Number Theory

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Problems – April 13

- *Wilson's Theorem.* If p is a prime, then $(p-1)! \equiv -1 \pmod{p}$.
- *Chinese Remainder Theorem.* Let $n_1, n_2, \dots, n_k$ be pairwise coprime positive integers and let $a_1, a_2, \dots, a_k$ be any integers. Then the system of congruences
$$\begin{cases} x \equiv a_1 & \pmod{n_1} \\ x \equiv a_2 & \pmod{n_2} \\ \quad \cdots\cdots & \\ x \equiv a_k & \pmod{n_k} \end{cases}$$
has a unique solution modulo $n_1 n_2 \cdots n_k$.
- A reduced residue system modulo n has $\varphi(n)$ elements, where φ is the *Euler totient function*: $\varphi(n) = n \prod (1 - \frac{1}{p_i})$, where the product goes over all prime divisors p_i of n.

1. Find all x such that $x \equiv 1 \pmod{2019}$, $x \equiv 2 \pmod{2021}$ and $x \equiv 3 \pmod{2023}$.
2. Find all positive integers n for which $\varphi(n)$ divides n.
3. Show that there are 1000 consecutive positive integers, none of which is a perfect power.
4. Prove that there exists a positive integer x such that the smallest prime divisor of $1000x + 1$ is 601.
5. Prove that there exist 200 consecutive positive integers, each of which has at least one prime divisor not exceeding 103.
6. Is there a positive integer n such that n, $2n$ and $3n$ are perfect powers? Find the smallest such n if it exists.
7. Find all positive integers n with the following property: Whenever $n \mid xy + 1$ for some integers x, y, it also holds that $n \mid x + y$.
8. Let $p = 4k + 1$ be a prime and let $x = (2k)!$. Prove that $p \mid x^2 + 1$.
9. (a) Find the number of different remainders that numbers $1^2, 2^2, 3^2, \dots$ give when divided by n, where (a) n is a prime; (b) n is a product of two primes.
10. Let $a_1 < a_2 < \cdots < a_{\varphi(n)}$ be the positive integers not exceeding n and coprime to n. Find all n for which none of the sums $a_i + a_{i+1}$ is divisible by 3.