2018~2019

一、

(1) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 3 \end{pmatrix}$

$$(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1)=(\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3)\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 3 \end{pmatrix}$$

(2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$$\mathcal{A}(\alpha_1,\alpha_2,\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$(\alpha_1,\alpha_2-\alpha_3,\alpha_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}=(\alpha_1,\alpha_2,\alpha_3)P$$

$$\begin{aligned}\mathcal{A}(\alpha_1,\alpha_2-\alpha_3,\alpha_3)=\mathcal{A}(\alpha_1,\alpha_2,\alpha_3)P=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}P \\ =(\alpha_1,\alpha_2-\alpha_3,\alpha_3)P^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}P\end{aligned}$$

(3) $x=0,y=3$

$$Tr(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & x \end{pmatrix})=3+x=Tr(\begin{pmatrix} -1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix})=y$$

$$det(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & x \end{pmatrix})=-3=det(\begin{pmatrix} -1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix})=-y$$

(4) $4n^2$

$$(\pm\frac{1}{2n})^2\times k=\frac{k}{4n^2}=1$$

(5) $t>0,\begin{vmatrix} t & \sqrt{2} \\ \sqrt{2} & 2 \end{vmatrix}>0,\begin{vmatrix} t & \sqrt{2} & t-1 \\ \sqrt{2} & 2 & 0 \\ t-1 & 0 & 2 \end{vmatrix}>0$

$$Q=\begin{pmatrix} t & \sqrt{2} & t-1 \\ \sqrt{2} & 2 & 0 \\ t-1 & 0 & 2 \end{pmatrix}$$

$$t>0,\begin{vmatrix} t & \sqrt{2} \\ \sqrt{2} & 2 \end{vmatrix}>0,\begin{vmatrix} t & \sqrt{2} & t-1 \\ \sqrt{2} & 2 & 0 \\ t-1 & 0 & 2 \end{vmatrix}>0$$

二、

(1) ✖

$$A_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=B_1=B_2,A_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(2) ✔

$$A\sim\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}\sim B$$

(3) ✔

$$\text{by definition}$$

(4) ✖

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\text{is real symmetric matrix,}$$

$$\exists P,\text{such that }\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}=P^T\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}P$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\text{congruent to}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{if} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{congruent to} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{then}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{congruent to} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{then exist a inversible matrix Q,}$$

$$that \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = Q^T I Q = Q^T Q,$$

$$\text{but } \lambda(Q^TQ) \geq 0, \text{which leads a contradiction!}$$

三、

(i) $Ax = 0$

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \\ -1 \end{pmatrix}$$

(ii) $Ax_p = \beta$

$$x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

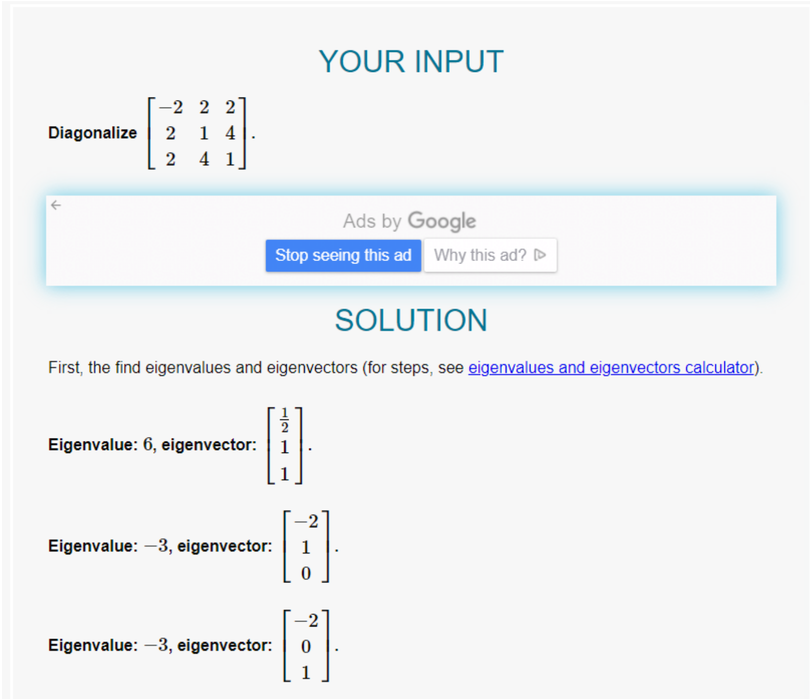
Hence

$$x = x_p + t_1x_1 + t_2x_2 (t_1、t_2 \in F)$$

四、

(1)

$$Q = \begin{pmatrix} -2 & 2 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$$




Hence

$$P = \begin{pmatrix} 1/3 & -2/\sqrt{5} & -2/\sqrt{5} \\ 2/3 & 1/\sqrt{5} & 0 \\ 2/3 & 0 & 1/\sqrt{5} \end{pmatrix}$$

then

$$Q = P^T \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} P$$

(2) 双曲面

$$6x^2 - 3y^2 - 3z^2 = -1$$

五、

proof:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} n & 0 & \cdots & 0 \\ n-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

六、

proof:

$$\text{Let } Q=(\alpha_1,\alpha_2,\cdots,\alpha_n),\text{ then } A=Q^TQ$$
$$rank(A)=rank(Q^TQ)=rank(Q)$$

**列空间**

矩阵$A_{n\times m}=(\mathbf{a}_1,\ldots,\mathbf{a}_m)$的列空间$C(A)$为$A$的各列张成的线性空间：

$$C(A)=L(A)=\mathrm{Img}(A)=L(\mathbf{a}_1,\ldots,\mathbf{a}_m)=\left\{\sum_{i=1}^{m}\mathbf{a}_i x_i \mid x_1,\ldots,x_m\in R\right\}$$
$$=\left\{A\mathbf{x}\mid \mathbf{x}\in R^m\right\}.$$

-

**行空间**

$A_{n\times m}$的行空间为各个行向量张成的空间，即$A^{\mathsf T}$的列空间

$$C(A^{\mathsf T})=\mathrm{Img}(A^{\mathsf T})=\left\{A^{\mathsf T}\mathbf{y}\mid \mathbf{y}\in R^n\right\}$$

**零/核空间**

矩阵$A_{n\times m}$的核空间(kernel space)或零空间(null space):

$$N(A)=\ker(A)=\left\{\mathbf{x}\in R^m\mid A\mathbf{x}=\mathbf{0}\right\},$$

$N(A)$为与$A$的各行正交的向量构成的子空间。

**核空间是行空间的正交补**

$$N(A)=L(A^{\mathsf T})^{\perp},\ N(A)\bigcap L(A^{\mathsf T})=\{\mathbf{0}\},\text{记作}R^m=L(A^{\mathsf T})\oplus N(A)$$

证明：对$\forall\mathbf{x}\in N(A), A\mathbf{x}=\mathbf{0}$. 对$\forall\mathbf{y}\in L(A^{\mathsf T})$,存在$\mathbf{b}$使得
$\mathbf{y}=A^{\mathsf T}\mathbf{b}$, 则$\mathbf{x}^{\mathsf T}\mathbf{y}=\mathbf{x}^{\mathsf T}A^{\mathsf T}\mathbf{b}=(A\mathbf{x})^{\mathsf T}\mathbf{b}=\mathbf{0}^{\mathsf T}\mathbf{b}=0$,所以$N(A)\subset L(A^{\mathsf T})^{\perp}$.
反之，若$\mathbf{x}\in L(A^{\mathsf T})^{\perp},\mathbf{x}\perp L(A^{\mathsf T})$,特别地正交于$A^{\mathsf T}$的每一列,$A$的每一行，

**$A$ 与$AA^{\mathsf T}$**
**$A^{\mathsf T}$与$A^{\mathsf T}A$**
**张成相同空间**

定理5: $C(AA^{\mathsf T})=C(A),\ C(A^{\mathsf T}A)=C(A^{\mathsf T})$, ;
$\mathrm{rank}(AA^{\mathsf T})=\mathrm{rank}(A^{\mathsf T}A)=\mathrm{rank}(A)$.

证明：$A_{n\times m}$,设$\mathbf{y}\in C(A)$,存在$\mathbf{x}\in R^m$,使得$\mathbf{y}=A\mathbf{x}$,
由定理2，$R^m=C(A^{\mathsf T})\oplus N(A)$,存在$\mathbf{x}_{row}\in C(A^{\mathsf T}),\mathbf{x}_0\in N(A), A\mathbf{x}_0=0$,使得
$\mathbf{x}=\mathbf{x}_{row}+\mathbf{x}_0$.所以$\mathbf{y}=A\mathbf{x}=A\mathbf{x}_{row}+A\mathbf{x}_0=A\mathbf{x}_{row}$
$\mathbf{x}_{row}\in C(A^{\mathsf T})\Rightarrow$ 存在$\mathbf{t}\in R^n$,使得$\mathbf{x}_{row}=A^{\mathsf T}\mathbf{t}$,
$\Rightarrow \mathbf{y}=A\mathbf{x}_{row}=AA^{\mathsf T}\mathbf{t}\in C(AA^{\mathsf T})\ \ \Rightarrow C(A)\subseteq C(AA^{\mathsf T})$

另一方面，$C(AA^{\mathsf T})\subseteq C(A)$,这是因为任何$\mathbf{y}\in C(AA^{\mathsf T})$,存在
$\mathbf{t}\in R^n,\mathbf{y}=AA^{\mathsf T}\mathbf{t}=A\mathbf{u}\in C(A)$.