Saudi Arabia – Online Math Camp
April 2021. – Level L2

Number Theory

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Problems – April 22

1. If p is an odd divisor of $a^{2^n}+1$, prove that $p \equiv 1 \pmod{2^{n+1}}$.
2. Let p be a prime and let n be coprime to $p-1$. Prove that th numbers $1^n, 2^n, \dots, (p-1)^n$ are all distinct modulo p.
3. If $a>1$, prove that $\varphi(a^n-1)$ is divisible by n.
4. (a) Find all positive integers n such that n divides 2^n-1.

   (b) Find all positive integers n such that n divides $2^{n-1}+1$.
5. Find (a) the order of 2 modulo 3^n; the order of 3 modulo 2^n.
6. Find the largest (a) power of 2; (b) power of 11 that divides $3^{100!}-1$.
7. Let p be an odd prime divisor of $a-b$, not dividing a or b.

   (a) If $p \nmid n$, prove that $p \nmid \frac{a^n-b^n}{a-b}$.

   (b) $\frac{a^p-b^p}{a-b}$ is divisible by p, but not by p^2.

- Let a and b be integers with $|a| \neq |b|$. Prime p divides $a-b$, but not a and b. Then
$$v_p(a^n-b^n) = v_p(a-b) + v_p(n),$$
provided that either $p>2$, or $p=2$ and $4 \mid a-b$. *(Lifting the exponent Lemma)*

8. What is the largest power of 5 dividing $2^{299}+2^{199}-1$?
9. Find all positive integers n for which n^2 divides 2^n+1.
10. Find all positive integers x, y and n for which $x^3+y^3=3^n$.
11. Let $n>2$ be a positive integer. Prove that the number $2^{2^n-1}-2^n-1$ is composite.