概统作业（第九次）

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Ex.1

$$P(V \leqslant v) = P(X - Y \leqslant v)$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\, I[x-y \leqslant v]\, dy\, dx$$

$$= \int_{-\infty}^{+\infty} \int_{x-v}^{+\infty} f(x,y)\, dy\, dx$$

对 v 求导，得 $f_V(v) = \int_{-\infty}^{+\infty} f(x, x-v)\, dx$

X、Y 独立时，$f(x, x-v) = f_X(x) f_Y(x-v)$

故 $f_V(v) = \int_{-\infty}^{+\infty} f_X(x) f_Y(x-v)\, dx$

Ex.2.

$X = \frac{U+V}{2}$，$Y = \frac{U-V}{2}$.

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| = \left|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix}\right| = \frac{1}{2}$$

故 $dxdy = J\,dudv = \frac{1}{2}dudv$

故 $P(U=u, V=v) = P(X+Y=u, X-Y=v)$

$$= P\left(X=\frac{u+v}{2}, Y=\frac{u-v}{2}\right)$$

$$= f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) dxdy$$

$$= \frac{1}{2} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) dudv$$

故 $f_{(U,V)}(u,v) = \frac{1}{2} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$.

Ex 3. $f_{X_1}(x_1)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left[-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right]$

$f_{X_2}(x_2)=\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left[-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right]$

$Z_1=aX_1$，$Z_2=bX_2$，$Z_3=c$.

$F_{Z_1}(z_1)=P(Z_1\leq z_1)=P(aX_1\leq z_1)=P(X_1\leq \frac{z_1}{a})$

$=\int_{-\infty}^{\frac{z_1}{a}} f_{X_1}(x_1)dx_1$.

对z_1求导，$f_{Z_1}(z_1)=\frac{1}{a\sqrt{2\pi}\sigma_1}\exp\left[-\frac{(\frac{z_1}{a}-\mu_1)^2}{2\sigma_1^2}\right]$

同理 $f_{Z_2}(z_2)=\frac{1}{b\sqrt{2\pi}\sigma_2}\exp\left[-\frac{(\frac{z_2}{b}-\mu_2)^2}{2\sigma_2^2}\right]$

即 $Z_1\sim N(a\mu_1, a^2\sigma_1^2)$，$Z_2\sim N(b\mu_2, b^2\sigma_2^2)$

由"X+Y"的卷积公式，设 $Z=Z_1+Z_2$.

$$l(z)=\frac{1}{2\pi ab\sigma_1\sigma_2}\int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2}\left(\frac{(z_1-a\mu_1)^2}{a^2\sigma_1^2}+\frac{(z-z_1-b\mu_2)^2}{b^2\sigma_2^2}\right)\right]dz_1$$

而 $\frac{(z_1-a\mu_1)^2}{a^2\sigma_1^2}+\frac{(z-z_1-b\mu_2)^2}{b^2\sigma_2^2}=\frac{(z-a\mu_1-b\mu_2)^2}{a^2\sigma_1^2+b^2\sigma_2^2}+(kz_1-m)^2$,

这里 $k=\frac{\sqrt{a^2\sigma_1^2+b^2\sigma_2^2}}{ab\sigma_1\sigma_2}$，$m=\frac{ab\sigma_1\sigma_2}{\sqrt{a^2\sigma_1^2+b^2\sigma_2^2}}\left[\frac{\mu_1}{a\sigma_1^2}+\frac{(z-b\mu_2)}{b^2\sigma_2^2}\right]$

(展开即证)

故 $l(z)=\frac{1}{2\pi ab\sigma_1\sigma_2}\exp\left[-\frac{(z-a\mu_1-b\mu_2)^2}{a^2\sigma_1^2+b^2\sigma_2^2}\right]\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(kz_1-m)^2}dz_1$

令 $t=kz_1-m$. $\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(kz_1-m)^2}dz_1=\int_{-\infty}^{+\infty}e^{-\frac{1}{2}t^2}dt/k=\frac{\sqrt{2\pi}}{k}$.

$$故\ \ell(z)=\frac{1}{2\pi\cancel{ab\sigma_1\sigma_2}}\cdot\sqrt{2\pi}\cdot\frac{\cancel{ab\sigma_1\sigma_2}}{\sqrt{a^2\sigma_1^2+b^2\sigma_2^2}}\exp\left[-\frac{(z-a\mu_1-b\mu_2)^2}{2(a^2\sigma_1^2+b^2\sigma_2^2)}\right]$$

$$=\frac{1}{\sqrt{2\pi}\sqrt{a^2\sigma_1^2+b^2\sigma_2^2}}\exp\left[-\frac{(z-a\mu_1-b\mu_2)^2}{2(a^2\sigma_1^2+b^2\sigma_2^2)}\right]$$

故 $Z\sim N(a\mu_1+b\mu_2,\ a^2\sigma_1^2+b^2\sigma_2^2)$.

令 $Y=Z+c$. 则 $f_Y(y)=\ell(y-c)$

$$=\frac{1}{\sqrt{2\pi}\sqrt{a^2\sigma_1^2+b^2\sigma_2^2}}\exp\left[-\frac{(y-c-a\mu_1-b\mu_2)^2}{2(a^2\sigma_1^2+b^2\sigma_2^2)}\right]$$

故 $Y\sim N(a\mu_1+b\mu_2+c,\ a^2\sigma_1^2+b^2\sigma_2^2)$.