Homework4

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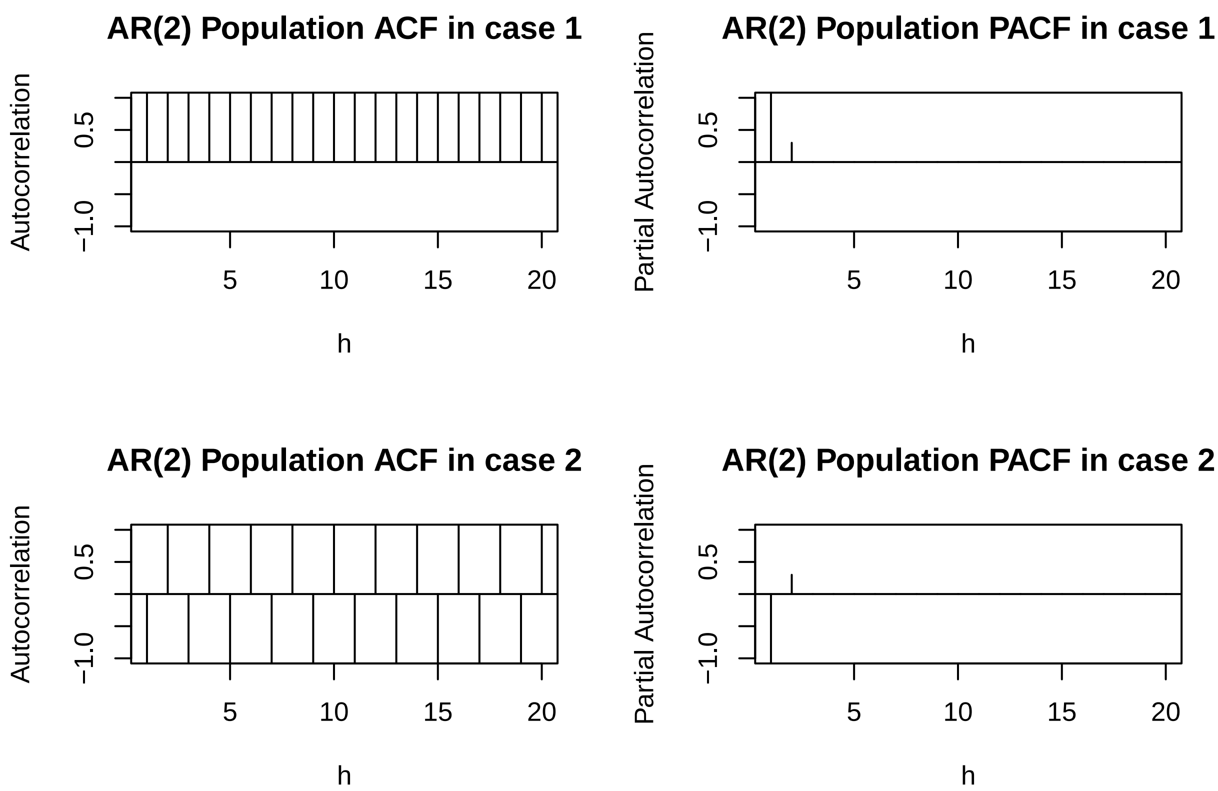
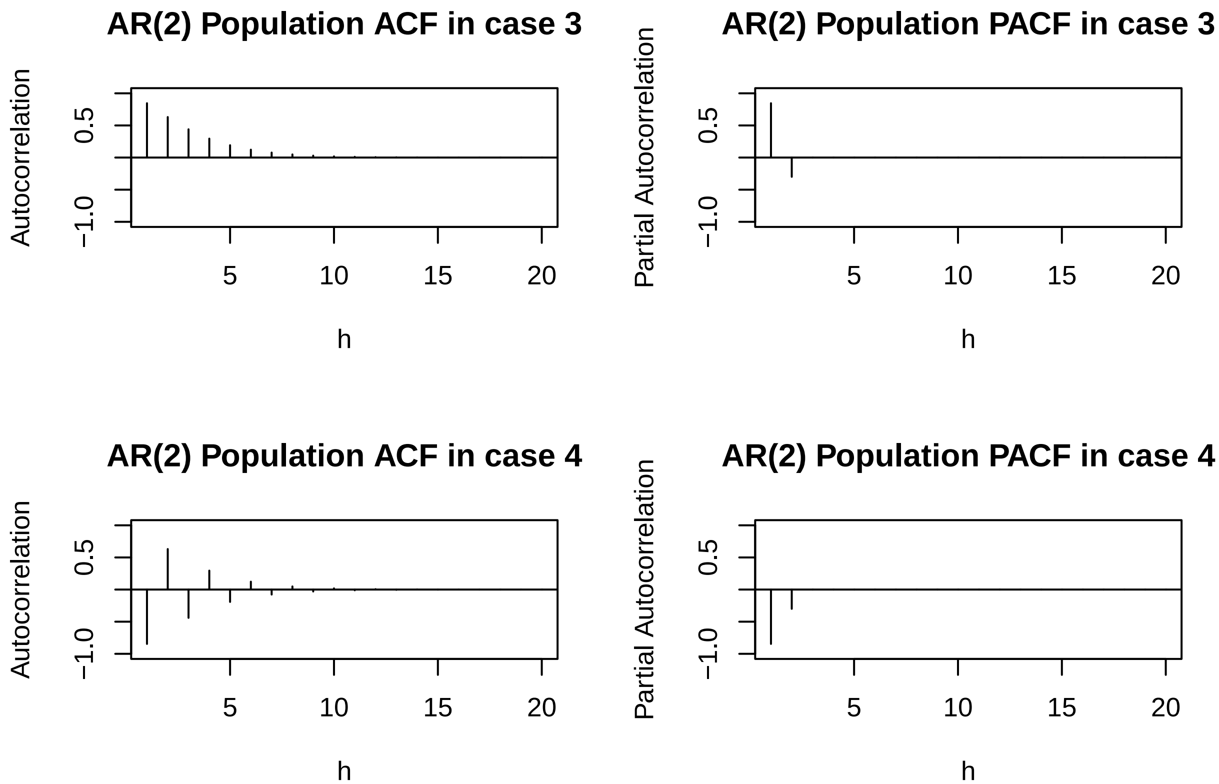
Problem 1

For each of the following scenarios, choose Φ1 and Φ2 so that the resulting AR(2) process is causal. In each case use R to plot ACF and PACF of the AR(2) process and discuss the characteristics of the plots. (i) Φ1 > 0 and Φ2 > 0 (ii) Φ1 < 0 and Φ2 > 0 (iii) Φ1 > 0 and Φ2 < 0 (iv) Φ1 < 0 and Φ2 < 0

```
par(mfrow=c(2,2))
#1
y = ARMAacf(ar = c(1.1,0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
y = ARMAacf(ar = c(1.1, 0.3), lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "AR(
abline(h = 0)
#2
y = ARMAacf(ar = c(-1.1,0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
y = ARMAacf(ar = c(-1.1, 0.3), lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "AR(
abline(h = 0)
```



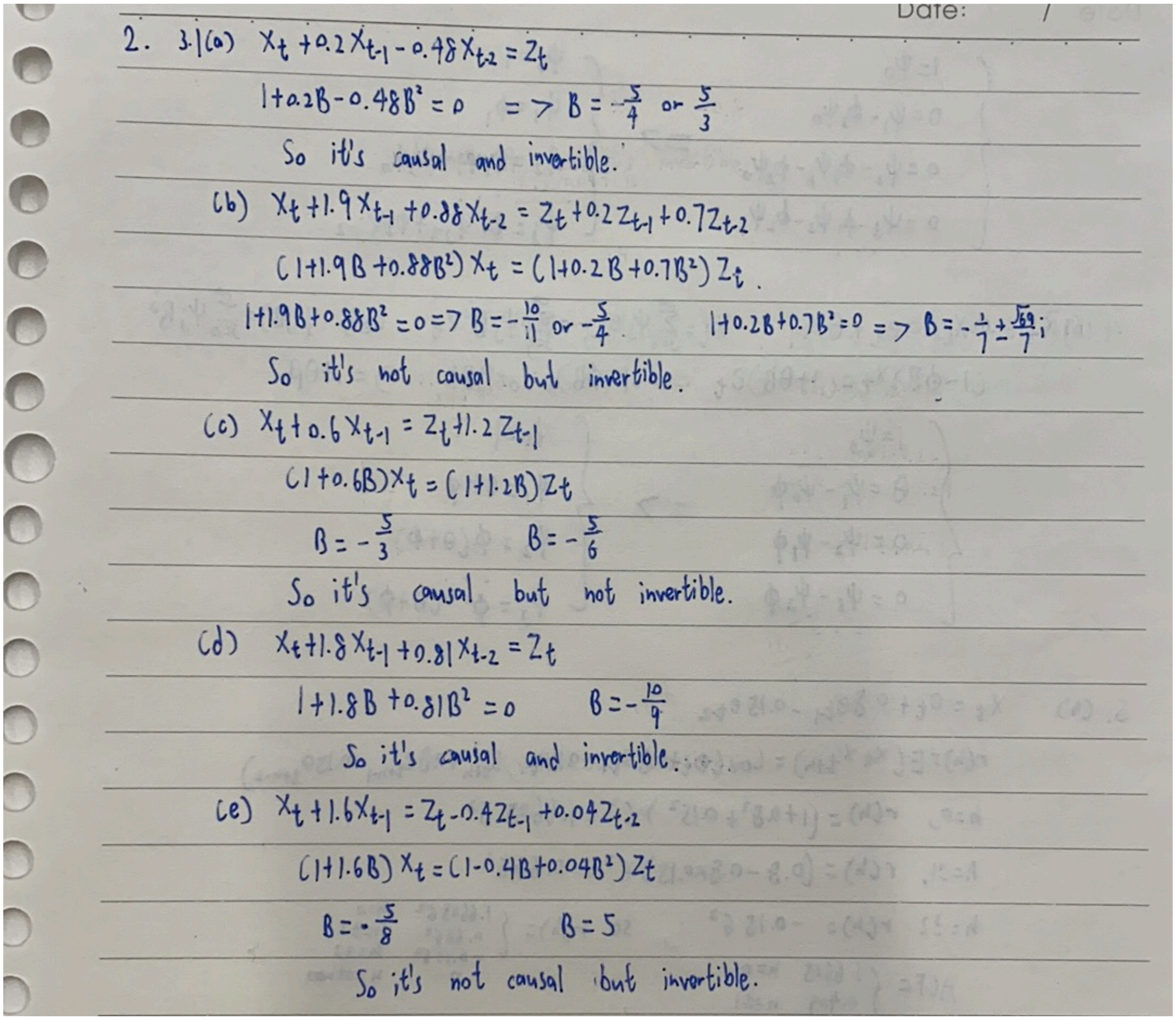

```
#3
y = ARMAacf(ar = c(1.1,-0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
y = ARMAacf(ar = c(1.1,-0.3), lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "AR(
abline(h = 0)
#4
y = ARMAacf(ar = c(-1.1,-0.3), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "AR(2) Popula
abline(h = 0)
y = ARMAacf(ar = c(-1.1, -0.3), lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "AR(
abline(h = 0)
```



A: Two cases the ACF are always positive and the other 2 is half positive and half negative. There are only two lines in PACF which implies that when $h>2$, the PACF is 0.

Problem 2

3.1


Date: /

2. 3.1(a) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$

$1+0.2B-0.48B^2=0 \Rightarrow B=-\frac{5}{4}$ or $\frac{5}{3}$

So it's causal and invertible.

(b) $X_t+1.9X_{t-1}+0.88X_{t-2} = Z_t+0.2Z_{t-1}+0.7Z_{t-2}$

$(1+1.9B+0.88B^2)X_t = (1+0.2B+0.7B^2)Z_t$.

$1+1.9B+0.88B^2=0 \Rightarrow B=-\frac{10}{11}$ or $-\frac{5}{4}$. $1+0.2B+0.7B^2=0 \Rightarrow B=-\frac{1}{7}\pm\frac{\sqrt{69}}{7}i$

So it's not causal but invertible.

(c) $X_t+0.6X_{t-1}=Z_t+1.2Z_{t-1}$

$(1+0.6B)X_t=(1+1.2B)Z_t$

$B=-\frac{5}{3}$ $B=-\frac{5}{6}$

So it's causal but not invertible.

(d) $X_t+1.8X_{t-1}+0.81X_{t-2}=Z_t$

$1+1.8B+0.81B^2=0$ $B=-\frac{10}{9}$

So it's causal and invertible.

(e) $X_t+1.6X_{t-1}=Z_t-0.4Z_{t-1}+0.04Z_{t-2}$

$(1+1.6B)X_t=(1-0.4B+0.04B^2)Z_t$

$B=-\frac{5}{8}$ $B=5$

So it's not causal but invertible.

Figure 1: Problem 2 (3.1)

3.2

For those processes in Problem 3.1 that are causal and invertible, compute and graph their ACF and PACF using the program R.

```
#Let the maximum lag=10
par(mfrow=c(2,2))
#(a)
ARMAacf(ar=c(-0.2,0.48),lag.max=10)
```

```
##          0          1          2          3          4          5
```

```
##  1.00000000 -0.38461538  0.55692308 -0.29600000  0.32652308 -0.20738462
##           6           7           8           9          10
##  0.19820800 -0.13918622  0.12297708 -0.09140480  0.07730996
```

```r
ARMAacf(ar=c(-0.2,0.48),lag.max=10,pacf=T)
```

```
##  [1] -3.846154e-01  4.800000e-01  0.000000e+00  8.465242e-17  7.759805e-17
##  [6] -4.873234e-17 -4.147969e-17 -4.232621e-18  1.015829e-17 -2.349577e-33
```

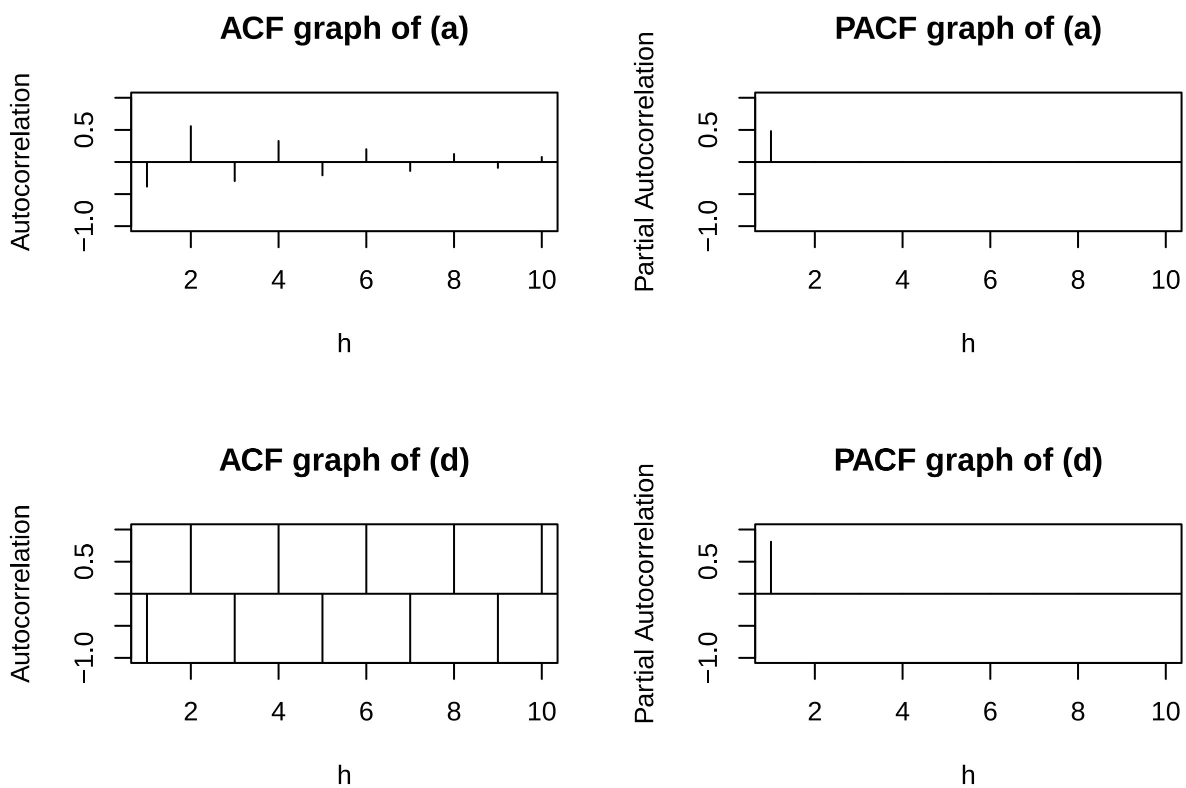
```r
y = ARMAacf(ar = c(-0.2,0.48), lag.max = 10)
y = y[2:11]
plot(y, x = 1:10, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "ACF graph o
abline(h = 0)
y = ARMAacf(ar = c(-0.2,0.48), lag.max = 10, pacf = T)
y = y[2:11]
plot(y, x = 1:10, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "PAC
abline(h = 0)
#(d)
ARMAacf(ar=c(-1.8,0.81),lag.max=10)
```

```
##           0           1           2           3           4           5
##    1.000000   -9.473684   17.862632  -39.826421   86.156289 -187.340722
##           6           7           8           9          10
##  406.999894 -884.345795 1921.492345 -4175.006314 9071.420164
```

```r
ARMAacf(ar=c(-1.8,0.81),lag.max=10,pacf=T)
```

```
##  [1] -9.473684e+00  8.100000e-01 -2.910022e-17  6.723301e-17  4.985305e-16
##  [6] -1.976861e-15 -1.680829e-15  1.517868e-15 -8.976836e-15 -1.780468e-14
```

```r
y = ARMAacf(ar = c(-1.8,0.81), lag.max = 10)
y = y[2:11]
plot(y, x = 1:10, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "ACF graph o
abline(h = 0)
y = ARMAacf(ar = c(-1.8,0.81), lag.max = 10, pacf = T)
y = y[2:11]
plot(y, x = 1:10, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "PAC
abline(h = 0)
```



3.4

3.4 $X_t = 0.8X_{t-2} + Z_t$

$r(h) = E(X_{t+h}X_t) = 0.8r(h-2) \Rightarrow \rho(h) = 0.8\rho(h-2)$

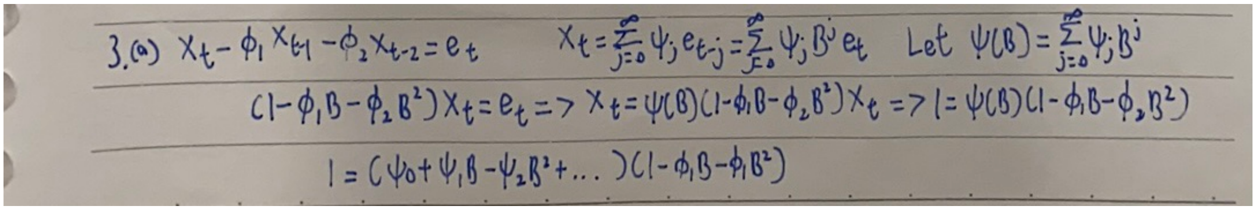
$h=1$, $\rho(1) = 0.8\rho(-1) \Rightarrow \rho(1) = 0$ $\quad h=2$, $\rho(2) = 0.8\rho(0) = 0.8$

So the ACF $= \begin{cases} 0 & h \text{ is odd} \\ 0.8^{\frac{|h|}{2}} & h \text{ is even.} \end{cases}$

$\phi_{11} = \rho(1) = 0$, $\phi_{22} = \phi_2 = 0.8$, so the PACF $= \begin{cases} 0.8 & h=2 \\ 0 & h=\text{others} \end{cases}$

Figure 2: Problem 2 (3.4)

Problem 3

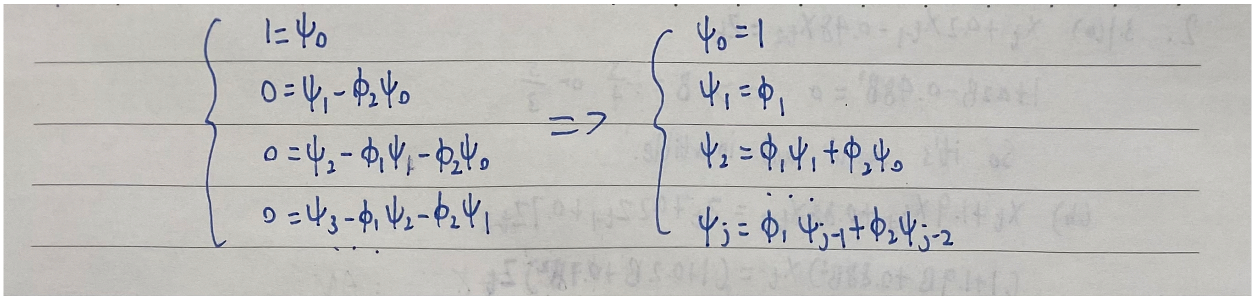


3.(a) $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = e_t$ $\quad X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = \sum_{j=0}^{\infty} \psi_j B^j e_t$ $\quad$ Let $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$

$$(1-\phi_1 B - \phi_2 B^2)X_t = e_t \Rightarrow X_t = \psi(B)(1-\phi_1 B-\phi_2 B^2)X_t \Rightarrow 1 = \psi(B)(1-\phi_1 B-\phi_2 B^2)$$

$$1 = (\psi_0 + \psi_1 B - \psi_2 B^2 + \dots)(1-\phi_1 B - \phi_1 B^2)$$

Figure 3: Problem 3(a)



$$\begin{cases} 1 = \psi_0 \\ 0 = \psi_1 - \phi_2 \psi_0 \\ 0 = \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0 \\ 0 = \psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1 \\ \dots \end{cases} \Rightarrow \begin{cases} \psi_0 = 1 \\ \psi_1 = \phi_1 \\ \psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 \\ \dots \\ \psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2} \end{cases}$$

Figure 4: Problem 3(a)

```
#b
ARMAtoMA(c(1.1,0.3),0,5)
```

```
## [1] 1.10000 1.51000 1.99100 2.64310 3.50471
```

```
#c
ARMAacf(ar=c(1.1,0.3),lag.max=5)
```

```
##        0        1        2        3        4        5
## 1.000000 1.571429 2.028571 2.702857 3.581714 4.750743
```

```
#d
ARMAacf(ar=c(1.1,0.3),lag.max=5,pacf=T)
```

```
## [1] 1.571429e+00 3.000000e-01 8.302950e-17 1.806069e-16 3.681784e-17
```

Problem 4

4. (a) $X_t - \phi X_{t-1} = e_t + \theta e_{t-1}$ $\quad X_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = \sum_{j=0}^{\infty} \psi_j B^j e_t$ $\quad$ Let $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$

$(1-\phi B) X_t = (1+\theta B) e_t \Rightarrow (1-\phi B)(\psi_0 + \psi_1 B + \dots) = 1 + \theta B$

$$\begin{cases} 1 = \psi_0 \\ \theta = \psi_1 - \psi_0 \phi \\ 0 = \psi_2 - \psi_1 \phi \\ 0 = \psi_3 - \psi_2 \phi \\ \dots \end{cases} \Rightarrow \begin{cases} \psi_0 = 1 \\ \psi_1 = \theta + \phi \\ \psi_2 = \phi(\theta + \phi) \\ \psi_j = \phi^{j-1}(\theta + \phi) \end{cases}$$

Figure 5: Problem 4(a)

```
#b
ARMAtoMA(0.6,-0.2,5)
```

```
## [1] 0.40000 0.24000 0.14400 0.08640 0.05184
```

Problem 5

5. (a) $X_t = e_t + 0.8 e_{t-1} - 0.15 e_{t-2}$

$r(h) = E(X_t X_{t+h}) = Cov(e_t + 0.8 e_{t-1} - 0.15 e_{t-2},\ e_{t+h} + 0.8 e_{t+h-1} - 0.15 e_{t+h-2})$

$h=0$, $r(h) = (1 + 0.8^2 + 0.15^2)\sigma^2 = 1.6625\sigma^2$

$h=\pm 1$, $r(h) = (0.8 - 0.8 \times 0.15)\sigma^2 = 0.68\sigma^2$

$h = \pm 2$ $\quad r(h) = -0.15\sigma^2$, $\quad$ so $r(h) = \begin{cases} 1.6625\sigma^2 & h=0 \\ 0.68\sigma^2 & h=\pm 1 \\ -0.15\sigma^2 & h=\pm 2 \\ 0 & h = \text{others} \end{cases}$

$$ACF = \begin{cases} 1.6625 & h=0 \\ 0.409 & h=\pm 1 \\ -0.090 & h=\pm 2 \\ 0 & h=\text{others.} \end{cases}$$

(b) $\phi_{11} = \rho(1) = 0.409$ $\quad \phi_{22} = \frac{\rho(2) - \rho^2(1)}{1 - \rho^2(1)} = -0.309$ $\quad \phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = 0.535$

$\phi_{33} = \frac{\rho(3) - \phi_{21}\rho(2) - \phi_{22}\rho(1)}{1 - \phi_{21}\rho(1) - \phi_{22}\rho(2)} = 0.232$ $\quad \phi_{31} = \phi_{21} - \phi_{33}\phi_{22} = 0.607$ $\quad \phi_{32} = \phi_{22} - \phi_{33}\phi_{21} = -0.433$

$\phi_{44} = \frac{\rho(4) - \phi_{31}\rho(3) - \phi_{32}\rho(2) - \phi_{31}\rho(1)}{1 - \phi_{31}\rho(1) - \phi_{32}\rho(2) - \phi_{33}\rho(3)} = -0.188$ $\quad \phi_{41} = \phi_{31} - \phi_{44}\phi_{33} = 0.651$ $\quad \phi_{42} = \phi_{32} - \phi_{44}\phi_{32} = -0.514$

$\phi_{43} = \phi_{33} - \phi_{44}\phi_{31} = 0.346$ $\quad \phi_{55} = \frac{\rho(5) - \phi_{41}\rho(4) - \phi_{42}\rho(3) - \phi_{43}\rho(2) - \phi_{44}\rho(1)}{1 - \phi_{41}\rho(1) - \phi_{42}\rho(2) - \phi_{43}\rho(3) - \phi_{44}\rho(4)} = 0.157.$

Figure 6: Problem 5(a)

```r
#b
ARMAacf(ma=c(0.8,-0.15),lag.max=5,pacf=T)
```

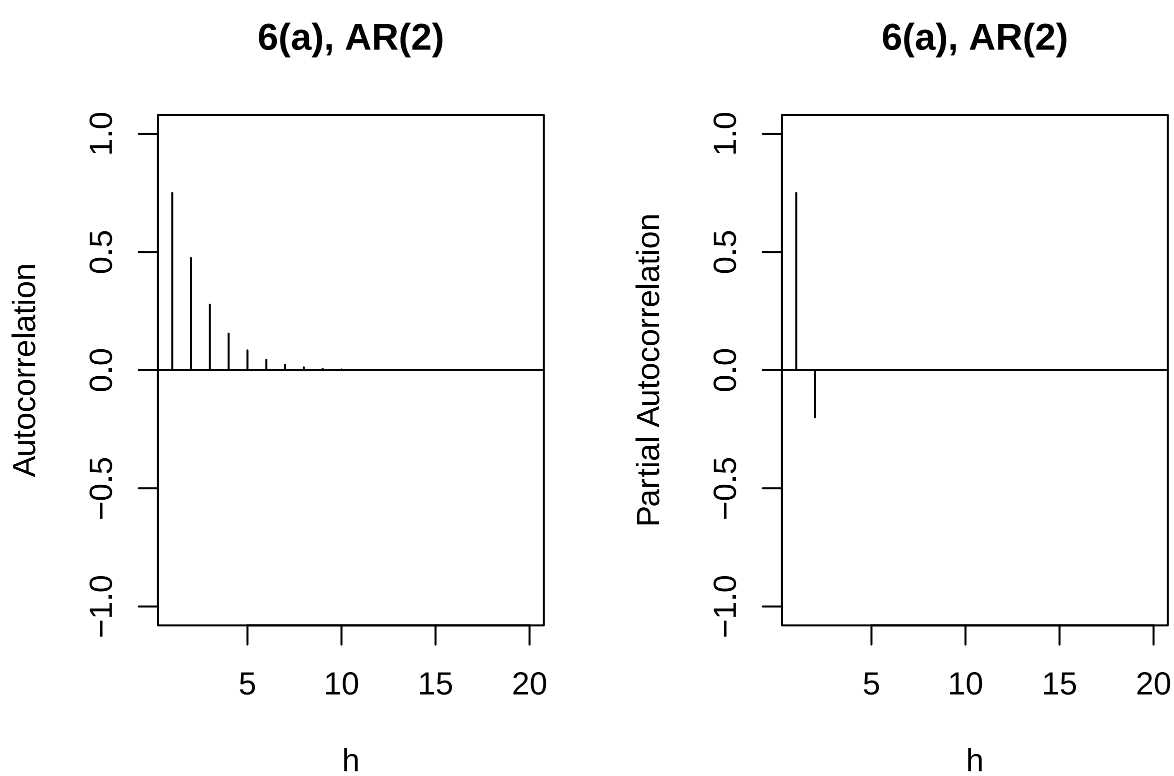
```
## [1]  0.4090226 -0.3092649  0.2321388 -0.1881747  0.1574740
```

Problem 6

(a)

A:By looking at the plot, I know that it's a AR(2) model, and according the plot of PACF, Φ22=Φ2=-0.2, so Φ2=-0.2, and Φ11=Φ1/(1-Φ2)=0.75, so Φ1=0.9,and I plot this model and it looks the same.

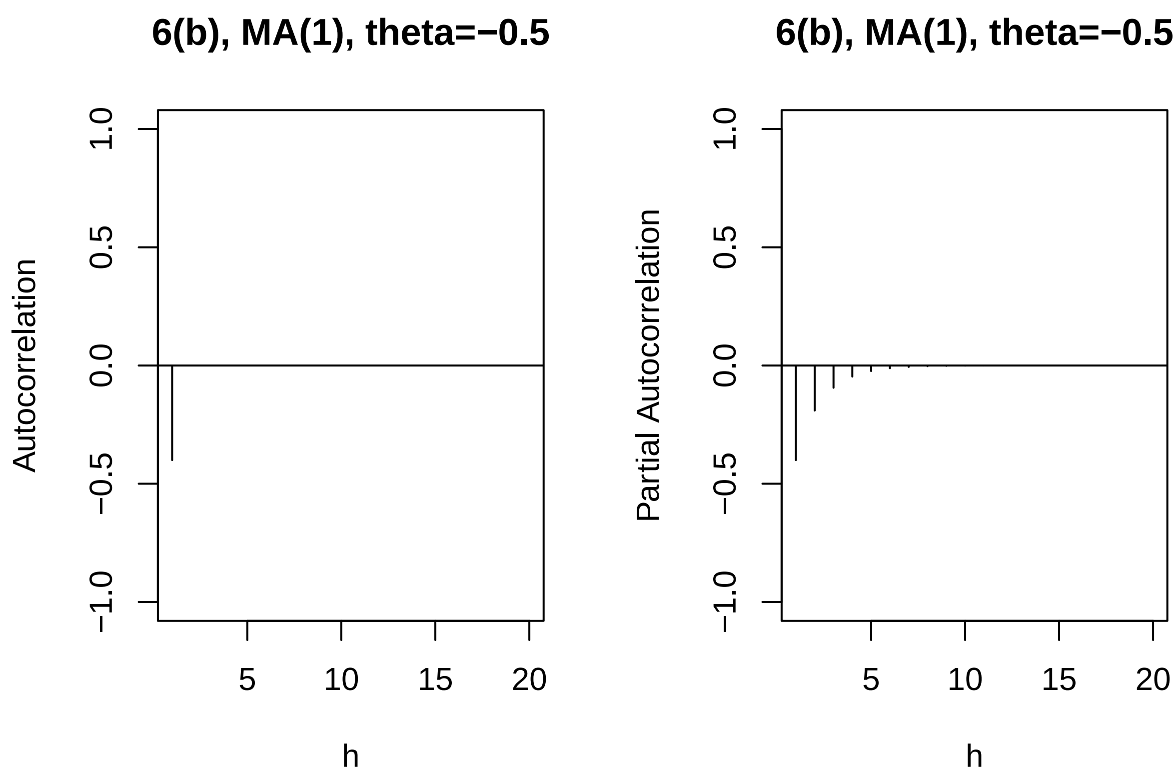
```r
par(mfrow=c(1,2))
y = ARMAacf(ar = c(0.9,-0.2), lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "6(a), AR(2)
abline(h = 0)
y = ARMAacf(ar = c(0.9, -0.2), lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "6(a
abline(h = 0)
```



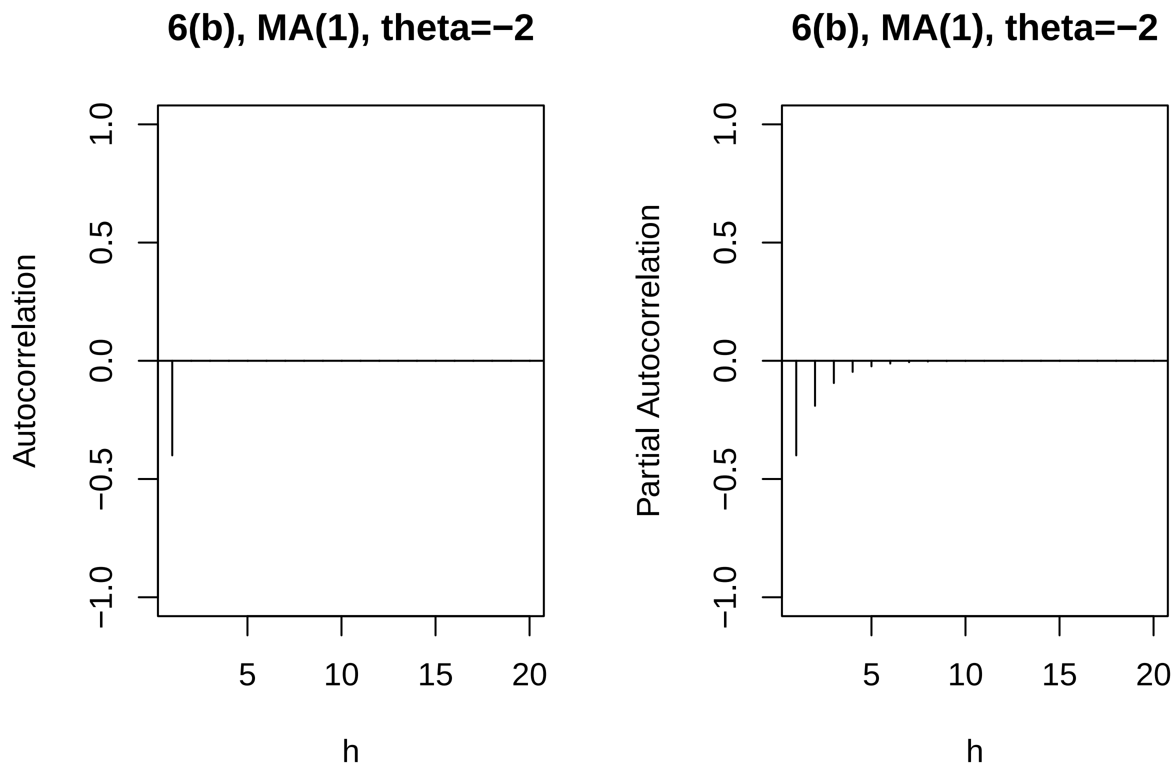
(b)

A:By looking at the plot, I know that it's a MA(1) model, and according the plot of ACF, p(1)=$\theta/(1+\theta$^2)=-0.4, so θ=-2or -0.5,,and I plot this model and it looks the same.

```
par(mfrow=c(1,2))
y = ARMAacf(ma = -0.5, lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "6(b), MA(1)
abline(h = 0)
y = ARMAacf(ma = -0.5, lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "6(b
abline(h = 0)
```




```
y = ARMAacf(ma = -2, lag.max = 20)
y = y[2:21]
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Autocorrelation", main = "6(b), MA(1)
abline(h = 0)
y = ARMAacf(ma = -2, lag.max = 20,pacf=T)
plot(y, x = 1:20, type = "h", ylim = c(-1,1), xlab = "h", ylab = "Partial Autocorrelation", main = "6(b
abline(h = 0)
```


6(b), MA(1), theta=−2
Autocorrelation
1.0
0.5
0.0
−0.5
−1.0
5
10
15
20
h
6(b), MA(1), theta=−2
Partial Autocorrelation
1.0
0.5
0.0
−0.5
−1.0
5
10
15
20
h