$$a^n + b^n = c^n.$$

ok?

HELLO

Find all integers

1) $$x^3 + 3y^3 + 9z^3 - 3xyz = 0$$

2) Find all rationals a, b such that $a, b \in \mathbb{Q}$

$$a^2 + ab + b^2 = 2$$

3) Solve in integers $$y^4 = x^3 + 7$$

Fermat descent method (Revision)

$$\boxed{x^3 + 3y^3 + 9z^3 - 3xyz = 0} \quad (*)$$, so $3 \mid x^3 \Rightarrow 3 \mid x$

so $x = 3x_1$, put

$$27x_1^3 + 3y^3 + 9z^3 - 9x_1yz = 0 \quad | :3$$

$$9x_1^3 + y^3 + 3z^3 - 3x_1yz = 0$$, so $3 \mid y^3 \Rightarrow y = 3y_1$

put $y = 3y_1$,

$$9x_1^3 + 27y_1^3 + 3z^3 - 9x_1y_1z = 0 \quad | :3$$

$$3x_1^3 + 9y_1^3 + z^3 - 3x_1y_1z = 0$$, so $z = 3z_1$

$$3x_1^3 + 9y_1^3 + 27z_1^3 - 9x_1y_1z_1 = 0 \quad | :3$$

$$\boxed{x_1^3 + 3y_1^3 + 9z_1^3 - 3x_1y_1z_1 = 0} .$$

(x, y, z) is sol. of $(*)$ then $\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$ is sol of $(*)$

$$a^2+ab+b^2=2 \qquad y \neq 0$$

HINT $\boxed{a=\frac{x}{y}\ ,\quad b=\frac{z}{y}}$ you can find x, y, z

$\frac{x}{y}, \frac{z}{t} = \frac{xt}{yt}, \frac{zy}{yt} \Rightarrow \left(\frac{x}{y}\right)^2+\frac{x}{y}\cdot\frac{z}{y}+\left(\frac{z}{y}\right)^2=2 \mid y^2$

$$\boxed{x^2+xz+z^2=2y^2}$$

odd odd.2 + z_1^2

even even

I Continue (x, z, y) =

II $|x|+|y|+|z|$ and

$(x,y,z)=(0,0,0)$

	x_1	z
	0	0, 1
	1	1, 0

modul 2

so $2 \mid x \Rightarrow x = 2\cdot x_1$

$2 \mid z \Rightarrow z = 2\cdot z_1$

$$4x_1^2+2x_1\cdot 2z_1+4z_1^2=2y^2 \ \mid :2$$

$$2x_1^2+2x_1z_1+2z_1^2=y^2 \Rightarrow 2 \mid y^2 \Rightarrow y=2y_1$$

$$2\cdot x_1^2+2x_1z_1+2z_1^2=4y_1^2 \ \mid :2 \qquad \boxed{x_1^2+x_1z_1+z_1^2=2y_1^2}$$

I Take the solution (x,y,z) with smallest sum $|x|+|y|+|z|$: smallest possible.

then $\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$ is solution <u>but</u> !

$$\left|\frac{x}{3}\right| + \left|\frac{y}{3}\right| + \left|\frac{z}{3}\right| = \frac{1}{3}\left(|x|+|y|+|z|\right) < |x|+|y|+|z|$$

<u>contradiction</u>.

$$|x|+|y|+|z| = 0 \Rightarrow \underline{x=y=z=0}$$

II $(x,y,z) \longrightarrow \left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \longrightarrow \left(\frac{x}{3^2}, \frac{y}{3^2}, \frac{z}{3^2}\right) \longrightarrow \cdots$

are divisible by arbitrary large power of <u>3</u>

$\Downarrow$

$\boxed{x=y=z=0}$

$y^4 \equiv x^3 + 7$

Residues of x^3 modulo 13?

x	0	1	2	3	4	5	6	7	8	9	10	11	12	
x^3 mod 13	$\boxed{0}$	$\boxed{1}$	$\boxed{8}$	1	12	8	8	5	5	1	12	$\boxed{5}$	$\boxed{12}$	5 diff residues.
(x^3+7) mod 13	7	8	12	8	6	2	12	12	12	8	6	12	6	4 nons

y	0	1	2	3	4	5	6	7	8	9	10	11	12	
(y^4) mod 13	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	3	$\boxed{9}$	1	9	9	1	9	3	3	1	4 diff resid. 3 nons

NO SOLUTIONS.

Scanned with CamScanner

$x^5 = y^2 + 4$

$y^2 \bmod 11 \Rightarrow 0, 1, 3, 4, 5, 9$

$y^2 + 4 \bmod 11 \Rightarrow 2, 4, 5, 7, 8, 9$

$x^5 \bmod 11 \Rightarrow -1, 0, 1 \bmod 11.$

NO SOLUTIONS

$5 \mid p-1$ — take 11.
$2 \mid p-1$

Theorem Take prime $p > 2$, $k \geq 0$ integer. Then

$1^k, 2^k, 3^k, \dots, (p-1)^k$ modulo p

give $\dfrac{p-1}{\gcd(p-1, k)}$ different residues modulo p.

$\boxed{\downarrow \text{small}}$ by chance that we will get coincidences.

$\dfrac{p-1}{\gcd(p-1,k)}$ is small if $\gcd(p-1,k)$ is big.

$3, 4 \rightsquigarrow 3 \cdot 4 + 1.$ $\gcd(13, 4) = 4$

$\gcd(13, 3) = 3$.

$$x^2 + x + 1 = y^2$$ ("Square between squares")

if $x > 0$

$$(x+1)^2 > \underbrace{x^2 + x + 1}_{y^2} > x^2 \Rightarrow \text{contradiction.}$$

~~there~~

< 0

if $x \leq -2$ then

$$x^2 > \underbrace{x^2 + x + 1}_{y^2} > \underset{x^2+2x+1}{\underset{\parallel}{(x+1)^2}} \Rightarrow \text{no solution.}$$

therefore $\underline{x = 0, -1}$

$x = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1 \vee -1$
$x = -1 \Rightarrow y^2 = 1 \Rightarrow y = 1 \vee -1$.

5) Solve in integers

$$x^2 + x + 1 = y^2.$$

6) Solve in integers

$$x^4 + y = x^3 + y^2$$

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7) Find all positive integers (a, b) for which $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are cubes.

$$x^4 + y = x^3 + y^2$$

HINT : $x^4 + y = x^3 + y^2 \Rightarrow x^4 - x^3 = y^2 - y \;\Big|\cdot 4 \Rightarrow$

$\Rightarrow 4x^4 - 4x^3 = 4y^2 - 4y \Rightarrow 4x^4 - 4x^3 + 1 = 4y^2 - 4y + 1 \Rightarrow$

$\Rightarrow \underline{4x^4 - 4x^3 + 1 = (2y-1)^2}$.

If $x \geq 2$ or $x \leq -2$ i.e. $|x| \geq 2$, then

$(2x^2 - x - 1)^2 < 4x^4 - 4x^3 + 1 < (2x^2 - x)^2$

$4x^4 - 4x^3 + x^2$

$\cancel{4x^4} + x^2 + 1 - \cancel{4x^3} - 4x^2 + 2x < \cancel{4x^4} - \cancel{4x^3} + 1$. $x^2 > 1$.

$\boxed{-3x^2 + 2x + 1 < 0}$ so $|x| < 2 \Rightarrow x = 0, 1, -1$

$(x,y) = (0,0), (0,1), (1,0), (1,1), (-1,2), (-1,-1)$.

NLOG $a \leq b$, then

$b^3 < \underbrace{b^3 + 6ab + 1}_{\text{is cube}} \leq b^3 + 6b^2 + 1 < b^3 + 6b^2 + 12b + 8 = (b+2)^3$

$\cancel{b^3} + 6ab + \cancel{1} = (b+1)^3 = \cancel{b^3} + 3b^2 + 3b + \cancel{1} \Rightarrow$

$2ab = b(b+1) \Rightarrow \boxed{b = 2a - 1}$

When $a^3 + 6ab + 1$ is cube if $b = 2a - 1$. If put $b := 2a - 1$

$a^3 + 6a(2a-1) + 1 = a^3 + 12a^2 - 6a + 1 \Leftarrow$ cube.

$a^3 \leq a^3 + 6a^2 - 6a < a^3 + 12a^2 - 6a + 1 < a^3 + 12a^2 + 48a + 64 = (a+4)^3$

$b = -1$ or $b = 1$

so we ha posolutive:

$a^3 + 12a^2 - 6a + 1 = (a+1)^3 \Rightarrow 9a^2 - 9a = 0 \Rightarrow a = 0$ or $a = 1$

$a^3 + 12a^2 - 6a + 1 = (a+2)^3 \Rightarrow 6a^2 - 18a - 7 = 0 \rightarrow$ no sol.

$a^3 + 12a^2 - 6a + 1 = (a+3)^3 \Rightarrow 3a^2 - 33a - 26 = 0$ — no sol