Email training, N4

Level 4, October 4-10

Problem 4.1. Let a, b, c, d be real numbers such that

$$a^4 + b^4 + c^4 + d^4 = 16.$$

Prove the inequality

$$a^5 + b^5 + c^5 + d^5 \leq 32.$$

Solution 4.1. We have $a^4 \leq a^4 + b^4 + c^4 + d^4 = 16$, i.e. $a \leq 2$ from which it follows that $a^5 \leq 2a^4$. Similarly we obtain $b^5 \leq 2b^4$, $c^5 \leq 2c^4$ and $d^5 \leq 2d^4$. Hence

$$a^5 + b^5 + c^5 + d^5 \leq 2(a^4 + b^4 + c^4 + d^4) = 32.$$

equality holds if one of a, b, c, d is equal 2 and the rest are equal 0.

Problem 4.2. Consider the positive numbers $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{x_i}.$$

Prove that

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$

Solution 4.2. Let $y_i = \frac{1}{n-1+x_i}$ and let assume for the purpose of contradiction that $\sum_{i=1}^{n} y_i > 1$. Note that

$$x_i = \frac{1-(n-1)y_i}{y_i}$$

Denote $a_i = 1-(n-1)y_i$, and our assumption becomes $S = \sum a_i < 1$ and we have $x_i = \frac{(n-1)a_i}{1-a_i}$. Since y_i's are positive then we have $y_i < \frac{1}{n-1}$, from which will follow that $0 < a_i < 1$. The problem hypothesis becomes the following equality

$$(n-1)\sum \frac{a_i}{1-a_i} = \sum x_i = \sum \frac{1}{x_i} = \sum \frac{1-a_i}{a_i(n-1)}.$$

But observe that

$$\sum \frac{1-a_i}{a_i} > \sum \frac{S-a_i}{a_i} = \sum_{k \neq i} \frac{a_k}{a_i} =$$

$$\sum_i a_i\Big(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{i-1}} + \frac{1}{a_{i+1}} + \ldots + \frac{1}{a_n}\Big) \geq$$

$$\sum_i a_i \frac{(n-1)^2}{a_1 + a_2 + \ldots + a_{i-1} + a_{i+1} + \ldots + a_n} =$$

$$(n-1)^2 \sum_i \frac{a_i}{S-a_i} > (n-1)^2 \sum_i \frac{a_i}{1-a_i}.$$

We got contradiction.

Problem 4.3. Find all pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{lcm(x,y)} + \frac{1}{gcd(x,y)} = \frac{1}{2}.$$

Solution 4.3. We put $x = du$ and $y = dv$ where $d = gcd(x, y)$. So we have $(u, v) = 1$. From the conclusion of the problem we have

$$\frac{1}{du} + \frac{1}{dv} + \frac{1}{d} + \frac{1}{duv} = \frac{1}{2},$$

$$u + v + uv + 1 = \frac{duv}{2},$$

or

$$2(u + 1)(v + 1) = duv.$$

Since $gcd(v, v + 1) = 1$ therefore v divides $2(u + 1)$.

Case 1. $u = v$. Then $u = v = 1$ and we get $d = 2(1+1)(1+1) = 8$ which leads $x = y = 8$.

Case 2. $u < v$. Then $u + 1 \leq v$ so $2(u + 1) \leq 2v$ so $\frac{2(u+1)}{v}$ is equal either 1 or 2.

If $\frac{2(u+1)}{v} = 1$ then we have $(d - 2)u = 3$ which means $(d, u) = (3, 3)$ or $(d, u) = (5, 1)$. So we get $(x, y) = (9, 24)$ or $(x, y) = (5, 20)$.

If $\frac{2(u+1)}{v} = 2$ then we have $(d - 2)u = 4$ which means $(d, u) = (3, 4)$ or $(d, u) = (4, 2)$ or $(d, u) = (6, 1)$. So we get $(x, y) = (12, 15)$ or $(x, y) = (8, 12)$ or $(x, y) = (6, 12)$.

Case 3. $u > v$. This is identical to the case 2.

Answer: $(8, 8), (9, 24), (24, 9), (5, 20), (20, 5), (12, 15),$
$(15, 12), (8, 12), (12, 8), (6, 12), (12, 6).$

Problem 4.4. Find all integer numbers m and n such that

$$(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n.$$

Solution 4.4. Note that if (m, n) satisfies then $(-m, -n)$ satisfies as well, so we may assume that $m, n > 0$, since $m = n = 0$ satisfies. We may write

$$(5 + 3\sqrt{2})^m a + b\sqrt{2}, \qquad (3 + 5\sqrt{2})^n = c + d\sqrt{2}$$

with

$$a = 5^m + 5^{m-2} \cdot 18\binom{m}{2} + \dots$$

$$c = 3^n + 5^{n-2} \cdot 50\binom{n}{2} + \dots$$

These must be equal, and it is obvious that 5 doesn't divide c and 3 doesn't divide a, which may happen only if the expansion of a will end with the term $5^0 \cdot 18^{\frac{m}{2}}$. This implies that m is even and similarly n is even. Then, by extracting square root out of the relation $(5+3\sqrt{2})^m = (3 + 5\sqrt{2})^n$. we get that

$$(5 + 3\sqrt{2})^{\frac{m}{2}} = (3 + 5\sqrt{2})^{\frac{n}{2}}.$$

This process can be continues infinitely long and we conclude that the order of 2 in m is infinite. So the only solution is $m = n = 0$.

Answer: $m = n = 0$.

Problem 4.5. Let $1 \leq r \leq n$. We consider all r-element subsets of $(1, 2, \dots, , n)$. Each of them has a minimum. Prove that the average of these minima is $\frac{n+1}{r+1}$.

Solution 4.5. There are exactly $\binom{n-k}{r-1}$ subsets with minimal element equal k (chose k and the rest $r - 1$ elements arbitrary from the set $\{k + 1, \dots, n\}$. So the total sum of minimal elements is equal

$$\sum_{k=1}^{n} k\binom{n-k}{r-1} = \sum_{k=1}^{n} \binom{k}{1}\binom{n-k}{r-1}$$

Let there are $n + 1$ ball among a line and we need to chose any $r + 1$ of them. For some value of k between 1 and n, inclusive, we say that the second ball will occur in the $(k + 1)$th place. Clearly, there are $\binom{k}{1}$ ways to arrange the bits coming before the second 1, and $\binom{n-k}{r-1}$ ways to arrange the bits after the second 1. So there are $\sum_{k=1}^{n}\binom{k}{1}\binom{n-k}{r-1}$ ways to chose any $r+1$ balls, which is eventually equal to $\binom{n+1}{r+1}$. So the average is equal

$$\frac{\binom{n+1}{r+1}}{\binom{n}{r}} = \frac{n+1}{r+1}.$$

Problem 4.6. Twenty children are queueing for ice cream that is sold at SR5 per cone. Ten of the children have a SR5 coin, the others want to pay with a R10 bill. At the beginning, the ice cream man does not have any change. How many possible arrangements of the twenty kids in a queue are there so that the ice cream man will never run out of change?

Solution 4.6. Let us consider a diagram in which the amount of change left after each child is shown. If a child pays with a coin 5, the amount increases by 1, otherwise it decreases by 1. Our requirement is equivalent to the condition that the amount of change stays non-negative throughout the process.

Thus, we have Catalan numbers (https://www.youtube.com/watch?v=GlI17WaMrtw). So there are

$$\frac{1}{11}\binom{20}{10}$$

ways to draw such a graph. Since there are 10! ways to arrange children with 5 coins and 10! ways to arrange children with 10SR worth, then there is in total

$$10!^2 \cdot \frac{1}{11}\binom{20}{10} = \frac{20!}{11}$$

ways to organize the queue.

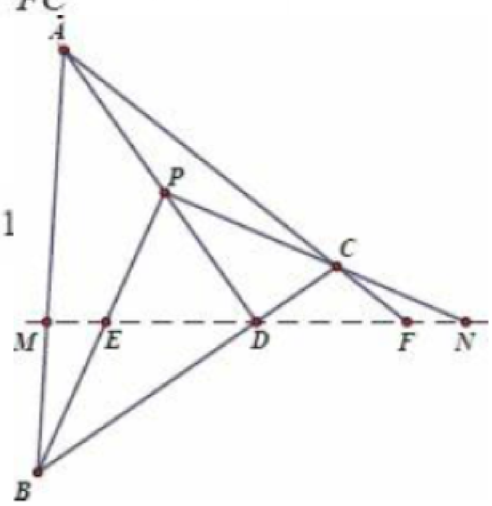
Answer: $\frac{20!}{11}$.

Problem 4.7. Given ΔABC, D is a point on BC and P is on AD. A line ℓ is passing through D intersects AB, PB at M, E respectively, and intersects AC extended and PC extended at F, N respectively. Let $DE = DF$. Prove that $DM = DN$.

Solution 4.7. Apply Menelaus' Theorem to ΔAMD intersected by BP, ΔAMF intersected by BC, and ΔADF intersected by PN:

$$\frac{AB}{MB}\cdot\frac{DP}{AP}\cdot\frac{ME}{DE}=1 \quad (1) \qquad \frac{MB}{AB}\cdot\frac{FD}{MD}\cdot\frac{AC}{FC}=1 \quad (2)$$

$$\frac{AP}{DP}\cdot\frac{DN}{FN}\cdot\frac{FC}{AC}=1 \quad (3)$$



Multiplying (1), (2), (3) gives $\dfrac{ME\cdot FD\cdot DN}{DE\cdot MD\cdot FN}=1$

Since $DE = DF$, we have $\dfrac{DM}{EM}=\dfrac{DN}{FN}$, i.e.,

$$\frac{DE}{EM}+1=\frac{DF}{FN}+1.$$

It follows that $EM = FN$ and hence, $DM = DN$.