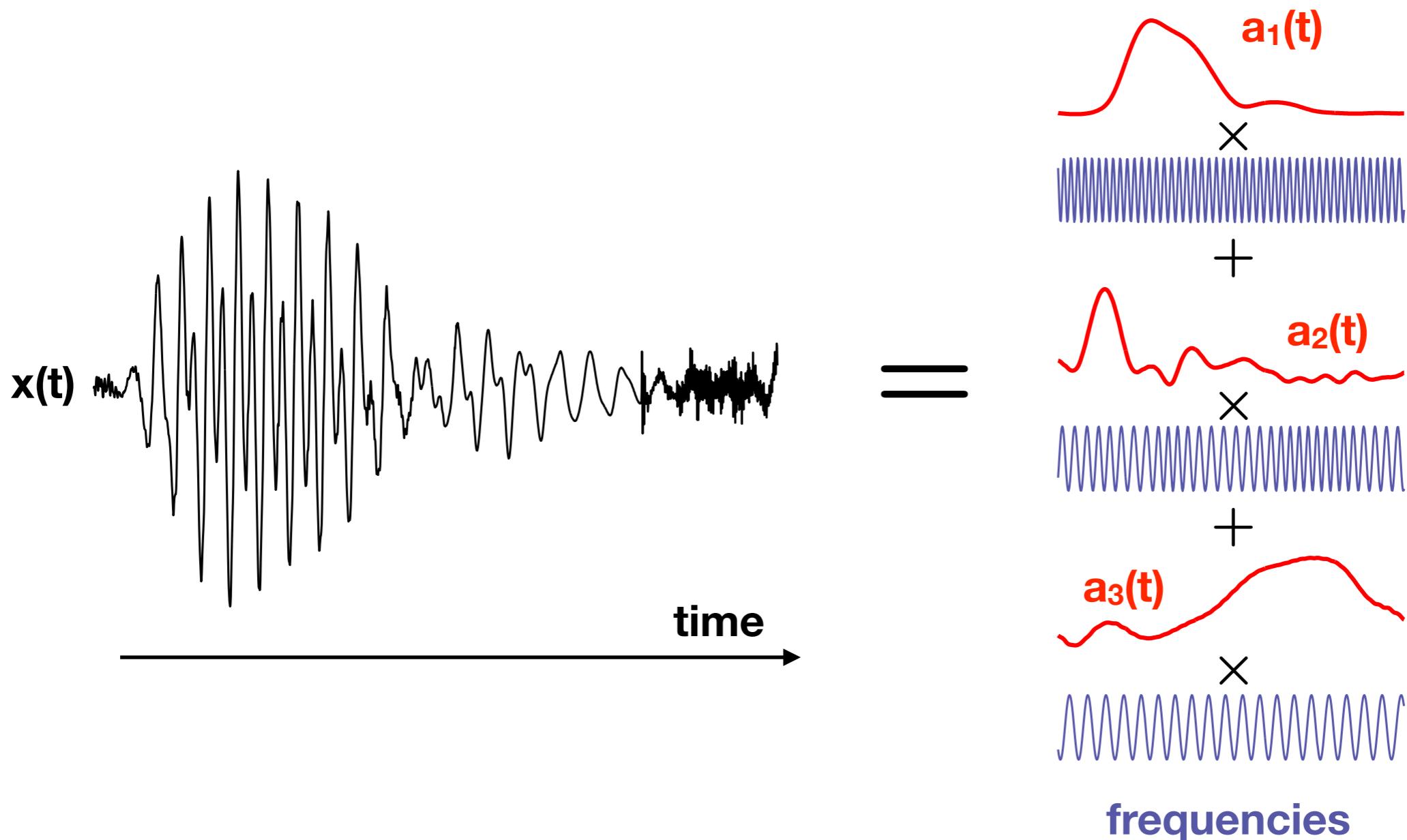


DS-GA 3001.008 Modeling time series data

L12. Spectral methods

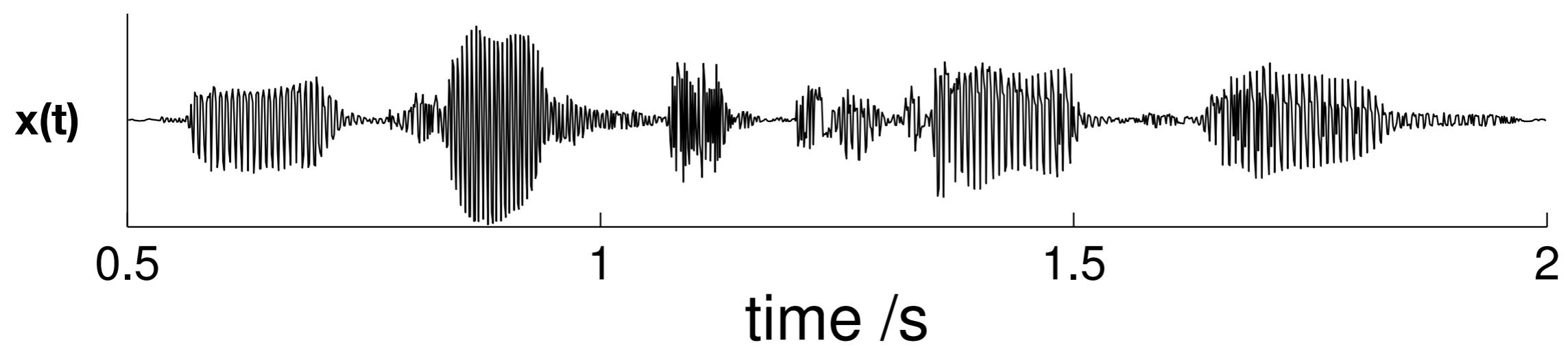
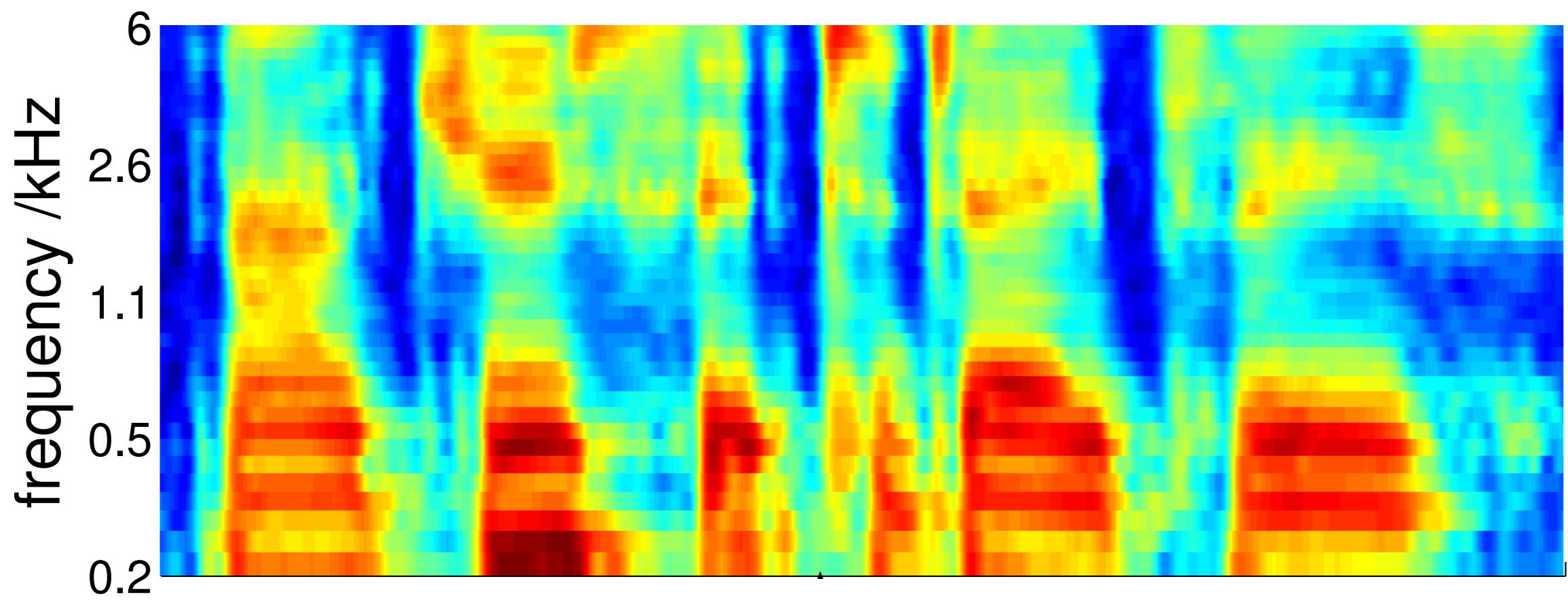
Instructor: Cristina Savin
NYU, CNS & CDS

Change of representation: from time to frequencies



Motivation:

periodic structure is by def. predictable
if it's there in the data then we should take advantage of it



Overview:

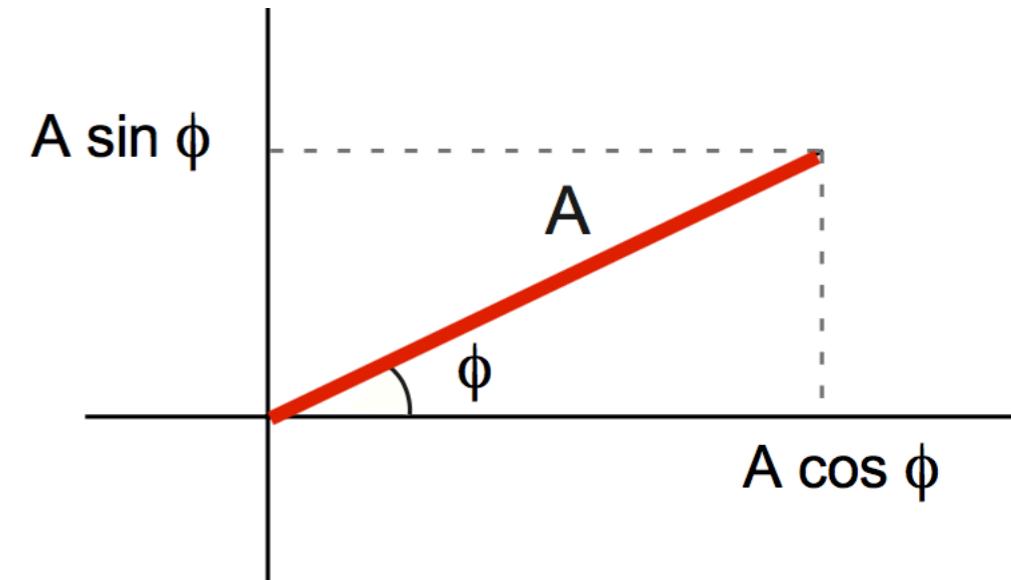
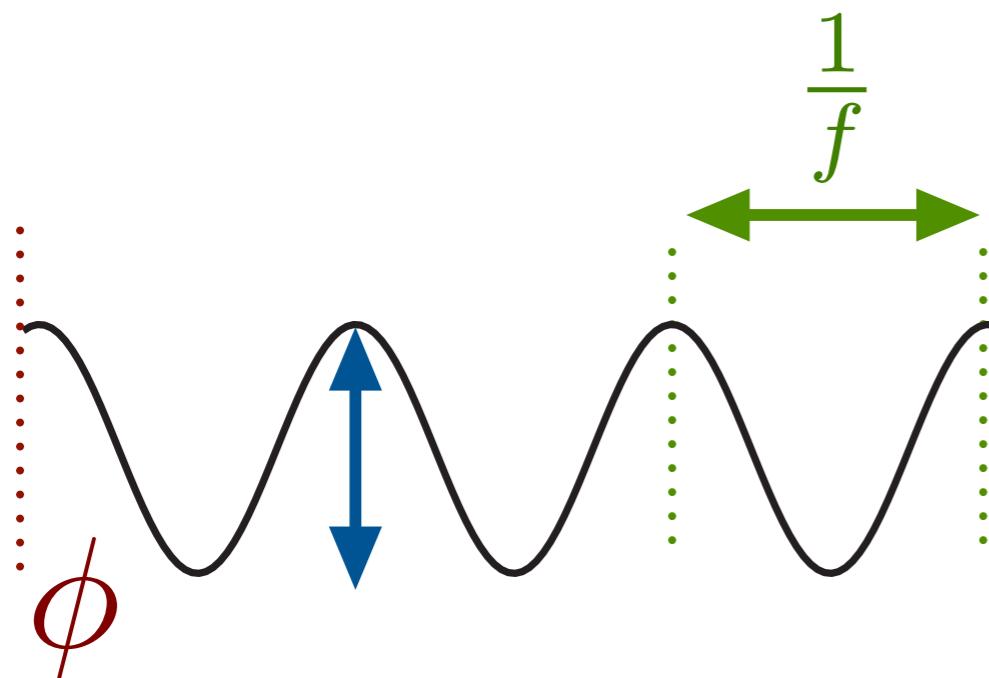
The traditional signal processing view: Fourier transform

Stats perspective (chp. 4 in tsa4.pdf)

Probabilistic spectral analysis

Modeling real-world data

Reminder basics frequency analysis



$$x_t = A \cos(2\pi\omega t + \phi)$$

amplitude frequency phase

alternative forms on the board

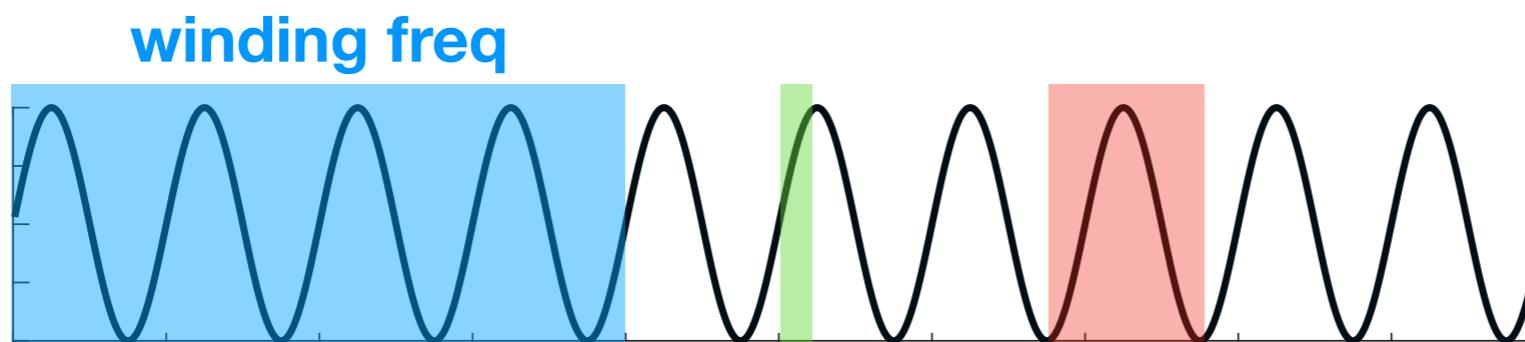
The Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} x(t)e^{-2\pi i \omega t} dt$$

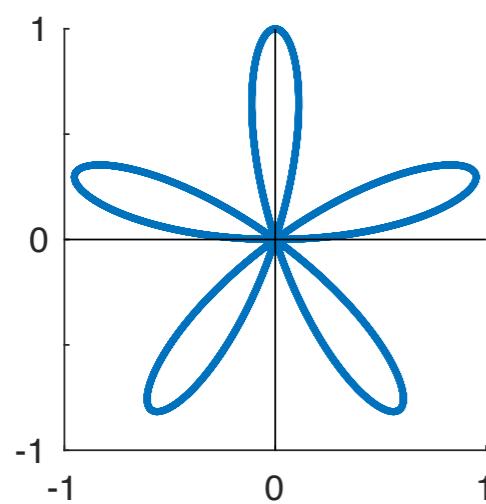
***inverse**

**mathematically convenient

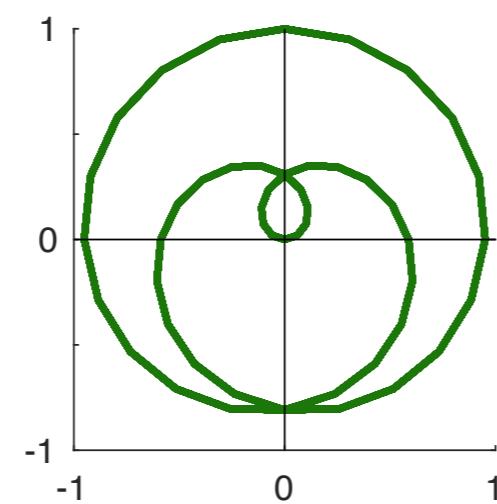
***discrete vs continuous



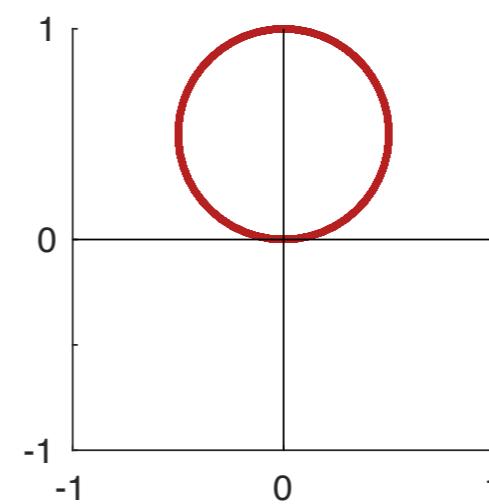
slow



fast



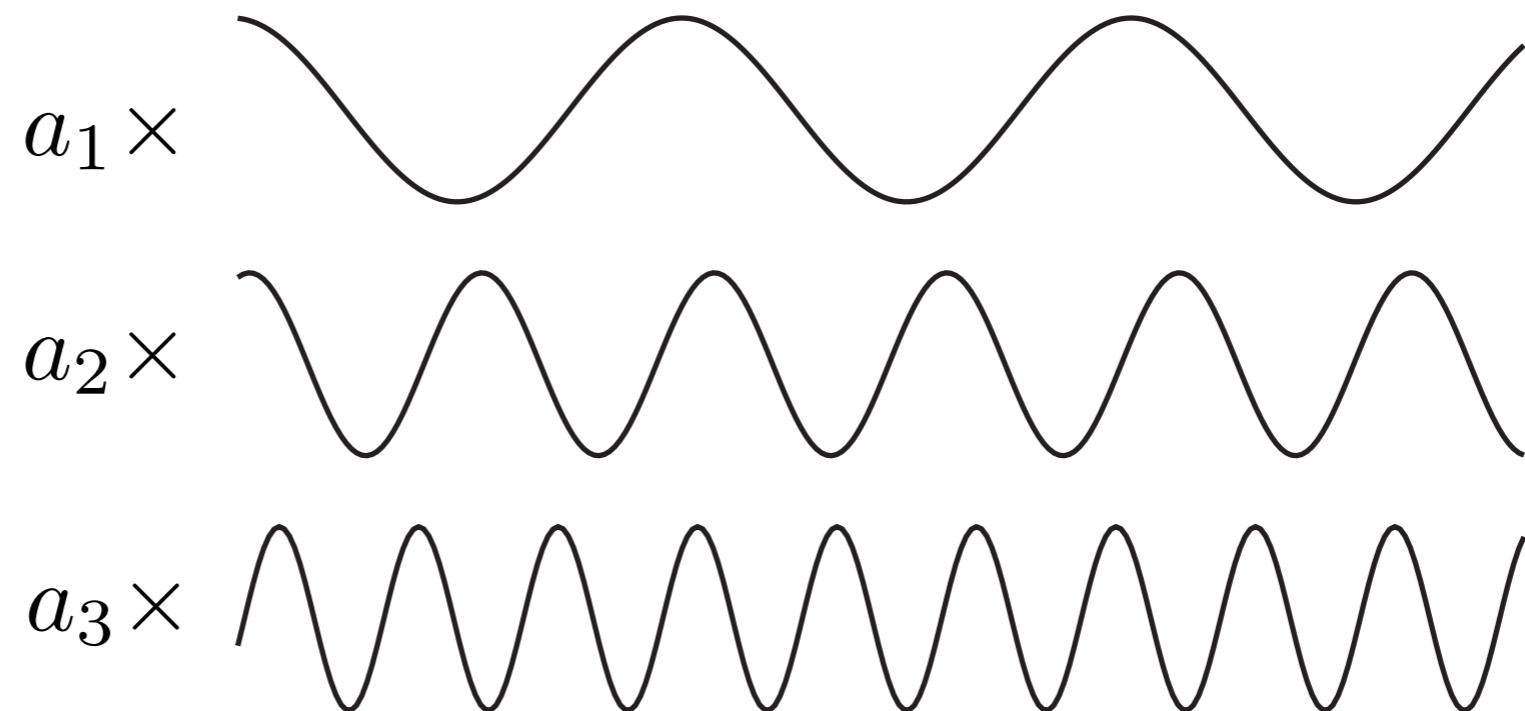
just right



Stationary time series



frequency domain

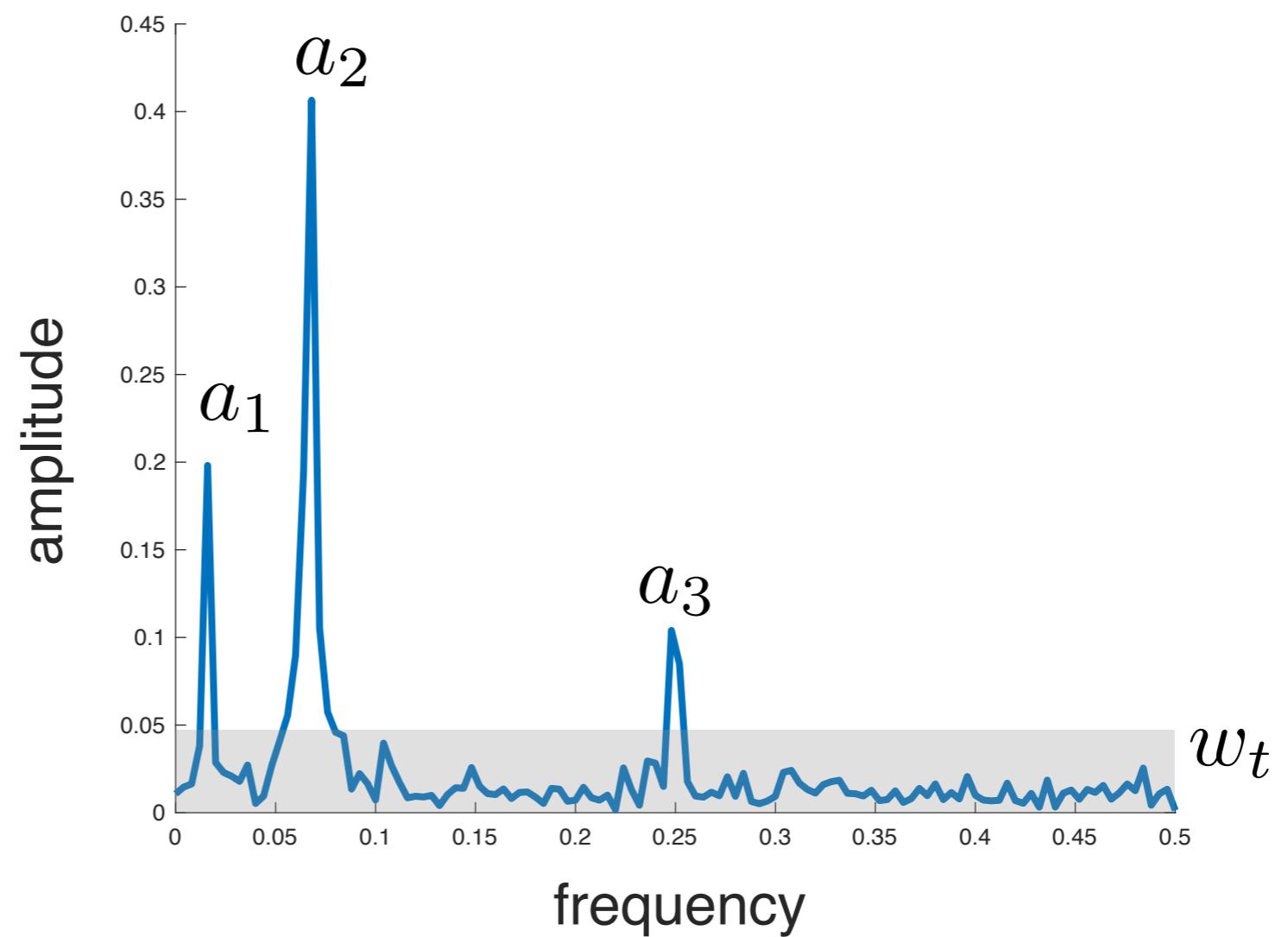


+ white noise

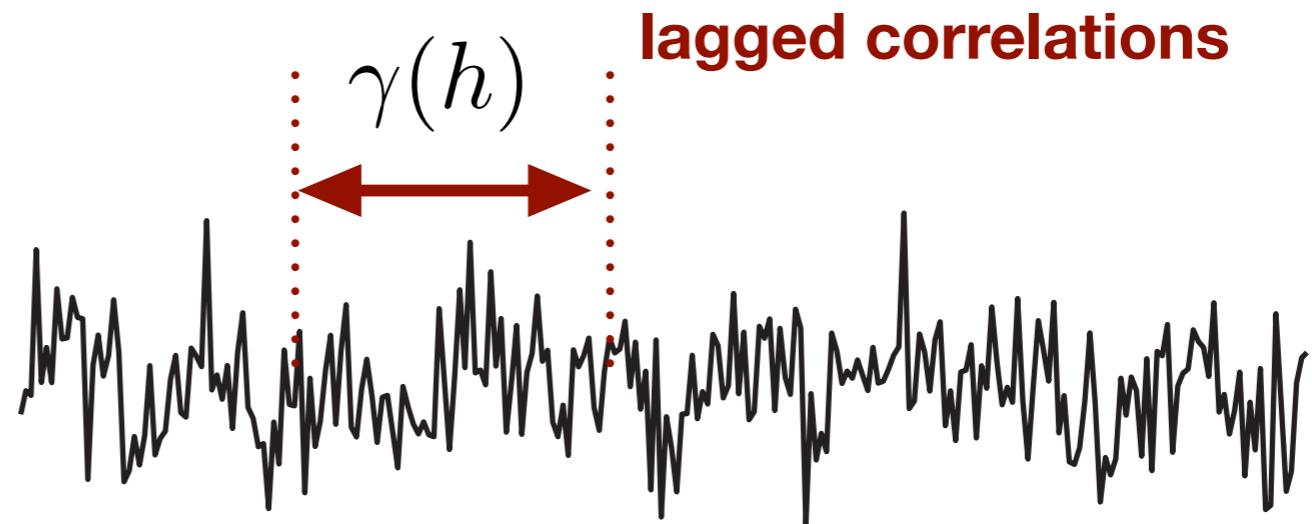
Stationary time series



frequency domain



Stationary time series



ARIMA-style

$$x_t = \sum_{k=1}^q [U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t)],$$

$$\gamma_x(h) = \sum_{k=1}^q \sigma_k^2 \cos(2\pi\omega_k h),$$

Discrete **time**-discrete **frequencies**

$$x_t = a_0 + \sum_{j=1}^{(n-1)/2} [a_j \cos(2\pi t j/n) + b_j \sin(2\pi t j/n)],$$

Nyquist freq/
folding freq

$$a_j = \frac{2}{n} \sum_{t=1}^n x_t \cos(2\pi t j/n) \quad \text{and} \quad b_j = \frac{2}{n} \sum_{t=1}^n x_t \sin(2\pi t j/n). \quad \mathcal{O}(n^2)$$

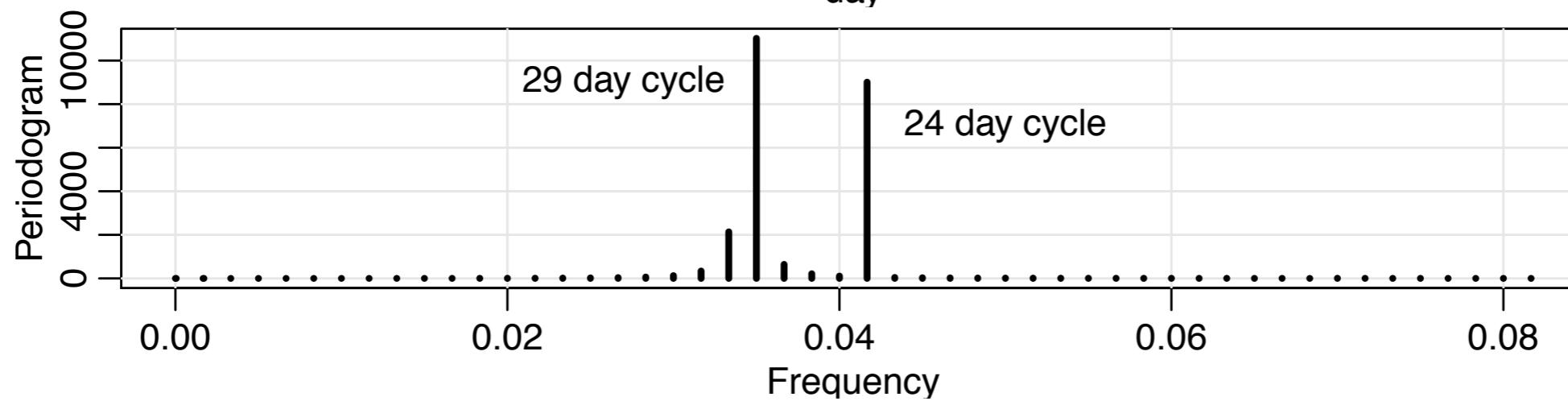
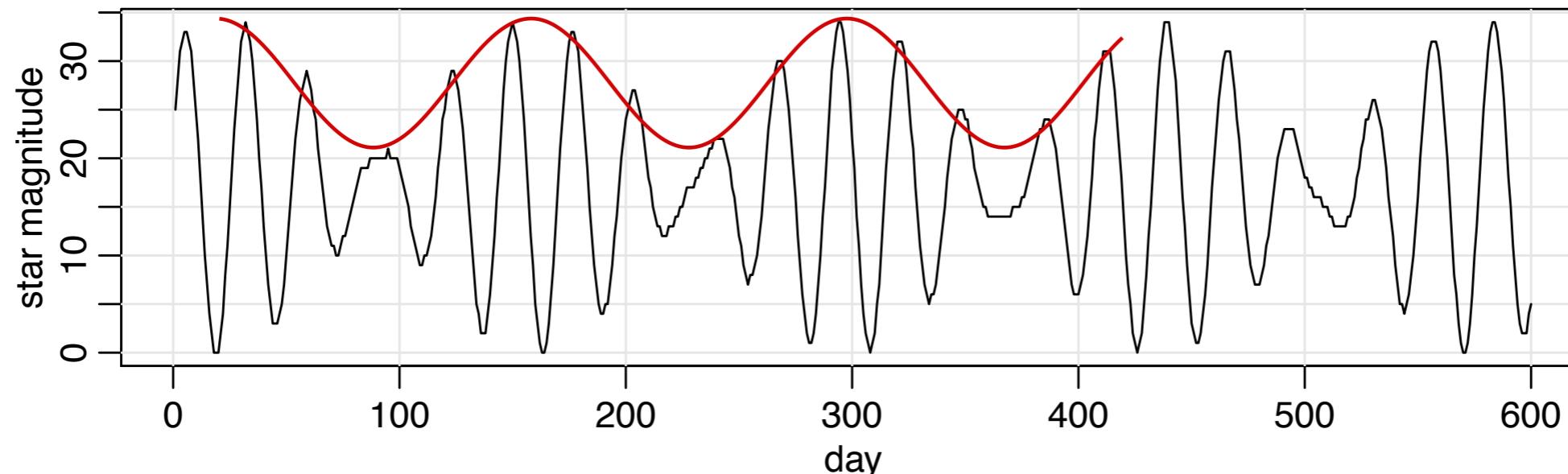
**Scaled
periodogram**

$$P(j/n) = a_j^2 + b_j^2 = \frac{4}{n} |d(j/n)|^2.$$

computed efficiently using **discrete Fourier transform DFT**

$$\begin{aligned} d(j/n) &= n^{-1/2} \sum_{t=1}^n x_t \exp(-2\pi i t j/n) \\ &= n^{-1/2} \left(\sum_{t=1}^n x_t \cos(2\pi t j/n) - i \sum_{t=1}^n x_t \sin(2\pi t j/n) \right), \end{aligned} \quad \mathcal{O}(n \log(n))$$

amplitude modulated signal



Spectral density

$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t)$$

U_1 and U_2 are uncorrelated zero-mean random variables with equal variance σ^2 .

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i \omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i \omega_0 h} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \omega h} dF(\omega)$$

spectral distribution function

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \leq \omega < \omega_0, \\ \sigma^2 & \omega \geq \omega_0. \end{cases}$$

**~cumulative variance
always exists, unique**

$$dF(\omega) = f(\omega) d\omega,$$

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty, \quad (4.15)$$

then it has the representation

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \omega h} f(\omega) d\omega \quad h = 0, \pm 1, \pm 2, \dots \quad (4.16)$$

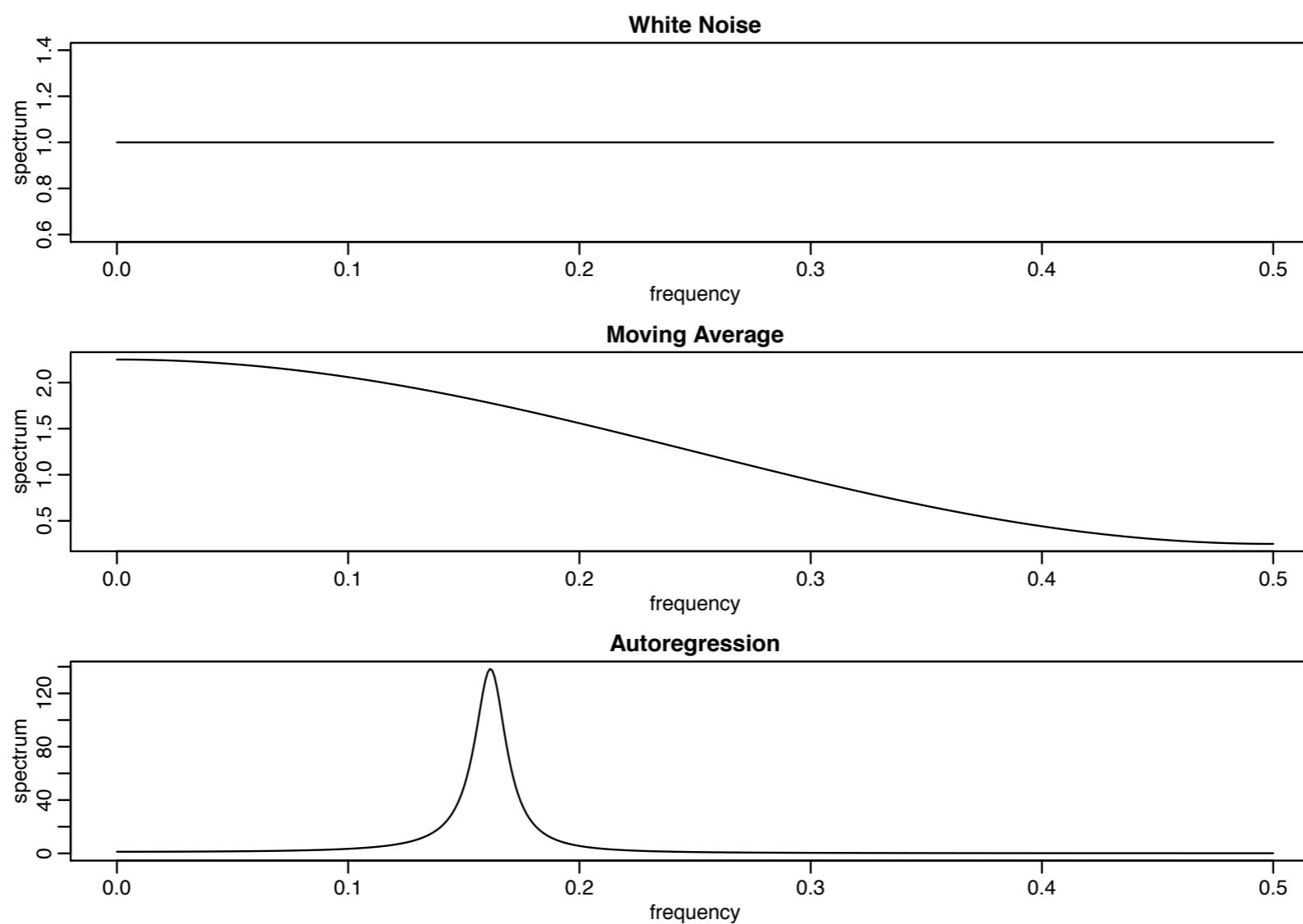
as the inverse transform of the spectral density,

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h} \quad -1/2 \leq \omega \leq 1/2. \quad (4.17)$$

**One-to-one map covariance function and f, invertible
Spectral analogue of a probability density**

Example: white noise

$$f_W(\omega) = \sigma_w^2 \quad \text{flat spectrum}$$



check at home for AR(1), MA(1)

Linear combinations, convolutions, filters

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \quad \sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

Impulse
response

Fourier
Transform

frequency response

$$A(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i \omega j},$$

spectral
density

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega),$$

proof on the board

Spectral analysis as PCA

$$\text{cov}(X) = \Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix}.$$

For n sufficiently large, the eigenvalues of Γ_n are

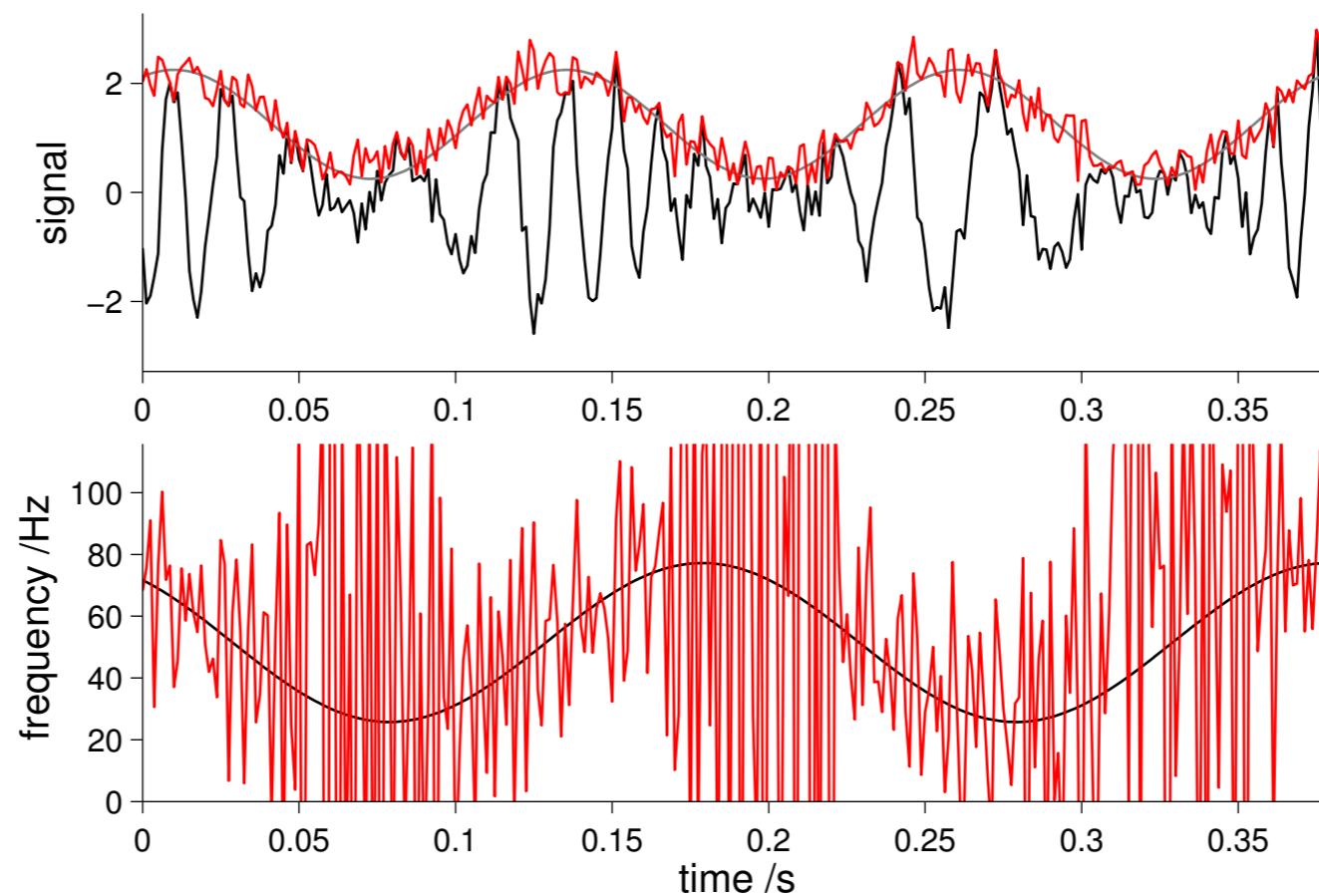
$$\lambda_j \approx f(\omega_j) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i h j / n},$$

A Bayesian perspective

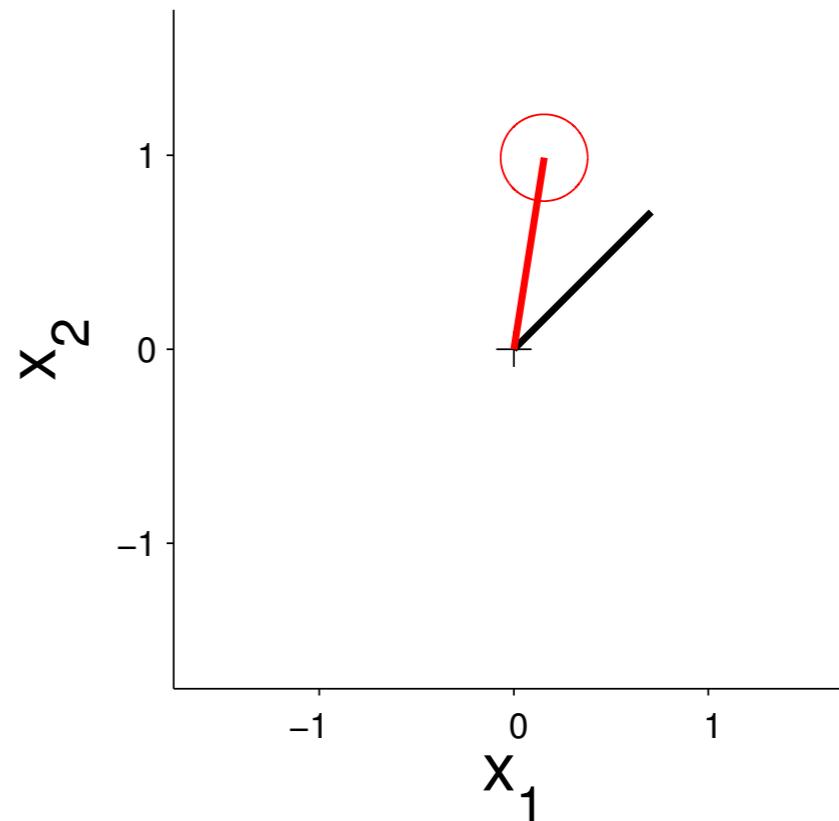
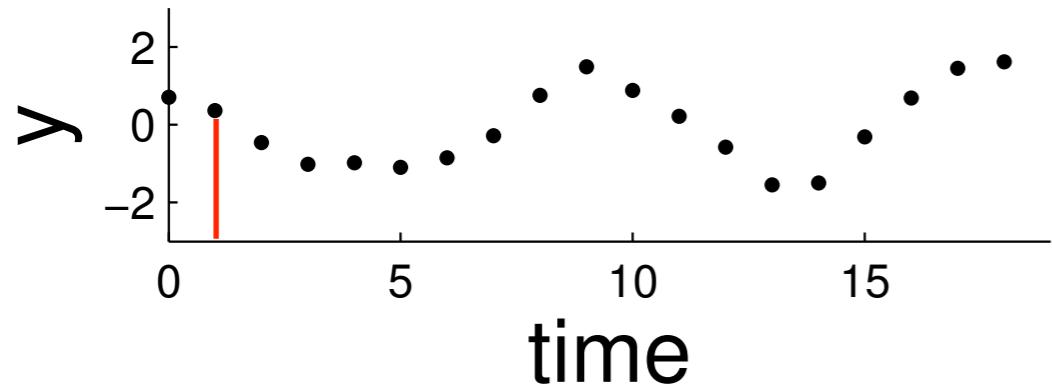
Why bayesian:

No explicit notion of measurement noise

No way to deal with missing data



A Bayesian perspective



$$y(t) = \Re(a(t) \exp(i\phi(t)))$$

$$\mathbf{y}_t = [1, 0] \mathbf{x}_t + \sigma_y \eta_t$$

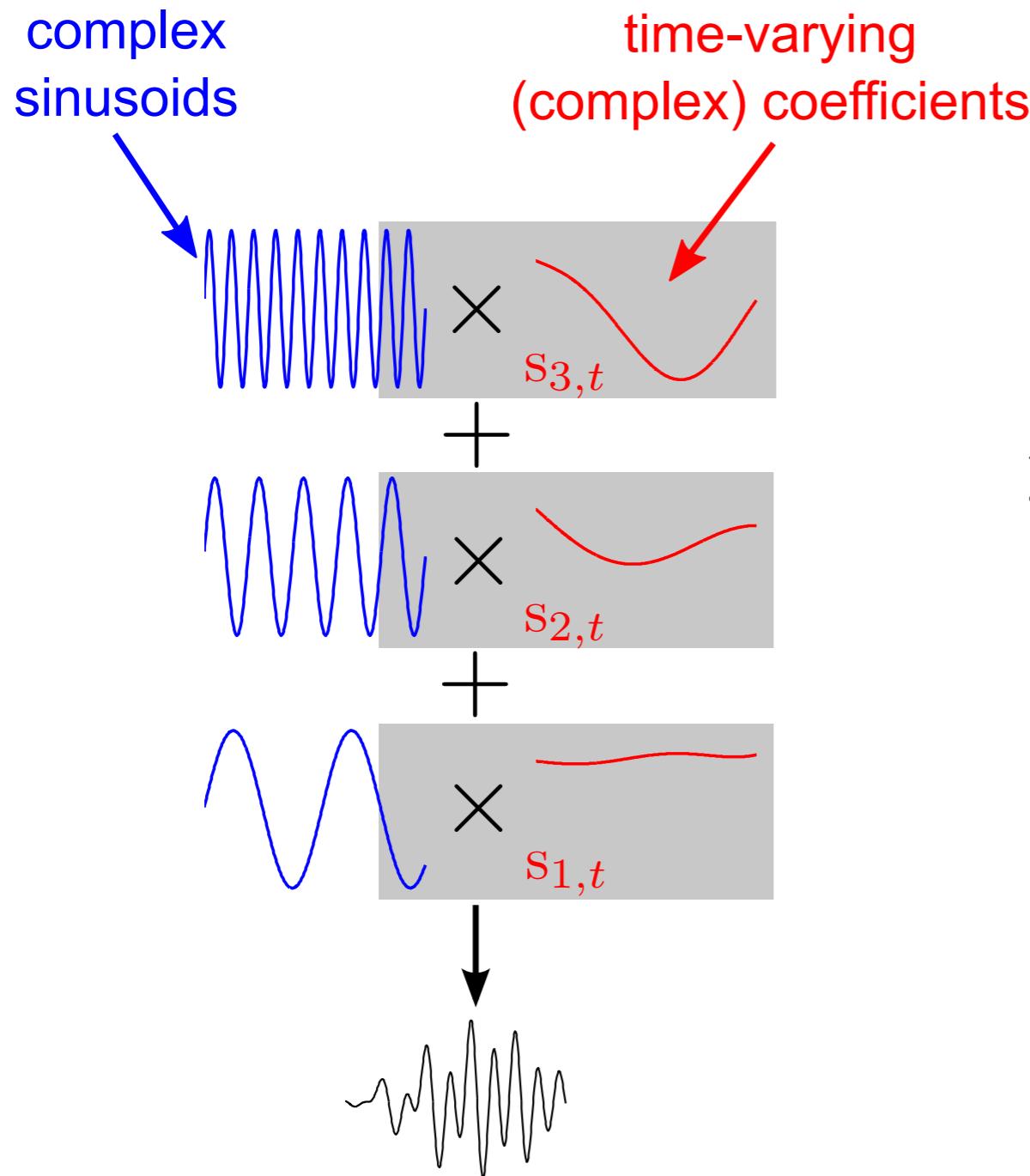
$$\mathbf{x}_t = \lambda \mathbf{R}(\bar{\omega}) \mathbf{x}_{t-1} + \sigma_x^2 \boldsymbol{\epsilon}_t$$

**Just an extended
Kalman filter**

Rotation matrix: $R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Qi and Minka 2002, Cemgil and Godshill 2005.

Generative model in complex domain

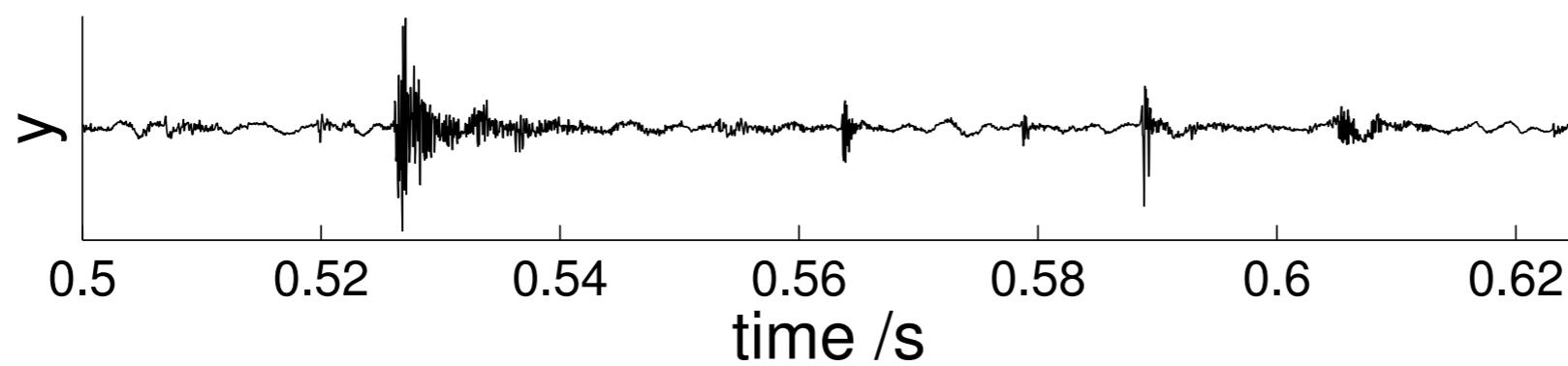
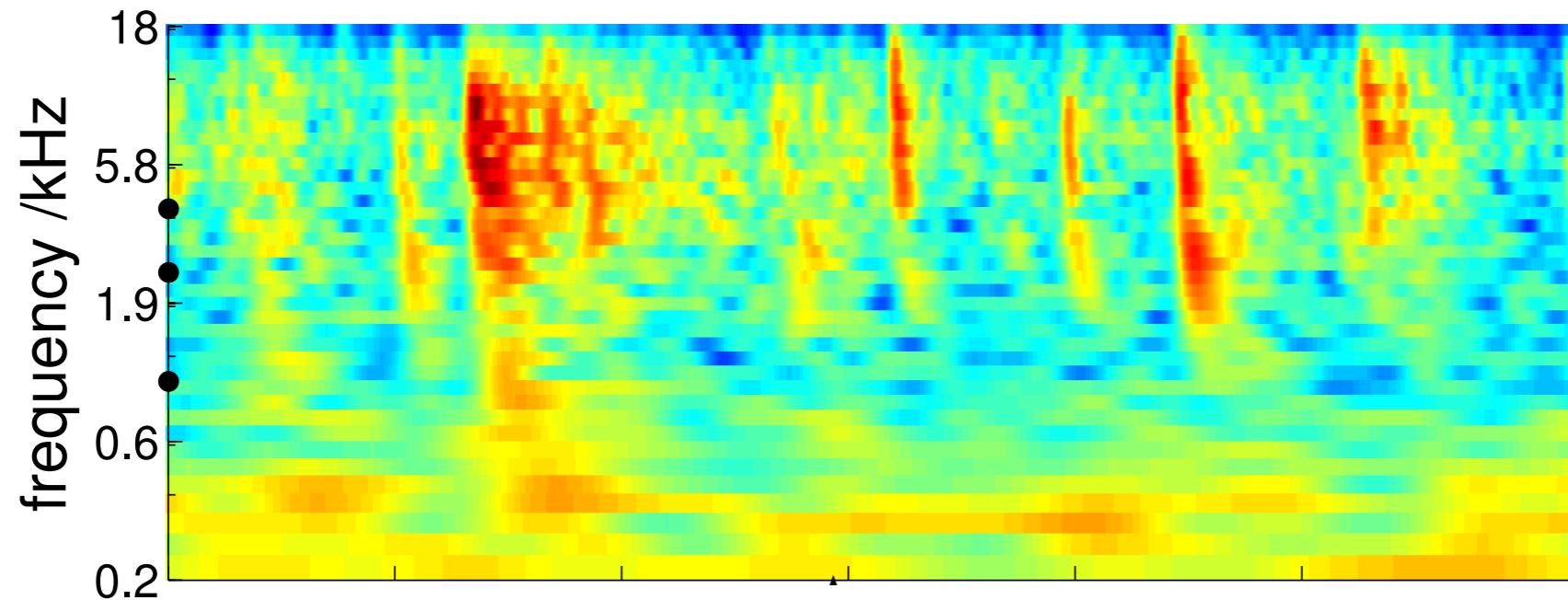


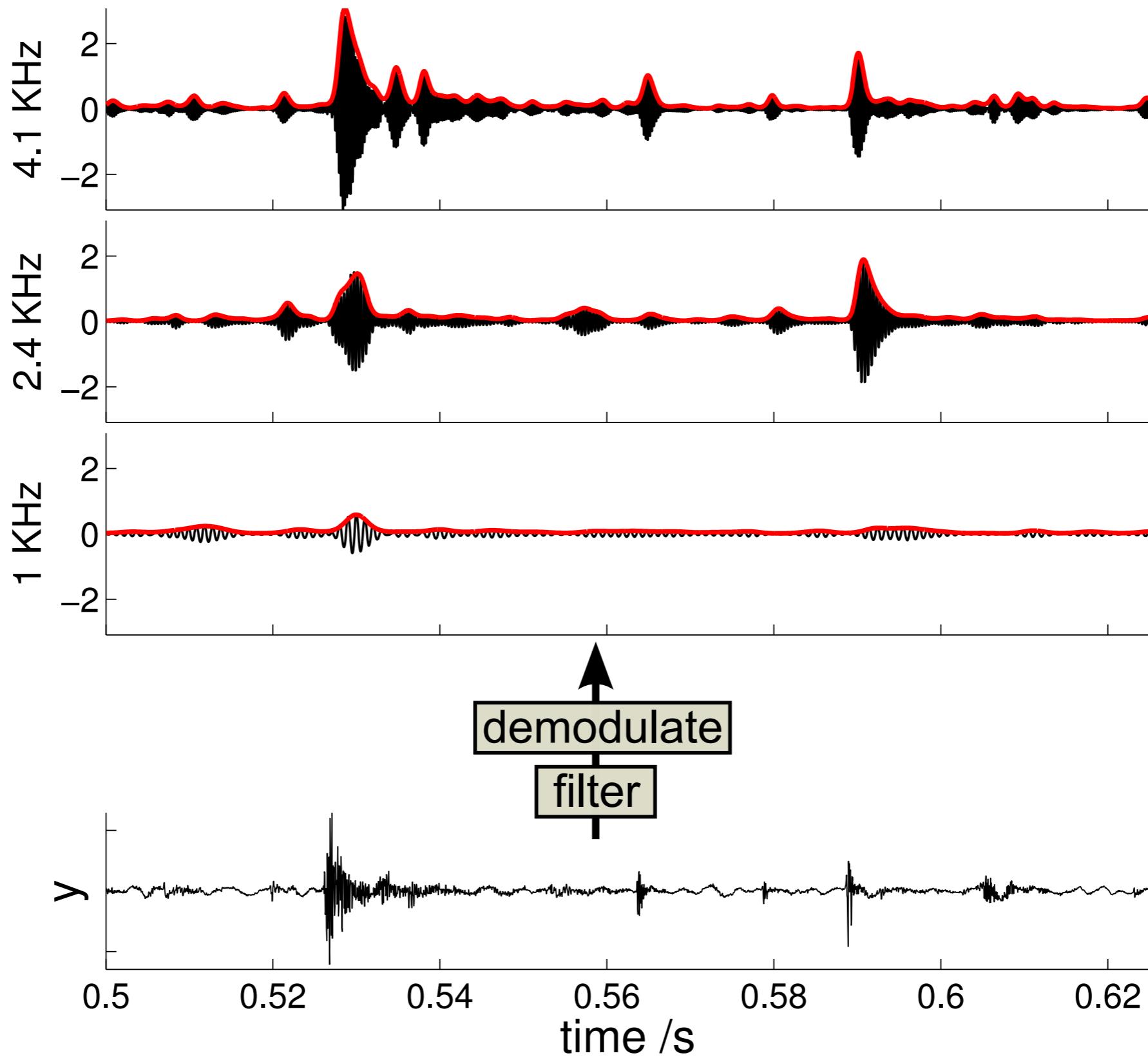
$$y_t = \sum_d \Re(e^{i\omega_d t} s_{d,t}) + \sigma_y \eta_t$$

$$\Re(s_{d,t}) \sim \mathcal{GP}(0, \Gamma)$$

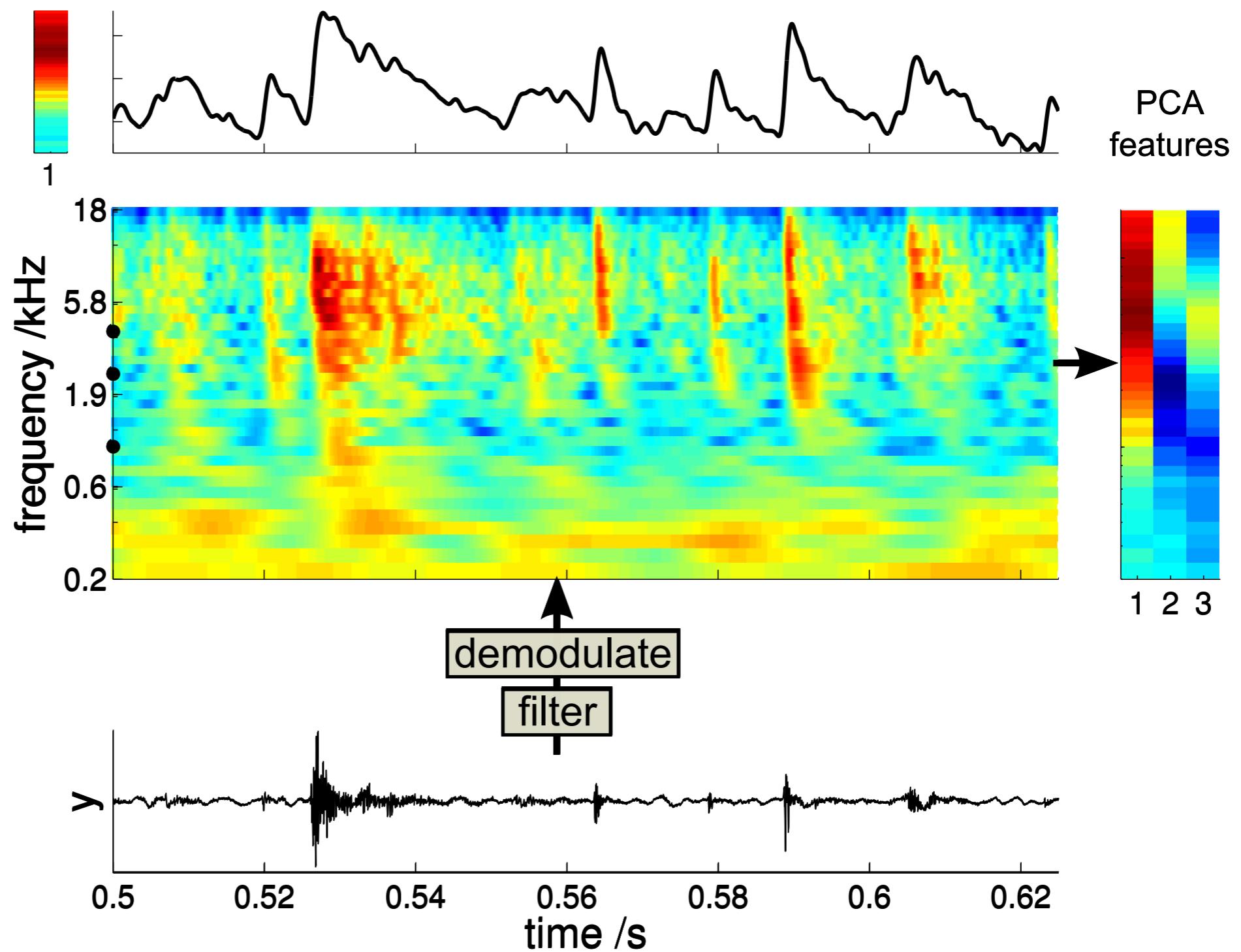
$$\Im(s_{d,t}) \sim \mathcal{GP}(0, \Gamma)$$

The interesting cases are not stationary

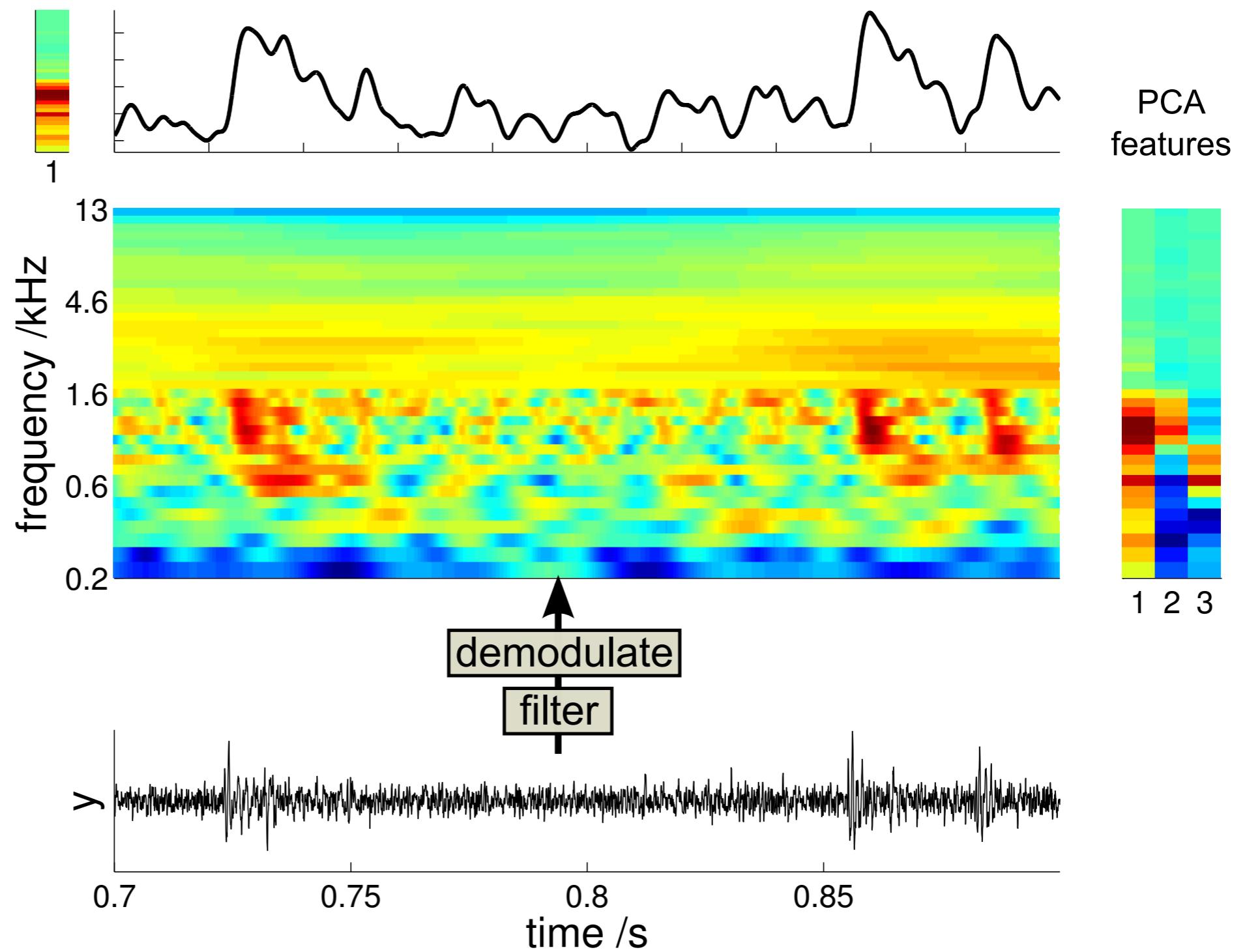




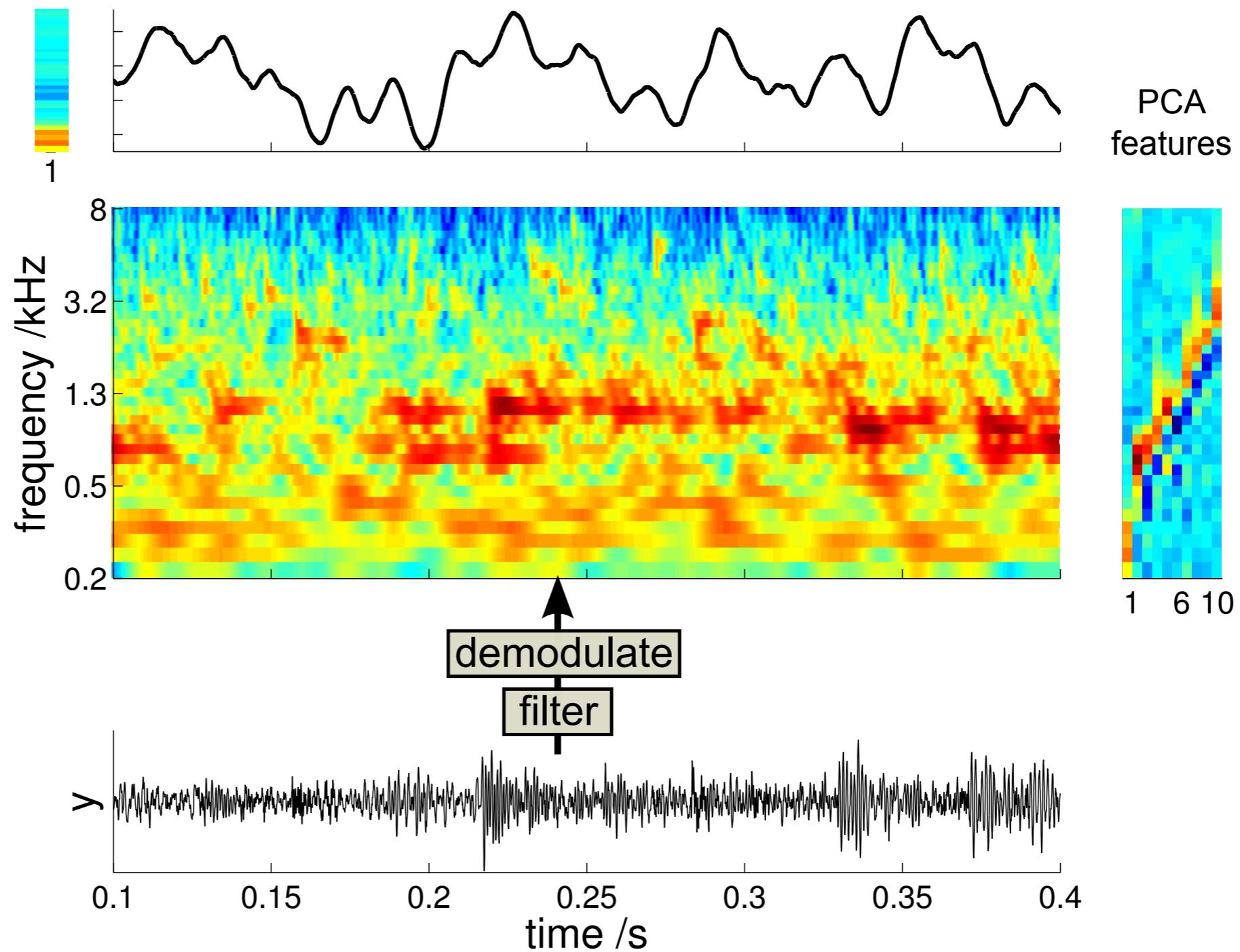
Fire



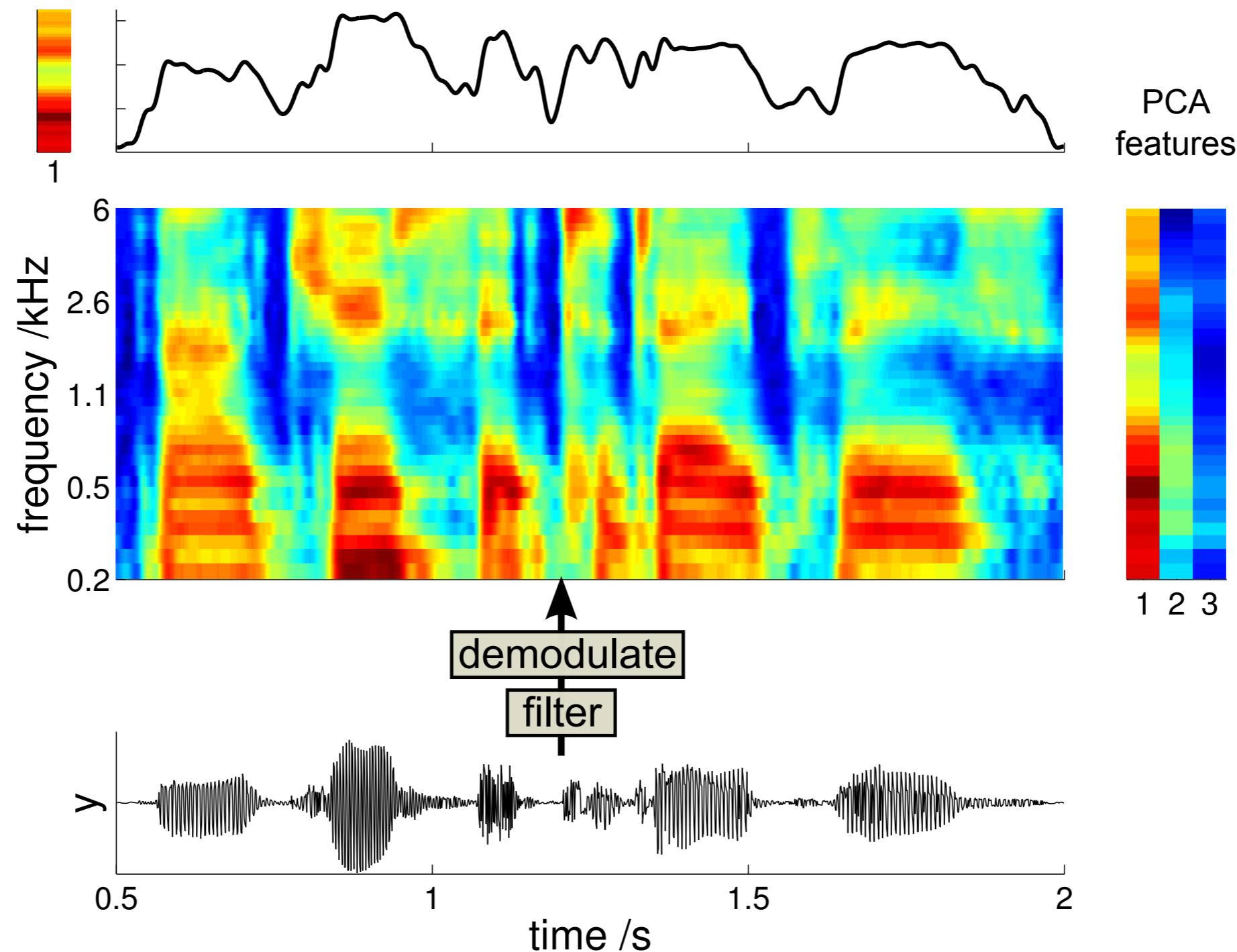
Wind



Water



Speech

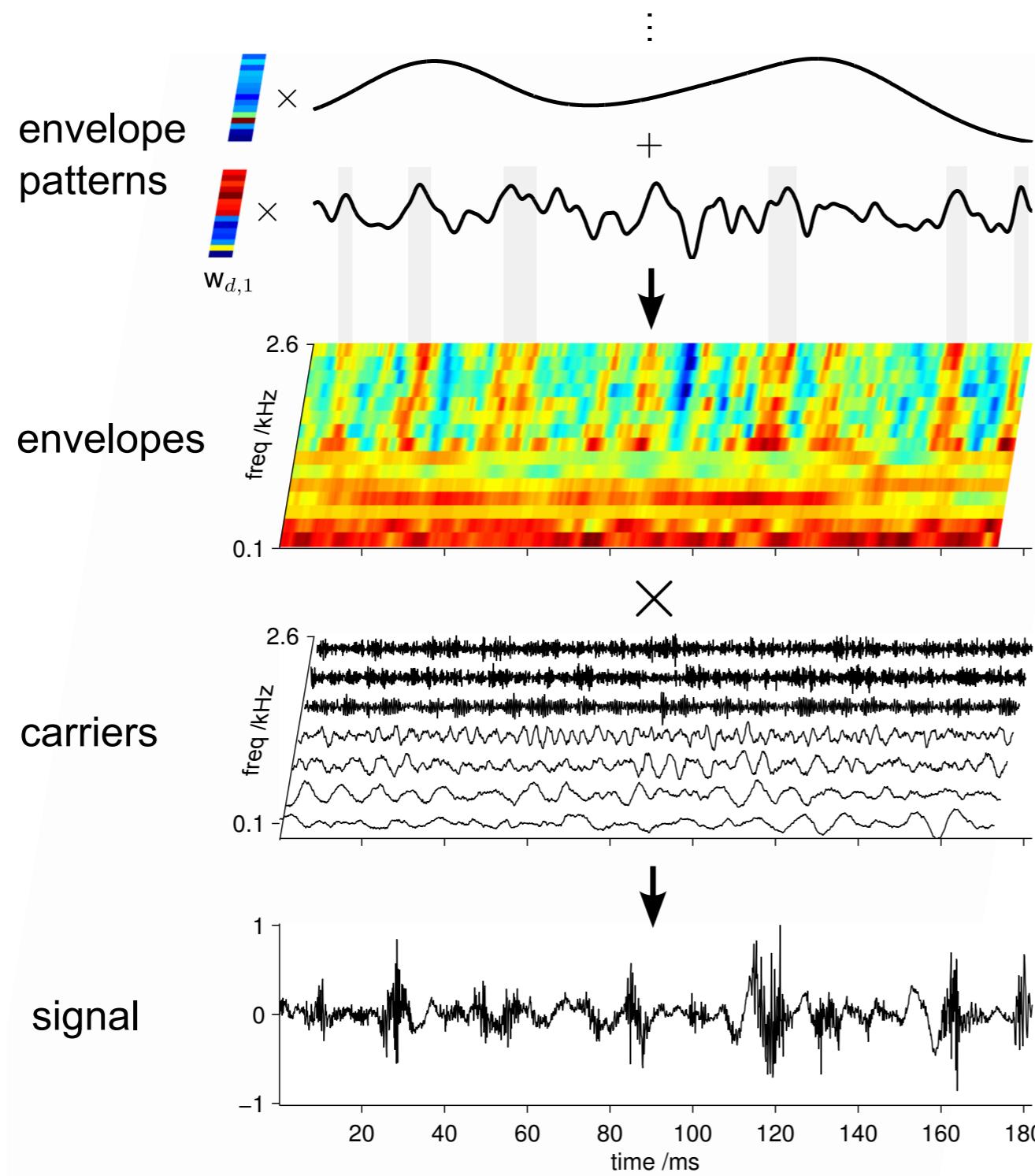


Spectral analysis reveals a lot of potentially useful statistical regularities in real-world data

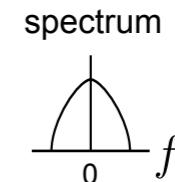
energy in sub-bands (power-spectrum)
patterns of co-modulation
time-scale of the modulation
depth of the modulation (sparsity)

**We can build probabilistic models
that capture such phenomena and fit them to data**

Statistical Model

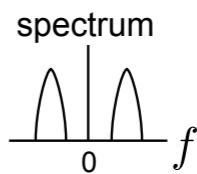


$x_k(t)$ = lowpass Gaussian noise



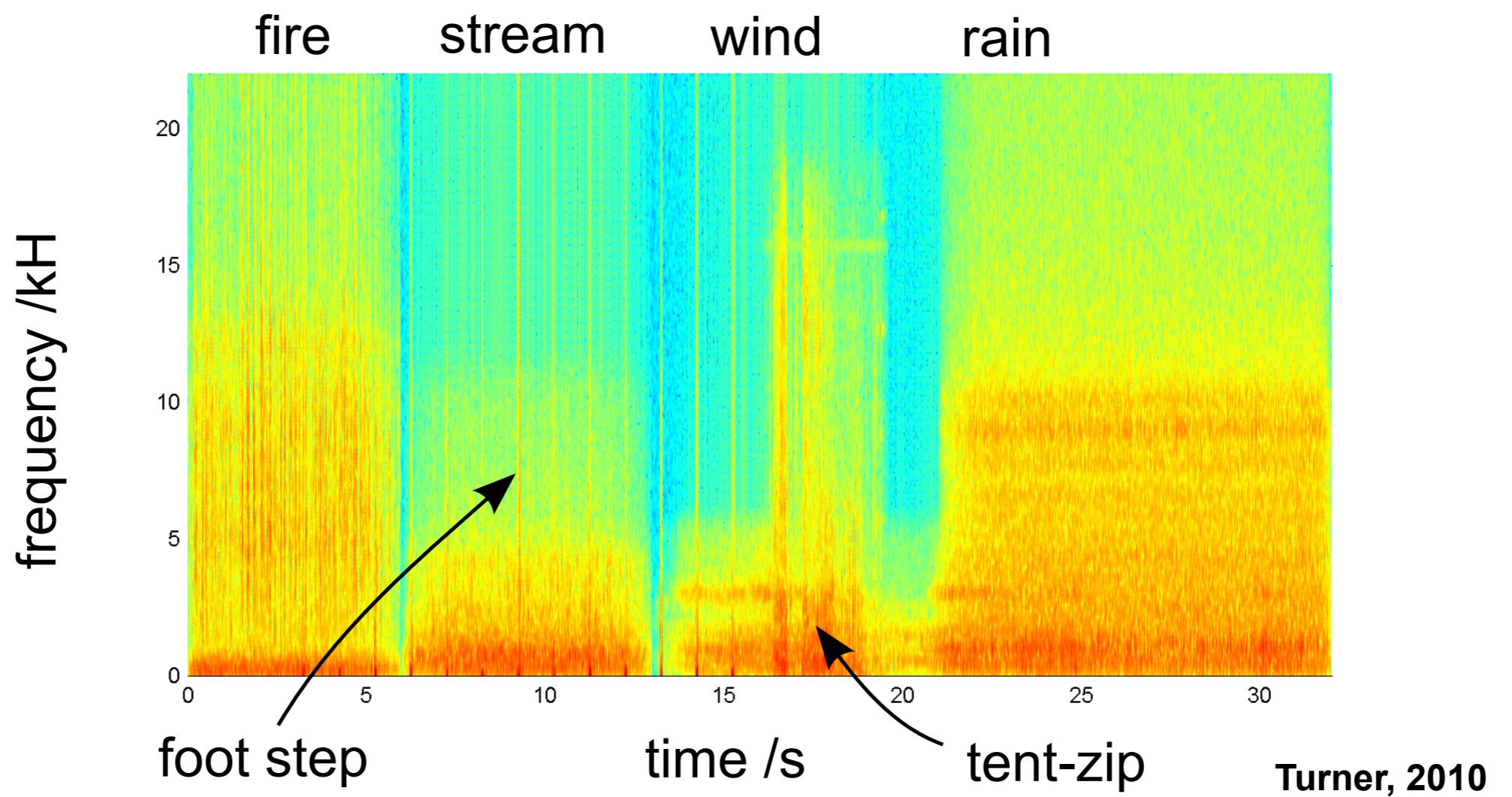
$$a_d(t) = g_+ \left(\sum_{k=1}^K w_{d,k} x_k(t) \right)$$

$c_d(t)$ = bandpass Gaussian noise

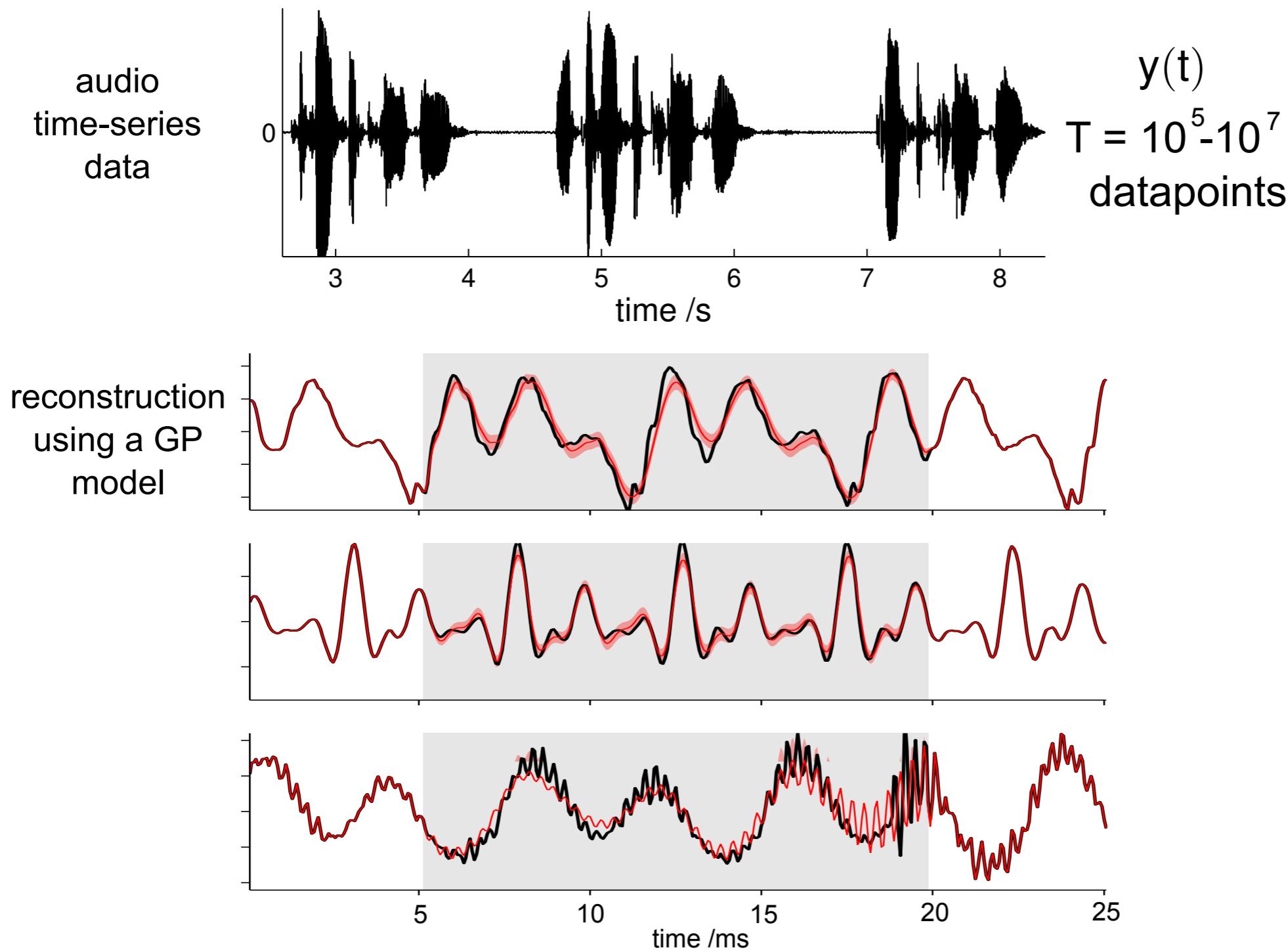


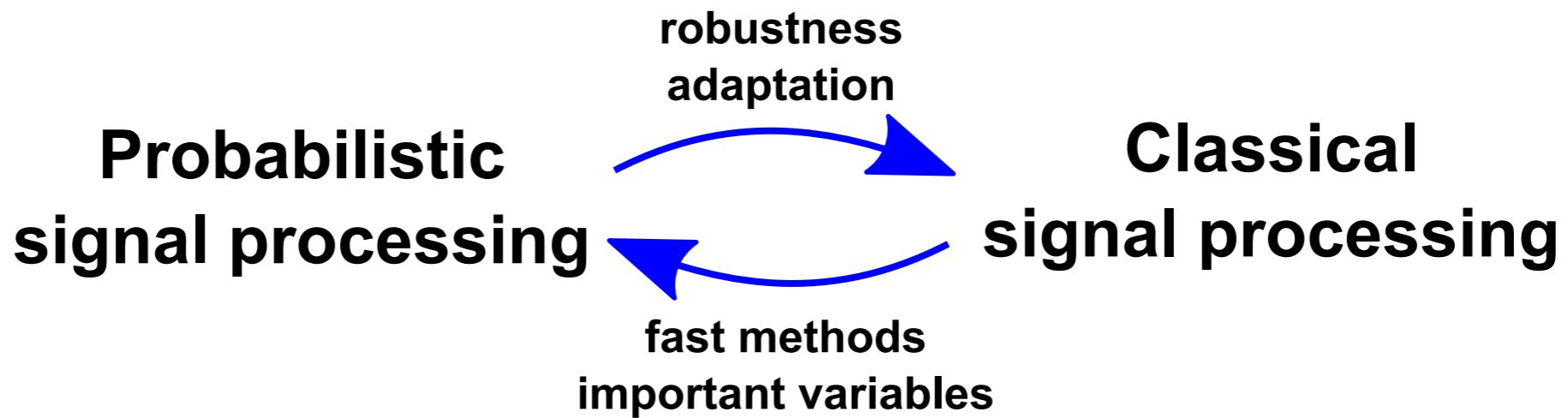
$$y(t) = \sum_{d=1}^D c_d(t) a_d(t)$$

Sound Generation Demo



Application: speech imputation





extra reading:

Turner, R. E. and Sahani, M.
"Time-frequency analysis as probabilistic inference"
IEEE Transactions on Signal Processing. 2014

Turner, R. and Sahani M. "Probabilistic amplitude and frequency demodulation."
Advances in Neural Information Processing Systems. 2011.

Turner, R. E., and Sahani, M.
"Demodulation as probabilistic inference."
IEEE Transactions on Audio, Speech, and Language Processing 19.8 (2011): 2398-2411.