Level 2 – Test 12

Wednesday, 24 June 2020

Time: 4 $^1/_2$ hours (9:00 – 13:30)

Part 1: *Short problems (the answer to each problem is an integer between 000 and 999. Only write the final answer). Each problem worth 2 points.*

Problem 1:

Find the number of positive integers less than or equal to 2020 that are no more than 10 away from a perfect square.

Problem 2:

Let a and b be real numbers and let $f(x) = ax + b,\ g(x) = x^2 - x$. Assume that $g\big(f(3)\big) = 0, g\big(f(2)\big) = 2$, and $g\big(f(4)\big) = 6$. Find $g\big(f(5)\big)$.

Problem 3:

Majid and Hamza are planning to read a 400 pages book. It takes Majid 50 seconds to read one page while it takes Hamza one minute. If Hamza started reading 30 minutes before Majid, which page will both start reading at the same time?

Problem 4:

Find the largest positive integer n satisfying that 5^n divides $1442! - 1441! - 1440!$.

Problem 5:

Khalid drove from home to a store with a constant speed. When he passed by his school, he noticed that in 20 minutes he will have completed one fourth of the distance to the store, and in 45 minutes, he will have completed one third of the distance to the store. Find the number of minutes that Khalid needs to get to the store from his school.

Problem 6:

Let $k > 0$. The two lines $ky = 8x + 544$ and $50x + ky = 1240$ intersect forming a right angle at (m, n). Find the value of $m + n$.

Problem 7:

The segment $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$, where C is the midpoint of $\overline{AB}$. It is given that $AB = 64$ and $CD = 60$. Let $\mathcal{R}$ be the set of points P which are the midpoints of the segments $\overline{XY}$, in which X lies on the segment $\overline{AB}$ and Y on the segment $\overline{CD}$. Find the area of $\mathcal{R}$.

Problem 8:

The rectangle $ABCD$ has $AB = 6$ and $BC = 8$. If we choose an arbitrary point inside the rectangle, the probability of it being closer to the diagonal $\overline{AC}$ than all sides of the rectangle is $\frac{\mathrm{m}}{\mathrm{n}}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 9:

Find the number of coefficients that are divisible by 3 in the polynomial

$$P(x) = x(x+1)(x+2)\cdots(x+239)$$

Problem 10:

We have 4 different colors and two hats of each color. In how many ways can we distribute the hats over 8 people sitting on a round table so that no adjacent people have the same hat color?

Part 2: *Long Problems (write the full solution). Each problem worth 10 points.*

Problem 11:

Find all integers n such that $3^n - n$ is divisible by 17.

Problem 12:

Let a, b, c be sidelengths of a triangle. Prove the following inequality

$$\frac{a^3+b^3+c^3}{3} \geq abc + 2|(a-b)(b-c)(c-a)|$$

Problem 13:

The numbers $0, 3, 4, 5$ are written on a board. In each step we erase a, b and write $\frac{a+b}{\sqrt{2}}$, $\frac{|a-b|}{\sqrt{2}}$.

After a finite number of steps, can the numbers on the board be:

a) $1, 2, 3, 6$?

b) $1, 2\sqrt{2}, 3\sqrt{2}, 5$?