Winter Camp 2021

Algebra Level 2

Day 3 (2)

2 January 2021

$2p^q - q^p = 7$ $\quad$ p, q primes

<u>q odd</u> $\quad$ $p=2:\ 2^{q+1} - q^2 = 7$

$$\Big(\underbrace{2^{\frac{q+1}{2}} - q}_{1}\Big)\Big(\underbrace{2^{\frac{q+1}{2}} + q}_{7}\Big) = 7 \rightarrow q = 3$$

$p=q:\ p^p = 7$

$\bmod p:\ -q^p \equiv -q \equiv 7\ [p]$ $\quad$ فرما

$\bmod q:\ 2p^q \equiv 2p \equiv 7\ [q]$

$$\Rightarrow \boxed{\underset{\text{فرد}}{p} \,|\, \underbrace{q+7}_{\text{زوج}}},\quad q \,|\, \underbrace{2p-7}$$

$p=3:\ q \,|\, -1 \rightarrow q=1 \Rightarrow\Leftarrow$

$\boxed{p \geq 5}:\ p \leq q+7,\quad q \leq 2p-7 \rightarrow \frac{q+7}{2} \leq p$

$2p \,|\, q+7$

$\rightarrow p \leq \frac{q+7}{2}$

$\Rightarrow p = \frac{q+7}{2} \rightarrow q = 2p-7$

$$\Rightarrow 2p^{2p-7} = \boxed{(2p-7)^p + 7 < (2p)^p} \qquad p^{2p-7} - p^p$$

$$(2p)^p = (2p-7+7)^p > (2p-7)^p + 7^p$$

$$2p^{2p-7} < (2p)^p = 2^p p^p$$

$$\rightarrow \quad p^{p-7} < 2^{p-1}$$

$p=11$: $\quad 10^4 < 11^4 > 2^{10} = 1024$

$\Rightarrow \quad p < 11 \quad \Rightarrow \quad p \in \{5, 7\}$

$p=5 \rightarrow q=3 \qquad 2p^q - q^p = 250 - 243 = 7$ ✓

$p=7 \rightarrow q=7 \qquad 7^7 \neq 7 \quad \times$

$\Rightarrow \quad (p,q) = (2,3), (5,3)$

$$2p^q - q^p = 7$$

$$p^q + \underbrace{(p^q - q^p)}_{>0} = 7$$

$p = q \rightarrow \Rightarrow\Leftarrow$

$\Rightarrow \quad p^q < 7$ $\qquad$??

$p^q - q^p < 0 \rightarrow p^q < q^p \longrightarrow p > q$

$$a^b \overset{?}{<} b^a \qquad a > b$$

$a = bx \ , \ x > 1$

$$(bx)^b \overset{?}{<} b^{bx}$$

$$bx \overset{?}{<} b^x$$

$$x \overset{?}{<} b^{x-1} \qquad \forall b \geq 3$$

يكفي

$$b^{x-1} \geq \underline{3^{x-1} > x}$$

<u>حل الاستقراء</u>

نثبت b و الاستقراء على $a \geq b+1$

$$b^{b+1} \overset{?}{>} (b+1)^b = b^b + \sum_{i=1}^{b-1} \binom{b}{i} b^{b-i} + 1 < b^b (b) = b^{b+1}$$

$$\binom{b}{i} < b^i$$

$$i > 1$$

- $p = 2$ or $q = 2$ $\longrightarrow$
- $p > q \geq 3$ $\qquad p \mid q+7, \quad q \mid 2p-7$

$$q < p \leq q+7$$

- $q = 3 \to 3 < p \leq 10 \to p = 5$
- $q \equiv 1 [6] \longrightarrow p = q+4$ or $q+6$
- $q \equiv -1 [6] \longrightarrow p = q+2$ or $q+6$

Problem 1. $f : \mathbb{R}/\{0\} \to \mathbb{R}$. If f is additive and $f(x) = x^2 f(\frac{1}{x})$, then prove $f(x) = f(1)x$ for all $x \in \mathbb{R}/\{0\}$.

$x \to x+1 \qquad f(x+1) = (x+1)^2 f\left(\frac{1}{x+1}\right)$

- $f\left(\frac{1}{x+1}\right) = f(1) - \underbrace{f\left(\frac{x}{x+1}\right)}$

$$f\left(\frac{x}{x+1}\right) = \frac{x^2}{(x+1)^2} f\left(\frac{x+1}{x}\right) = \frac{x^2}{(x+1)^2} f\left(1+\frac{1}{x}\right)$$

$$= \frac{x^2}{(x+1)^2}\left(f(1) + f\left(\frac{1}{x}\right)\right)$$

$$= \frac{x^2}{(x+1)^2}\left(f(1) + \frac{f(x)}{x^2}\right)$$

$$f(x+1) = f(x) + f(1) = (x+1)^2\left[f(1) - \frac{x^2}{(x+1)^2}\left(f(1) + \frac{f(x)}{x^2}\right)\right]$$

$$f(x) + f(1) = (\underbrace{(x+1)^2 - x^2}_{2x+1})f(1) - f(x)$$

$$2f(x) = 2x\,f(1) \longrightarrow f(x) = x\,f(1)$$

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition:

$$f(x^2 - y) = xf(x) - f(y)$$

for all x, y from $\mathbb{R}$

$x=y=0$: $f(0) = -f(0) \rightarrow f(0)=0$

$y=0$: $f(x^2) = x f(x) \rightarrow f(x)=kx$

$x=0$: $f(-y) = -f(y) \rightarrow f$ odd

$f(x^2-y) = xf(x) - f(y) = f(x^2) + f(-y)$

$a = x^2 \geq 0$

$b = -y \in \mathbb{R}$

$\rightarrow \left.\begin{array}{l} f(a) + f(b) = f(a+b) \\ f(\underbrace{-a}_{<0}) + f(-b) = f(-a-b) \end{array}\right\} \rightarrow f$ additive $\rightarrow f(x)=kx$

Problem 3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^4 + f(y)) = y + f^4(x) \qquad \forall x, y \in \mathbb{R}$$

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(2x^2 + y + f(y)) = 2y + 2f^2(x) \qquad \forall x, y \in \mathbb{R}$$

Problem 5. Find all functions $f : \mathbb{R}/\{0\} \to \mathbb{R}/\{0\}$ such that

$$f(x+y) = x^2 f(\frac{1}{x}) + y^2 f(\frac{1}{y})$$

Problem 6. $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y + xy) = f(x) + f(y) + f(xy)$$

Prove that f is additive.

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Problem 7. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(m+n) = f(m) + f(n) + mn$$

$$\boxed{f(m) = mf(1) + \frac{m(m-1)}{2}}$$

$n=1$: $f(m+1) = f(m) + f(1) + m$

$f(1) = k \rightarrow f(2) = 2k+1$

$f(3) = f(2) + k + 2 = 3k+3$

$f(4) = f(3) + k + 3 = 4k + 6 = 4k + \underbrace{(3+2+1)}_{\frac{4\cdot 3}{2}}$

$f(n) = nk + \frac{n(n-1)}{2}$

استقراء

$$f(n+1) = f(n) + k + n$$

$$= (n+1)k + \frac{n(n-1)}{2} + n = (n+1)k + \frac{(n+1)n}{2}$$

<u>check</u>:

$$\left.\begin{aligned} f(m+n) &= (m+n)k + \frac{(m+n)(m+n-1)}{2} \\ f(m)+f(n)+mn &= mk + \frac{m(m-1)}{2} + nk + \frac{n(n-1)}{2} + mn \end{aligned}\right\} \text{“}=\text{”}$$

Problem 8. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

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$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad \forall x, y \in \mathbb{Q}$$

$f(x) = kx^2$ التوقع

$x = y = 0 \to f(0) = 0$

$x = 0 \to f(y) + f(-y) = 2f(y) \to f(y) = f(-y) \to f$ even

$$\begin{cases} g(x) = f(x) - kx^2 \\ g(x+y) + g(x-y) = 2g(x) + 2g(y) \end{cases}$$

و $\underline{f}$

و $\underline{f + kx^2}$

$y = x: \ f(2x) = 4f(x)$

$f(nx) \overset{?}{=} n^2 \underline{f(x)} \quad n \in \mathbb{Z}$

$f(nz) = n^2 f(z) \quad z \in \mathbb{Q}$

$$\boxed{y = 1: \ f(x+1) + f(x-1) = 2f(x) + 2f(1)}$$

$x=1: \quad f(2)=4f(1)$

$n\in\mathbb{Z}^+ \quad \boxed{f(n)=n^2 f(1)}$ استقرای

$f(n+1)+(n-1)^2 f(1)=2n^2 f(1)+2f(1)$

$f(n+1)=(2n^2-n^2+2n-1+2)f(1)$

$=(n+1)^2 f(1)$

$y=z$

$x\to xz$ $\quad \boxed{f((x+1)z)+f((x-1)z)=2f(zx)+2f(z)}$

$\underline{f(nz)=n^2 f(z)}$ $\quad n=2 \quad f(2z)=4f(z)$

استقرای روی x

$n\in\mathbb{Z}$

$z\in\mathbb{Q}$

$f((n+1)z)+(n-1)^2 f(z)=2n^2 f(z)+2f(z)$

$f((n+1)z)=(2n^2-n^2+2n-1+2)f(z)$

$=(n+1)^2 f(z)$

$f(nx)=n^2 f(x) \quad n\in\mathbb{Z}, \ x\in\mathbb{Q}.$

$x=\frac{p}{q}: \ f\left(n\cdot\frac{p}{q}\right)=n^2 f\left(\frac{p}{q}\right)$

$n=q: \quad f(p)=q^2 f\left(\frac{p}{q}\right)$

$\to f\left(\frac{p}{q}\right)=\frac{f(p)}{q^2}=\frac{p^2}{q^2}f(1) \quad \to f(x)=x^2 f(1)$

$\forall x\in\mathbb{Q}$

Problem 9. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x+y)f(x-y)) = x^2 - yf(y) \qquad \forall x, y \in \mathbb{R}$$

Problem 10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y \qquad \forall x, y \in \mathbb{R}$$

Problem 11. Find all $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m) + f(n) | m + n \qquad \forall m, n \in \mathbb{N}$$