$\boxed{17}$

$a_1, a_2, \dots, a_k, \quad b_1, b_2, \dots, b_k \in \mathbb{Z}_+$

$\gcd(a_i, b_i) = 1 \quad$ for any $\quad i \in \{1 \dots k\}$.

$m = \operatorname{lcm}(b_1, \dots, b_k)$

$$\gcd\left(\frac{a_1 m}{b_1}, \dots, \frac{a_k m}{b_k}\right) = \gcd(a_1, a_2, \dots, a_k).$$

$v_p(a_i) = c_i$

$v_p(b_i) = d_i$

Properties of v_p

$$v_p(\operatorname{lcm}(a,b)) = \max\{v_p(a), v_p(b)\}$$

$$v_p(\gcd(a,b)) = \min\{v_p(a), v_p(b)\},$$

if $a \mid b$ then $v_p(a) \leq v_p(b)$

if we want check if $a = b$

it suffices to prove that

$v_p(a) = v_p(b)$

if $B \mid A$

$$\boxed{v_p\left(\frac{A}{B}\right) = v_p(A) - v_p(B)}$$

17

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$m = \operatorname{lcm}(b_1, \dots, b_k)$

$$\boxed{\gcd\left(\frac{a_1 m}{b_1}, \dots, \frac{a_k m}{b_k}\right)} = \gcd(a_1, a_2, \dots, a_k)$$

$v_p(a_i) = c_i$

$v_p(b_i) = d_i$

• $\boxed{\min(c_i, d_i) = 0}$

$\boxed{c_i = 0 \text{ or } d_i = 0}$

$$v_p(\operatorname{lcm}(b_1, b_2, \dots, b_k)) = \max(d_1, d_2, \dots, d_k) = m'$$

$$\boxed{v_p\left(\gcd\left(\frac{a_1 m}{b_1}, \dots, \frac{a_k m}{b_k}\right)\right)} =$$

$$v_p(LHS) = \min(c_1 + m' - d_1, \; c_2 + m' - d_2, \dots, c_k + m' - d_k) = 0$$

$$v_p(RHS) = \min(C_1, \dots, C_k) = 0$$

• $c_i = 0 \implies m' = 0 \implies$ all d_i's are $d_i = 0$

$c_i - d_i + m' = 0$

$m' \neq 0 \qquad m' = d_j \neq 0$

$c_j - d_j + m' = 0 - d_j + d_j = 0$

$\min(c_j, d_j) = 0 \Rightarrow c_j = 0$

- All $c_i \neq 0 \Rightarrow d_i = 0 \ \forall i$

$m' = \max(d_1, \dots, d) = 0$

$$v_p(LHS) = \min(c_1, c_2, \dots, c_k)$$
$$\parallel$$
$$v_p(RHS) \qquad \square$$

$\boxed{18}$ $a, b, c, d \qquad ab = cd$

$\gcd(a,c) \cdot \gcd(a,d) = a \cdot \gcd(a,b,c,d)$

p – arbitrary prime:

$v_p(a) = a', \quad v_p(b) = b', \quad v_p(c) = c', \quad v_p(d) = d'.$

$$v_p(\gcd(a,c)\cdot\gcd(a,d)) = \min(a',c') + \min(a',d')$$

$$v_p(a\cdot\gcd(a,b,c,d)) = v_p(a) + v_p(\gcd(a,b,c,d)) =$$

$$= a' + \min(a',b',c',d')$$

$$v_p(ab) = v_p(a)+v_p(b) = a'+b' = v_p(cd) = c'+d'$$

Numbers: $a', b', c', d' \geq 0$

$$a'+b' = c'+d'$$

Prove

$$\min(a',c') + \min(a',d') = a' + \underbrace{\min(a',b',c',d')}$$

WLOG $c' \leq d'$

Case 1 $a' \leq c' \leq d'$

$$b' = c'+d'-a' \geq a'+a'-a' = a'$$

$$a'+a' = a'+a'$$

$$c' = \min(c',a',d')$$

Case 2 $c' \leq a' \leq d'$

$$c'+a' = a' + \min(b',c') = a'+c'$$

Suppose $b' < c'$.

$$c'+d' = b'+a' \leq b'+d' < c'+d' \rightarrow \text{contradiction!}$$

so $b' \geq c'$, $\min(b', c') = c'$

<u>Case 3</u> $c' \leq d' \leq a'$

$$\underset{c'}{\underset{\|}{\min(a', c')}} + \underset{d'}{\underset{\|}{\min(a', d')}} = \underset{a'}{\underset{\|}{a'}} + \underbrace{\min(a', b', c', d')}_{\substack{\| \\ \min(b', c') \\ \| \\ b'}}$$

✓

Suppose $c' < b'$

$a' + b' \geq d' + b' > d' + c' \rightarrow$ contradiction

so $c' \geq b'$ so $\min(b', c') = b'$

P

$$\boxed{\begin{array}{l} a, b, c \in \mathbb{Z}_+ : \\ \quad a^b \mid b^c \\ \quad a^c \mid c^b \\ \text{Prove} : \quad a^2 \mid bc \end{array}}$$