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Homework 3 Hsueh-Pin Liu

1\. 1.4 (a) $X_t = a + bZ_t + cZ_{t-2}$

$E(X_t) = a$ $\quad \sigma_{t,t} = E((X_t-\mu_t)^2) = (b^2+c^2)\sigma^2$ $\quad \sigma_{t,t+1} = E((X_t-\mu_t)(X_{t+1}-\mu_{t+1})) = 0$

$\sigma_{t,t+2} = E((X_t-\mu_t)(X_{t+2}-\mu_{t+2})) = bc\sigma^2$ When $h>2$, $\sigma_{t,t+h} = 0$.

So (a) is stationary, mean is a, $\sigma_{t,t+h} = \begin{cases} (b^2+c^2)\sigma^2 & h=0 \\ bc\sigma^2 & h=2 \\ 0 & \text{others.} \end{cases}$

(b) $X_t = Z_1\cos(ct) + Z_2\sin(ct)$

$E(X_t) = 0$, $\sigma_{t,t} = E((X_t-\mu_t)^2) = 0$ $\quad \sigma_{t,t+h} = E((X_t-\mu_t)(X_{t+h}-\mu_{t+h})) = \sigma^2\cos(ch)$

So (b) is stationary, mean is 0, $\sigma_{t,t+h} = \begin{cases} 0 & h=0 \\ \sigma^2\cos(ch) & \text{others} \end{cases}$

(c) $X_t = Z_t\cos(ct) + Z_{t-1}\sin(ct)$

$E(X_t) = 0$ $\quad \sigma_{t,t} = 0$ $\quad \sigma_{t,t+1} = \sigma^2\sin(c(t+1))\cos(ct)$ When $h>2$ $\sigma_{t,t+2} = 0$.

So (c) is stationary when $c = k\pi$, k is an integer, mean is 0, $\sigma_{t,t+h} = \begin{cases} \sigma^2\sin(c(t+1))\cos(ct) & h=1 \\ 0 & \text{others.} \end{cases}$

(d) $X_t = a + bZ_0$

$E(X_t) = a$, $\sigma_{t,t} = b^2\sigma^2$ $\quad \sigma_{t,t+h} = 0$

So (d) is stationary, mean is a, $\sigma_{t,t+h} = \begin{cases} b^2\sigma^2 & h=0 \\ 0 & \text{other} \end{cases}$

(e) $X_t = Z_0\cos(ct)$

$E(X_t) = 0$, $\sigma_{t,t+h} = \sigma^2\cos(ct)\cos(ct+ch)$

So (e) is stationary when $c = k\pi$, k is an integer, mean is 0, $\sigma_{t,t+h} = \sigma^2\cos(ct)\cos(ct+ch)$

(f) $X_t = Z_t Z_{t-1}$

$E(X_t) = 0$, $\sigma_{t,t} = E(Z_t^2 Z_{t-1}^2) = \sigma^4$ $\quad \sigma_{t,t+h} = 0$.

So (f) is stationary, $\sigma_{t,t+h} = \begin{cases} \sigma^4 & h=0 \\ 0 & \text{others.} \end{cases}$

1.5 a. $X_t = Z_t + 0.8Z_{t-2}$

$E(X_t) = 0$, $\sigma_{t,t} = 1.64$ $\quad \sigma_{t,t+2} = 0.8$ $\quad \Rightarrow \sigma_{t,t+h} = \begin{cases} 1.64 & h=0 \\ 0.8 & h=2 \\ 0 & \text{others} \end{cases}$

$\rho_k = \begin{cases} 1 & k=0 \\ \frac{20}{41} & k=2 \\ 0 & \text{others} \end{cases}$

b. $Var(\frac{1}{4}(X_1+X_2+X_3+X_4)) = \frac{1}{16}(4Var(X) + 2Cov(X_1,X_3) + 2Cov(X_2,X_4)) = \frac{1}{16}(6.56+3.2) = 0.61$

c. When $\theta = -0.8$, $Var(\frac{1}{4}(X_1+X_2+X_3+X_4)) = \frac{1}{16}(6.56-3.2) = 0.21$

1.8 $X_t = \begin{cases} Z_t & t \text{ is even} \\ (Z_{t-1}^2-1)/\sqrt{2} & t \text{ is odd.} \end{cases}$

a. When t is even, $E(X_t) = E(Z_t) = 0$ When t is odd $E(X_t) = E(\frac{Z_{t-1}^2-1}{\sqrt{2}}) = 0$.

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When t is even, $\sigma_{t,t} = E(Z_t^2) = 1$, when t is odd $\sigma_{t,t} = E\left(\frac{(Z_{t-1}^2-1)^2}{2}\right) = 1$

$\sigma_{t,t+1} = \frac{1}{\sqrt{2}}(Z_t^3 - Z_t) = 0$ so $X_t \sim WN(0,1)$.

When t is odd, $X_t = \frac{Z_{t-1}^2-1}{\sqrt{2}}$ $X_{t-1} = Z_{t-1}$, they are dependent, so they're not iid noise.

b. When n is odd, $E(X_{n+1} | X_1 \dots X_n) = Z_{t+1} = 0$. When n is even, $E(X_{n+1} | X_1 \dots X_n) = \frac{X_{n-1}^2}{\sqrt{2}}$.

1.11 (a) $\sum_{j=-q}^{q} a_j m_{t-j} = \frac{(2q+1)c_0 + c_1 \cdot (2q+1)t}{2q+1} = c_0 + c_1 t = m_t$.

(b) $E(A_t) = \frac{\sum E(Z_{t-j})}{2q+1} = 0$ $\quad Var(A_t) = \frac{\sum Var(Z_{t-j})}{(2q+1)^2} = \frac{(2q+1)\sigma^2}{(2q+1)^2} = \frac{\sigma^2}{2q+1}$

2.21 $X_t = Z_t + \theta Z_{t-1}$ $\{Z_t\} \sim WN(0, \sigma^2)$ $\quad (1+\theta^2)\sigma^2 \quad \theta\sigma^2$

(a) Assume $Y = aX_1 + bX_2$ $\quad E(X_1(X_3 - aX_1 - bX_2)) = 0 \Rightarrow E(X_1X_3) = aE(X_1^2) + bE(X_1X_2)$

$E(X_2(X_3 - aX_1 - bX_2)) = 0 \Rightarrow E(X_2X_3) = aE(X_1X_2) + bE(X_2^2)$ $\Rightarrow \begin{cases} a = \frac{r^2(1)}{r^2(1)-r^2(0)} \\ b = \frac{-r(0)r(1)}{r^2(1)-r^2(0)} \end{cases}$

$\Rightarrow \begin{cases} a = \frac{-\theta^2}{\theta^4+\theta^2+1} \\ b = \frac{\theta^3+\theta}{\theta^4+\theta^2+1} \end{cases}$, so $\hat{X}_3 = \frac{-\theta^2}{\theta^4+\theta^2+1} X_1 + \frac{\theta+\theta^3}{\theta^4+\theta^2+1} X_2$

(b) Use X_5 to replace X_1, X_4 to replace X_2 in (a). $\hat{X}_3 = \frac{\theta+\theta^3}{\theta^4+\theta^2+1} X_4 - \frac{\theta^2}{\theta^4+\theta^2+1} X_5$.

2. $Corr(XY, Y) = \frac{Cov(XY, Y)}{\sqrt{Var(XY)Var(Y)}}$ $\quad Cov(XY, Y) = E(XY^2) - E(X)E(Y)^2 = E(X)Var(Y) = \mu_X \sigma_Y^2$

$Var(XY) = E(X^2)E(Y^2) - E(X)^2E(Y)^2 = \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2$

So $Corr(XY, Y) = \frac{\mu_X \sigma_Y}{\sqrt{\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2}}$

3. $Cov(X_1+X_2, X_2+X_3) = Cov(X_2, X_2) = Var(X_2) = \sigma^2$.

$Cov(X_1+X_2, X_1-X_2) = E(X_1^2 - X_2^2) = 0$.

4. $Corr(X, Y) = Corr(X, X+Z) = \frac{Cov(X, X+Z)}{\sqrt{Var(X)Var(X+Z)}}$ $\quad Cov(X, X+Z) = Cov(X, X) = 1$ $\quad Var(X+Z) = 1.01$

$Corr(X, X+Z) = \frac{1}{\sqrt{1.01}}$

5. $Var(U) = Cov(U, U) = Cov(3X-2Y, 3X-2Y) = 9 - 12\cdot(-1) + 4\cdot 2 = 29$

$Var(V) = Cov(V, V) = Cov(X+2Y, X+2Y) = 1 + 4\cdot(-1) + 4\cdot 2 = 5$

$Cov(U, V) = Cov(3X-2Y, X+2Y) = 3 + 4\cdot(-1) - 4\cdot 2 = -9$

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6. (a) $E(T_N) = E(N)E(X_i) = \theta(15+50+40) = 105\theta$.

$Var(T_N) = E(N) \cdot Var(X_i) + (105\theta)^2$ $\quad$ $Var(X_i) = E(X_i^2) - E(X_i)^2 = (50^2 \cdot 0.3) + (100^2 \cdot 0.5) + (200^2 \cdot 0.2) - 105^2 = 2725$

so $Var(T_N) = 2725\theta + 11025\theta = 13750\theta$

(b) See the RMD pdf.

Homework3

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Problem 6(b)

```
set.seed(1)
n <- 10000
theta <- 50
N <- rpois(n,theta)
sample <- sample(c(50,100,200),n,replace=T,prob=c(0.3,0.5,0.2))
TN <- rep(0,10000)
for(i in 1:10000)
{
TN[i]<- sum(sample[1:N[i]])
}
mean(TN)
```

```
## [1] 4898.165
```

```
var(TN)
```

```
## [1] 497193.1
```

The sample mean should be 5250 and the sample variance should be 687500, we can see that the result has some differences. I also tried to delete "set.seed(1)" and run the R code several times, the result is sometimes larger and sometimes smaller, so the result seems reasonable.