Email training, N2

September 18-23

Problem 2.1. Simplify

$$\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}.$$

Solution 2.1.

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}} =& \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2+2\sqrt{3}}{\sqrt{4+2\sqrt{3}}} \\ =& \frac{2(1+\sqrt{3})}{\sqrt{1+2\sqrt{3}+3}} = \frac{2(1+\sqrt{3})}{1+\sqrt{3}} = 2:\end{aligned}$$

Problem 2.2. Let $a \neq 0$ and let x_1 and x_2 are the roots of the equation

$$x^2+ax-\frac{1}{2a^2}=0.$$

Prove that

$$x_1^4+x_2^4 \geq 2+\sqrt{2}.$$

Solution 2.2. According to Viet theorem one has $x_1+x_2=-a$ and $x_1x_2=-\frac{1}{2a^2}$. Therefore one may write

$$\begin{gathered}x_1^4+x_2^4=(x_1^2+x_2^2)^2-2x_1^2x_2^2= \\ ((x_1+x_2)^2-2x_1x_2)^2-\frac{1}{2a^4}= \\ \left(a^2+\frac{1}{a^2}\right)^2-\frac{1}{2a^4}=2+a^4+\frac{1}{2a^4}.\end{gathered}$$

To get the desired result it's enough to use the inequality $x+y \geq 2\sqrt{xy}$ and conclude that

$$a^4+\frac{1}{2a^4} \geq 2\sqrt{a^4\cdot\frac{1}{2a^4}}=\sqrt{2}$$

Problem 2.3. Find all integer solutions to the equation

$$x^2-6xy+13y^2=100.$$

Solution 2.3. Rewrite the equation in the following form

$$x^2-6xy+13y^2=(x-3y)^2+(2y)^2=100.$$

Since 100 can be written as sum of squares in the following ways

$$100=8^2+6^2=10^2+0^2$$

, therefore we have the following options

$$x-3y=8, 2y=6$$

$$x-3y=-8, 2y=6$$

$$x-3y=6, 2y=8$$

$$x-3y=-6, 2y=-8$$

$$x-3y=10, 2y=0$$

$$x-3y=-10, 2y=0$$

$$x-3y=0, 2y=10$$

$$x-3y=0, 2y=-10$$

From these cases we get solutions

Answer: $(17,3),(1,3),(18,4),(-18,-4),(10,0),(-10,0),(15,5),(-15,-5)$.

Problem 2.4. Prove that $lcm(1,2,3,\dots,2n) = lcm(n+1,n+2,\dots,2n)$, where lcm is the least common multiplier.

Solution 2.4. At first it's obvious that

$$lcm(n+1,\dots,2n) \mid lcm(1,\dots,2n).$$

On the other side $d \mid lcm(n+1,\dots,2n)$ for all $1 \le d \le 2n$. Therefore one has

$$lcm(1,2,\dots,2n) \mid lcm(n+1,\dots,2n).$$

From $a|b$ and $b|a$ follows that $a = b$.

Problem 2.5. In the school more than 90% of the students speak both English and Armenian, more than 90% of the students speak both English and Arabic. Prove that within a students that speak both Armenian and Arabic more than 90% speak English.

Solution 2.5. Let a is the number of students that know all three languages, b is the number of students that know only English and Armenian, c only English and Arabic, d only Armenian and Arabic.

According to the conditions one has

$$a+b > 9(c+d), \quad a+c > 9(b+d).$$

By taking the half-sum one gets

$$a > 9d + 4(b+c),$$

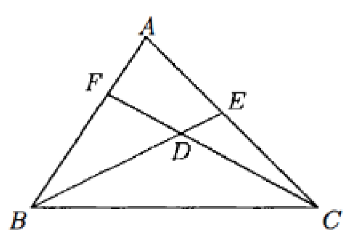
from which follows that $a > 9d$, which was requested to prove.

Problem 2.6. The endpoints of N arcs split the circle into $2N$ equal arcs of length 1. It is known that each arc splits the circle into 2 parts of even length. Prove that N is even.

Solution 2.6. Let's paint the endpoints in clockwise order by switching black and white. Since arc lengths are even, then endpoints of arcs will have the same color. It means that N white endpoints are split into pairs. So N is even.

Problem 2.7. Through vertices B and C of triangle ABC are constructed two lines which divide the triangle into four regions (three triangles and one quadrilateral). It is known that three of them have equal area. Prove that one of these three regions is the quadrilateral.

Solution 2.7. Let the lines be BE and CF intersecting at D.

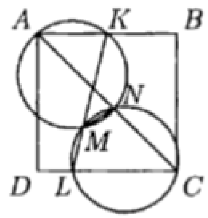


If AEDF is not one of the three regions of equal area, then each of each of BFD, BDC and DCE has the same area. However , this means that FD = DC and BD = DE, so that BCEF will be a parallelogram. This is a contradiction .

Problem 2.8. We are given the square $ABCD$. On sides AB and CD we are given points K and L respectively, and on segment KL we are given point M. Prove that the second point of intersection (i.e. the one other than M) of the points of intersection of circles circumscribed about triangles AKM and MLC lies on the diagonal AC.

Solution 2.8. -

Let N be the other point of intersection.



We have $\angle ANM = \angle AKM = \angle CLM$ since $AKNM$ is cyclic and AK is parallel to LC. Since $NMLC$ is cyclic, $\angle CLM + \angle CNM = 180°$. Hence $\angle ANM + \angle CNM = 180°$ so that A, N and C are collinear.