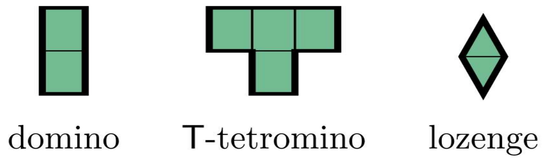
— Combinatorics for L2 —

— January Camp, 2022 — Board Colorings (1): Checkerboard —

Warm-up.

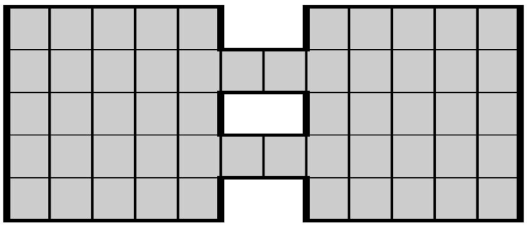
- Get familiar with the following shapes:



- Can an 8×8 board with two opposite corners removed be tiled with dominoes?
- Can an 8×8 chessboard with any two cells of different colors removed be tiled with dominoes?

1. Can a 10×10 board be tiled with T-tetrominoes?

2. Can the figure shown in the picture below be covered with dominoes?



3. Can a $7 \times 7 \times 7$ cube with central cell removed be built from $1 \times 1 \times 2$ bricks?

4. A 7×7 square is tiled with 24 dominoes and one 1×1 square. Determine all possible positions of the single 1×1 tile.

5. In this problem cells are equilateral triangles of side length 1 (*unit triangles*). *Lozenge* is a figure consisting of two adjacent cells. Suppose that an equilateral triangle of side length n is tiled with a collection of unit triangles and lozenges. Find the smallest possible number of unit triangles used in such a tiling.

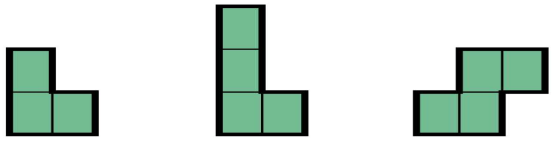
6. Can a $6 \times 6 \times 6$ cube be built from $1 \times 2 \times 4$ bricks?

— Combinatorics for L2 —

— January Camp, 2022 — Board Colorings (2): Classics —

Warm-up.

- Get familiar with the following shapes:



L-trimino L-tetromino S-tetromino

- Find all n for which a $2n \times 2n$ board can be tiled with equally many vertical and horizontal dominoes.

- An 8×8 board is covered by twenty-one 1×3 tiles and one unit square. Find all possible positions of the unit square.

- Suppose that 1×4 and 2×2 tiles are covering figure $\mathcal{F}$. Prove that it's impossible to replace one of these tiles with a tile of the other kind and then reassemble all tiles in a way that they will still be covering $\mathcal{F}$.

7. Can a 10×10 board be tiled with 1×4 rectangles?

8. Call an $a \times b$ tile with both a and b even an *even rectangle*. Suppose that an $n \times n$ board, where n is odd, is tiled with even rectangles and unit squares. Determine the smallest possible number of unit squares used in such a tiling.

9. We are at disposal of the following tiles: S-tetrominoes and L-triminoes. Determine the smallest number of such tiles needed to cover a 7×7 board.

10. A rectangle is tiled with L-tetrominoes and S-tetrominoes. Prove that the number of used L-tetrominoes is even.

11.

(a) Find all positive integers n with the property that $n \times n$ square can be covered with 2×2 and 3×3 tiles.

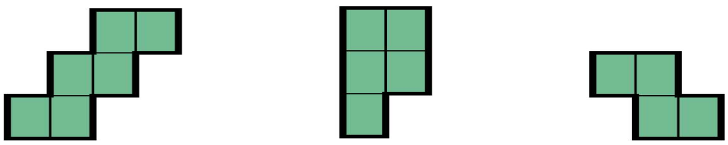
(b) For each such n determine the smallest possible number of used 3×3 squares.

— Combinatorics for L2 —

— January Camp, 2022 — Board Colorings (3): Advanced —

Warm-up.

- Get familiar with the following shapes:



snake P-pentomino Z-tetromino

- Several 1×4 tiles, 2×2 tiles and S-tetrominoes cover an 8×8 board. Prove that the number of used 2×2 tiles is even.

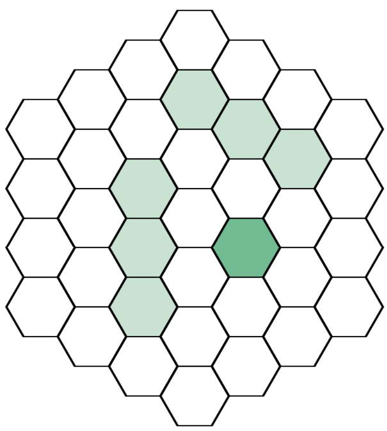
12. *Snake* is a figure composed of 6 unit cells presented in the picture (or any figure obtained from it by reflection or rotation). Find the greatest number of snakes that can be cut from a 6×11 piece of paper dissected into unit squares (each snake should contain six whole such cells).

13. A *honeycomb* composed of 37 hexagonal cells is to be dissected into 12 sticks of three neighboring (collinear) cells and a single cell. Find all possible positions of the single cell.

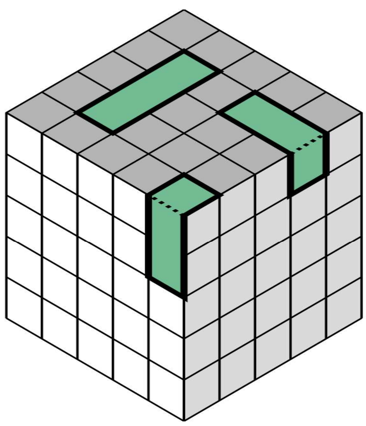
14. A 7×7 board is to be covered with one-sided P-pentominoes (i.e. they can be rotated, but cannot be flipped over) in such a way that exactly one cell is covered with two tiles and all the remaining cells are covered with exactly one tile. Determine all possible placements of the "doubly-covered" cell.

15. Consider three faces of a $5\times5\times5$ cube having a common vertex, each divided into cells. Can they be covered with twenty-five 1×3 stickers (tiles that can be folded), each covering whole cells?

16. In this problem we don't allow flipping tiles over (so S- and Z-tetrominoes are considered different). Suppose that a figure $\mathcal{F}$ can be tiled with S-tetrominoes. Prove than no matter how we tile $\mathcal{F}$ using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.



honeycomb



cube and stickers

— Combinatorics for L2 —

— January Camp, 2022 — Induction in Combinatorics —

Warm-up.

- Prove that a $2^n \times 2^n$ board with an arbitrary 1×1 cell removed can be tiled with L-triminoes.
- Prove that an L-shaped hexagon of consecutive sides of lengths n, n, n, n, $2n$, $2n$ can be tiled with L-triminoes.
- Given is an $n \times n$ chessboard with n odd and black corners. Find all n's for which all black cells can be covered with L-triminoes and for each such n determine the smallest number of L-triminoes needed.
- Find the number of regions into which n lines divide the plane (assuming that no two lines are parallel and no three lines are concurrent).
- In a country there are n cities and each two joined by a one-way road (in only one direction). Prove that there exists a city from which you can travel to every other city using at most 2 roads.

17. Suppose that in a group of $n \geq 4$ people everyone has a piece of information, and all these pieces are different. If two people make a phone call, they share all the information that they have at that moment. Prove that $2n-4$ calls are enough to make everyone know everything.

18. In a tournament with $n \geq 3$ participants every two played exactly one game and there were no ties. After the tournament all players took seats at a round table in such a way that everyone defeated their direct right-neighbor. Prove that there exist players A, B, C such that A defeated B, B defeated C, and C defeated A.

Bonus: Prove that there are at least $n-2$ such triples of players.

19. Some cells of an $m \times n$ chessboard are occupied by *rooks*. One rook *attacks* another if they are placed in the same row or column and there are no other rooks between them. Suppose that each rook is attacked by *at most two* other rooks. Find (in terms of m, n) the largest number of rooks for which such situation is possible.

20. At a party there are $n \geq 2$ people with hats. Every two of them greeted each other, where each greeting had a form of exchanging the hats (worn by the greeters at the moment). Prove that if n is divisible by 4, then it could happen that after all of the greetings everyone has their own hat back.