Email training, N5

September 22-28, 2019

Problem 5.1. Show that for positive reals a, b, c we have $abc = 1$ if and only if
$$\frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} = 1.$$

Problem 5.2. Let $p > 3$ be a prime such that $p \equiv 3[4]$. Given a positive integer a_0 define the sequence $a_0, a_1, \dots$ of integers by $a_n = a_{n-1}^{2n}$ for all $n = 1, 2, \dots.$ Prove that it is possible to choose a_0 such that the subsequence $a_N, a_{N+1}, a_{N+2}, \dots$ is not constant modulo p for any positive integer N.

Problem 5.3. Three pairwise distinct positive integers a, b, c with $gcd(a; b; c) = 1$, satisfy
$$a \mid (b-c)^2; \quad b \mid (c-a)^2 \quad \text{and} \quad c \mid (a-b)^2.$$
Prove that there does not exist a non-degenerate triangle with side lengths a, b, c.

Problem 5.4. Prove that any sequence of $n^2 + 1$ real numbers contains a subsequence of length $n + 1$ which is either increasing or decreasing.

Problem 5.5. There are n integers, each of them equal to 1 written on a blackboard. At each step, you erase any two numbers a and b and replace them with $\frac{a+b}{4}$. After $n - 1$ steps, there is only one number left on the blackboard. Prove that this number is at least $\frac{1}{n}$.

Problem 5.6. Is it true that in any convex n-gon with $n > 3$, there exists a vertex and a diagonal passing through this vertex such that the angles of this diagonal with both sides adjacent to this vertex are acute?

Problem 5.7. Circles ω_1 and ω_2 have centres O_1 and O_2, respectively. These two circles intersect at points X and Y. AB is common tangent line of these two circles such that A lies on ω_1 and B lies on ω_2. Let tangents to ω_1 and ω_2 at X intersect O_1O_2 at points K and L, respectively. Suppose that line BL intersects ω_2 for the second time at M and line AK intersects ω_1 for the second time at N. Prove that lines AM, BN and O_1O_2 concur.

Problem 5.8. Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remains at H, inradius of triangle ABC remains unchanged.

Solution submission deadline September 28, 2019