Number Theory

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Problems – April 14

1. (a) Find the number of different remainders that numbers $1^2, 2^2, 3^2, \dots$ give when divided by n, where (a) n is a prime; (b) n is a product of two primes; (c) $n = 2^k$.

- A reduced residue system modulo n has $\varphi(n)$ elements, where φ is the *Euler totient function*: $\varphi(n) = n \prod(1 - \frac{1}{p_i})$, where the product goes over all prime divisors p_i of n.

2. Find all positive integers n for which $\varphi(n)$ divides n.

3. Prove that for every positive integer n there exists a positive integer x such that $\varphi(x) = n!$.

4. Let $a_1 < a_2 < \cdots < a_{\varphi(n)}$ be the positive integers not exceeding n and coprime to n. Find all n for which none of the sums $a_i + a_{i+1}$ is divisible by 3.

5. Suppose that n is odd and both $\varphi(n)$ and $\varphi(n+1)$ are powers of 2. Prove that either $n = 5$, or $n + 1$ is itself a power of two.

- The number of divisors of a positive integer $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is $\tau(n) = \prod_{i=1}^{k}(r_i + 1)$.

- The sum of divisors of a positive integer $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ is $\sigma(n) = \prod_{i=1}^{k}(1 + p_i + p_i^2 + \cdots + p_i^{r_i})$.

6. Let $1 = d_1 < d_2 < d_3 < \cdots < d_k = 4n$ be all divisors of $4n$, where n is a positive integer. Prove that there exists i such that $d_i - d_{i-1} = 2$.

7. Given a positive integer n, define a sequence (a_k) by $a_0 = n$ and $a_{k+1} = \tau(a_k)$. Find all n for which no term a_k is a perfect square.

8. If $a \mid b$ and $a < b$, prove that $\frac{\sigma(a)}{a} < \frac{\sigma(b)}{b}$.

9. Prove that there are infinitely many pairs of different positive integers m and n such that $\sigma(m^2) = \sigma(n^2)$?

10. A positive integer n is *perfect* if its sum of divisors $\sigma(n)$ (including itself) equals $2n$. Prove that every even perfect number is of the form $n = 2^{k-1}(2^k - 1)$, where k is a positive integer.

11. For $n \in \mathbb{N}$, denote by $f(n)$ the smallest positive integer having exactly n divisors. Thus e.g. $f(5) = 16$ and $f(6) = 12$. Prove that, for any $k \in \mathbb{N}$, $f(2^k)$ divides $f(2^{k+1})$.