NT constructions

Lesson by Senya, group L4+

Problem 1. Let k be a positive integer. Prove that there is an integer number t such that for any n, m that are coprime with t that satisfy $m^m \equiv n^n (\text{mod } t)$ it is true that $m \equiv n (\text{mod } t)$ k

Problem 2. Prove that there are infinitely many triples (m, n, k) of positive integers larger than 1 such that

$$m! \cdot n! = k!$$

Problem 3. Do there exist 2022 pairwise different positive integers such that for any two of them their sum is divisible by their difference.

Problem 4. Prove that there are infinitely many triples (a, b, c) such that

$$2a^2 + 3b^2 - 5c^2 = 2015.$$

Problem 5. Show that there exist 4 integers whose absolute values are larger than 1000000 such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{abcd}$$

Problem 6. Prove that there are infinitely many positive integers n such that the following equation has at least one solution in positive integers:

$$a^2 + b^2 + 1 = 3^n$$

Problem 7. Show that it is possible to write positive integers in the cells of a 2022×2022 board in such a way that for any rectangle on this board, the sum of the numbers placed in it is a perfect square if and only if the rectangle is square itself.