

复变函数 B 作业 W6

习题 8

因为 $f(z)$ 是解析函数, 所以可以在 $z = z_0$ 处展开为

$$f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m$$

同理 $\varphi(z)$ 可以在 $z = z_0$ 处展开为

$$\varphi(z) = \sum_{m=0}^{\infty} \frac{\varphi^{(m)}(z_0)}{m!} (z - z_0)^m$$

由于 z_0 是 $f(z)$ 的至少 n 级零点, 所以对 $0 \leq k < n$, 都有

$$f^{(k)}(z_0) = 0$$

同理, 对 $0 \leq k < n$, 都有

$$\varphi^{(k)}(z_0) = 0$$

所以当 $\varphi^{(n)}(z_0) \neq 0$ 时, 有

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z)}{\varphi(z)} &= \lim_{z \rightarrow z_0} \frac{\sum_{m=0}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m}{\sum_{m=0}^{\infty} \frac{\varphi^{(m)}(z_0)}{m!} (z - z_0)^m} \\ &= \lim_{z \rightarrow z_0} \frac{\sum_{m=n}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m}{\sum_{m=n}^{\infty} \frac{\varphi^{(m)}(z_0)}{m!} (z - z_0)^m} \\ &= \lim_{z \rightarrow z_0} \frac{\sum_{m=n}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z - z_0)^{m-n}}{\sum_{m=n}^{\infty} \frac{\varphi^{(m)}(z_0)}{m!} (z - z_0)^{m-n}} \\ &= \lim_{z \rightarrow z_0} \frac{f^{(n)}(z_0)}{\varphi^{(n)}(z_0)} = \frac{f^{(n)}(z_0)}{\varphi^{(n)}(z_0)} \end{aligned}$$

习题 9

由题可知, 存在在 $z = z_0$ 处解析且值不为 0 的两个函数 $f_1(z), g_1(z)$ 使得

$$f(z) = (z - z_0)^m f_1(z) \quad g(z) = (z - z_0)^n g_1(z)$$

(1) 可知

$$f(z)g(z) = (z - z_0)^{m+n} f_1(z)g_1(z)$$

$f_1(z)g_1(z)$ 在 $z = z_0$ 解析且值不为 0, 所以 $f(z)g(z)$ 在 $z = z_0$ 处有 $m+n$ 级零点。

(2) 可知

$$\begin{aligned}f(z) + g(z) &= (z - z_0)^m f_1(z) + (z - z_0)^n g_1(z) \\&= (z - z_0)^n [(z - z_0)^{m-n} f_1(z) + g_1(z)]\end{aligned}$$

当 $m > n$ 时, $(z - z_0)^{m-n} f_1(z) + g_1(z)$ 在 $z = z_0$ 处解析且值不为 0, 所以有 n 级零点; 当 $m = n$ 时, $f(z) + g(z) = (z - z_0)^n [f_1(z) + g_1(z)]$, 由于不能确定 $f_1(z) + g_1(z)$ 是否有因式 $(z - z_0)$, 只能确定 $f(z) + g(z)$ 有至少 n 级的零点。

(3) 可知

$$f(z)/g(z) = (z - z_0)^{m-n} f_1(z)/g_1(z)$$

当 $m > n$ 时, 由于 $f_1(z)/g_1(z)$ 在 $z = z_0$ 处解析且值不为 0, 所以有 $m - n$ 级零点; 当 $m = n$ 时, $z = z_0$ 是可去奇点。

习题 10

(1)

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} \sum_{n=0}^{\infty} z^n = \sum_{n=-2}^{\infty} z^n$$

(2)

$$z^2 \exp\left(\frac{1}{z}\right) = z^2 \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} z^{-n}$$

习题 11

对于前三个小问首先裂项处理:

$$\frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right)$$

(1) 当 $0 \leq |z| < |a|$ 时有

$$\begin{aligned}\frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) &= \frac{1}{a-b} \left(\frac{1}{b} \cdot \frac{1}{1 - \frac{z}{b}} - \frac{1}{a} \cdot \frac{1}{1 - \frac{z}{a}} \right) \\&= \frac{1}{a-b} \left[\frac{1}{b} \sum_{n=0}^{\infty} \left(\frac{z}{b} \right)^n - \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a} \right)^n \right] \\&= \frac{1}{a-b} \sum_{n=0}^{\infty} \left(\frac{1}{b^{n+1}} - \frac{1}{a^{n+1}} \right) z^n\end{aligned}$$

(2) 当 $|a| < |z| < |b|$ 时有

$$\begin{aligned}\frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) &= \frac{1}{a-b} \left(\frac{1}{z} \cdot \frac{1}{1-\frac{a}{z}} + \frac{1}{b} \cdot \frac{1}{1-\frac{z}{b}} \right) \\ &= \frac{1}{a-b} \left[\frac{1}{z} \sum_{n=0}^{+\infty} \left(\frac{a}{z} \right)^n + \frac{1}{b} \sum_{n=0}^{+\infty} \left(\frac{z}{b} \right)^n \right] \\ &= \frac{1}{a-b} \sum_{n=0}^{+\infty} \left(\frac{a^n}{z^{n+1}} + \frac{z^n}{b^{n+1}} \right)\end{aligned}$$

(3) 当 $|b| < |z| < \infty$ 时有

$$\begin{aligned}\frac{1}{a-b} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) &= \frac{1}{a-b} \left(\frac{1}{z} \cdot \frac{1}{1-\frac{a}{z}} - \frac{1}{z} \cdot \frac{1}{1-\frac{b}{z}} \right) \\ &= \frac{1}{a-b} \left[\frac{1}{z} \sum_{n=0}^{+\infty} \left(\frac{a}{z} \right)^n - \frac{1}{z} \sum_{n=0}^{+\infty} \left(\frac{b}{z} \right)^n \right] \\ &= \frac{1}{a-b} \sum_{n=1}^{+\infty} (a^{n-1} - b^{n-1}) \frac{1}{z^n}\end{aligned}$$

(4) 当 $0 < |z-a| < |z-b|$ 时有

$$\begin{aligned}\frac{1}{(z-a)(z-b)} &= -\frac{1}{z-a} \cdot \frac{1}{(b-a)-(z-a)} \\ &= -\frac{1}{(b-a)(z-a)} \cdot \frac{1}{1-\frac{z-a}{b-a}} \\ &= -\frac{1}{(b-a)(z-a)} \sum_{n=0}^{+\infty} \left(\frac{z-a}{b-a} \right)^n \\ &= -\sum_{n=-1}^{+\infty} \frac{(z-a)^n}{(b-a)^{n+2}}\end{aligned}$$

(5) 当 $|b-a| < |z-a| < \infty$ 时有

$$\begin{aligned}\frac{1}{(z-a)(z-b)} &= \frac{1}{z-a} \cdot \frac{1}{(z-a)-(b-a)} \\ &= \frac{1}{(z-a)^2} \cdot \frac{1}{1-\frac{b-a}{z-a}} \\ &= \frac{1}{(z-a)^2} \sum_{n=0}^{+\infty} \left(\frac{b-a}{z-a} \right)^n \\ &= \sum_{n=2}^{+\infty} \frac{(b-a)^{n-2}}{(z-a)^n}\end{aligned}$$

(6) 当 $0 < |z - b| < |a - b|$ 时有

$$\begin{aligned}\frac{1}{(z-a)(z-b)} &= -\frac{1}{z-b} \cdot \frac{1}{(a-b)-(z-b)} \\ &= -\frac{1}{(a-b)(z-b)} \cdot \frac{1}{1-\frac{z-b}{a-b}} \\ &= -\frac{1}{(a-b)(z-b)} \sum_{n=0}^{+\infty} \left(\frac{z-b}{a-b}\right)^n \\ &= -\sum_{n=1}^{+\infty} \frac{(z-b)^n}{(a-b)^{n+2}}\end{aligned}$$

(7) 当 $|a-b| < |z-b| < \infty$ 时有

$$\begin{aligned}\frac{1}{(z-a)(z-b)} &= \frac{1}{z-b} \cdot \frac{1}{(z-b)-(a-b)} \\ &= \frac{1}{(z-b)^2} \cdot \frac{1}{1-\frac{a-b}{z-b}} \\ &= \frac{1}{(z-b)^2} \sum_{n=0}^{+\infty} \left(\frac{a-b}{z-b}\right)^n \\ &= \sum_{n=2}^{+\infty} \frac{(a-b)^{n-2}}{(z-b)^n}\end{aligned}$$