Problem 20 second case $\}$ ✓

$\boxed{P21}$

P, Q have a real root

$P(a)=0$, $Q(b)=0$ $a, b \in \mathbb{R}$

$$\boxed{P(1+x+Q(x)^2) = Q(1+x+P(x)^2)}$$

$$\Downarrow$$

$$\boxed{P \equiv Q}$$

Prove there is common point of P^2 and Q^2

$\exists\ x_0$: $\boxed{P^2(x_0) = Q^2(x_0)}$

$P(a) = 0$

$Q(b) = 0$

$$(P^2-Q^2)(a) \leq 0 \leq (P^2-Q^2)(b)$$

$P^2(c) = Q^2(c)$

$$\Downarrow$$

$$\boxed{(P^2-Q^2)c = 0}$$

Take common $P^2(c) = Q^2(c)$

$$P(\underbrace{1+c+Q(c)^2}_{d}) = Q(\underbrace{1+c+P(c)^2}_{d})$$

$$\boxed{P(d) = Q(d)}$$

$$\downarrow$$

$e > d$ ⋮ . . . $\boxed{\underbrace{1+d+Q(d)^2}_{d}}$

$$\boxed{P \equiv Q}$$

$\boxed{19}$

$$\begin{cases} x_2 + x_1^2 = 4x_1 \\ x_3 + x_2^2 = 4x_2 \\ x_4 + x_3^2 = 4x_3 \\ \vdots \\ x_n + x_{n-1}^2 = 4x_{n-1} \\ x_1 + x_n^2 = 4x_n \end{cases}$$

hint: Prove that you can put

$$\boxed{\begin{array}{c} x_1 = 4 \cdot \sin^2\alpha \\ \alpha \in \langle 0, \frac{\pi}{2} \rangle \end{array}}$$

$x_1 \leq 4$

$$\boxed{0 \leq x_1 \leq 4}$$

$$\boxed{x_1(x_1 - 4) = -x_2 \leq 0}$$

$x_1 \geq 0$

$\boxed{x_2} = ?$ $\quad x_1$

$$\boxed{\sin^2\alpha + \cos^2\alpha = 1}$$

$$x_2 = x_1(4 - x_1) = 4\sin^2\alpha\,(\underbrace{4 - 4\sin^2\alpha}) =$$
$$= 4\sin^2\alpha \cdot 4 \cdot \cos^2\alpha = 4\sin^2 2\alpha$$

$$\boxed{\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha}$$

$$x_1 = 4\sin^2\alpha$$
$$x_2 = 4\sin^2 2\alpha$$
$$x_3 = 4\sin^2 4\alpha$$
$$\vdots$$
$$x_i = 4\sin^2(2^{k-1}\alpha)$$

$$\sin^2\alpha = \sin^2(2^n\alpha)$$
$$\Downarrow$$
$$\sin\alpha = \pm\sin(2^n\alpha)$$
$$\Updownarrow$$
$$2^n\alpha - \alpha = k\pi \quad \text{or}$$
$$2^n\alpha + \alpha = k\pi$$
for some $k \in \mathbb{Z}$.

$$\sin\alpha = \sin\beta$$
$$\Uparrow$$
$$\alpha + \beta = k\pi$$

$(2^n-1)\alpha$ or $(2^n+1)\alpha$ is multiply of π

so

$\alpha =$

$0, \frac{k\pi}{2^n+1}, \frac{l\pi}{2^n-1}$

2^n

solutions

$k \in \{1 \dots 2^{n-1}\}$

$l \in \{1 \dots 2^{n-1}-1\}$

$\alpha = \frac{k\pi}{2^n+1}$ for some k

$0 \quad \frac{\pi}{2}$

$0 \le \frac{k\pi}{2^n+1} \le \frac{\pi}{2}$

$0 \le k \le \frac{2^n+1}{2}$

$\boxed{0 \le k \le 2^{n-1}}$

$\frac{k\pi}{2^n+1} \quad k \in \{0, 1, \dots, 2^{n-1}\}$

$$0 \le \frac{k\pi}{2^n-1} \le \frac{\pi}{2}$$

$$0 \le k \le \frac{2^n-1}{2} = 2^{n-1}-1$$

$$0 \le k \le 2^{n-1}-1$$

$$\frac{k\pi}{2^n-1} \quad k \in \{0, 1, \ldots, 2^{n-1}-1\}$$

$$0, \quad + 2^{n-1} + 2^{n-1}-1 = 2^n$$

$$22, 23, 24$$