We will consider combinatorial problems concerning geometrical objects. Before we start some specific methods let's recall some basic solving ideas which are used everywhere: extremal principle, Dirichlet's principle, double-counting and induction.

Short intro

1. **[Extremal,Dirichlet's, induction]** There were 31 kids in the room on a party. All distances between them were pairwise distinct. At some moment each kid threw a piece of cake to the nearest to him person. Prove that there is a person in whom the cake did not fly.
2. **[Dirichlet's, double-counting]** Prove that it's impossible to draw a convex 2020-gon with integer angle measures of all angles.
3. **[Induction]** Prove that a square can be partitioned into any fixed number $n \geqslant 6$ of smaller squares.
4. **[False induction]** Inside the triangle there were marked n points. All these points were connected by non-intersecting segments with each other and with the verices of the triangle so that the triangle was partitioned into m smaller triangles (each marked point is a vertex of some triangles and cannot lie on the side of the triangles). Find all possible values of m.

Warm-up

5. From some point inside the convex polygon the altitudes to all sides are drawn. Prove that at least one foot of these altitudes lies on the side (not on the extension).
6. Prove that in each convex $2n$-gon there exists a diagonal which is not parallel to any side of this $2n$-gon.
7. On the sides of the convex quadrilateral as the diameters the four circles are constructed. Prove that these circles cover the whole quadrilateral.
8. Given n points on the plane such that the area of each triangle with vertices at these points doesn't exceed 1. Prove that all given points can be covered by the triangle of the area 4.
9. Prove that there is no convex polyhedron, all faces of which have different numbers of sides.
10. A polyline of length 200 is drawn inside the square with sidelength 1. Prove that there exists a line parallel to one of the sides of the square such that it intersects the polyline at at least 101 points.
11. Find all $n > 3$ such that it's possible to draw n points on the plane and connect some of them by segments so that each point is the endpoint of exactly 3 segments and no two segments intersect.

Warm-up

1. Each of the given 9 lines partitions the square into two quadrilaterals with the the ratio of it's areas 2 : 3. Prove that at least three of these lines pass through the same point.
2. Find the minimal possible number of points which it is enough to mark inside a convex n-gon so that each triangle which vertices are vertices of the n-gon contains inside at least one marked point.
3. There are $2n$ points in general position in the space. One drew $n^2 + 1$ segments between these points. Prove that at least one triangle occurred.

Classic problems pt. 1

4. A finite number of points is marked on the plane so that for any two marked points the line passing through them contains at least one other marked point. Prove that all marked points lie on the line.
5. Given n points on the plane. Not all of these points lie on the same line. Prove that there exist at least n different lines connecting these points.
6. From each vertex of the polygon the altitudes to all sides (for which this vertex is not an endpoint) are drawn. Prove that at least one foot of these altitudes lies on the side (not on the extension).

Training problems

1. A finite number of pairwise non-parallel lines is drawn on the plane so that for any two lines the point of their intersection lies on at least one other line. Prove that all lines passes through one point.
2. A convex polygon P lies inside another convex polygon Q. Prove that the perimeter of Q is greater thatn the perimeter of P.

Classic problems pt. 2

3. Given $n \geqslant 3$ points on the plane so that not all of them lie on the same line. Prove that there exists a circle passing through some three of these points and doesn't containing inside any other given point.
4. Prove that in every convex n-gon it's impossible to choose more than n diagonals such that any two of them have a common point.
5. Given a convex polygon $A_1A_2 \dots A_n$. Prove that the circumcircle of some triangle $A_iA_{i+1}A_{i+2}$ contains the whole polygon.

Challenging problems pt. 1

6.[1] The square is partitioned into several convex polygons with pairwise distinct numbers of sides. Prove that one of these polygons is a triangle.

[1] Sharygin Geometry Olympiad 2012

Classic problems pt. 3

1. Given $n \geqslant 3$ points on the plane so that not all of them lie on the same line. Prove that there exists a circle passing through some three of these points and doesn't containing inside any other given point.
2. Prove that in every convex n-gon it's impossible to choose more than n diagonals such that any two of them have a common point.
3. Given a convex polygon $A_1A_2\dots A_n$. Prove that the circumcircle of some triangle $A_iA_{i+1}A_{i+2}$ contains the whole polygon.

Challenging problems pt. 2

4.[2] Let n be a fixed odd positive integer. Consider $m+2$ distinct points $P_0, P_1, \dots, P_{m+1}$ on the Cartesian plane (m is a non-negative integer) such that:
   (a) $P_0 = (0,1)$, $P_{m+1} = (n+1, n)$ and for each i, $1 \leqslant i \leqslant m$ both coordinates of the point P_i are integers from $[1, n]$;
   (b) for each i, $0 \leqslant i \leqslant m$ the line P_iP_{i+1} is parallel to Ox-axis if i is even and parallel to Oy-axis if i is odd;
   (c) for each pair (i, j), $0 \leqslant i < j \leqslant m$ the segments P_iP_{i+1} and P_jP_{j+1} have at most one common point.

   Find the maximal possible m.

[2] APMO 2011

Challenging problems pt. 3

1.[3] Ann and Bob play the game making alternate moves. Initially Ann chooses the set V of 2020 vectors with zero sum and draw one of them on the board. Then, starting from Bob each player chooses vector from V which wasn't drawn before and draw it on the board. Each player must put the beginning of his vector at the end of the previous drawn vector. After all 2019 moves are performed the closed polyline is obtained. Ann wins if it is non-intersecting otherwise Bob wins. Determine, who has a winning strategy.

2.[4] Is it true that in any convex n-gon with $n > 3$, there exists a vertex and a diagonal passing through this vertex such that the angles of this diagonal with both sides adjacent to this vertex are acute?

3.[5] There are 2021 lines in a plane such that no 3 of them go through the same point. Turbo the snail can slide along the lines in the following fashion: she initially moves on one of the lines and continues moving on a given line until she reaches an intersection of 2 lines. At the intersection, she follows her journey on the other line turning left or right, alternating the direction she chooses at each intersection point she passes. Can it happen that she slides through a line segment for a second time in her journey but in the opposite directions she did for the first time?

4.[6] Given $n \geqslant 2$ lines on the plane. These lines divide the plane into parts and some parts are colored such that no two colored parts share a side. Prove the the number of colored parts doesn't exceed $\frac{1}{3}(n^2+n)$.

5.[7] The set L consists of 2020 lines in general position on the plane. Let's say that the line $\ell_1 \in L$ *bounds* another line $\ell_2 \in L$ if all points of intersection of ℓ_2 with other lines from L lies on one same half-plane with respect to ℓ_1. Prove that there exist lines ℓ and ℓ' in L such that ℓ bounds ℓ' and ℓ' doesn't bound ℓ.

[3] Iran 2018
[4] IGO 2019
[5] EGMO 2017
[6] USSR 1985
[7] SRMC 2006

Challenging problems pt. 3

1.[8] For $n \geqslant 5$ let $A_1A_2\ldots A_n$ be a convex n-gon for which all internal angles are obtuse. For each i, $1 \leqslant i \leqslant n$ let O_i be the circumcenter of the triangle $A_{i-1}A_iA_{i+1}$ (indices modulo n). Prove that the polyline $O_1O_2\ldots O_n$ in not a convex n-gon.

2.[9] Given a convex $2n$-gon H with pairwise parallel opposite sides.
   (a) Prove that there exists a pair of the opposite sides of H such that one can draw a line which is perpendicular to both of these sides and intersects both of them;
   (b) Prove that for each $n \geqslant 2$ there exists a $2n$-gon H for which there is exactly one pair of sides from (a).
   (c) Construct a non-convex hexagon for which the statement of (a) is false.

Convex hull and orderings

In many problems we cannot use pure extremal principle but it's useful to consider some specific orderings: direction on arbitrary line, Cartesian plane, convex hull, e.t.c. *Convex hull* of set M of points is the smallest convex set that contains M. If M is finite then it's convex hull is convex polygon with the vertices at M.

Warm-up

3. On the checkered plane n cells were marked. Prove that it's possible to choose at least $n/4$ of them such that they don't have common points.
4. Given n segments on the line such that any two of them have common point. Prove that all segments have common point.
5. Given n points on the plane such that any 4 of them are vertices of a convex polygon. Prove that all given points are vertices of a convex n-gon.
6. Is it possible to draw 2020 segments on the plane so that the endpoint of each segment lies inside some other segment?

Classic problems pt. 1

7. **(Helly's theorem)** Given 4 convex sets on the plane such that any three of them has a common point. Prove that all these sets have a common point.
8. Give an example of four non-convex sets on the plane for which the statement of Helly's theorem is false.

[8] Turkey 2016
[9] Belarus 2017

Test (Day 7) discussion

1.[10] Let ABC be an acute triangle. Let H be the foot on AB of the altitude through C. Suppose that $AH = 3BH$. Let M and N be the midpoints of the segments AB and AC respectively. Let P be a point such that $NP = NC$ and $CP = CB$ and such that B and P lie on the opposite sides of the line AC. Prove that $\angle APM = \angle PBA$.

2.[11] Prove that every positive integer can be expressed as a sum of powers of 3, 4 and 7 in such a way that the representation doesn't contain two powers with the same base and the same exponent. For example, $2 = 7^0 + 7^0$ and $22 = 3^2 + 3^2 + 4^1$ are not valid sums, but $2 = 3^0 + 7^0$ and $22 = 3^2 + 3^0 + 4^1 + 4^0 + 7^1$ are valid.

3.[12] A sequence $a_1, a_2, a_3, \ldots$ of positive integers satisfies $a_{n+1} = a_n + 2d(n)$ for every $n \geqslant 1$, where $d(n)$ is the number of different positive divisors of n. Is it possible that two consecutive terms of this sequence are perfect squares?

4.[13] Let n be a positive integer, and consider a square of dimensions $2^n \times 2^n$. We cover this square by a number of (at least 2) rectangles, without overlaps, and in such a way that every rectangle has integer dimension and a power of two as area. Show that two of the rectangles used must have the same dimensions. (Two rectangles are said to have the same dimensions if they have the same height and the same width, without rotating them.)

Classic problems pt. 2

5. **(Helly's theorem)** Given $n \geqslant 4$ convex sets on the plane such that any three of them has a common point. Prove that all these sets have a common point.
6. Given n points on the plane such that any three of them can be covered by a circle of radius R. Prove that all these points can be covered by a circle of radius R.
7. **(Young's theorem)** Given n points on the plane such that pairwise distances between them don't exceed 1. Prove that all these points can be covered by a circle of radius $\frac{1}{\sqrt{3}}$.

Exercises

8. Prove that in each convex polygon (except parallelogram) it's possible to choose 3 sides such that the triangle formed by intersections of the lines passing through these sides contains the whole polygon.
9. Given a convex polygon such that for any 3 sides exists a point O such that the feet of perpendiculars drawn from O to this sides lie inside the sides. Prove such point exists for all sides.
10. The plane is lightened by $n \geqslant 3$ lamps. Each lamp lightens a halfplane. Prove that we can leave 3 of them so that they lighten the whole plane.

[10] Netherlands 2016

[11] Olimpiada del Cono Sur 2018

[12] Bulgaria 2012

[13] Netherlands 2016

Exercises pt. 2

1. Given n points in the general position on the plane. The lines passing through each pair of points are drawn. Find the minimal possible number of pairwise nonparallel lines among all of them.
2. Given $2n+3$ points in the general position on the plane such that no 4 of them lie on a circle. Prove that it is always possible to choose 3 so that n of the remaining points lie inside their circumcircle and n other — outside it.
3. Prove that inside each convex heptagon (7-gon) exists a point which doesn't belong to any quadrilateral formed by 4 consecutive vertices.
4. During the day 100 people visited the library (each one visited it exactly once). Among any 3 visitors one there exist two which met at the library. Prove that is was possible to make 2 announcements during the day so that each visitor heard it.
5. Given n points in general position on the plane. Prove that it is always possible to connect them by non-selfintersecting polyline.

Challenging problems pt. 1

6.[14] There are $n \geqslant 4$ points on the plane, no three of which are colinear. Each pair of points is connected by a segment. Find the maximal possible number of segments having no intersection with other segments except its endpoints.

7. Given $n \geqslant 3$ lines in general position on the plane. Mark all their points of intersection. Call marked point A to be *internal* if for each line ℓ containing it on both there exist points B and C on ℓ such that A lies between B and C. Prove that the number of all internal points is at least $\frac{(n-2)(n-3)}{2}$.

[14] Turkey 2016

Warm-up

1. Given n points in general position on the plane. Is it always possible to connect them by a closed non-selfintersecting polyline.
2. Given $2n$ points in general position on the plane. n nonintersecting segments connect these points in pairs. Is it always possible to draw n more segments to connect all points into closed non-selfintersecting polyline.

Challenging problems pt. 2

3.[15] Let $N \geqslant 2$ be an integer. $N(N+1)$ soccer players, no two of the same height, stand in a row i some order. Coach Ralph wants to remove $N(N-1)$ people from this row so that in the remaining row of $2N$ players, no one stands between two tallest ones, no one stands between the third and the fourth tallest ones, ..., and finally no one stands between the two shortest ones. Show that it is always possible.

4.[16] Let $n > 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps. **a)** Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point. **b)** Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

5.[17] A diagonal of a regular 2006-gon is called odd if its endpoints divide the boundary into two parts, each composed of an odd number of sides. Sides are also regarded as odd diagonals. Suppose the 2006-gon has been dissected into triangles by 2003 nonintersecting diagonals. Find the maximum possible number of isosceles triangles with two odd sides.

NT homework from November

6.[18] Determine all pairs (x, y) of integers such that $1 + 2^x + 2^{2x+1} = y^2$.

[15] IMO 2017
[16] IMO 2016
[17] IMO 2006
[18] IMO 2006

Test (Day 11) discussion

1.[19] The square 20×20 was cut along the grid lines into several smaller (not necessarily different) squares. For each square it's sidelength was written on the board. Determine the maximal possible amount of different numbers on the board.

2.[20] Let $n > 2$ be an integer such that n does not divide any element of the set $\{2^n - 1, 3^n - 1, \ldots, (n-1)^n - 1\}$. **(a)** Prove that n is a square-free number; **(b)** Does it necessarily follow than n is a prime number?

3.[21] Determine all pairs of polynomials (P, Q) with real coefficients satisfying

$$P(x + Q(y)) = Q(x + P(y))$$

for all real numbers x and y.

4.[22] Point M lies on the side AB of circumscribed quadrilateral $ABCD$. Points I_1, I_2 and I_3 are incenters of $\triangle MBC$, $\triangle MCD$ and $\triangle MDA$. Prove that the points M, I_1, I_2 and I_3 lie on a circle.

Exercises

5. Let ABC be a non-isosceles triangle ($AB \neq BC$). Point B_1 is the midpoint of arc AC containing B of the circumcircle of ABC. Points A_0 and C_0 are marked on the segments BC and BA. Prove that points B, B_1, C_0 and A_0 are concyclic iff $AC_0 = CA_0$. Consider all possible positions of points C_0 and A_0 on the lines BC and BA.
6. Let ABC be a non-isosceles triangle ($AB \neq BC$). Denote the incenter of triangle ABC by I. Points A_0 and C_0 are marked on the segments BC and BA. Prove that points B, I, C_0 and A_0 are concyclic iff $AC_0 + CA_0 = AC$. Consider all possible positions of points C_0 and A_0 on the lines BC and BA.
7. Let ABC be a non-isosceles triangle ($AB \neq BC$) and M be the midpoint of AC. Denote by I_A and I_C the incenters of the triangles MAB and MCB. Point B_1 is the midpoint of arc AC containing B of the circumcircle of ABC. Points A_0 and C_0 are marked on the segments BC and BA. Prove that points B, B_1, I_A and I_C are concyclic.

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[19] Belarus 2020

[20] Romania 2016

[21] MEMO 2017

[22] Bulgaria 2018

Warmup equations

1. Find all pairs of integers (x, y) such that $y^2(x^2+y^2-2xy-x-y) = (x+y)^2(x-y)$.
2.[23] Find all $n \in \mathbb{N}$ and prime numbers p, q such that $n^3 = p^3 + 2p^2q + 2pq^2 + q^3$.

Squeeze with squares

3. Find all pairs of integers (m, n) such that $n^{n-1} = 4m^2 + 2m + 3$.
4. Distinct prime numbers p, q, r satisfy $rp^3 + p^2 + p = 2rq^2 + q^2 + q$. Find pqr.
5. Show that 2017^{2018} cannot be represented as the sum of two cubes of positive integers.
6.[24] **a)** Do there exist positive integers a and b such that $a \cdot 2^n + b \cdot 5^n$ is a square for each positive integer n?
   b) Do there exist positive integers a, b and c such that $a \cdot 2^n + b \cdot 5^n + c$ is a square for each positive integer n?

Exercises

7. **a)** Let $n^2 + 1 = ab$ for positive integers n, a, b, $a > b$. Prove that $a - b \geqslant \sqrt{4n-3}$.
   b) Determine all positive integers n for which the inequality from (a) can turn to the equality.
8. Find all pairs (p, q) of prime numbers such that $p^8 - p^4 = q^5 - q$.

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[23] Ukraine 2018

[24] Belarus 2006

Homework discussion

1. Find all positive integers m and n such that $n^5 + n^4 = 7^m - 1$.
2. For every positive integer n by $d(n)$ denote the number of all divisors of n and by $s(n)$ denote the number of all divisors d of n such that $d + 1$ divides $n + 1$. Consider the set $M = \{2s(1) - d(1), 2s(2) - d(2), 2s(3) - d(3), \ldots\}$. Prove that M has maximal elements and find it's value.

3.[25] Given a polynomial $P(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_2x^2 + a_0$, $m \geqslant 2$, with positive integer coefficients. Consider the sequence (b_n) denoted by $b_1 = a_0$ and $b_{n+1} = P(b_n)$ for $n \geqslant 1$. Prove that for each $n \geqslant 2$ the number b_n has prime divisor which doesn't divide the product $b_1 b_2 \ldots b_{n-1}$.

Chinese Remainder Theorem.

For any pairwise co-prime integers $n_1, n_2, \ldots, n_k$ and k-tuple of integers $x_1, x_2, \ldots, x_k$ there exists a unique integer x between 0 and $n_1 \cdot n_2 \cdot \ldots \cdot n_k - 1$ such that $x \equiv x_i \pmod{n_i}$, $i = \overline{1, k}$. There are two basic ways to prove this fact: non-constructive and constructive:

1. (Counting) Each number x define exactly one k-tuple of remainders, each k-tuple $x_1, x_2, \ldots, x_k$ corresponds to not more than one x between 0 and $n_1 \cdot n_2 \cdot \ldots \cdot n_k - 1$, and the amounts of x and of k-tuples are equal. Though this prove is easy, it doesn't give any information on x.

2. (Interpolation) There is a solution in the form $x = a_1 \cdot \frac{N}{n_1} + a_2 \cdot \frac{N}{n_2} + \ldots + a_k \cdot \frac{N}{n_k}$ where $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$ and $a_1, a_2, \ldots, a_k$ are integers.

CRT is usually used when we need to construct an integer with required remainders.

Exercises

4. Do there exist positive integers a, b, c, d such that $a^2 < b^3 < c^4 < d^5$ is an arithmetic progression?
5. The Director wants to arrange students in identical squares on holidays. He doesn't know how many of them are ill and will not come. But he knows that the amount of ill students is between 1 and 37. Prove that there can be such fixed amount of all students in the school that Director will manage in any case.
6. Prove that for any positive integer n there exist a non-constant arithmetic progression consisting of n powers (with exponent greater than one) of integers. Prove it both by induction and providing an explicit construction.

Challenging problems

7.[26] Find all positive integers $n > 1$ such that there exist positive integers $b_1, b_2, \ldots, b_n$ (some of them can be equal but not all) such that for all positive integers k the product $(b_1 + k)(b_2 + k) \ldots (b_n + k)$ is a power of positive integer? (The exponent may depend on k but must exceed 1.)

[25] Bulgaria 2018
[26] Russia 2008

Homework discussion

1.[27] Consider polynomial $P(n) = n^2 + n + 1$. For any positive integers a and b, the set $\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$ is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. **a)** Determine the smallest size of a fragrant set. **b)** Prove that a there exists a fragrant set of each size greater than minimal.

Definitions

Function $f\colon \mathbb{N} \to \mathbb{Z}$ is multiplicative if $\gcd(m, n) = 1$ implies $f(mn) = f(m)f(n)$. The study of such functions is reduced to the study of their values for powers of primes. Trivial examples: $d(n)$, $\sigma(n)$ and more generally $\sigma_x = \sum_{d|n} d^x$. One more example: $\varphi(n)$. The minimal positive integer d such that $a^d \equiv 1 \pmod{n}$ is called the order of a modulo n ($\gcd(a, n) = 1$) and it is denoted by $\operatorname{ord}_n(a)$. If $a^d \equiv 1 \pmod{n}$ then $\operatorname{ord}_n(a) \mid d$.

Useful facts

2. Prove for all positive integers n the equality $\sum\limits_{d|n} \varphi(d) = n$.
3. Let $h(x)$ be a polynomial with integer coefficients and let p be a prime number. Prove that the congruence $h(x) \equiv 0 \pmod{p}$ has at most $\deg h$ solutions.
4. Let $\gcd(a, b) = 1$ and p be a prime number. Then all prime divisors (except p) of the fraction $\frac{a^p - b^p}{a-b}$ are congruent to 1 modulo p.

Primitive roots

A number a such that $\operatorname{ord}_n(a) = \varphi(n)$ is called primitive root modulo n. The definition implies that every co-prime with n number is congruent to some power of a. Clearly primitive roots exist modulo 2, 4. For prime modules existence of primitive roots follows from facts 2 and 3.

5. Prove that primitive roots exist modulo p^k where p is an odd prime.
6. Prove that primitive roots exist modulo $2p^k$ where p is an odd prime.
7. Prove that $n^{2^{k-2}} \equiv 1 \pmod{2^k}$ for any positive integer $k \geqslant 3$ and odd n holds. Whence there is no primitive root modulo 2^k, $k > 2$.
8. Suppose n doesn't equal to 2^k, p^k or $2p^k$ where p is an odd prime. Prove that $\operatorname{ord}_n(a) \mid \frac{\varphi(n)}{2}$ for each a, $\gcd(a, n) = 1$. Hence there's no primitive root modulo n.

[27] IMO 2016

Problems

1.[28] Ann and Bob play the game making an alternative moves: a move consists of choosing the index $i \in \{0, 1, 2, \ldots, p-1\}$ which wasn't chosen before and a digit a_i ($p > 2$ is fixed prime). The game ends when all indices are chosen. Ann makes the first move and she wins if the number $M = a_0 + 10 \cdot a_1 + 10^2 \cdot a_2 + \ldots + 10^{p-1} a_{p-1}$ is divisible by p otherwise Bob wins. Prove that Ann has a winning strategy.

2. Prove that there is no integer $n > 1$ such that $n \mid 2^n - 1$.

3. Does there exist an integer $n > 1$ such that $n \mid 2^{n-1} + 1$?

4.[29] Find all positive integers n for which the fraction $\frac{n^{3n-2} - 3n + 1}{3n-2}$ is an integer.

5.[30] Let (a_n) be a sequence of positive integers and let (p_n) be a sequence of prime numbers such that for any $n \geqslant 1$ we have $p_n | a_n$ and $a_{n+1} = \frac{a_n}{p_n}(p_n^{1009} - 1)$. Show that this sequence contains a_n which is a multiple of 2018.

6.[31] Prove that there exist infinitely many positive integers n such that at least one of the numbers $2^{2^n} + 1$ and $2018^{2^n} + 1$ is composite.

7.[32] Find all pairs (p, n) of prime number p and positive integer n, $n \neq p$, such that the fraction $\frac{n^p+1}{p^n+1}$ is integer.

[28] IMO 2017 Shortlist N2

[29] Belarus 2018

[30] Mongolia 2018

[31] Serbia 2018

[32] APMO 2012

Homework discussion

1.[33] Let $n \geqslant 2$ be a positive integer. Prove that $n \mid (1^{n-1} + 2^{n-1} + \ldots + (n-1)^{n-1}) + 1$ if and only if for each prime divisor p of n both $p \mid (\frac{n}{p} - 1)$ and $(p-1) \mid (\frac{n}{p} - 1)$.

2.[34] Show that for each square-free integer $n > 1$ there exist prime divisor $p \mid n$ and integer number m such that $n \mid p^2 + p \cdot m^p$.

Lifting the exponent lemma

Suppose that $p > 2$ is a prime number and a and b are integers such that $p \not| a$, $p \not| b$ and $p \mid (a - b)$.

3. Prove that $v_p(a^n - b^n) = v_p(a - b) + v_p(n)$ for any positive integer n.

If n is odd, this lemma also shows how to count $v_p(a^n + b^n)$. And in if $p = 2$ we need to consider special cases: if $4 \mid x - y$ then $v_2(a^n - b^n) = v_2(a - b) + v_2(n)$; if $2 \mid n$ then $v_2(a^n - b^n) = v_2(a - b) + v_2(a + b) + v_2(n) - 1$.

Exercises pt. 1

4. Find all positive integers n for which there exist positive integers x, y and $k > 1$ such that $\gcd(x, y) = 1$ and $x^k + y^k = 3^n$.

Homework discussion using LTE

5.[35] Find all positive integers n such that $2n^2 \mid a^n - 1$ for any positive integer a coprime with n.

6.[36] Find all prime numbers p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$.

[33] SRMC 2005
[34] Turkey 2015
[35] Turkey 2015
[36] Balkan MO 2018

Exercises pt. 2

1. Find $v_3(2^{3^n}+1)$ for positive integer n.
2. Prove that n^7+7 is not a perfect square for any integer n.
3. Find all positive integers n for which n^2 divides 2^n+1.

LEFT AS HOMEWORK

4. Find all primes p and positive integers (x, y) such that $x^{p-1}+y$ and $y^{p-1}+x$ are powers of p.
5. Find all positive integers k, n such that $k! = (2^n-1)(2^n-2)(2^n-4)\dots(2^n-2^{n-1})$.
6. Find all triples (x, y, z) of positive integers such that $2^x+1 = 7^y+2^z$.
7. Prove that for each positive integer m one can find m consecutive positive integers n such that $(1^3+2018^3)\cdot(2^3+2018^m)\cdot\ldots\cdot(n^3+2018^3)$ is not a perfect power.
8. Find all triples (a, b, k), $k \geqslant 2$, of positive integers such that $(a^k+b)(b^k+a)$ is a power of 2.