Intensive Training
Geometry

Day 10 and 11
24 April 2021

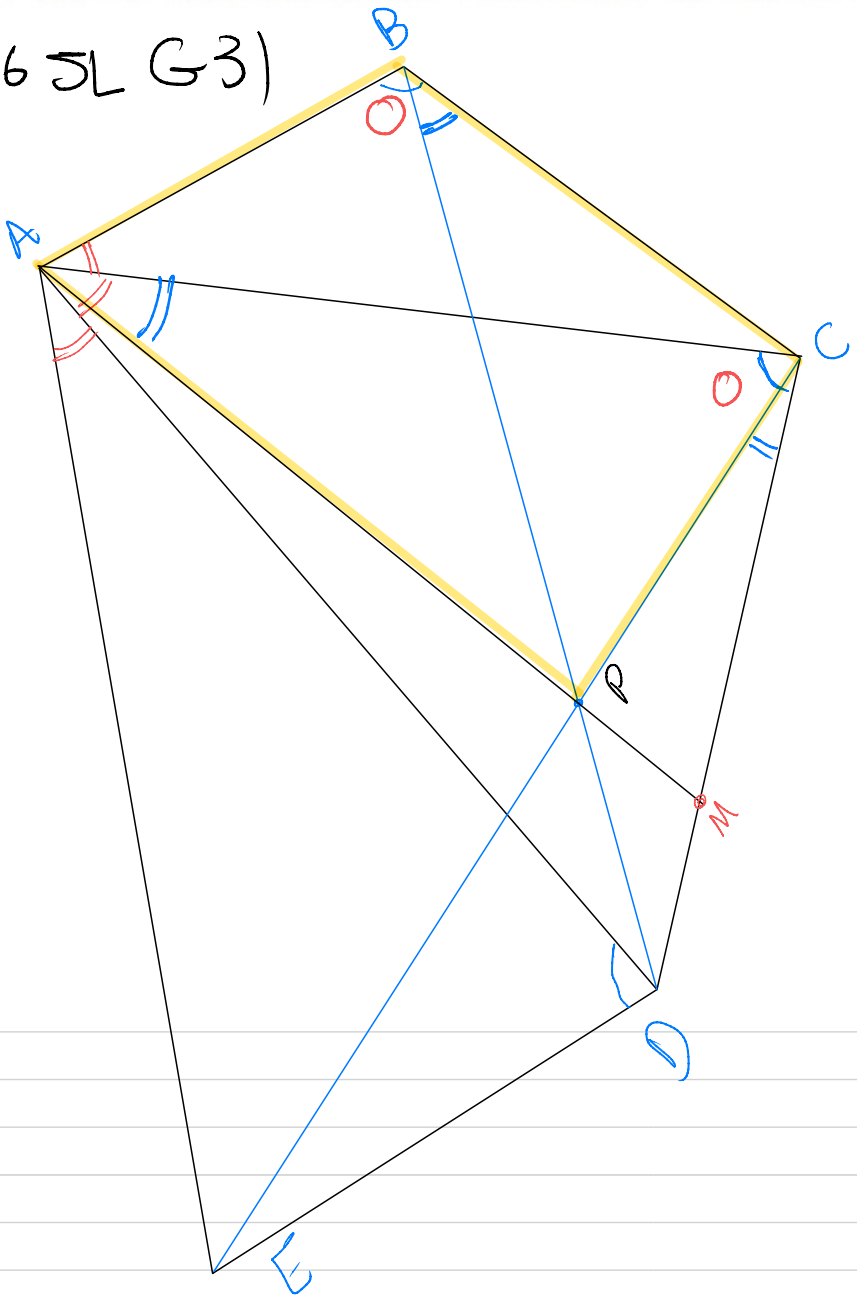
Includes solutions for:
IMO 2006 Shortlist G3
JBMO Shortlist 2019 G3
JBMO Shortlist 2019 G5
BMO 2015 P2
IMO 2014 P4

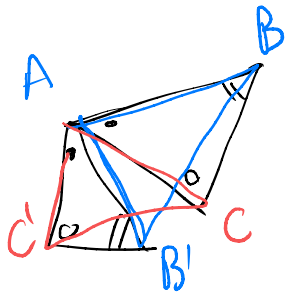
Problem 4. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

(IMO 2006 SL G3)





$\triangle ABC \sim \triangle ACD \sim \triangle ADE$

By Spiral Similarity lemma on $\triangle ABC$, $\triangle ADE$:

$\Rightarrow \triangle ABD \sim \triangle ACE$

$\Rightarrow \angle ABD = \angle ACE \Rightarrow ABCP$ cyclic

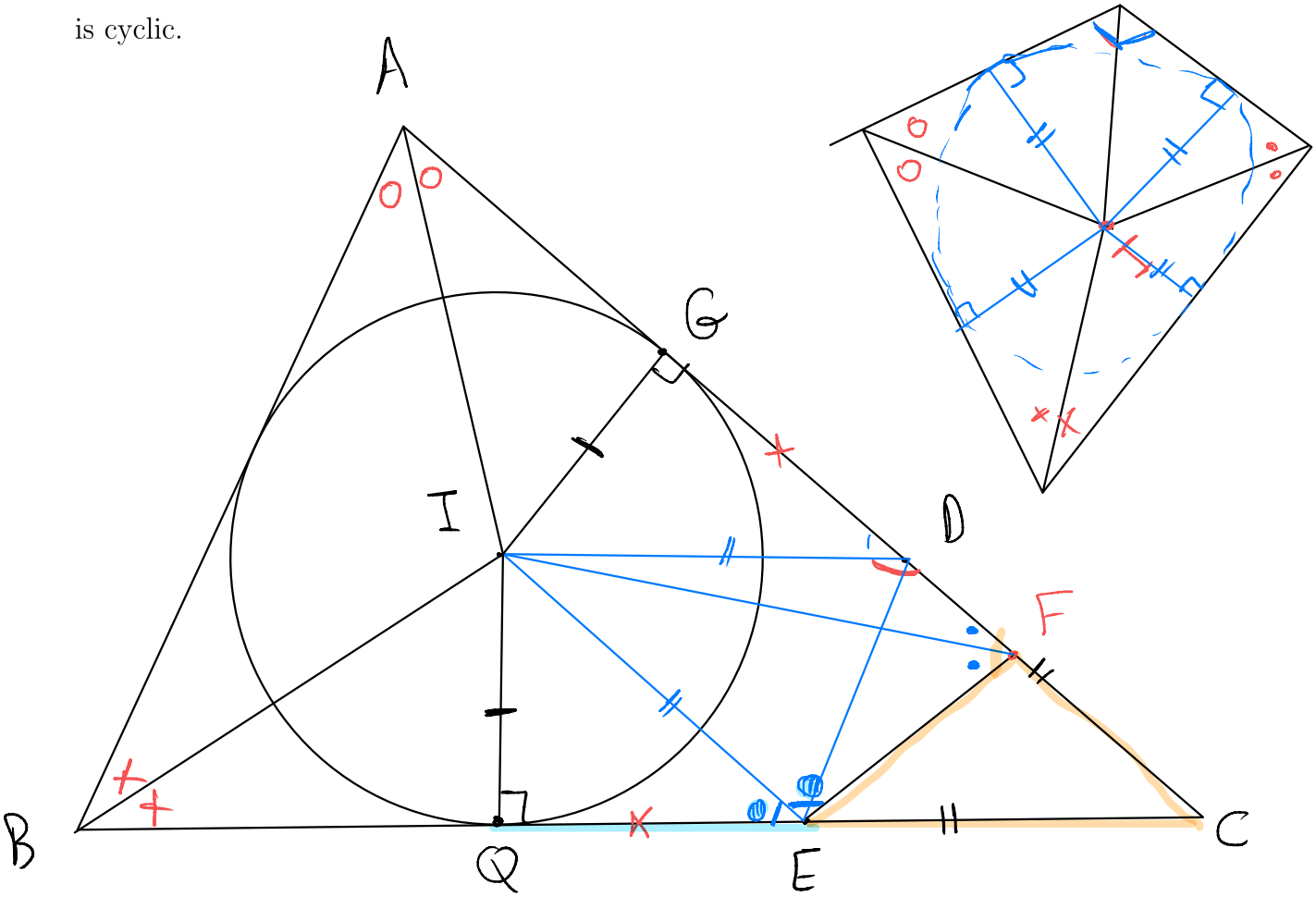
$\Rightarrow \angle ECD = \angle ACD - \angle ACP$

$= \angle ABC - \angle ABP = \angle DBC = \angle PBC = \angle PAC$

Therefore, MC is tangent to $\triangle ACP \Rightarrow MC^2 = MP \cdot MA$

Similarly, $MD^2 = MP \cdot MA \Rightarrow M$ is the midpoint of DC $\square$

1. Let ABC be a triangle with incenter I. The points D and E lie on the segments CA and BC respectively, such that $CD = CE$. Let F be a point on the segment CD. Prove that the quadrilateral $ABEF$ is circumscribable if and only if the quadrilateral $DIEF$ is cyclic.



if DIEF is acyclic:

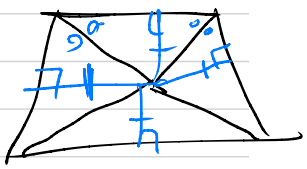
$IQ = IG$, $GD = QE \Rightarrow ID = IE$

In $IDFE$: $\angle DFI = \angle IFE$

CI bisects C, FI bisects $\angle AFE$

$\Rightarrow$ I excenter $\triangle FCE$

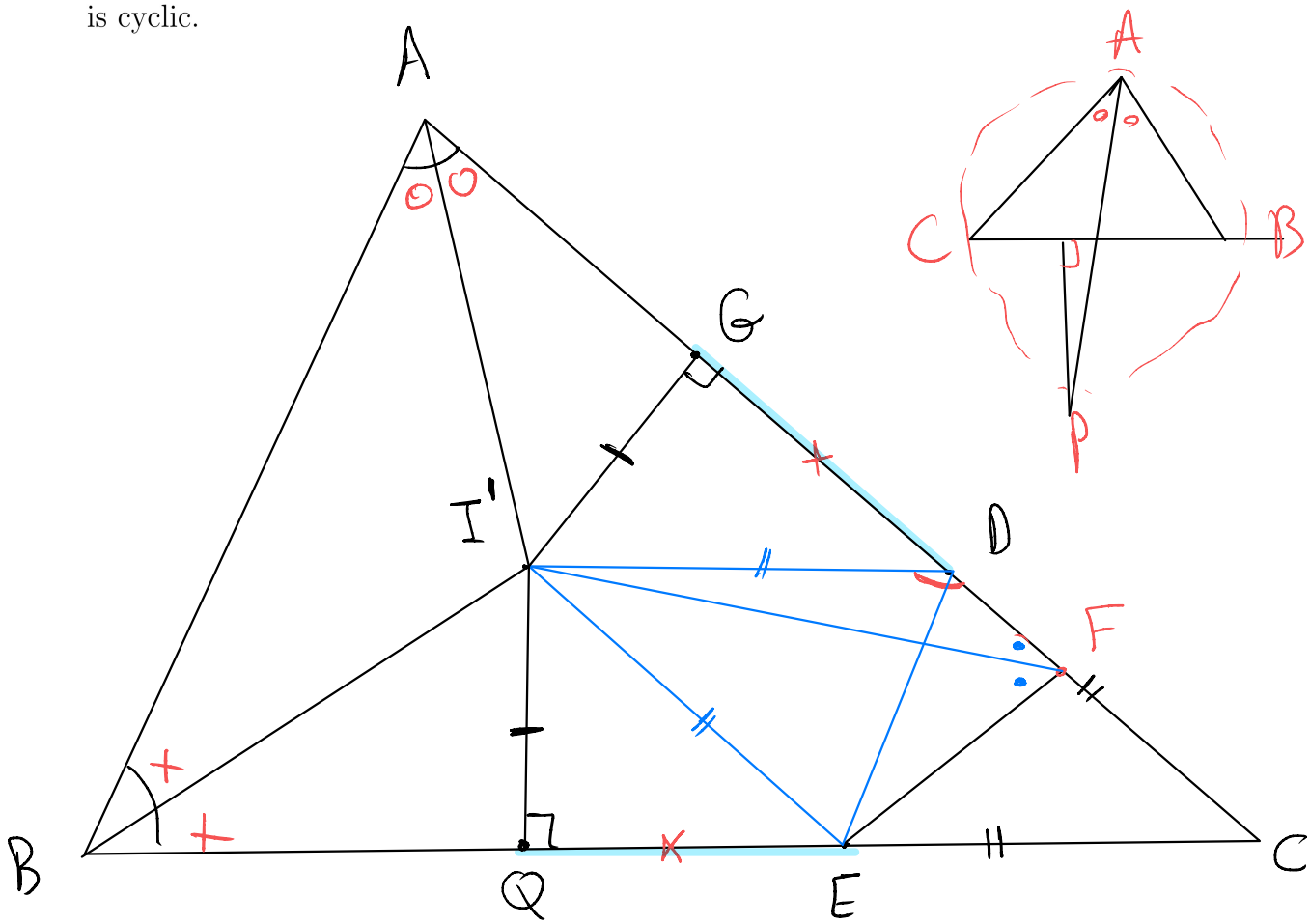
$\Rightarrow$ I bisects $\angle BEF$



I bisects $\angle A, \angle B, \angle AFE, \angle FEB$

$\Rightarrow$ $ABEF$ circumscribable $\square$

1. Let ABC be a triangle with incenter I. The points D and E lie on the segments CA and BC respectively, such that $CD = CE$. Let F be a point on the segment CD. Prove that the quadrilateral $ABEF$ is circumscribable if and only if the quadrilateral $DIEF$ is cyclic.



If ABEF is circumscribable:

Let I' be the center of the circle

$\Rightarrow \quad I'G = I'Q \Rightarrow I'D = I'E$

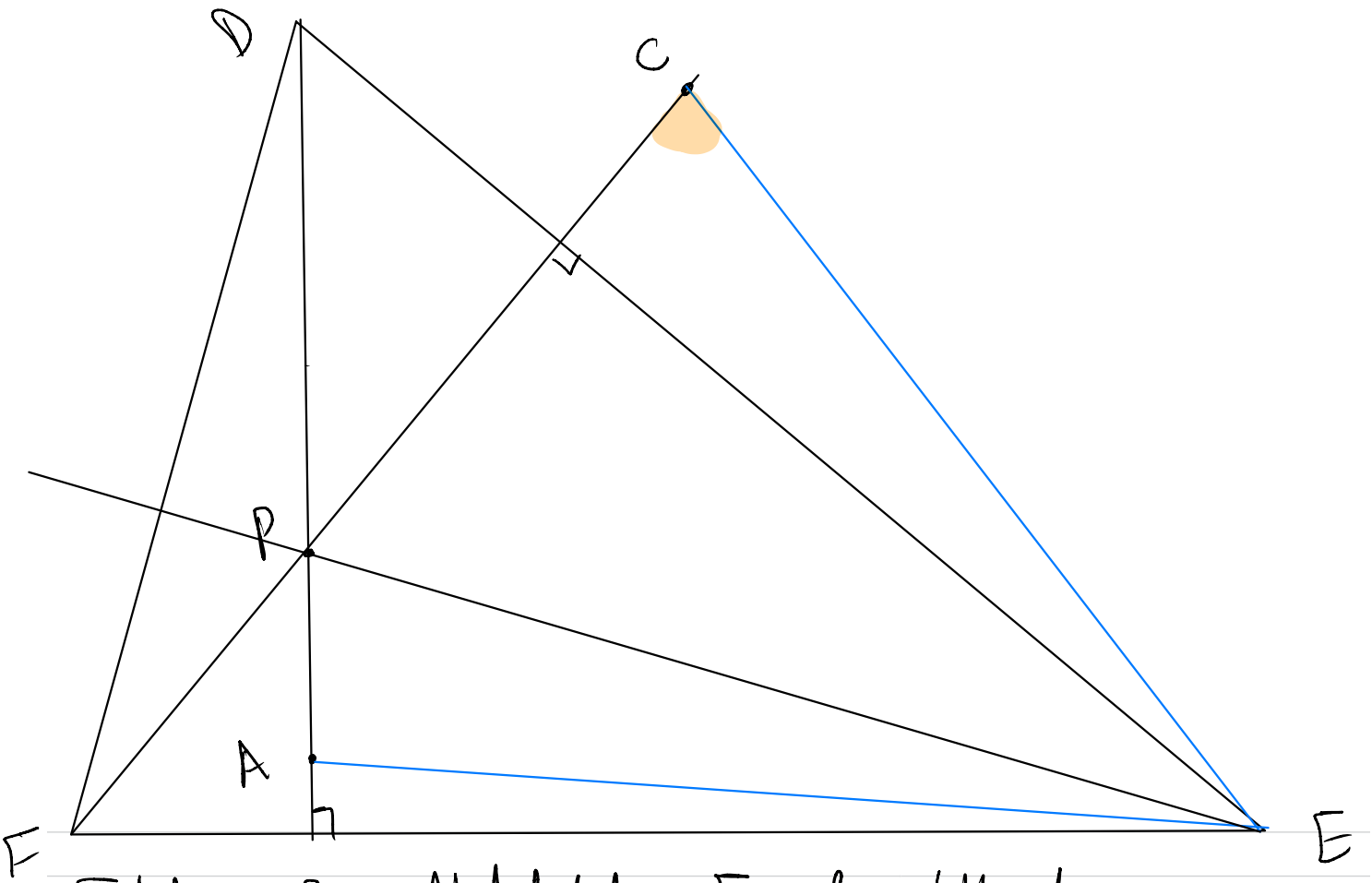
Also FI bisects $\angle AFE \Rightarrow$

$$\left.\begin{array}{l} FI \text{ bisects } \angle DFE \\ I'D = I'E \end{array}\right\} \xrightarrow[\text{lemma}]{\text{by}} I'DFE \text{ cyclic}$$

Also $$\left.\begin{array}{l} I'A \text{ bisects } \angle A \\ I'B \text{ bisects } \angle B \end{array}\right\} \Rightarrow I' = I \Rightarrow IDFE \text{ cyclic}$$

$\square$

2. Let P be a point in the interior of a triangle ABC. The lines AP, BP and CP intersect again the circumcircles of the triangles PBC, PCA and PAB at D, E and F respectively. Prove that P is the orthocenter of the triangle DEF if and only if P is the incenter of the triangle ABC.



Solution of : Abdulelah, Eyad and Haider

Let P be a point in the interior of a triangle ABC.
The lines AP, BP and CP intersect again the circumcircles of the triangles PBC, PCA and PAB at D, E and F respectively . Prove that P is the orthocenter of the triangle DEFif and only if P is the incenter of the triangle ABC.

If P is the incenter of ABC, then $\angle BPD = \angle ABP + \angle BAP = \frac{\hat{A}}{2} + \frac{\hat{B}}{2}$

, and

$$\angle BDP = \angle BCP = \frac{\hat{C}}{2}$$

. From triangle BDP, it follows that $\angle PBD = 90\circ$, i.e. that EB is one of the altitudes of the triangle DEF. Similarly, AD and CF are altitudes, which means that P is the orhocenter of DEF.

If P is the orthocenter of DEF.
Then the quadrilaterals $B'PA'F$, $C'PB'D$, and $A'PC'E$ are cyclic, which means that
$\angle B'FA' = 180\circ - \angle B'PA' = 180\circ - \angle BPA = \angle BFA$.
Similarly $\angle C'DB' = \angle CDB$, *and* $\angle A'EC' = \angle AEC$.

If $B \in Ext(\Delta FPD)$, *then* $A \in Int(\Delta EPF)$, $C \in Ext(\Delta DPE)$, *and thus* $B \in Int(\Delta FPD)$, *contradiction.*

- *If* $B \in Int(\Delta FPD)$, *then* $A \in Ext(\Delta EPF)$, $C \in Int(\Delta DPE)$, *and thus* $B \in Ext(\Delta FPD)$, *contradiction*

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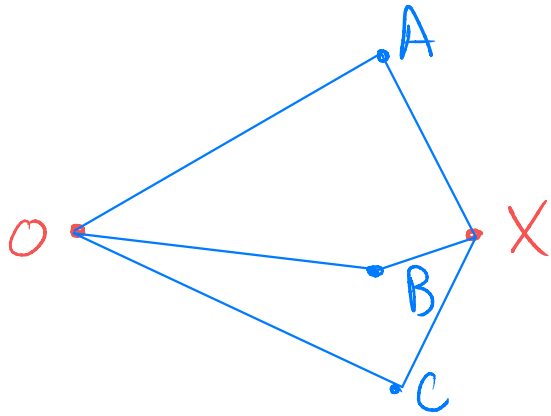
Lemma: If ω is a circle and X is a point such that

$$\begin{cases} \rho_\omega(A) = AX^2 \\ \rho_\omega(B) = BX^2 \\ \rho_\omega(C) = CX^2 \end{cases}$$

Then A, B, C are collinear (on the radical axis of $(X), \omega$)



$$\rho_\omega(A) = AO^2 - R^2 \implies AO^2 - AX^2 = R^2$$
$$\rho_\omega(B) = BO^2 - R^2 \implies BO^2 - BX^2 = R^2$$
$$\rho_\omega(C) = CO^2 - R^2 \implies CO^2 - CX^2 = R^2$$

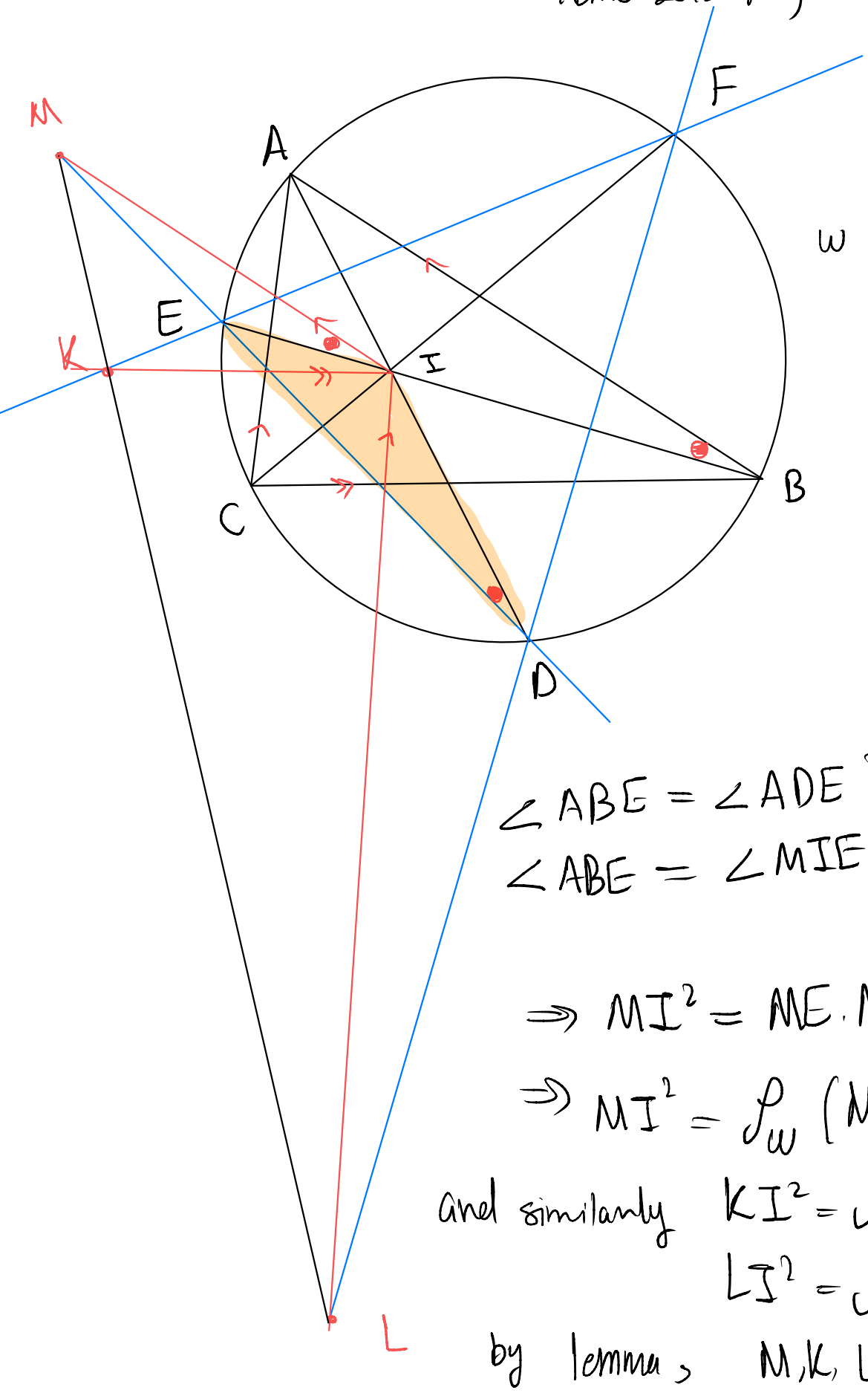
Then $AO^2 - AX^2 = BO^2 - BX^2$

$\implies$ by Cannot's lemma $AB \perp OX$

Similarly, $BC \perp OX$, $CA \perp OX$

Therefore, A, B, C are collinear

4. Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear. (BMO 2015 P2)



$$\left.\begin{array}{l}\angle ABE = \angle ADE \\ \angle ABE = \angle MIE\end{array}\right\} \Rightarrow MI \text{ tangent to } \triangle DEI$$

$$\Rightarrow MI^2 = ME \cdot MD$$

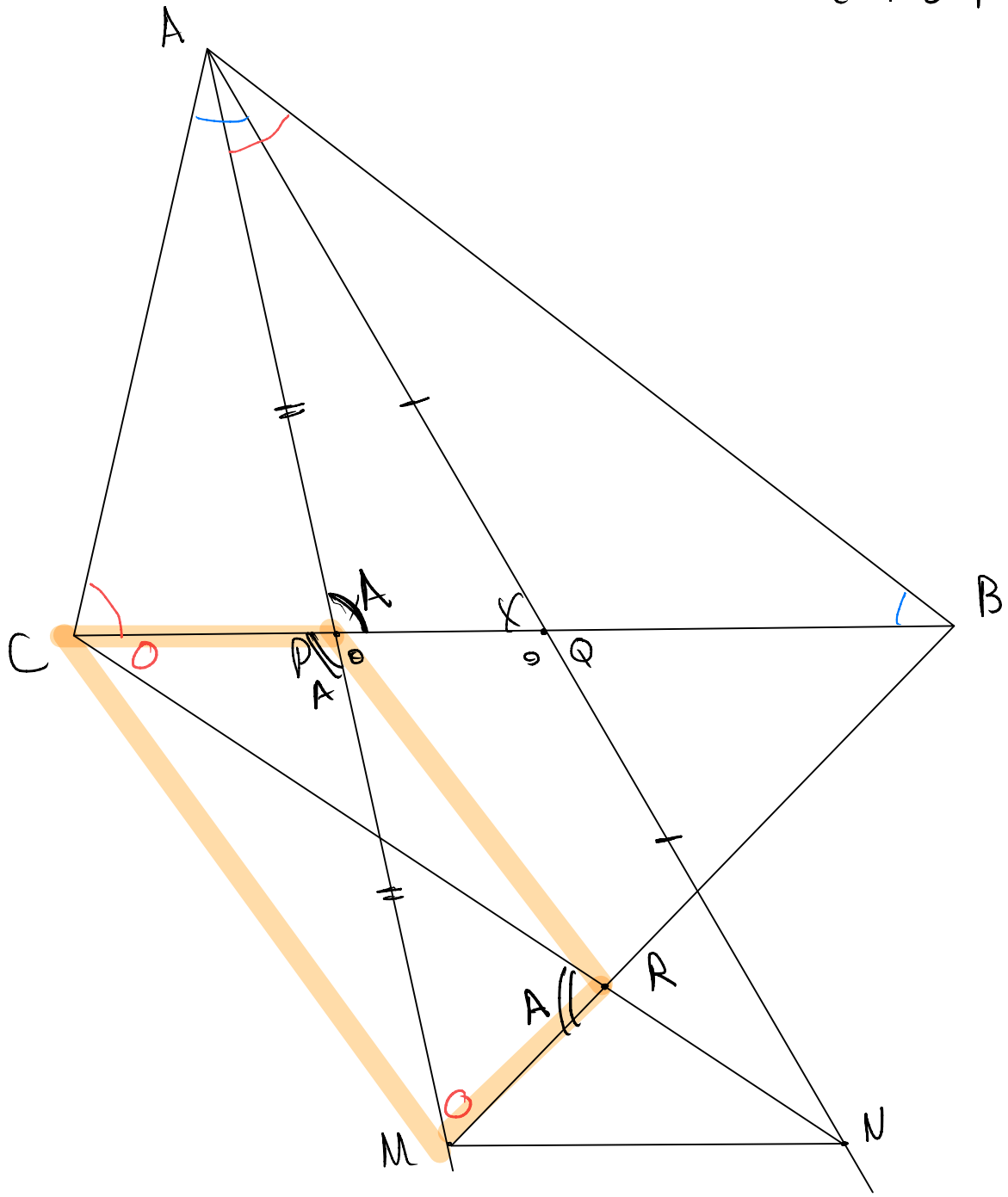
$$\Rightarrow MI^2 = \mathcal{P}_\omega(M)$$

and similarly $KI^2 = \mathcal{P}_\omega(K)$

$LI^2 = \mathcal{P}_\omega(L)$

by lemma, M, K, L collinear $\square$

5. Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC. (IMO PY 2014)



$$\triangle APB \sim \triangle AQC \sim \triangle ABC \Rightarrow \begin{cases} \angle NQC = \angle BPM \\ \dfrac{AP}{PB} = \dfrac{AQ}{QC} \Rightarrow \dfrac{PM}{PB} = \dfrac{QN}{QC} \end{cases}$$

$\Rightarrow \triangle MPB \sim \triangle NQC \Rightarrow \angle PMB = \angle QNC \Rightarrow CPRM$ cyclic

$\Rightarrow \angle MRC = \angle MPC = \angle BPA = \angle A \Rightarrow ACRB$ cyclic $\square$