Email training, N4

September 15-21, 2019

Problem 4.1. Find the largest possible value of $[7x + 11y] - 7[x] - 11[y]$ where x and y are real numbers.

$[x]$ is the floor function, the greatest integer less than or equal to x.

Problem 4.2. In the fraction below, there are n radicals in the numerator and $n - 1$ radicals in the denominator. Prove that

$$\frac{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}} > \frac{1}{4}.$$

Problem 4.3. Determine whether there exist rational numbers r and q such that $r^2 + q^2 = 15$.

Problem 4.4. Let a, m, n be positive integers. Prove that

$$gcd(a^m - 1, a^n - 1) = a^{gcd(m,n)} - 1.$$

Problem 4.5. A rectangle is divided by segments parallel to its sides into small rectangles. We call a point a T-point if it is a vertex of exactly two such small rectangles. Prove that the number of T-points is even.

Problem 4.6. Let S be a set of 10 distinct positive real numbers. Show that there exist $x, y \in S$ such that

$$0 < x - y < \frac{(1 + x)(1 + y)}{9}.$$

Problem 4.7. Let $ABCD$ is a square, P is an inner point such that $PA : PB : PC = 1 : 2 : 3$. Find $\angle APB$ in degrees.

Problem 4.8. Let ABC is a right-angled triangle with hypotenuse AB. A square $ABDE$, such that C and D are on opposite sides of AB. The bisector of $\angle C$ cuts DE at F. Let $AC = 1$, $BC = 3$. Find $\frac{DF}{EF}$.

Solution submission deadline September 21, 2019