— Number theory for L3 —

— May 2, 2019 — Remainders and divisibility —

Revision

R1. Prove there are infinitely many positive integers that cannot be expressed as the sum of three perfect squares.

Theory

The Chinese remainder theorem: Given pairwise coprime positive integers $m_1, m_2, \ldots, m_k$ and some integers $r_1, r_2, \ldots, r_k$. Then there exists such N that $N \equiv r_i \ (mod\ m_i)$ for all $i \in \{1, 2, \ldots, k\}$. Moreover any two such N differ by a multiple of $m_1 m_2 \ldots m_k$.

T1. Do you remember how to prove the Chinese remainder theorem?

Problems

20. Find the remainder of **a)** 22^{22} mod 15 **b)** 17^9 mod 210.

21. Given positive integer $n > 1$. For any divisor d of the number $n + 1$, Ahmed has written a remainder of n mod d in his notebook, and has written a quotient of n mod d on the board. Prove that sets (more precisely: multisets) of integers on the board and in the notebook are the same.

22. Prove that there exist 2019 consecutive positive integers such that any of them is not a power (bigger than 1) of some integer.

23. Prove that any number of the form 10^{3k+1} cannot be expressed as the sum of two cubes.

24. For which $n > 1$ there exist such positive integers $b_1, b_2, \ldots, b_n$ (not all of them are equal), that for all positive integers k a number $(b_1 + k)(b_2 + k)\ldots(b_n + k)$ is a power (bigger than 1) of some integer number?

25. Prove that all positive integers k for which $k^k + 1$ is divisible by 30 form an arithmetic progression. Find this arithmetic progression.

26. Prove that for any prime p there exist two integers a and b such that $a^2 + b^2 + 1 \vdots p$.

27. A teacher is waiting for a group of children from the grade 10 "M" in which there can be 1 or 2 or 3 .... or 30 people. Next to a teacher there are n different children who have already come. Prove that n can be such a number that no matter how many children from 10 "M" come, it will always be possible to divide the whole group into several same number of ppl carets with more that 1 child (i.e. smaller groups in which the number of children is a perfect square, that is bigger than 1).