Competition Preparation for Saudi Arabia Team 2021: Level 4

Nikola Petrović

Lesson 5

Social strategies

Problems:

1. An emperor has told 100 wise men that tomorrow he will assemble them in a row where each wise man will only be able to see in front of him and place on each one a cap of one of k given colors. Each minute one wise man is supposed to guess the color of his hat. The emperor will award all the wise men who guess correctly the color of their cap. What strategy can the wise men come up to maximize the guaranteed number of awarded wise men and what is this number?

2. A guard has assembled 100 prisoners and told them he will place a number on each prisoner's cap from 1 to 100. He may place the same number on more than one prisoner. The prisoner can see only the number on other prisoners' cap but not his. He is not allowed to communicate with the other prisoners once the caps are placed. The prisoner is to write on a piece of paper (that the other prisoners cannot see) what he guesses the number on his cap to be and hand it to the guard. If at least one prisoner is correct in his guess the guard will release all prisoners. The prisoners have time to discuss a strategy but the guard will listen in and exploit any strategy that is not fool-proof. With what strategy can the prisoners ensure their release?

3. You and your five friends have entered a magical cave that opens once every thousand years, where there are 60 bottles on display. One bottle contains a magical potion that grants perfect health and long life to those who drink it, but turns your skin blue for a few hours the morning after you drink it. The other 59 bottles contain water. The cave closes the next day, so there is only one chance to identify the potion, and you can leave the cave with just one bottle. Assuming the magical potion cannot be distinguished from water by taste or smell, how will the six of you find the magic potion with certainty and leave with it before the cave closes?

4. A guard leads 100 prisoners from their mutually isolated cells together into a room with just one switch, initially off. The guard explains to the prisoners that from this point he will lead them one-by-one into the room so that each prisoner will at every point eventually be returned to the room, but that each prisoner will have no way of knowing which prisoners or how many prisoners visited the room in between his visits. While in the room, a prisoner is only allowed to either change the position of the switch or let it remain in the same position he encountered. If at one point a prisoner is sure that all prisoners have visited the room, he is supposed to tell it to the guard. If he is correct, all the prisoners will be released, if not, the prisoners lose the game and all will remain captive. The prisoners have time to discuss a strategy before returning to their cells, but the guard will listen in and exploit any strategy that is not fool-proof. With what strategy can the prisoners ensure their release?

5. A gang of n pirates have found a treasure chest with 100 coins and now have to split the coins amongst themselves. The pirates are all ordered by strength and everyone is aware of the ordering. Under the pirate rules, the strongest pirate suggests a way to split the coins. If at least half of the pirates vote for the split, the coins will be split in that way. If less than half vote for the split, they'll be a mutiny, the strongest pirate will be thrown overboard and the next strongest pirate will become the captain and suggest the split. The pirates behave rationally and know that others also behave rationally. The first priority of each pirate is to remain alive, then to maximize the number of coins he gets and finally the third priority is to eliminate rivals. How many pirates will survive and how will be the coins divided amongst the survivors if:

(a) $n = 100$,

(b) $n = 2021$?

6. A group of n spies has gathered to share a code for their mission. Each spy must remember a number from 1 to n so that: the numbers the spies have are all different from each other and no spy has any knowledge of any other spies number, i.e. they cannot guess it with a probability larger than $\frac{1}{n-1}$. In order to assist them with this task, the spies will lock themselves in n different rooms and use phones to communicate with each other so that only two spies talk to each other at one time. Before going to their rooms the spies have a chance to discuss their strategy. What strategy will the spies use to achieve their goal? (Assume each room is equipped with a perfect random-number generator for any range of integers.)

7. A group of 2020 prisoners is gathered by the guard and told that they will receive freedom in the following way: Each will be given a hat of one of five colors and be given a chance to see everyone else's hat except their own. Then they will be arranged randomly in a line and one-by-one each will be interrogated by the guard. First they will be asked if they know the color of their hat. If they answer 'no', they will be returned to their cell. If they answer 'yes' they must whisper the guess in the warden's ear so the other prisoners don't hear it. If the guess is correct the prisoner is freed, otherwise he is returned to his cell. (Prisoners not yet interrogated can see whether someone is freed or returned to their cell.) As usual, the prisoners are allowed to discuss their strategy beforehand but the guard will listen in and exploit any strategy not fool-proof. Find the largest possible number of prisoners that are guaranteed to be released and with which strategy the prisoners can achieve this.