Classical combinatorics

1. Find all functions $f: N \to N$ satisfying to the condition $3f(n) - 2f\big(f(n)\big) = n$.

2. Each of n parts of an encyclopedia are placed either on their position, or on the neighboring positions. Find the number of possible permutations.

3. There are n lines on a plane. Into how many parts they may divide the plane (find the maximal number)?

4. There are n infinite “angles” on a plane. Into how many parts they may divide the plane (find the maximal number)?

5. There are n lines on a plane. Into how many bounded parts they may divide the plane (find the maximal number)?

6. Find a closed-form formula for the sequence x_n is defined as follows: $x_1 = 1$, $x_2 = 3$, and

   a) $x_{n+2} = 3x_{n+1} + 2x_n$
   b) $x_{n+2} = 3x_{n+1} + 2x_n + 2$

7. Find all functions $f: R^+ \to R^+$ satisfying to the relation $f\big(f(x)\big) = -f(x) + 6x$.

8. Find all functions $f: N \to N$ satisfying to the relation $f\Big(f\big(f(n)\big)\Big) + f\big(f(n)\big) + f(n) = 3n$.

9. Let n be a positive integer. Harry has n coins lined up on his desk, each showing heads or tails. He repeatedly does the following operation: if there are k coins showing heads and $k > 0$, then he flips the k-th coin over; otherwise, he stops the process. Write $l(C)$ for the number of steps needed before all coins show T. Show that this number $l(C)$ is finite and determine its average value over all 2^n possible initial configurations C.

10. Suppose that a sequence of positive numbers $a_1, a_2, \ldots$ satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

Prove that $a_1 + \cdots + a_n \geq n$ for every $n \geq 2$.

11. For a finite set A of positive integers, we call a partition of A into two disjoint nonempty subsets A_1 and A_2 good if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

12. Find the number of lattice paths from $(0,0)$ to (n, n), which are below the main diagonal.

13. Find the generating series of the following polynomials:

    a) $(1,1,1,1,1,0,0,0,0,0,0,0 \dots)$
    b) $(1,1,0,0,1,1,0,0,1,1,0,0,1, \dots)$
    c) $(1,1,1,1,1,1,1,1,1,1,1,1,1 \dots)$

14. Find the coefficient of x^n is the following characteristic functions:
    a) $\frac{1}{1+3x}$
    b) $\frac{1}{(1+4x)^2}$
    c) $\frac{1}{(1-x)^2(1+x)}$

15. The sequence x_n is defined as follows: $x_1 = 1$, $x_{2n} = 2x_n - 1$ and $x_{2n+1} = 2x_n + 1$. Find a closed-form formula for x_n.

16. The sequence x_n is defined as follows: $x_1 = \alpha$, $x_{2n} = 2x_n + \beta$ and $x_{2n+1} = 2x_n + \gamma$. Find a closed-form formula for x_n.

17. Lukas numbers are defined in the following way: $L_0 = 2,\ L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$. Find a closed form formula for the Lucas numbers.

18. Prove the following relations between Lucas and Fibonacci ($F_0 = 0, F_1 = 1$) numbers:

    a) $L_n = F_{n-1} + F_{n+1}$
    b) $5F_n = L_{n-1} + L_{n+1}$

19. Let A and E be two opposite vertices of a regular octagon $ABCDEFGH$. A frog starts at vertex A and jumps along one of the edges at every move. Once the frog reaches E, it stays there. Determine the number of possible paths from A to E of length n (i.e., sequences of points, starting at A and ending at E, such that all points except for the last one are different from E and any two consecutive points of the sequence are adjacent).

20. A coin is tossed until we obtain a sequence of an odd number of "heads", followed by one "tail", for the first time. How many possible sequences of n tosses are there? (One possible sequence of length 13 would be $HHHHTTHHTHHHT$).

21. How many 1000-digit numbers are there, such that all digits are odd, and the difference between any two subsequent digits is exactly 2?

22. How many sequences of length 1997 formed by the letters A, B, C are there, for which the number of A's and the number of C's are both odd?

23. A sequence of n points is given on a line. How many possibilities are there to colour them with two colours (red and blue) in such a way that for any subsequence of consecutive points the number of red points in this subsequence differs from the number of blue points by at most 2?

24. Twelve people sit around a round table. In how many ways can they shake hands in six pairs, if no two of the pairs may cross?

25. For a positive integer n, let p_n be the number of words of length n using only the letters A and B which do not contain $AAAA$ or BBB as a subword. Determine

$$\frac{p_{2004} - p_{2002} - p_{1999}}{p_{2000} + p_{2001}}$$