Email training, N2

September 18-23

Problem 2.1. Let x_1 and x_2 are the roots of the equation $x^2+5x-11$. Find a quadratic polynomial which roots are x_1x_2 and $x_1^2x_2^2$.

Solution 2.1. Since $x_1$and x_2 are solution of $x^2+5x-11=0$ the, according to Vieta theorem one has $x_1+x_2=-5$ and $x_1x_2=-11$. If x_1x_2 and $x_1^2x_2^2$ are solution of $x^2+ax+b=0$ then $-a=x_1x_2+x_1^2x_2^2=x_1x_2(x_1x_2+1)=-110$ and $b=x_1^3x_2^3=-1331$. So

$$x^2-110x-1331=0$$

Answer: $x^2-110x-1331=0$.

Problem 2.2. Simplify

$$\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}.$$

Solution 2.2.

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}&=\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{2+2\sqrt{3}}{\sqrt{4+2\sqrt{3}}}\\&=\frac{2(1+\sqrt{3})}{\sqrt{1+2\sqrt{3}+3}}=\frac{2(1+\sqrt{3})}{1+\sqrt{3}}=2:\end{aligned}$$

Problem 2.3. Find all positive integers n for which n^2+3n is perfect square.

Solution 2.3. Note that

$$n^2<n^2+3n<n^2+4n+4=(n+2)^2.$$

Since n^2+3n is perfect square, then it is equal to $(n+1)^2$.

$$n^2+3n=n^2+2n+1,$$

$$n=1.$$

Answer: $n=1$.

Problem 2.4. Find all integer solutions to the equation

$$x^2-6xy+13y^2=100.$$

Solution 2.4. Rewrite the equation in the following form

$$x^2-6xy+13y^2=(x-3y)^2+(2y)^2=100.$$

Since 100 can be written as sum of squares in the following ways

$$100=8^2+6^2=10^2+0^2$$

, therefore we have the following options

$$x-3y=8, 2y=6$$

$$x-3y=-8, 2y=6$$

$$x-3y=6, 2y=8$$

$$x-3y=-6, 2y=-8$$

$$x-3y=10, 2y=0$$

$$x-3y=-10, 2y=0$$

$$x-3y=0, 2y=10$$

$$x - 3y = 0, 2y = -10$$

From these cases we get solutions

Answer: $(17,3), (1,3), (18,4), (-18,-4), (10,0), (-10,0), (15,5), (-15,-5)$.

Problem 2.5. Find the number of 7-digit positive integers that all digits are ordered in

a) strictly increasing order,

b) strictly decreasing order.

Solution 2.5. a) part is equivalent to write expression 123456789 and remove any 2 digits. It can be done in $\binom{9}{2}$ ways.

b) part is equivalent to write expression 9876543210 and remove any 3 digits. It can be done in $\binom{10}{3}$ ways.

Answer: a) $\binom{9}{2}$, b) $\binom{10}{3}$.

Problem 2.6. A triple $(1,1,1)$ is given. On each step one chooses 2 of them and increases by 1. Is it possible after some steps get numbers $(2016, 2016, 2016)$.

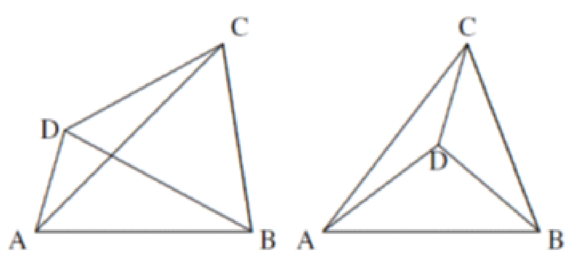
Solution 2.6. Note that the sum of written numbers is equal $1+1+1=3$. After first step the total sum will be equal $3+2=5$, after second step $5+2=7$ and so on, after nth step the total sum will be $3+2n$ which is odd. However, at the end we want to have 3 numbers which equal 2016, so their sum must be $3 \cdot 2016 = 6048$ which is even. We have already shown that the sum must be always odd.

Answer: Not possible.

Problem 2.7. There are four points A, B, C, D on the plane, such that any three points are not collinear. Prove that in the triangles ABC, ABD, ACD and BCD there is at least one triangle which has an interior angle not greater than 45^o.

Solution 2.7. -

It suffices to discuss the two cases indicated by the following figures:



(a) (b)

For case (a), since $\angle DAB + \angle ABC + \angle BCD + \angle CDA = 360°$, at least one of them is not less than $90°$. Assuming $\angle CDA \geq 90°$, then in $\triangle CDA$, $\angle DCA + \angle CAD \leq 90°$, so one of them is not greater than $45°$.

For case (b), since $\angle ADB + \angle ADC + \angle BDC = 360°$, one of the three angles is greater than $90°$, say $\angle ADB > 90°$, then $\angle DAB + \angle DBA < 90°$, so one of $\angle DAB$ and $\angle DBA$ is less than $45°$.

Problem 2.8. Triangles ABC and ABD are isosceles with $AB = AC = BD$, and AC intersects BD at E. If AC is perpendicular to BD. Find $\angle C + \angle D$ on degrees.

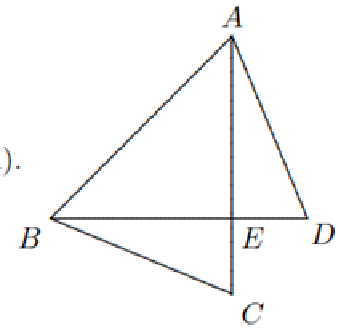
Solution 2.8. -

In $\triangle ABC$ and $\triangle ABD$, since $AB = AC = BD$, we have

$$\angle C = \frac{1}{2}(180^\circ - \angle BAC),$$

$$\angle D = \tfrac{1}{2}(180^\circ - \angle DBA),$$

$$\therefore \angle C + \angle D = 180^\circ - \frac{1}{2}(\angle BAC + \angle DBA).$$



$$\because \angle BAC + \angle DBA = 90^\circ,$$
$$\therefore \angle C + \angle D = 180^\circ - 45^\circ = 135^\circ,$$