Practice Problems

23 June, 2020

Level 2

Score Board

	P1	P2	P3	P4	P5	P16	P17	P18	P19	P20	P21	P22
28	*	*	✓	✓			✓	✓	✓			
1001	*	*	✓	✓	✓	✓	✓	✓			✓	
Rlm			✓	*		*		*	*			
8480		✓	✓	✓	✓	✓	✓	✓	✓			
Ash	✓	*		✓	✓	✓		✓				
54		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	*
3310		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
xd		✓	✓		✓	✓	✓	✓		*		
na	✓		✓		✓	✓	✓	✓	✓			
		2	1	2	4	3	4	2	5	6	7	7
		5	1	3	4	4	4	3	6	7	7	8

1. Let 1, 4, . . . and 9, 16, . . . be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

$\rightarrow A = \{a_d = 1 + 3d \mid d \in \mathbb{Z},\ d \geq 0,\ a_d \leq 1 + 3 \cdot 2003 = 6010\}$

$\rightarrow B = \{b_d = 9 + 7d \mid d \in \mathbb{Z},\ d \geq 0,\ b_d \leq 9 + 7 \cdot 2003 = 14022\}$

$\rightarrow$ Find $|A \cup B|$

- $C = A \cap B$

Hint: We want to find $|A \cap B|$

$\# k$, $\begin{cases} k \equiv 1 \pmod 3 \\ k \equiv 9 \pmod 7 \end{cases} \Rightarrow \begin{cases} k \equiv 16 \pmod 3 \\ k \equiv 16 \pmod 3 \end{cases}$

: CRT coo

$\Rightarrow \boxed{k \equiv 16 \pmod{21}}$

if $c \in C \Rightarrow c = 21m + 16$, $m \in \mathbb{Z}$

1. Let 1, 4, . . . and 9, 16, . . . be two arithmetic progressions. The set S is the union of the first 2004 terms of each sequence. How many distinct numbers are in S?

$$21m+16 \leq 6010 \Rightarrow \boxed{m \leq 285}, \quad m \geq 0$$

$\Rightarrow$ يوجد 286 عدد يحقق
وجوده في كل المتتابعتين

عدد الأعداد في S هو :
$$\Rightarrow |A|+|B|-|A\cap B|$$

$$=(2004+2004)-286$$

$$=3722$$

"متزايدة تماما"

2. Given a sequence of six strictly increasing positive integers such that each number (besides the first) is a multiple of the one before it and the sum of all six numbers is 79, what is the largest number in the sequence?

$$\begin{cases} a_1, a_2, a_3, \dots, a_6 \\ a_1 + a_2 + a_3 + \dots + a_6 = 79 \\ a_1 | a_2,\ a_2 | a_3,\ \dots,\ a_5 | a_6 \end{cases}$$

$$a_i | a_{i+1} \Rightarrow \begin{cases} a_{i+1} = k a_i \\ a_{i+1} > a_i \end{cases} \Rightarrow a_{i+1} \geq 2 a_i \quad \forall i \in \{1, \dots, 5\}$$

$$a_4,\ a_5 \geq 2a_4,\ a_6 \geq 4a_4 \Rightarrow a_4 + a_5 + a_6 \geq 7a_4$$

$$\Rightarrow 79 \geq 7a_4 \Rightarrow \boxed{11 \geq a_4}$$

$\boxed{a_4 \leq 8}$ ⇐ $a_1|a_2|a_3|a_4$; a_1, a_2, a_3 قواسم مختلفة 3 القسمة على تقبل a_4

$a_1 \geq 1 \Rightarrow a_2 \geq 2 \Rightarrow a_3 \geq 4,\ a_4 \geq 8$ *, $a_5 = m a_4$, $a_6 = n a_5 = nm a_4$

2. Given a sequence of six strictly increasing positive integers such that each number (besides the first) is a multiple of the one before it and the sum of all six numbers is 79, what is the largest number in the sequence?

$\Rightarrow a_4 = 8 \Rightarrow a_3 = 4, \ a_2 = 2, \ a_1 = 1$

$\Rightarrow a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 15 + 8m + 8mn = 79$

$\Rightarrow 8m + 8mn = 64$

$\Rightarrow 8m(1+n) = 64$

$\Rightarrow m(1+n) = 8$

$1+n > 2$

$\Rightarrow m = 2, \ 1+n = 4 \Rightarrow \begin{cases} m = 2 \\ n = 3 \end{cases}$

$\Rightarrow a_6 = 48$

3. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

$$n+10 \mid n^3 + 100$$

$$\Rightarrow n^3 + 100 \equiv 0 \pmod{n+10}$$

$$n \equiv -10 \pmod{n+10}$$

$$\Rightarrow n^3 + 100 \equiv (-10)^3 + 100 \equiv -900 \pmod{n+10}$$

$$\Rightarrow n+10 \mid 900$$

$$\Rightarrow n+10 \leq 900 \Rightarrow n \leq 980.$$

ولكن 980 تحقق الشرط.

$\Leftarrow$ أكبر قيمة ممكنة لـ n هي 980

4. Those irreducible fractions!

(1) Let n be an integer greater than 2. Prove that among the fractions

$$\frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n-1}{n},$$

an even number are irreducible.

(2) Show that the fraction

$$\frac{12n+1}{30n+2}$$

is irreducible for all positive integers n.

1) $\gcd(n,k)=1 \iff \gcd(n,n-k)=1$

$\underline{k \neq n-k}$, if $k_1, k_2 \leq \frac{n-1}{2}$

$\{k_1, n-k_1\} \cap \{k_2, n-k_2\} = \emptyset$ if $k_1 \neq k_2$

⇐ عدد الأعداد الأولية نسبياً مع n زوجي

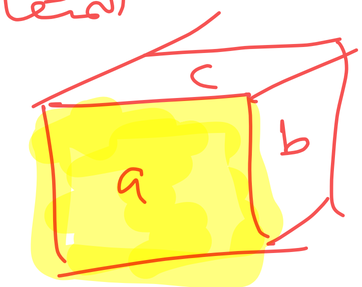
يوجد قاسم أولي فردي للعدد n

$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i}(p_i - 1) \rightarrow$ زوجي

5. A positive integer is written on each face of a cube. Each vertex is then assigned the product of the numbers written on the three faces intersecting the vertex. The sum of the numbers assigned to all the vertices is equal to 1001. Find the sum of the numbers written on the faces of the cube.

$(a, f), (b, d), (c, e)$

أزواج الأوجه المتقابلة



$1001 = abc + abe + acd + bcf + bef + cdf + def$

$= (a+f)(b+d)(c+e)$

$1001 = 7 . 11 . 13$

$a+f, b+d, c+e > 1$

$\Rightarrow a+f+b+d+c+e = 7+11+13 = 31$

16. Let p be a prime of the form $3k+2$ that divides a^2+ab+b^2 for some integers a and b. Prove that a and b are both divisible by p.

$p \mid a^2+ab+c^2 \Rightarrow p \mid a^3-b^3$

$\Rightarrow a^3 \equiv b^3 \pmod p$

نفرض أن $p \nmid a, b \Rightarrow 1 = \gcd(a,p) = \gcd(p,b)$

.الطريقة الأولى:

نعرف $c = ab^{-1} \Rightarrow c^3 \equiv 1 \pmod p$, $c \not\equiv 1 \pmod p$ $\Rightarrow \operatorname{ord}_p c = 3$

$\Rightarrow 3 \mid p-1$

$\Rightarrow$ تناقض

.الطريقة الثانية:

$p-1 = 3k+1$

$a^{3k+1} \equiv b^{3k+1} \pmod p$

$\Rightarrow (a^3)^k \cdot a \equiv (b^3)^k b \pmod p \Rightarrow (a^3)^k \cdot a \equiv (a^3)^k b \pmod p$

$\Rightarrow a \equiv b \pmod p$

16. Let p be a prime of the form $3k + 2$ that divides $a^2 + ab + b^2$ for some integers a and b. Prove that a and b are both divisible by p.

$\Rightarrow \underbrace{a^2+ab+b^2}_{\equiv 0} \equiv 3a^2 \pmod{p} \Rightarrow p \mid 3a^2 \Rightarrow p \mid a^2 \Rightarrow p \mid a$

$p \neq 3$

تناقض

17. The number 27000001 has exactly four prime factors. Find their sum.

$$27000\,000 + 1 = 300^3 + 1 = 301\underline{(300^2 - 300 + 1)}$$

$3 \cdot 300 = 900$ مربع كامل

$$300^2 - 300 + 1 = 300^2 + 2 \cdot 300 + 1 - 3 \cdot 300$$

$$= (301)^2 - 900 = \underline{301^2 - 30^2}$$

$$\Rightarrow 27000001 = 301(331)(271) = 7 \cdot 43 \cdot 271 \cdot 331$$

حاصل جمع القواسم الأولية ال 4 :

$$7 + 43 + 271 + 331 = 652$$

18. Find all positive integers n for which $\overset{5}{n}! + 5$ is a perfect cube.

الحل الأول:

if $n \geq 10$

$n! + 5 = 5(5k+1)$

$\Rightarrow v_5(n!) \geq 2 \Rightarrow v_5(n!+5) = 1$

$v_5(5) = 1$

$\Rightarrow$ يستحيل أن يكون مكعب $n!+5$

كاملا لـ $n \geq 10$

$\Rightarrow n < 9 \Rightarrow$ الوحيد الحل هو 5

الحل الثاني:

if $n \geq 7$

$n! + 5 \equiv 5 \pmod 7$

ولكن بواقي x^3 $(\bmod 7)$ هي $0, \pm 1$

$\Rightarrow$ تناقض

19. Find all primes p such that the number $p^2 + 11$ has exactly six different divisors (including 1 and the number itself).

$3 \mid p^2+11$ if $p \neq 3$ $\quad (11 \equiv -1 \pmod 3))$

$4 \mid p^2+11$ if $p \neq 2$

$\Rightarrow \quad 12 \mid p^2+11$ if $p \neq 2$

ولكن قواسم الـ 12 :

$\{1, 2, 3, 4, 6, 12\}$

$\Leftarrow$ قواسم p^2+11 تحتوي

$\{\underbrace{1, 2, 3, 4, 6, 12, p^2+11}_{>7 \text{ قواسم على الأقل}}\}$

$\Leftarrow$ تناقض

$\Leftarrow \quad p \leq 3$

$\Leftarrow \quad p = 3$ حل

20. Call a positive integer N a *7-10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

N : 7-10 double if

$N = \overline{a_n a_{n-1} \cdots a_1}_{7} \Rightarrow \boxed{N = 7^k a_n + 7^{k-1} a_{n-1} + \cdots + 1}$ *

$0 \leq a_i \leq 6$

$2N = \overline{a_n a_{n-1} \cdots a_1}_{10}$

$\Rightarrow 2N = 10^k a_n + 10^{k-1} a_{n-1} + \cdots + a_1$ **

بالتعويض من * :

$2.7^k a_n + 2.7^{k-1} a_{n-1} + \cdots + 2.7 a_1 + 2a_0 = 10^k a_n + \cdots + 10 a_1 + a_0$

فكرة الحل : بشكل عام $2.7^k < 10^k$ و $k \geq 3$

20. Call a positive integer N a *7-10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

$$(10^k-2\cdot7^k)a_k + (10^{k-1}-2\cdot7^{k-1})a_{k-1} + \dots + (10-2\cdot7)a_2 + (1-2)a_0 = 0$$

$10^i-2\cdot7^i>0$ ، لكل $i\geq 3$

الإثبات:

$$10^i>2\cdot7^i \iff \left(\frac{10}{7}\right)^i>2$$

$$\iff \left(\frac{10}{7}\right)^3\cdot\left(\frac{10}{7}\right)^{i-3}>2$$

ولكن

$$\left(\frac{10}{7}\right)^{i-3}\geq 1 \text{ ، } \left(\frac{10}{7}\right)^3>2$$

$$\Rightarrow \frac{10^i}{7^i}>2$$

20. Call a positive integer N a *7-10 double* if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

$10-14$

$100-98$

$$0 = \underbrace{(10^k - 2\cdot 7^k)a_k + \cdots + (10^3 - 2\cdot 7^3)a_3}_{\text{كل حد أكبر من أو يساوي } 314} + (10^2 - 2\cdot 7^2)a_2 + \cdots - a_1 - a_0$$

$+2a_2 - 4a_1 - a_0$

$\geq +2\cdot 0 - 4\cdot 6 - 6$

-30

$10^3 - 2\cdot 7^3 = 314$

$10^k - 2\cdot 7^k \geq 10^3 - 2\cdot 7^3$ ✓ ≥ 314

$\Leftrightarrow \quad 10^k - 10^3 \geq 2\cdot 7^k - 2\cdot 7^3$

$\Leftrightarrow \quad 10^3(10^{k-3} - 1) \geq 2\cdot 7^3(7^{k-3} - 1)$

$\boxed{k \leq 3} \Rightarrow$

$$2a_2 = 4a_1 + a_0$$

$a_2 = 6, \; a_1 = 3, \; a_0 = 0 \Rightarrow \overline{630}_7 = 315$

21. If $a \equiv b \pmod{n}$, show that $a^n \equiv b^n \pmod{n^2}$. Is the converse true?

$a^n \equiv b^n \pmod{n^2} \Leftrightarrow n^2 \mid a^n - b^n$

$\Leftrightarrow n^2 \mid (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$

ولكن $n \mid a-b$ ، نريد إثبات أن $n \mid a^{n-1} + a^{n-2}b + \cdots + b^{n-1}$

$\Leftrightarrow a^{n-1} + a^{n-2}b + \cdots + b^{n-1} \equiv 0 \pmod{n}$

ولكن $a \equiv b \pmod{n}$

$\Leftrightarrow a^{n-1} + a^{n-2}a + \cdots + a^{n-1} \equiv 0 \pmod{n}$

$\Leftrightarrow n\,a^{n-1} \equiv 0 \pmod{n}$

وهذه العلاقة صحيحة

العكس غير صحيح

$3^4 \equiv 1^4 \pmod{4}$, $3 \not\equiv 1 \pmod{4}$

22. Let p be a prime, and let $1 \le k \le p-1$ be an integer. Prove that

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

$$\binom{p-1}{k} = \frac{(p-1)!}{(p-1-k)!\ k!} \equiv (-1)^k \pmod{p}$$

$$\gcd((p-1-k)!\ k!,\ p) = 1$$

$$\Leftrightarrow \quad (p-1)! \equiv (-1)^k \left[(p-1-k)!\ k!\right] \pmod{p}$$

$$\Longleftrightarrow \quad (p-1)! \overset{?}{\equiv} (-1)^k (p-1-k)(p-1-k-1) \cdots (p-1-(p-2))\ k! \pmod{p}$$

$$\overset{?}{\equiv} (-1)^k \underbrace{(-1-k)(-2-k) \cdots (-p+1)}_{p-1-(k+1)+1}\ k! \pmod{p}$$

$$\equiv$$

$$(k+1)(k+2) \cdots (p-1)\ k! \pmod{p}$$

$$= (p-1)! \pmod{p}$$