组合数学习题课-期中

HW1

2、4、6、8、10

2

Let Q(x, y) denote the statement “x is the capital of y.”

What are these truth values?

- **Q(“Hangzhou (杭州)”, “Zhejiang (浙江)”)** T
- **Q(“Shenzhen (深圳)”, “Guangdong (广东)”)** F, 广东的省会是广州
- **Q(“Qingdao (青岛)”, “Shandong (山东)”)** F, 山东的省会是济南
- **Q(“Yinchuan (银川)”, “Ningxia (宁夏)")** T

4

Use truth tables to verify these equivalences.

i. $p \wedge T \equiv p$

真值表：

p	p∧T
T	T
F	F

4

Use truth tables to verify these equivalences.

ii. $p \vee F \equiv p$

真值表：

p	p∨F
T	T
F	F

4

Use truth tables to verify these equivalences.

iii. p∧F ≡F

真值表：

p	p∧F
T	F
F	F

4

Use truth tables to verify these equivalences.

iv. $p \vee T \equiv T$

真值表：

p	$p \vee T$
T	T
F	T

4

Use truth tables to verify these equivalences.

v. $p \vee p \equiv p$

真值表：

p	$p \vee p$
T	T
F	F

4

Use truth tables to verify these equivalences.

vi. $p \wedge p \equiv p$

真值表：

p	p∧p
T	T
F	F

6

Use truth tables to verify the commutative laws

i. $p \wedge q \equiv q \wedge p$

真值表：

p	q	p∧q	q∧p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

6

Use truth tables to verify the commutative laws

ii. $p \vee q \equiv q \vee p$

真值表：

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

8

Use a truth table to verify the distributive law

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

真值表：

p	q	r	$(p \wedge q)$	$(p \wedge r)$	$(q \vee r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

10

Show that each of these implications is a tautology by using truth tables.

i. $(p \wedge q) \rightarrow p$

真值表：

p	q	$(p \wedge q)$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

10

Show that each of these implications is a tautology by using truth tables.

ii. $p \to (p \vee q)$

真值表：

p	q	$(p \vee q)$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

10

Show that each of these implications is a tautology by using truth tables.

iii. $\neg p \rightarrow (p \rightarrow q)$

真值表：

p	q	$\neg p$	$(p \rightarrow q)$	$\neg p \rightarrow (p \rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

10

Show that each of these implications is a tautology by using truth tables.

iv. $(p \wedge q) \rightarrow (p \rightarrow q)$

真值表：

p	q	$(p \wedge q)$	$(p \rightarrow q)$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

10

Show that each of these implications is a tautology by using truth tables.

v. $\neg(p \rightarrow q) \rightarrow p$

真值表：

p	q	$(p \rightarrow q)$	$\neg(p \rightarrow q)$	$\neg(p \rightarrow q) \rightarrow p$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

10

Show that each of these implications is a tautology by using truth tables.

vi. $\neg(p \to q) \to \neg q$

真值表：

p	q	$(p \to q)$	$\neg(p \to q)$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

HW2

18、20、22、24、26

18

Let P(x) be the statement “x can speak Russian” and let Q(x) be the statement “x knows the computer language C++”. Express each of these sentences in terms of P(x), Q(x), quantifiers, and logical connectives. The universe of discourse for quantifiers consists of all students at your school.

P(x): x can speak Russian,

Q(x): x knows the computer language C++

18

i. **There is a student at your school who can speak Russian and who knows C++.**

   $\exists x(P(x) \land Q(x))$

ii. **There is a student at your school who can speak Russian but who doesn't know C++.**

   $\exists x(P(x) \land \neg Q(x))$

iii. **Every student at your school either can speak Russian or knows C++**

   $\forall x(P(x) \lor Q(x))$

iv. **No student at your school can speak Russian or knows C++**

   $\forall x(\neg P(x) \land \neg Q(x))$

   $\forall x \neg (P(x) \lor Q(x))$

   $\neg \exists x (P(x) \lor Q(x))$

20

What rule of inference is used in each of these arguments?

i. Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

ii. Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

iii. If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

iv. If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

v. If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, If I go swimming, then I will sunburn

20

Rule of inference	Tautology	Name
$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$	$p \to (p \vee q)$	Addition
$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$	$(p \wedge q) \to p$	Simplification
$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

20

i. Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

$p \rightarrow (p \vee q)$ Addition (附加规则)

ii. Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

$(p \wedge q) \rightarrow p$ Simplification (化简规则)

iii. If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

$[p \wedge (p \rightarrow q)] \rightarrow q$ Modus ponens (假言推理)

20

iv. If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$ Modus tollens (拒取式)

v. If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, If I go swimming, then I will sunburn

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ Hypothetical syllogism (假言三段论)

22

Let A, B, and C be sets. Show that

I. **$A \cup (B \cup C) = (A \cup B) \cup C$.**

II. **$A \cap (B \cap C) = (A \cap B) \cap C$.**

III. **$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.**

I. $x \in A \cup (B \cup C) \equiv x \in A \vee x \in (B \cup C)$

$$\equiv x \in A \vee (x \in B \vee x \in C)$$

$$\equiv x \in A \vee x \in B \vee x \in C$$

$$\equiv (x \in A \vee x \in B) \vee x \in C$$

$$\equiv x \in (A \cup B) \vee x \in C$$

$$\equiv x \in (A \cup B) \cup C$$

22

Let A, B, and C be sets. Show that

I. $A \cup (B \cup C) = (A \cup B) \cup C$.

II. $A \cap (B \cap C) = (A \cap B) \cap C$.

III. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

II. $$
\begin{aligned}
x \in A \cap (B \cap C) &\equiv x \in A \wedge x \in (B \cap C) \\
&\equiv x \in A \wedge (x \in B \wedge x \in C) \\
&\equiv x \in A \wedge x \in B \wedge x \in C \\
&\equiv (x \in A \wedge x \in B) \wedge x \in C \\
&\equiv x \in (A \cap B) \wedge x \in C \\
&\equiv x \in (A \cap B) \cap C
\end{aligned}
$$

22

Let A, B, and C be sets. Show that

I. $A \cup (B \cup C) = (A \cup B) \cup C$.

II. $A \cap (B \cap C) = (A \cap B) \cap C$.

III. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

III. $x \in A \cup (B \cap C) \equiv x \in A \vee x \in (B \cap C)$

$\equiv x \in A \vee (x \in B \wedge x \in C)$

$\equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

$\equiv (x \in A \cup B) \wedge (x \in A \cup C)$

$\equiv x \in (A \cup B) \cap (A \cup C)$

24

Determine whether the function f: $Z \times Z \rightarrow Z$ is onto of

- i. $f(m, n) = m + n$
- ii. $f(m, n) = \mathrm{m}^2 + n^2$
- iii. $f(m, n) = m$
- iv. $f(m, n) = |n|$
- v. $f(m, n) = m - n$

Definition

A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

- I. onto，对任意整数x，可以由f(x, 0)=x得到
- II. not onto，不存在两个整数平方数之和小于0
- III. onto，对任意整数x，可以由f(x, y)=x得到
- IV. not onto，不存在整数的绝对值小于0
- V. onto，对任意整数x，可以由f(x, 0)=x得到

26

(1) $\sum_{k=1}^{5}(k+1) = 5 + \sum_{k=1}^{5} k = 5 + \frac{(1+5)*5}{2} = 20$

(2) $\sum_{j=0}^{4}(-2)^j = \frac{1-(-2)^5}{1-(-2)} = 11$

(3) $\sum_{i=1}^{10} 3 = 30$

(4) $\sum_{j=0}^{8}\left(2^{j+1} - 2^j\right) = -2^0 + 2^1 - 2^1 + \cdots + 2^9 = -2^0 + 2^9 = 511$

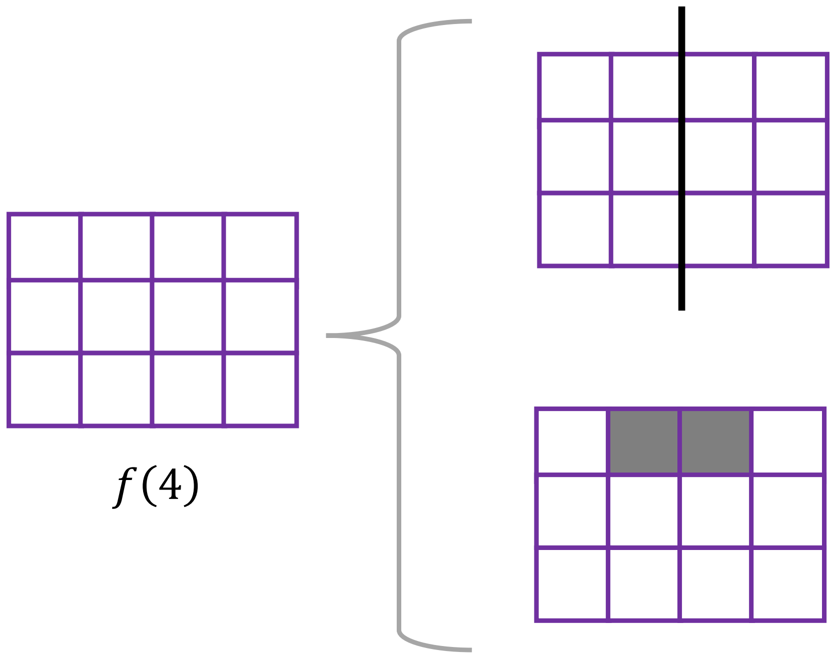
HW3

31、41、52、54、57

31

Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

设 $f(n)$为$3 * \mathrm{n}$的完美覆盖个数



$f(4)$

1. 可以完美切开一个3*2

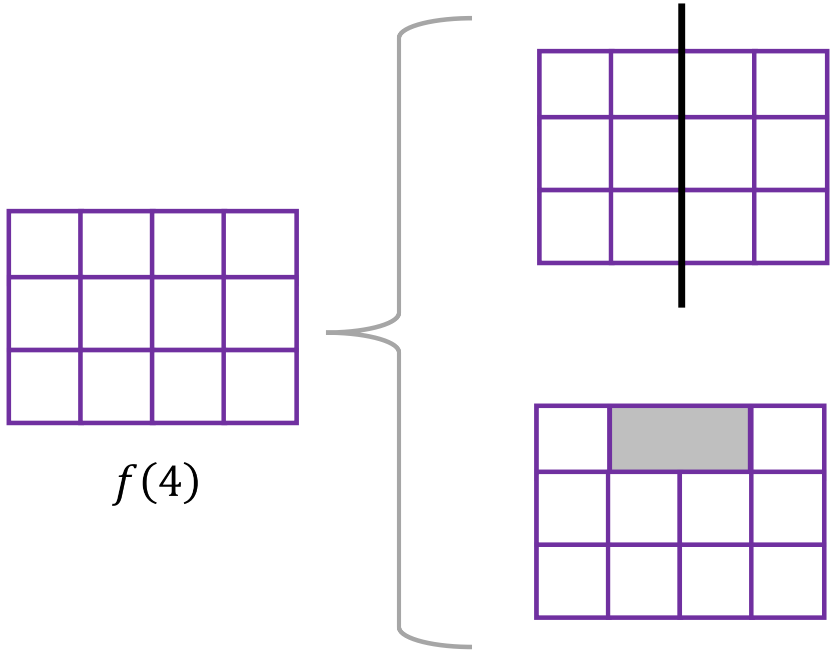
2. 不可以完美切开一个3*2

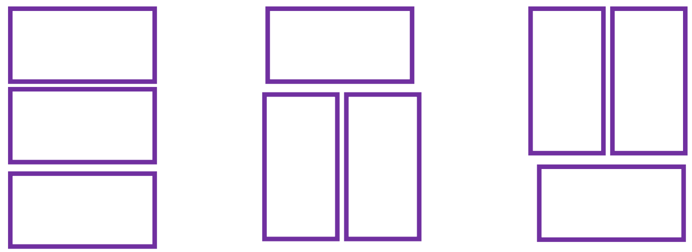
31

Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

设 $f(n)$为$3*\text{n}$的完美覆盖个数

1. 可以完美切开一个3*2，数量为$f^2(2)=9$



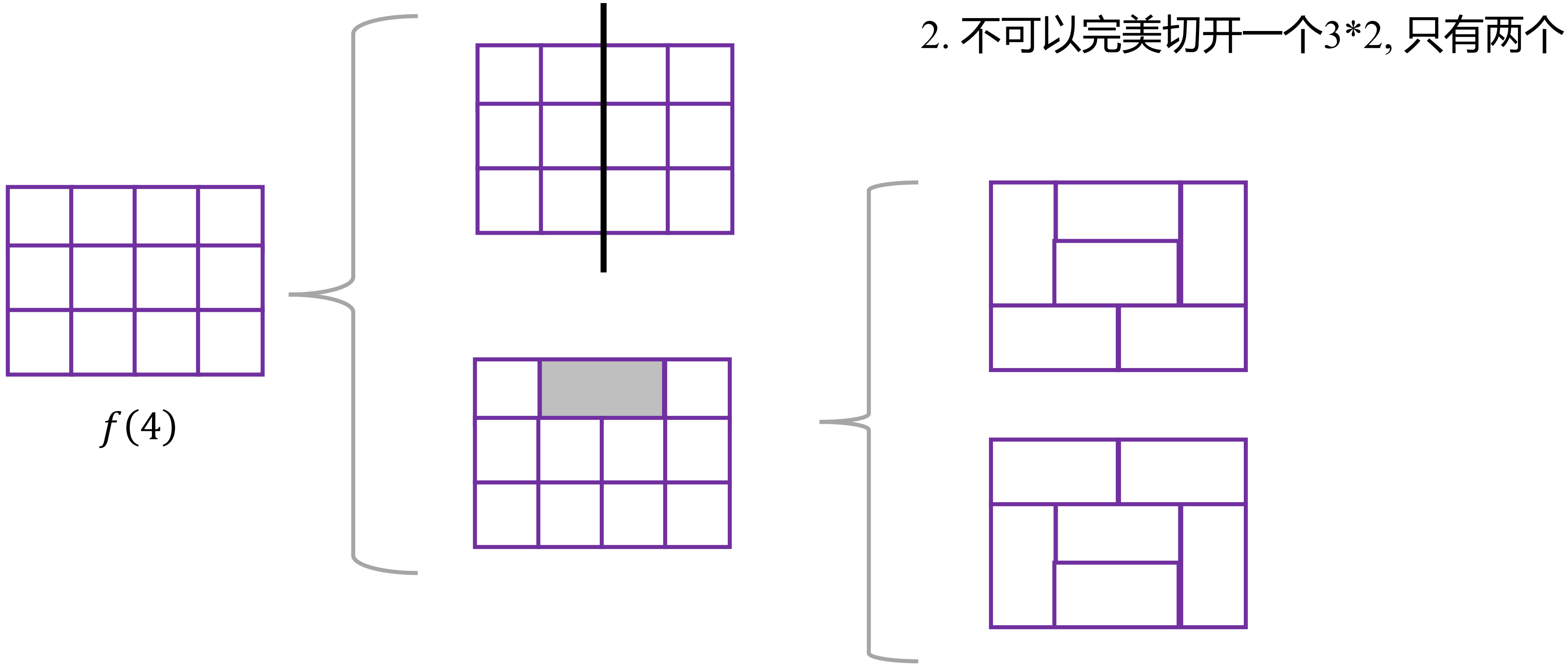


$f(2)$ 的三种

31

Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

设 $f(n)$为$3*$ n的完美覆盖个数, $f(4)=3*3+2=11$



31

Find the number of different perfect covers of a 3-by-4 chessboard by dominoes.

设 $f(n)$为$3*\mathrm{n}$的完美覆盖个数, $f(0)=1$

$$f(n)=3f(2)+2\sum_{i=2}^{\frac{n}{2}} f(n-2i)$$

41

Show that a magic square of order 3 must have a 5 in the middle position. Deduce that there are exactly 8 magic squares of order 3.

a	b	c
d	e	f
g	h	i

$$\left\{\begin{array}{l} a + e + i = 15 \\ b + e + h = 15 \\ c + e + g = 15 \\ d + e + f = 15 \\ a + b + c + d + e + f + g + h + i = 45 \end{array}\right. \Rightarrow e = 5$$

41

Show that a magic square of order 3 must have a 5 in the middle position. Deduce that there are exactly 8 magic squares of order 3.

接下来研究1的位置，假设1的位置为a，则 i=9 且 b + c = d + g = 14

因为b, c, d, g 互不相等且不为9，可得此时无解。

因为对称，1的位置不能为a, c, g, i

a	b	c
d	5	f
g	h	i

1	b	c
d	5	f
g	h	9

（2）

41

Show that a magic square of order 3 must have a 5 in the middle position. Deduce that there are exactly 8 magic squares of order 3.

假设1的位置为b，则 h=9 且 a + c = 14

因为a, c互不相等且不为9，可得 a=6或8

可得图（3）和图（4）两种结果

因为对称，1的位置可以是b, d, f, h，每种位置两个解法，所以一种有8种

a	b	c
d	5	f
g	h	i

a	1	c
d	5	f
g	9	i

（2）

6	1	8
7	5	3
2	9	4

（3）

8	1	6
3	5	7
4	9	2

（4）

52

A 6-by-6 chessboard is perfectly covered with 18 dominoes. Prove that it is possible to cut it either horizontally or vertically into two nonempty pieces without cutting through a domino; that is, prove that there must be a fault-line.

反证法：假设不存在一条线，能在不切割骨牌的情况下分割棋盘。

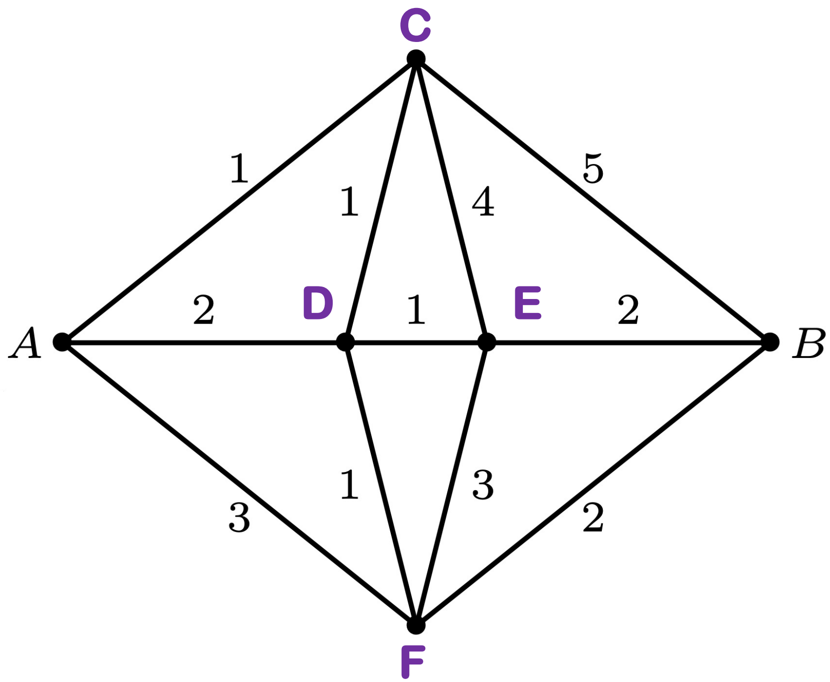
即每条分割线至少分割一个骨牌，又因为分开的两个部分均为偶数，所以每条分割线至少分割两个骨牌。

在6*6的棋盘中有5+5条分割线，则至少有10*2个骨牌，这与18个骨牌的事实矛盾！

所以一定存在一条线，在不切割骨牌的情况下分割棋盘

54

Determine all shortest routes from A to B in the system of intersections and streets (graph) in the figure shown. The numbers on the streets represent the lengths of the streets measured in terms of some unit.



最短路径为5，共有5条

(1) A→C→D→E→B

(2) A→C→D→F →B

(3) A→D→E→B

(4) A→D→F →B

(5) A→F →B

57

Consider 5-pile Nim with heaps of sizes 10, 20, 30, 40, and 50. Is this game balanced? Determine a first move for player I.

	2^5	2^4	2^3	2^2	2^1	2^0
10			1		1	
20		1		1		
30		1	1	1	1	
40	1		1			
50	1	1			1	
	2	3	3	2	3	

不全为偶数，是不公平的。最大的非平衡位为2^4

57

Consider 5-pile Nim with heaps of sizes 10, 20, 30, 40, and 50. Is this game balanced? Determine a first move for player I.

	2^5	2^4	2^3	2^2	2^1	2^0
10			1		1	
20		1		1		
30		0	0	1	0	
40	1		1			
50	1	1			1	
	2	2	2	2	2	

最大的非平衡位为2^4，所以要从30的堆里取走$2^4+2^3+2^1$（26）个让其平衡。

HW4

74、83、86、101、122

74

Use the pigeonhole principle to prove that the decimal expansion of a rational number m/n eventually is repeating. For example,

34, 478/99, 900 = .34512512512…

$$\frac{m}{n} = a_0 + \frac{b_0}{n}$$
$$\frac{b_0}{n} = \frac{1}{10} * (\frac{10 * b_0}{n}) = \frac{1}{10} * (a_1 + \frac{b_1}{n})$$
$$\frac{b_1}{n} = \frac{1}{10} * (\frac{10 * b_1}{n}) = \frac{1}{10} * (a_2 + \frac{b_2}{n})$$
$$\ldots$$
$$\frac{b_i}{n} = \frac{1}{10} * (\frac{10 * b_i}{n}) = \frac{1}{10} * (a_{i+1} + \frac{b_{i+1}}{n})$$

74

$$\frac{m}{n} = a_0 + \frac{b_0}{n}$$
$$\frac{b_0}{n} = \frac{1}{10} * (\frac{10 * b_0}{n}) = \frac{1}{10} * (a_1 + \frac{b_1}{n})$$
$$\frac{b_1}{n} = \frac{1}{10} * (\frac{10 * b_1}{n}) = \frac{1}{10} * (a_2 + \frac{b_2}{n})$$
$$\dots$$
$$\frac{b_i}{n} = \frac{1}{10} * (\frac{10 * b_i}{n}) = \frac{1}{10} * (a_{i+1} + \frac{b_{i+1}}{n})$$

$a_0, a_1, a_2, \dots, a_i$就是$\frac{m}{n}$的每一位数字的十进制表示，即$\frac{m}{n} = a_0.a_1a_2a_3\dots a_i\dots$

$b_0, b_1, b_2, \dots, b_i(0 < b_i < n)$则是运算过程中产生的余数。那么对序列b使用鸽笼原理，必存在i, j使得$b_i = b_j$.

为什么开始循环：因为每个b_i可以确定性的计算出a_{i+1}和b_{i+1}，一旦相同，则说明之后也一定相同。

83

There are 100 people at a party. Each person has an even number (possibly zero) of acquaintances. Prove that there are three people at the party with the same number of acquaintances.

每个人认识的人的数量的取值范围为0，2，4，…，98（共50个数）。

由鸽巢原理可知，有着相同的熟人数量的人的数量，至少为2。

如果有相同的熟人数量最多为2，那么必有两个人，认识的人的数量为0，这也就意味着对于任意一个人，其认识的人的数量最多为97（除去自身）。

这与一开始的结论相矛盾，所以至少有三个人，有相同的熟人数量。

86

Prove that $r(3, 3, 3) \le 17$

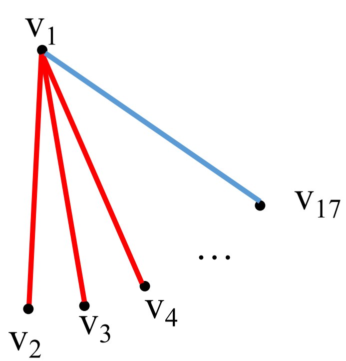
The *Ramsey's number* $r(m, n)$ is the smallest integer p such that $K_p \to K_m, K_n$.

只需证明$K_{17} \to K_3,K_3,K_3$

用三种颜色对K_{17}进行着色，则K_{17}内部要么有一个红色的K_3，要么有一个黄色K_3，要么有个蓝色K_3.

任选一个顶点，记作v_1，有16条边与其相连，这16条边着三种颜色之一。由鸽巢原理可知，至少由6条边着同一个颜色 ，不妨设为红色。相应的顶点记作v_2到v_7。

1）如果v_2~v_7这六个顶点之间存在i，j满足e_{ij}是一条红边，那么v_1，v_i，v_j构成一个红色K_3



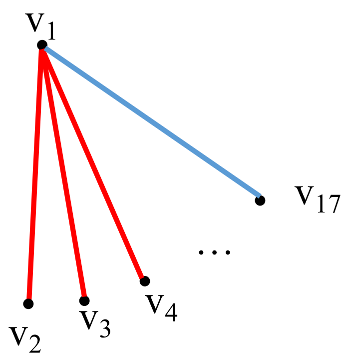
86

2） 如果v_2~v_7六个顶点之间不存在红边，那么v_2~v_7构成的K_6，只有两种颜色着色。

选择v_2，与其相连的五条边，至少由三条是同一种颜色，不妨设为黄色。相应的点不妨设为v_3~v_5。

2.1）如果v_3~v_5之间存在黄边，则会和v_2一起构成一个黄色K_3。

2.2）v_3~v_5之间不存在黄边，即v_3~v_5之间全部由蓝色相连，那么v_3~v_5构成一个蓝色K_3



101

Determine the number of poker hands of the following types,

I. full houses (3 cards of one rank and 2 of a different rank).
II. straights (5 consecutive ranks).
III. flushes (5 cards of the same suit).
IV. straight flushes (5 consecutive cards of the same suit).
V. exactly two pairs (2 cards of one rank, 2 cards of another rank, and 1 card of a third rank).
VI. exactly one pair (2 cards of one rank, and 3 cards of three other and different ranks).

101

I. **full houses (3 cards of one rank and 2 of a different rank).**

13 x $\binom{4}{3}$ x 12 x $\binom{4}{2}$ =3744

II. **straights (5 consecutive ranks).**

10(9) x 4^5

III. **flushes (5 cards of the same suit).**

$\binom{13}{5}$ x 4=5148

IV. **straight flushes (5 consecutive cards of the same suit).**

10(9) x 4

101

V. exactly two pairs (2 cards of one rank, 2 cards of another rank, and 1 card of a third rank).

$$\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{11}{1} \times \binom{4}{1} = 123552$$

VI. exactly one pair (2 cards of one rank, and 3 cards of three other and different ranks).

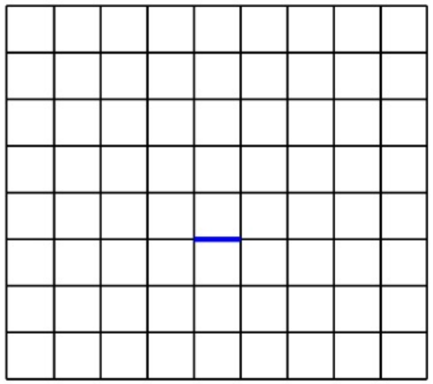
$$\binom{13}{1} \times \binom{4}{2} \times \binom{12}{3} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1} = 1098240$$

122

A secretary works in a building located 9 blocks east and 8 blocks north of his home. Every day he walks 17 blocks to work (the map shown).

i. How many different routes are possible for him?

ii. How many different routes if the street in the easterly direction, which begins 4 blocks east and 3 blocks north of his home, is under water (and he cant swim)? (Hint: Count the routes that use the block under water.)



1. 一共经过17个街区，向东街区9个，向北街区8个，组合。$\binom{17}{9}$=24310
2. 如题意，图中蓝色部分街区被淹没，减去需要经过蓝色街区的路径。

   从家到蓝色街区（左端点）$\binom{7}{4}$种走法，

   从蓝色街区（右端点）到公司$\binom{9}{4}$种走法，

   故总方法数为$\binom{17}{9}$- $\binom{7}{4}$ $\binom{9}{4}$=19900

HW5

127、132、149 、152、156、164

127

Determine the number of 10-permutations of the multiset $S = \{3\cdot a, 4\cdot b, 5\cdot c\}$.

S一共12个元素，共选10个元素，每次会有2个元素未被选择，一共6种情况：{a, a} {a, b} {a, c} {b, b} {b, c} {c, c}。对应的排列数有：

$\{1\cdot a, 4\cdot b, 5\cdot c\}$: $\frac{10!}{1!4!5!}$

$\{2\cdot a, 3\cdot b, 5\cdot c\}$: $\frac{10!}{2!3!5!}$

$\{2\cdot a, 4\cdot b, 4\cdot c\}$: $\frac{10!}{2!4!4!}$

$\{3\cdot a, 2\cdot b, 5\cdot c\}$: $\frac{10!}{3!2!5!}$

$\{3\cdot a, 3\cdot b, 4\cdot c\}$: $\frac{10!}{3!3!4!}$

$\{3\cdot a, 4\cdot b, 3\cdot c\}$: $\frac{10!}{3!4!3!}$

132

How many integral solutions of $x_1 + x_2 + x_3 + x_4 = 30$ satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$

设 $y_1 = x_1 - 2, y_2 = x_2, y_3 = x_3 + 5, y_4 = x_4 - 8$

得 $y_1 + y_2 + y_3 + y_4 = 25$ 且 $y_1, y_2, y_3, y_4 \geq 0$

由公式可得$\binom{25+4-1}{25} = \binom{28}{25}$

149

How many permutations are there of the letters in the word PNEUMONOULTRAMICROSCOPICSILICOVOLCANOCONIOSIS? This word is, by some accounts, the longest word in the English language.

一共45个字母，统计其成分：75 98 22

{A:2, C:6, E:1, I:6, L:3, M:2, N:4, O:9, P:2, R:2, S:4, T:1, U:2, V:1}

带入公式可得其排列数为：

$$\frac{45!}{2!\,6!\,1!\,6!\,3!\,2!\,4!\,9!\,2!\,2!\,4!\,1!\,2!\,1!}$$

152

Use the algorithm of Section 4.1 to generate the permutations 12345 , starting with $\overleftarrow{1}\,\overleftarrow{2}\,\overleftarrow{3}\,\overleftarrow{4}\,\overleftarrow{5}$

120	$\overrightarrow{4}\overleftarrow{5}\overleftarrow{1}\overleftarrow{3}\overleftarrow{2}$	$\overleftarrow{3}\overleftarrow{1}\overleftarrow{4}\overrightarrow{5}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{5}\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}$	$\overleftarrow{4}\overleftarrow{2}\overrightarrow{3}\overrightarrow{5}\overleftarrow{1}$
$\overleftarrow{1}\overleftarrow{2}\overleftarrow{3}\overleftarrow{4}\overleftarrow{5}$	$\overleftarrow{5}\overrightarrow{4}\overleftarrow{1}\overleftarrow{3}\overleftarrow{2}$	$\overleftarrow{3}\overleftarrow{1}\overleftarrow{4}\overleftarrow{2}\overrightarrow{5}$	$\overleftarrow{5}\overrightarrow{3}\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}$	$\overleftarrow{4}\overleftarrow{2}\overrightarrow{3}\overleftarrow{1}\overrightarrow{5}$
$\overleftarrow{1}\overleftarrow{2}\overleftarrow{3}\overleftarrow{5}\overleftarrow{4}$	$\overrightarrow{5}\overleftarrow{1}\overrightarrow{4}\overleftarrow{3}\overleftarrow{2}$	$\overleftarrow{3}\overleftarrow{4}\overleftarrow{1}\overleftarrow{2}\overleftarrow{5}$	$\overrightarrow{5}\overrightarrow{3}\overleftarrow{2}\overleftarrow{1}\overrightarrow{4}$	$\overrightarrow{4}\overleftarrow{2}\overleftarrow{1}\overrightarrow{3}\overleftarrow{5}$
$\overleftarrow{1}\overleftarrow{2}\overleftarrow{5}\overleftarrow{3}\overleftarrow{4}$	$\overleftarrow{1}\overrightarrow{5}\overrightarrow{4}\overleftarrow{3}\overleftarrow{2}$	$\overleftarrow{3}\overleftarrow{4}\overleftarrow{1}\overleftarrow{5}\overleftarrow{2}$	$\overrightarrow{3}\overrightarrow{5}\overleftarrow{2}\overleftarrow{1}\overrightarrow{4}$	$\overrightarrow{4}\overleftarrow{2}\overleftarrow{1}\overleftarrow{5}\overrightarrow{3}$
$\overleftarrow{1}\overleftarrow{5}\overleftarrow{2}\overleftarrow{3}\overleftarrow{4}$	$\overleftarrow{1}\overrightarrow{4}\overrightarrow{5}\overleftarrow{3}\overleftarrow{2}$	$\overleftarrow{3}\overleftarrow{4}\overleftarrow{5}\overleftarrow{1}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{2}\overrightarrow{5}\overleftarrow{1}\overrightarrow{4}$	$\overrightarrow{4}\overleftarrow{2}\overleftarrow{5}\overleftarrow{1}\overrightarrow{3}$
$\overleftarrow{5}\overleftarrow{1}\overleftarrow{2}\overleftarrow{3}\overleftarrow{4}$	$\overleftarrow{1}\overrightarrow{4}\overleftarrow{3}\overrightarrow{5}\overleftarrow{2}$	$\overleftarrow{3}\overleftarrow{5}\overleftarrow{4}\overleftarrow{1}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{2}\overleftarrow{1}\overrightarrow{5}\overrightarrow{4}$	$\overrightarrow{4}\overleftarrow{5}\overleftarrow{2}\overleftarrow{1}\overrightarrow{3}$
$\overrightarrow{5}\overleftarrow{1}\overleftarrow{2}\overleftarrow{4}\overleftarrow{3}$	$\overleftarrow{1}\overrightarrow{4}\overleftarrow{3}\overleftarrow{2}\overrightarrow{5}$	$\overleftarrow{5}\overleftarrow{3}\overleftarrow{4}\overleftarrow{1}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{2}\overleftarrow{1}\overrightarrow{4}\overrightarrow{5}$	$\overleftarrow{5}\overrightarrow{4}\overleftarrow{2}\overleftarrow{1}\overrightarrow{3}$
$\overleftarrow{1}\overrightarrow{5}\overleftarrow{2}\overleftarrow{4}\overleftarrow{3}$	$\overleftarrow{1}\overleftarrow{3}\overrightarrow{4}\overleftarrow{2}\overleftarrow{5}$	$\overrightarrow{5}\overleftarrow{4}\overleftarrow{3}\overleftarrow{1}\overleftarrow{2}$	$\overleftarrow{2}\overrightarrow{3}\overleftarrow{1}\overleftarrow{4}\overleftarrow{5}$	$\overrightarrow{5}\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}\overrightarrow{3}$
$\overleftarrow{1}\overleftarrow{2}\overrightarrow{5}\overleftarrow{4}\overleftarrow{3}$	$\overleftarrow{1}\overleftarrow{3}\overrightarrow{4}\overleftarrow{5}\overleftarrow{2}$	$\overleftarrow{4}\overrightarrow{5}\overleftarrow{3}\overleftarrow{1}\overleftarrow{2}$	$\overleftarrow{2}\overrightarrow{3}\overleftarrow{1}\overleftarrow{5}\overleftarrow{4}$	$\overleftarrow{2}\overrightarrow{5}\overrightarrow{4}\overleftarrow{1}\overrightarrow{3}$
$\overleftarrow{1}\overleftarrow{2}\overleftarrow{4}\overrightarrow{5}\overleftarrow{3}$	$\overleftarrow{1}\overleftarrow{3}\overleftarrow{5}\overrightarrow{4}\overleftarrow{2}$	$\overleftarrow{4}\overleftarrow{3}\overrightarrow{5}\overleftarrow{1}\overleftarrow{2}$	$\overleftarrow{2}\overrightarrow{3}\overleftarrow{5}\overleftarrow{1}\overleftarrow{4}$	$\overleftarrow{2}\overrightarrow{4}\overrightarrow{5}\overleftarrow{1}\overrightarrow{3}$
$\overleftarrow{1}\overleftarrow{2}\overleftarrow{4}\overleftarrow{3}\overrightarrow{5}$	$\overleftarrow{1}\overleftarrow{5}\overleftarrow{3}\overrightarrow{4}\overleftarrow{2}$	$\overleftarrow{4}\overleftarrow{3}\overleftarrow{1}\overrightarrow{5}\overleftarrow{2}$	$\overleftarrow{2}\overleftarrow{5}\overrightarrow{3}\overleftarrow{1}\overleftarrow{4}$	$\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}\overrightarrow{5}\overrightarrow{3}$
$\overleftarrow{1}\overleftarrow{4}\overleftarrow{2}\overleftarrow{3}\overleftarrow{5}$	$\overleftarrow{5}\overleftarrow{1}\overleftarrow{3}\overrightarrow{4}\overleftarrow{2}$	$\overleftarrow{4}\overleftarrow{3}\overleftarrow{1}\overleftarrow{2}\overrightarrow{5}$	$\overleftarrow{5}\overleftarrow{2}\overrightarrow{3}\overleftarrow{1}\overleftarrow{4}$	$\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}\overrightarrow{3}\overrightarrow{5}$
$\overleftarrow{1}\overleftarrow{4}\overleftarrow{2}\overleftarrow{5}\overleftarrow{3}$	$\overrightarrow{5}\overleftarrow{1}\overleftarrow{3}\overleftarrow{2}\overrightarrow{4}$	$\overrightarrow{4}\overrightarrow{3}\overleftarrow{2}\overleftarrow{1}\overleftarrow{5}$	$\overrightarrow{5}\overleftarrow{2}\overrightarrow{3}\overleftarrow{4}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{1}\overrightarrow{4}\overrightarrow{3}\overleftarrow{5}$
$\overleftarrow{1}\overleftarrow{4}\overleftarrow{5}\overleftarrow{2}\overleftarrow{3}$	$\overleftarrow{1}\overrightarrow{5}\overleftarrow{3}\overleftarrow{2}\overrightarrow{4}$	$\overrightarrow{4}\overrightarrow{3}\overleftarrow{2}\overleftarrow{5}\overleftarrow{1}$	$\overleftarrow{2}\overrightarrow{5}\overrightarrow{3}\overleftarrow{4}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{1}\overrightarrow{4}\overleftarrow{5}\overrightarrow{3}$
$\overleftarrow{1}\overleftarrow{5}\overleftarrow{4}\overleftarrow{2}\overleftarrow{3}$	$\overleftarrow{1}\overleftarrow{3}\overrightarrow{5}\overleftarrow{2}\overrightarrow{4}$	$\overrightarrow{4}\overrightarrow{3}\overleftarrow{5}\overleftarrow{2}\overleftarrow{1}$	$\overleftarrow{2}\overrightarrow{3}\overrightarrow{5}\overleftarrow{4}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{1}\overleftarrow{5}\overrightarrow{4}\overrightarrow{3}$
$\overleftarrow{5}\overleftarrow{1}\overleftarrow{4}\overleftarrow{2}\overleftarrow{3}$	$\overleftarrow{1}\overleftarrow{3}\overleftarrow{2}\overrightarrow{5}\overrightarrow{4}$	$\overrightarrow{4}\overleftarrow{5}\overrightarrow{3}\overleftarrow{2}\overleftarrow{1}$	$\overleftarrow{2}\overrightarrow{3}\overleftarrow{4}\overrightarrow{5}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{5}\overleftarrow{1}\overrightarrow{4}\overrightarrow{3}$
$\overrightarrow{5}\overleftarrow{4}\overleftarrow{1}\overleftarrow{2}\overleftarrow{3}$	$\overleftarrow{1}\overleftarrow{3}\overleftarrow{2}\overrightarrow{4}\overrightarrow{5}$	$\overleftarrow{5}\overrightarrow{4}\overrightarrow{3}\overleftarrow{2}\overleftarrow{1}$	$\overleftarrow{2}\overrightarrow{3}\overleftarrow{4}\overleftarrow{1}\overrightarrow{5}$	$\overleftarrow{5}\overleftarrow{2}\overleftarrow{1}\overrightarrow{4}\overrightarrow{3}$
$\overleftarrow{4}\overrightarrow{5}\overleftarrow{1}\overleftarrow{2}\overleftarrow{3}$	$\overleftarrow{3}\overleftarrow{1}\overleftarrow{2}\overleftarrow{4}\overleftarrow{5}$	$\overrightarrow{5}\overrightarrow{3}\overrightarrow{4}\overleftarrow{2}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{4}\overrightarrow{3}\overleftarrow{1}\overleftarrow{5}$	$\overrightarrow{5}\overleftarrow{2}\overleftarrow{1}\overrightarrow{3}\overrightarrow{4}$
$\overleftarrow{4}\overleftarrow{1}\overrightarrow{5}\overleftarrow{2}\overleftarrow{3}$	$\overleftarrow{3}\overleftarrow{1}\overleftarrow{2}\overleftarrow{5}\overleftarrow{4}$	$\overrightarrow{3}\overrightarrow{5}\overrightarrow{4}\overleftarrow{2}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{4}\overrightarrow{3}\overleftarrow{5}\overleftarrow{1}$	$\overleftarrow{2}\overrightarrow{5}\overleftarrow{1}\overrightarrow{3}\overrightarrow{4}$
$\overleftarrow{4}\overleftarrow{1}\overleftarrow{2}\overrightarrow{5}\overleftarrow{3}$	$\overleftarrow{3}\overleftarrow{1}\overleftarrow{5}\overleftarrow{2}\overleftarrow{4}$	$\overrightarrow{3}\overrightarrow{4}\overrightarrow{5}\overleftarrow{2}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{4}\overleftarrow{5}\overrightarrow{3}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{1}\overrightarrow{5}\overrightarrow{3}\overrightarrow{4}$
$\overleftarrow{4}\overleftarrow{1}\overleftarrow{2}\overleftarrow{3}\overrightarrow{5}$	$\overleftarrow{3}\overleftarrow{5}\overleftarrow{1}\overleftarrow{2}\overleftarrow{4}$	$\overrightarrow{3}\overrightarrow{4}\overleftarrow{2}\overrightarrow{5}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{5}\overleftarrow{4}\overrightarrow{3}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{1}\overrightarrow{3}\overrightarrow{5}\overrightarrow{4}$
$\overrightarrow{4}\overleftarrow{1}\overleftarrow{3}\overleftarrow{2}\overleftarrow{5}$	$\overleftarrow{5}\overleftarrow{3}\overleftarrow{1}\overleftarrow{2}\overleftarrow{4}$	$\overrightarrow{3}\overrightarrow{4}\overleftarrow{2}\overleftarrow{1}\overrightarrow{5}$	$\overleftarrow{5}\overleftarrow{2}\overleftarrow{4}\overrightarrow{3}\overleftarrow{1}$	$\overleftarrow{2}\overleftarrow{1}\overrightarrow{3}\overrightarrow{4}\overrightarrow{5}$
$\overrightarrow{4}\overleftarrow{1}\overleftarrow{3}\overleftarrow{5}\overleftarrow{2}$	$\overrightarrow{5}\overleftarrow{3}\overleftarrow{1}\overleftarrow{4}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{2}\overrightarrow{4}\overleftarrow{1}\overleftarrow{5}$	$\overrightarrow{5}\overleftarrow{4}\overleftarrow{2}\overrightarrow{3}\overleftarrow{1}$	
$\overrightarrow{4}\overleftarrow{1}\overleftarrow{5}\overleftarrow{3}\overleftarrow{2}$	$\overleftarrow{3}\overrightarrow{5}\overleftarrow{1}\overleftarrow{4}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{2}\overrightarrow{4}\overleftarrow{5}\overleftarrow{1}$	$\overleftarrow{4}\overrightarrow{5}\overleftarrow{2}\overrightarrow{3}\overleftarrow{1}$	
	$\overleftarrow{3}\overleftarrow{1}\overrightarrow{5}\overleftarrow{4}\overleftarrow{2}$	$\overrightarrow{3}\overleftarrow{2}\overleftarrow{5}\overrightarrow{4}\overleftarrow{1}$	$\overleftarrow{4}\overleftarrow{2}\overrightarrow{5}\overrightarrow{3}\overleftarrow{1}$	

152

156

Construct the permutations of {1, 2, …, 8} whose inversion sequences are

i. 2, 5, 5, 0, 2, 1, 1, 0

ii. 6, 6, 1, 4, 2, 1, 0, 0

i.

第八个数为0，代表8左侧有0个比8大的数字，先将8加入排列，{8}
第七个数为1，代表7左侧有1个比7大的数字，只有8大于7，所以7在8右侧，{8, 7}；
第六个数为1，代表6左侧有1个比6大的数字，只有7，8大于6，所以6在8和7中间，{8, 6, 7}；
第五个数为2，代表5左侧有2个比5大的数字，所以5在已有排列第三位，{8, 6, 5, 7}；
第四个数为0，代表4左侧有0个比4大的数字，所以4在已有排列第一位，{4, 8, 6, 5, 7}；
第三个数为5，代表3左侧有5个比3大的数字，所以3在已有排列第六位，{4, 8, 6, 5, 7, 3}；
第二个数为5，代表2左侧有5个比2大的数字，所以2在已有排列第六位，{4, 8, 6, 5, 7, 2, 3}；
第一个数为2，代表1左侧有2个比1大的数字，所以1在已有排列第三位，**{4, 8, 1, 6, 5, 7, 2, 3}**；

156

Construct the permutations of {1, 2, ..., 8} whose inversion sequences are

i. 2, 5, 5, 0, 2, 1, 1, 0

ii. 6, 6, 1, 4, 2, 1, 0, 0

ii.

第八个数为0，代表8左侧有0个比8大的数字，先将8加入排列，{8}
第七个数为1，代表7左侧有0个比7大的数字，只有8大于7，所以7在8左侧，{7, 8}；
第六个数为1，代表6左侧有1个比6大的数字，只有7，8大于6，所以6在8和7中间，{7, 6, 8}；
第五个数为2，代表5左侧有2个比5大的数字，所以5在已有排列第三位，{7, 6, 5, 8}；
第四个数为4，代表4左侧有4个比4大的数字，所以4在已有排列第五位，{7, 6, 5, 8, 4}；
第三个数为1，代表3左侧有1个比3大的数字，所以3在已有排列第二位，{7, 3, 6, 5, 8, 4}；
第二个数为6，代表2左侧有6个比2大的数字，所以2在已有排列第七位，{7, 3, 6, 5, 8, 4, 2}；
第一个数为6，代表1左侧有6个比1大的数字，所以1在已有排列第七位，**{7, 3, 6, 5, 8, 4, 1, 2}**；

164

For each of the following combinations of {x7, x6, …, x1, x0} , determine the combination that immediately follows it by using the base 2 arithmetic generating scheme,

i. {x4, x1, x0}

ii. {x7, x5, x3}

iii. {x7, x5, x4, x3, x2, x1, x0}

iv. {x0}

i. {x4, x1, x0}对应00010011，下一个为00010100，对应{x4, x2}
ii. {x7, x5, x3}对应10101000，下一个为10101001，对应{x7, x5, x3, x0}
iii. {x7, x5, x4, x3, x2, x1, x0}对应10111111，下一个为11000000，对应{x7, x6}
iv. {x0}对应00000001，下一个为00000010，对应{x1}

HW6

172、192、199、 220、235、254

172

Determine the immediate successors of the following 9-tuples in the reflected Gray code of order 9

反射格雷码的生成规则：

（1）若1的个数为偶数，则改变最右边的位元

（2）若1的个数为奇数，改变右起第一个为1的位元的左边位元

i. 010100110, 1的个数为4，用规则（1），为010100111

ii. 110001100, 1的个数为4，用规则（1），为110001101

iii. 111111111，1的个数为9，用规则（2），为111111101

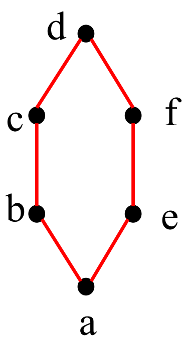
192

Let X = {a, b, c, d, e, f} and let the relation R on X be defined by *aRb, bRc, cRd, aRe, eRf, fRd*. Verify that R is the cover relation of a partially ordered set, and determine all the linear extensions of this partial order.

偏序集可以用几何的方法表示。为了叙述几何表示方法，需要定义偏序集$(X,\leqslant)$的覆盖关系。令 a 和 b 是 X 中的元素。如果 $a<b$ 并且没有元素 c 能够夹在 a 和 b 之间，那么 a 就被 b 覆盖（也说成 b 覆盖 a），记为 $a<_c b$；就是说，不存在元素 c，使得 $a<c$ 和 $c<b$ 同时成立。如果 X 是一个有限集，则由传递性可知，偏序$\leqslant$被它的覆盖关系唯一确定。因此，覆盖关系是描述偏序的有效方法。由定理 4.5.1 可知，如果$(X,\leqslant)$是全序集，则 X 的元素可以列成 x_1，x_2，…，x_n，使得 $x_1<_c x_2<_c \cdots <_c x_n$。正是由于这种原因，全序集也叫做线性有序集。

192

Let X = {a, b, c, d, e, f} and let the relation R on X be defined by *aRb, bRc, cRd, aRe, eRf, fRd*. Verify that R is the cover relation of a partially ordered set, and determine all the linear extensions of this partial order.

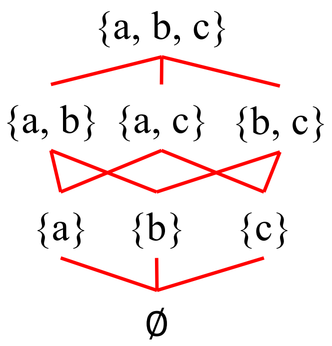


所有的线性扩展为：

a b c e f d
a b e c f d
a b e f c d
a e f b c d
a e b f c d
a e b c f d

199

Consider the partially ordered set (X,⊆) of subsets of the set X= {a,b,c}of 3 elements. How many linear extensions are there?



无论怎么排序，第一位永远是∅，最后一位都是{a, b, c}

(1)若只按集合元素个数来排，则有 ${A_3^3}^2$=36种

(2) {a}/ {b}/ {c} 其中一个排列在{a,b}/{a,c}/{b,c}中

比如 {b}{c}{b,c}{a}{a,b}{a,c},其中{b}{c}和{a,b}{a,c}可以调换顺序，则一共有3*2*2=12种

综上所示，一共有36+12=48种

220

Prove, that for every integer n > 1,

$\binom{n}{1}$-2 $\binom{n}{2}$+3 $\binom{n}{3}$+…+(-1)$^{n-1}$n $\binom{n}{n}$=0

根据二项式定理：$(x+1)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$

同时对等式两边取x的倒数，得到：$n(x+1)^{n-1}=\sum_{k=1}^{n}\binom{n}{k}kx^{k-1}$

令x=-1，即可得到：0= $\sum_{k=1}^{n}\binom{n}{k}k(-1)^{k-1}$，展开，即可得到等式

220

Prove, that for every integer n > 1,

$\binom{n}{1}$-2 $\binom{n}{2}$+3 $\binom{n}{3}$+...+(-1)$^{n-1}$n $\binom{n}{n}$=0

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$$

等式变为： $n\binom{n-1}{0}$-n $\binom{n-1}{1}$+n $\binom{n-1}{2}$+...+(-1)$^{n-1}$n $\binom{n-1}{n-1}$

= n[$\binom{n-1}{0}$- $\binom{n-1}{1}$+ $\binom{n-1}{2}$+...+ (-1)$^{n-1}$ $\binom{n-1}{n-1}$]

$=n * 0 = 0$

235

Find and prove a formula for $\sum_{\substack{r,s,t\geq 0\\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$ where the summation extends over all nonnegative integers r, s and t with sum r+ s+ t= n.

组合意义：从总共 (m_1+m_2+m_3) 个物品中选取n个，总的方法数为 $\binom{m_1+m_2+m_3}{n}$。

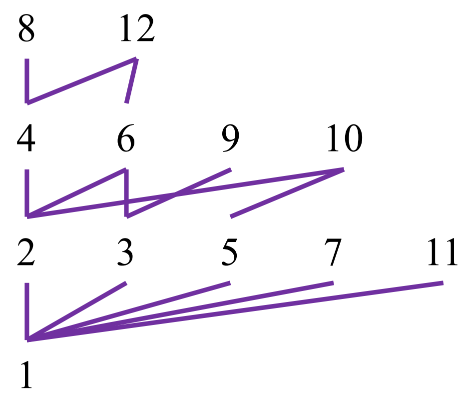
其也等价于从m_1个物品中选取r个，m_2个物品中选取s个，m_3个物品中选取t个，并且r+s+t=n

254

Consider the partially ordered set $(X, |)$ on the set $X = \{1, 2, \cdots, 12\}$ of the first 12 positive integers, partially ordered by "is divisible by."

i. Determine a chain of largest size and a partition of X into the smallest number of antichains.

最长链为$\{1, 2, 4, 8\}$。
可以划分为：
$\{1\}$
$\{2, 3, 5, 7, 11\}$
$\{4, 6, 9, 10\}$
$\{8, 12\}$



254

Consider the partially ordered set (X, |) on the set X = {1, 2, · · · , 12} of the first 12 positive integers, partially ordered by “is divisible by.”

ii. Determine an antichain of largest size and a partition of X into the smallest number of chains.

最长反链为{7, 8, 9, 10, 11, 12}。

可以划分为：

{1, 2, 4, 8}

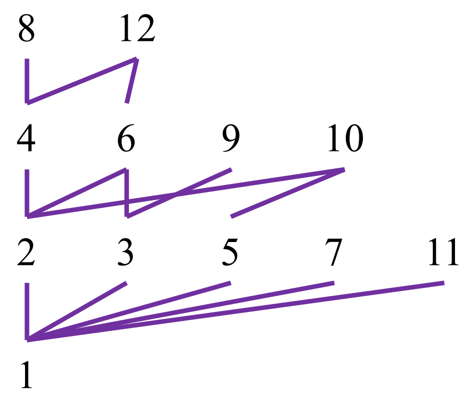
{3, 6, 12}

{9,}

{5, 10}

{7}

{11}



组合数学习题课-作业7

256, 258, 262, 270, 272, 279.

HW7

256、258、262、270、 272、 279

256

Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 5, or 6.

设A_1为能被4整除的数, A_2为能被5整除的数, A_3为能被4整除的数, 根据容斥原理：

$$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$$

$$= S - (A_1 + A_2 + A_3) + (A_1 \cap A_2) + (A_1 \cap A_3) + (A_2 \cap A_3) - (A_1 \cap A_2 \cap A_3)$$

= 10000 – (2500 + 2000 + 1666) + 500 + 833 + 333 - 166

= 5334

258

Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

设A_1为完全平方数, A_2为完全立方数，根据容斥原理：

$$\overline{A_1} \cap \overline{A_2} = S - (A_1 + A_2) + (A_1 \cap A_2)$$

$= 10000 - (\lfloor \sqrt[2]{10000} \rfloor + \lfloor \sqrt[3]{10000} \rfloor) + \lfloor \sqrt[6]{10000} \rfloor$

$= 10000\text{-}(100+21) + 4$

$= 9883$

262

Determine the number of solutions of the equation x1 + x2 + x3 + x4 = 14 in nonnegative integers x1, x2, x3, and x4 not exceeding 8.

若无x4的限制，则有公式可得解法数目为：

$$\binom{14+4-1}{14} = 680$$

现在先求x4超过8的情况，设y4 = x4-9，则有 x1+x2+x3+y4 = 5，且均非负，由公式可得解法数目为：

$$\binom{5+4-1}{5}=224$$

则x4不超过8的解数为：680-224 = 456

270

At a party seven gentlemen check their hats. In how many ways can their hats be returned so that

i. no gentleman receives his own hat?

ii. at least one of the gentlemen receives his own hat?

iii. at least two of the gentlemen receive their own hats?

(1) 7个的错排：D_7

(2) 全集减去完全错排： $7! - D_7$

(3) 全集减去完全错排和一个人帽子对的情况： $7! - D_7 - 7D_6$

272

Determine the number of permutations of the multiset S = {3 · a, 4 · b, 2 · c} , where, for each type of letter, the letters of the same type do not appear consecutively. (Thus abbbbcaca is not allowed, but abbbacacb is.)

设A_1为出现 aaa, A_2为出现 bbbb, A_3为出现 cc, 根据容斥原理：

$$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} = S - (A_1 + A_2 + A_3) + (A_1 \cap A_2 + A_1 \cap A_3 + A_2 \cap A_3) - (A_1 \cap A_2 \cap A_3)$$

$$= \frac{9!}{3!4!2!} - \left(\frac{7!}{1!4!2!} + \frac{6!}{3!1!2!} + \frac{8!}{3!4!1!}\right) + \left(\frac{4!}{1!1!2!} + \frac{6!}{1!4!1!} + \frac{5!}{3!1!1!}\right) - \frac{3!}{1!1!1!}$$

$$= 871$$

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What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?

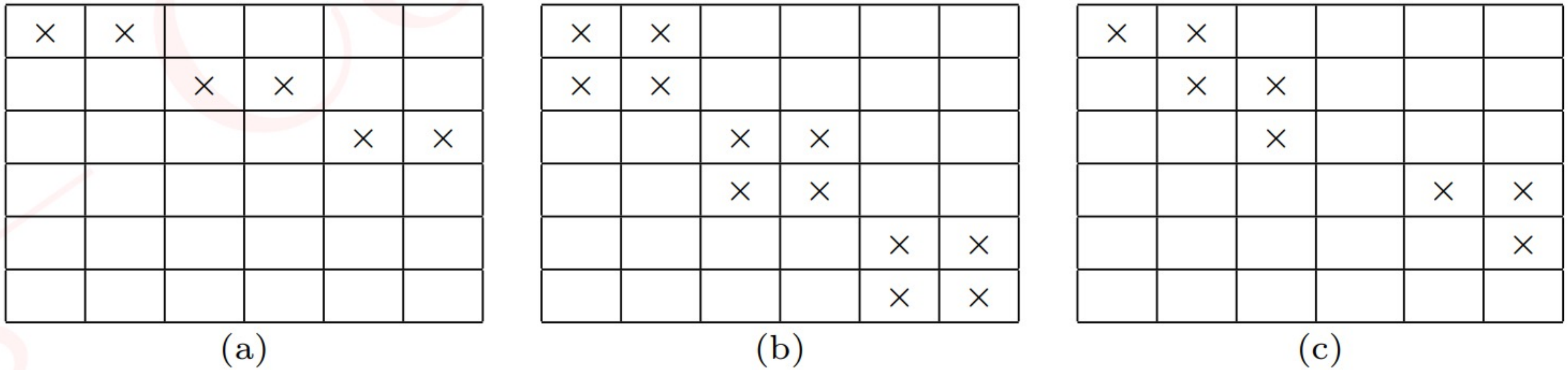

Figure in Question 279

279
如何解决棋盘上有禁止模式的排列问题？

定理 6.4.1 将 n 个非攻击型不可区分的车放到带有禁止放置位置的 n 行 n 列棋盘上的放置方法数等于

$$n! - r_1(n-1)! + r_2(n-2)! - \cdots + (-1)^k r_k(n-k)! + \cdots + (-1)^n r_n \qquad \square$$

r_k 是把 k 个非攻击型车放到 n 行 n 列棋盘上的这样一种方法数，其中这 k 个车中的每个都处在禁止放置的位置上（$k=1, 2, \cdots, n$）。

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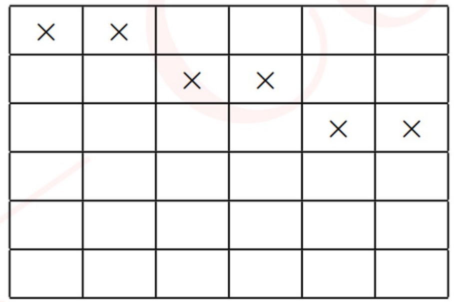
$r_1 = 6$, $r_2 = 3 \times 2 \times 2 = 12$

$r_3 = 2 \times 2 \times 2 = 8$, $r_4 = r_5 = r_6 = 0$

所以总的放置的方法数是：

$6! – 6 \times 5! + 12 \times 4! – 8 \times 3! = 240$



(a)

r2和r3的计算需要仔细分析，有没有更简便的方法？

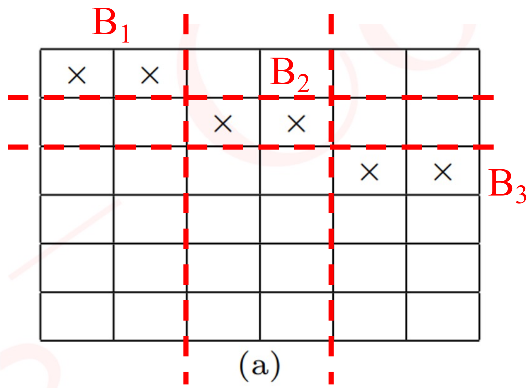
279

引理 4.4.1 如果棋盘 B 分解成两个不相交的子棋盘 B_1 和 B_2，则

$$r_k(B) = \sum_{i=0}^{k} r_i(B_1) r_{k-i}(B_2).$$

对于棋盘 B，定义棋子多项式 $R(x,B)$ 为数列 $\{r_0(B), r_1(B), \cdots\}$ 的生成函数，即

$$R(x,B) = \sum_{i=0}^{\infty} r_i(B) x^i,$$



(a)

引理 4.4.2 若 B_1, B_2 是棋盘 B 的两个不相交的子棋盘，则

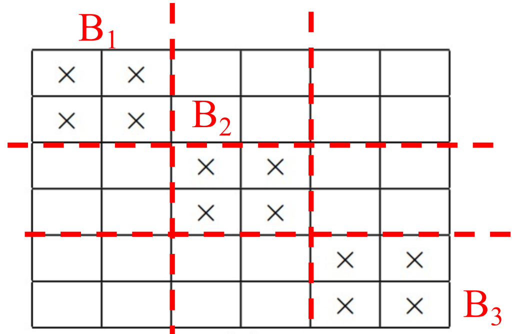
$$R(x,B) = R(x,B_1) \cdot R(x,B_2).$$

$$\begin{aligned} R(x, B) &= R(x, B_1) \times R(x, B_2) \times R(x, B_3) \\ &= (1 + 2x) \times (1 + 2x) \times (1 + 2x) \\ &= 1 + \underbrace{6x}_{r_1} + \underbrace{12x^2}_{r_2} + \underbrace{8x^3}_{r_3} \end{aligned}$$

279

$$R(x, B) = R(x, B_1) \times R(x, B_2) \times R(x, B_3)$$
$$= (1 + 4x + 2x^2)^3$$
$$= 1+12x+54x^2+112x^3+108x^4+48x^5+8x^6$$

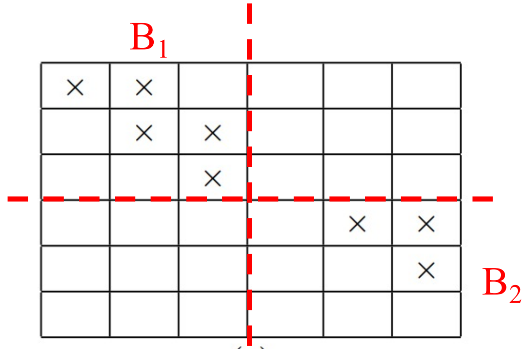
排列数为：$6!-12\times5!+54\times4!-112\times3!+108\times2!-48+8=80$



(b)

$$R(x, B) = R(x, B_1) \times R(x, B_2)$$
$$= (1+5x+6x^2+x^3)\times(1+3x+x^2)$$
$$= 1+8x+22x^2+24x^3+9x^4+x^5$$

排列数为：$6!-8\times5!+22\times4!-24\times3!+9\times2!-1=161$



(c)