# 随机过程B Week 10&11

黄瑞轩 PB20111686

## Ch4 T12

因为 $\int_{-\infty}^{\infty} R(\tau)d\tau < \infty$，则均值遍历性成立，则

$$E\bar{X} = E\left(\frac{1}{T}\int_0^T X(s)ds\right) = \frac{1}{T}\int_0^T EX(s)ds = m \tag{1}$$

$$\begin{aligned}\mathrm{Var}(\bar{X}) &= E(\bar{X}^2) - [E(\bar{X})]^2 \\ &= \frac{1}{4T^2}\iint_{[-T,T]\times[-T,T]} E[(X(t)-m)(X(s)-m)]dtds \\ &= \frac{1}{4T^2}\iint_{[-T,T]\times[-T,T]} R(t-s)dtds \\ &(\text{令}\tau = t-s, u = t+s) \\ &= \frac{1}{4T^2}\frac{1}{2}\iint_D R(\tau)d\tau du \\ &= \frac{1}{4T^2}\int_{2T}^{2T} R(\tau)(2T-|\tau|)d\tau \\ &= \frac{1}{T}\int_0^{2T} R(\tau)(1-\frac{\tau}{2T})d\tau \\ &= \frac{1}{T}\int_0^{2T} ae^{-b\tau}(1-\tau/2T)d\tau \\ &= \frac{2a}{bT}[1-\frac{1}{bT}(1-e^{-bt})]\end{aligned}$$

## Ch4 T16

$EX_0 = \int_0^1 2x^2dx = \frac{2}{3}$，给定 $X_0$ 时，$EX_1 = E[E(X_1|X_0)] = E(1-\frac{X_0}{2}) = \frac{2}{3}$；

不妨用归纳法，假设 $EX_n = 2/3$，同上面的计算过程，可知 $EX_{n+1} = 2/3$，所以对于任意 $n$，$EX_n$ 是常数。

$EX_0^2 = \int_0^1 2x^3dx = 1/2$，所以 $DX_0 = EX_0^2 - (EX_0)^2 = 1/18$。假设 $EX_n^2 = 1/2$，则 $EX_{n+1}^2 = E[E(X_{n+1}^2|X_n)] = E[\int_{1-X_n}^1 \frac{1}{X_n}x_{n+1}^2dx_{n+1}] = E[X_n^2/12 + (2-X_n)^2/4] = 1/2$，所以对于任意 $n$，$EX_n^2$ 是常数。

$R_X(n+t,n) = E(X_{n+t}-2/3)(X_n-2/3) = E(X_{n+t}X_n) - 4/9$

$E(X_{n+t}X_n) = E[E(X_{n+t}X_n|X_{n+t-1},\ldots,X_0)] = E[X_nE(X_{n+t}|X_{n+t-1},\ldots,X_0)]$

$= E[X_n(1-\frac{1}{2}X_{n+t-1})] = \frac{2}{3} - \frac{1}{2}E(X_{n+t-1}X_n) = \ldots = \frac{2}{3}\sum_{i=0}^{t-1}(-\frac{1}{2})^i + \frac{1}{2}(-\frac{1}{2})^t = \frac{4}{9} + \frac{1}{18}(-\frac{1}{2})^t$

所以 $R_X(n+t,n) = \frac{1}{18}(-\frac{1}{2})^t = R(t)$。所以 $\{X_n\}$ 是平稳的。

又因为 $R(t)\to 0(t\to\infty)$，知均值遍历性成立。

# Ch4 T17

$EX_n = \sum_{k=0}^{\infty} \alpha^k E\varepsilon_{n-k} = 0$，

$R_X(n+t,n) = EX_{n+t}X_n = E(\sum_{k=0}^{\infty} \alpha^k \varepsilon_{n+t-k})(\sum_{l=0}^{\infty} \alpha^l \varepsilon_{n-l}) = E(\sum_{\infty} \alpha^a \varepsilon_b \varepsilon_c)$

对于 $b \neq c$，由白噪声性质，有 $E\varepsilon_b\varepsilon_c = 0$，所以

$R_X(n+t,n) = E(\sum_{k=0}^{\infty} \alpha^{2k-t} \varepsilon_{n+t-k}^2) = \frac{\sigma^2}{\alpha^t} \cdot \frac{1}{1-\alpha^2} = R(t)$

所以过程 $\{X_n\}$ 是平稳的，且 $R(t) \to 0 (t \to \infty)$，知均值遍历性成立。

# Ch4 T22

$$
\begin{aligned}
\int_{\mathbb{R}} \cos\omega_0\tau \cdot \exp(-j\omega\tau) d\tau &= \int_{\mathbb{R}} \frac{1}{2}[\exp(j\omega_0\tau) + \exp(-j\omega_0\tau)]\exp(-j\omega\tau)d\tau \\
&= \int_{\mathbb{R}} \frac{1}{2}[\exp(j\tau(\omega-\omega_0)) + \exp(j\tau(\omega+\omega_0))]d\tau \\
&= \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \\
\int_{\mathbb{R}} \exp(-a|\tau|)\exp(-j\omega\tau)d\tau &= \frac{2a}{a^2+\omega^2}
\end{aligned}
$$

所以

$$
S(\omega) = \frac{a^2\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \frac{2ab^2}{a^2+\omega^2} \tag{2}
$$

# Ch4 T23

由平方检波，知 $R_Y(\tau) = 2R_X^2(\tau) = 2A^2\exp(-2a|\tau|)\cos^2\beta\tau = A^2\exp(-2a|\tau|)(1+\cos 2\beta\tau)$。

又

$$
\begin{aligned}
\int_{\mathbb{R}} \exp(-a|\tau|)\exp(-j\omega\tau)d\tau &= \frac{2a}{a^2+\omega^2} \\
\int_{\mathbb{R}} \exp(-a|\tau|)\cos\beta\tau\exp(-j\omega\tau)d\tau &= \frac{a}{a^2+(\omega+\beta)^2} + \frac{a}{a^2+(\omega-\beta)^2}
\end{aligned} \tag{3}
$$

所以

$$
S(\omega) = \frac{4aA^2}{4a^2+\omega^2} + \frac{2aA^2}{4a^2+(\omega+2\beta)^2} + \frac{2aA^2}{4a^2+(\omega-2\beta)^2} \tag{4}
$$

# Ch4 T24

$$
R(\tau) = \frac{1}{\pi}\int_0^{\infty} S(\omega)\cos\omega\tau d\omega \tag{5}
$$

当 $\tau = 0$ 时，即为方差 $D = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+\omega^2} = \frac{1}{\pi}\arctan\omega \mid_0^{\infty} = \frac{1}{2}$。

则 $X(t) \sim N(0, \frac{1}{2})$，所以 $\sqrt{2}X(t) \sim N(0,1)$，
$P(\frac{1}{2} \leq X(t) \leq 1) = P(\frac{\sqrt{2}}{2} \leq \sqrt{2}X(t) \leq \sqrt{2}) = \Phi(\sqrt{2}) - \Phi(\frac{\sqrt{2}}{2})$，这里 $\Phi(x)$ 是标准 Gauss 分布的分布函数。

# Ch4 T28

### (1) $S(\omega) = \frac{\omega^2+64}{\omega^4+29\omega^2+100}$

$R(\tau) = \frac{1}{2\pi}\int \frac{\omega^2+64}{\omega^4+29\omega^2+100}\exp(j\omega\tau)d\omega$

由留数定理（对 $\tau \geq 0$ 时用上半平面围道，$\tau < 0$ 时用下半平面上的围道）可算得

$R(\tau) = j \cdot \{\mathrm{Res}[f(z), 2j] + \mathrm{Res}[f(z), 5j]\} = \frac{5}{7}\exp(-2|\tau|) - \frac{13}{70}\exp(-5|\tau|)$

### (2) $S(\omega) = \frac{1}{(1+\omega^2)^2}$

$R(\tau) = \frac{1}{2\pi}\int \frac{1}{(1+\omega^2)^2}\exp(j\omega\tau)d\omega$

由留数定理（对 $\tau \geq 0$ 时用上半平面围道，$\tau < 0$ 时用下半平面上的围道）可算得

$R(\tau) = j \cdot \mathrm{Res}[f(z), j] = \frac{1}{4}\exp(-|\tau|)(1+|\tau|)$