Email training, N5

October 9-16

Problem 5.1. Let $a^2 + b^2 > a + b$ with $a > 0$ and $b > 0$. Prove that

$$a^3 + b^3 > a^2 + b^2.$$

Problem 5.2. Let the sequence x_n is given such that $0 < x_1 < 1$ and $x_{k+1} = x_k - x_k^2$ for all $k \geq 1$. Prove that for all n one has

$$x_1^2 + x_2^2 + \ldots + x_n^2 < 1.$$

Problem 5.3. Find the maximum value of expression $\sqrt{x^2 + y^2}$ if it's known that

$$\{-4 \leq y - 2x \leq 2, \quad 1 \leq y - x \leq 2\}.$$

Problem 5.4. Prove that for any numbers a, b, $c > 0$ the following inequality holds

$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \frac{2}{a} + \frac{2}{b} - \frac{2}{c}.$$

Problem 5.5. Prove the inequality

$$\sqrt{a+1} + \sqrt{2a-3} + \sqrt{50-3a} \leq 12.$$

Problem 5.6. Let the parabola $y = x^2 + px + q$ is given, which intersects coordinate axes in 3 different points. Consider the circumcircle of the triangle having vertices these 3 points. Prove that there is a point that belongs to that circle, regardless of values p and q. Find that point.

Problem 5.7. -

$\triangle ABC$ is an isosceles triangle with $AB = AC = 2$. There are 100 points $P_1 + P_2 + \ldots + P_{100}$ on the side BC. Write $m_i = AP_i^2 + BP_i \times P_iC$ $(i = 1, 2, 3, \ldots, 100)$.

Find the value of $m_1 + m_2 + \ldots + m_{100}$.

Problem 5.8. -

Let a and b be two sides of a triangle. How Should the third side c be chosen so that the points of contact of the incircle and excircle with side c divide that side into three equal segments? (The excircle corresponding to the side c is the circle which is tangent to the side c and the extensions of the sides a and b.)

Solution submission deadline October 16, 2022