形式化方法导引 HW3

使用 Simplex Method，求解如下问题：

1. Maximize $z = 1 + 2x_1 + 3x_2 + 6x_3$ satisfying the constraints:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 10 \\ x_1 - x_3 &\leq 3 \\ -x_2 + 2x_3 &\leq 5 \end{aligned} \tag{1}$$

where $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

Slack form:

- $y_1 = 10 - x_1 - 2x_2 - 3x_3$
- $y_2 = 3 - x_1 + x_3$
- $y_3 = 5 + x_2 - 2x_3$

Goal: Maximize $z = 1 + 2x_1 + 3x_2 + 6x_3$ while keeping $x_i, y_i \geq 0$

Basic solution

$x_1 = x_2 = x_3 = 0, y_1 = 10, y_2 = 3, y_3 = 4$

Choose x_3 to increase:

- $y_1 = 10 - 3x_3 \geq 0$ only OK if $x_3 \leq \frac{10}{3}$
- $y_2 = 3 + x_3 \geq$ OK if x_3 increases
- $y_3 = 5 - 2x_3 \geq 0$ only OK if $x_3 \leq \frac{5}{2}$

So $y_i \geq 0$ only holds for all i if $x_3 \leq \frac{5}{2}$

Pivot: swap x_3, y_3

- $2x_3 = 5 + x_2 - y_3 \Rightarrow x_3 = \frac{5}{2} + \frac{1}{2}x_2 - \frac{1}{2}y_3$

Slack form:

- $y_1 = \frac{5}{2} - x_1 - \frac{7}{2}x_2 + \frac{3}{2}y_3$
- $y_2 = \frac{11}{2} - x_1 + \frac{1}{2}x_2 - \frac{1}{2}y_3$

- $x_3 = \frac{5}{2} + \frac{1}{2}x_2 - \frac{1}{2}y_3$

Goal: $z = 16 + 2x_1 + 6x_2 - 3y_3$

Temporary solution

$x_1 = x_2 = y_3 = 0, y_1 = \frac{5}{2}, y_2 = \frac{11}{2}, x_3 = \frac{5}{2}, z = 16$

Choose x_2 to increase:

- $y_1 = \frac{5}{2} - \frac{7}{2}x_2 \geq 0$ only OK if $x_2 \leq \frac{5}{7}$
- $y_2 = \frac{11}{2} + \frac{1}{2}x_2 \geq 0$ OK if x_2 increases
- $x_3 = \frac{5}{2} + \frac{1}{2}x_2$ OK if x_2 increases

Pivot: swap x_2, y_1

- $\frac{7}{2}x_2 = \frac{5}{2} - x_1 - y_1 + \frac{3}{2}y_3 \Rightarrow x_2 = \frac{5}{7} - \frac{2}{7}x_1 - \frac{2}{7}y_1 + \frac{3}{7}y_3$

Slack form:

- $x_2 = \frac{5}{7} - \frac{2}{7}x_1 - \frac{2}{7}y_1 + \frac{3}{7}y_3$
- $y_2 = \frac{41}{7} - \frac{8}{7}x_1 - \frac{1}{7}y_1 - \frac{2}{7}y_3$
- $x_3 = \frac{20}{7} - \frac{1}{7}x_1 - \frac{1}{7}y_1 - \frac{2}{7}y_3$

Goal: $z = \frac{142}{7} + \frac{2}{7}x_1 - \frac{12}{7}y_1 - \frac{3}{7}y_3$

Temporary solution:

$x_1 = y_1 = y_3 = 0, x_2 = \frac{5}{7}, y_2 = \frac{41}{7}, x_3 = \frac{20}{7}, z = \frac{142}{7}$

Choose x_1 to increase:

- $x_2 = \frac{5}{7} - \frac{2}{7}x_1 \geq 0$ only OK if $x_1 \leq \frac{5}{2}$
- $y_2 = \frac{41}{7} - \frac{8}{7}x_1 \geq 0$ only OK if $x_1 \leq \frac{41}{8}$
- $x_3 = \frac{20}{7} - \frac{1}{7}x_1 \geq 0$ only OK if $x_1 \leq 20$

Pivot: swap x_2, x_1

- $\frac{2}{7}x_1 = \frac{5}{7} - x_2 - \frac{2}{7}y_1 + \frac{3}{7}y_3 \Rightarrow x_1 = \frac{5}{2} - \frac{7}{2}x_2 - y_1 + \frac{3}{2}y_3$

Slack form:

- $x_1 = \frac{5}{2} - \frac{7}{2}x_2 - y_1 + \frac{3}{2}y_3$

- $y_2 = 3 + 4x_2 + y_1 - 2y_3$
- $x_3 = \frac{5}{2} + \frac{1}{2}x_2 - \frac{1}{2}y_3$

Goal: $z = 21 - x_2 - 2y_1$

Temporary solution:

$x_2 = y_1 = y_3 = 0, x_1 = \frac{5}{2}, y_2 = 3, x_3 = \frac{5}{2}$

Pivot ends, $z \leq 21 \Rightarrow \max(z) = 21$ where $x_2 = y_1 = y_3 = 0, x_1 = \frac{5}{2}, y_2 = 3, x_3 = \frac{5}{2}$.

2. Fine values $x, y \geq 0$ satisfying

$$\begin{aligned} x - y &\leq -3 \\ 2x + y &\leq 7 \\ -x - 2y &\leq -9 \end{aligned} \tag{2}$$

Solution

Extend the set of inequalities to

- $x - y - z \leq -3$
- $2x + y - z \leq 7$
- $-x - 2y - z \leq -9$

Slack form:

Maximize $-z$ satisfying

- $y_1 = -3 - x + y + z \geq 0$
- $y_2 = 7 - 2x - y + z \geq 0$
- $y_3 = -9 + x + 2y + z \geq 0$

Basic Solution:

$x = y = y_3 = 0, y_1 = 6, y_2 = 16, z = 9$

Pivot: swap z, y_3

- $z = 9 - x - 2y + y_3$

Slack form:

- $y_1 = 6 - 2x - y + y_3$
- $y_2 = 16 - 3x - 3y + y_3$

- $z = 9 - x - 2y + y_3$

Goal: Maximize $-9 + x + 2y - y_3$

From now on: proceed by simplex algorithm as before.

Pivot: swap x, y_1

- $x = 3 - \frac{1}{2}y_1 - \frac{1}{2}y + \frac{1}{2}y_3$

Slack form:

- $x = 3 - \frac{1}{2}y_1 - \frac{1}{2}y + \frac{1}{2}y_3$
- $y_2 = 7 + \frac{3}{2}y_1 - \frac{3}{2}y - \frac{1}{2}y_3$
- $z = 6 + \frac{1}{2}y_1 - \frac{3}{2}y + \frac{1}{2}y_3$

Goal: Maximize $-6 - \frac{1}{2}y_1 + \frac{3}{2}y - \frac{1}{2}y_3$

Pivot: swap y, z

- $y = 4 + \frac{1}{3}y_1 - \frac{2}{3}z + \frac{1}{3}y_3$

Slack form:

- $x = 1 - \frac{2}{3}y_1 + \frac{1}{3}z + \frac{1}{3}y_3$
- $y_2 = 1 + y_1 + z - y_3$
- $y = 4 + \frac{1}{3}y_1 - \frac{2}{3}z + \frac{1}{3}y_3$

Goal: Maximize 0

Resulting basic solution:

$y_1 = z = y_3 = 0, x = y_2 = 1, y = 4$

yields the optimal value $-z = 0$

The original set of inequalities is satisfiable: $x = 1, y = 4$

Construct a formula in CNF based on each of the following truth tables:

p	q	r	ϕ
1	1	1	1
1	1	0	0
1	0	1	0
0	1	1	1
1	0	0	0
0	1	0	0
0	0	1	1
0	0	0	0

(3)

Solution

$(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r)$

Apply algorithm HORN to each of these Horn formulas:

1. $(p \wedge q \wedge s \rightarrow \perp) \wedge (q \wedge r \rightarrow p) \wedge (\top \rightarrow s)$

Marked: $\top$ s return 'satisfiable'

2. $(p_5 \rightarrow p_{11}) \wedge (p_2 \wedge p_3 \wedge p_5 \rightarrow p_{13}) \wedge (\top \rightarrow p_5) \wedge (p_5 \wedge p_{11} \rightarrow \perp)$

Marked: $\top$ p_5 p_{11} $\perp$ return 'unsatisfiable'

3. $(\top \rightarrow q) \wedge (\top \rightarrow s) \wedge (w \rightarrow \perp) \wedge (p \wedge q \wedge s \rightarrow \perp) \wedge (v \rightarrow s) \wedge (\top \rightarrow s) \wedge (r \rightarrow p)$

Marked: $\top$ q s return 'satisfiable'