Email training, N2

August 31- September 7

Problem 2.1. Let $S(x)$ be the sum of digits of x. Solve the equation

$$x + S(x) + S(S(x)) = 2018.$$

Solution 2.1. According to the rule about divisibility by 3 one has $x \equiv S(x)[3]$. Therefore one can state that x, $S(x)$ and $S(S(x))$ give the same residue mod 3. So their sum is divisible by 3 and can't be equal 2018. It means that such an x does not exist.

Answer: Has no solution.

Problem 2.2. Find the maximum possible value of x^6+y^6 if it's known that $x^2+y^2=1$.

Solution 2.2.

$$x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = (x^2 + y^2)^2 - 3x^2y^2 = 1 - 3x^2y^2$$

So we need to estimate the minimal possible value of x^2y^2. Since $x^2y^2 \geq 0$, one has

$$x^6 + y^6 = 1 - 3x^2y^2 \leq 1$$

and the equality holds when $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$.

Answer: 1.

Problem 2.3. Solve the inequality

$$\frac{2x^2 - 5x - 2}{3x - x^2 - 7} \leq 1.$$

Solution 2.3.

$$\frac{2x^2 - 5x - 2}{3x - x^2 - 7} - 1 \leq 0,$$

$$\frac{3x^2 - 8x + 5}{3x - x^2 - 7} \leq 0,$$

$$\frac{(x-1)(3x-5)}{x^2 - 3x + 7} \geq 0.$$

Since $x^2 - 3x + 7 = (x - 1.5)^2 + 4.75 > 0$ so we get the equivalent inequality $(x-1)(3x-5) > 0$ which has solution $x \in (-\infty, 1) \cup (\frac{5}{3}, +\infty)$.

Answer: $x \in (-\infty, 1) \cup (\frac{5}{3}, +\infty)$.

Problem 2.4. Let $S(n)$ be the sum of divisors of n (for example $S(6) = 1+2+3+6 = 12$). Find all n for which $S(2n) = 3S(n)$.

Solution 2.4. Let $d_1, d_2, \ldots, d_k$ are all divisors of n. Then all divisors of $2n$ are from the multi-set $\{2d_1, 2d_2, \ldots, 2d_k, d_1, d_2, \ldots, d_k\}$ and the sum of all elements from this multi-set is equal to $3S(n)$. So to have $S(2n) = 3S(n)$ one needs to have all numbers from the multi-set pairwise different. It's possible if and only if all d_i's are odd, otherwise it would appear twice - as $2\frac{d_i}{2}$ and as d_i.

Answer: n is odd.

Problem 2.5. Is it possible to write numbers (each once) from 1 to 10 on edges and vertices of triangular pyramid in such a way, that any number on the edge is the arithmetical mean of the numbers written on the endpoints of that edge.

Solution 2.5. Assume it's possible and consider such an situation. Since 1 can' be written as average of two different integers between 1 and 10, so 1 is written on the vertex. Similarly 10 is written on some vertex. Since all vertices are pairwise connected then 1 and 10 are written on endpoints of some edge. So on that edge their average should be written, but the average of 1 and 10 is not integer.

Answer: Not possible.

Problem 2.6. Let numbers $(1, 2, 3, 4)$ are given. On each step one chooses 2 neighboring numbers (first and fourth numbers are considered as neighboring) and increases by 1. Is it possible after some steps get numbers $(2015, 2016, 2017, 2016)$?

Solution 2.6. For the 4-ple (a, b, c, d) consider the value of expression $a - b + c - d$. Note, that if we increase two neighbor numbers by 1 then for the new 4-ple the value of expression defined as above will not change. It means that during the process the value of expression will not change. Not that at the beginning it is equal $1 - 2 + 3 - 4 = -2$, but at the end we want to have it equal $2015 - 2016 + 2017 - 2016 = 0$.

Answer: Not possible.

Problem 2.7. Find the number of acute triangles that has perimeter less than 100 and sides are 3 consecutive positive integers.

Solution 2.7. let the sides of the triangle are $n, n+1, n+2$. Since the perimeter is at most 100, then

$$n + (n+1) + (n+2) \leq 100,$$

which means $1 \leq n \leq 32$. According to the triangle inequality we have $n + (n+1) > (n+2)$, which yields $n \geq 2$. Finally, since the triangle is acute, then

$$n^2 + (n+1)^2 > (n+2)^2,$$

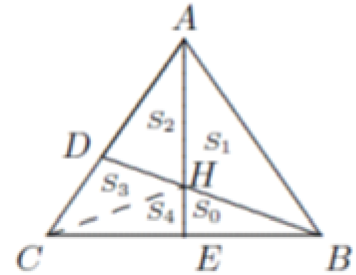
which gives $n > 3$. So $4 \leq n \leq 32$, so there are $32 - 4 + 1 = 29$ options for n.

Answer: 29.

Problem 2.8. In the triangle ABC one has $\angle A = 70^o$. The point D is chosen on the segment AC such that the bisector AE intersects BD at point H and $AH : HE = 3 : 1$. as well $BH : HD = 5 : 3$. Find $\angle C$.

Solution 2.8. -

Connect CH. As shown in the digram, let the areas of triangles be $S_0, S_1, \cdots, S_4$. Without loss of genrality we may assume that $S_0 = 1$. Since $AH/HE = 3$ yields $S_1 = 3$, then $\dfrac{BH}{HD} = \dfrac{5}{3}$ implies that $S_2 = \dfrac{3}{5}S_1 = \dfrac{9}{5}$.



Since $\dfrac{S_2+S_3}{S_4} = \dfrac{3}{1}$, so $S_4 = \dfrac{1}{3}(S_2+S_3)$.

$$\therefore \frac{S_3}{S_4+S_0} = \frac{S_3}{\frac{1}{3}(S_2+S_3)+S_0} = \frac{3}{5},$$
$$\frac{S_3}{\frac{1}{3}(\frac{9}{5}+S_3)+1} = \frac{3}{5}.$$

Hence, $S_3 = \dfrac{6}{5}$, $S_4 = \dfrac{1}{3}\left(\dfrac{9}{5}+\dfrac{6}{5}\right) = 1$, so $S_0+S_1 = 4$, $S_2+S_3+S_4 = 4$, i.e. $CE = BE$, the triangle ABC is isosceles. Thus,

$$\angle C = \frac{1}{2}(180^\circ - \angle A) = \frac{1}{2}(180^\circ - 70^\circ) = 55^\circ.$$