Email training, N5

September 22-28

Problem 5.1. Let a, b, c be real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 1.$$

Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} = 0.$$

Solution 5.1. By multiplying the equality by $a+b+c$ we get

$$\frac{a(a+b+c)}{b+c} + \frac{b(a+b+c)}{c+a} + \frac{c(a+b+c)}{a+b} = a+b+c.$$

Separating the terms we want from the terms we don't, we simplify to

$$\frac{a^2}{b+c} + \frac{b^2}{a+c} + \frac{c^2}{a+b} + \frac{a(b+c)}{b+c} + \frac{b(a+c)}{a+c} + \frac{c(a+b)}{a+b} = a+b+c.$$

The last three terms of the left side is simply $a+b+c$ and cancelling them gives us the required result.

Problem 5.2. The set $\{1, 2, ..., 10\}$ is partitioned to three subsets A, B and C. For each subset the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.

Is it possible that A alone has the largest sum of elements, B alone has the largest product of elements, and C alone has the largest sum of digits?

Solution 5.2. Let $A = \{1, 9, 10\}$, $B = \{2, 4, 5, 6\}$ and $C = \{3, 7, 8\}$. Then the sum of number is biggest in A (namely 20), the product is biggest in B (namely 240) and the sum of digits is biggest in C (namely 18).

Problem 5.3. Find all positive integers n for which

$$3x^n + n(x+2) - 3 \geq nx^2$$

holds for all real numbers x.

Solution 5.3. We show that the inequality holds for even n and only for them. If n is odd, then for $x = -1$ the left hand side of the inequality equals $n-6$ while the right hand side is n. So the inequality is not true for $x = -1$ for any odd n. So now assume that n is even. Since $|x| \geq x$, it is enough to prove $3x^n + 2n - 3 \geq nx2 + n|x|$ for all x or equivalently that

$$3x^n + (2n-3) \geq nx^2 + nx$$

for $x \geq 0$. Now the AM-GM-inequality gives

$$2x^n + (n-2) = x^n + x^n + 1 + 1 + \ldots + 1 \geq n\left(x^n \cdot x^n \cdot 1^{n-2}\right)^{1/n} = nx^2.$$

Similarly
$$x^n + (n-1) \geq n\big(x^n \cdot 1^{n-1}\big)^{1/n} = nx.$$
Adding these two inequalities gives the claim.

Problem 5.4. Denote by $P(n)$ the greatest prime divisor of n. Find all integers $n \geq 2$ for which
$$P(n) + [\sqrt{n}] = P(n+1) + [\sqrt{n+1}].$$
(Note: $[x]$ denotes the greatest integer less than or equal to x.)

Solution 5.4. The equality holds only for $n = 3$. It is easy to see that $P(n) \neq P(n+1)$. Therefore we need also that
$$[\sqrt{n}] \neq [\sqrt{n+1}].$$
This is only possible if $n+1$ is a perfect square. In this case $[\sqrt{n}] + 1 = [\sqrt{n+1}]$, and hence
$$P(n) = P(n+1) + 1.$$
As both $P(n)$ and $P(n+1)$ are primes, it must be that $P(n) = 3$ and $P(n+1) = 2$.

It follows that $n = 3^a$ and $n+1 = 2^b$, and we are required to solve the equation $3^a = 2^b - 1$. Calculating modulo 3, we find that b is even. Put $b = 2c$.
$$3^a = (2^c - 1)(2^c + 1).$$
As both factors cannot be divisible by 3 (their difference is 2), $2^c - 1 = 1$. From this we get $c = 1$, which leads to $n = 3$.

Problem 5.5. Two players play the following game. At the outset there are two piles, containing 10.000 and 20.000 tokens, respectively. A move consists of removing any positive number of tokens from a single pile or removing $x > 0$ tokens from one pile and $y > 0$ tokens from the other, where $x + y$ is divisible by 2015. The player who cannot make a move loses. Which player has a winning strategy?

Solution 5.5. The first player wins. He should present his opponent with one of the following positions: $(0;0); (1;1); (2;2); :::; (2014;2014)$: All these positions have different total numbers of tokens modulo 2015. Therefore, if the game starts from two piles of arbitrary sizes, it is possible to obtain one of these positions just by the first move. In our case
$$10.000 + 20.000 \equiv 1790[2015],$$
and the first player can leave to his opponent the position $(895; 895)$. Now the second type of move can no longer be carried out. If the second player removes n tokens from one pile, the first player may always respond be removing n tokens from the other pile

Problem 5.6. Find all quadrilaterals $ABCD$ such that all four triangles DAB, CDA, BCD and ABC are similar to one-another.

Solution 5.6. -

First assume that $ABCD$ is a concave quadrilateral. Without loss of generality one can assume $\angle D > 180°$, in other words D lies inside of triangle ABC. Again without loss of generality one can assume that $\angle ABC$ is the maximum angle in triangle ABC. Therefore

$$\angle ADC = \angle ABC + \angle BAD + \angle BCD > \angle ABC.$$

Thus $\angle ADC$ is greater than all the angles of triangle ABC, so triangles ABC and ADC cannot be similar. So it is concluded that $ABCD$ must be convex.

Now let $ABCD$ be a convex quadrilateral. Without loss of generality one can assume that the $\angle B$ is the maximum angle in the quadrilateral. It can be written that

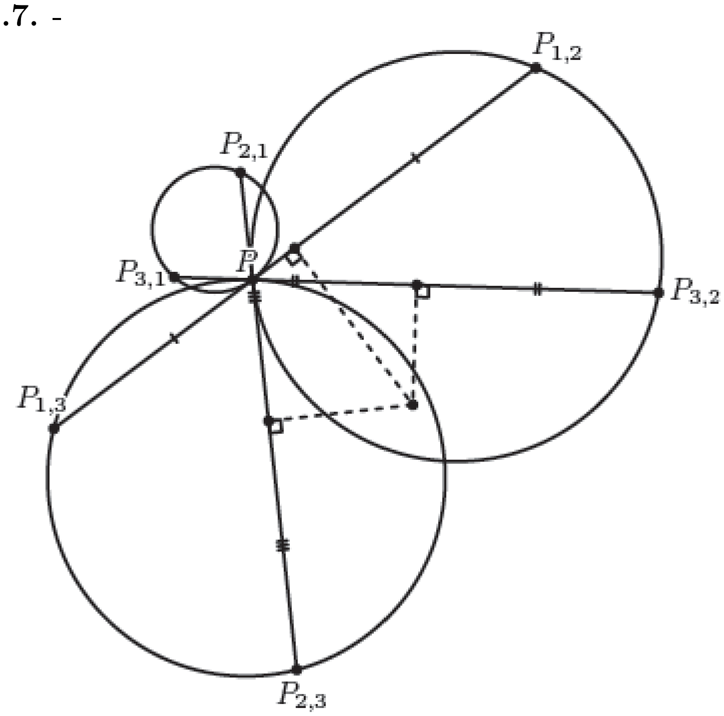
$$\angle ABC > \angle DBC, \quad \angle ABC \geq \angle ADC \geq \angle BCD.$$

Since triangles ABC and BCD are similar, it is implied that $\angle ABC = \angle BCD$ and similarly, all the angles of $ABCD$ are equal; Meaning $ABCD$ must be a rectangle. It is easy to see that indeed, all rectangles satisfy the conditions of the problem.

Answer. All rectangles.

Problem 5.7. Three circles ω_1, ω_2 and ω_3 pass through one common point, say P. The tangent line to ω_1 at P intersects ω_2 and ω_3 for the second time at points $P_{1,2}$ and $P_{1,3}$, respectively. Points $P_{2,1}$, $P_{2,3}$, $P_{3,1}$ and $P_{3,2}$ are similarly defined. Prove that the perpendicular bisector of segments $P_{1,2}P_{1,3}$, $P_{2,1}P_{2,3}$ and $P_{3,1}P_{3,2}$ are concurrent.

Solution 5.7. -



First assume that no two of the lines $\ell_1 \equiv P_{2,1}P_{3,1}$, $\ell_2 \equiv P_{1,2}P_{3,2}$ and $\ell_3 \equiv P_{1,3}P_{2,3}$ are parallel; Consider triangle XYZ made by intersecting these lines, where

$$X \equiv \ell_2 \cap \ell_3,$$
$$Y \equiv \ell_1 \cap \ell_3,$$
$$Z \equiv \ell_2 \cap \ell_1.$$

Note that

$$\angle P_{3,2}P_{1,2}P = \angle P_{3,2}PP_{2,3} = \angle PP_{1,3}P_{2,3},$$

meaning $XP_{1,2} = XP_{1,3}$. Similarly, it is implied that $YP_{2,1} = YP_{2,3}$ and $ZP_{3,1} = ZP_{3,2}$. Therefore, the angle bisectors of angles YXZ, XYZ and YZX are the same as the perpendicular bisectors of segments $P_{1,2}P_{1,3}$, $P_{2,1}P_{2,3}$ and $P_{3,1}P_{3,2}$; Thus, these three perpendicular bisectors are concurrent at the incenter of triangle XYZ, resulting in the claim of the problem.

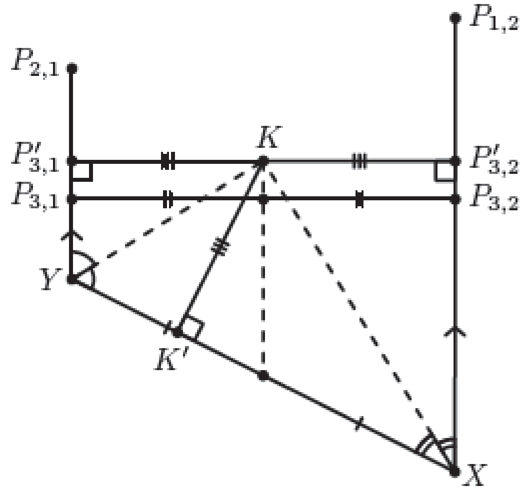
Now assume that at least two of the lines $\ell_1 = P_{2,1}P_{3,1}$, $\ell_2 = P_{1,2}P_{3,2}$ and $\ell_3 = P_{1,3}P_{2,3}$ are parallel; Without loss of generality assume that ℓ_1 and ℓ_2 are parallel. Similar to the previous case,

$$\angle P_{1,2}P_{3,2}P = \angle P_{1,2}PP_{2,1} = \angle P_{2,1}P_{3,1}P.$$

But since $\ell_1 \parallel \ell_2$, it is also true that

$$\angle P_{1,2}P_{3,2}P + \angle P_{2,1}P_{3,1}P = 180°,$$

Hence $\angle P_{1,2}P_{3,2}P = \angle P_{2,1}P_{3,1}P = 90°$. This equation immediately implies $\ell_3 \nparallel \ell_2$, because otherwise it would be deduced that $\ell_3 \perp P_{1,3}P_{1,2}$ and $\ell_2 \perp P_{1,3}P_{1,2}$, resulting in $P_{1,3}P_{1,2} \parallel P_{3,1}P_{3,2}$; Which is clearly not possible. Now consider trapezoid $XYP_{2,1}P_{1,2}$. The problem is now equivalent to show that the angle bisector of $\angle X$, angle bisector of $\angle Y$ and the perpendicular bisector of $P_{3,1}P_{3,2}$ concur. Note that ℓ_1 and ℓ_2 are parallel to the perpendicular bisector of $P_{3,1}P_{3,2}$, and in fact, the perpendicular bisector of $P_{3,1}P_{3,2}$ connects the midpoints of XY and $P_{3,1}P_{3,2}$. Now the claim of the problem is as simple as follows.



Claim. *In trapezoid $XYP_{2,1}P_{1,2}$, the angle bisector of $\angle X$, the angle bisector of $\angle Y$, and the mid-line of the trapezoid are concurrent.*

Proof. Let K be the intersection of the angle bisector of $\angle X$ and the angle bisector of $\angle Y$. Let $P'_{3,2}$, $P'_{3,1}$ and K' be the foot of perpendicular lines from K to lines $P_{1,2}X$, $P_{2,1}Y$ and XY, respectively. Since K lies on the angle bisector of $\angle X$, it is deduced that $KP'_{3,2} = KK'$, and similarly since K lies on the angle bisector of $\angle Y$, $KP'_{3,1} = KK'$; Thus $KP'_{3,1} = KP'_{3,2}$, meaning K lies on the mid-line of trapezoid $XYP_{2,1}P_{1,2}$.

This result leads to the conclusion of the problem.

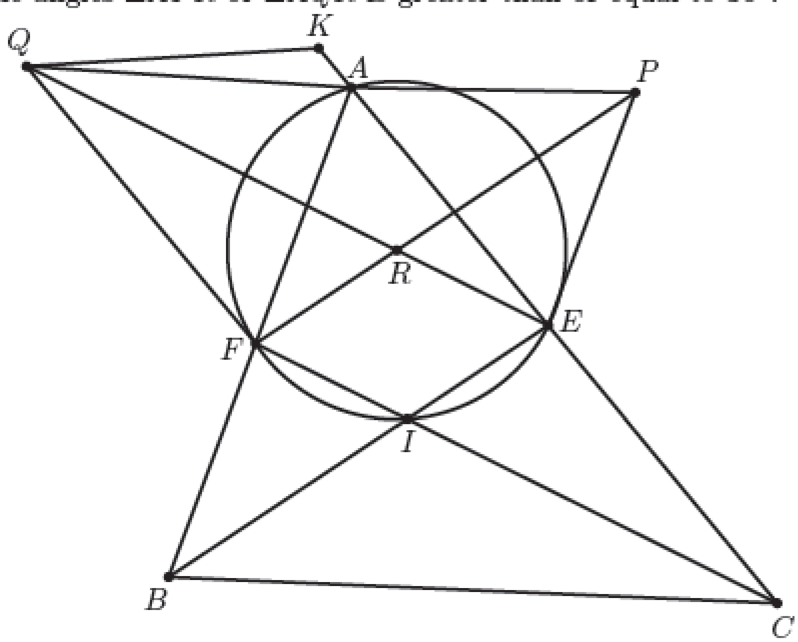
Problem 5.8. Let ABC be a triangle with $\angle A = 60^o$. Points E and F are the foot of angle bisectors of vertices B and C respectively. Points P and Q are considered such that quadrilaterals $BFPE$ and $CEQF$ are parallelograms. Prove that $\angle PAQ > 150^o$. (Consider the angle PAQ that does not contain side AB of the triangle.)

Solution 5.8. -

Let I and be the intersection point of lines BE and CF, and let R be the intersection point of lines QE and PF. It is easy to see that $\angle BIC = 120°$. Thus $AEIF$ is a cyclic quadrilateral and so

$$CE \cdot CA = CI \cdot CF \tag{1}$$

Also $\angle PRQ = \angle BIC = 120°$, therefore it suffices to show that at least on of the angles $\angle APR$ or $\angle AQR$ is greater than or equal to $30°$.



Assume the contrary, meaning both of these angles are less than $30°$. Hence there exists a point K on the extension of ray CA such that $\angle KQE = 30°$. Since $\angle IAC = 30°$ $\angle ACI = \angle KEQ$, it is deduced that $\triangle AIC \sim \triangle QKE$. This implies

$$\frac{CI}{CA} = \frac{KE}{QE} > \frac{AE}{CF} \implies AE < \frac{CF \cdot CI}{CA} \overset{(1)}{=} CE.$$

Similarly, it is obtained that $AF < BF$. On the other hand at least on of the angles $\angle ABC$ or $\angle ACB$ are not less than $60°$. Without loss of generality one can assume that $\angle ABC \geq 60°$ thus $AC \geq BC$ and according to angle bisector theorem it is obtained that $AF \geq BF$, which is a contradiction. Hence the claim of the problem.