# Competition Preparation for Saudi Arabia Team 2021: Level 4

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## Lesson 2

# Problems with coins

**Problems:**

1. There are $n$ coins on a pile and two players $A$ and $B$ take turns taking either 1, 2 or 6 coins from the pile. The winner is the person who takes the last coin. Player $A$ plays first. Determine for which $n$ does player $A$ have a winning strategy and for which $n$ does player $B$ have a winning strategy.

2. There are $n$ coins in one pile and two players $A$ and $B$ take turns with the following game. In each move a player whose turn it is may either take a coin from a pile and discard it, or divide a pile into (at least 2) equal piles. Player $A$ starts first. When a player takes the last coin from a pile it gone. The winner is the player who takes the last coin off the table. For which $n$ does player $A$ have a winning strategy and for which $n$ does player $B$ have a winning strategy.

3. On each square of a $3 \times n$ board a black and white coin is placed with the black side facing up. In each move we select a square and flip all its adjacent squares. A square is adjacent to another square if it has at least one common vertex. A square is not adjacent to itself. For which $n$ is it possible to end up with all coins having the white side facing up.

4. On an infinite one-dimensional board there are $n(n+1)/2$ coins. In each move one is allowed to take all coins from a square and distribute the coins to the right, one coin per square until there are no squares left. Prove that after a finite number of moves the coins will on $n$ consecutive squares be distributed as follows: $n$ coins, $n-1$ coins, $\ldots$, 2 coins, 1 coin.

5. Let $n$ be a given integer $f(x,y)$ be a function on non-negative integers such that $f(0,i) = f(i,0) = 0$, $i \in \mathbb{N}_0$, $f(1,1) = n$, $f(i,j) = \lfloor f(i-1,j)/2 \rfloor + \lfloor f(i,j-1)/2 \rfloor$, $ij > 1$, $i,j \in \mathbb{N}$. Find the number of odd values of $n$.

6. On a $1 \times (m+1)$ board the squares are labeled from 0 to $m$. Initially there are $n$ stones at position zero. In each move we can stone to the right up to as many squares as the number of stones on the square the stone is currently occupying (e.g. a lone stone can only be moves one square). Show that the number of moves needed for all stones to reach square $m$ is at least $\lceil \frac{m}{1} \rceil + \lceil \frac{m}{2} \rceil + \cdots + \lceil \frac{m}{n} \rceil$.

7. We are given a natural number $k$. Let us consider the following game on an infinite one-dimensional board. At the start of the game, we distribute $n$ coins on the fields of the given board (one field can have multiple coins on itself). After that, we have two choices for the following moves:

   (i) We choose two non-empty fields next to each other and we transfer all the coins from one of the fields to the other.

   (ii) We choose a field with at least 2 coins on it and we transfer one coin from the chosen field to the $k-$th field on the left , and one coin from the chosen field to the $k-$th field on the right.

   (a) If $n \leq k+1$, prove that we can play only finitely many moves.

   (b) For which values of $k$ we can choose a natural number $n$ and distribute $n$ coins on the given board such that we can play infinitely many moves.