Number Theory

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Problems – April 12

1. Let a and b be integers of different parity. Prove that there exists an integer c such that the three numbers $c+a$, $c+b$ and $c+ab$ are perfect squares.
2. Let $a,b\in\mathbb{N}$ be such that $a!b!$ is divisible by $a!+b!$. Prove that $3a\geqslant 2b+2$.
3. Find all prime numbers p such that $\frac{p^2-p-2}{2}$ is a perfect cube.
4. Prove that a perfect cube is always congruent to -1, 0 or 1 modulo 9.
5. If p is a prime number and $x\equiv y \pmod p$, prove that $x^p\equiv y^p \pmod{p^2}$.
6. Find all pairs of positive integers m,n for which $mn-1$ divides n^3+1.
7. Let a and b be coprime positive integers.
   (a) Prove that $ab-a-b$ cannot be expressed as $ax+by$, where $x,y\geqslant 0$ are integers.
   (b) Prove that all integers greater than $ab-a-b$ can be expressed in this way.

   Chinese Remainder Theorem. Let $n_1,n_2,\dots,n_k$ be pairwise coprime positive integers and let $a_1,a_2,\dots,a_k$ be any integers. Then the system of congruences
   $$\begin{cases} x\equiv a_1 \pmod{n_1}\\ x\equiv a_2 \pmod{n_2}\\ \quad\cdots\cdots\\ x\equiv a_k \pmod{n_k}\end{cases}$$
   has a unique solution modulo $n_1n_2\cdots n_k$.
8. Find all x such that $x\equiv 1 \pmod{2019}$, $x\equiv 2 \pmod{2021}$ and $x\equiv 3 \pmod{2023}$.
9. Show that there are 1000 consecutive positive integers, none of which is a perfect power.
10. Prove that there exist 200 consecutive positive integers, each of which has at least one prime divisor not exceeding 103.
11. Is there a positive integer n such that n, $2n$ and $3n$ are perfect powers? Find the smallest such n if it exists.
12. Find all positive integers n with the following property: Whenever $n\mid xy+1$ for some integers x,y, it also holds that $n\mid x+y$.