概统第十次作业　　PB20111686　黄瑞轩

EX 1.

$$Var(\sum X_i) = E(\sum X_i)^2 - (E\sum X_i)^2$$

$$= E(\sum X_i)^2 - (\sum EX_i)^2$$

$$= E\left(\sum X_i^2 + 2\sum_{i\neq j} X_iX_j\right) - \left(\sum (EX_i)^2 + 2\sum_{i\neq j} EX_i EX_j\right)$$

$$= \sum EX_i^2 + 2\sum_{i\neq j} EX_iX_j - \sum (EX_i)^2 - 2\sum_{i\neq j} EX_i EX_j$$

因为 X_i 与 X_j 独立 $(i\neq j)$，故 $EX_iEX_j = EX_iX_j$.

故 $Var(\sum X_t) = \sum EX_i^2 - \sum (EX_i)^2 = \sum (EX_i^2 - (EX_i)^2) = \sum Var X_i$

EX. 2. (4.1)

(1) $$EX = \sum_{i=1}^{\infty} iP(X=i) = \sum_{i=1}^{\infty} ip(1-p)^{i-1} = \sum_{i=1}^{\infty} (1-q)(q^i)' = (1-q)\left(\sum_{i=1}^{\infty} q^i\right)'$$

$$= (1-q)\left(\frac{q}{1-q}\right)' = (1-q)\frac{1-q+q}{(1-q)^2} = \frac{1}{p}.$$

$$EX^2 = E[X(X-1)] + EX = \sum_{j=1}^{\infty} j(j-1)p(1-p)^{j-1} + \frac{1}{p}$$

$$= \sum_{j=1}^{\infty} pq(q^j)'' + \frac{1}{p} = \left(\sum_{j=1}^{\infty} q^j\right)'' pq + \frac{1}{p}$$

$$= \left(\frac{q}{1-q}\right)'' pq + \frac{1}{p} = \left[\frac{1}{(1-q)^2}\right]' pq + \frac{1}{p} = \frac{2}{(1-q)^3}pq + \frac{1}{p} = \frac{2q}{p^2} + \frac{1}{p}$$

$$VarX = EX^2 - (EX)^2 = \frac{2-2p}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

(2) $$EX = \sum_{i=r}^{\infty} iP(X=i) = \sum_{i=r}^{\infty} iC_{i-1}^{r-1} p^r (1-p)^{i-r}$$

$$= \sum_{i=r}^{\infty} \frac{i!}{(r-1)!(i-r)!} p^r (1-p)^{i-r} = rp^r \sum_{i=r}^{\infty} \frac{i!}{r!(i-r)!}(1-p)^{i-r} = rp^r \sum_{i=r}^{\infty} C_i^r (1-p)^{i-r}$$

令 $i-r=t$， $EX = rp^r \sum_{t=0}^{\infty} C_{t+r}^{r}(1-p)^t = rp^r[1-(1-p)]^{-(r+1)} = \frac{r}{p}$

$$EX^2 = \sum_{i=r}^{\infty} i^2 P(X=i) = \sum_{i=r}^{\infty} i^2 C_{i-1}^{r-1} p^r (1-p)^{i-r} = \sum_{i=r}^{\infty} i(i+1) C_{i-1}^{r-1} p^r (1-p)^{k-r} - EX$$

$= r(r+1)p\sum_{i=r}^{\infty} C_{i+1}^{r+1}(1-p)^{i-r} - EX$，令 $t=i-r$

$$EX^2 = r(r+1)p^r \sum_{t=0}^{\infty} C_{r+t+1}^{r+1}(1-p)^{k-r} - EX = r(r+1)p^r\left[1-(1-p)\right]^{-(r+2)} - \frac{r}{p}$$

$$= r(r+1)p^r/p^{r+2} - r/p = \frac{r(r+1)}{p^2} - \frac{r}{p} = \frac{r^2+r-pr}{p^2}$$

$$\operatorname{Var}X = EX^2 - (EX)^2 = \frac{r^2+r-pr}{p^2} - \frac{r^2}{p^2} = \frac{r-pr}{p^2}$$

EX 3. (4.20)

设第i次摸球摸出白球的个数为 X_i.

$$EX_1 = 0\cdot\frac{b}{a+b} + 1\cdot\frac{a}{a+b} = \frac{a}{a+b}.$$

有递推公式：$EX_i = 0\cdot\frac{b+(1-EX_{i-1})}{a+b+1} + 1\cdot\frac{a+EX_{i-1}}{a+b+1}$

$$(\underbrace{a+b+1}_{:=c})EX_i = a + EX_{i-1} \Rightarrow c\left(EX_i - \frac{a}{a+b}\right) = EX_{i-1} - \frac{a}{a+b}$$

$EX_1 - \frac{a}{a+b} = 0$. 故 $EX_i = \frac{a}{a+b}$.

$$EW_n = E(X_1+\cdots+X_n) = EX_1+\cdots+EX_n = \frac{na}{a+b}$$

EX 4. (4.28)

设 $x\in(-\frac{\pi}{2},\frac{\pi}{2})$, $f(x)=t$, 故 $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\,dx = \pi t = 1$.　$t=\frac{1}{\pi}$

$$E(\sin X) = \int_{-\infty}^{+\infty}\sin x\cdot f(x)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\pi}\sin x\,dx = 0$$

$$E(\cos X) = \int_{-\infty}^{+\infty}\cos x\cdot f(x)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\pi}\cos x\,dx = \frac{1}{\pi}(\sin x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$E(X\cos X) = \int_{-\infty}^{+\infty}x\cos x\, f(x)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\pi}x\cos x\,dx = 0.$$

EX 5. (4.37)

(1) $f_Y(y) = \sum_{k=0}^{2}P(X=k)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-k)^2}{2}\right) = \frac{1}{3}\sum_{k=0}^{2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-k)^2}{2}\right)$

$$EY = \int_{-\infty}^{+\infty}y f_Y(y)\,dy = \frac{1}{3}\int_{-\infty}^{+\infty}y\sum_{k=0}^{2}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-k)^2}{2}\right)dy = \frac{1}{3}(0+1+2) = 1$$

(2) 令 $U=X+Y$. $P(U\le u) = \sum_{k=0}^{2}P(Y\le u-k\mid X=k)P(X=k)$

$= \sum_{k=0}^{2}P(Y\le u-k\mid X=k)\cdot\frac{1}{3}$, 若令 Φ 为标准正态分布函数.

$P(Y\le u-k\mid X=k) = \Phi(u-2k)$, 故 $F_U(u) = \frac{1}{3}\sum_{k=0}^{2}\Phi(u-2k)$.

(3) $EX=1.\ \ EY=1.\quad Cov(X,Y)=E(XY)-(EX)(EY)=E(XY)-1$

$P(XY=t)=P(t=0)\cdot\frac{1}{3}+P(t=Y)\cdot\frac{1}{3}+P(Y=\frac{t}{2})\cdot\frac{1}{3}$

在$t=0$处，XY不连续.

故先令$t\neq 0$，$P(XY=t)=\frac{1}{3}f_Y(t)\,dy+\frac{1}{3}f_Y(\frac{t}{2})\,dy.$

故$f_{XY}(t)=\frac{1}{3}f_Y(t)+\frac{1}{3}f_Y(\frac{t}{2}).\quad t\neq 0.$

若$t=0$，$P(XY=0)=\frac{1}{3}+c\,dy.\quad c\in\mathbb{R}.\quad P(XY=0)\to\frac{1}{3}.$

故$EXY=0\cdot P(XY=0)+\int_{t\neq 0}tf_{XY}(t)\,dt.$

$$=\int_{-\infty}^{+\infty}\left[\frac{1}{3}tf_Y(t)+\frac{1}{3}\frac{t}{2}f_Y(\frac{t}{2})\cdot 2\right]dt$$

$$=\frac{1}{3}EY+\frac{1}{3}\int_{-\infty}^{+\infty}\frac{t}{2}f_Y(\frac{t}{2})\cdot 4\cdot d\frac{t}{2}\ =\ \frac{5}{3}EY=\frac{5}{3}$$

故$Cov(X,Y)=\frac{5}{3}-1=\frac{2}{3}.$