Primi Composti: a Tabletop Game Analysis

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***Abstract*—This project has the objective of providing a game theoretic analysis of a real-world scenario, that is the tabletop game Primi Composti. In particular, we are interested into checking whether there is an inherent advantage into going first or second by considering all combinations of different practical strategies from a set we define. This project tackles some of the challenges that arise when applying game theoretic tools to real-world practical scenarios, like dealing with the random assignments of cards to the players and the high computational costs of standard best-response strategies.**

I. Introduction

In this work, we want to apply game theoretic tools to the analysis of a real-life tabletop game, called Primi Composti.

Our objective is first of all to understand whether this specific game is unfair, in the sense that it favours one player over another independently of their ”skill” level. Additionally, we also want to study the efficacy of approximated strategies for players, taking inspiration from fundamental concepts like security strategies but adapting them to a real-world scenario, where computational power is limited.

However, before delving into the specifics of those topics, we want to first present the game rules and then introduce our approach to its modelling.

A. Game's rules

In this work we focus on the 2 players scenario, where the rules are as follows.

There are 24 different cards whose value vary between 2 and 25, for a total of 9 primes and 15 composites in play. The 24 cards get distributed at random to the players, building 2 hands of 12 cards; the player that received the 2 gets to play first.

The game is turn-based: at each turn a player has to play a card from its hand, receiving a certain amount of points that gets added to its score, and then pass the turn to its opponent, until they both run out of cards in the hand.

Each of the players has two piles of cards in front of himself, one for cards of prime values and one for composites; when a card is played, it gets placed on the top of its player's primes or composites pile depending on its value.

To conclude, Table I-A shows how the points are assigned to a certain play. At the end of the game, whoever obtained the highest score wins.

It is easy to see that, under this formulation, ties can happen. However, we consider a tie as an invalid result that needs to be repeated, in order to simplify the analysis.

Column 1	Column 2	Column 3
C	1	Whoever plays a composite card, whose value is not the result of any operation among the visible cards on the table, gets 1 point.
P	2	Whoever plays a prime card, whose value is not the result of any operation among the visible cards on the table, gets 2 points.
C = C # C	3	Whoever plays a composite card, whose value can be seen the result of an operation between two composite cards visible on the table, gets 3 points.
C = P # C	4	Whoever plays a composite card, whose value can be seen the result of an operation between a composite card and a prime card, both visible on the table, gets 4 points.
P = C # C	4	Whoever plays a prime card, whose value can be seen the result of an operation between two composite cards visible on the table, gets 4 points.
P = P # C	5	Whoever plays a prime card, whose value can be seen the result of an operation between a composite card and a prime card, both visible on the table, gets 5 points
P = P # P	6	Whoever plays a composite card, whose value can be seen the result of an operation between two prime cards visible on the table, gets 6 points.

TABLE I
Rules for assigning points to a player after it plays one of its cards. Symbol # indicates an arithmetic operation, while P stands for prime and C for composite.

B. Modelling approach

It is straightforward to see that this game can essentially be modelled as a finite dynamic Stackelberg game. In particular, it is interesting to notice how this game is of complete imperfect information, since the content of the other player's hand is determined given one's hand, that is a Nature's choice.

However, it is worth observing how, after Nature's choice, the game becomes of complete perfect information. This is interesting because, after observing Nature's choice, each player can theoretically ”solve” the game by always playing a best response through backward induction on the resulting decision tree.

Nevertheless, the backward induction approach, i.e. a strategy choice that would lead to a SPE, is infeasible to implement in practice, given that the resulting decision tree would have $2^24 - 1$ nodes and a total of $24!$ possible different sequences of cards played (number of leaves). This major practical limitation hinders a rigorous analysis of the game, limiting us to only consider sub-optimal strategies due to the computational infeasibility of computing strategies that would yield subgame-perfect Nash equilibria.

Additionally, another major problem in analysing this game arises when considering the randomness of the hands: Nature has a total of ${}_{24}C_{12}$ possible choices, meaning that the number of games to consider would be well beyond practical feasibility. However, this problem is easily solvable since Nature's choice involves randomness, by just using an high enough number of trials to obtain results that are good in approximation.

Another interesting aspect of this game is that it can be modelled also as a zero-sum game: it is straightforward to prove that if we do not only add the points of a play to the score of the relative player, but also subtract them from the score of its opponent, the final result, i.e. winning or losing the tabletop game, remains the same. More formally, it is easy to see how player 1 playing a card granting x points gives partial utilities $u_1 = x$ and $u_2 = -x = -u_1$. This aspect is particularly interesting when we consider the possible strategies for our players, something that we will delve into in Section II.

Finally, we want to spend some words on potential error that would come up when trying to model this game. First of all, one would be inclined to model it as a multistage game, however this would be an error since at each stage the hands of the players and the cards visible on the table depend on the outcomes of the previous stages, meaning that the stage games would not be independent of each other. And on this note it is also important to recall that, given the impossibility to model the game as multistage, there is no way to obtain a SPE equilibrium aside for applying backward induction, which is infeasible in practice.

II. Strategies

As already mentioned, finding strategies to obtain SPE equilibria is a computationally expensive task that is not feasible for our game. However, we still want to test the evolution of the game under reasonable strategies, in order to see the distribution of wins and check whether any of them outperform the others.

In particular, we implemented the following strategies:

- random strategy: the player plays a random card from their hand;
- max strategy: the player plays the card with the maximum number written on it;
- min strategy: the player plays the card with the minimum number written on it;
- prime_first strategy: the player plays at first only prime numbers cards, if any, chosen at random, otherwise he plays composite numbers card, again chosen in random order;
- composite_first strategy: the player plays at first only composite numbers cards, if any, chosen at random, otherwise he plays prime numbers card, again chosen in random order;
- max_val strategy: play the card that gives the highest amount of points, given the current set of visible cards on the table;

Additionally, as mentioned before, we can leverage the zero-sum characteristics of the game and take inspiration from the concepts of security strategies and minimax to define our own approximated versions. The idea is essentially to see each turn and the following one as a static stage-like game; factoring in the zero-sum modelling of the game, this allows the player to choose at each turn a card to play as it would be dictated by the maximin and minimax paradigm in pure strategies. It is needless to say that this is a huge simplification: we are just considering the current turn and the opponent response as a static game, even though it is actually a Stackelberg relation and just a small component of the whole picture of the game. Given this idea, we define the two following strategies:

- security strategy: compute the utility matrix of all possible outcomes of the current turn and the next one as a static game (as described above), then choose the card that maximises the minimum amount of points achievable, i.e. the payoff in the worst case of opponent's play.
- minimax strategy: compute the utility matrix of all possible outcomes of the current turn and the next one as a static game, then choose the card that minimizes the maximum amount of points achievable, i.e. the minimum amount of points achievable by the player if it could perfectly predict its opponent's move.

Building on this idea, one may be able to obtain better results by applying instead "best-response"-like strategies using backward induction on the small stage-like Stackelberg modelling of the current turn and the next one, however they would incur again in the practical infeasibility of simulating a significant number of games using such a strategy. Additionally, one may also be able to generalize those two strategies to the general case of more players by also building a belief system and keep the computational costs contained; however, this idea is not explored further since it goes out of the focus of our analysis, but it may prove interesting as future work.

III. Simulation

In this section, we are going to refer to player 1 as P1 and player 2 as P2 for simplicity.

In order to understand if there is an inherit advantage into going first or second, we run several simulations and compare the outcomes of the games under all the combinations of two strategies from the ones explained in section II. Then, we consider two cases:

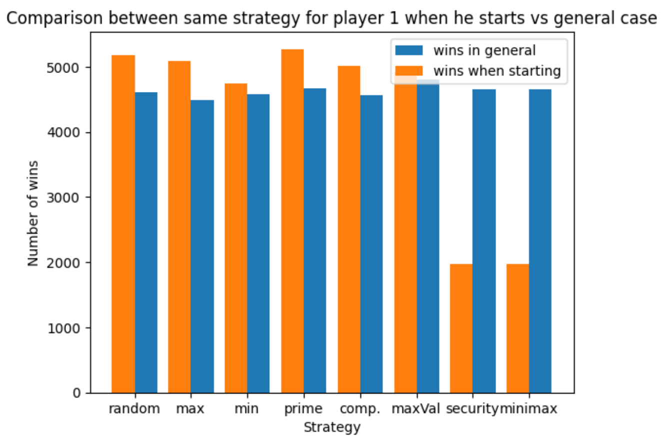

Fig. 1. Wins of player 1 in the two cases when the two players play the same strategy.

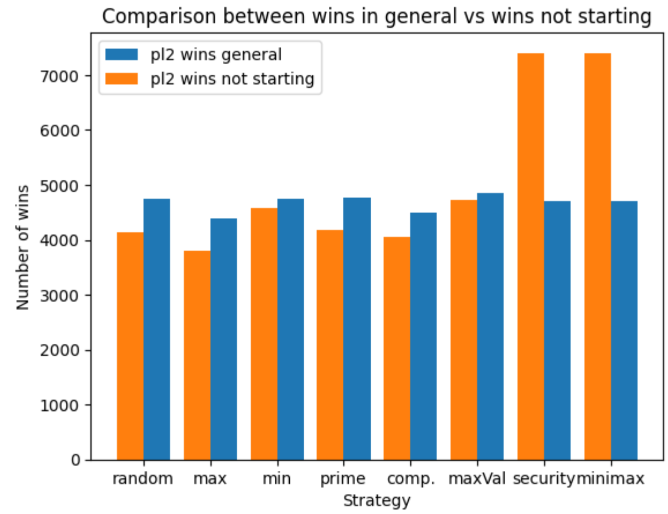

Fig. 2. Wins of player 2 in the two cases when the two players play the same strategy.

- General case: whoever starts between P1 and P2 is decided by which player has the card 2 in its hand.
- P1 always starts: player P1 always gets the hand containing the 2, thus playing first.

To achieve our objective, we first generate n pairs of random hands, one for each player. Then for each pair of hands, we simulate the game on every possible combination of two strategies from Section II.

A. Results

To have enough data to make statistical analyses but also to have limited computation time, we decided to do $n = 10000$ samples of possible assignments; then we simulate the different strategies.

By looking at figure 1, that plots the win of player 1 when the two players play the same strategy, we can see that there is some small advantage for some strategies (random, max, min, prime, comp, max_val) versus a great worsening for two particular strategies (security, minimax).

This is also reflected in the opposite plot, so the one for player 2 reported in figure 2.

After these, we have computed the probability that player 1 wins the game for every possible pair of strategies. The probability is computed as follows:

$$prob = \frac{number-of-wins}{number-of-total-matches} \tag{1}$$

To visualize the results in these two cases, we decided to use the heat-maps; the plots are reported in figure 3 and in figure 4.

From the plot in figure 3 we can see that in the general case the probability of winning for player 1 is very high when these combinations arise:

1) max_val vs random
2) security vs random
3) minimax vs random
4) max_val vs min
5) security vs min
6) minimax vs min
7) security vs prime first
8) minimax vs prime first

The first 3 were somehow obvious because the second player is choosing the card randomly without a strategy, so a good strategy for player 1 will very likely beat player 2. The other five results instead are more interesting. Since the probability of winning for player 2 is $1-probability-winning-player-1$ we can see the dark part at the top right of the heat-map.

By looking at figure 4 we can see what we have pointed out before using figure 1 and 2 that is the fact that we have a significant drop in the chance of winning for the strategies in the bottom right part of the heat-map, but we have an increasing probability for all the other combinations.

To visualize in a better way this last observation we plot also the difference between the two sets of probabilities and we obtained the heat-map in 5, using this plot we can see that we have a 5 % increment in the top left but a 25 % decrease in the bottom right. So given these values we can say that maybe the player that does not start has an advantage because in certain pairs of strategies the probability of the starting player increases by a small percentage but it decreases a lot for other strategies so the not-starting player has in general a greater chance to win.

For all possible simulations that we have run, we also collected the different scores of each match. These results are used in order to plot a box plot for each pair of strategies in the two cases that we have analyzed. These can be seen in figure 6 and figure 7. From the comparison between these plots, we can see again the bottom left part where we have a worsening of the scores of player 1 from general to the case where it starts confirming again what we have said before. Also, we can see that, in general, we have very small boxes but very large whiskers; this means that we have a large variance of

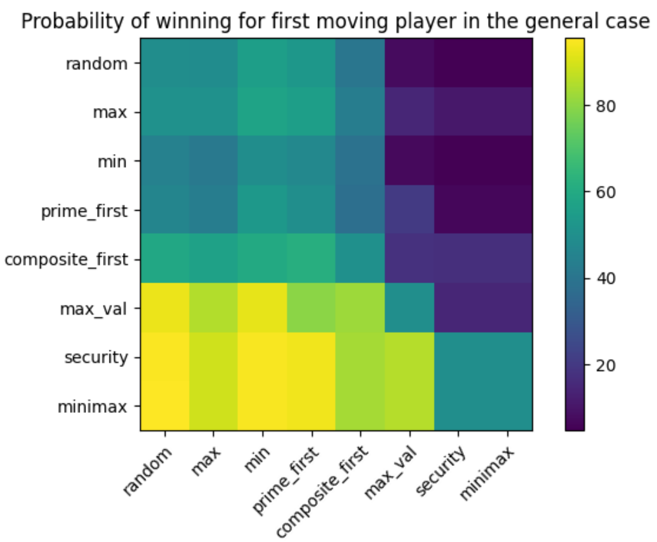

Fig. 3. Probability of winning for player 1 in general case (strategy of player 1 on the rows).

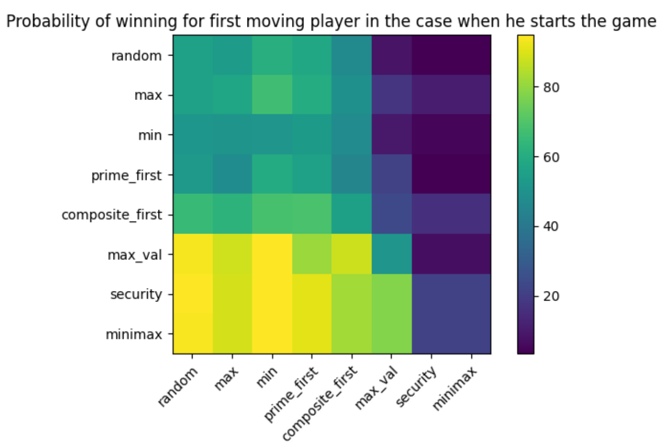

Fig. 4. Probability of winning for player 1 in the case where he always starts (strategy of player 1 on the rows).

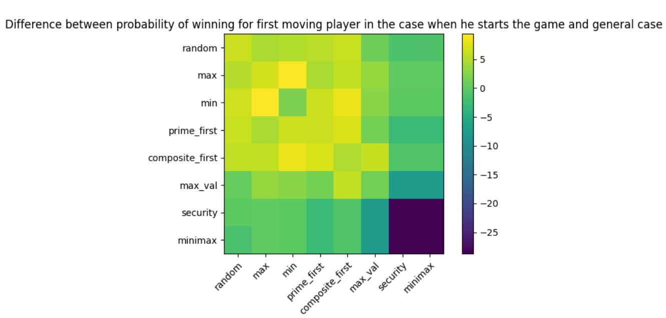

Fig. 5. Difference between probability of winning for player 1 when it starts and the general case.

Fig. 6. Plot that reports the box plots of the different score for each pair of strategies in the general case.

Fig. 7. Plot that reports the box plots of the different score for each pair of strategies when player 1 starts.

the scores but also that in the 2nd and 3rd quantiles we have data that is very concentrated. We can also see that there are a lot of outliers pretty much everywhere. In particular, we can see that when player 1 starts, the number of outliers increases a lot for both players.

At the end of the code, there is also a small part that uses an optimization method to find the binomial distribution that best fits the data in the two cases. The distributions found are very similar to the ones computed before using the equation 1.

IV. Conclusions

Acknowledgment

References