Email training, N5

Level 2, October 11-17

Problem 5.1. Let a and b are divisors of n with $a > b$. Prove that $a > b + \frac{b^2}{n}$.

Solution 5.1. Since a and b are divisors of n, therefore $\frac{n}{a}$ and $\frac{n}{b}$ are divisors of n as well. So

$$1 \leq \frac{n}{b} - \frac{n}{a} = \frac{(a-b)n}{ab} < \frac{(a-b)n}{b^2}.$$

After multiplication by $\frac{b^2}{n}$ one gets

$$\frac{b^2}{n} < a - b.$$

Problem 5.2. Do there exist 3 real numbers a, b and c such that the following inequalities hold simultaneously

$$|a| < |b-c|, \quad |b| < |c-a|, \quad |c| < |a-b|.$$

Solution 5.2. From $|a| < |b-c|$ follows $a^2 < (b-c)^2$ or equivalently

$$(a-b+c)(a+b-c) < 0.$$

Applying the same procedure for other conditions one gets

$$(b-c+a)(b+c-a) < 0$$

and

$$(c-a+b)(c+a-b) < 0.$$

By taking the product all 3 inequalities one gets

$$(a-b+c)^2(a+b-c)^2(b+c-a)^2 < 0,$$

which is impossible.

Answer: Not possible.

Problem 5.3. Prove that for $n \geq 1$ the following inequality holds

$$1 + \frac{5}{6n-5} \leq 6^{1/n} \leq 1 + \frac{5}{n}.$$

Solution 5.3. Let's apply Bernoulli inequality.

$$\left(1 + \frac{5}{n}\right)^n > 1 + n \cdot \frac{5}{n} = 6,$$

therefore

$$1 + \frac{5}{n} > 6^{1/n}.$$

Also

$$\left(1 + \frac{-5}{6n}\right)^n > 1 + n \cdot \frac{-5}{6n} = \frac{1}{6},$$

$$\left(\frac{6n-5}{6n}\right)^n > \frac{1}{6},$$

$$6 > \left(\frac{6n}{6n-5}\right)^n,$$

$$6^{1/n} > \frac{6n}{6n-5} = 1 + \frac{5}{6n-5}.$$

Problem 5.4. Let a, b, c are positive and less than 1. Prove that

$$1 - (1-a)(1-b)(1-c) > k,$$

where $k = max(a, b, c)$.

Solution 5.4. Since $0 < 1 - a, 1 - b, 1 - c < 1$ therefore one may state that

$$1 - k > (1 - a)(1 - b)(1 - c),$$

since in right side one multiplier is equal to $1 - k$ and two others are positive and less than one. From that inequality immediately follows that

$$1 - (1 - a)(1 - b)(1 - c) > k.$$

Problem 5.5. Let $x, y, z \geq 0$ and $x + y + z = 3$. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

Solution 5.5. One has

$$3(x + y + z) = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx).$$

Hence it follows that

$$xy + yz + zx = \frac{1}{2}(3x - x^2 + 3y - y^2 + 3z - z^2).$$

Then

$$\sqrt{x} + \sqrt{y} + \sqrt{z} - (xy + yz + zx) =$$

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \frac{1}{2}(x^2 - 3x + y^2 - 3y + z^2 - 3z)$$

$$= \frac{1}{2}\sum_{cyc}(x^2 - 3x + 2\sqrt{x}) = \frac{1}{2}\sum_{cyc}\sqrt{x}(\sqrt{x} - 1)^2(\sqrt{x} + 2) \geq 0.$$

Problem 5.6. Let $a, b, c > 0$. Prove that

$$\frac{a + b}{a^2 + b^2} + \frac{b + c}{b^2 + c^2} + \frac{c + a}{c^2 + a^2} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution 5.6. By applying the AM-GM for the denominator one gets

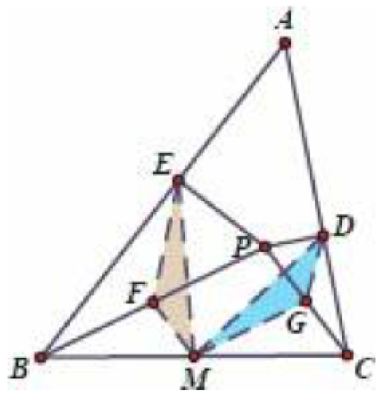
$$\frac{a + b}{a^2 + b^2} \leq \frac{a + b}{2ab} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

By applying the same estimation for 2 other expressions of the left side and by taking the sum we get the desired statement.

Problem 5.7. Let P be a point inside ΔABC such that $\angle PBA = \angle PCA$. Draw $PD \perp AB$ at D and $PE \perp AC$ at E. Show that the perpendicular bisector of DE passes through the midpoint of BC.

Solution 5.7. -

: Refer to the diagram on the below. Let M be the midpoint of BC. Let F, G be the midpoints of BP, CP respectively.



In the right angled triangle ΔBEP, $EF = \frac{1}{2}BP$.

In ΔBCP, MG is a midline and hence, $MG = \frac{1}{2}BP$ and $MG // BP$. It follows that $EF = MG$.

Similarly, $FM // CP$ and $FM = DG$. Now $FPGM$ is a parallelogram.
Notice that $\angle EFM = \angle EFP + \angle PFM = 2\angle PBA + \angle PFM$.
Similarly, $\angle MGD = 2\angle PCA + \angle PGM$. Since $\angle PFM = \angle PGM$ (in the parallelogram $FPGM$) and given that $\angle PBA = \angle PCA$, we must have $\angle EFM = \angle MGD$.
Now $\Delta EFM \cong \Delta MGD$ (S.A.S.), which implies $MD = ME$. It follows that M lies on the perpendicular bisector of DE.