Email training, N5

October 9-16

Problem 5.1. Prove the inequality

$$\sqrt{a+1}+\sqrt{2a-3}+\sqrt{50-3a}\leq 12.$$

Problem 5.2. Let the parabola $y=x^2+px+q$ is given, which intersects coordinate axes in 3 different points. Consider the circumcircle of the triangle having vertices these 3 points. Prove that there is a point that belongs to that circle, regardless of values p and q. Find that point.

Problem 5.3. Find all integer polynomials P for which $(x^2+6x+10)P^2(x)-1$ is the square of an integer polynomial.

Problem 5.4. a) Find the minimum number of elements that must be deleted from the set $\{1,2\ldots,2018\}$ such that the set of the remaining elements does not contain two elements together with their product. b) Does there exist, for any k, an arithmetic progression with k terms in the infinite sequence

$$\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n},\ldots.$$

Problem 5.5. Prove that not all zeros of a polynomial of the form $x^n+2nx^{n-1}+2n^2x^{n-2}+\ldots$ can be real.

Problem 5.6. Let the polynomial $P(x)=a_{2n}x^{2n}+a_{2n-1}x^{2n-1}+\ldots+a_1x+a_0$ with all coefficients $a_i\in[100,101]$ is given. Find the minimal possible value of n for which $P(x)$ has a root.

Problem 5.7. -

The chord AC and BD of a circle with centre O intersects at the point K. The circumcenters of triangles AKB and CKD are M and N respectively. Prove that $OM=KN$.

Problem 5.8. -

A convex quadrangle ABCD is inscribed in a circle with the center O. The angles ∠ AOB, ∠BOC, ∠COD and ∠DOA, taken in some order, are of the same size as the angles of quadrangle ABCD. Prove that ABCD is a square.

Solution submission deadline October 16, 2022