Email training, N4

October 2-8

Problem 4.1. Let a, b, c are solutions of equation $x^3+x^2-3x-1=0$. Construct an equation which roots are $a+1$, $b+1$ and $c+1$.

Solution 4.1. Since a, b, c are solutions of the equation $x^3+x^2-3x-1=0$ then according to Viet's theorem one has

$$\begin{cases} a+b+c=-1 \\ ab+ac+bc=-3 \\ abc=1 \end{cases}.$$

If $a+1$, $b+1$ and $c+1$ are solutions of the equation $x^3+px^2+qx+r=0$ then, according to Viet's theorem one has

$$\begin{cases} p= & -(a+1+b+1+c+1)=-(-1+3)=-2 \\ q= & (a+1)(b+1)+(a+1)(c+1)+(b+1)(c+1) \\ & =ab+bc+ac+2(a+b+c)+3=-2 \\ r= & -(a+1)(b+1)(c+1)=-(abc+ab+bc+ac+a+b+c+1)=2 \end{cases}$$

So one gets the equation $x^3-2x^2-2x+2=0$.

Solution 2. Note that if the roots of the equation $P(x)=0$ are a, b, c, then the roots of the equation $P(x-1)=0$ will be $a+1$, $b+1$ and $c+1$. It remains to conclude that

$$P(x-1)=(x-1)^3+(x-1)^2-3(x-1)-1=$$

$$x^3-3x^2+3x-1+x^2-2x+1-3x+3-1=$$

$$x^3-2x^2-2x+2.$$

Answer: $x^3-2x^2-2x+2=0$.

Problem 4.2. Let a, b and c are pairwise different numbers. Solve the system of equations

$$\begin{cases} z+ay+a^2x+a^3=0 \\ z+by+b^2x+b^3=0 \\ z+cy+c^2x+c^3=0. \end{cases}$$

Solution 4.2. Consider the polynomial t^3+xt^2+yt+z. The numbers a, b, c are the roots of the polynomial. According to Viet's theorem one has

$$\begin{cases} a+b+c=-x \\ ab+bc+ca=y \\ abc=-z \end{cases}$$

Answer: $x=-a-b-c,\ y=ab+bc+ca,\ z=-abc.$

Problem 4.3. Solve equation in integers

$$x!+13=y^2.$$

Solution 4.3. Note that for the numbers bigger than 4 their factorial is divisible by 10, so their last digit is 0. Therefore for $x \geq 5$ the last digit of the left side of the equation is 3 so it can't be perfect square. So $x \leq$. By verifying each case $x=1,2,3,4$ we conclude that the equation has no solution in integers.

Problem 4.4. Let numbers $x_1, x_2, \ldots, x_n$ are given and each of them is equal either $+1$ or -1. Prove that if

$$x_1x_2+x_2x_3+\ldots+x_nx_1=0$$

then n is divisible by 4.

Solution 4.4. There are n terms in the sum, and half of them are $+1$ and half of them are equal -1, so n is even. Now lets prove that there are even number of -1's. From this we will conclude the statement of the problem. To obtian -1 we need consecutive written numbers $+1$ and -1. Lets write numbers $x_1, x_2, \ldots, x_n$ around the circle and by starting from $x_!$ go around the circle. Note that we move from $+1$ to -1 as many times as we move from -1 to $+1$. So there are even number of terms -1 in the sum $x_1x_2+x_2x_3+\ldots+x_nx_1$.

Problem 4.5. In the cells of infinite grid are written positive integers such, that each number is equal to the arithmetical mean of the 4 neighbor numbers. Prove that all numbers are equal.

Solution 4.5. Choose the smallest number written in the grid. Denote it a. It's equal to the arithmetical mean of it's four neighbors. Each of them has value at least a and their average is equal to a, so we may conclude that all of them are equal a. By repeating the same argument for these 4 neighbors we conclude that their neighbors are equal to a as well and so on we conclude that all numbers on the board are equal to a.

Problem 4.6. Let 10 pairwise different numbers are written on the board. Ali writes on his paper the square of difference ($(a-b)^2$) for all possible pairs , and Bob writes on his paper the absolute value of difference of squares ($|a^2 - b^2|$) for all possible pairs. May it happen that Ali and Bob have the same collection of numbers?

Solution 4.6. Note that if we change all signs in the sequence then the statement of the problem will not change. Therefore we may assume that the biggest by absolute value number is positive. Denote it by x. Let y is the smallest number within the numbers. If y is negative then we get

$$x^2 < (x-y)^2 = z^2 - t^2 < x^2.$$

We get contradiction. So $y \geq 0$ which means all numbers have the same sign. Now we have

$$x^2 - y^2 = (x-y)(x+y) = (z-t)^2,$$

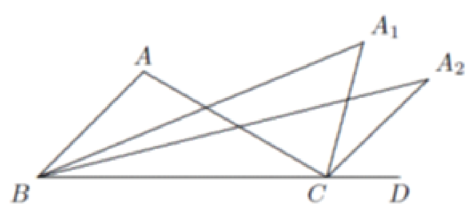
where $x \geq z > t \geq y \geq 0$ so $x+y \geq x-y \geq z-t$. The only possibility is $y = t = 0$ and $x = z$. So 0 is written on the board. By using 0 we get the same collection in $(a-0)^2$ and $|a^2 - 0^2|$. So we may just erase 0 from the board and the conditions of the problem will not change. Then, by repeating the same arguments we may conclude that again 0 is written on the board which is not possible, since all numbers must be pairwise different.

Answer: Not possible.

Problem 4.7. In triangle ABC, $\angle A = 96^o$. Extend BC to an arbitrary point D. The angle bisectors of angle ABC and ACD intersect at A_1, and the angle bisectors of A_1BC and A_1CD intersect at A_2, and so on. The angle bisectors of A_4BC and A_4CD intersect at A_5 . Find the size of $\angle A_5$ in degrees.

Solution 4.7. -

Since A_1B and A_1C bisect $\angle ABC$ and $\angle ACD$ respectively, $\angle A = \angle ACD - \angle ABC = 2(\angle A_1CD - \angle A_1BC) = 2\angle A_1$, therefore $\angle A_1 = \frac{1}{2}\angle A$.



Similarly, we have $A_{k+1} = \frac{1}{2}A_k$ for $k = 1, 2, 3, 4$. Hence

$$A_5 = \frac{1}{2}A_4 = \frac{1}{4}A_3 = \frac{1}{2^3}A_2 = \frac{1}{2^4}A_1 = \frac{1}{2^5}A = \frac{96^\circ}{32} = 3^\circ.$$

Problem 4.8. Let $ABCD$ is a parallelogram. A point M is found on the side AB or its extension such that $\angle MAD = \angle AMO$ where O is the point of intersection of the diagonals of the parallelogram. Prove that $MD = MC$.

Solution 4.8. -

Extend MO to cut CD at N. Since $\angle MAD = \angle AMN$, $AMND$ is an isosceles trapezoid. By symmetry, $AM = NC$ so that $AMCN$ is a parallelogram. Hence $\angle MDC = \angle AND = \angle MCD$ and therefore $MC = MD$.