Number Theory

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Problems – April 19

1. Prove that every even perfect number is of the form $n = 2^{k-1}(2^k - 1)$, where k is a positive integer. Also prove that this is indeed a perfect number if $2^k - 1$ is a prime.

2. Prove that $\sigma(n) \cdot \varphi(n) < n^2$.

3. Suppose a and b are positive integers such that $\gcd(an+2, bn+3) > 1$ for every positive integer n. Prove that $b = \frac{3}{2}a$.

4. Positive integers a, b, c are such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $d = \gcd(a, b, c)$. Prove that $abcd$ is a perfect square.

5. Find all pairs of positive integers a and b such that $\operatorname{lcm}[a+1, b+1] = a^2 - b^2$.

6. If a, b, c are positive integers, prove that $\gcd(a, b-1) \cdot \gcd(b, c-1) \cdot \gcd(c, a-1) \leqslant ab+bc+ca-a-b-c+1$. Show that equality occurs for infinitely many triples (a, b, c).

7. If a and b are positive integers such that $\operatorname{lcm}[a, b] + \operatorname{lcm}[a+2, b+2] = 2 \cdot \operatorname{lcm}[a+1, b+1]$, prove that $a \mid b$ or $b \mid a$.

8. Suppose that n is odd and both $\varphi(n)$ and $\varphi(n+1)$ are powers of 2. Prove that either $n = 5$, or $n+1$ is itself a power of two.

9. Let S be the set of all numbers that can be written in the form $x^2 + 2y^2$ for some integers x, y. If $3n \in S$ for some integer n, prove that $n \in S$.

10. Let p be a prime and let $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} = \frac{a}{b}$.
    (a) If $p \geqslant 3$, prove that $p \mid a$;
    (b) If $p \geqslant 5$, prove that $p^2 \mid a$.

11. If $p \geqslant 5$ is a prime number, prove that $\binom{2p}{p} \equiv 2 \pmod{p^2}$. *(Wolstenholme's theorem)*

12. Find all positive integers n such that $(n-1)! + 1$ is a power of n.