Saudi Arabia 2022 – Math Camp

Level 3

Geometry – Power of a Point and Radical Axis

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Definition (Power of a Point) The power of a point P with respect to a circle Γ of center O and radius r is defined by

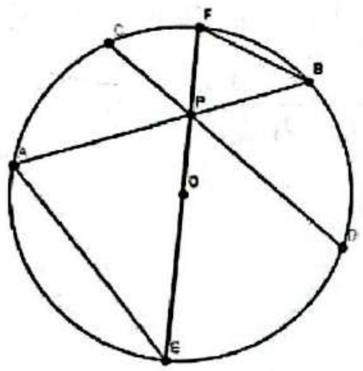
$$Pot_\Gamma P = PO^2 - r^2 \quad .$$

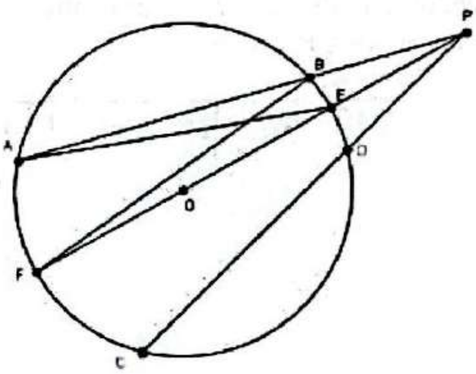
Theorem 3. If P is inside the circle Γ and a line through P cuts Γ at A and B, then

$$PA \cdot PB = -Pot_\Gamma P$$

If P is outside the circle Γ, a line through P cuts Γ at A and B and a line through P is tangent to Γ at T, then

$$PA \cdot PB = PT^2 = Pot_\Gamma P$$





Proof.

The line OP cross the circle on points E and F.

On the first case we have $PE = r + PO$ and $PF = r - PO$. By the Chrod Theorem

$$PA \cdot PB = PE \cdot PF = (r + PO)(r - PO) = r^2 - PO^2 = -Pot_\Gamma P$$

The second case is analogous.

Problems

7. (AMC/2013-10A) In ΔABC, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects BC at points B and X. Moreover BX and CX have integer lengths. What is BC?

(A) 11 ~~**(B) 28**~~ **(C) 33** **(D) 61** ~~**(E) 72**~~

8. (AIME I/2019) Let AB be a chord of a circle ω, and let P be a point on the chord AB. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. (Euler's relation) In a triangle with circumcenter O, incenter I, circumradius R, and inradius r, prove that

$$OI^2 = R^2 - 2Rr$$

10. (Tuymaada/2012) Point P is taken in the interior of the triangle ABC, so that

$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C)$$

Let L be the foot of the angle bisector of $\angle B$. The line PL meets the circumcircle of ΔAPC at point Q. Prove that QB is the angle bisector of $\angle AQC$.

11. (China/2013) Two circles K_1 and K_2 of different radii intersect at two points A and B, let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.

i) Show that l_1 and l_2 have a unique common point (denoted by P).

ii) Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

12. (IMO Shortlist/2011) Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0$$