随机过程B Week 1

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Ch1 T2

按期望定义和期望的线性性，

$$E[X(t)]=E\left[\frac{1}{n}\sum_{k=1}^{n}I\left(t,U_k\right)\right]=E\left[I\left(t,U_1\right)\right]$$

由于$U_1\sim\mathcal{U}(0,1)$，所以

$$E\left[I\left(t,U_1\right)\right]=1*P\left[I\left(t,U_1\right)=1\right]+0*P\left[I\left(t,U_1\right)=0\right]=t$$

于是在任意两个时刻$t_1,t_2(t_1,t_2>0)$，有

$$E[X(t_1)]=t_1,E[X(t_2)]=t_2$$

按协方差函数定义，

$$\begin{aligned}
\operatorname{Cov}(X\left(t_1\right),X\left(t_2\right))&=\frac{1}{n^2}\operatorname{Cov}\left(\sum_{k=1}^{n}I\left(t_1,U_k\right),\sum_{k=1}^{n}\left(t_2,U_k\right)\right)\\
&=\frac{1}{n^2}\sum_{k=1}^{n}\sum_{m=1}^{n}\operatorname{Cov}(I\left(t_1,U_k\right),I\left(t_2,U_m\right))\\
&=\frac{1}{n}\operatorname{Cov}(I\left(t_1,U_1\right),I\left(t_2,U_1\right))\\
&=\frac{1}{n}[E\left(I\left(t_1,U_1\right)I\left(t_2,U_1\right)\right)-t_1t_2]\\
&=\frac{1}{n}[P\left(I\left(t_1,U_1\right)I\left(t_2,U_1\right)=1\right)-t_1t_2]\\
&=\frac{1}{n}[P\left(u_1\le t_1,u_1\le t_2\right)-t_1t_2]\\
&=\frac{1}{n}[\min\left\{t_1,t_2\right\}-t_1t_2]
\end{aligned}$$

Ch1 T4

按期望定义和条件，

$$E[X(t)]=E[X(t)-X(0)]=\lambda t$$

按协方差函数定义，在任意两个时刻$t_1,t_2(t_1,t_2>0)$，有

$$\operatorname{Cov}(X\left(t_1\right),X\left(t_2\right))=E\left(X\left(t_1\right)X\left(t_2\right)\right)-\lambda^2t_1t_2$$

计算等号后的第一项，记$m=\min(t_1,t_2),M=\max(t_1,t_2)$，按条件(iii)有

$$\begin{aligned}
E(X(t_1)X(t_2))&=E\{[X(m)-X(0)][X(M)-X(m)+X(m)-X(0)]\}\\
&=\lambda m\cdot\lambda(M-m)+E[X(m)-X(0)]^2\\
&=\lambda^2(t_1t_2-m^2)+\operatorname{Var}[X(m)-X(0)]+\{E[X(m)-X(0)]\}^2\\
&=\lambda^2(t_1t_2-m^2)+\lambda m+\lambda^2m^2\\
&=\lambda^2t_1t_2+\lambda m
\end{aligned}$$

所以，

$$\operatorname{Cov}(X(t_1), X(t_2)) = \lambda \min(t_1, t_2)$$

由于均值函数不是常数，此过程不是宽平稳的。

Ch1 T5

按期望线性性，有

$$\begin{aligned} E(Y(t)) &= E(X(t+1) - X(t)) \\ &= E(X(t+1) - X(0)) - E(X(t) - X(0)) \\ &= \lambda \end{aligned}$$

计算协方差函数，在任意两个时刻$t_1, t_2 (t_1, t_2 > 0)$，利用上题结论及$\min(x, y) = \dfrac{x + y - |x - y|}{2}$

$$\begin{aligned} \operatorname{Cov}(Y(t_1), Y(t_2)) &= E(Y(t_1) Y(t_2)) - \lambda^2 \\ &= E(X(t_1+1)X(t_2+1)) - E(X(t_1+1)X(t_2)) - E(X(t_1)X(t_2+1)) + E(X(t_1)X(t_2)) - \lambda^2 \\ &= \lambda[2\min(t_1, t_2) - \min(t_1+1, t_2) - \min(t_1, t_2+1)] \\ &= \lambda\left(\frac{|t_1 - t_2 + 1| + |t_1 - t_2 - 1|}{2} - |t_1 - t_2|\right) \\ &\sim f(t_1 - t_2) \end{aligned}$$

由协方差函数知二阶矩存在，所以此过程是宽平稳的。

Ch1 T9

由几何概型，

$$P(X^2 + Y^2 \geq \frac{3}{4} | X > Y) = \frac{P(X^2 + Y^2 \geq \frac{3}{4}, X > Y)}{P(X > Y)} = \frac{\frac{1}{2}(1 - \frac{3}{4})\pi}{\frac{1}{2}\pi} = \frac{1}{4}$$

Ch1 T14

$$\begin{aligned} P(X_1 + X_2 = n) &= \sum_{k=0}^{\infty} P(X_1 = k, X_2 = n - k) \\ &= \sum_{k=0}^{\infty} P(X_1 = k) P(X_2 = n - k) \\ &= \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} e^{-\lambda_1} \frac{\lambda_2^{n-k}}{(n-k)!} e^{-\lambda_2} \\ &= \frac{e^{-\lambda_1 - \lambda_2}}{n!} \sum_{k=0}^{\infty} C_n^k \lambda_1^k \lambda_2^{n-k} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n \end{aligned}$$

这说明

$$X_1 + X_2 \sim \mathcal{P}(\lambda_1 + \lambda_2)$$

条件概率

$$
\begin{aligned}
P\left(X_1=k \mid X_1+X_2=n\right) &= \frac{P\left(X_1=k, X_2=n-k\right)}{P\left(X_1+X_2=n\right)} \\
&= \frac{\frac{\lambda_1^k}{k!} e^{-\lambda_1} \frac{\lambda_2^{n-k}}{(n-k)!} e^{-\lambda_2}}{\frac{\left(\lambda_1+\lambda_2\right)^n}{n!} e^{-\left(\lambda_1+\lambda_2\right)}} \\
&= C_n^k \frac{\lambda_1^k \lambda_2^{n-k}}{\left(\lambda_1+\lambda_2\right)^n}
\end{aligned}
$$

这里$k=0,1,\ldots,n$。

Ch1 T15

$X_1, X_2, \ldots$ i.i.d.，则$\sum_{i=1}^n X_i \sim \Gamma(n, \lambda)$，即

$$
f_{Y|N}(y|n)=\frac{y^{n-1} \lambda^n e^{-\lambda y}}{(n-1)!}, y \geq 0
$$

故而

$$
\begin{aligned}
f_Y(y) &= \sum_{i=1}^n f(y|n) P(N=n) \\
&= \sum_{i=1}^n \frac{y^{n-1} \lambda^n e^{-\lambda y}}{(n-1)!} \beta(1-\beta)^n \\
&= \lambda \beta e^{-\lambda \beta y}
\end{aligned}
$$

即

$$
Y \sim \mathcal{E}(\lambda \beta)
$$

Ch2 T2

$$
\begin{aligned}
E(N(s+t) N(t)) &= E\{[N(t)-N(0)][N(s+t)-N(t)+N(t)-N(0)]\} \\
&= \lambda t \cdot \lambda(s+t-t)+E[N(t)-N(0)]^2 \\
&= \lambda^2 s t+\operatorname{Var}[N(t)-N(0)]+\{E[N(t)-N(0)]\}^2 \\
&= \lambda^2 s t+\lambda t+\lambda^2 t^2
\end{aligned}
$$

Ch2 T4

由Poisson过程定义条件(iii)可解。

(i) $N(1)=N(1)-N(0) \sim \mathcal{P}(2)$，则$P(N(1) \leq 2)=P(N(1)=0,1,2)=5/e^2$;

(ii) $N(1)=N(1)-N(0) \sim \mathcal{P}(2), N(2)=N(2)-N(1) \sim \mathcal{P}(2)$，则

$$
\begin{aligned}
P(N(1)=1, N(2)=3) &= P(N(1)-N(0)=1, N(2)-N(1)=2) \\
&= P(N(1)-N(0)=1) P(N(2)-N(1)=2) \\
&= \frac{2}{1} e^{-2} \frac{2^2}{2!} e^{-2}=\frac{4}{e^4}
\end{aligned}
$$

(iii) $P(N(1) \geq 2|N(1) \geq 1)=\frac{P(N(1) \geq 2, N(1) \geq 1)}{P(N(1) \geq 1)}=\frac{1-P(N(1)=0,1)}{1-P(N(1)=0)}=\frac{e^2-3}{e^2-1}$