复变函数 B 作业 W2

习题 1

设 $z=x+iy$,$w=u+iv$，则 $w=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}$，则 $u=\frac{x}{x^2+y^2}$、$v=-\frac{y}{x^2+y^2}$。
(2) 代入 $y=0$，得 $u=\frac{1}{x}$,$v=0(x\neq 0)$。
(4) 带入 $x^2+y^2=4$，得 $u^2+v^2=\frac{1}{4}$，是一个原点为圆心、半径是 1/2 的圆。

习题 3

当 y 沿着 $y=kx$ 的方向趋向 0 时

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{kx^2}{(k^2+1)\,x^2}=\frac{k}{k^2+1}$$

由于 k 不确定，所以原函数在 0 处没有极限，所以不连续。

习题 5

(1) 首先，u,v 在 $z=0$ 处不是实可微的，所以在这一点不可导。其次，在 z 平面其他任意一点 $z=x+iy$ 处，

$$f(z)=|x+iy|=\sqrt{x^2+y^2}$$
$$\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}\neq\frac{\partial v}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

不满足 CR 方程，所以处处不可导。
(3) 在 z 平面任意一点 $z=x+iy$ 处，

$$f(z)=\frac{1}{\bar{z}}=\frac{1}{x-iy}=\frac{1}{x^2+y^2}$$

$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\neq\frac{\partial v}{\partial y}=\frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2}$$

不满足 CR 方程，所以处处不可导。

习题 6

（2）

$$f(z)=\begin{cases}\sqrt{x^2+y^2}(x+iy), & x^2+y^2<1\\ x^2+2xyi-y^2, & x^2+y^2\geqslant 1\end{cases}$$

在区域 I: $\dfrac{\partial u}{\partial x}=\sqrt{x^2+y^2}+x\cdot\dfrac{x}{\sqrt{x^2+y^2}}=\dfrac{2x^2+y^2}{\sqrt{x^2+y^2}}\neq\dfrac{\partial v}{\partial y}=\dfrac{2y^2+x^2}{\sqrt{x^2+y^2}}$

在区域 II: $\dfrac{\partial u}{\partial x}=2x=\dfrac{\partial v}{\partial y}=2x\dfrac{\partial u}{\partial y}=-2y=-\dfrac{\partial u}{\partial x}=-2y.$

并且在区域 II，u,v 实可微条件满足，所以仅在区域 II（$|z|\geq 1$）上解析。

习题 7

（2）首先，u,v 都满足实可微条件。又

$$\frac{\partial u}{\partial x}=e^x(x\cos y-y\sin y+\cos y)=\frac{\partial v}{\partial y}=e^x(\cos y-y\sin x+x\cos y)$$

$$\frac{\partial u}{\partial y}=e^x(-x\sin y-\sin y-y\cos y)=\frac{\partial v}{\partial x}=-e^x(y\cos y+x\sin y+\sin y)$$

所以函数在全平面上解析，导数为

$$f'(z)=e^x(x\cos y-y\sin y+\cos y)-ie^x(y\cos y+x\sin y+\sin y)$$

习题 8

首先，因为 $f(z)$ 解析，设 $z=x+yi,f(z)=a(x,y)+b(x,y)i$，则

$$\frac{\partial a}{\partial x}=\frac{\partial b}{\partial y}\quad\frac{\partial a}{\partial y}=-\frac{\partial b}{\partial x}$$

（2）由于 $\overline{f(z)}$ 解析，则

$$\frac{\partial a}{\partial x}=-\frac{\partial b}{\partial y}\quad\frac{\partial a}{\partial y}=\frac{\partial b}{\partial x}$$

则

$$\frac{\partial b}{\partial y}=\frac{\partial b}{\partial x}=0\quad\frac{\partial a}{\partial x}=\frac{\partial a}{\partial y}=0$$

则 $a(x,y)\equiv a,b(x,y)\equiv b$，即 $f(z)$ 为常数。

（4）

$$\operatorname{Im}f(z)\equiv b\Rightarrow\frac{\partial a}{\partial x}=\frac{\partial a}{\partial y}=0\Rightarrow a(x,y)\equiv a\Rightarrow f(z)\equiv a+bi$$

（6）因为 $\arg f(z) \equiv \phi_0 \in [-\pi,\pi]$，所以 $\dfrac{b}{a} \equiv k_0$，$b = k_0 a$，则

$$\frac{\partial a}{\partial x} = k_0 \frac{\partial a}{\partial y}$$

$$\frac{\partial a}{\partial y} = -k_0 \frac{\partial a}{\partial x}$$

则

$$\frac{\partial a}{\partial y} = -k_0^2 \frac{\partial a}{\partial y}$$

由于 $k_0 \in \mathbb{R}$，所以只能 $\frac{\partial a}{\partial y} = 0$，所以有

$$\frac{\partial b}{\partial y} = \frac{\partial b}{\partial x} = 0 \quad \frac{\partial a}{\partial x} = \frac{\partial a}{\partial y} = 0$$

则 $a(x,y) \equiv a, b(x,y) \equiv b$，即 $f(z)$ 为常数。

习题 9

极坐标变换：$x = r\cos\theta, y = r\sin\theta$，且

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial y}(-r\sin\theta) + \frac{\partial u}{\partial x}(r\cos\theta)$$

代入题中所给式子发现恒成立，证毕。

1. $f(z) = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$

$\frac{\partial v}{\partial \theta} = nr^n\cos n\theta$

$\frac{\partial u}{\partial \theta} = -nr^n\sin n\theta$

$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta$

经验证满足 CR 方程。

2. $f(z) = \ln z = \ln r + i\theta$

$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial u}{\partial \theta} = 1$

$\frac{\partial u}{\partial \theta} = 0 \quad \frac{\partial v}{\partial r} = 0$

经验证满足 CR 方程。3. $f(z) = r(\cos\theta + i\sin\theta)$

$\frac{\partial u}{\partial r} = \cos\theta \quad \frac{\partial v}{\partial r} = \sin\theta$

$\frac{\partial u}{\partial \theta} = -r\sin\theta \quad \frac{\partial v}{\partial \theta} = r\cos\theta$

经验证满足 CR 方程。

习题 10

（1） $\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}=\frac{(x-2)-yi}{(x-2)^2+y^2}-\frac{(x-1)-yi}{(x-1)^2+y^2}$
显然在 (2,0),(1,0) 不可微，在其他地方都是实可微。

$$\frac{\partial u}{\partial x}=\frac{y^2-(x-2)^2}{((x-2)^2+y^2)^2}+\frac{y^2-(x-1)^2}{((x-1)^2+y^2)^2}=\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y}=-\frac{2y(x-2)}{\left((x-2)^2+y^2\right)^2}+\frac{2y(x-1)}{\left((x-1)^2+y^2\right)^2}=-\frac{\partial v}{\partial x}$$

所以在除了 (2,0),(1,0) 的邻域外的其他区域都是解析的，其微商是

$$f'(z)=\frac{y^2-(x-2)^2}{((x-2)^2+y^2)^2}+\frac{y^2-(x-1)^2}{((x-1)^2+y^2)^2}+i\left[\frac{2y(x-2)}{\left((x-2)^2+y^2\right)^2}-\frac{2y(x-1)}{\left((x-1)^2+y^2\right)^2}\right]$$