Preparation for Saudi Arabia Team 2021

May/June Session: Level 4

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Lesson 2

Inversion in geometry

Problems:

1. Let A be an intersection point of circles k and l. Circle Γ_1 is externally tangent to k and l (respectively) in points B_1 and C_1, while circle Γ_1 is externally tangent to k and l in points B_2 and C_2. Prove that the circumcircles of $\triangle AB_1C_1$ and $\triangle AB_2C_2$ are tangent to each other.

2. Let PQ be the diameter of k and let B be a point on the diameter. A line perpendicular to PQ from B intersects k at A. A circle l is internally tangent to k and is also tangent to segment AB and to segment PB at point C. Prove that AC is the symmetry line of $\measuredangle PAB$.

3. Let k_1 and k_2 be circles with radii $r_1 < r_2$ respectively and let AD be a common tangent, d_1, where $A \in k_1$ and $D \in k_2$. Let d_2 be a (different) line parallel to d_1 which is tangent to k_1. A line through D intersects d_2 in B and k_2 the second time in C, where $B \neq C$. Prove that the circumcircle of $\triangle ABC$ is tangent to d_1.

4. A diameter AB of circle k is given and on it point C. Let l be a circle of diameter AC and let circle t be externally tangent to l at T, internally tangent to k and tangent to a line through C perpendicular to AB. Prove that BT is a tangent to l.

5. In triangle ABC a point M is given such that $\measuredangle AMB - \measuredangle ACB = \measuredangle AMC - \measuredangle ABC$. Prove that $AB \cdot CM = AC \cdot BM$.

6. Let $\triangle ABC$ be given such that $\measuredangle ACB = 2\measuredangle CAB$. Let circle s with center S be tangent to the circumcircles of $\triangle ADC$, BDC and line AC, where point D is the intersection of the angle bisector of $\measuredangle ACB$ witgh AB. Prove that $CS \perp AB$.

7. Let ABC be a right-angled triangle with the right angle at vertex A and such that $AB > AC$. Let the tangent at A of the circumcircle k of $\triangle ABC$ intersect BC at D. Let E be the point symmetrical to A with respect to BC, X the foot of the perpendicular from A to BE, Y the midpoint of AX and Z the second point of intersection of k and BY. Prove that BD is the tangent of the circumcircle of ADZ.

8. Let A_1, B_1 and C_1 be the respective midpoints of the sides of $\triangle ABC$. Let O be the circumcenter of $\triangle ABC$. Circumcenters of BOC and $A_1B_1C_1$ intersect in X and Y. Prove that $\measuredangle BAX = \measuredangle CAY$.

9. Prove that the nine-point circle touches the incircle and all three excircles.

10. A triangle ABC is given. Let D, E, F be respectively the feet of perpendiculars from A, B and C and let M be the midpoint of AB. Let DE intersect AB in P and let a line through F parallel with DE intersect AC and BC respectively in Q and R. Prove that $PQMR$ is cyclic.

11. Let D be the midpoint of BC of $\triangle ABC$. Let k be the circumcircle of ABD. On the arc AB of k not containing D we notice a point E so that $\measuredangle EDB = \measuredangle DAC$. Let a perpendicular line from A to AD intersect BC in F. Let G be the second intersection point of FE with k unless FE is a tangent of k in which case we define $G \equiv E$. Prove that $DG = DB$.

12. Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.