Introduction to Algorithms

Topic 2 : Asymptotic Mark and Recursive Equation

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Fall Semester 2022

Outline of Topics

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渐进的

只关心$n \to \infty$的情况

Asymptotic Notation: $O-$notation

O-notation: upper bounds

We write $f(n) = O(g(n))$ if there exist constants $c > 0, n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

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functions,
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functions, not values

funny, "one-way" equality

Set Definition of O-notation

$$O(g(n)) = \{f(n) : \text{there exist constants } c > 0, n_0 > 0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}.$$

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Example: $2n^2 \in O\left(n^3\right)$

Macro Substitution

Convention: A set in a formula represents an anonymous function in the set.

Example: $f(n) = n^3 + O\left(n^2\right)$
means
$f(n) = n^3 + h(n)$
for some $h(n) \in O\left(n^2\right)$.

Asymptotic Notation: Ω-notation

O-notation is an upper-bound notation.
The Ω-notation provides a lower bound.

Set definition of Ω-notation

$$\Omega(g(n)) = \{f(n) : \text{there exist constants } c > 0, n_0 > 0 \text{ such that}$$
$$0 \leq c \cdot g(n) \leq f(n) \text{ for all } n \geq n_0\}$$

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Example: $\sqrt{n} = \Omega(\lg n)$

Asymptotic Notation: Θ-notation

Θ-notation: tight bounds

We write $f(n) = \Theta(g(n))$ if there exist constants $c_1 > 0, c_2 > 0, n_0 > 0$ such that $c_2 g(n) \geq f(n) \geq c_1 g(n) \geq 0$ for all $n \geq n_0$.

$$\Theta(g(n)) = \mathrm{O}(g(n)) \bigcap \Omega(g(n))$$

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Example: $\frac{1}{2}n^2 - 2n = \Theta\left(n^2\right)$

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Example:

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$\Theta(n^0)$ or $\Theta(1)$

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$$\Theta(g(n)) = \mathrm{O}(g(n)) \bigcap \Omega(g(n))$$

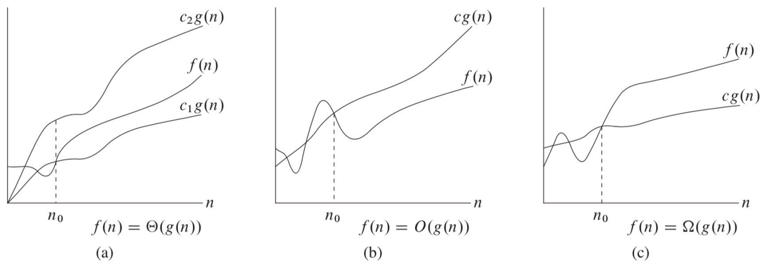
Example: $\frac{1}{2}n^2 - 2n = \Theta\left(n^2\right)$

$\Theta(n^0)$ or $\Theta(1)$

Theorem:

The leading constant and low order terms do not matter.

Graphic Examples of the Θ, O, Ω



Other Asymptotic Notations

o-notation

$o(g(n)) = \{f(n)$: for all $c > 0$, there exist constants $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$.

Other equivalent definition $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

ω-notation

$\omega(g(n)) = \{f(n)$: for all $c > 0$, there exist constants $n_0 > 0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0\}$.

Other equivalent definition $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

A Helpful Analogy

$f(n) = O(g(n))$ is similar to $f(n) \le g(n)$.

$f(n) = o(g(n))$ is similar to $f(n) < g(n)$.

$f(n) = \Theta(g(n))$ is similar to $f(n) = g(n)$.

$f(n) = \Omega(g(n))$ is similar to $f(n) \ge g(n)$.

$f(n) = \omega(g(n))$ is similar to $f(n) > g(n)$.

Transitivity

$$f(n) = \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \text{ imply } f(n) = \Theta(h(n)).$$

$$f(n) = O(g(n)) \text{ and } g(n) = O(h(n)) \text{ imply } f(n) = O(h(n)).$$

$$f(n) = \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \text{ imply } f(n) = \Omega(h(n)).$$

$$f(n) = o(g(n)) \text{ and } g(n) = o(h(n)) \text{ imply } f(n) = o(h(n)).$$

$$f(n) = \omega(g(n)) \text{ and } g(n) = \omega(h(n)) \text{ imply } f(n) = \omega(h(n)).$$

Reflexivity

$$f(n) = \Theta(f(n))$$

$$f(n) = O(f(n))$$

$$f(n) = \Omega(f(n))$$

Symmetry & Transpose Symmetry

Symmetry

$$f(n) = \Theta(g(n)) \text{ if and only if } g(n) = \Theta(f(n)).$$

Transpose Symmetry

$$f(n) = O(g(n)) \text{ if and only if } g(n) = \Omega(f(n)).$$
$$f(n) = o(g(n)) \text{ if and only if } g(n) = \omega(f(n)).$$

Non-completeness

Non-completeness of O, Ω, and Θ notations

For real numbers a and b, we know that either $a < b$, or $a = b$, or $a > b$ is true.

However, for two functions $f(n)$ and $g(n)$, it is possible that neither of the following is true: $f(n) = O(g(n))$, or $f(n) = \Theta(g(n))$, or $f(n) = \Omega(g(n))$. For example, $f(n) = n$, and $g(n) = n^{1-\sin(n\pi/2)}$.

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Floors and Ceilings

Floor

For any real number x, we denote the greatest integer less than or equal to x by $\lfloor x \rfloor$ (read "the floor of x")

Ceiling

For any real number x, we denote the least integer greater than or equal to x by $\lceil x \rceil$ (read "the ceiling of x")

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil \leq x + 1.$$

For any integer n, $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

For any real number $x \geq 0$ and integers $a, b > 0$,

$$\left\lceil \frac{\lceil x/a \rceil}{b} \right\rceil = \left\lceil \frac{x}{ab} \right\rceil, \left\lfloor \frac{\lfloor x/a \rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor, \left\lceil \frac{a}{b} \right\rceil \leq \frac{a+(b-1)}{b}, \left\lfloor \frac{a}{b} \right\rfloor \geq \frac{a-(b-1)}{b},$$

Modular Arithmetic

Mod

For any integer a and any positive integer n, the value a mod n is the remainder (or residue) of the quotient a/n:

$$a \mod n = a - n\lfloor a/n \rfloor.$$

Equivalent

If $(a \mod n) = (b \mod n)$, we write $(a \equiv b) \mod n$ and say that a is equivalent to b, modulo n.

Exponentials

$\forall\, a > 0, \quad a^0 = 1; \quad (a^m)^n = (a^n)^m = a^{mn}; \quad a^m a^n = a^{m+n}$

When $a > 1$, $\lim_{n\to\infty} \frac{n^b}{a^n} = 0$. That is, $n^b = o(a^n)$.

For all real x, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$
When $|x| \le 1$, $1 + x \le e^x \le 1 + x + x^2$
When $x \to 0$, $e^x = 1 + x + \Theta(x^2)$
For all x, $\lim_{n\to\infty}(1 + \frac{x}{n})^n = e^x$

Logarithms

$\lg n = \log_2 n$; $\ln n = \log_e n$; $\lg^k n = (\lg n)^k$; $\lg \lg n = \lg(\lg n)$

For all real $a, b, c > 0$, and n,
$a = b^{\log_b a}$; $\log_c(ab) = \log_c a + \log_c b$;
$\log_b a^n = n \log_b a$; $\log_b a = \frac{\lg a}{\lg b}$; $a^{\log_b c} = c^{\log_b a}$

When $a > 0$, $\lim_{n\to\infty} \frac{\lg^b n}{(2^a)^{\lg n}} = \lim_{n\to\infty} \frac{\lg^b n}{n^a} = 0$. That is, $\lg^b n = o(n^a)$.

When $|x| \leq 1$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$
For $x > -1$, $\frac{x}{1+x} \leq \ln(1+x) \leq x$

Factorials

$$n! = \begin{cases} 1 & \text{if} \quad n = 0 \\ n \cdot (n-1)! & \text{if} \quad n > 0 \end{cases}$$

$n! \leq n^n$. A better bound:

Stirling's approximation

$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)$

Functional iteration

functional iteration

We use the notation $f^{(i)}(n)$ to denote the function $f(n)$ iteratively applied i times to an initial value of n. Formally, let $f(n)$ be a function over the reals. For non-negative integers i, we recursively define

$$f^{(i)}(n) = \begin{cases} n & \text{if } i = 0, \\ f(f^{(i-1)}(n)) & \text{if } i > 0, \end{cases}$$

Example: if $f(n) = 2n$, then $f^{(i)}(n) = 2^i n$.

The iterated logarithm function

We use the notation $\lg^* n$ to denote the iterated logarithm.

$$\lg^* n = min\{i \geq 0 : \lg^{(i)} n \leq 1\}.$$

Example:

$$\lg^* 2 = 1,$$
$$\lg^* 4 = 2,$$
$$\lg^* 16 = 3,$$
$$\lg^* (2^{65536}) = 5.$$

Fibonacci Numbers

Fibonacci numbers

We define the Fibonacci numbers by the following recurrence:

$$F_0 = 0,$$
$$F_1 = 1,$$
$$F_i = F_{i-1} + F_{i-2}, \quad for\ i \geq 2.$$

Each Fibonacci number is the sum of the two previous ones, yielding the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

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Solving Recurrences

Recurrences go hand in hand with the divide-and-conquer paradigm.
A recurrence is an equation or inequality that describes a function in terms of its value on smaller inputs.
Three methods for solving recurrences

- substitution method: guess a bound and use mathematical induction to prove the guess correct.
- recursion-tree method: converts the recurrence into a tree and use techniques for bounding summations.
- master method: provides bounds of the form $T(n) = a \cdot T(\frac{n}{b}) + f(n)$.

Substitution Method

The most general method

1. Guess the form of the solution.
2. Verify by induction.
3. Solve for constants.

- This method only works if we can guess the form of the answer.
- The method can be used to establish either upper or lower bounds on a recurrence.

Example of Substitution

Example: $T(n) = 4T(n/2) + n$

- Assume that $T(1) = \Theta(1)$.
- Guess $T(n) = O(n^3)$. (Note that if we guess Θ, we need prove O and Ω separately.)
- Assume that $T(k) \leq ck^3$ for $k < n$ and some constant $c > 0$.
- Prove $T(n) \leq cn^3$ by induction.

Example of Substitution

$$\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \quad \longleftarrow \text{desired} - \text{residual} \\
&\leq cn^3 \quad \longleftarrow \text{desired}
\end{aligned}$$

whenever $(c/2)n^3 - n \geq 0$, for example, if $c \geq 2$ and $n \geq 1$.

residual

Example (Continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- **Base:** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \leq n < n_0$, we have "$\Theta(1)$" $\leq cn^3$, if we pick c big enough.

Example (Continued)

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This bound is not tight!

A Tighter Upper Bound?

We shall prove that $T(n) = O(n^2)$.

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Assume that $T(k) \leq ck^2$ for $k < n$:

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^2 + n \\ &= cn^2 + n \\ &= O(n^2) \end{aligned}$$

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Wrong! We must prove the I.H.

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$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^2 + n \\
&= cn^2 + n \\
&\cancel{= O(n^2)} && \text{Wrong! We must prove the I.H.} \\
&= cn^2 - (-n) && \text{[desired } - \text{ residual]} \\
&\le cn^2 && \text{for } \textit{no} \text{ choice of } c > 0\text{. Lose!}
\end{aligned}
$$

A Tighter Upper Bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.
  Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$

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  Inductive hypothesis: $T(k) \leq c_1 k^2 - c_2 k$ for $k < n$

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\leq 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1 n^2 - 2c_2 n + n \\
&= c_1 n^2 - c_2 n - (c_2 n - n) \\
&\leq c_1 n^2 - c_2 n \text{ if } c_2 \geq 1
\end{aligned}
$$

Pick c_1 big enough to handle the initial conditions.

A Tighter Lower Bound

We shall prove that $T(n) = \Omega(n^2)$.

A Tighter Lower Bound

We shall prove that $T(n) = \Omega(n^2)$.

Assume that $T(k) \geq ck^2$ for $k < n$, and for some chosen constant c.

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\geq 4c(n/2)^2 + n \\ &= cn^2 + n \\ &\geq cn^2 \end{aligned}$$

Recursion-tree Method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable.
- The recursion tree method is good for generating guesses for the substitution method.

Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of Recursion Tree

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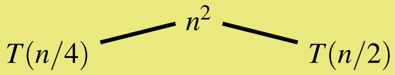

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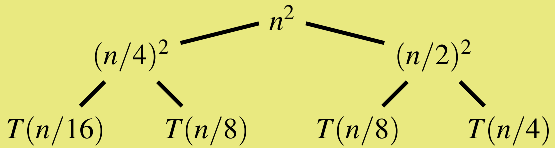

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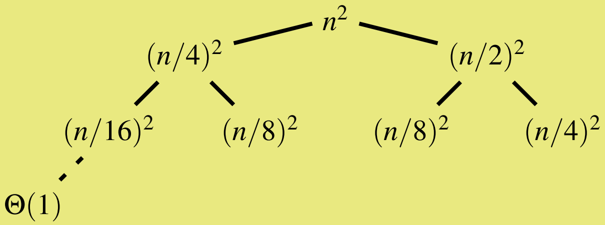

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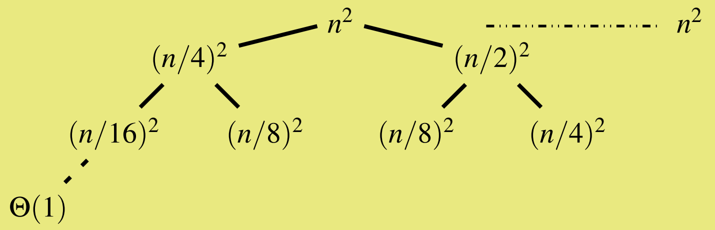

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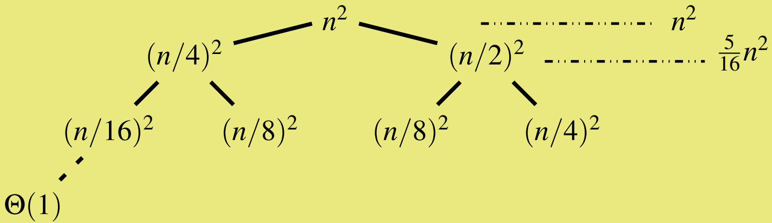

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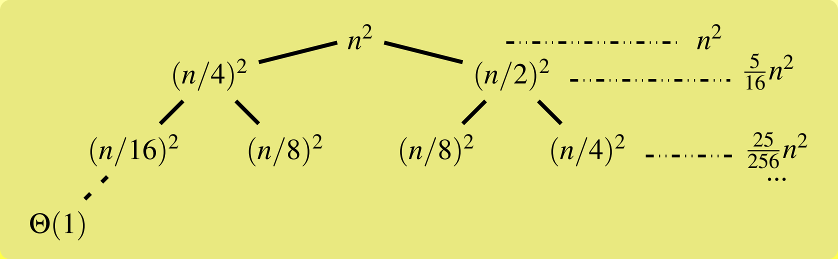

Example of Recursion Tree

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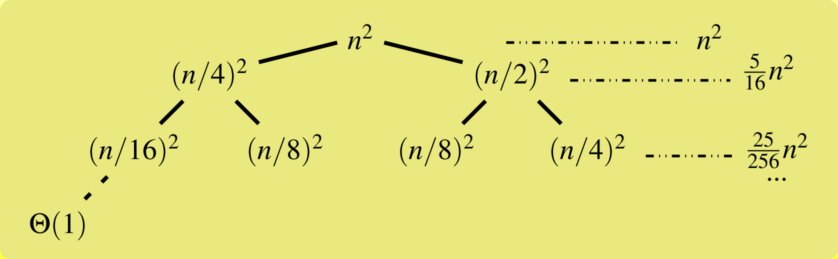

$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right) = \Theta(n^2)$$

(geometric series)

The Master Method

Master method

The master method applies to recurrences of the form

$$T(n) = aT(\frac{n}{b}) + f(n)$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Three Common Cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for some constant $\varepsilon > 0$
    - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).
      Solution: $T(n) = \Theta\left(n^{\log_b a}\right)$.

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   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).
     Solution: $T(n) = \Theta\left(n^{\log_b a}\right)$.
2. $f(n) = \Theta\left(n^{\log_b a} \lg^k n\right)$ for some constant $k \geq 0$
   - $f(n)$ and $n^{\log_b a} \lg^k n$ grow at similar rates.
     Solution: $T(n) = \Theta\left(n^{\log_b a} \lg^{k+1} n\right)$.

Three Common Cases

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega\left(n^{\log_b a+\varepsilon}\right)$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor), **and** $f(n)$ satisfies the **regularity condition** that $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.
     Solution: $T(n) = \Theta(f(n))$.

Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n.$

Case 1: $f(n) = O\left(n^{2-\varepsilon}\right)$ for $\varepsilon = 1$

$\therefore T(n) = \Theta\left(n^2\right).$

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Ex. $T(n) = 4T(n/2) + n^2$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^2.$

Case 2: $f(n) = \Theta\left(n^2 lg^0 n\right)$, that is, $k = 0$.

$\therefore T(n) = \Theta\left(n^2 lg n\right).$

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3$.

Case 3: $f(n) = \Omega\left(n^{2+\varepsilon}\right)$ for $\varepsilon = 1$

and $4(n/2)^3 \leq cn^3$(reg. cond.) for $c = 1/2$.

$\therefore T(n) = \Theta\left(n^3\right)$.

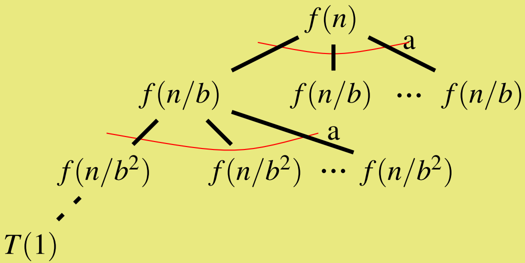
Examples

Ex. $T(n) = 4T(n/2) + n^3$
$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2; f(n) = n^3$.
Case 3: $f(n) = \Omega\left(n^{2+\varepsilon}\right)$ for $\varepsilon = 1$
and $4(n/2)^3 \leq cn^3$(reg. cond.) for $c = 1/2$.
$\therefore T(n) = \Theta\left(n^3\right)$.

Ex. $T(n) = 4T(n/2) + n^2/\lg n$
$a = 4, b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n$.
Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^\varepsilon = \omega(\lg n)$.

Idea of Master Theorem

$T(n) = aT(\frac{n}{b}) + f(n)$. ***Recursion tree:***



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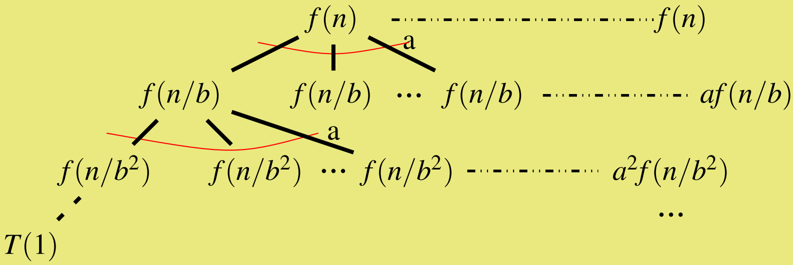

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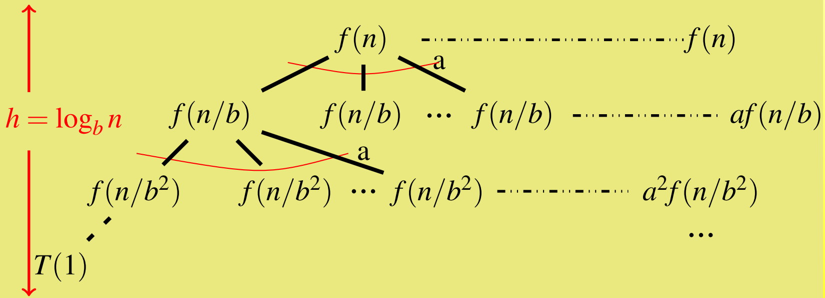

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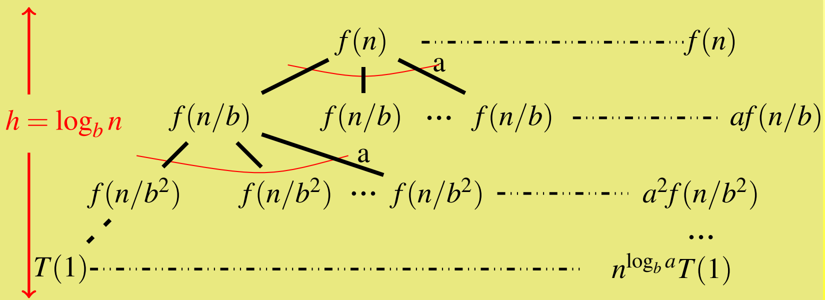

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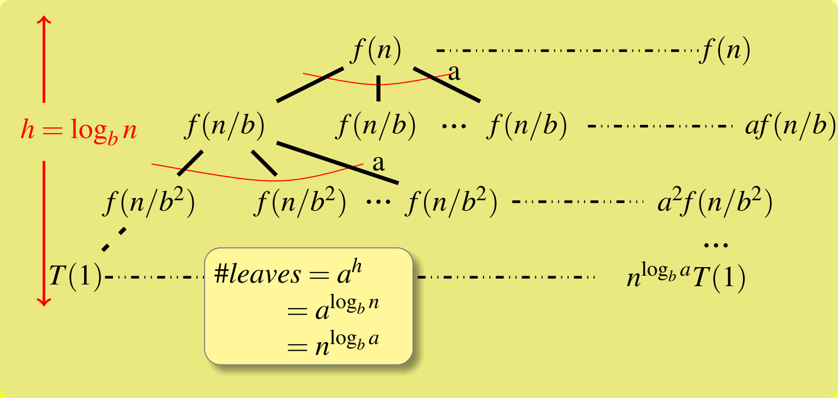

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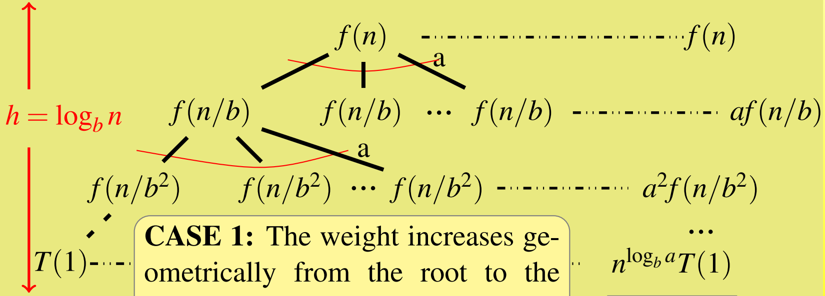

CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

$\boxed{\Theta(n^{\log_b a})}$

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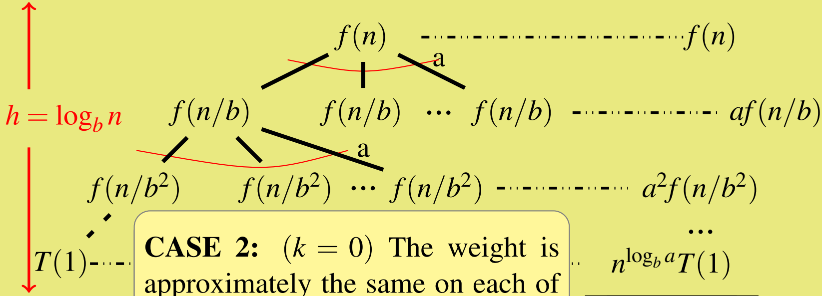

CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$\boxed{\Theta(n^{\log_b a} \lg n)}$

Idea of Master Theorem

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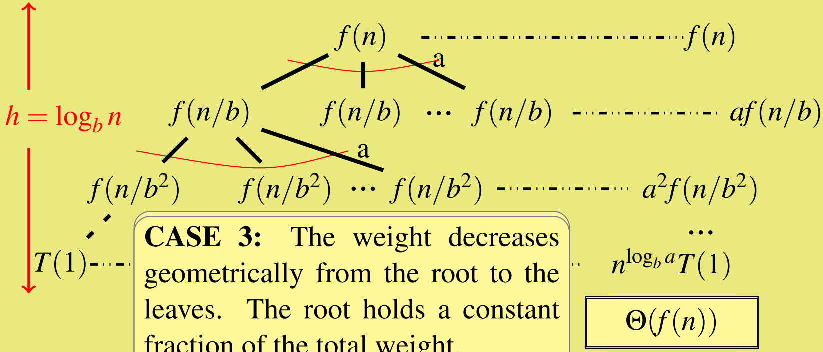

CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Appendix: Geometric Series

$$1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x} \quad \textit{for} \quad x \neq 1$$

$$1+x+x^2+\cdots=\frac{1}{1-x} \qquad \textit{for} \quad |x|<1$$