January Camp 2022
Problems

Geometry – L3

Steiner line

Problem 1. Points X and Y lie on segments AB and AC of acute-angled triangle ABC, such that $AX = AY$ and the orthocenter of triangle ABC lies on line XY. Tangents to circumcircle of triangle AXY at X and Y intersect at P. Show that points A, B, C, P are concyclic.

Problem 2. Given is a parallelogram $ABCD$ and point F on segment CD. Points O_1, O_2 and O_3 are circumcenters of triangles ABF, BCF and ADF, respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on AB.

Problem 3. Given is an equilateral triangle ABC and line ℓ passing through point A. Circle ω_b, centred at I_b, is tangent to segment AC at B_1 and lines ℓ and BC. Analogously circle ω_c, centred at I_c, is tangent to segment AB at C_1 and lines ℓ and BC. Prove that the orthocenter of triangle AC_1B_1 lies on line I_bI_c.

Problem 4. Point O is the circumcenter of triangle ABC. Circle passing through points A, O intersects lines AB, AC at P, Q, respectively. Prove that the orthocenter of triangle OPQ lies on BC.

Problem 5. Given is a triangle ABC and line ℓ passing through point A, but not intersecting segment BC. Point O_1 is center of circle tangent to segment AB, line BC and line ℓ, however it is outside the interior of triangle ABC. Analogously point O_2 is center of circle tangent to segment AC, line BC and line ℓ, however it is outside the interior of triangle ABC. Point O_3 is A-excenter of triangle ABC. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on BC.

Problem 6. Point H is the orthocenter of triangle ABC. Given is a line ℓ passing through point H and point P on this line. Points P_1, P_2 are the reflections of point P across lines BC and AC, respectively. Point S is defined as intersection of the reflections of ℓ about lines AB, AC. Prove that points P_1, P_2, C and S are concyclic.

Problem 7. Let ABC be an acute-angled triangle in which no two sides have the same length. The reflections of the centroid G and the circumcentre O of ABC in its sides BC, CA, AB are denoted by G_1, G_2, G_3 and O_1, O_2, O_3, respectively. Show that the circumcircles of triangles G_1G_2C, G_1G_3B, G_2G_3A, O_1O_2C, O_1O_3B, O_2O_3A and ABC have a common point.

Problem 8$^\bigstar$. Quadrilateral $ABCD$ is circumscribed about a circle. Line ℓ passing through point A intersects BC at M and ray DC at N. Points I_1, I_2 and I_3 are incenters of triangles ABM, MNC i NDA, respectively. Prove that the orthocenter of triangle $I_1I_2I_3$ lies on ℓ.

Incenter and excenter

Problem 9. Let $ABCD$ be a cyclic quadrilateral such that incircles of triangles ABD and ABC are congruent. Decide whether incircles of triangles CDB and CDA are also congruent.

Problem 10. A trapezoid $ABCD$ in which $AB \parallel CD$ and $AB > CD$, is circumscribed. The incircle of the triangle ABC touches the lines AB and AC at the points M and N, respectively. Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

Problem 11. Points A, B lie on circle ω. Points C and D are moved on the arc AB, such that CD has constant length. Points I_1, I_2 are incenters of ABC and ABD, respectively. Prove that line I_1I_2 is tangent to some fixed circle.

Problem 12. Let K and L be two points on the arcs AB and BC of the circumcircle of a triangle ABC, respectively, such that $KL \parallel AC$. Show that the incenters of triangles ABK and CBL are equidistant from the midpoint of the arc AC, containing point B, of the circumcircle of triangle ABC.

Problem 13. Let ABC be a triangle with $\sphericalangle BAC = 60^\circ$. Let D and E be the feet of the perpendiculars from A to the external angle bisectors of ABC and ACB, respectively. Let O be the circumcenter of the triangle ABC. Prove that the circumcircles of the triangles ADE and BOC are tangent to each other.

Problem 14. The circle Γ has centre O, and BC is a diameter of Γ. Let A be a point which lies on Γ such that $\sphericalangle AOB < 120^\circ$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets Γ at E and at F. Prove that I is the incentre of the triangle CEF.

Problem 15$^\bigstar$. Let AD be altitude in acute-angled triangle ABC. Points M and N are projections of point D onto AB and AC. Lines MN and AD intersect circumcirlce ω of triangle ABC respectively at points P, Q and A, R. Prove that D is incenter of PQR.

Problem 16$^\bigstar$. Let I be the incenter of $\triangle ABC$. Denote by D, $S \neq A$ intersections of AI with BC and circumcircle ω of ABC, respectively. Let K, L be incenters of triangles DSB and DCS. Let P be a reflection of I with respect to KL. Prove that $BPC = 90^\circ$.

Midarc

Problem 17. Let ABC be a triangle and M be the midpoint of arc BAC. Let X and Y lie on AB and AC such that $BX = CY$. Prove that $AXYM$ are concyclic.

Problem 18. In triangle ABC point I is its incenter. The circle passing through A and I intersects AB and AC at X and Y, respectively. Prove that $BX + CY = BC$.

Problem 19. Let ω be circumcircle of an acute triangle ABC. Point X lies inside ABC, such that $\sphericalangle BAX = 2\sphericalangle XBA$ and $\sphericalangle XAC = 2\sphericalangle ACX$. M is midpoint of arc BC of ω, which contains point A. Show that $XM = XA$.

Problem 20. Let I be incenter of triangle ABC. Let M and N be midarc point of arc BAC and midpoint of BC. Prove that $\sphericalangle AMI = \sphericalangle INB$.

Problem 21$^\bigstar$. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcenter of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

Problem 22. Let M be the midpoint of side BC of triangle ABC. Let I and J be incenters of triangles ABM and AMC. Prove that circumcircle of triangle AIJ passes through midarc BAC.

Problem 23. Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Inversion

Problem 24. Let ω be a circle internally tangent to circle Ω at S. Let SA and SB be diameters of ω, Ω, respectively. Let o be the circle tangent to AB at C and tangent to ω and Ω. Prove that

$$\frac{2}{SC} = \frac{1}{SA} + \frac{1}{SB}.$$

Problem 25. Let ω be a circle with center A. Let B be a point on ω. Consider circle Ω tangent to perpendicular bisector of AB, ω, and line AB at D. Prove that $AB = BD$.

Problem 26. Let ABC na a right triangle with $\sphericalangle BAC = 90°$. Let o_1 be circle with diameter AC. Let o_2 be the circle tangent to BC at D, to segment AB and externally to o_1. Prove that $AC = DC$.

Problem 27. Let o_1, o_2 with radii r_1 and r_2 be externally tangent at A. Let o_3, o_4 with radii r_3 and r_4 be externally tangent at A, but the are not tangent to circles o_1, o_2. Prove that there exists circle tangent to o_1, o_2, o_3, o_4 iff

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}.$$

Problem 28. Let A, B, C, D be arbitrary points on a plane. Prove that

$$AC \cdot BD \leq AD \cdot BC + AB \cdot CD.$$

Problem 29. Let P be a point inside a triangle ABC such that

$$\sphericalangle APB - \sphericalangle ACB = \sphericalangle APC - \sphericalangle ABC.$$

Let D, E be the incenters of triangles APB andAPC, respectively. Show that the lines AP, BD, CE meet at a point.

Problem 30. Circles k_1, k_2, k_3, k_4 are such that k_2 and k_4 each touch k_1 and k_3. Show that the tangency points are collinear or concyclic.

Problem 31. Let ω be the semicircle with diameter PQ. A circle k is tangent internally to ω and to segment PQ at C. Let AB be the tangent to k perpendicular to PQ, with A on ω and B on segment CQ. Show that AC bisects the angle $\sphericalangle PAB$.

Problem 32. Given is a triangle ABC. Consider such transformation ϕ which is composition of inversion with respect to circle with center A and radius $\sqrt{AB \cdot AC}$ and symmetry about the angle bisector of angle BAC. Show that
(a) $\phi(B) = C$, $\phi(C) = B$;
(b) $\phi(\omega) = BC$, $\phi(BC) = \omega$, where ω is circumcircle of triangle ABC.

Problem 33. Let ω be a circle tangent internally to circle Ω. Let A be a point on Ω, and let AX and AY be tangents to ω. Consider circles ω_1, ω_2 tangent internally to Ω and tangent to ω at X and Y, respectively. Prove that there exists common exterior tangent of circles ω, ω_1, ω_2.

Problem 34. Let p be the semiperimeter of triangle ABC. Points E and F are on line AB such that $|CE| = |CF| = p$. Prove that the circumcircle of triangle CEF is tangent to the excircle of triangle ABC with respect to the side AB.

Problem 35. Points A, B, C are given on a line in this order. Semicircles ω, ω_1, ω_2 are drawn on AC, AB, BC respectively as diameters on the same side of the line. A sequence of circles (k_n) is constructed as follows: k_0 is the circle determined by ω_2 and k_n is tangent to ω, ω_1, k_{n-1} for $n \geq 1$. Prove that the distance from the center of k_n to AB is $2n$ times the radius of k_n.

Problem 36. Given is a triangle ABC. Consider such transformation ϕ which is composition of inversion with respect to circle with center A and radius $\sqrt{\frac{1}{2}AB \cdot AC}$ and symmetry about the angle bisector of angle BAC. Prove that

(i) $\phi(O) = H_A$, where O is circumcenter of ABC, and H_A is the base of altitude from vertex A;
(ii) circumcircle of triangle BOC is mapped to nine point circle of ABC.

Problem 37. Let Ω be circumcirle of triangle ABC. Denote by o circle tangent to AB, AC and internally tangent to ω at point T. Point D is tangency point of A-excircle with line BC. Prove that $\sphericalangle BAT = \sphericalangle DAC$.

Problem 38. Let ABC be a triangle and A', B', C' the symmetrics of vertex about opposite sides. The intersection of the circumcircles of triangles ABB' and ACC' is A_1. Points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1 and CC_1 are concurent.

Problem 39*. Given is triangle ABC. Points D i E lie on line AB in order D, A, B, E. Moreover $AD = AC$ and $BE = BC$. Angle bisector of angles at BAC and ABC intersect BC, AC at P and Q, and circumcircle of ABC at M and N, respectively. Line connecting A with circumcenter of BME and line connecting B with circumcenter of AND intersect at X. Prove that $CX \perp PQ$.

Spiral Similarity

Problem 40. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 41. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

Problem 42. In triangle ABC point M is a midpoint of AB. Point D lies inside this triangle and satisfies conditions

$$\sphericalangle DAC = \sphericalangle ABC \quad \text{and} \quad \sphericalangle DCA = \sphericalangle BCM.$$

Show that $DM \parallel BC$.

Problem 43★. Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90°$ and $\measuredangle DAX = \measuredangle CBX < 90°$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

Problem 44. Let $ABCDE$ be a convex pentagon such that

$$\sphericalangle BAC = \sphericalangle CAD = \sphericalangle DAE \qquad \text{and} \qquad \sphericalangle ABC = \sphericalangle ACD = \sphericalangle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Problem 45. A circle with center O passes through the vertices A and C of triangle ABC and intersects segments AB and BC again at distinct points K and N, respectively. The circumcircles of triangles ABC and KBN intersects at exactly two distinct points B and M. Prove that $\sphericalangle OMB = 90°$.

Problem 46. Consider a circle with diameter AB and center O, and let C and D be two points on this circle. The line CD meets the line AB at a point M satisfying $MB < MA$ and $MD < MC$. Let K be the point of intersection (different from O) of the circumcircles of triangles AOC and DOB. Show that $\sphericalangle MKO = 90°$.

Symmedians

Problem 47. Let ABC be an acute-angled triangle and M be the midpoint of the side BC. Let N be a point in the interior of the triangle ABC such that $\measuredangle NBA = \measuredangle NAC$ and $\measuredangle NCA = \measuredangle NAB$. Prove that $\measuredangle NAB = \measuredangle MAC$.

Problem 48. Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\measuredangle PAB = \measuredangle PBC$. If M is the midpoint of AB, then show that $\measuredangle APM + \measuredangle BPC = 180°$.

Problem 49$^\bigstar$. The tangents at B and A to the circumcircle of an acute angled triangle ABC meet the tangent at C at T and U respectively. AT meets BC at P, and Q is the midpoint of AP; BU meets CA at R, and S is the midpoint of BR. Prove that $\measuredangle ABQ = \measuredangle BAS$. Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

Problem 50$^\bigstar$. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

Problem 51. The altitudes AA_1, BB_1, CC_1 of an acute triangle ABC concur at H. The perpendicular lines from H to B_1C_1, A_1C_1 meet rays CA, CB at P, Q respectively. Prove that the line from C perpendicular to A_1B_1 passes through the midpoint of PQ.