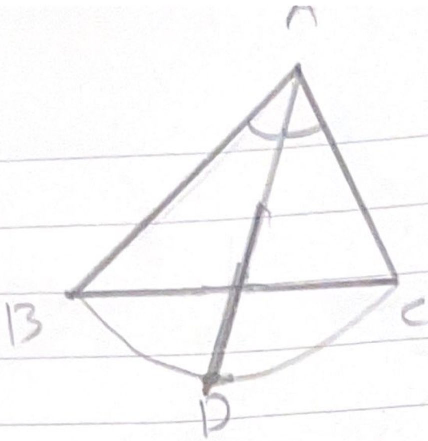
start

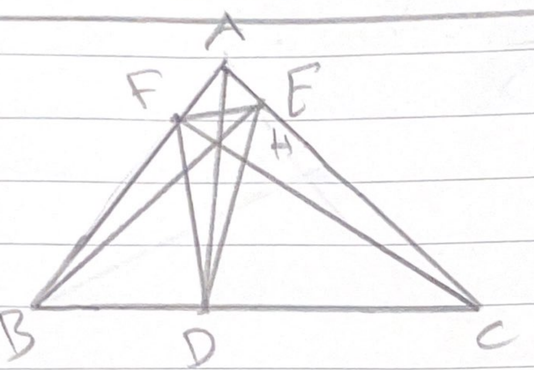
P.1)



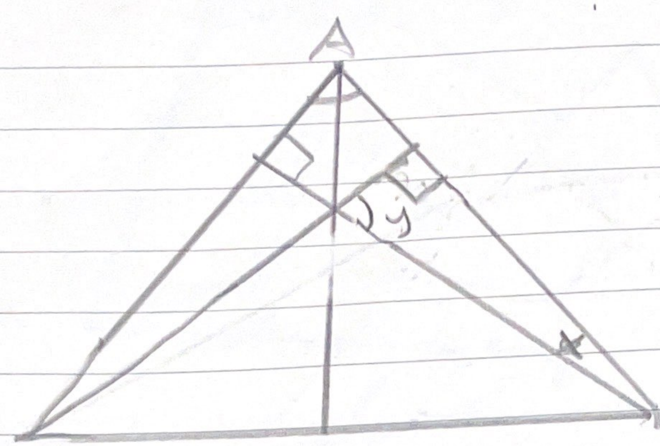
Let D be intersection of angle bisector and circumcircle

so $\overset{\frown}{BD}=\overset{\frown}{DC}$ (mid $\overset{\frown}{BC}$) and perpendicular bisector of BC will cut $\overset{\frown}{BC}$ at it mid

P.2)



$BFEC$ is cyclic $\Rightarrow \angle DEF = \angle AEF \Rightarrow \angle FEB = \angle BED \Rightarrow BE$ is angle bisector of FED



$x + \angle A = 90 \Rightarrow y = \angle A \Rightarrow \angle BHC = 180 - \angle A$

B, C, H, I on circle $\Rightarrow$ $BHC = BIC \Leftrightarrow 180 - \angle A = 90 + \frac{\angle A}{2}$

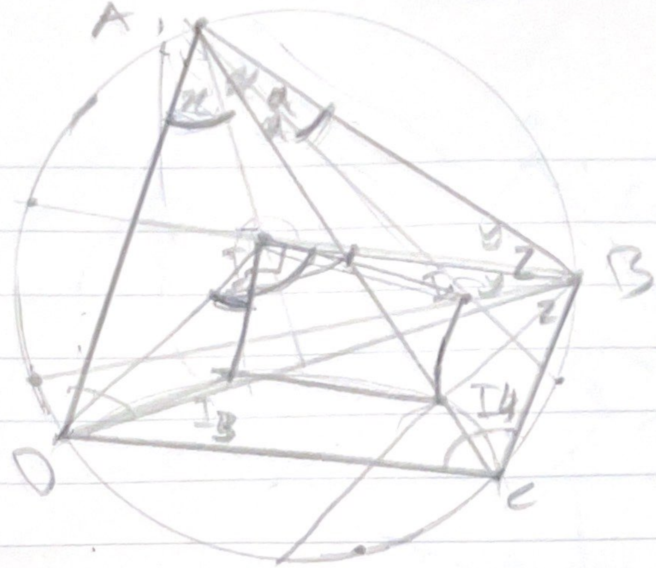
$\Leftrightarrow 90 = \frac{3}{2}\angle A \Rightarrow \angle A = 60$, $\Rightarrow \angle BOC = 120 = 180 - \angle A = BIC$

$\angle BQC = 60$?

$\angle BQC = 180 - (x+y)$, $BCA = 180 - 2x$, $CBA = 180 - 2y \Rightarrow \angle A =$

$2(x+y) - 180$ $\Rightarrow 180 - (x+y) = 90 - \frac{\angle A}{2} = 60$

P.4



$\angle ADB = \angle ACB \Rightarrow 90 + \frac{\angle ADB}{2} = 90 + \frac{\angle ACB}{2} \Rightarrow \angle AI_2B = \angle AI_1B$

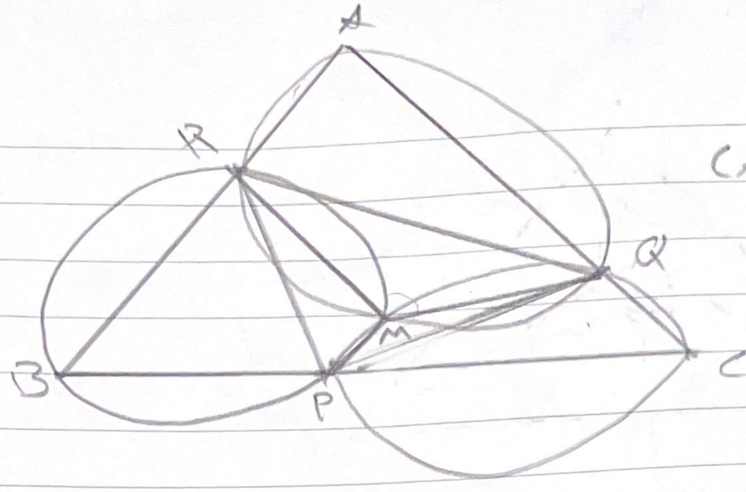
$\Rightarrow AI_1I_2B$ is cyclic (a)

$\angle DI_2B = 90 + \frac{\angle A}{2}$, $\angle DI_2I_3 = \angle DAI_2 \Rightarrow \frac{\angle DAC}{2} = DI_2I_3$

same way $\frac{\angle CAB}{2} = BI_2I_1 \Rightarrow \angle I_3I_2I_1 = 90 + \frac{\angle A}{2} - \angle DI_2I_3$

$- \angle BI_2I_1 = 90 + \frac{\angle A}{2} - \frac{\angle A}{2} = 90$

P. 5



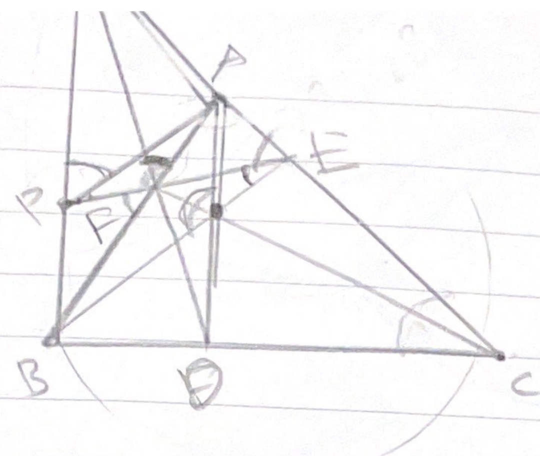
$(ARQ) \cap (CQP) = M$

$\angle RMQ + \angle QMP = 180 - \angle A + 180 - \angle C = 360 - (\angle A + \angle C)$

$\Rightarrow \angle RMP = 360 - (360 - (\angle A + \angle C)) = \angle A + \angle C = 180 - \angle B \Rightarrow$

M lies on $(R.B.P)$

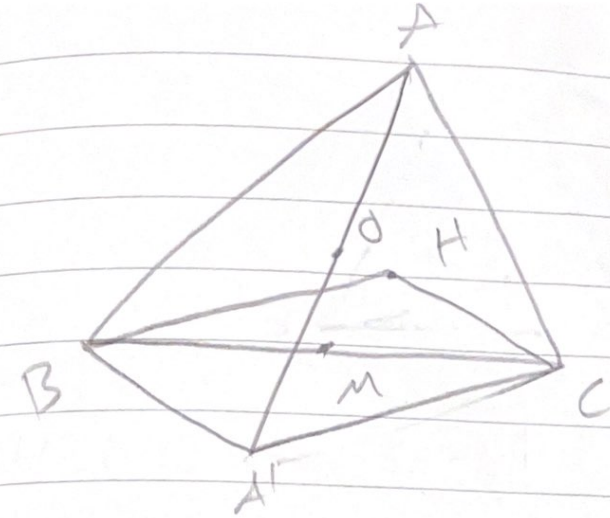
8)



$\angle APQ = \angle C$, $CDFA$ cyclic $\Rightarrow \angle AFQ = \angle C \Rightarrow AFPQ$ is cyclic

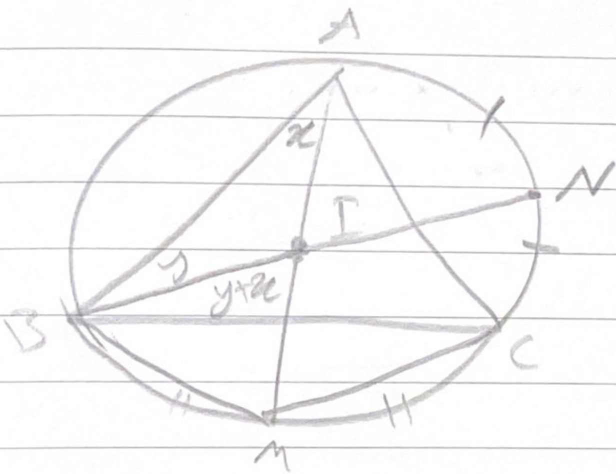
$\Rightarrow \angle AQP = 180 - \angle AFP = \angle AFE = \angle C \Rightarrow \angle APQ = \angle AQP \Rightarrow AP = AQ$

9)



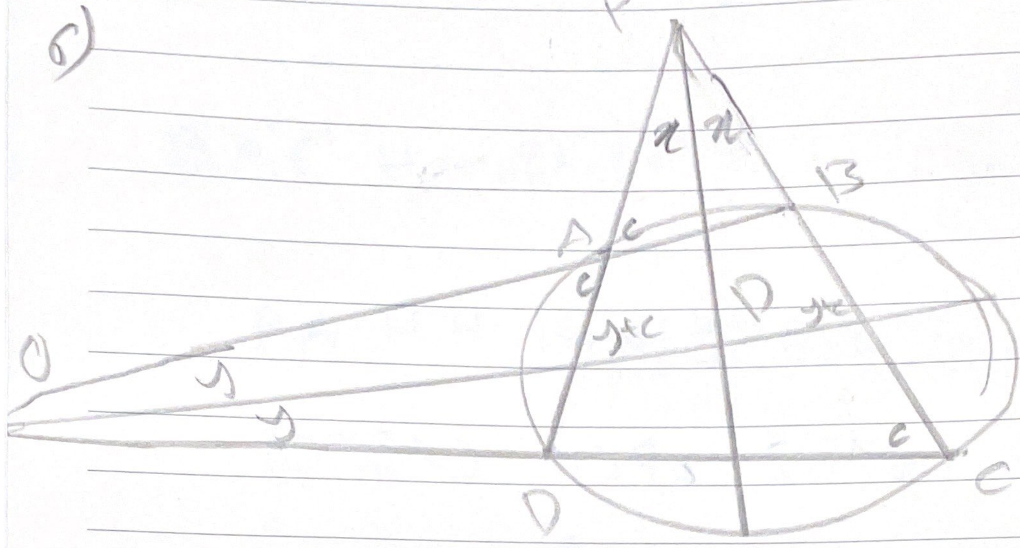
$A'B \perp AB$, $AB \perp CH \Rightarrow CH \parallel A'B$, $BH \perp AC$, $A'C \perp AC$

$\Rightarrow A'C \parallel BH \Rightarrow BA'CH$ ▱ $\Rightarrow A'$ invert H around M

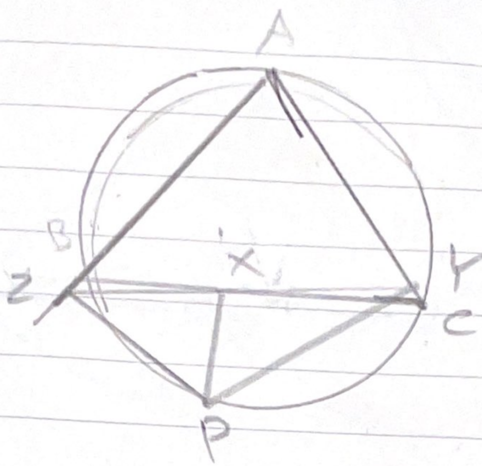


$\angle BIM = \frac{1}{2}\angle BAC + \frac{1}{2} ABC = \angle IBM$

б)



1.2



$\angle PXY = 180 - \angle PCY$

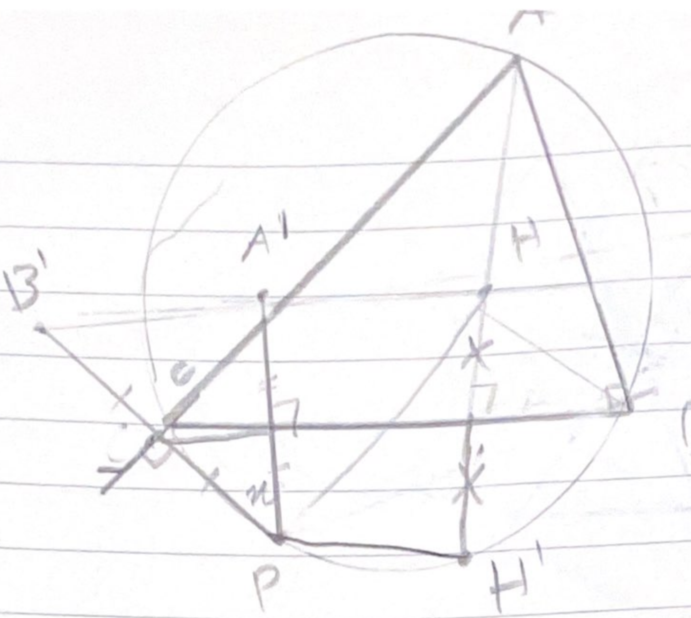
$\angle PXZ = \angle PBZ$

$\Rightarrow \angle PXZ + \angle PXY = 180 + \angle PBZ - \angle PCY$

$\angle PBZ = \angle PCY$ (ABPC)

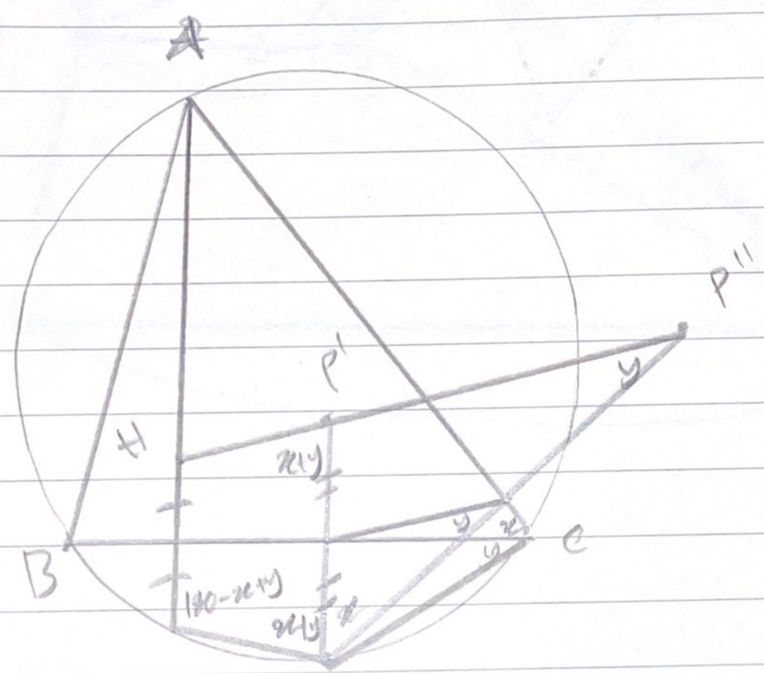
$\Rightarrow \angle PXZ + \angle PXY = 180$

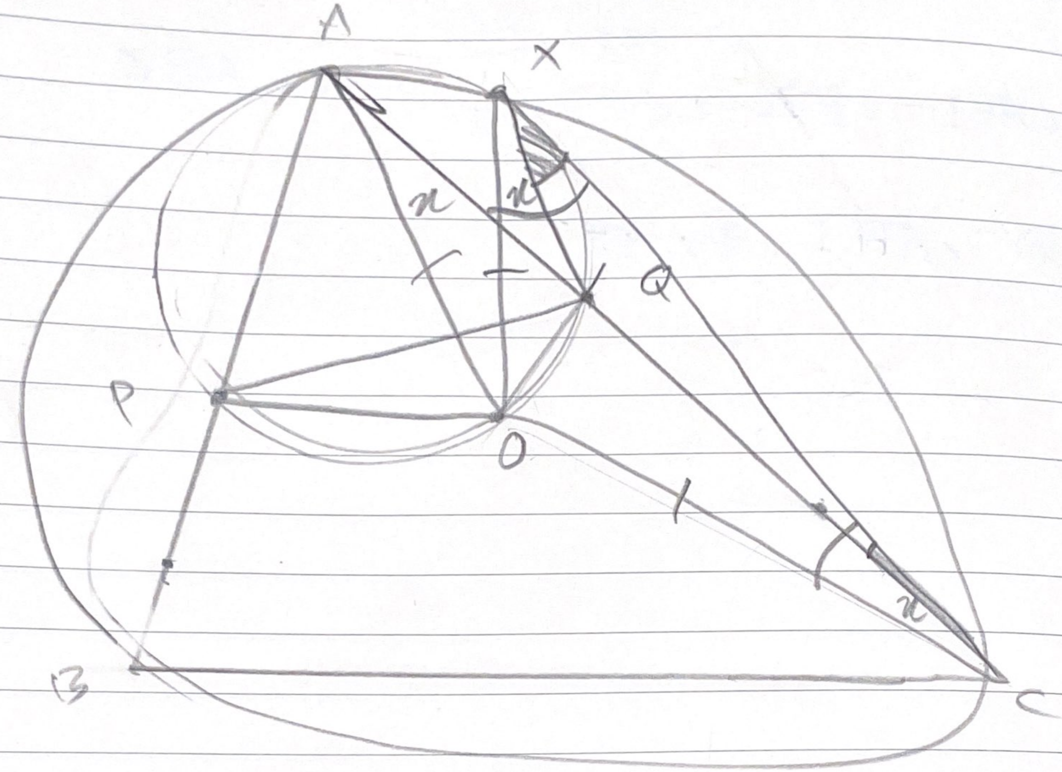
13)



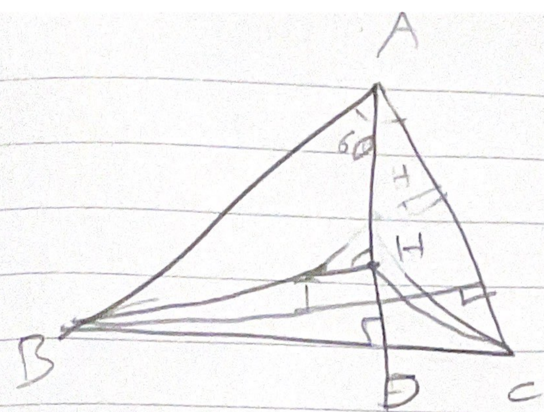
B, A', C' Homothety

PA'HH isoceles trapzoid





A
X
Q
P
O
B
C
x
x
x



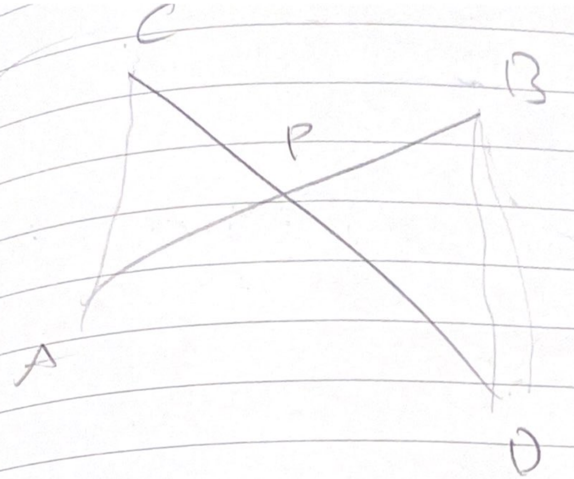
BIHC is cyclic

$$\Rightarrow \angle AHI = 180 - \angle IHD$$

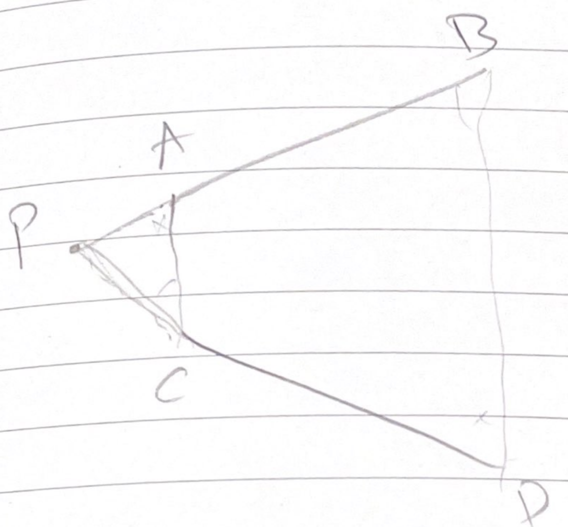
$$= 180 - (\underbrace{\angle IHC}_{180 - \angle IBC} - \angle CHD)$$

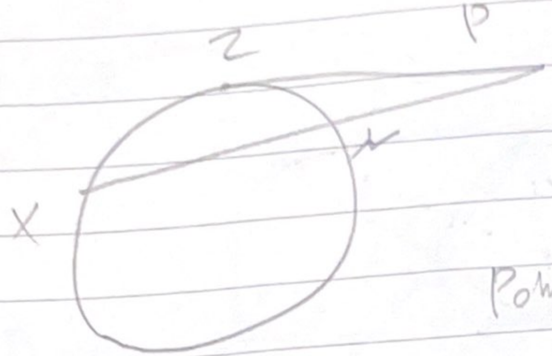
$$\Rightarrow \angle AHI = \angle IBC + \angle CHD = \frac{1}{2} ABC + \angle ABC$$

$$\Rightarrow \angle AHI = \frac{3}{2} \angle ABC$$

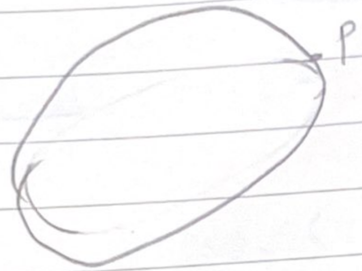


$CP \cdot PD = BP \cdot PA$ if and only if $ACBD$ is cyclic

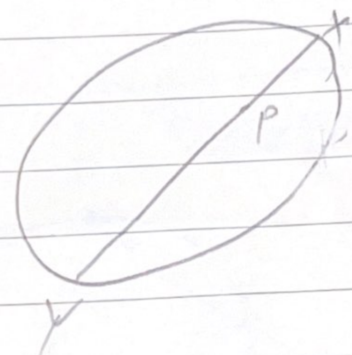




$Pow(P,\omega) = PX \cdot PY = PZ^2 > 0$

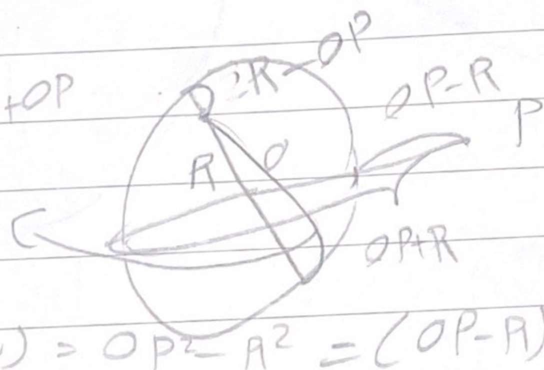


$Pow(P,\omega) = 0$



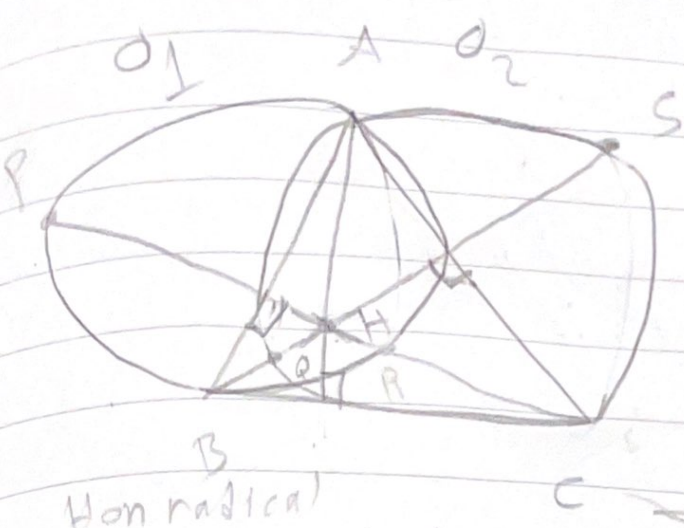
$Pow(P,\omega) = -PX \cdot PY < 0$

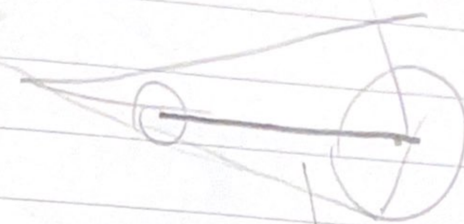
$2R-(R-OP) = R+OP$



$Pow(P,\omega) = OP^2 - R^2 = (OP-R)(OP+R)$

$Pow(P,\omega) = OP^2 - R^2$



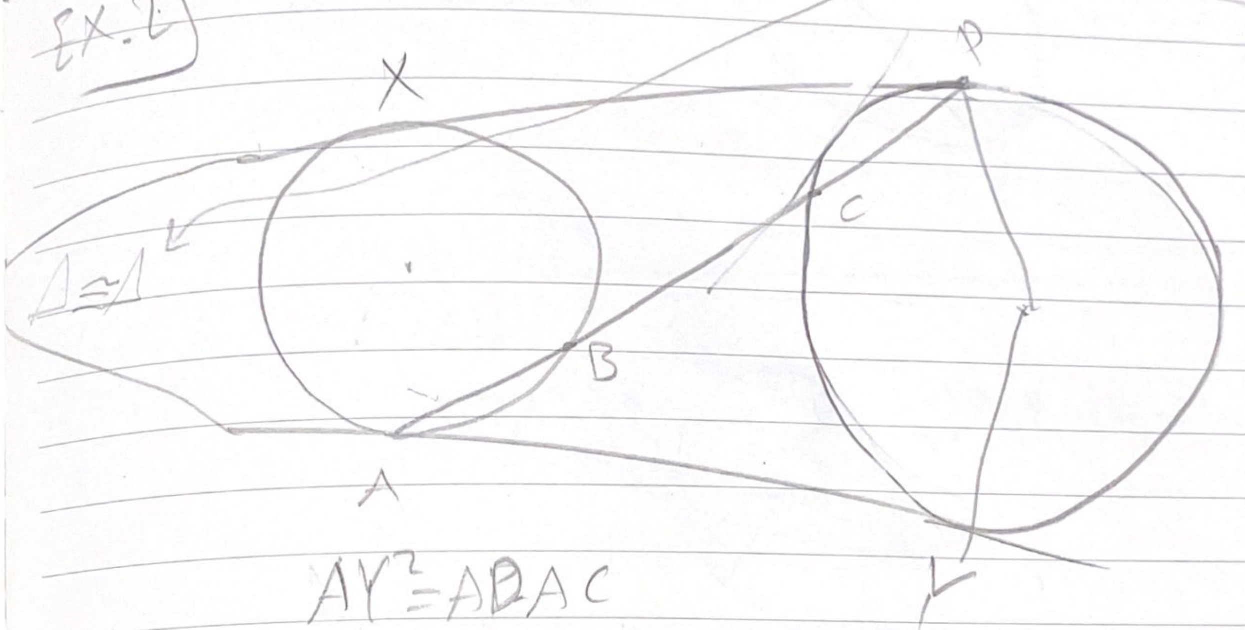


H on radical

$\Rightarrow Pow(H, O_1) = POW(H, O_2)$

$\Leftrightarrow HP \cdot HR = HQ \cdot HS \Rightarrow$ cyc

Ex.2)

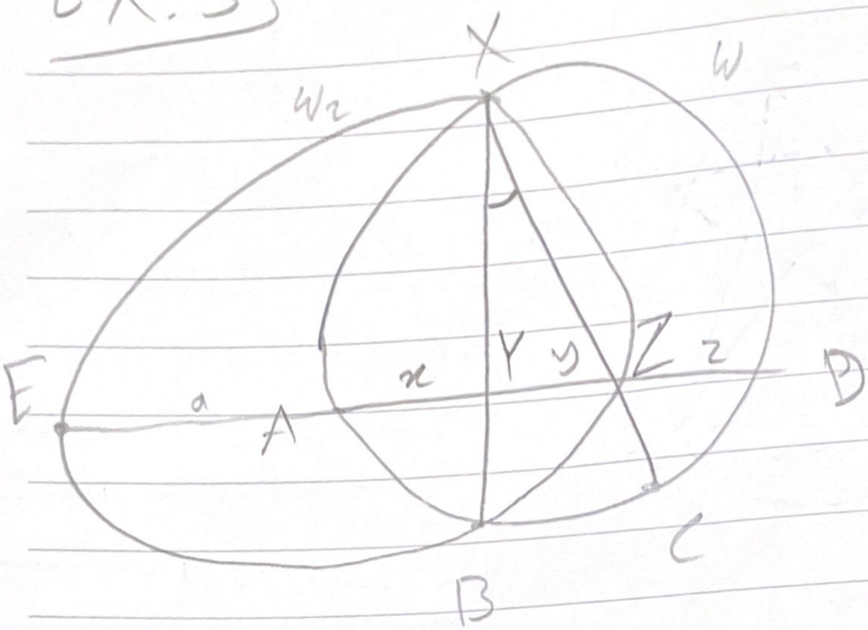


$AY^2 = AB \cdot AC$

$XP^2 = AD \cdot BD$

$AY = XP \Rightarrow AC = BD \Rightarrow AB = CD$

EX. 3)



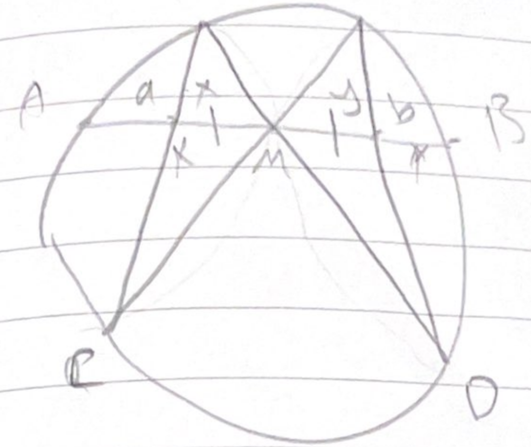
$$\frac{AY \cdot ZD}{YZ}$$

$x(y+z) = -\operatorname{Pow}(Y, w) = BY \cdot YX = -\operatorname{Pow}(Y, w_2) = y(a+x$

$$\Rightarrow a = \frac{xz}{y}$$

$\angle BXC = \angle DEB \Rightarrow \angle DEB$ is constant $\Rightarrow E$ is constant
constant

Ex.4



M mid AB

MX = MY

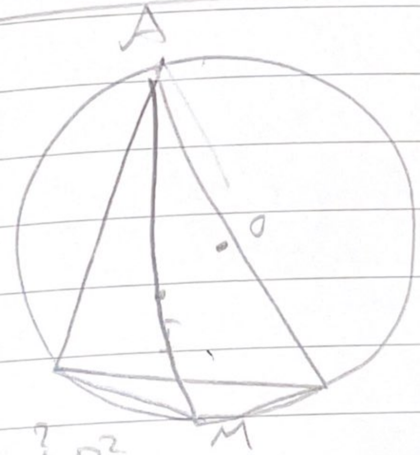
apply Ex3

$$\Rightarrow \frac{a\cancel{(y+b)}}{x} = \frac{\cancel{(a+x)}b}{y}$$

$$a = m - x, \quad b = m - y$$

$$(m-x)\,y = (m-y)\,x \Rightarrow x = y$$

Ex.5



$$OI^2 \overset{?}{=} R^2 - 2Rr$$

$$OI - R^2 = Pow(I, \Omega) \overset{?}{=} 2Rr$$

$Pow(I,\Omega) = -AI \cdot IM = AI \cdot BM$

$2Rr \stackrel{?}{=} AI \cdot BM$

$\Leftrightarrow MN \cdot ID = AI \cdot BM$

$\frac{MN}{BM} \stackrel{?}{=} \frac{AI}{ID}$

$OI^2 \geq 0$

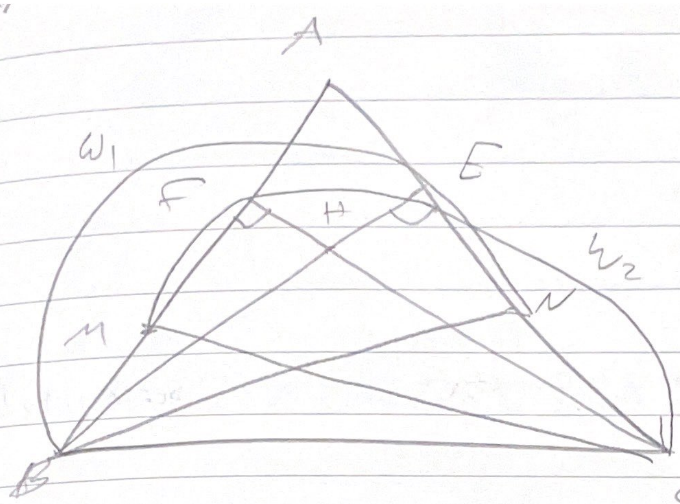
$\Downarrow$

$\boxed{R \geq 2r}$ $\Leftarrow \triangle NBM \stackrel{?}{\sim} \triangle ADI$ ✓ from $\angle NBM = 90 = ADI$, $\angle BAM = \angle BNM$

$OI = d$

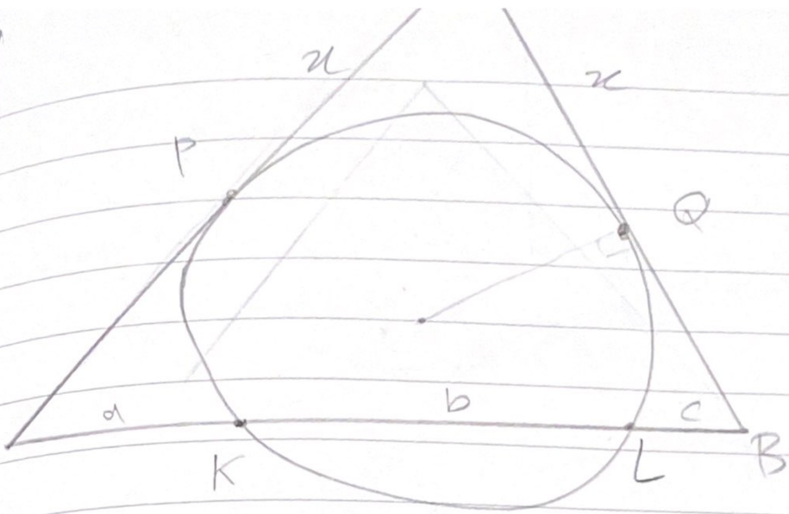
$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}$$

6)



$P(H, \omega_2) = -HF \cdot HC$
$P(H, \omega_1) = -HE \cdot HB$ "=" BFEC is cyclic $\Rightarrow$ H on radical.

17



$AK \cdot AL = AP^2$ $\qquad$ $BL \cdot BK = BQ^2$

$AK \cdot AL = (AC - x)^2$ $\qquad$ $BL \cdot BK = (BC - x)^2$

$\cancel{AK \cdot AL - BK \cdot BL = (AC - BC)(AC + BC + 2x)}$

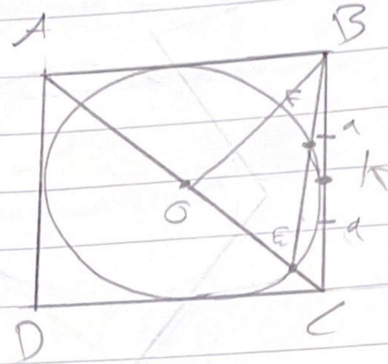
$\cancel{AP^2 - BQ^2 = a(a+b) - c(b+c)} = (a-c)(a+b+c)$

$$AK - BL = \frac{AX^2 - BY^2}{AB} = \frac{(AX - BY)(AX + BY)}{AB} = \frac{(AC - BC)(AX + BX)}{AB}$$

$$\leq AC - BC$$

Because $AX + BY \leq AB$

18)

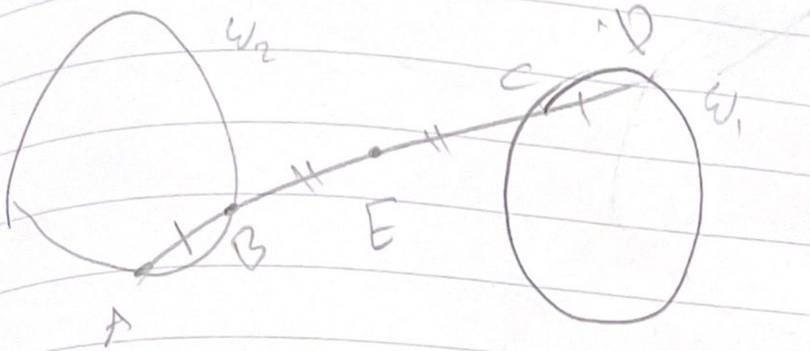


$BE \cdot BF = a^2$

$BE^2 = BO^2 + OE^2 = 2a^2 + a^2 \Rightarrow BE = \sqrt{3}\,a$

$BF = \frac{\sqrt{3}}{3}a \Rightarrow EF = BE - BF = \frac{3\sqrt{3} - \sqrt{3}}{3}a = \frac{2\sqrt{3}}{3}a$

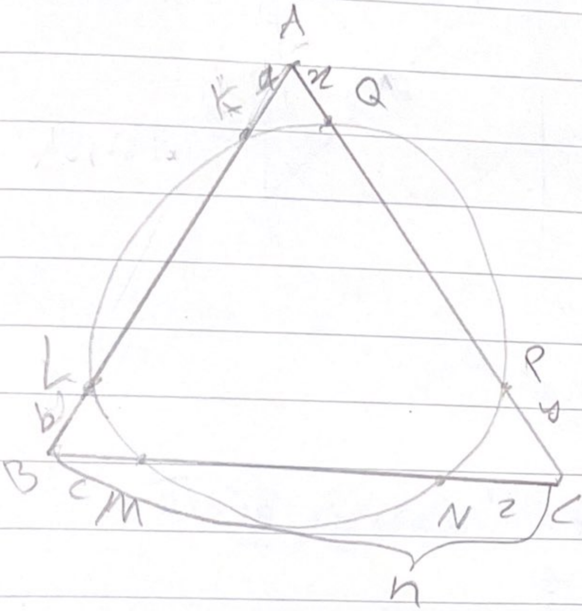
19)



$Pow(E, w_1) = EC \cdot ED = EB \cdot EA = Pow(E, w_2)$

$\Rightarrow$ E on radical

20)



$\cancel{AK \cdot AL = AQ}$

$\cancel{z \cdot a = y} =$

$an + ab = xn - yx$

$-bn + ba = -cn + cz$

$-zn + zc = -yn + yx$

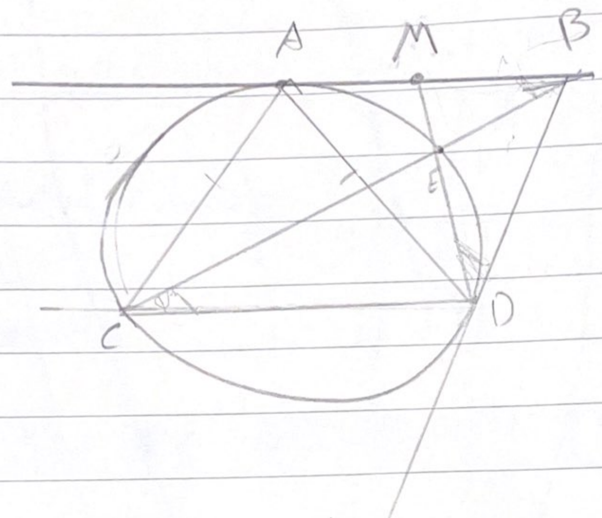
$an + cn + yn = xn + bn + zn$

$\div n$

$AK + BM + CP = BL + CN + AQ$

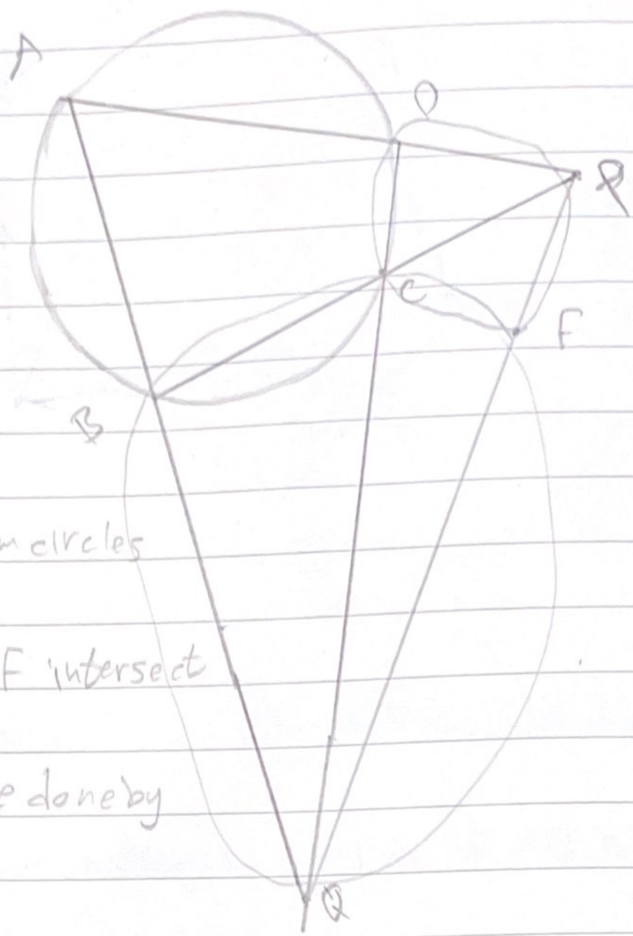
---

21)



$\angle MBC = \angle BCD = \angle MDB \Rightarrow \triangle BDM \sim \triangle MBE \Rightarrow MB^2 = ME \cdot MD = MA^2$

$\Rightarrow MB = MA$



Label F s.t circumcircles

ABD, PDF, QBF intersect

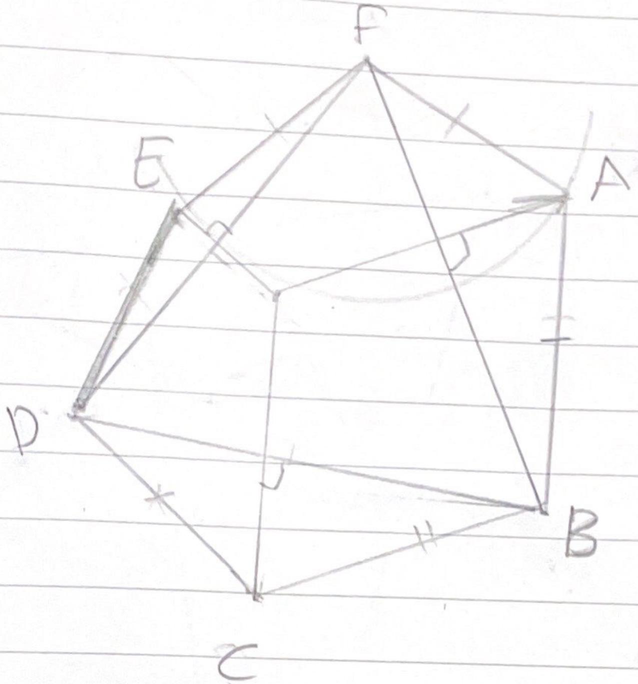
at C which can be done by

Miquel Theorm

$\Rightarrow Pow(P,\Omega) = QC \cdot QD = Pow(P, (PDF)) = QF \cdot QP,$

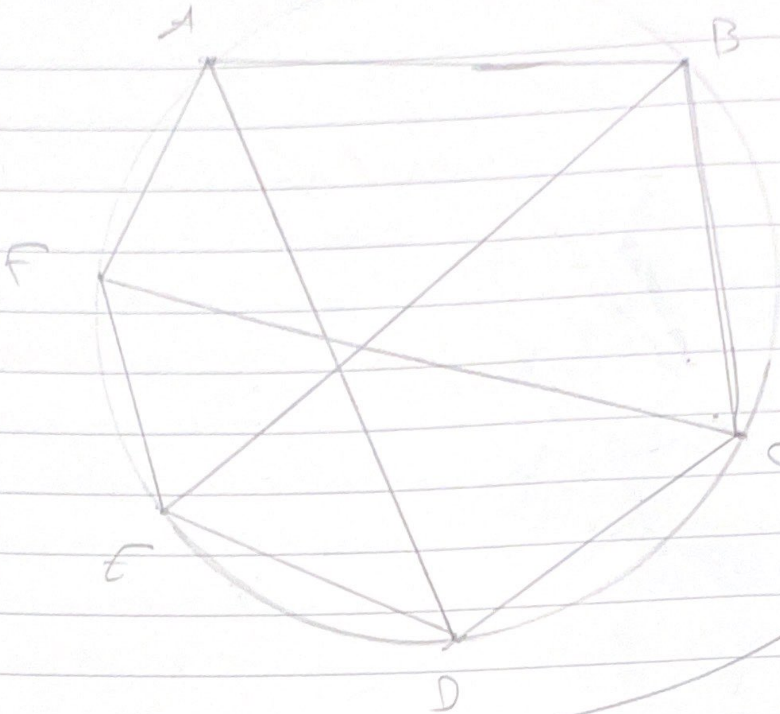
$Pow(Q,\Omega) = PC \cdot PB = Pow(Q,(QBCF)) = PF \cdot QP$

$\Rightarrow Pow(P,\Omega) + pow(Q,\Omega) = QP \cdot QP + PF \cdot QP = QP(PF+QP) = QP^2$



the 3 altitudes will be radical axe circle with circumcenter F, B, D

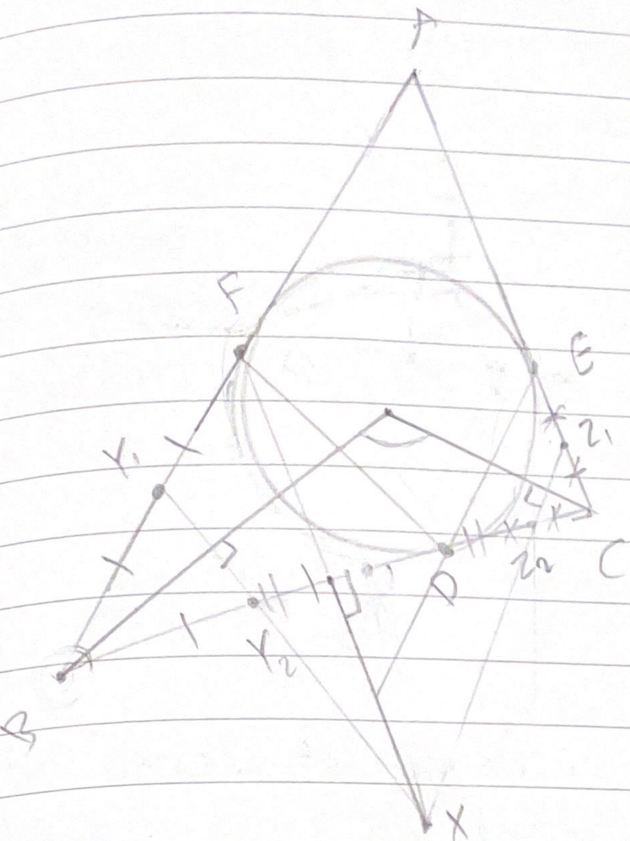
and radius FA, BA, DC => they will meet at one point



ABCD
EFAB
CDEF

are cyclic => they intersect in one point if the circles are diffrent, but hey dont => 2 heve the same circle => A, B, C, D, E, F
on one ci

25

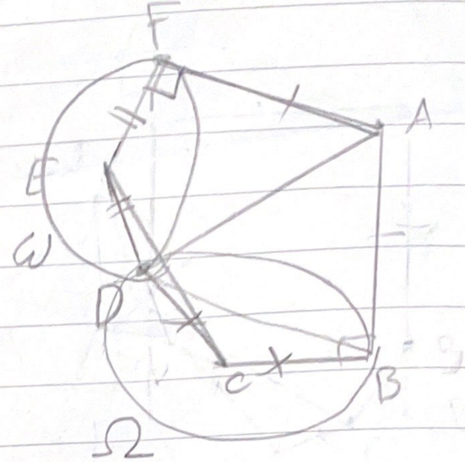


Y_1Y_2 is radical B and (FDE), Z_1Z_2 is radical C and (FED)

$\Rightarrow \cancel{\angle Z_2 X Y_2 = 90 - \frac{1}{2}A}$

$\Rightarrow$ XM is radical B and C $\Rightarrow XM \perp BC$

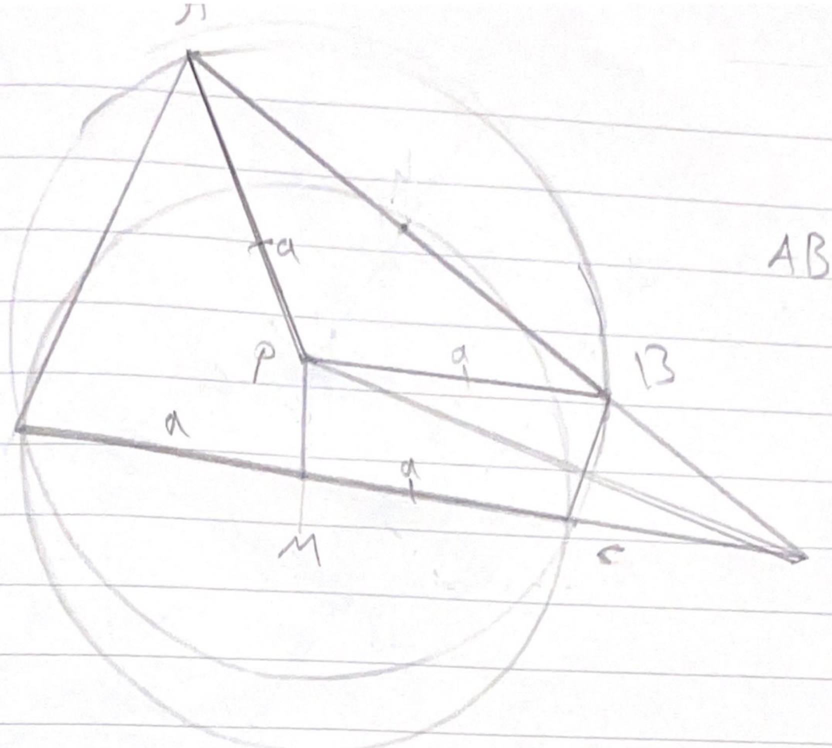
26



$P(A,\omega)=AF^2$, $P(A,\Omega)=AB^2 \Rightarrow P(A,\omega)=P(A,\Omega) \Rightarrow$ A on radical a

$\Rightarrow AD \perp CE$ (خط المركزين)

27)



• A

3 circles

$AB \cap CD = X$

circle with M center and MC radius and P center and PB radius

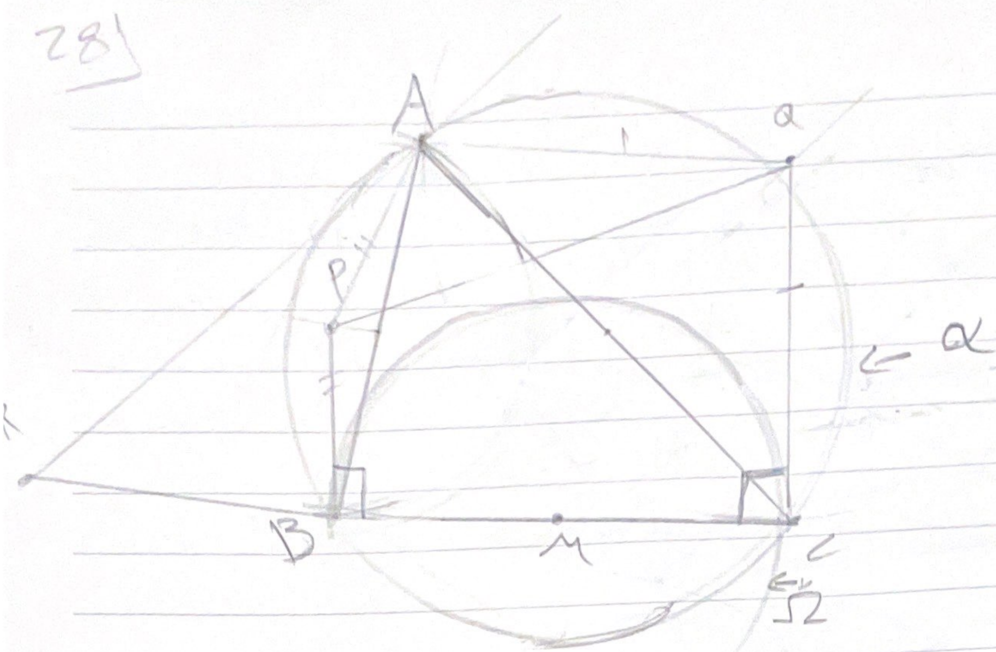
are $\cong$ so there radical is the perpendicular bisector of there

center line. $\Rightarrow$ perpendicular bisector of MP is the radical.

ABCD cyclic $\Rightarrow$ $XB \cdot XA = XC \cdot XD \Rightarrow P(X,M) = P(X,P) \Rightarrow$ X on radical

or 3 circles and 3 radical axis so 1 concurrent

28)



$P(P,A) = PA^2$, $P(P,\Omega) = PA^2$, $P(Q,A) = QA^2$, $P(Q,\Omega) = QC^2$

$\Downarrow$ P on radical A, Ω

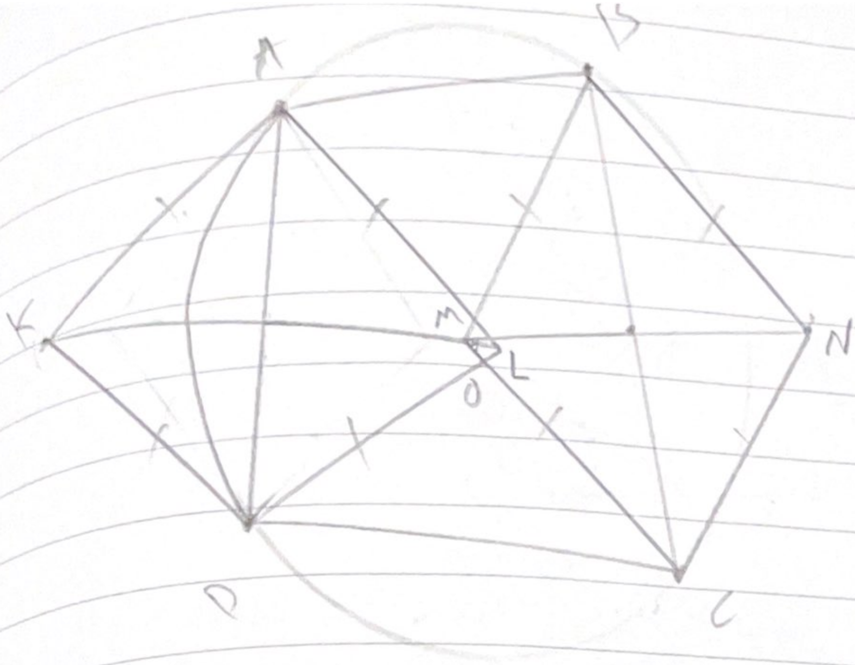
$\Downarrow$ Q on radical A, Ω

$P(R,\alpha) = RA^2 = RB \cdot RC = P(R,\Omega)$

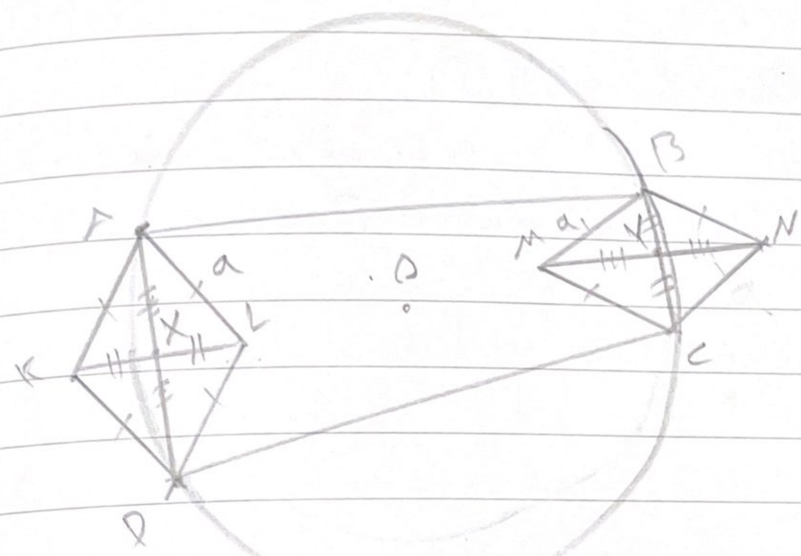
$\Rightarrow$ R on radical Ω, α (but radical Ω, α is same radical A, Ω

$\Rightarrow$ P, Q, R colliner

29)



O center (ABCD)



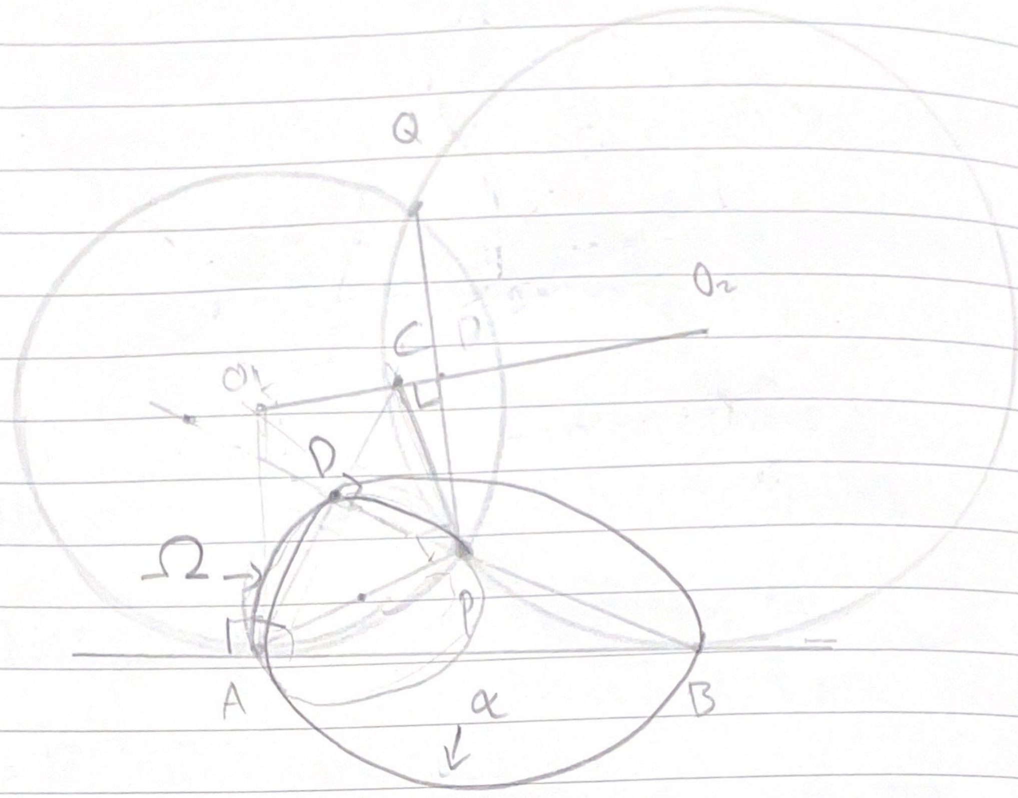
$OL \cdot OK = (OX + XK)(OX - XL) = OX^2 - XL^2 = OX^2 - AL^2 + AX^2$

$= (OX^2 - AX^2) - R^2 - a^2$

same way

$OM \cdot ON = R^2 - a^2$ $\Rightarrow OL \cdot OK = OM \cdot ON \Rightarrow K, L, M, N$ on a circle

30)



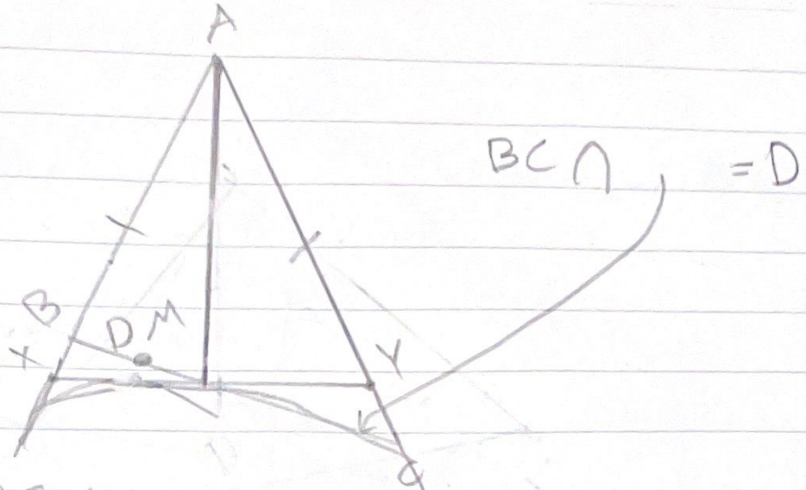
AD is radical (α, Ω), O_1O_2 is radical (P, α) because

$Pow(O_1, P) = O_1P^2$, $Pow(O_1, \Omega) = O_1A^2$, $\Rightarrow$ 3 radical of Ω, α, P

Meet at one point which is perpendicular of AP to P because it is the radical

of (P, Ω)

31)



$AB + BC + CA = 4$

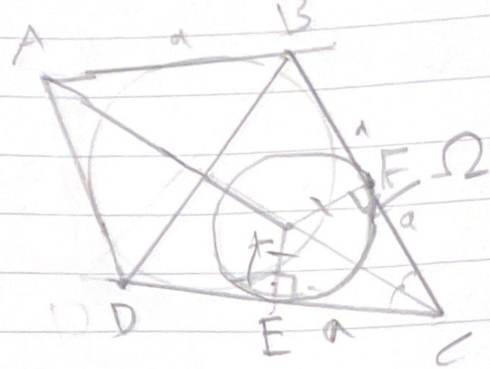
$AC + CM + MA \stackrel{?}{=} 2 = \frac{AB+BC+CA}{2}$

XY radical ω, ex-circle

$\Rightarrow MA^2 = MD^2$

$AC + CM + MA = AC + CM + MD = AC + CD = b + s - b = s = 2$

32



take E st $XE \perp DC$

Ω circle with X center and XF radius

$AB + AD = CD \Rightarrow AD = DE$

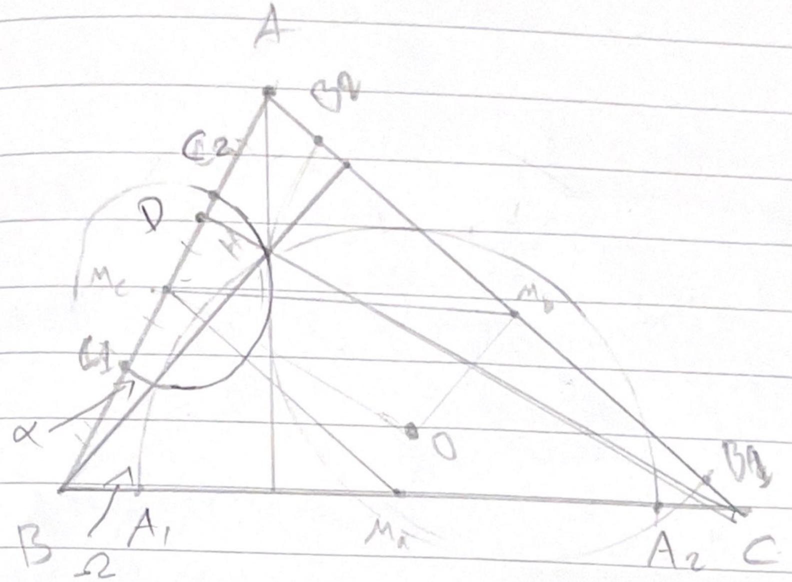
$Pow(D, \Omega) = DE^2 = AD^2 = Pow(D, A)$

$Pow(B, \Omega) = BF^2 = BA^2 = Pow(B, A)$

$\Rightarrow$ BD radical Ω and A

$\Rightarrow BD \perp AX$

33



CD DH

∥

~~$OC_1 = OC_2$, $OB_1 = OB_2$, $OA_1 = OA_2$~~

~~$OC_2 \overset{?}{=} OB_2$~~

BH is a radical of α and Ω because $BH \perp AC$, $AC \parallel M_aM_c \Rightarrow BH \perp M_aM_c$

and H is one of the radical points $\Rightarrow BA_1 \cdot BA_2 = BC_1 \cdot BC_2$

$\Rightarrow C_2C_1A_1A_2$ is cyclic, Problem 21 Do rest

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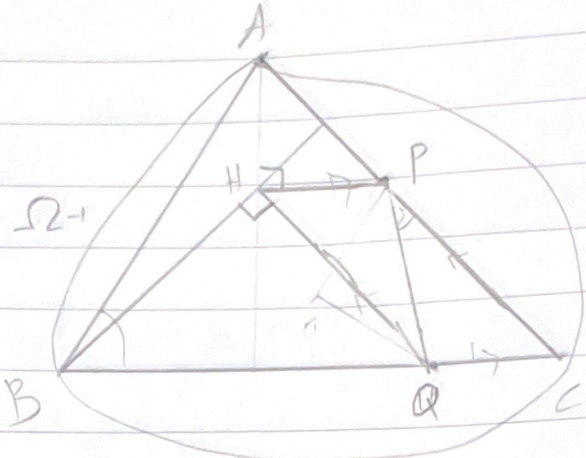

$$\stackrel{?}{=}\begin{cases} P(P,\Omega) = OP^2 - R^2 \\ P(Q,\Omega) = OQ^2 - R^2 \end{cases}$$

$$-PC \cdot PA \stackrel{?}{=} -BQ \cdot QC$$

$$\frac{PC}{QC} \stackrel{?}{=} \frac{BQ}{PA} \Leftrightarrow$$

$$\triangle BHQ \sim \triangle AHP \Rightarrow \frac{BQ}{AP} = \frac{HQ}{HP} = \frac{PC}{QC} \checkmark$$

35) $(BCEF) = \Omega$

$\angle QAF = 180 - \angle B = \angle QEA \Rightarrow \Delta QAE \sim \Delta QFE \Rightarrow QA^2 = QE \cdot QF$

$\Rightarrow P(Q, A) = P(Q, \Omega) \Rightarrow Q$ on radical $A, \Omega \Rightarrow PQ$ is radical A, Ω

$\Rightarrow AM \perp PQ$

$PA^2 = PB \cdot PC = Pow(P, (ABC))$

$P(P, A) = P(P, \Omega) \Rightarrow P$ on radical A, Ω

36) Ω = ex-circle

$P(B, \Omega) = (s-b)^2, \; P(B, S) = (s-b)^2$

$P(D, \Omega) = a^2, \; P(D, S) = a^2$

$\Rightarrow BD$ is radical S, ex-circle $\Rightarrow P(G, \text{ex-circle}) = P(G, S)$

Same way

$P(G, \text{ex-circle}) = P(G, R) \Rightarrow P(G, R) = P(G, S) \Rightarrow GR = GS$