LOGIC, SETS AND REALS

Constructing The Real Numbers Using Dedekind Cuts

by

Terri Lee Stover

Candidate, MS Mathematics

Department Mathematics and Computer Science

Texas Woman's University, Denton, Texas 76204

March 22, 2018

Contents

0.1 **Introduction** 3

0.2 **Historical Background** 4

0.3 **Review of Related Literature** 5

0.3.1 Numbers and Mathematics by Clayton W. Dodge 5

0.3.2 An Introduction to the History of Mathematics, Sixth Edition, by Howard Eves 6

0.4 **Construction of the Real Numbers Using Dedekind Cuts** 6

0.4.1 The Idea Behind a Cut 7

0.4.2 The Definition of a Cut 7

0.4.3 Dedekind Cut by A Rational Number 8

0.4.4 A Cut Made by an Irrational Number 8

0.5 **Conclusion** 9

List of Figures

1 Real Numbers via Flow Chart 3

2 Incommensurable quantity 5

3 Cut at real number 'n' 7

4 Division of the reals by $\sqrt{2}$ 9

0.1 Introduction

Throughout history, mathematicians have pondered the idea of a number. In the beginning, the "emergence in primitive man of the concept of *number* and the process of *counting*" gave us the natural numbers or counting numbers [Eves, p.7]. Through the discovery of rational and irrational numbers, the real number system was formed.
Figure 1 shows the composition of the real number system. The development of numbers and the properties associated with them continue to evolve.

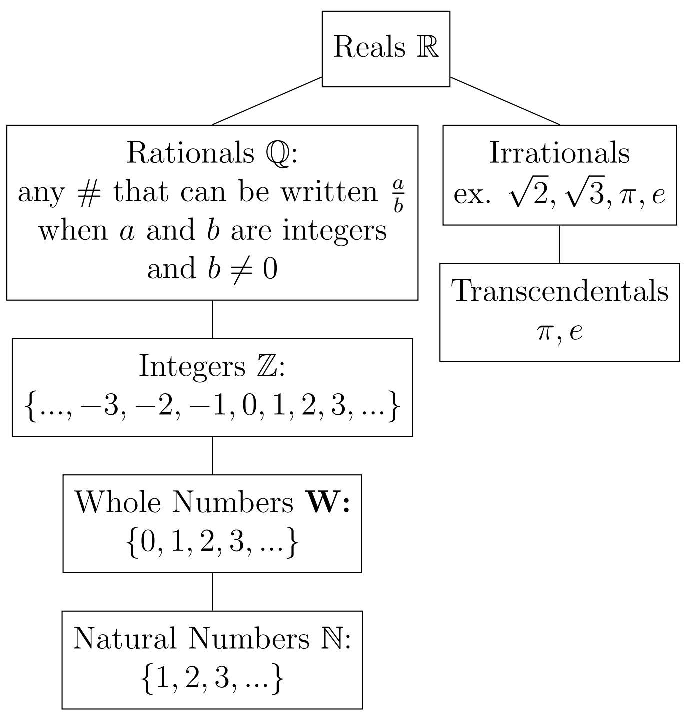

Figure 1: Real Numbers via Flow Chart

Through the 18th century, advances were made in algebra, geometry, and other areas of mathematics. During the 19th century, though, there was a renewed interest to the basic ideas of a number and number sense. The question was raised, how does one construct the real number set using the rational numbers? Two approaches for this construction were discovered. One is called Dedekind cuts and the other Cauchy sequences. This paper will focus on the Dedekind cut method for constructing the reals.

The Dedekind cut method divides the rational numbers into two disjoint sets. Every rational number determines a unique Dedekind cut. The cut may also be made by an irrational number. There are criteria that need to be met in order to identify a true Dedekind cut. Together, the rational and irrational numbers characterize the set of real numbers. The cuts made by both rational and irrational numbers will be discussed in more detail below.

0.2 Historical Background

Richard Dedekind (1831-1916), the man behind the method of Dedekind cuts, was a German mathematician in the nineteenth century. He developed the idea of Dedekind cuts in 1858 but did not publish his findings until fourteen years later in 1872. In the published article he wrote,

As professor in the Polytechnic School in Zurich I found myself for the first time obliged to lecture upon the ideas of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitufe to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continuously but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences.... This feeling *of dissatisfaction was so overpowering that I made the fixed resolve to keep mediating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis* [O'Connor, September 2001].

0.3 Review of Related Literature

0.3.1 Numbers and Mathematics by Clayton W. Dodge

This book delves into the development of the number system. It tells the story of a Pythagorean named Hippasus (ca. 470 B.C.) who became outspoken about some numbers not being rational. The Greeks had already discovered that the side and diagonal lengths of a square were not commensurable quantities. For a square with sides of length one unit measure the diagonal length would be $\sqrt{2}$.

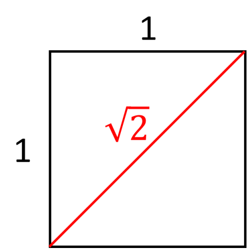

Figure 2: Incommensurable quantity

This outspoken theory of Hippasus so devastated the Pythagorean's assumption that all things evolved around the whole numbers and their ratios, that Hippasus was thrown off a ship into the Mediterranean Sea and drowned. It took many years before mathematicians fully accepted the reality of the irrational numbers. Another Pythagorean, Eudoxus (408-355 B.C.) developed a theory of proportions that took into account the irrational numbers. This theory was received well.

It was not until the nineteenth century, 2200 years after Exodus, that more in depth discoveries were made about numbers. In 1872, Richard Dedekind's theory of Dedekind cuts, inspired from the ideas of Eudoxus, took into account both the rational and irrational numbers. This theory from Dedekind formed a way to discuss the entire Real Number System and to prove that it is a complete ordered field.

0.3.2 An Introduction to the History of Mathematics, Sixth Edition, by Howard Eves

In early mathematics, the world evolved around the natural numbers and the ratios of those numbers. For sometime $\sqrt{2}$ was the only known irrational number and for years was a "scandal" in the math community[Eves, p 82]. Like Dodge, Eves traced the development and knowledge of rational and irrational numbers from the early Greeks, to Exodous, and then to Dedekind. Dedekind had a geometrically inspired analysis of the rational numbers by placing all of them on a number line and realizing the irrational numbers filled in the gaps. Since a number line is a make up of all existing numbers, it can be visually seen as an ordered field.

0.4 Construction of the Real Numbers Using Dedekind Cuts

The set of real numbers contain all points that compose a number line. Every point on the number line designates either a rational number or an irrational number. Dedekind's method constructs the real numbers from the set of rational numbers. It is geometric in nature. The set of real numbers are defined by compiling all the Dedekind cuts.

0.4.1 The Idea Behind a Cut

The idea behind the construction of a Dedekind cut is that every real number will cut the number line into two subsets of the rational numbers. One set is all numbers less than the number "n" at the cut point. The other set is all numbers greater than or equal to the number "n" at the cut point. If the two sets of rational numbers are known, then the real number "n" of the cut can be found. There are properties that need to be satisfied.

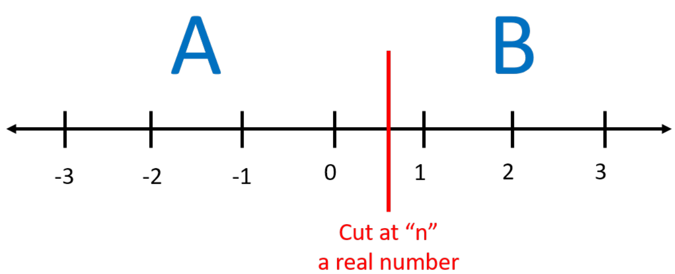

Figure 3: Cut at real number 'n'

0.4.2 The Definition of a Cut

A simple definition of a Dedekind cut in mathematical terms:

A Dedekind cut is a pair of subsets $A, B \in \mathbb{Q}$ satisfying the following properties.

1. $A \cup B = \mathbb{Q}$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$
2. $a < b$, $\forall a \in A$ and $b \in B$
3. A has no largest element

The following is a summary of the meaning of the mathematical defintion:

A Dedekind cut (A, B) is a division of the rational numbers into two subsets A and B that do not have any elements in common. Together A and B contain all the rational numbers. All elements in set A are greater than every element in set B. We can even define addition and multiplication on the set of Dedekind cuts. Therefore, Dedekind cuts will satisfy the properties of the real numbers.

0.4.3 Dedekind Cut by A Rational Number

Given a real number r, an example of a cut in the form

$$[r] = (\{q \in \mathbb{Q} : q < r\}, \{q \in \mathbb{Q} : q \geq r\})$$

would represent a cut made by a rational number.

0.4.4 A Cut Made by an Irrational Number

Since a Dedekind cut relies on two subsets of rational elements, it becomes a bit tricky when a cut is made by an irrational number. The most famous irrational cut is the one at $\sqrt{2}$.

For a Dedekind cut (A, B) to be $\sqrt{2}$, the two subsets must be defined by the following:

$A = \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\}$ and $B = \{x \in \mathbb{Q} \mid x \geq 0 \text{ or } x^2 \geq 2\}$

These two sets will satisfy the properties of Dedekind cuts and define the cut at $\sqrt{2}$.

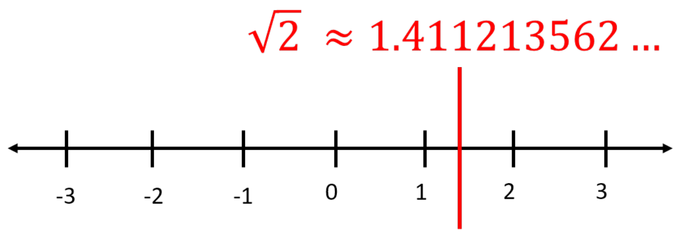

Figure 4: Division of the reals by $\sqrt{2}$

0.5 Conclusion

The power of a number is evident in the history of its evolution. Even with all the higher order mathematical discoveries, all that is math really boils down to numbers and number sense. With Dedekind cuts, the real numbers come alive on a number line through the two sets of rational numbers. The irrational numbers then come in to fill the gaps made between the rational elements on the line. In doing so, all Dedekind cuts together form the real number system which serves as the foundation of all that is Math.

Bibliography

[1] Clayton W.Dodge, *Numbers and Mathematics*, Prindle, Weber & Schmidt, Incorporated, Boston, 1969.

[2] Howard Eves, *An Introduction to the History of Mathematics*, Sixth Edition, Brooks/Cole-Thomson Learning, 1990.

[3] Standford Encyclopedia of Philosophy, Dedekind's Contributions to the Foundations of Mathematics, `https://plato.stanford.edu/entries/dededekind-foundations/`.

[4] John O'Connor, "Dedekind cuts", September 2001, `http://www-history.mcs.st-and.ac.uk/~john/analysis/Lectures/A3.html`.