Email training, N4

September 15-21, 2019

Problem 4.1. In the fraction below, there are n radicals in the numerator and $n-1$ radicals in the denominator. Prove that

$$\frac{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} > \frac{1}{4}.$$

Problem 4.2. Find the largest k such that from $a, b, c > 0$ and $kabc > a^3 + b^3 + c^3$ implies that a, b and c are the side-lengths of a triangle.

Problem 4.3. Let $n, p > 1$ be positive integers and p be prime. Given that $n \mid p-1$ and $p \mid n^3 - 1$, prove that $4p-3$ is a perfect square.

Problem 4.4. Does there exist a 100-term sequence of rational numbers a_1, a_2, a_3, $\ldots$, a_{100} such that all terms are of the form $\frac{1}{n}$ for some positive integer n and for all $3 \leq k \leq 100$, we have $a_k = a_{k-2} - a_{k-1}$?

Problem 4.5. Let S be a set of 10 distinct positive real numbers. Show that there exist $x, y \in S$ such that

$$0 < x - y < \frac{(1+x)(1+y)}{9}.$$

Problem 4.6. For a partition π of $\{1, 2, 3, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. For example, for the partition π given by $\{1, 4, 6, 7\} \cup \{2, 8\} \cup \{3\} \cup \{5, 9\}$, we have $\pi(5) = 2$, $\pi(6) = 4$, and $\pi(3) = 1$. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, \ldots, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

Problem 4.7. Let AD, BE and CF are the altitudes of triangle ABC. Let K, M and N are the orthocentres of triangles AEF, BFD and CDE respectively. Prove that KMN and DEF are congruent triangles.

Problem 4.8. In triangle ABC let X is a point on AB and Y is a point on BC. The segments AY and CX intersect at Z. Let $AY = YC$ and $AB = ZC$. Prove that the points B, X, Y and Z lie on a circle.

Solution submission deadline September 21, 2019