Email training, N5

September 22-28, 2019

Problem 5.1. Let a, b, c be real numbers such that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1.$$

Prove that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=0.$$

Problem 5.2. The set $\{1, 2, ..., 10\}$ is partitioned to three subsets A, B and C. For each subset the sum of its elements, the product of its elements and the sum of the digits of all its elements are calculated.

Is it possible that A alone has the largest sum of elements, B alone has the largest product of elements, and C alone has the largest sum of digits?

Problem 5.3. Find all positive integers n for which

$$3x^n + n(x+2) - 3 \geq nx^2$$

holds for all real numbers x.

Problem 5.4. Denote by $P(n)$ the greatest prime divisor of n. Find all integers $n \geq 2$ for which

$$P(n) + [\sqrt{n}] = P(n+1) + [\sqrt{n+1}].$$

(Note: $[x]$ denotes the greatest integer less than or equal to x.)

Problem 5.5. Two players play the following game. At the outset there are two piles, containing 10.000 and 20.000 tokens, respectively. A move consists of removing any positive number of tokens from a single pile or removing $x > 0$ tokens from one pile and $y > 0$ tokens from the other, where $x + y$ is divisible by 2015. The player who cannot make a move loses. Which player has a winning strategy?

Problem 5.6. Find all quadrilaterals $ABCD$ such that all four triangles DAB, CDA, BCD and ABC are similar to one-another.

Problem 5.7. Three circles ω_1, ω_2 and ω_3 pass through one common point, say P. The tangent line to ω_1 at P intersects ω_2 and ω_3 for the second time at points $P_{1,2}$ and $P_{1,3}$, respectively. Points $P_{2,1}$, $P_{2,3}$, $P_{3,1}$ and $P_{3,2}$ are similarly defined. Prove that the perpendicular bisector of segments $P_{1,2}P_{1,3}$, $P_{2,1}P_{2,3}$ and $P_{3,1}P_{3,2}$ are concurrent.

Problem 5.8. Let ABC be a triangle with $\angle A = 60^o$. Points E and F are the foot of angle bisectors of vertices B and C respectively. Points P and Q are considered such that quadrilaterals $BFPE$ and $CEQF$ are parallelograms. Prove that $\angle PAQ > 150^o$. (Consider the angle PAQ that does not contain side AB of the triangle.)

Solution submission deadline September 28, 2019