Email training, N10

November 17 - 23, 2019

Many problems require a generalization of the idea of an "invariant." Even if we cannot identify some function of the state of a process that never changes, we may be able to identify a function that always changes in the same direction. In fact, this information can be invaluable. Consider the following fact:

If there is some positive integral function that decreases at each step of a process, the process must eventually terminate.

This is trivial: if a positive integral function starts at n and decreases with each step, the process certainly cannot continue for more than $n-1$ steps. Yet despite its apparent obviousness, this principle is perhaps the most powerful way for us to draw conclusions about how a process behaves.

Example 1. In the Senate of Kazakhstan, each member has at most three enemies. A member cannot be his own enemy, and enmity is mutual. Prove that the Senate can be divided into two factions such that each Senator has at most one enemy within his faction.

First, we separate the members arbitrarily into two factions. Let H be the sum of all the enemies each member has in his own faction. Suppose one member (let's call him Bob) has at least two enemies in his own faction. Then if Bob switches factions, H will decrease. Let this process continue for all members in the same situation as Bob. Since H is a positive integral function that decreases at each step of the process, the process must terminate. At this point, no Senator can have more than one enemy in his own faction, because otherwise the process (by definition) would not have terminated. Thus we have found the desired division of Senators.

The extremal principle rests on three important facts (from Engel):

1. Every finite nonempty set A of nonnegative integers or real numbers has a minimal element $\min A$ and a maximal element $\max A$, which need not be unique.
2. Every nonempty subset of positive integers has a smallest element.

Warm-up 2. There are 13 gray, 15 red and 17 blue chameleons living on the island. Whenever two chameleons of different colors meet, they change their color to the third one (if gray and blue meet then both of them become red). Is it possible that at some moment all chameleons on the island have the same color?

Warm-up 3. Let a convex 10-gon is given, where no any three diagonals intersect at one point. At each vertex of the 10-gon and at each intersection point of two diagonals the numbers $+1$ are written. At each step one may choose any diagonal and any side and change change all signs on that diagonal/side. Is it possible that after some steps all numbers are equal -1.

Warm-up 4. Let several numbers are written on the board (not all of them equal 0). At each step one may choose two numbers a and b and replace them by $a - \frac{b}{2}$ and $b + \frac{a}{2}$. Prove that one can't achieve a situation when all numbers are equal 0.

Warm-up 5. There are three piles with n tokens each. In every step we are allowed to choose two piles, take one token from each of those two piles and add a token to the third pile. Using these moves, is it possible to end up having only one token?

Problem 10.1. Positive integers 1 and 2 are written on the board in laboratory. Every morning professor Ali erases the written numbers from the board and writes their arithmetical and harmonic means. It occurs that

a) At some moment $\frac{941664}{665857}$ was written on the board. Determine either it is written as arithmetical or harmonic mean and determine the other number written on the board.

b Determine if some moment the number $\frac{35}{24}$ can be written on the board.

Problem 10.2. Let non-regular polygon is inscribed to the circle. Each step one may choose a vertex A which divides the arc between two neighbour to the A vertices into non equal parts and move the point A to the midpoint of that arc. Is it possible that after 100 moves one gets a polygon which is equal to the original one?

Problem 10.3. Let 10 students stay around the circle. During the step is allowed that one student changes his position and stays somewhere else around the circle. Find the minimal number of moves that needed that students stay in increasing order by height in clockwise (from shortest to tallest) independent from the original positions.

Problem 10.4. Let the cake has a form of regular pentagon (5-gon). There is a nuts in the center of it. At each step one is allowed to cut a piece of the cake which has a form of triangle and the line which cuts the triangle doesn't contain vertex of the polygon (originally it was pentagon). Is it possible to achieve to the nuts?

General problems.

Problem 10.5. Let several pairwise different positive number are written on the board. It is known that the sum of any two of them is power of 2. Find the maximal possible number of numbers written on the board.

Problem 10.6. Let 100 apples are put in a row such that the difference of weights of neighbour apples is less than 1 gram. Prove that one may combine them in 50 groups, each group having 2 apples such, and put them in a row such, that the difference of weights of neighbor groups is less than 1 gram.

Problem 10.7. Let the triangle is divided into 3 similar triangles. Prove that the original triangle can be divided into 2 similar triangles.

Solution submission deadline November 23, 2019