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Number Theory & Algebra – L3

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Problems

Congruences, divisibility and diophantine equations

Problem 1. Let a, b be a positive integers and p, q distinct primes such that $aq \equiv 1 \pmod{p}$ and $bp \equiv 1 \pmod{q}$. Prove that

$$\frac{a}{p} + \frac{b}{q} > 1.$$

Problem 2. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a and b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Problem 3. Determine all positive integer x,y,z such that

$$1 = \frac{2}{x^2} + \frac{3}{y^2} + \frac{4}{z^2}.$$

Problem 4. Prove that $n \geqslant 2$ is composite if and only if there exist integers $a, b, x, y \geqslant 1$ such that

- $a + b = n$,
- $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Problem 5. Two natural numbers d and d', where $d' > d$, are both divisors of n. Prove that $d' > d + \dfrac{d^2}{n}$.

Problem 6. Let $n \geqslant 3$ and consider pairwise coprime numbers $p_1, p_2, \ldots, p_n$. Suppose that for any $k \in \{1, 2, \ldots, n\}$ the residue of $\prod\limits_{i \neq k} p_i$ modulo p_k equals r. Prove, that $r \leqslant n - 2$.

Problem 7. Let x, y, z, t be a positive integers such that

$$x^2 + y^2 + z^2 + t^2 = 2018!.$$

Prove that $x, y, z, t > 10^{250}$.

Problem 8$^\bigstar$. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Problem 9. Let a, b be positive integers such that

$$2a^2 + a = 3b^2 + b.$$

Prove that $a-b$ and $2a+2b+1$ are perfect squares. What if we drop positivity of a and b?

Problem 10. Let $k > 1$ be the integer. Sum of a divisor of k and a divisor of $k-1$ is equal to ℓ and $\ell > k+1$. Prove that at least one number: $\ell - 1$ or $\ell + 1$ is composite.

Problem 11. Find all primes p for which the following sequence $1, 2, \ldots, p$ (in that order) can be divided into some (more than one) blocks of consecutive numbers, such that the sum of numbers in each block is the same.

Problem 12. Let a, b, c, n be positive integers such that the following conditions hold

(i) numbers a, b, c, $a+b+c$ are pairwise coprime,
(ii) number $(a+b)(b+c)(c+a)(a+b+c)(ab+bc+ca)$ is a perfect n-th power.

Prove, that the product abc can be expressed as a difference of two perfect n-th powers.

Problem 13. Let a, b, and c be odd positive integers such that a is not a perfect square and

$$a^2 + a + 1 = 3(b^2 + b + 1)(c^2 + c + 1).$$

Prove that at least one of the numbers $b^2 + b + 1$ and $c^2 + c + 1$ is composite.

Problem 14. Let x, y be natural numbers greater than 1. Suppose that $x^2 + y^2 - 1$ is divisible by $x + y - 1$. Prove that $x + y - 1$ is a composite number.

Problem 15. Let m, n be two unequal positive integers such that

$$\operatorname{lcm}(m, n) = m^2 - n^2 + mn.$$

Prove that mn is a perfect cube.

Problem 16. Find all primes p and q such that $p+q$, $p+q^2$, $p+q^3$, $p+q^4$ are primes.

Problem 17. Find all integer triples (a, b, c) satisfying the equation

$$5a^2 + 9b^2 = 13c^2.$$

Problem 18. Show that no non-zero integers a, b, x, y satisfy

$$\begin{cases} ax - by = 16, \\ ay + bx = 1. \end{cases}$$

Problem 19. Let $a, b, c > 1$ be distinct integers such that $\gcd(a, b, c) = 1$. Find all possible values of

$$\gcd(a^2b + b^2c + c^2a, ab^2 + bc^2 + ca^2, a + b + c).$$

Problem 20. Three pairwise distinct positive integers a, b, c, with $\gcd(a, b, c) = 1$, satisfy

$$a \mid (b - c)^2, \quad b \mid (a - c)^2, \quad c \mid (a - b)^2.$$

Prove that there does not exist a non-degenerate triangle with side lengths a, b, c.

Problem 21. Find all the triples of positive integers (a, b, c) for which the number

$$\frac{(a+b)^4}{c} + \frac{(b+c)^4}{a} + \frac{(c+a)^4}{b}$$

is an integer and $a + b + c$ is a prime.

Problem 22. A Pythagorean triple (a, b, c) is a triple of integers satisfying the equation $a^2 + b^2 = c^2$. We say that such a triple is primitive if $\gcd(a, b, c) = 1$. Let d be an odd integer with exactly n prime divisors. Show that there exist exactly 2^{n-1} primitive Pythagorean triples where d is the first element of the triple. For example if $p = 15$ then $(15, 8, 17)$ and $(15, 112, 113)$ are the primitive Pythagorean triples.

Problem 23. Let p and q are distinct odd primes. Prove that $(pq + 1)^4 - 1$ has at least 4 distinct prime divisors.

Problem 24. Let n be a positive integer. Prove that the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{n}$$

has a solution (x, y) with x, y positive integers, iff n is divisible by some m^2, where $m > 1$ is an integer.

Problem 25. All the prime numbers are written in order $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on. Find all pairs of positive integers a and b with $a - b \geq 2$ such that $p_a - p_b$ divides $2(a - b)$.

Problem 26. Find the smallest prime number that cannot be written in the form $|2^a - 3^b|$ with nonnegative integers a, b.

Problem 27. Find all pairs (m, n) of integers that satisfy the equation

$$(m - n)^2 = \frac{4mn}{m + n - 1}.$$

Inequalities

Problem 28. Prove that for any non-negative real numbers the following inequality holds

$$x_1 + 2x_2 + \ldots + nx_n \leq \frac{n(n-1)}{2} + x_1 + x_2^2 + \ldots + x_n^n.$$

Problem 29. For given integer $n \geq 1$ find the smallest value of

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \ldots + \frac{x_n^n}{n}$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers that

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = n.$$

Problem 30. Let $a, b, c \in (0,1)$. Prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

Problem 31. Prove that for any positive real numbers a, b, c the following inequality holds

$$a\sqrt{b^2 - bc + c^2} + c\sqrt{a^2 - ab + b^2} \geq b\sqrt{a^2 + ac + c^2}.$$

Problem 32. Let a, b, c, d be positive real numbers. Prove that

$$\sqrt[3]{(a+b+c)(a+b+d)} \geq \sqrt[3]{ac} + \sqrt[3]{bd}.$$

Problem 33. Prove that for any positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$\frac{a_1^3}{a_1^2 + a_1a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2a_3 + a_3^2} + \ldots + \frac{a_n^3}{a_n^2 + a_na_1 + a_1^2} \geq \frac{a_1 + a_2 + \ldots + a_n}{3}.$$

Problem 34. Let a, b, c be real numbers. Prove that

$$\sqrt{2(a^2+b^2)} + \sqrt{2(b^2+c^2)} + \sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2 + 3(b+c)^2 + 3(c+a)^2}.$$

Problem 35. Let a_1, a_2, $\ldots$, a_n be positive real numbers such that

$$a_1^2 + 2a_2^3 + \cdots + na_n^{n+1} < 1.$$

Prove that

$$2a_1 + 3a_2^2 + \cdots + (n+1)a_n^n < 3.$$

Solutions

Congruences, divisibility and diophantine equations

Problem 1. Let a, b be a positive integers and p, q distinct primes such that $aq \equiv 1 \pmod{p}$ and $bp \equiv 1 \pmod{q}$. Prove that

$$\frac{a}{p} + \frac{b}{q} > 1.$$

Solution. Given conditions imply that $aq = pk + 1$ and $bp = ql + 1$ for some positive integers k and l. Since p, q are primes we see that $pq \mid pk + ql + 1$. Therefore

$$\frac{a}{p} + \frac{b}{q} = \frac{aq + bp}{pq} = \frac{pk + ql + 1}{pq} + \frac{1}{pq} \geqslant 1 + \frac{1}{pq} > 1.$$

□

Problem 2. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a and b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Solution. It suffices to show that one of $2d-1, 5d-1, 13d-1$ is not a perfect square. We will prove the result by contradiction.

Suppose that all subsets pairs of $\{2, 5, 13, d\}$ contain two elements, a and b, that $ab-1$ is a perfect square. The quadratic residues modulo 16 are $0, 1, 4, 9$. It follows that $2d, 5d, 13d$ must all be either $1, 2, 5, 10 \pmod{16}$. Solving these congruences give us the following results:

- $2d \equiv 1, 2, 5, 10 \implies d \equiv 1, 5, 9, 13,$
- $5d \equiv 1, 2, 5, 10 \implies d \equiv 1, 2, 10, 13,$
- $13d \equiv 1, 2, 5, 10 \implies d \equiv 2, 5, 9, 10.$

Clearly no residue satisfies all three conditions, thus we have our contradiction.

□

Problem 3. Determine all positive integer x,y,z such that

$$1 = \frac{2}{x^2} + \frac{3}{y^2} + \frac{4}{z^2}.$$

Solution. Obviously $x, y \geqslant 2$ and $z \geqslant 3$. Consider the following cases

- $x = 2$, then $\dfrac{3}{y^2} + \dfrac{4}{z^2} = \dfrac{1}{2}$, so $4y^2 + 3z^2 = \frac{1}{2}y^2z^2$ i.e.

$$(y^2 - 6)(z^2 - 8) = 48 \quad (y \geqslant 2,\ z \geqslant 3).$$

One can see that there are no (x, y, z) satisfying these conditions.

- $y = 2$, then $\dfrac{2}{x^2} + \dfrac{4}{z^2} = \dfrac{1}{4}$ or equivalently

$$(x^2 - 8)(z^2 - 16) = 128 \quad (x \geqslant 2, y \geqslant 3).$$

Easy to see that $x = 3$ and $z = 12$ are only solution.

- $x, y \geqslant 3$; then $z \geqslant 3$ so

$$\frac{2}{x^2} + \frac{3}{y^2} + \frac{4}{z^2} \leqslant \frac{2}{9} + \frac{3}{9} + \frac{4}{9} = 1,$$

where equality holds only for $x = y = z = 3$.

Finally all solution (x, y, z) are $(3, 2, 12)$ and $(3, 3, 3)$. □

Problem 4. Prove that $n \geqslant 2$ is composite if and only if there exist integers $a, b, x, y \geqslant 1$ such that

- $a + b = n$,
- $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Solution. Let n be a composite number i.e. $n = qr$, $q \geqslant 2$, $r \geqslant 2$. Take

$$a = r, \qquad b = (q-1)r, \qquad x = 1, \qquad y = (q-1)(r-1),$$

then

$$a + b = n, \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 1.$$

In other direction suppose that $n \geqslant 2$ and there exist $a, b, x, y \geqslant 1$ satisfying two problems conditions. We see that $x < a$, $y < b$ and

$$ay = b(a - x).$$

Let $d = \gcd(a, b)$ and take $a = \alpha d$, $b = \beta d$, where α and β are coprime. Also we have a relation $n = (\alpha + \beta)d$. Since $\alpha + \beta > 1$ we are left to prove that $d > 1$.

Now from $\alpha y = \beta(a - x)$ follows that $\beta \leqslant y$. Moreover $y < b$, thus $\beta < b$ and consequently $d = \dfrac{b}{\beta} > 1$. □

Problem 5. Two natural numbers d and d', where $d' > d$, are both divisors of n. Prove that $d' > d + \dfrac{d^2}{n}$.

Solution. Let $k = \dfrac{n}{d}$, $k' = \dfrac{n}{d'}$ with $k > k'$. We have

$$k^2 > (k+1)(k-1) \geqslant (k+1)k'$$

so $\dfrac{1}{k'} > \dfrac{1}{k} + \dfrac{1}{k^2}$. Therefore

$$d' = \frac{n}{k'} > \frac{n}{k} + \frac{n}{k^2} = d + \frac{d^2}{n}.$$

$\square$

Problem 6. Let $n \geqslant 3$ and consider pairwise coprime numbers $p_1, p_2, \ldots, p_n$. Suppose that for any $k \in \{1, 2, \ldots, n\}$ the residue of $\prod\limits_{i \neq k} p_i$ modulo p_k equals r. Prove, that $r \leqslant n - 2$.

Solution. If $r = 0$ then the problem is obvious. Let $r > 0$ and let us define

$$P = p_1 p_2 \ldots p_n, \quad P_i = \frac{P}{p_i} \text{ and } S = P_1 + P_2 + \ldots + P_n - r.$$

Of course $p_i > r$. We have $p_1 \mid S = (P_1 - r) + P_2 + \ldots + P_n$, so for all $1 \leqslant i \leqslant n$ we have that $p_i \mid S$. Because $p_1, p_2, \ldots, p_n$ are pairwise coprime, then $P \mid S$. Moreover $S > 0$, so $S \geqslant P$ and

$$P_1 + P_2 + \ldots + P_n = S + r > P.$$

Therefore for some i, $P_i > \dfrac{P}{n}$ and $p_i \leqslant n - 1$, so $r \leqslant n - 2$ because $p_i > r$. $\square$

Problem 7. Let x, y, z, t be a positive integers such that

$$x^2 + y^2 + z^2 + t^2 = 2018!.$$

Prove that $x, y, z, t > 10^{250}$.

Solution. Since any even square of integers gives 0 or 4 $\pmod 8$, and square of odd numbers gives 1 $\pmod 8$ we see that x, y, z and t are even.

Therefore $x = 2x_1$, $y = 2y_1$, $z = 2z_1$ and $t = 2t_1$ for some integers x_1, y_1, z_1 i t_1. Therefore

$$x_1^2 + y_1^2 + z_1^2 + t_1^2 = \frac{2018!}{4},$$

which is analogous equation to the given in problem assumption.

Moreover $2^{2000} \mid 2018!$, indeed by *Legendre's formula*

$$v_2(2018!) = \left\lfloor \frac{2018}{2} \right\rfloor + \left\lfloor \frac{2018}{4} \right\rfloor + \left\lfloor \frac{2018}{8} \right\rfloor + \ldots + \left\lfloor \frac{2018}{2^i} \right\rfloor + \ldots > 2000.$$

Repeating the above argument 900 times (we can since $2000 > 2 \cdot 900$) we infer that x, y, z and t are divisible by 2^{900}. It remains to observe that

$$2^{900} = \left(2^{10}\right)^{90} > \left(10^3\right)^{90} = 10^{270} > 10^{250},$$

hence all of x, y, z and t are greater than 10^{250}.

$\square$

Problem 8★. Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution. Note that $x \geqslant -1$ since otherwise the LHS would not be an integer. If $x = -1$ then we have $2 = y^2$ which is clearly impossible. If $x = 0$ then we have the solutions $(0, \pm 2)$. Now assume $x > 0$. We have

$$2^x(1 + 2^{x+1}) = (y + 1)(y - 1).$$

Note that once factor of the RHS must be congruent to 2 (mod 4) and the other must be congruent to 0 (mod 4). Therefore, we have $y = 2^{x-1} \cdot k \pm 1$ where k is an odd integer.

- Case 1: $y = 2^{x-1} \cdot k + 1$. Then we have

$$2^x(1 + 2^{x+1}) = (2^{x-1} \cdot k + 2)(2^{x-1} \cdot k) \implies 1 + 2^{x+1} = (2^{x-2} \cdot k + 1) \cdot k.$$

  This rearranges as $k - 1 = 2^{x-2} \cdot (8 - k^2)$. Since $k - 1 \geqslant 0$, we must have $8 - k^2 \geqslant 0$ so the only possible value of k is 1. This yields $2^{x-2} = 0$ which is clearly impossible.
- Case 2: $y = 2^{x-1} \cdot k - 1$. Then we have

$$2^x(1 + 2^{x+1}) = 2^{x-1} \cdot k(2^{x-1} \cdot k - 2) \implies 1 + 2^{x+1} = k(2^{x-2} \cdot k - 1).$$

  This rearranges as $\dfrac{1 + k}{k^2 - 8} = 2^{x-2}$. If $x = 1$ then $k^2 - 2k - 10 = 0$ which has no integral solution. Therefore, assume $x \geqslant 2$ so the RHS is an integer. Then the only possible value for k is 3 since if $k = 1$ then $\dfrac{1 + k}{k^2 - 8} < 0$ and if $\geqslant 5$ then $\dfrac{1 + k}{k^2 - 8} < 1$. Then we have $4 = 2^{x-2}$ so $x = 4$. This yields the solutions $(4, \pm 23)$.

Hence, the only solutions are $(0, \pm 2), (4, \pm 23)$. $\square$

Problem 9. Let a, b be positive integers such that

$$2a^2 + a = 3b^2 + b.$$

Prove that $a - b$ and $2a + 2b + 1$ are perfect squares. What if we drop positivity of a and b?

Solution. Notice that

$$b^2 = 2a^2 + a - 2b^2 - b = (a-b)(2a+2b+1). \tag{1}$$

If prime p divides $a-b$ and $2a+2b+1$, then $p \mid b$ and so

$$p \mid (2a+2b+1-2(a-b)) - 4b = 4b+1-4b = 1.$$

Therefore $\gcd(a-b, 2a+2b+1) = 1$.

If $a-b<0$, then $a-b=-k^2$ and $2a+2b+1=-\ell^2$ for some integers k, l. The number $3b^2+b=b(3b+1)$ is even, so $2a^2+a$ too and so a is even.

Since $a = \dfrac{1}{4}(-2k^2-\ell^2-1)$, we see that ℓ is odd, i.e. $\ell = 2n+1$ for some integer n. Then $a = \dfrac{-k^2-2n(n+1)-1}{2}$. Therefore k is odd, so $k=2m+1$ for some integer m. Thus

$$a = \frac{-2(4m^2+4m+1)-4n(n+1)-2}{4} = -2m^2-2m-n(n+1)-1,$$

hence a is odd – contradiction.

Therefore $a-b>0$ and so from 1 the problem follows. □

Problem 10. Let $k>1$ be the integer. Sum of a divisor of k and a divisor of $k-1$ is equal to ℓ and $\ell > k+1$. Prove that at least one number: $\ell-1$ or $\ell+1$ is composite.

Solution. Any divisor of N is equal to N or is at most $N/2$. Therefore if sum of divisors of k and $k-1$ is greater then $k+1$ then one of them is equal to k or $k-1$.

If one is equal to k, then $\ell-1 = k-1+d$, where $d>1$ and $d \mid k-1$. Thus $d \mid \ell-1$, so $\ell-1$ is composite.

If one of them is $k-1$, then $\ell+1 = k+e$, for some $e>1$ and $e \mid k$. Then $e \mid \ell+1$, so $\ell+1$ is composite. □

Problem 11. Find all primes p for which the following sequence $1, 2, \ldots, p$ (in that order) can be divided into some (more than one) blocks of consecutive numbers, such that the sum of numbers in each block is the same.

Solution. Easy to see that $p>2$. Let b, where $1<b<p$ be the number of blocks, and suppose that the sum of numbers in each block equals S. Then

$$bS = 1+2+\ldots+p = \frac{p(p+1)}{2} \equiv 0 \pmod{p},$$

thus $p \mid S$, becouse $p>b$.

Now consider the following block: $1, 2, \ldots, k$ for some $k<p$ with sum S equal to $\frac{k(k+1)}{2}$. Then $p \mid S$ iff $p \mid (k+1)$ (since $k<p$), therefore applying

again inequality $k < p$, we get $k = p - 1$. Thus the next block contains only one number: p, hence from equality of sums we optain $p = \frac{(p-1)p}{2}$, so $p = 3$.

Easy to see that the partition $\{1, 2\}, \{3\}$ of $1, 2, 3$ satisfies problem condition. $\square$

Problem 12. Let a, b, c, n be positive integers such that the following conditions hold

(i) numbers a, b, c, $a + b + c$ are pairwise coprime,
(ii) number $(a + b)(b + c)(c + a)(a + b + c)(ab + bc + ca)$ is a perfect n-th power.

Prove, that the product abc can be expressed as a difference of two perfect n-th powers.

Solution. Note that, $(a + b + c, a) = (a + b + c, b + c) = 1$. Moreover, we also have, $(a+b+c, a+b) = (a+b+c, c) = 1$ and $(a+b+c, a+c) = (a+b+c, b) = 1$. Therefore, $(a + b)(a + c)(b + c)$ and $a + b + c$ are coprime.

Next, let $p \mid a + b$ be a prime number. We shall prove that $p \nmid ab + bc + ca$. Assume the converse. Let $p \mid ab + bc + ca$. Then $p \mid a + b \Rightarrow p \mid ac + bc$, and thus, $p \mid ab$. Thus, either $p \mid a$, in which case $p \mid a + b$ yields $p \mid b$, contradicting with the coprimality of a and b. Similar holds for $p \mid b$. Thus, $\gcd((a + b)(a + c)(b + c), ab + bc + ca) = 1$.

Now, this yields that, $\gcd((a+b)(a+c)(b+c), (a+b+c)(ab+bc+ca)) = 1$. Since the product is a perfect power, it therefore holds that for some m, k integers, $(a + b)(a + c)(b + c) = m^n$ and $(a + b + c)(ab + bc + ca) = k^n$. Thus,

$$abc = (a + b + c)(ab + bc + ca) - (a + b)(a + c)(b + c) = k^n - m^n,$$

as claimed. $\square$

Problem 13. Let a, b, and c be odd positive integers such that a is not a perfect square and

$$a^2 + a + 1 = 3(b^2 + b + 1)(c^2 + c + 1).$$

Prove that at least one of the numbers $b^2 + b + 1$ and $c^2 + c + 1$ is composite.

Solution. WLOG $b \geq c$ and let $p = b^2 + b + 1$. Thus

$$p \mid (a^2 + a + 1) - (b^2 + b + 1) = (a - b)(a + b + 1)$$

which means $p \mid a - b$ or $p \mid a + b + 1$.

Consider the following cases.

- $a - b = p$, then $a = b^2 + 2b + 1 = (b + 1)^2$ – contradiction.
- $a + b + 1 = p$, then $a = b^2$ – contradiction.

- $a+b+1 \geq 2p \implies a \geqq 2b^2+b+1$, then since $b \geq c$, we get
$$(2b^2+b+1)^2+(2b^2+b+1)+1 \leq 3(b^2+b+1)^2,$$
which expands to $b(b-3)(b^2+b+1) \leq 0$. Thus $b \leq 3$.
  - If $b=1$, then $c=1$ which means $a^2+a+1=27$ – contradiction.
  - If $b=3$, then either $c=1$ or $c=3$. This means $a^2+a+1 \in \{117, 507\}$ which yields $a=22$ – contradiction to a odd.
- $a-b \geq 2p$ implies that $a+b+1 \geq a-b \geq 2p$, so we have the previous case.

□

Problem 14. Let x, y be natural numbers greater than 1. Suppose that x^2+y^2-1 is divisible by $x+y-1$. Prove that $x+y-1$ is a composite number.

Solution. Note that the number
$$x^2+y^2-1+2xy=(x+y-1)(x+y+1)$$
is divisible by $x+y-1$. Therefore $2xy$ is also divisible by $x+y-1$. If $x+y-1$ is prime, then one of 2, x and y is divisible by $x+y-1$ – contradiction. □

Problem 15. Let m, n be two unequal positive integers such that
$$\operatorname{lcm}(m,n)=m^2-n^2+mn.$$
Prove that mn is a perfect cube.

Solution. Suppose that $\operatorname{lcm}(m,n)=m^2-n^2+mn$. Let $d=\gcd(m,n)$ and $(m,n)=(ds,dt)$ for coprime integers s,t. Since $\operatorname{lcm}(m,n)=dst$, we have $st=d\left(s^2-t^2+st\right)$. So we have $s \mid dt^2$ and $t \mid ds^2$, which implies $st \mid d$ because $\gcd(s,t)=1$. So we can take $k \in \mathbb{N}^*$ such that $d=kst$. Thus, we have $1=k\left(s^2-t^2+st\right)$, which implies $|k|=1$. Therefore, $mn=d^2st=(kst)^2st=k^2(st)^3=(st)^3$. □

Problem 16. Find all primes p and q such that $p+q$, $p+q^2$, $p+q^3$, $p+q^4$ are primes.

Solution. From parity easy to see that p or q is equal to 2.

If $p=2$, then we require $2+q$, $2+q^2$, $2+q^3$ and $2+q^4$ to be primes. If $q \neq 3$, then $3 \mid 2+q^2$ – contradiction. Thus $q=3$, and easy to see that $(2,3)$ satisfies problem assumptions.

If $q=2$, then we require $p+2$, $p+4$, $p+8$ and $p+16$ to be primes. Suppose $p>3$, then $p \equiv \pm 1 \pmod 6$. If $p \equiv 1 \pmod 6$, then $3 \mid p+8$, otherwise $p \equiv -1 \pmod 6$, so $3 \mid p+4$. Therefore $p=3$. Easy to see that $(3,2)$ satisfies problem assumptions. □

Problem 17. Find all integer triples (a, b, c) satisfying the equation
$$5a^2 + 9b^2 = 13c^2.$$

Solution. Taking $\pmod 5$ we see that $5 \mid b,\ c$. Define $b = 5b'$, $c = 5c'$. We have $a^2 + 45b'^2 = 65c'^2$ and $5 \mid a$. Define $a = 5a'$. We have $5a'^2 + 9b'^2 = 13c'^2$. By infinite descent we get $a = b = c = 0$. $\square$

Problem 18. Show that no non-zero integers a, b, x, y satisfy
$$\begin{cases} ax - by = 16, \\ ay + bx = 1. \end{cases}$$

Solution. Notice that
$$257 = 16^2 + 1 = (ax - by)^2 + (ay + bx)^2 = (a^2 + b^2)(x^2 + y^2).$$
Since 257 is prime we see that either $a^2 + b^2$ or $x^2 + y^2$ is 1, but it cannot happen for non-zero integers a, b, x, y. $\square$

Problem 19. Let $a, b, c > 1$ be distinct integers such that $\gcd(a, b, c) = 1$. Find all possible values of
$$\gcd(a^2b + b^2c + c^2a, ab^2 + bc^2 + ca^2, a + b + c).$$

Solution. Let $d = \gcd(K := a^2b + b^2c + c^2a, L = ab^2 + bc^2 + ca^2)$ and $M := a + b + c$. Suppose that some prime p divides a and d then $p \mid K$ and $d \mid M$. Thus $p \mid K - a(ab + c^2) = b^2c$, so at least one number b or c is divisible by p. Thus $M = a + b + c$ and its two summands are divisible by p. Thus a, b, c are divisible by p, which contradicts to $\gcd(a, b, c) = 1$. Therefore $\gcd(a, d) = 1$ and $\gcd(b, d) = \gcd(c, d) = 1$.

On the other hand $d \mid (ab + bc + ca)M - K - L = 3abc$ and since a, b, c are coprime with d, so $d \mid 3$ i.e. $d = 1$ or $d = 3$. Easy to see that both cases are possible:

- for $(a, b, c) = (2, 3, 4)$ we have $\gcd(K, L, M) = 1$,
- for $(a, b, c) = (2, 5, 8)$ we have $\gcd(K, L, M) = 3$.

$\square$

Problem 20. Three pairwise distinct positive integers a, b, c, with $\gcd(a, b, c) = 1$, satisfy
$$a \mid (b - c)^2, \quad b \mid (a - c)^2, \quad c \mid (a - b)^2.$$
Prove that there does not exist a non-degenerate triangle with side lengths a, b, c.

Solution. Easy to see a, b, c are pairwise coprime and that $abc \mid M$, where

$$M = 2(ab + bc + ca) - (a^2 + b^2 + c^2).$$

Suppose that a triangle with sides a, b, c exists. Then $a < b+c$, so $a^2 < ab+ac$. Similarly $b^2 < bc + ba$ and $c^2 < ca + cb$. Therefore $M > 0$. On the other hand $a^2 + b^2 + c^2 > ab + bc + ca$, so $M < ab + bc + ca$. WLOG $a > b > c$. Since $abc \mid M$, we get $c = 1$ or $c = 2$. If $c = 1$ then $b < a < b + 1$ – contradiction. If $c = 2$ we get $b < a < b + 2$, so $a = b + 1$. But then $1 = (a - b)^2$ is not divisible by $c = 2$. □

Problem 21. Find all the triples of positive integers (a, b, c) for which the number

$$\frac{(a+b)^4}{c} + \frac{(b+c)^4}{a} + \frac{(c+a)^4}{b}$$

is an integer and $a + b + c$ is a prime.

Solution. Set $a + b + c = p$ and conclude that

$$p^4 \left(\frac{ab + bc + ca}{abc} \right) \in \mathbb{Z},$$

as $a, b, c < p$ and p is a prime, we get that $abc \mid ab + bc + ca$, but this is trivial to solve using bounding argument. All triples are the following: $(1, 1, 1), (1, 2, 2), (2, 3, 6)$. □

Problem 22. A Pythagorean triple (a, b, c) is a triple of integers satisfying the equation $a^2 + b^2 = c^2$. We say that such a triple is primitive if $\gcd(a, b, c) = 1$. Let d be an odd integer with exactly n prime divisors. Show that there exist exactly 2^{n-1} primitive Pythagorean triples where d is the first element of the triple. For example if $p = 15$ then $(15, 8, 17)$ and $(15, 112, 113)$ are the primitive Pythagorean triples.

Solution. We may write

$$d = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n},$$

where p_1, p_2, …, p_n are distinct odd primes, and e_1, e_2, …, e_n are positive integers. We seek all primitive Pythagorean triples (d, b, c). For such a triple $d^2 + b^2 = c^2$, so $(c + b)(c - b) = d^2$. Suppose p_i divides both $c + b$ and $c - b$ for some i. Then p_i divides $(c + b) + (c - b) = 2c$, and $(c + b) - (c - b) = 2b$, so $p_i \mid c, b$. Then (d, b, c) fails to be a primitive Pythagorean triple since p_i divides all three numbers d, b, c. Thus for each i, the prime p_i divides at most one of $c + b$ and $c - b$.

This implies that in the factorization $(c + b)(c - b) = d^2$ for each i all the factors of p_i must reside in $c + b$ or $c - b$. In other words for each i we can make one of two choices of where to place all the factors of p_i for a total of

2^n factorizations. However half of them must be discarded since $c+b$ must be the greater number. Each of the other half however does lead to a unique solution, so there are 2^{n-1} such Pythagorean triples. $\square$

Problem 23. Let p and q are distinct odd primes. Prove that $(pq+1)^4-1$ has at least 4 distinct prime divisors.

Solution. Let $x := pq+1$. Observe that

$$(pq+1)^4-1 = pqx(pqx+2).$$

Now $x \equiv 2 \pmod{p}$ and $x \equiv 2 \pmod{q}$. Since $p, q \geq 3$ we get $p \nmid x$ and $q \nmid x$. Moreover $x > p, q$.

If x is composite, then of course we are done. If x is prime, then let $y := pqx+2$. Then $y \equiv 2 \pmod{p}$ and $y \equiv 2 \pmod{q}$ and $y \equiv 2 \pmod{x}$, so $p \nmid y$, $q \nmid y$ and $x \nmid y$, which leads us to the conclusion from the thesis. $\square$

Problem 24. Let n be a positive integer. Prove that the equation

$$\sqrt{x}+\sqrt{y}=\sqrt{n}$$

has a solution (x, y) with x, y positive integers, iff n is divisible by some m^2, where $m > 1$ is an integer.

Solution. If $n = am^2$ for some integers a, m with $a \geq 1$ and $m > 1$, then the pair of positive integers $(x, y) = (a(m1)^2, a)$ is a solution to $\sqrt{x}+\sqrt{y}=\sqrt{n}$.

Conversely, we suppose that $\sqrt{x}+\sqrt{y}=\sqrt{n}$ where x and y are positive integers. Clearly, we must have $n \geq 4$. Let $d = \gcd(x, y, n)$ and x', y', n' be defined by $x = dx'$, $y = dy'$, $n = dn'$. Then $\gcd(x', y', n') = 1$ and $\sqrt{x'}+\sqrt{y'} = \sqrt{n'}$. Thus, we may (and will) suppose that $\gcd(x, y, n) = 1$.

From the equation, we deduce $n = x+y+2\sqrt{xy}$ and then $4xy = (nxy)^2$ which rewrites as

$$(xy)^2 = n(2x+2yn). \tag{2}$$

If n is a power of 2, say $n = 2^\alpha$ where $\alpha \geq 2$ (since $n \geq 4$), then n is divisible by 2^2. Otherwise, n is divisible by an odd prime p. From (2), p also divides xy.

If p divide $2x+2y-n$, p would divide $2(x+y)$, hence $x+y$ and also x, y (as an odd divisor of both $x-y$ and $x+y$), in contradiction to $\gcd(x, y, n) = 1$. It follows that p^2, which divides $(xy)^2$, must divide n. Thus n is also divisible by a square greater than 1 in this case. $\square$

Problem 25. All the prime numbers are written in order $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on. Find all pairs of positive integers a and b with $a-b \geq 2$ such that $p_a - p_b$ divides $2(a-b)$.

Solution. Suppose (a,b) is any solution. If $b=1$ then $p_b=2$, and p_a-p_b is an odd number that divides $a-b$. Moreover, $p_a \geq 2a-1$ and $a>2$. Hence

$$a-b \geq p_a - p_b \geq 2a-3 > a-1 = a-b,$$

a contradiction.

Thus the numbers p_a and p_b are odd primes, which implies that $p_a - p_b \geq 2(a-b)$. We obtain $p_a - p_b = 2(a-b)$ so that all odd numbers between p_b and p_a are primes. Then, the numbers p_b, p_{b+2}, p_{b+4} are primes. But one of thse numbers is divisible by 3. It follows $p_b = 3$. Since $a-b \geq 2$ and all odd numbers between 3 and p_a are primes, we have $p_a = 7$. So $(4,2)$ is the only solution. $\square$

Problem 26. Find the smallest prime number that cannot be written in the form $|2^a - 3^b|$ with nonnegative integers a, b.

Solution. Easy to check that all primes smaller than 41 can be expressed in the required form. Suppose $2^a - 3^b = 41$. Then $a > 2$ and $3^b \equiv 2^a - 41 \equiv 7 \pmod 8$, contradiction, because 3^b is congruent to 1 or 3 modulo 8.

Thus $3^b - 2^a = 41$. Easy to see that $a > 1$ and $b > 0$. Also $2^a \equiv 3^b - 41 \equiv 1 \pmod 3$ and $3^b \equiv 2^a + 41 \equiv 1 \pmod 4$. Thus a and b are even, so a=2m and b=2n for some positive integers m, n. Thus $41 = 3^b - 2^a = (3^n - 2^m)(3^n + 2^m)$, which easily implies there are no solutions. $\square$

Problem 27. Find all pairs (m,n) of integers that satisfy the equation

$$(m-n)^2 = \frac{4mn}{m+n-1}.$$

Solution. If $m=n$ then $m=n=0$. So assume m, n are distinct. Suppose $m>n$ then $(m-n)^2 \mid 4mn$. Let $d = \gcd(m,n)$ then $m = da$, $n = db$, where $\gcd(a,b) = 1$. We get $(a-b)^2 \mid 4ab$ but $\gcd(a-b, ab) = 1$, so $a-b \mid 2$ hence $a-b \in \{1,2\}$.

- If $a-b=1$, so $a=b+1$, thus $d^2(d(a+b)-1) = 4d^2ab$, so $d(a+b)-1 = 4ab$ and $d(2b+1)-1 = 4b(b+1)$ i.e $d = 2b+1$. Finally $m = (2b+1)(b+1)$, and $n = (2b+1)b$.
- If $a-b=2$, so $a=b+2$, thus $4d^2(d(a+b)-1) = 4d^2ab$, and $d(2b+2)-1 = (b+2)b$ thus $d = \dfrac{b+1}{2}$. Finally $m = \dfrac{(b+2)(b+1)}{2}$, and $n = \dfrac{b(b+2)}{2}$.

$\square$

Inequalities

Problem 28. Prove that for any non-negative real numbers the following inequality holds

$$x_1 + 2x_2 + \ldots + nx_n \leq \frac{n(n-1)}{2} + x_1 + x_2^2 + \ldots + x_n^n.$$

Solution. By AM-GM inequality we have

$$x_k{}^k + \underbrace{1 + 1 + \ldots + 1}_{k-1} \geq kx_k,$$

so

$$\sum_{x=1}^{n} x_k{}^k + \binom{n}{2} = \sum_{x=1}^{n} \Big(x_k{}^k + (k-1) \Big) \geq \sum_{i=1}^{n} kx_k.$$

$\square$

Problem 29. For given integer $n \geq 1$ find the smallest value of

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \ldots + \frac{x_n^n}{n}$$

where $x_1, x_2, \ldots, x_n$ are positive real numbers that

$$\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = n.$$

Solution. Using AM-GM inequality we see that

$$\frac{1}{1+k} \left(x^k + \underbrace{\frac{1}{x} + \ldots + \frac{1}{x}}_{k} \right) \geq \left(x^k \cdot \underbrace{\frac{1}{x} \cdot \ldots \cdot \frac{1}{x}}_{k} \right)^{1/(k+1)} = 1,$$

so

$$x^k + \frac{k}{x} \geq k + 1.$$

Therefore

$$\sum_{k=1}^{n} \frac{x_k^k}{k} + \sum_{k=1}^{n} \frac{1}{x_k} \geq \sum_{k=1}^{n} \left(1 + \frac{1}{k} \right) = n + \sum_{k=1}^{n} \frac{1}{k},$$

hence

$$\sum_{k=1}^{n} \frac{x_k^k}{k} \geq \sum_{k=1}^{n} \frac{1}{k},$$

equality holds iff $x_i = 1$. $\square$

Problem 30. Let $a, b, c \in (0, 1)$. Prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

Solution. Note that $\sqrt{x} < \sqrt[3]{x}$ for $x \in (0, 1)$, so by AM-GM

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{abc} + \sqrt[3]{(1-a)(1-b)(1-c)} \le$$
$$\le \frac{a+b+c}{3} + \frac{(1-a)+(1-b)+(1-c)}{3} = 1.$$

□

Problem 31. Prove that for any positive real numbers a, b, c the following inequality holds

$$a\sqrt{b^2 - bc + c^2} + c\sqrt{a^2 - ab + b^2} \ge b\sqrt{a^2 + ac + c^2}.$$

Solution. We will square both sides. We get

$$a^2(b^2 - bc + c^2) + c^2(a^2 - ab + b^2) + 2ac\sqrt{(b^2 - bc + c^2)(a^2 - ab + b^2)} \ge b^2(a^2 + ac + c$$
$$\iff 2\sqrt{(b^2 - bc + c^2)(a^2 - ab + b^2)} \ge b^2 + ab + cb - 2ac \iff$$
$$\iff 4(b^2 - bc + c^2)(a^2 - ab + b^2) \ge (b^2 + ab + cb - 2ac)^2 \iff (a - b + c)^2 \ge 0.$$

□

Problem 32. Let a, b, c, d be positive real numbers. Prove that

$$\sqrt[3]{(a+b+c)(a+b+d)} \ge \sqrt[3]{ac} + \sqrt[3]{bd}.$$

Solution. Note that

$$\frac{a}{a+b} + \frac{c}{a+b+c} + \frac{a+b}{a+b+d} \ge 3\sqrt[3]{\frac{ac}{(a+b+c)(a+b+d)}}$$

and similarly

$$\frac{b}{a+b} + \frac{d}{a+b+d} + \frac{a+b}{a+b+c} \ge 3\sqrt[3]{\frac{bd}{(a+b+c)(a+b+d)}}.$$

Adding these two inequalities proves the desired inequality. □

Problem 33. Prove that for any positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$\frac{a_1^3}{a_1^2 + a_1a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2a_3 + a_3^2} + \ldots + \frac{a_n^3}{a_n^2 + a_na_1 + a_1^2} \ge \frac{a_1 + a_2 + \ldots + a_n}{3}.$$

Solution. Let
$$A_i := \frac{a_i^3}{a_i^2 + a_ia_{i+1} + a_{i+1}^2} \text{ and } B_i := \frac{a_{i+1}^3}{a_i^2 + a_ia_{i+1} + a_{i+1}^2}.$$
Then
$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i.$$
Moreover
$$\frac{x^3 + y^3}{x^2 + xy + y^2} \geq \frac{x+y}{3}$$
so
$$2 \cdot \left(\frac{a_1^3}{a_1^2 + a_1a_2 + a_2^2} + \frac{a_2^3}{a_2^2 + a_2a_3 + a_3^2} + \ldots + \frac{a_n^3}{a_n^2 + a_na_1 + a_1^2} \right) =$$
$$= \sum_{i=1}^{n} (A_i + B_i) = \sum_{i=1}^{n} \frac{a_i^3 + a_{i+1}^3}{a_i^2 + a_ia_{i+1} + a_{i+1}^2} \geq \sum_{i=1}^{n} \frac{a_i + a_{i+1}}{3} = 2 \cdot \frac{\sum\limits_{i=1}^{n} a_i}{3}.$$
□

Problem 34. Let a, b, c be real numbers. Prove that
$$\sqrt{2(a^2+b^2)} + \sqrt{2(b^2+c^2)} + \sqrt{2(c^2+a^2)} \geq \sqrt{3(a+b)^2 + 3(b+c)^2 + 3(c+a)^2}.$$
Solution. Squaring both sides we get
$$\sum_{cyc} 2(a^2 + b^2) + 2 \sum_{cyc} \sqrt{2(a^2+b^2)}\sqrt{2(a^2+c^2)} \geq 6 \sum_{cyc} a^2 + ab.$$
$6\sum_{cyc} a^2 + ab \leq 8\sum_{cyc} a^2 + 4\sum_{cyc} ab \iff \sum_{cyc}(a-b)^2 \geq 0$ is true. So we need only to prove: $\sum_{cyc} \sqrt{a^2+b^2}\sqrt{a^2+c^2} \geq \sum_{cyc} a^2 + ab$ But by Cauchy-Schwarz: $\sqrt{a^2+b^2}\sqrt{a^2+c^2} \geq a^2 + bc$. Summing up we get the desired. □

Problem 35. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that
$$a_1^2 + 2a_2^3 + \cdots + na_n^{n+1} < 1.$$
Prove that
$$2a_1 + 3a_2^2 + \cdots + (n+1)a_n^n < 3.$$
Solution. Apply AM-GM on k copies of a_k^{k+1} and 1 copy of $\frac{1}{2^{k+1}}$ to get
$$\frac{1}{2}a_k^k \leq \frac{ka_k^{k+1} + \frac{1}{2^{k+1}}}{k+1} \implies (k+1)a_k^k \leq 2ka_k^{k+1} + \frac{1}{2^k}.$$

Summing over all k, we get

$$\sum_{k=1}^{n}(k+1)a_k^k \leq 2\sum_{k=1}^{n} k a_k^{k+1} + \sum_{k=1}^{n}\frac{1}{2^k} < 2 \cdot 1 + 1 = 3.$$

$\square$

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