Level 2 E-training, week 5

Due to 23:59, Friday, 9 October 2020

Problem 1. Determine all pairs of (a, b) of non negative integers such that:

$$\frac{a+b}{2} - \sqrt{ab} \;=\; 1$$

Problem 2. Let $a_1, a_2, a_3, ...$ be a sequence of positive real numbers that satisfies $a_1 = 1$ and $a_{n+1}^2 + a_{n+1} = a_n$ for every natural number n. Prove that $a_n \geq \frac{1}{n}$ for every natural number n.

Problem 3. In a soccer league with 2020 teams every two team have played exactly once and no game has ended in a draw. The participating teams are ordered first by their points (3 points for a win, 0 points for a loss) then by their goal difference (goals scored minus goals against) in a normal soccer table. Suppose that no two teams share the same total points and the same goal difference. Is it possible for the goal difference in such table to be nondecreasing from the top to the bottom?

Problem 4. Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.

Problem 5. Let a, b, c be positive integers. Consider the set G consisting of all of their possible geometric means (i.e, $G = \{a, b, c, \sqrt{ab}, \sqrt{bc}, \sqrt{ca}, \sqrt[3]{abc}\}$, notice that we don't assume that a, b, c are distinct, so G could contain less than 7 elements). Suppose that

$$\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2} \in G$$

Show that $a = b = c$.

Problem 6. Let $x \geq y \geq z > 0$ with $x + y + z = 1$. Prove that

$$(x + 2y + 3z)(x^2 + y^2 + z^2) < 1$$

Can we replace the "1" in the RHS by a better constant?

Problem 7. Consider an integer $n > 1$, and a set $\mathcal{S}$ of n points in the plane such that the distance between any two different points in $\mathcal{S}$ is at least 1. Show that there is a line ℓ separating $\mathcal{S}$ such that the distance from any point of $\mathcal{S}$ to ℓ is at least $\frac{1}{2n}$.

(A line ℓ *separates* a set of points $\mathcal{S}$ if some segment joining two points in $\mathcal{S}$ crosses ℓ.)

Problem 8. Given four distinct points A, B, E, P so that P is the midpoint of AE and B lies on the segment AP. Let k_1 and k_2 be two circles passing through A and B. Let t_1 and t_2 be the tangents of k_1 and k_2, respectively, at A. Let C be the second intersection of t_2 and k_1 and Q be the 2^{nd} intersection of t_2 and the circumcircle of ECB. Let D be the second intersection of t_1 and k_2 and R be the 2^{nd} intersection of t_1 and the circumcircle of BDE. Prove that P, Q, R are collinear.