Primi Composti: a Tabletop Game Analysis

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***Abstract*—This project has the objective of providing a game theoretic analysis of a real-world scenario, that is the tabletop game Primi Composti. In particular, we are interested in checking whether there is an inherent advantage in going first or second by considering all combinations of different practical strategies from a set we define. This project tackles some of the challenges that arise when applying game theoretic tools to real-world practical scenarios, like dealing with the random assignments of cards to the players and the high computational costs of standard best-response strategies. So in order to deal with these aspects, we have tried to find a balance over the number of samples and the number of strategies in order to have statistical significance and also to not have too much computational time required to run all the computations. After the simulations, some analyses are performed in order to understand if there are some advantages and if the game is fair enough for both players. Finally a minimization method is used to find the best binomial distribution that best fits the data resulting by the simulations.**

I. Introduction

In this work, we want to apply game theoretic tools to the analysis of a real-life tabletop game, called Primi Composti.

Our objective is first of all to understand whether this specific game is unfair, in the sense that it favours one player over another independently of their ”skill” level. Additionally, we also want to study the efficacy of approximated strategies for players, taking inspiration from fundamental concepts like security strategies but adapting them to a real-world scenario, where computational power is limited.

However, before delving into the specifics of those topics, we want to first present the game rules and then introduce our approach to its modelling.

A. Game’s rules

In this work we focus on the 2 players scenario, where the rules are as follows.

There are 24 different cards whose value vary between 2 and 25, for a total of 9 primes and 15 composites in play. The 24 cards get distributed at random to the players, building 2 hands of 12 cards; the player that received the 2 gets to play first.

The game is turn-based: at each turn a player has to play a card from its hand, receiving a certain amount of points that gets added to its score, and then pass the turn to its opponent, until they both run out of cards in the hand.

Each of the players has two piles of cards in front of himself, one for cards of prime values and one for composites; when a card is played, it gets placed on the top of its player’s primes or composites pile depending on its value.

To conclude, Table I-A shows how the points are assigned to a certain play. At the end of the game, whoever obtained the highest score wins.

It is easy to see that, under this formulation, ties can happen. However, we consider a tie as an invalid result that needs to be repeated, in order to simplify the analysis.

Column 1	Column 2	Column 3
C	1	Whoever plays a composite card, whose value is not the result of any operation among the visible cards on the table, gets 1 point.
P	2	Whoever plays a prime card, whose value is not the result of any operation among the visible cards on the table, gets 2 points.
C = C # C	3	Whoever plays a composite card, whose value can be seen the result of an operation between two composite cards visible on the table, gets 3 points.
C = P # C	4	Whoever plays a composite card, whose value can be seen the result of an operation between a composite card and a prime card, both visible on the table, gets 4 points.
P = C # C	4	Whoever plays a prime card, whose value can be seen the result of an operation between two composite cards visible on the table, gets 4 points.
P = P # C	5	Whoever plays a prime card, whose value can be seen the result of an operation between a composite card and a prime card, both visible on the table, gets 5 points
P = P # P	6	Whoever plays a composite card, whose value can be seen the result of an operation between two prime cards visible on the table, gets 6 points.

TABLE I
RULES FOR ASSIGNING POINTS TO A PLAYER AFTER IT PLAYS ONE OF ITS CARDS. SYMBOL # INDICATES AN ARITHMETIC OPERATION, WHILE P STANDS FOR PRIME AND C FOR COMPOSITE.

B. Modelling approach

It is straightforward to see that this game can essentially be modelled as a finite dynamic game. In particular, it is interesting to notice how this game is of complete imperfect information, since the hands are given at random by Nature's choice.

However, it is worth observing how, after Nature's choice, the game becomes of complete perfect information, since the content of the other player's hand is determined given a player's hand, and thus common knowledge. Additionally, it is worth pointing out how, after Nature's move, the game can be related to a Stackelberg game. This is interesting because, after observing Nature's choice, each player can theoretically "solve" the game by always playing a best response through backward induction on the resulting decision tree.

Nevertheless, the backward induction approach, i.e. a strategy choice that would lead to a SPE, is infeasible to implement in practice, given that the resulting decision tree would have $2^24 - 1$ nodes and a total of $24!$ possible different sequences of cards played (number of leaves). This major practical limitation hinders a rigorous analysis of the game, limiting us to only consider sub-optimal strategies due to the computational infeasibility of computing strategies that would yield subgame-perfect Nash equilibria.

Additionally, another major problem in analysing this game arises when considering the randomness of the hands: Nature has a total of ${}_{24}C_{12}$ possible choices, meaning that the number of games to consider would be well beyond practical feasibility. However, this problem is easily solvable since Nature's choice involves randomness, by just using an high enough number of trials to obtain results that are good approximations with high probability.

Another interesting aspect of this game is that it can be modelled also as a zero-sum game: it is straightforward to prove that if we do not only add the points of a play to the score of the relative player, but also subtract them from the score of its opponent, the final result, i.e. winning or losing the tabletop game, remains the same. More formally, it is easy to see how player 1 playing a card granting x points gives partial utilities $u_1 = x$ and $u_2 = -x = -u_1$. This aspect is particularly interesting when we consider possible computationally-practical strategies for our players, something that we will delve into in Section II.

Finally, we want to spend some words on potential error that would come up when trying to model this game. First of all, one would be inclined to see it as a multistage game, however this would be an error since at each stage the hands of the players and the cards visible on the table depend on the outcomes of the previous stages, meaning that the stage games would not be independent of each other. And on this note it is also important to recall that, given the impossibility to model the game as multistage, there is no way to obtain a SPE equilibrium aside for applying backward induction, which is infeasible in practice.

II. Strategies

As already mentioned, finding strategies to obtain SPE equilibria is a computationally expensive task that is not feasible for our game. However, we still want to test the evolution of the game under reasonable strategies, in order to see the distribution of wins and check whether any of them outperform the others.

In particular, we implemented the following strategies:

- random strategy: the player plays a random card from their hand;
- max strategy: the player plays the card with the maximum number written on it;
- min strategy: the player plays the card with the minimum number written on it;
- prime_first strategy: the player plays at first only prime numbers cards, if any, chosen at random, otherwise he plays composite numbers card, again chosen in random order;
- composite_first strategy: the player plays at first only composite numbers cards, if any, chosen at random, otherwise he plays prime numbers card, again chosen in random order;
- max_val strategy: play the card that gives the highest amount of points, given the current set of visible cards on the table;

Additionally, as mentioned before, we can leverage the zero-sum characteristics of the game and take inspiration from the concepts of security strategies and minimax to define our own approximated versions. The idea is essentially to see each turn and the following one as a static stage-like game; factoring in the zero-sum modelling of the game, this allows the player to choose at each turn a card to play as it would be dictated by the maximin and minimax paradigm in pure strategies. It is needless to say that this is a huge simplification: we are just considering the current turn and the opponent response as a static game, even though it is actually a Stackelberg relation and just a small component of the whole picture of the game. Given this idea, we define the two following strategies:

- security strategy: compute the utility matrix of all possible outcomes of the current turn and the next one as a static game (as described above), then choose the card that maximises the minimum amount of points achievable, i.e. the payoff in the worst case of opponent's play.
- minimax strategy: compute the utility matrix of all possible outcomes of the current turn and the next one as a static game, then choose the card that minimizes the maximum amount of points achievable, i.e. the minimum amount of points achievable by the player if it could perfectly predict its opponent's move.

Building on this idea, one may be able to obtain better results by applying instead "best-response"-like strategies using backward induction on a small stage-like Stackelberg modelling of the current turn and one or more turns in the future, however they would incur again in the practical infeasibility of simulating a significant number of games using such a strategy.

On a side note, one may also be able to generalize those two strategies to the general case of more players by also building a belief system and keep the computational costs contained;

however, this idea is not explored further since it goes out of the focus of our analysis, but it may prove interesting as future work.

III. Simulation2

We now cover the simulation work that we have done in order to analyse the fairness of the game and the performance of the proposed practical strategies.

For the sake of clarity, in this section we are going to refer to a generic player 1 as P1 and to player 2 as P2 for simplicity.

In order to understand if there is an inherit advantage into going first or second, we run several simulations and compare the outcomes of the games under all the combinations of two strategies from the ones explained in section II. In particular, we consider two cases:

- General case: whoever starts between P1 and P2 is decided by which player gets the card 2 in its hand.
- P1 always starts: player P1 always gets the hand containing the 2, thus playing first.

To achieve our objective, we first generate n pairs of random hands, one for each player. Then for each pair of hands, we simulate the game on every possible combination of two strategies from Section II. We make the simplifying assumption of players sticking to their selected strategy. Obviously, this abstracts us from the human-playing scenario, however it is an abstraction without loss of generality, since our focus is in finding evidence of an intrinsic unfairness of the game that must remove itself of any human-like factors.

Specifically, the simulations have been run on Google Colab with default configuration[1], for a total of $n = 10000$ simulations on random hands for all possible combinations of strategies. This number of runs has been chosen in order to guarantee both an high enough sample size and a feasible computational time.

A. Outcome analysis

In this section we cover the analysis of the game outcomes (win, lose, tie), both under the fairness aspect and the performance of strategies.

Our analysis starts by considering Figures 1 and 2 where the two players play the same strategies. Those figures highlight the following points about the fairness of the game:

- There is an advantage in going first for some strategies, i.e. random, max, min, prime, comp;
- The situation is more or less balanced when playing the max_value strategy;
- There is a significant advantage in going second when both players play a either the security or minimax strategy;

A combined observation of the first and second point is quite interesting: it seems that there is an advantage in going first when using stupid primitive strategies, but for more advanced ones the situation is flipped. An accurate analysis of this behaviour is quite difficult, given in particular the random

[1]Intel(R) Xeon(R) CPU @ 2.20GHz, 13 GB of RAM

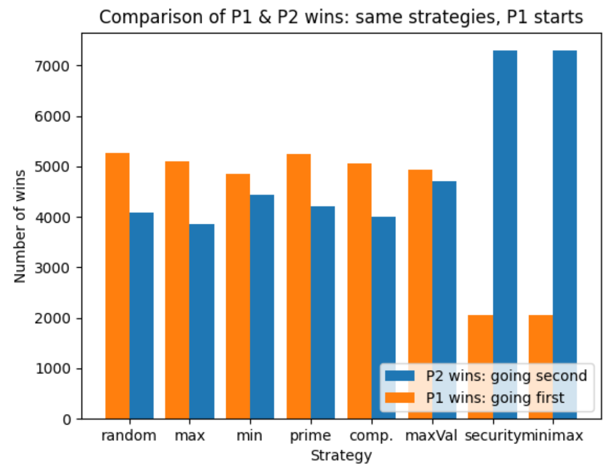

Fig. 1. Number of wins of P2 vs P1 when P1 starts; both players play the same strategy.

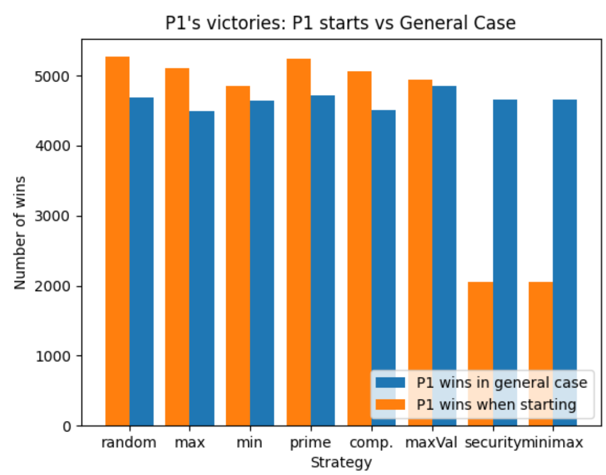

Fig. 2. Number of wins of P1 when starting vs the general case; both players play the same strategy.

nature of some of the strategies, however we propose an intuition to motivate it: a player using more advanced strategies and going second takes more opportunities to strategically form arithmetical operations with the cards visible on the table, considering also the potential opponent response, and thus achieve a greater number of points right from the first turns. This intuition is also supported by the more or less balanced behaviour of the max_val pair, where each player tries to get the maximum partial payoff out of the current game state but without strategically looking ahead on the consequences of its action.

Figures 3 and 4 strength those observations and expand on them, by allowing to see the distribution of wins for each player under all possible strategies combinations, in the general and P1-first cases respectively. In particular, it is easy to see that there is an evident advantage in using zero-sum-

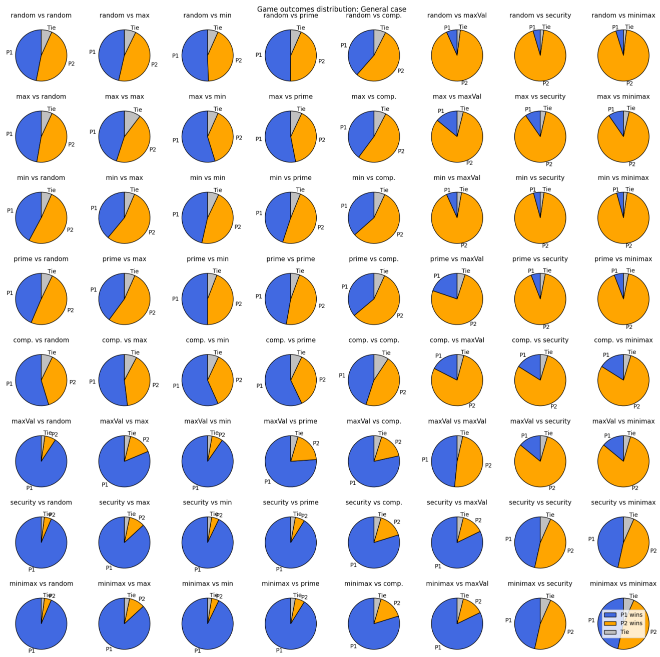

Fig. 3. Outcomes distribution under all possible strategies combinations in the general case.

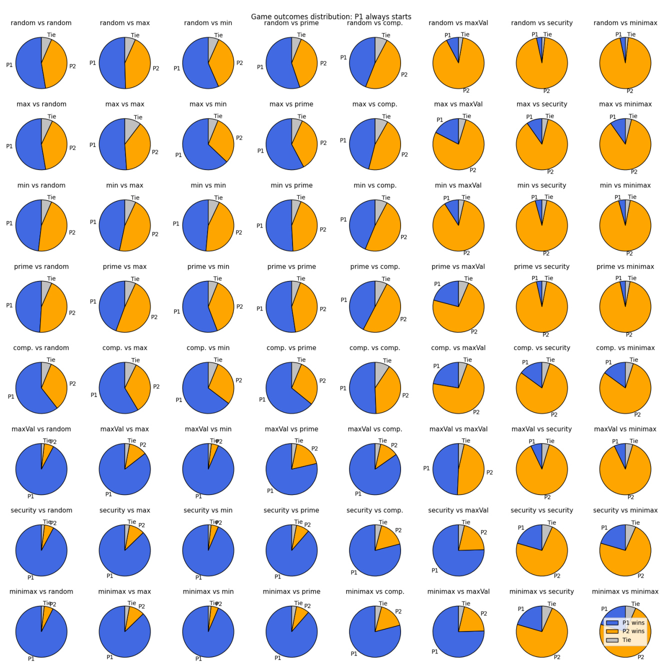

Fig. 4. Outcomes distribution under all possible strategies combinations when P1 starts.

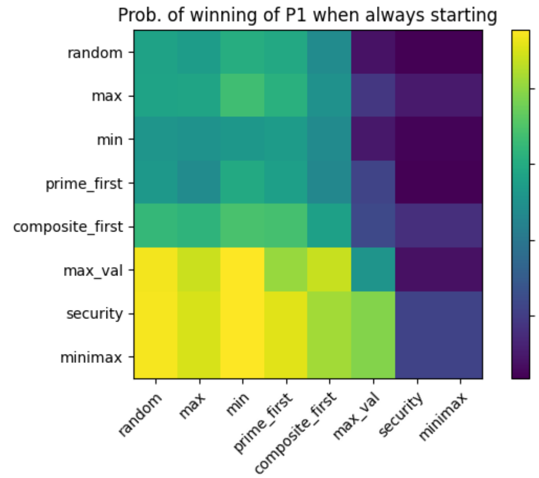

Fig. 5. Heatmap of the probability of P1 winning when always going first.

like strategies in both cases. This was expected, since those two practical strategies are the ones that approximate more closely actual best-response strategies. Additionally, we can see again the advantage in always going second when playing such strategies.

On the same note, also the max_val strategy performs quite well against the more primitive strategies, being based on the idea of "best play in a vacuum"; however, it gets outperformed by the zero-sum strategies, an expected result given their nature.

More importantly, those two figures allow us to discuss the fairness of the game on the whole: when considering the diagonals of the two figures (in other words players playing the same strategy), we clearly see that in the general case the situation is always more or less balanced, something that we do not see when we force P1 to go first. This allows us to conclude that the game is fair on the whole, but this fairness is more related to the randomized nature of whoever starts, i.e. whoever gets the 2 card, rather than the intrinsic characteristics of the game.

In Figures 6 and 5 we show the probability of P1 winning under all the different strategy combinations as heat-maps for the two study cases. The situation that we are presented with is quite similar to the one that we presented in Figures 3 and 4, additionally highlighting the symmetrical situations across the diagonals **(typical of zero-games CHECK THIS).**

However, the most interesting result is presented in Figure 7, where the difference in victory probability of P1 between the P1-always-starts case and the general case is shown. The key takeaways are:

- When considering simple primitive strategies, there is a slight advantage by always going first;
- When considering the more advanced strategies, there is a great difference in probability of victory between always starting and the general case, favouring going second;

To wrap up the outcomes analysis:

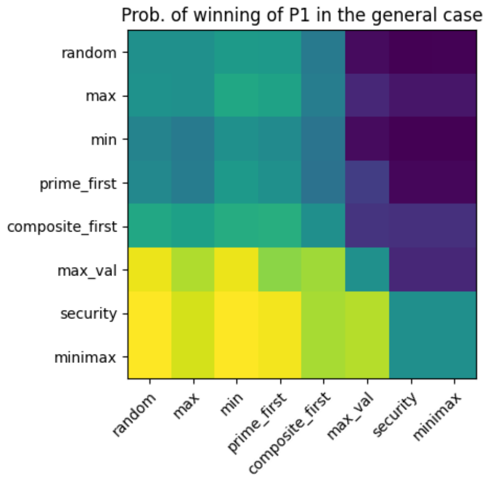

Fig. 6. Heatmap of the probability of P1 winning in the general case.

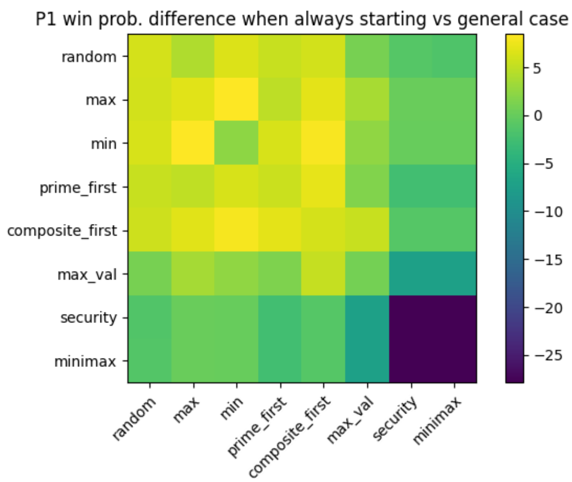

Fig. 7. Difference in probability of P1 winning when always going first vs in the general case.

- We have shown that more skilled players, i.e. players using advanced strategies, significantly outperform players using more simple strategies;
- Interestingly enough, we have seen that between low skilled players using primitive strategies the first one to move is favoured, however, the situation flips as soon as skilled players with advanced strategies come into play, favouring going second;
- Randomizing the hands, and thus whoever starts (general case), makes the game fair on the whole when considering players of ”same skill level” (in other words, strategies of comparable complexity); however, this has more to do with the properties of randomization than the innate characteristics of the game.

Fig. 8. Plot that reports the box plots of the different score for each pair of strategies in the general case.

B. Scores analysis

In our simulations, we have not only considered the outcomes of the simulated games but also the scores achieved by the players. We then used these results in order to build box plots for each pair of strategies in the two cases that we have analysed, which can be seen in Figure 8 and 9. In all these plots, the first box shows the scores distribution for P1, while the second one of P2.

In order to better visualize the distribution of the box plots, we restricted the strategies to random, max_val, security and minimax, since they are the most significant. The results can bee seen in Figures 10 and 11.

WRONG PLOTS

From the comparison between those restricted plots, we can see again reflected many of the results discussed in Section **??**. We can see again the bottom left part where we have a worsening of the scores of player 1 from general to the case where it starts confirming again what we have said before. Also, we can see that, in general, we have very small boxes but very large whiskers; this means that we have a large variance of the scores but also that in the 2nd and 3rd quantiles we have data that is very concentrated. We can also see that there are a lot of outliers pretty much everywhere. In particular, we can see that when player 1 starts, the number of outliers increases a lot for both players.

At the end of the code, there is also a small part that uses an optimization method to find the binomial distribution that best fits the data in the two cases. The distributions found are very similar to the ones computed before using the equation **??**.

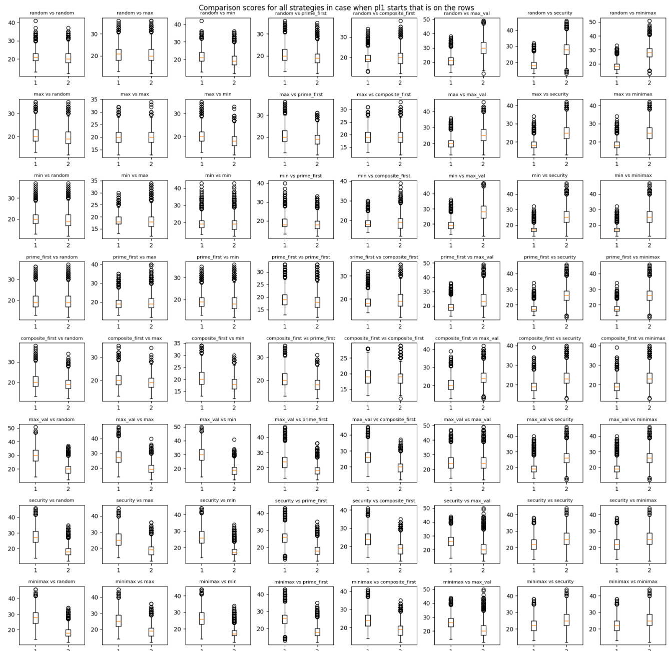

Fig. 9. Plot that reports the box plots of the different score for each pair of strategies when player 1 starts.

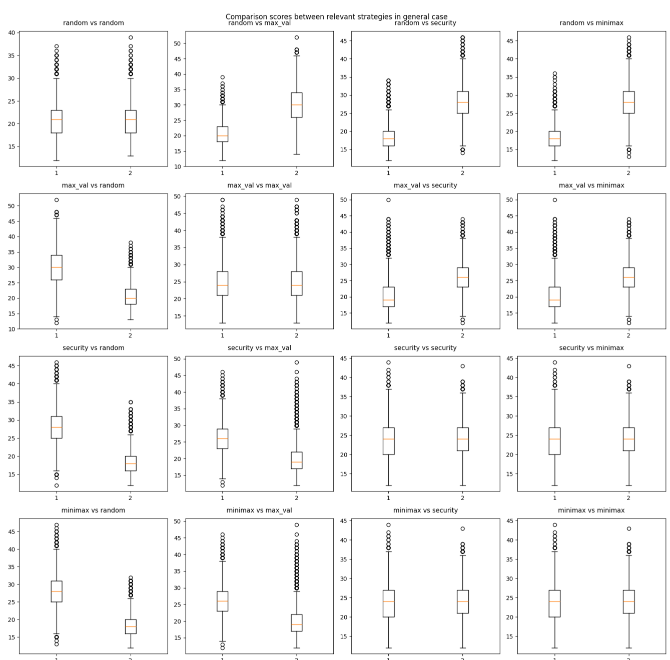

Fig. 10. Plot that reports the box plots of the different score for each pair of strategies in the general case.

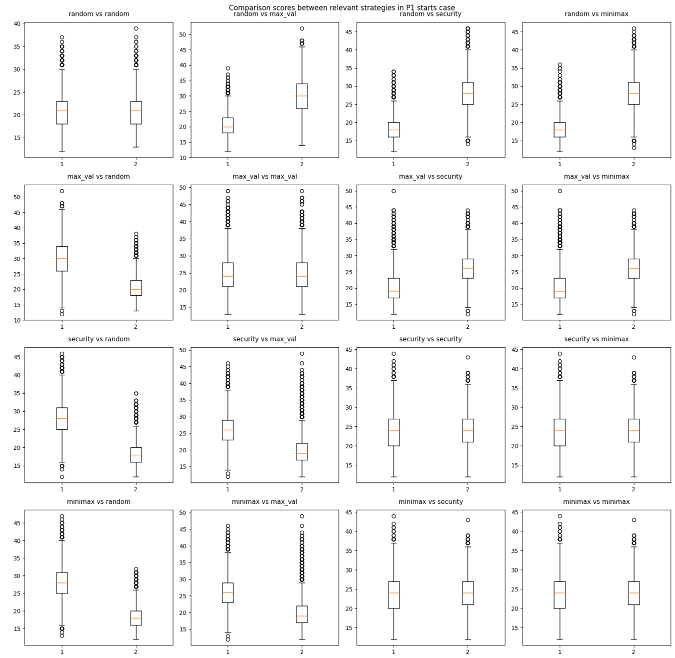

Fig. 11. Plot that reports the box plots of the different score for each pair of strategies when player 1 starts.

IV. CONCLUSIONS

Our analysis indicates that the game, in its general form, exhibits a balanced design. The observed winning probabilities for the majority of strategies converge towards a 50/50 distribution as you can see in figure 6, suggesting a fair game. While certain strategies deviate from this balance, leading to unbalanced probabilities, these instances are confined to a narrow subset of strategies that are less realistic.

To investigate potential first-mover or second-mover advantages, we analyzed the game dynamics with a fixed starting player. Our findings suggest that such advantages are negligible. The observed variations in winning probabilities when comparing the general case with the scenario where Player 1 always starts are minimal, with a modest 5 % increase. Furthermore, there are some significant shifts in winning probability, 25 % decrease for the starting player, but these decreases are primarily observed within the unrealistic strategies considered before. So we can say that we don't have first-mover or second-mover advantages.

LINK

The code is available at *Repository*

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REFERENCES

- L. Badia, T. Marchioro, Game theory - A handbook of problems and exercises, Esculapio, 2022
- S. Tadelis, Game Theory: an introduction. Princeton, 2013.