Around GCD

Lesson by Senya, group L4

The problems abouy GCD or the idea of considering GCD are quite popular. One of the most important observations about GCD is that $(a, b) = (a - b, b)$ (recall that (a, b) stands for "GCD of a and b), the consequences of this fact (e.g boud onf GCD) also play a big role.

Problem 1. There are two numbers: one consists of n ones in its decimal representation, another one consists of m ones in its decimal representation. What is their gcd?

Problem 2. i) Prove that for any positive integers m and n there are integer numbers a, b such that $am + bn = (m, n)$. (This is actually a well-known theorem)

ii) Using the fact above, answer the following question: "Given integers a, b, c when does equation $ax + by = c$ have integer solutions? When it does have solutions, how do they look like?

Problem 3. With the help of the problem above, do the following problems:

i) Solve in integers: $2x + 3y = 11$. Note: there are indeed infinitely many solutions to this equation, so what you need to do is to describe them nicely;

ii) Solve in integers: $2x + 3y + 5z = 11$;

Problem 4. Numbers m and n satisfy the equation $lcm(m, n) + gcd(m, n) = m + n$. Prove that one of the numbers m or n is divisible by another.

Problem 5. Show that for any n the number $2^{2^n} - 1$ has at least n distinct positive divisors.

Problem 6. Find all positive integers m, n, k such that $m+n = (m, n)^2, m+k = (m, k)^2, n+k = (n, k)^2$.

Problem 7. There is a sequence of integer numbers a_i such that $(a_i, a_j) = (i, j)$. Does it mean that we must have $a_i = i$ for all i?

Problem 8. Positive integers x and y are such that $2x^2 - 1 = y^{15}$. Prove that if $x > 1$ then x is divisible by 5.

Problem 9. Solve in integers

$$(x + 2)^4 - x^4 = y^3$$

Problem 10. There are n distinct positive integers $a_1, ..., a_n$. It turned out that for any $1 \leq i, j \leq n$ the following holds: $(|a_i - a_j|, |i - j|) < 2013$. What is the largest possible value of n?