Taking the SVD of a Matrix

Oscar Alvarez
Texas Woman's University
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1 Solving for the SVD

1.1 The SVD of a matrix.

The SVD is defined as the singular value decomposition of a matrix.

We are going to be dealing with eigenvalues and eigenvectors so let us say that we do not have a perfectly square matrix. (We have an nxm instead of an nxn matrix). https://preview.overleaf.com/public/swznrzxzdmjf/images/a052cb2b9fb18e98742752a010a4ae4a6e4db884.jpeg

When we break down a matrix into the SVD, the matrix is divided into three different individual matrices. The equation would look like this:

$$A = U\Sigma V^T \tag{1}$$

1.2 Solving for the SVD of a matrix.

- The first step would be to find either A^TA or AA^T. We have the option to use either one in the case that we do not have a perfectly square matrix. If we began with a 2x3 matrix, we would rather solve for AA^T because the matrix that turns out will be a 2x2 - instead of a 3x3. For example if we had

$$C = \begin{bmatrix} 2 & -2 & 1 \\ 5 & 1 & 4 \end{bmatrix}$$

then,

$$CC^T = \begin{bmatrix} 2 & 2 & 1 \\ 5 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 24 \end{bmatrix}.$$

This 2x2 matrix will be a lot easier to work with when solving for its eigenvalues and eigenvectors.

1.2.1 Example using a 2x2 matrix

We will now solve for the SVD of A_{2x2}.
Let us say

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

We begin by solving for $A^T A$ or AA^T

$$A^T A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} = Q \tag{2}$$

We will now need to find the eigenvalues and vectors of Q. We will use this equation to solve for the eigenvalues first. Take note that this equation only works when solving for 2x2 matrices.

$$\lambda^2 - \text{trace}(Q)\lambda + \det(Q) = 0 \tag{3}$$

We will begin by finding the trace(Q). The trace(Q) is equal to the sum of the diagonals of Q.

$$Q = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \qquad \text{then,} \qquad \text{trace}(Q) = 5 + 5 = 10$$

Next, we will solve for the det(Q)

$$\det(Q) = (5 \cdot 5) - (3 \cdot 3) = 16$$

Using equation 3, we now have

$$\lambda^2 - 10\lambda + 16 = 0.$$

Solving the equation will give us our eigenvalues.

$$(\lambda - 2)(\lambda - 8) = 0.$$

$$\lambda_1 = 2 \qquad\qquad \lambda_2 = 8. \tag{4}$$

Once we have our eigenvalues, we need to solve for our eigenvectors.
Using

$$(Q - I_2\lambda) = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix},$$

with

$$\lambda_1 = 2 \qquad \text{and} \qquad \lambda_2 = 8$$

we get

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We will now normalize the eigenvectors.

$$\| v_1 \|_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \| v_2 \|_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Create a matrix with the eigenvectors. This matrix will represent our V matrix which will be transposed in equation 1.

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \tag{5}$$

We will need a different equation when solving for the U matrix. To do this we will have to use our eigenvalues that we have already solved for (4).

This is the equation that we will need to solve for U.

$$u_i = \frac{1}{\sigma} A v_i \tag{6}$$

To find σ, we take the square root of our respective eigenvalues.

$$\sigma_1 = \sqrt{2} \qquad \text{and} \qquad \sigma_2 = \sqrt{8} = 2\sqrt{2}$$

Now we plug in our respective values.

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$u_2 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

Our completed U matrix:

$$U = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \tag{7}$$

To create our Σ matrix, we simply place our respective σ_is on the diagonals

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

so that

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}. \tag{8}$$

If you look back at (7) (8) and (5), we now have all of our matrices that make up the SVD of A, our original matrix. If we look back at equation (1) we can now construct our equation.

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \tag{9}$$

We now have the SVD of a $2x2$ matrix.