the following game: Given a pile of s stones, the players take turn alternatively with A going first. On each turn the player is allowed to take either one stone, or a prime number of stones, or a positive multiple of n stones. The winner is the one who takes the last stone. Assuming both A and B play perfectly, for how many values of s the player A cannot win?

8. Alice and Bob play the following game. A pile of n coins is given. Alice divides the pile into either two or three piles, not necessarily equal. Thereafter, each player divides one of the piles obtained in the previous turn into two or three piles, not necessarily equal. The player who is unable to make a move loses. For which values of n does Alice have a winning strategy and for which values of n does Bob have a winning strategy?

9. Given an initial integer $n_0 > 1$, two players, A and B, choose integers $n_1, n_2, n_3, \ldots$ alternately according to the following rules :

A 2021

- Knowing n_{2k}, A chooses any integer n_{2k+1} such that $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$.
- Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that $\frac{n_{2k+1}}{n_{2k+2}}$ is a prime raised to a positive integer power.

Player A wins the game by choosing the number 2021, while player B wins by choosing the number 1. For which n_0 does :

(a) A have a winning strategy?

(b) B have a winning strategy?

(c) Neither player have a winning strategy?

10. An $n \times n$ square is divided into unit squares. Let a unit side be a side of a unit square. For which n can we cover the board with equilateral right-angles triangles of side $\sqrt{2}$ and hypothenuse 2 so that the vertices of all the triangles lie on the vertices of unit squares and all unit sides lie in the interior or on the boundary of exactly one triangle?

11. On an infinite square grid we place finitely many cars, which each occupy a single cell and face in one of the four cardinal directions. Cars may never occupy the same cell. It is given that:

    (a) the cell immediately in front of each car is empty, and

    (b) no two cars face towards each other (no right-facing car is to the left of a left-facing car within a row, etc.).

    In a move, one chooses a car and shifts it one cell forward to a vacant cell. Prove that there exists an infinite sequence of valid moves using each car infinitely many times. Is an infinite sequence of moves where each car is used infinitely many times still possible if we drop the first condition?

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Lesson 4
Graph theory

- A graph G is a collection of vertices whose some pairs are connected by edges.
- A graph is connected if it is possible to reach any vertex from any other going along the edges.
- A graph is simple if no vertex connects to itself and between any two vertices there is at most one edge.
- A degree of a vertex is the number of edges coming out it. The sum of the degrees is twice the number of edges.
- An Euler path is a path that passes through each edge exactly once. An Euler circuit is a path which ends in the same position it started.
- Only graphs with 2 or 0 vertices of odd degree have an Euler path. If the former case, the path starts and ends in the two vertices with an odd degree and in the latter case the path is a circuit.
- A graph is complete if each pair of vertices is connected by an edge.
- A graph is bipartite if it is possible to color the vertices in two colors so that no two vertices of the same color are connected by an edge.

Problems:

1. Find all pairs of $m \times n$ $(m, n \geq 2)$ such that it is possible for a knight to make a sequence of moves such that a move is made between each pair of knight-adjacent squares exactly once. Two squares are considered knight-adjacent if it possible for the knight to reach one from the other in one move.

2. In a table tennis tournament with $2n+1$ contestants, each pair of contestants played each other exactly once and none of the games resulted in a tie. We will call a triangle any collection of 3 contestants A, B and C, such that A defeated B, B defeated C and C defeated A. Let T be the number of triangles that occurred on the tournament. (Two different triangles may overlap in zero, one or two contestants.)

   (a) Find the smallest possible value of T.

   (b) Find the largest possible value of T.

3. Let a graph be given such that there are no closed paths and you can reach any vertex from any other going along the edges of the graph. Prove that you can color the vertices of the graph in 2 colors so that no two vertices of the same color are connected. Such a graph is called a bipartite graph.

4. Prove that a graph is bipartite if and only if it contains no odd cycles.

5. Let us select n vertices of a regular $2n$-gon and color then red, while coloring the remaining vertices blue. Let us sort by length all line segments (whether sides or diagonals) whose endpoints are both red and let us do the same for all the line segments whose endpoints are both blue. Show that the two sequences of line segments are identical. Two sequences of line segments are considered identical if each pair of corresponding line segments is of equal length.

6. Let A and B be two countries such that each city is connected to exactly $k \geq 2$ cities in the other country via an interstate road. Interstate roads connect a city in country A with a city in country B with no stops in-between and no pair of cities is connected by more than one interstate. Assuming each city is reachable from every other city using the interstate roads is it possible to destroy a single road so that this no longer holds?

7. In each of three schools there are n students. Each student knows at least $n+1$ students from the other two schools. Prove that there are three students from three schools who know each other.

8. Let a k-group be a collection of k people who all know each other (knowing someone is a symmetric relation). Assume at a party that there are no 5-groups and that no two 3-groups are disjoint, in other words every pair of three groups has at least one common person. Prove that there are two people whose departure would result in there being no 3-groups at the party.

9. Let n be a complete graph, i.e. a graph whose every pair of vertices is connected. In each move, we select a cycle of length 4, i.e. $A_1A_2A_3A_4$ such that A_1A_2, A_2A_3, A_3A_4 and A_4A_1 all belong to the graph and remove one of the edges, say A_1A_2. We continue making moves until there are no more cycles of length 4 in the graph. Find the lowest number of edges we can obtain.

10. A graph is called good if among any three vertices there are at least two which are connected with each other. A complete graph of order 3 will be called a triangle and also a windmill of order 1, and a windmill of order $n > 1$ is defined to be a collection of n triangles that only intersect in a unique vertex common to all the n triangles (in other words, a collection of $2n+1$ distinct vertices $a, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$ such that ab_i, ac_i and b_ic_i are all edges in the graph for every $i = 1, 2, \ldots, n$). Find $f(n)$, the smallest number such that every good graph with $f(n)$ vertices contains a windmill of order n.