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$f, g : (0,2) \longrightarrow (0,2)$

$f(g(x)) = g(f(x)) = x$

$f(x) + g(x) = 2x$

Prove : $f(1) = g(1)$

$\boxed{\text{Hint}}$ It is enough to prove that $f(1) = 1$ or $g(1) = 1$

Suppos WLOG $\boxed{f(1) > 1}$

$f(1) = 1 + c$ for some $\boxed{c \in (0,1)}$

Assum $c \notin 0$

What is

$f(1+c) = ? \; (1+2c) \in (0,2)$

$f(1+2c) = (1+3c) \in (0,2)$

$f(1+3c) = (1+4c) \in (0,2)$

$\vdots$

By induy. $\forall_{n \in \mathbb{N}} \quad 1 + nc \in (0,2)$

$$\boxed{f(1+nc) = 1+(n+1)c}$$

Suppose $f(1+kc) = 1+(k+1)c$ $\qquad \underline{k<n}$

$$g(f(1+nc)) = 1+nc$$

$$f(1+nc)+g(1+nc) = 2(1+nc) =$$

$$= 2+2nc$$

$$f(1+nc) = 2+2nc - g(1+nc) =$$

$$= 2+2nc - g(f(1+(n-1)c)) =$$

$$= 2+2nc - [1+(n-1)c] =$$

$$= 1+(n+1)c. \qquad \square$$

$$\forall_{n\in\mathbb{N}} \quad 1+nc \in (0,2) \quad !$$

$$\forall_{n\in\mathbb{N}} \quad 1+nc < 2$$

$$nc < 1$$

$c \in D$ $\qquad n < \frac{1}{c}$ $\qquad$ ↯

$\boxed{c=0}$ $\quad f(1)=1 \quad g(1)=1$

(18)

$x_1, x_2 \ldots x_n \in \mathbb{R}_{\geq 0}$.

$x_1+x_2+\ldots+x_n=1 \quad x_1+x_2+\ldots+x_n$

$$\max\{x_1 \ldots x_n\}\Big(1+2\underbrace{\sum_{1\leq i<j\leq n}' \min\{x_i;x_j\}}\Big) \geq 1$$

maxim null.

$(x_1+x_2 \ldots x_n)^2$

$$\sum_{1\leq i<j\leq 3}' \min(x_i;x_j) =$$

$$\underbrace{\min(x_1,x_2)+\min(x_1,x_3)+\min(x_2,x_3)} \quad \ldots$$

$\underline{n=2}$ $\quad x_1+x_2=1$

$$\max\{x_1,x_2\}(1+2\cdot\min\{x_1,x_2\}) \geq 1$$

$x_1 \leq x_2$ $\quad x_1+x_2$

$$x_2(1+2x_1) \geq 1 = (x_1+x_2)^2$$

$x_1+x_2=1$

$$x_2(x_1+x_2+2x_1) \geq x_1^2+2x_1x_2+x_2^2$$

$$x_2x_1+\cancel{x_2^2}+\cancel{x_2x_1\cdot 2} \geq x_1^2+\cancel{2x_1x_2}+\cancel{x_2^2}$$

$$x_2x_1 \geq x_1^2$$

$$x_2 \geq x_1$$

$n=3$

$$x_1, x_2, \dots, x_n \in \mathbb{R}_{\geq 0}.$$

$$x_1 + x_2 + \dots + x_n = 1 \qquad x_1 + x_2 + \dots + x_n$$

$$\underbrace{\max\{x_1, \dots, x_n\}}_{\text{maximum null.}} \underbrace{\left(1 + 2 \sum_{1 \leq i < j \leq n} \min\{x_i; x_j\}\right)}_{L} \geq \underbrace{1}_{(x_1 + x_2 \dots x_n)^2}$$

$$\max\{x_1 \dots x_n\} \left(\sum_{i=1}^{n} x_i + 2 \sum_{1 \leq i < j \leq n} \min\{x_i, x_j\} \right) \geq$$

$$\geq (x_1 + x_2 + \dots + x_n)^2 \qquad R$$

$$R = \sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \leq i < j \leq n} x_i \cdot x_j$$

$$L = \sum_{i=1}^{n} \max\{x_1 \dots x_n\} \cdot x_i + 2 \sum_{1 \leq i < j \leq n} \underline{\max\{x_1 \dots x_n\} \min\{x_i, x_j\}}$$

Wha $x_i^2 \leq \max\{x_1 \ldots x_n\} \cdot x_i$

$x_i \leq \max\{x_1 \ldots x_n\}$

$\max\{x_1 \ldots x_n\} \cdot \boxed{\min\{x_i, x_j\}} \geq x_i \cdot x_j$

$x_i \geq x_j$

$\max\{x_1 \ldots x_n\} \cdot x_j \geq x_i \cdot x_j$

trick $1 \rightsquigarrow (x_1 + x_2 + \cdots x_n)^{1 \text{ or } 2}$

$\boxed{20}$

$$\boxed{f : \mathbb{N}_{>0} \longrightarrow \mathbb{N}_{>0}}$$

$$\boxed{(n-1)^2 < f(n) f(f(n)) < n^2 + n}$$

- try $n=1$

$$0 < \underbrace{f(1) \cdot f(f(1))}_{\substack{\shortparallel \\ 1}} < 1^2 + 1 = 2$$

$$f(1) \cdot f(f(1)) = 1$$

$$\boxed{f(1) = 1}$$

Claim $\boxed{f(n) = n}$ Prove it by induction.

Suppose $f(k) = k$ for $k < n$

1° $f(n) \leq n-1$

2° $\boxed{f(n) \geq n+1}$

$f(n) = n$

$f(n) \le n-1$

By ind assy

$$f(\underbrace{f(n)}_{<n}) = \underbrace{f(n)}_{<n}$$

$$f(\underbrace{f(n)}_{k}) = \underbrace{f(n)}_{k}$$

$k < n$

$$\boxed{(n-1)^2 < f(n)f(f(n)) < n^2+n}$$

$$(n-1)^2 < f(n)^2$$

Second case $\boxed{\text{Homework}}$