Lecture on Digital Signal Processing

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Only for students of TH Lübeck

Prof. Dr. Lothar Vogt

Contents

1 Introduction

Digital Signal Processing today is one of the fundamental subjects in Electrical Engineering. Since the first text book was published back in the 70-ties and the first mass product was made (Compact Disc) digital signal processing had an enormous development and today all communication is based on digital signal processing. The technical developments in high speed digital signal processors [1] brought this topic to success in all modern communication systems, multimedia application as well as in digital image processing and many more. The great advantage of digital signal processing is that it makes use of all things known from mathematics. Especially numerical analysis is applicable.

In most cases it is not a good idea to try to transform systems known from analog processing into digital signal processing since the inherent advantages of digital signal processing will not be used efficiently.
Many things now can be done that were not feasible before. Some examples are:
-complete new transmission standards (OFDM, COFDM...),
-coding with error correction capabilities
-data encryption,
-data compression (MP3, AAC, JPEG...),
and many more.

Extensions led to 2-dimensional digital signal processing:
- image processing,
- object detection in digital images,
- computer tomography with digital image post- processing.

Further applications are in adaptive digital signal processing
- adaptive noise cancelling,
- speech synthesis.

Just to mention a few areas. Therefore it is very important that students get familiar with the conceptual thinking of digital signal processing. Here we focus on basic principles for one dimensional theory (such as for audio signals). Some well-known textbooks are: [10][14][11][12][7] There are also collections of exercises found in: [4][6] It is strongly recommended to do simulations with Matlab, SCILAB or OCTAVE. As a student of TH Lübeck you can use our campus license for MATLAB. These tools are today state-of-the art in any development but they also are very helpful in understanding and learning the subject, see e.g. [1]

[1] digital signal processors are optimized for calculations like mac(multiply / accumulate), not to be mixed with μC. Some examples are: TMS320 family from Texas Instrument or Blackfin from Analog Devices or dedicated processors for graphics (Nvidia, Fuji...)

2 Digital Signals

We will denote digital signals as sequences

$$x_n\,, \qquad (n \in \mathbb{Z})\,. \tag{2.0.1}$$

Let T denote a short time. Then we could generate a sequence by taking samples of an analog signal $x(t)$ at equally spaced times T, called the sampling period:

$$x_n = x(nT) \tag{2.0.2}$$

This notation is mathematically not perfect since we denote x_n as a sequence and at the same time it is the sample at a certain point in time nT which is not the same! More precise it should be written for the sequence

$$\{x_n\}_{n\in\mathbb{Z}}$$

T is measured in seconds $[s]$ - or fractions of it depending on the particular application. The inverse of T is as usual a frequency. We shall denote it throughout this text as $1/T = f_s$. The frequency is measured in Hertz $[Hz]$.
These sequences may be directly generated in a digital system (computer) or as sampled values of an continuous time signal $x(t)$. The process of digitisation consists of two steps:

1. take samples at points equally spaced in time. This will generate a discrete time signal $x_n := x(nT)$. This step is called **sampling**. It is despicted in 2.1

2. **quantize** these values according to some quantisation steps, i.e. depending on the number of bits used. Typical values are 16, 32 or 48 bits. This step is called **quantisation**, see 2.2

This is shown in 2.2 . Assume we have N bits for representation. Then we have 2^N different values available. So any signal value in between must go through a certain truncation or rounding scheme. In this lecture we will mainly focus on the sampling process, its consequences and how digital signals can be treated. Nevertheless it must be mentioned that quantisation is a very important thing to be considered in practice since the effect of cumulative rounding errors may be not tolerable for some applications.
It is obvious, that taking only samples and omitting all signal values in between is a very sever process and it must be investigated what the impact of this process is and if the samples still represent information contained in the signal. This will be thoroughly study later and will lead to the sampling theorem.

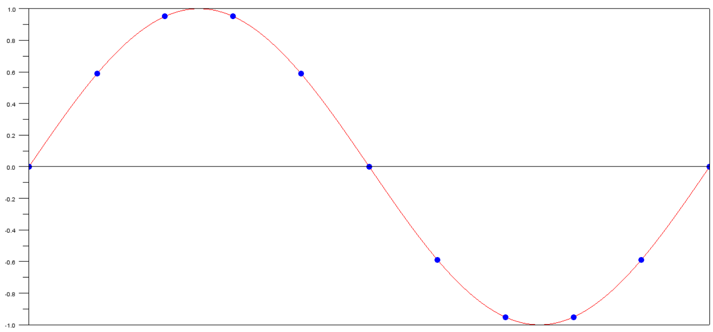

Figure 2.1: Sampling

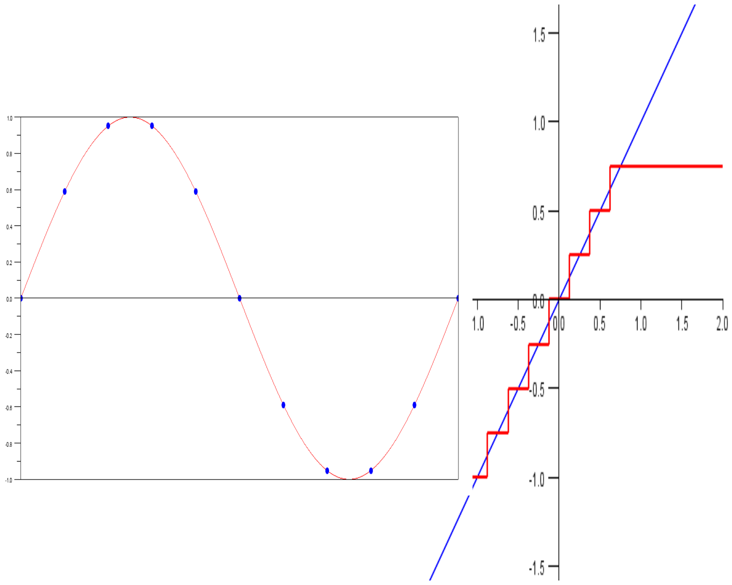

Figure 2.2: Quantisation

2.1 Special Signals

1. **Impulse Sequence**:

$$\delta_n = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0 \end{cases} . \tag{2.1.3}$$

2. The **Impuls Train** is defined as:

$$I_n = \sum_{k=-\infty}^{\infty} \delta_{n-k}. \tag{2.1.4}$$

3. **Unit Step Sequence**:

$$u_n = \begin{cases} 1 & n \geq 0. \\ 0 & n < 0 \end{cases} . \tag{2.1.5}$$

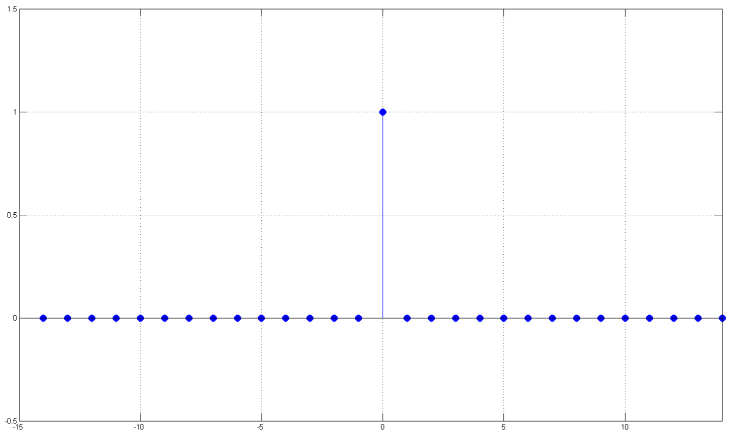

Figure 2.3: Delta Impulse

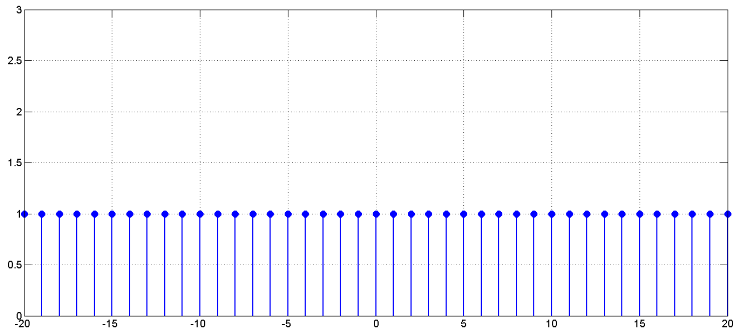

Figure 2.4: Impulse Train

It follows immediately that:

$$u_n - u_{n-N} = \delta_0, \delta_1, \ldots, \delta_{n-N-1} \tag{2.1.6}$$

4. **Exponential Sequence**: Let $a = |a|e^{j\phi} \quad \in \mathbb{C}$. Then

$$x_n = a^n = |a|^n \mathrm{e}^{j\phi n} \tag{2.1.7}$$

is called an exponential sequence. For $|a| > 1$ the sequence exceeds any limit. For $|a| = 1$ the sequence is bounded, if $|a| < 1$ it tends to 0 as n goes towards ∞

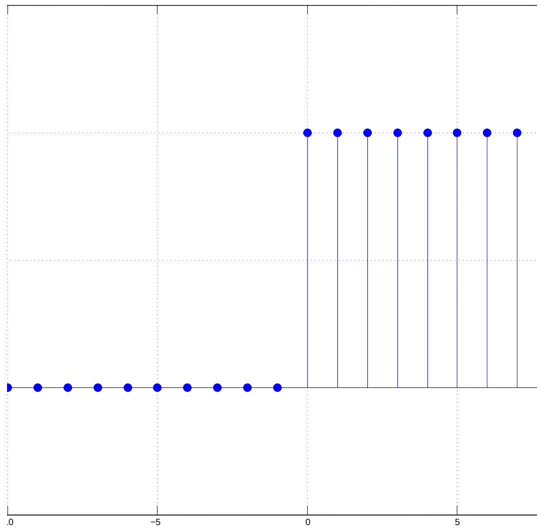

Figure 2.5: Unit step sequence

2.2 Basic Signal Properties

Definition 2.1
A digital signal is called

1. ***even*** $\Leftrightarrow x_{-n} = x_n$,
2. ***odd*** $\Leftrightarrow x_{-n} = -x_n$,
3. ***periodic with period*** $N \in \mathbb{N} \Leftrightarrow x_{n+N} = x_n$.
4. *A signal is said to be* ***bounded*** $\Leftrightarrow \exists K \in \mathbb{R}$ *so there holds :*

$$|x_n| \leq K \qquad \forall n \in \mathbb{Z}. \tag{2.2.8}$$

5. *A signal is an* ***Energy Signal*** $\Leftrightarrow$

$$E_x := \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty. \tag{2.2.9}$$

6. *A signal is a* ***Power Signal*** $\Leftrightarrow$

$$P_x := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_n|^2 < \infty. \tag{2.2.10}$$

2.3 Correlation and Auto-correlation

For two time discrete signals the correlation is defined:

Definition 2.2
Correlation
Let x_n and y_n be two sequences. Then the correlation is defined by:

$$r_{xy}(n) = \sum_{k=-\infty}^{\infty} \overline{x_k}\, y_{n+k}, \tag{2.3.11}$$

and the auto- correlation as:

$$r_{xx}(n) = \sum_{k=-\infty}^{\infty} \overline{x_k}\, x_{n+k}. \tag{2.3.12}$$

Correlation between two signals is a measure of similarity between the samples of the two signals delayed by the time nT with sampling period T.
With (2.3.12) the energy of a signal (if it exists) is:

$$E_x = r_{xx}(0) \tag{2.3.13}$$

3 Systems

We shall now look at operations we can do with digital signals. To this end we define as an operator $\mathcal{H}$ a mapping from one signal x_n (input) to another signal y_n (output) as:

$$y_n = \mathcal{H}(x_n) \qquad \mathcal{H} : x_n \to y_n \tag{3.0.1}$$

Note: $\mathcal{H}$ is not a function, since it maps a sequence to a sequence. Such a thing is called operator. Nevertheless for the purpose here it is sufficient to think of it as if it is a function.

3.1 LTI Systems

Definition 3.1
Linear System
If there holds with some $\alpha, \beta \in \mathbb{C}$:

$$\mathcal{H}(\alpha x_n + \beta y_n) = \alpha\mathcal{H}(x_n) + \beta\mathcal{H}(y_n). \tag{3.1.2}$$

then $\mathcal{H}$ is said to be linear.

Next an assertion is made concerning the behaviour over time.

Definition 3.2
Time Imvariance
A system is called time invariant

$$\Leftrightarrow$$

$$(y_n = \mathcal{H}(x_n) \;\Rightarrow \mathcal{H}(x_{n-k}) = y_{n-k})\,. \tag{3.1.3}$$

Combining these two properties then leads to:

Definition 3.3
Linear Time Imvariant(LTI)
A system $\mathcal{H}$ is called a LTI system $\Leftrightarrow$ it is ***linear(L) and time invariant(TI)****.*

The assumption for a system to be a LTI system is sufficient to analyse and describe the system behaviour completely. Note: in some other disciplines like adaptive systems time- invariant is not applicable but in most cases the time variations are kept small so that the LTI theory can be used in many cases as a good approximation.
The output of a LTI system when the input is the delta impulse signal is:

Definition 3.4
Impulse Response
The impulse response is given by

$$h_n = \mathcal{H}(\delta_n). \tag{3.1.4}$$

The impulse response is sufficient to determine the output of a LTI system for an arbitrary input x_n as will be shown now (note that the operator $\mathcal{H}$ only operates on n):

$$\begin{aligned}\mathcal{H}(x_n) = \mathcal{H}(\sum_{k=-\infty}^{\infty} x_k\delta_{n-k}) \\ &= \sum_{k=-\infty}^{\infty} \mathcal{H}(x_k\delta_{n-k}) \\ &= \sum_{k=-\infty}^{\infty} x_k\mathcal{H}(\delta_{n-k}) \\ &= \sum_{k=-\infty}^{\infty} x_k h_{n-k}.\end{aligned} \tag{3.1.5}$$

Use was first made of the linearity and then of the time-invariance. This is the relationship that shows how the output to an arbitrary input can be calculated.

Definition 3.5
(time-discrete) Convolution
For any two time- discrete signals x_n and y_n the convolution is defined as

$$(x_n * y_n) = \sum_{k=-\infty}^{\infty} x_k y_{n-k}. \tag{3.1.6}$$

Corollary 3.1
Commutativity of the convolution
For any two time- discrete signals x_n and y_n there holds

$$\sum_{k=-\infty}^{\infty} x_k y_{n-k} = \sum_{k=-\infty}^{\infty} x_{n-k} y_k\,. \tag{3.1.7}$$

Note:
The convolution is an operation on two sequences. Instead of writing $(x * y)_n$ which is correct but looks a little strange often one sees the writing $x_n * y_n$. Here we may use both writings.

Corollary 3.2
The correlation (2.3.11) *sequence may be expressed by a convolution:*

$$r(n) = \bar{x}_{-n} * y_n . \tag{3.1.8}$$

We summarize these results in

Theorem 3.1
Let $y_k = \mathcal{H}(x_k)$ *and* h_k *be the impulse response of a LTI System* $\mathcal{H}$*. Then*

$$y_k = \sum_{k=-\infty}^{\infty} x_k h_{n-k} = \sum_{k=-\infty}^{\infty} x_{n-k} h_k \tag{3.1.9}$$

3.2 Causal Systems

In general a LTI system will have a impulse response h_n for all values $n \in \mathbb{Z}$. Therefore the convolution sum of the input with the impulse response will be re-written:

$$\begin{aligned} y_n = \sum_{k=-\infty}^{\infty} x_{n-k} h_k &= \sum_{k=-\infty}^{-1} x_{n-k} h_k + \sum_{k=0}^{\infty} x_{n-k} h_k \\ &= \sum_{k=1}^{\infty} x_{n+k} h_{-k} + \sum_{k=0}^{\infty} x_{n-k} h_k \end{aligned} \tag{3.2.10}$$

where the index was substituted from k to $-k$ in the first sum. This now shows that in the first sum values of the input with higher index than n would be needed. This means that such a system would require samples being in the future and in the past looking from the point n. This is not feasible. LTI systems that don't require future samples are defined in

Definition 3.6
Causal Systems
Let h_n *be the impulse response of a LTI system* $\mathcal{H}$*. The system is said to be causal iff* $\forall\, n \leq 0$ *there holds*

$$h_n = 0 . \tag{3.2.11}$$

3.3 Stability

A very important property is that of **stability**.

Definition 3.7
BIBO Stability
*A system $\mathcal{H}$ is called bounded input- bounded output stable (**BIBO**) $\Leftrightarrow$*

$$|x_n| < K \qquad \Rightarrow \qquad |\mathcal{H}(x_n)| < M \qquad \forall\, n \in \mathbb{Z}\,. \tag{3.3.12}$$

with some constants $K\,, M \in \mathbb{Z}$.

In words: a bounded input returns a bounded output of the system. **Note: there are also other definitions on stability that cover different aspects (e.g. studying the behaviour of systems under certain initial condition. These address more the dynamical behaviour as in control theory)**. A first criteria to check BIBO- stability is given by

Theorem 3.2
BIBO Stability: (Criteria in time domain)
A system $\mathcal{H}$ is BIBO stable iff for its impulse response there holds:

$$\sum_{k=-\infty}^{\infty} |h_k| < \infty. \tag{3.3.13}$$

3.4 Frequency Response

Now let an exponential signal of the kind

$$x_n = e^{j2\pi fnT}$$

be the input to a LTI system. It is usual to define $f_n = fT = \frac{f}{f_s}$ as the **normalized frequency** and $\Omega = 2\pi f_n$ as the **normalized angular frequency**. Thus the sequence can be simply written as:

$$x_n = e^{jn\Omega}$$

. Now using 3.1.5 one gets:

$$y_n = \sum_{k=-\infty}^{\infty} h_k e^{j\frac{2\pi f}{f_s}(n-k)} = e^{j\frac{2\pi f}{f_s}n} \sum_{k=-\infty}^{\infty} h_k e^{-jk\frac{2\pi f}{f_s}}\,. \tag{3.4.14}$$

This unveils that the chosen signal sequence will be seen at the output but changed by the (*complex and frequency dependent!*) factor $\sum\limits_{k=-\infty}^{\infty} h_k e^{-jk\frac{2\pi f}{f_s}}$. This factor contains as characteristics of the system it impulse response!

Definition 3.8
Frequency Response
The frequency response of a LTI system $\mathcal{H}$ is defined as

$$H(f) = \sum_{k=-\infty}^{\infty} h_k e^{-jk\frac{2\pi f}{f_s}}\,. \tag{3.4.15}$$

3.5 Amplitude- and Phase Response, Group delay

Thus the system function can be written as:

$$H(f) = |H(f)|e^{j\varphi(f)}. \tag{3.5.16}$$

Definition 3.9
Amplitude- and Phase Response

1. *$|H(f)|$ is called the **Amplitude Response**,*
2. *$\varphi(f)$ is called **Phase Response**.*

The amplitude shows how much a input with a certain frequency f is amplified or attenuated. The output is then:

$$y_n = e^{jn\frac{2\pi f}{f_s}}|H(f)|e^{j\varphi(f)} = |H(f)|e^{j2\pi f(nT+\frac{\varphi(\Omega)}{2\pi f})}. \tag{3.5.17}$$

Note that $\frac{\varphi(\Omega)}{2\pi f}$ has the physical dimension of a time. Thus the result means, that the signal is *shifted in time* by exactly this amount. This gives the explanation for

Definition 3.10
Phase Delay
The phase delay of a system is given by

$$\tau_p := -\frac{\varphi(f)}{2\pi f}. \tag{3.5.18}$$

Note: the notation for the frequency response is sometimes different in the literature: Some use as variable the normalized angular frequency $\Omega = 2\pi\frac{f}{f_s}$ and thus name it $H(\Omega)$ even others use $H(j\Omega)$ or $H(e^{jk\Omega})$ can be found. These are all not mathematically and physically very meaningful, since the only true variable is the frequency f. All others are substitutions for maybe easier writing. Thus throughout this text the frequency f will be used as variable. Matlab® uses the normalized angular frequencies. Both the amplitude and phase response are plotted in a *Bode diagram* against the frequency. In discrete time processing it is common to plot against the normalised frequency $\frac{f}{f_s}$, Matlab plots against the normalised angular frequency. Investigating the effect of time delay caused by non linear phase response to a wave package one gets the definition of group delay.

Definition 3.11
Group Delay
Let $\varphi(f)$ be the phase response (3.5.16)*. Then the group delay is defined as:*

$$\tau_g(f) = -\frac{1}{2\pi}\frac{\mathrm{d}\varphi(f)}{\mathrm{d}f} = -\frac{1}{2\pi}\varphi'(f). \tag{3.5.19}$$

Thus one can approximate the phase response near a frequency f_0 as:

$$\varphi(f_0) + \varphi'(f_0)(f - f_0) \tag{3.5.20}$$

Or inserting the phase delay and group delay one gets:

$$\varphi(f) \approx -2\pi f_0 \tau_p(f_0) - 2\pi(f - f_0)\tau_g(f_0) \tag{3.5.21}$$

Example 3.1
Digital Differentiator
Let a System $\mathcal{H}$ be given by

$$y_k = x_k - x_{k-1} \quad \Leftrightarrow \quad h_0 = 1\,,\ h_1 = -1\,,\ h_k = 0\ \ (k \neq 0, 1)\,. \tag{3.5.22}$$

Then with 3.4.15 and setting for easier reading the normalised angular frequency $\Omega = 2\pi \frac{f}{f_s}$):

$$\begin{aligned} H(\Omega) &= \left(1 - e^{-j\Omega}\right) \\ &= e^{-j\Omega/2}\left(e^{j\Omega/2} - e^{-j\Omega/2}\right) \\ &= 2je^{-j\Omega/2}\sin(\frac{\Omega}{2}) \\ \\ &= 2\sin(\frac{\Omega}{2})e^{j\left(-\frac{\Omega}{2}+\frac{\pi}{2}\right)} \end{aligned} \tag{3.5.23}$$

Thus we get:

$$\begin{aligned} \textit{Amplitude Response: } |H(f)| &= 2|\sin\left(\pi\frac{f}{f_s}\right)| \\ \textit{Phase Response: } \varphi(f) &= -\pi\frac{f}{f_s} + \frac{\pi}{2} \\ \textit{Phase Delay: } \tau_p &= \frac{1}{2f_s} - \frac{1}{4f} \\ \textit{Group delay: } \tau_g &= \frac{1}{2f_s} \end{aligned} \tag{3.5.24}$$

The Amplitude and Phase Response are shown in 3.1 for $0 \le f \le \frac{f_s}{2}$.

The more common way of presenting the Amplitude Response is in Dezibels [dB]. This is defined as $20\log(|H(f)|)$.

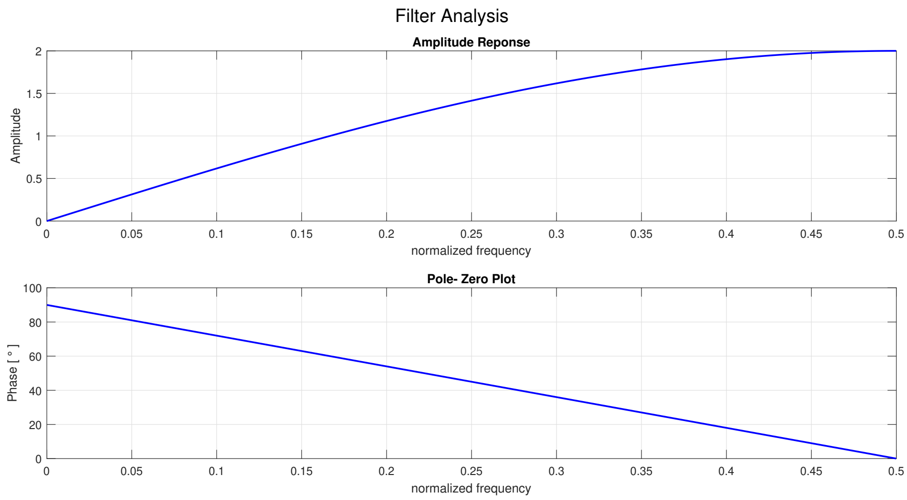

Figure 3.1: Amplitude and Phase Response

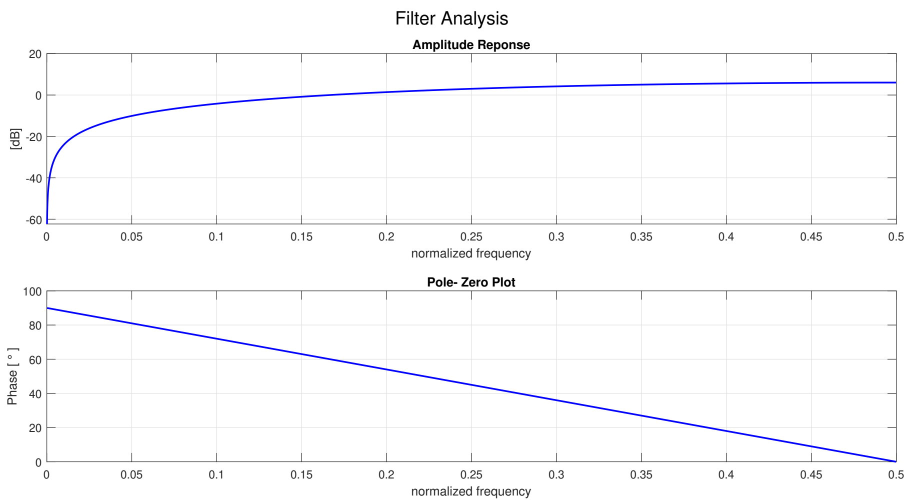

Figure 3.2: Example with Amplitude Response in dezibels

4 z- Transform

In analog signal processing the Laplace Transform was useful for further analysis of LTI systems. There is an equivalent for discrete time signals/sequences.

Definition 4.1
z- Transform

1. *For a sequence $\{x_n\}_{n\in\mathbb{Z}}$ the z- Transform is defined as*

$$X(z) = \sum_{k=-\infty}^{\infty} x_k z^{-k} \qquad z \in \mathbb{C} \tag{4.0.1}$$

2. *The region in $\mathbb{C}$ for which 4.0.1 converges is called region of convergence (ROC)*

Notation:

$$x_n \quad \underset{z}{\circ\!\!-\!\!\bullet} \quad X(z)$$

or

$$x_n \quad \leftrightarrow \quad X(z)$$

The z- transform has some important properties that are collected in

Corollary 4.1
Some Properties of the z-transform
Let $X(z)$ be the z-transform of a sequence $\{x_n\}_{n\in\mathbb{Z}}$. Then there holds:

1. ***(Time shifting)***

$$x_{n-i} \Rightarrow z^{-i}X(z), \tag{4.0.2}$$

2. ***(Convolution)*** *Let y_n be a second sequence then*

$$z_n = (x * y)_n \Rightarrow Z(z) = X(z)Y(z) \tag{4.0.3}$$

3. ***conjugation***

$$\bar{x_n} \underset{z}{\circ\!\!-\!\!\bullet} \bar{X}(\bar{z}) \tag{4.0.4}$$

4. ***time reversal***

$$x_{-n} \underset{z}{\circ\!\!-\!\!\bullet} X(z^{-1}) \tag{4.0.5}$$

5. ***differentiation in the z-domain***

$$nx_n \underset{z}{\circ\!\!-\!\!\bullet} -z\frac{\mathrm{d}}{\mathrm{d}z}X(z) \tag{4.0.6}$$

The proof is easily done by straight forward use of the definition of the z- transform and therefore left to the reader.

Definition 4.2
System Function
Let h_n be the impulse response of a LTI system. The system function $H(z)$ is given as the z-transform of the impulse response, i.e.

$$H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}. \tag{4.0.7}$$

Now applying the z-transform to the convolution of the impulse response with the input signal (3.1.5) one gets the

Corollary 4.2

$$Y(z) = H(z)X(z) \Rightarrow \; H(z) = \frac{Y(z)}{X(z)} \tag{4.0.8}$$

Example 4.1
Let $x_n = \delta_n$ then the z-transform is:

$$X(z) = \sum_{k=-\infty}^{\infty} \delta_k z^{-k} = 1 \tag{4.0.9}$$

since the delta impulse sequence is 1 for $k \neq 0$.

$$\delta_n \quad \underset{z}{\circ\!\!-\!\!\bullet} \quad 1 \quad \forall z \in \mathbb{C}$$

Example 4.2
Determine the z-transform of the exponential sequence $x_n = a^n u_n$. Using the definition and noting that $u_n = 0$ for $n < 0$ one gets:

$$\begin{aligned} X(z) &= \sum_{k=-\infty}^{\infty} a^k u_k z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k} \\ &= \sum_{k=-\infty}^{\infty} a^k u_k z^{-k} = \sum_{k=0}^{\infty} (az)^{-k} = \frac{1}{1 - az^{-1}} \\ &= \frac{z}{z-a} \text{ for } |az^{-1}| < 1 \end{aligned} \tag{4.0.10}$$

Thus we get

$$a^n u_n \quad \underset{z}{\circ\!\!-\!\!\bullet} \quad \frac{z}{z-a} \quad \text{ROC: } |a| < |z| \tag{4.0.11}$$

Example 4.3

Determine the z-transform of the exponential sequence $x_n = -a^n u_{-n-1}$. Using the definition and noting that $u_n = 0$ for $n \geq 0$ one gets:

$$\begin{aligned} X(z) &= -\sum_{k=-\infty}^{\infty} a^k u_{-k-1} z^{-k} = -\sum_{k=1}^{\infty} a^k z^{-k} \\ &= 1 - \sum_{k=0}^{\infty} a^{-k} z^k = 1 - \sum_{k=0}^{\infty} (a^{-1}z)^k = 1 - \frac{1}{1 - a^{-1}z} \\ &= \frac{z}{z-a} \ \textit{for}\ |a^{-1}z| < 1 \end{aligned} \tag{4.0.12}$$

Thus we now get

$$-a^n u_{-n-1} \quad \underset{z}{\circ\!\!-\!\!\bullet} \quad \frac{z}{z-a} \quad \textit{ROC: } |z| < |a| \tag{4.0.13}$$

In examples 4.2 and 4 the z-transform looks the same but it is not! Note the difference in ROC: in the last example its the inner of a circle with radius $|a|$ whereas before it was the outside of the circle with radius $|a|$.

Example 4.4

Determine the z-transform of the sequence

$$x_n = \frac{1}{5}^n u_n + \frac{1}{2}^n u_{-n-1} \tag{4.0.14}$$

This is called a two sided sequence, since it has values for negative and positive n. According to the previous examples the two sequences can be transformed as:

$$\frac{1}{5}^n u_n \ \underset{z}{\circ\!\!-\!\!\bullet} \ \frac{z}{z-\frac{1}{5}} \qquad \textit{ROC: } \frac{1}{5} < |z| \tag{4.0.15}$$

and

$$\frac{1}{2}^n u_{-n-1} \underset{z}{\circ\!\!-\!\!\bullet} \ -\frac{z}{z-\frac{1}{2}} \qquad \textit{ROC: } \frac{1}{2} > |z| \tag{4.0.16}$$

Thus combing the two results, one finally gets:

$$X(z) = \frac{z}{z-\frac{1}{5}} - \frac{z}{z-\frac{1}{2}}, \qquad \textit{ROC: } \frac{1}{5} < |z| < \frac{1}{2}. \tag{4.0.17}$$

In this case the regions of convergence f the two parts overlap and thus the ROC is a ring with inner radius $\frac{1}{5}$ and outer radius $\frac{1}{2}$ see 4.1. Note: this example is a two-sided sequence and thus is not causal.

In both last examples the z-transform looks the same but it is not! Note the difference in ROC: in the last example its the inner of a circle with radius $|a|$ whereas before it was the outside of the circle with radius $|a|$.

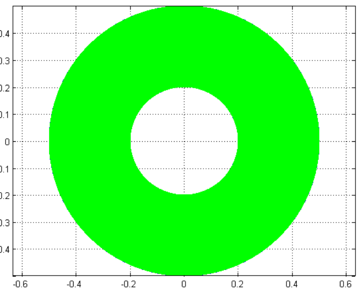

Figure 4.1: ROC for two sided sequence

4.1 Calculating the inverse z-Transform

In this section a method based algebraic calculations is introduced to determine the inverse of a given function $X(z)$. Use is made of the properties of the z-transform as well as of look-up tables of basic know transform pairs.

1. Any power of z i.e. z^i is identified with the equivalent time shift.
2. Use linearity:

$$x_n + y_n \quad \underset{z}{\circ\!\!-\!\!\bullet} \quad X(z) + Y(z) \quad \mathrm{ROC} = \mathrm{ROC}_x \cap \mathrm{ROC}_y \tag{4.1.18}$$

3. Use of known power series expansions. For example geometric series.
4. For rational functions: long division to reduce to a polynomial part and a proper rational function if necessary and then perform partial fractional expansion.

Example 4.5
For $a \in \mathbf{C}$ let

$$H(z) = \frac{z^3}{z-a} \quad |z| > |a|$$

1. *Instead of starting with $H(z)$ one better starts with (if the rational function is given in z, if it is given in z^{-1} this step can be omitted).*

$$\frac{H(z)}{z} = \frac{z^2}{z-a}$$

2. *Check if degree of nominator polynomial is great or equal degree of denominator polynomial. If so, perform long division. This is the case here.*

$$\frac{z^2}{z-a} = z + a + \frac{a^2}{z-a} \quad \Rightarrow \quad H(z) = z^2 + az + a^2 \frac{z}{z-a}$$

3. *Use known transform pairs and properties of z-transform. Here one uses for ROC : $|z| > |a|$:*

$$h_n = \delta_{n+2} + a\delta + a^{n+2} u_n$$

4.2 Stability Criteria in the z-Domain

Suppose $H(z)$ is the system function of a stable LTI system with impulse response Then it follows:

$$\sum_{k=-\infty}^{\infty} |h_k||e^{j2\pi \frac{f}{fs}}| = \sum_{k=-\infty}^{\infty} |h_k| \quad < \infty \tag{4.2.19}$$

From a well-know theorem it follows then that the series

$$\sum_{k=-\infty}^{\infty} h_k z^{-k} \qquad \text{converges for } z \text{ with} |z| = 1 \tag{4.2.20}$$

which means that the unit circle $|z| = 1$ must be contained in the ROC. [1] The requirement for a system to have a z- transform and stability are two different things. Stability requires absolute convergence of the z- transform on the unit circle, existence of the z-transform only requires the corresponding series to converge. Thus stability is a stronger condition.

Conclusion: Applying the above to the examples of exponential sequences one concludes (see 4.2 4.3)

[1] an absolutely convergent series is conditionally convergent

1. $$x_n = a^n u_n \quad \circ\!\!-\!\!\bullet \quad \frac{z}{z-a}, \quad (|a| < |z|)$$

   is stable for $|a| < 1$, i.e. the pole is inside the unit circle. The sequence is *causal.*

2. $$x_n = -a^n u_{-n-1} \quad \circ\!\!-\!\!\bullet \quad \frac{z}{z-a}, \quad (|a| > |z|)$$

   is stable for $|a| > 1$, i.e. the pole is outside the unit circle. The sequence then is *non-causal.*

Note: *Since of course the non-causal case is not truly realizable in practical application, one often finds the incomplete statement that a discrete time system is stable if the pole is inside the unit circle. This is a simplification which is not true. In practical applications it is at least approximately possible to also realize non- causal systems by applying a FIFO (first in first out) memory for the sequence and putting the time n e.g. in the middle of this memory. From that point it then is possible to define future and past elements of the sequence.*

4.3 Correlation Theorem

With the properties of the z-transform (4.0.3), (4.0.4) and (4.0.5) one can determine the z-transform of a correlation as:

$$r_{xy}(n) = \bar{x}_{-n} * y_n \quad \circ\!\!-\!\!\bullet \quad \bar{X}(\frac{1}{z})Y(z) \tag{4.3.21}$$

5 LTI System Function continued

LTI systems can be described by so called difference equations, as an example:

$$y_n = (x_n - x_{n-1}), \tag{5.0.1}$$

or a second example:

$$y_n = ay_{n-1} + bx_n \quad (a, b \in \mathbb{C}). \tag{5.0.2}$$

The general case then is given as:

$$\begin{gathered}\sum_{\mu=0}^{n} b_\mu x_{n-\mu} = \sum_{\nu=0}^{m} a_\nu y_{n-\nu} \\ \circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet \\ \sum_{\mu=0}^{n} b_\mu z^{-\mu} X(z) = \sum_{\nu=0}^{m} a_\nu z^{-\nu} Y(z)\end{gathered} \tag{5.0.3}$$

The z-transform was applied to both sides of the equation (noting that the z-transform is linear and using the time shifting property): From that follows the system function as the ratio (see (5.0.6)) of the output and input z-transforms:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum\limits_{\mu=0}^{n} b_\mu z^{-\mu}}{\sum\limits_{\nu=0}^{m} a_\nu z^{-\nu}}. \tag{5.0.4}$$

Note: *this is the usual way of notation in Matlab and Scilab/Octave as well as in most todays textbooks.* (5.0.4) shows that $H(z)$ is a rational function in z^{-1}. If $n < m$ then it is proper rational and a partial fractional decomposition can be carried out after finding the roots of the denominator. Otherwise i.e. the rational function is improper a *polynomial long division* is the appropriate way of treating such an expression leading to a polynomial part and a proper rational part. The proper rational part the can be treated by a partial rational expansion. Some special cases of (5.0.4) are important to notice.

1. all denominator coefficients are zero except for a_0. (5.0.4) can then always be normalised so that a_0 is 1. Thus the system function reduces to:

$$H(z) = \sum_{\mu=0}^{n} b_\mu z^{-\mu}. \tag{5.0.5}$$

This is a system function that has only zeros and no poles. Furthermore the impulse response is readily read off to be:

$$h_k = \begin{cases} b_k, & 0 \leq k \leq n, \\ 0. & else. \end{cases} \tag{5.0.6}$$

Definition 5.1
Finite Impulse Response (FIR) system
A LTI system for impulse response (5.0.6) *holds, is a finite impulse response (FIR) system.*

2. all nominator coefficients are zero except b_0. (5.0.4) then can always be normalised so that $b_0 = 1$. This is an all pole system, since it has only poles and no zeros. In this case the denominator can always be factorized (according to the Fundamental Theorem of Algebra it has m complex zeros, counting multiple zeros with their multiplicity) and be treated with a partial fractional expansion.

6 Spectral Representation of Time-Discrete Signals and Sequences

With any sequence x_n one can associate a series of the form:

$$X(f; f_s) := \sum_{k=-\infty}^{\infty} x_k e^{-jk\frac{2\pi f}{f_s}}. \tag{6.0.1}$$

This is a Fourier series -but in the frequency domain!- The function $X(f; f_s)$ is periodic with period f_s which is indicated by the notation. Later as soon as we know the right sample frequency this will be omitted. Thus the Fourier coefficients are given by:

$$x_n = \frac{1}{f_s} \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X(f; f_s) e^{j2\pi\frac{f}{f_s}} \mathrm{d}f \qquad (n \in \mathbb{Z}), \tag{6.0.2}$$

This means that the samples are the Fourier coefficients. This transform pair is also known as discrete time Fourier series:

Definition 6.1
DTFT
The discrete time Fourier Transform is given by the transform pairs:

1.

$$X(f; f_s) := \sum_{k=-\infty}^{\infty} x_k e^{-jk\frac{2\pi f}{f_s}} \tag{6.0.3}$$

2.

$$x_n = \frac{1}{f_s} \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X(f; f_s) e^{j2\pi\frac{f}{f_s}} \; \mathrm{d}f \qquad (n \in \mathbb{Z}) \tag{6.0.4}$$

The remaining open point is which sample frequency is to be taken? To this end the result is interesting and is depicted in the following examples: $f_s = 10$ and $f = 1$ 6.1 and using the same sample frequency but now with $f = 10$ 6.2. It is seen clear from the graphics that a sequence of samples may lead to complete different signal frequencies. So there is an ambiguity that needs to be resolved. An

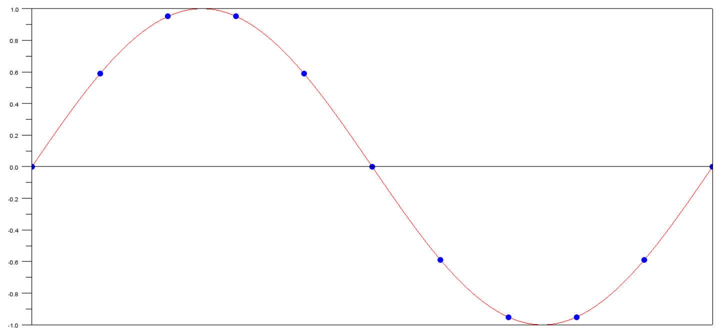

Figure 6.1: sampling a sine with frequency $f = 1$ and $f_s = 10$

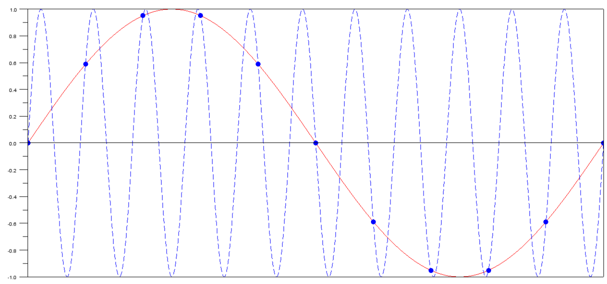

Figure 6.2: sampling a sine with frequency $f = 11$ and $f_s = 10$

additional information is obviously needed to decide which of the two examples leads to the correct signal. Or putting it in formulas: one needs to know in which region of the spectrum $[-\frac{f_s}{2}, \frac{f_s}{2}]$ or $[-n\frac{f_s}{2}, n\frac{f_s}{2}]$, $(n \in \mathbb{Z})$ the signal should be.

7 Sampling Theorem

7.1 The Sampling Process

It has been shown in 6 that with every sequence of samples there can be associated infinitely many signals depending on the sampling frequency. So what is the correct choice of the sampling frequency? The approach taken here to show the sampling theorem is different to the ones shown in most textbooks. Nevertheless those are all mathematically wrong since they abuse the Dirac- delta function. Let a analog signal $x(t)$ be give. This signal has the Fourier transform $X(f)$, (see 9, 9.2.3). Thus we have the transform pair:

$$x(t) \quad \underset{\mathcal{L}}{\circ\!\!-\!\!\bullet} \quad X(f). \tag{7.1.1}$$

Now consider the function given by:

$$X_{f_s}(f) := \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - kf_s). \tag{7.1.2}$$

This function is by construction periodic with period f_s, since

$$\begin{aligned} X_{fs}(f + f_s) : &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f + f_s - kf_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - (k-1)f_s) = X_{f_s}(f). \end{aligned} \tag{7.1.3}$$

Thus it can be expanded into a Fourier series

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - kf_s) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k \frac{f}{f_s}}. \tag{7.1.4}$$

Next it will be shown that the Fourier coefficients are exactly the samples of the signal. To this end the Fourier coefficients then are (note:$fs = 1/T$):

$$c_\nu = \frac{1}{f_s} \int_0^{f_s} \left(\frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - kf_s) \right) e^{j2\pi\nu \frac{f}{f_s}} \, \mathrm{d}f. \tag{7.1.5}$$

Substituting $\tilde{f} = f - kf_s$ one gets:

$$\begin{aligned} c_\nu &= \sum_{k=-\infty}^{\infty} \int_{-kf_s}^{(1-k)f_s} X(\tilde{f}) e^{j2\pi\nu \frac{\tilde{f}}{f_s}} \mathrm{d}\tilde{f} \\ &= \int_{-\infty}^{\infty} X(\tilde{f}) e^{j2\pi\nu \frac{\tilde{f}}{f_s}} \, \mathrm{d}\tilde{f} = x(\frac{\nu}{f_s}) = x(\nu T) = x_\nu. \end{aligned} \tag{7.1.6}$$

Now the spectral representation is more in a way to be interpreted easier. It is the spectrum

$$X_{f_s}(f) := \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{7.1.7}$$

that is associated with the sample sequence x_n. But this is periodic with period f_s. A result already seen in chapter 6. But now it can be analysed since 7.1.7 gives a relation between the spectrum X_T of the sampled signal and the spectrum of the signal $x(t)$ before sampling. To this end we assume that the spectrum before sampling is limited to a range B that is called bandwidth of the signal i.e.

Definition 7.1
band-limited signal
A signal is called band-limited iff

$$X(f) = 0 \quad \forall\, f \notin (-B, B) \tag{7.1.8}$$

The following graphics show three situations: $B = 1$, $f_s = 1,1$

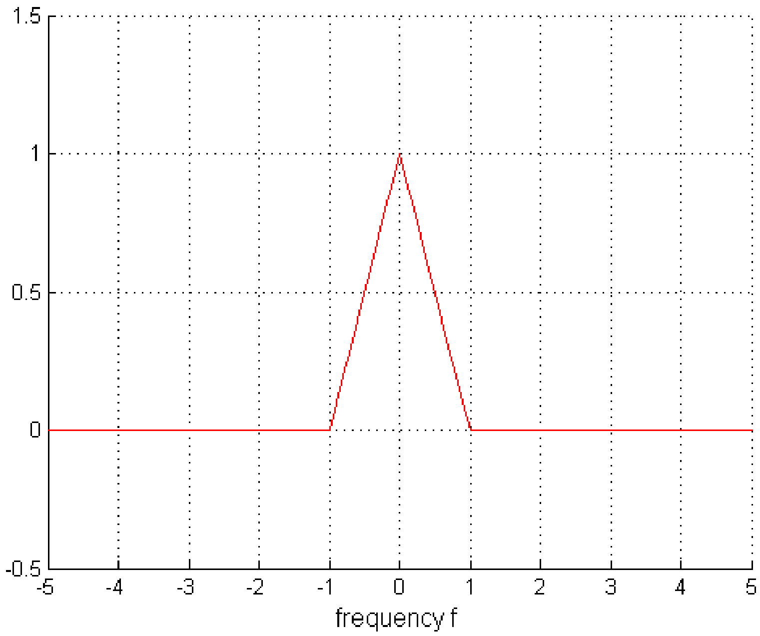

Figure 7.1: spectrum before sampling Bandwidth $B = 1$

It can clearly be seen that for a sampling frequency $f_s < 2B$ the spectra overlap. This is called **aliasing** . It *must* be avoided because it corrupts the information contained in the signal.
To summarize: in a sampling process it must be clear, which of the periodic

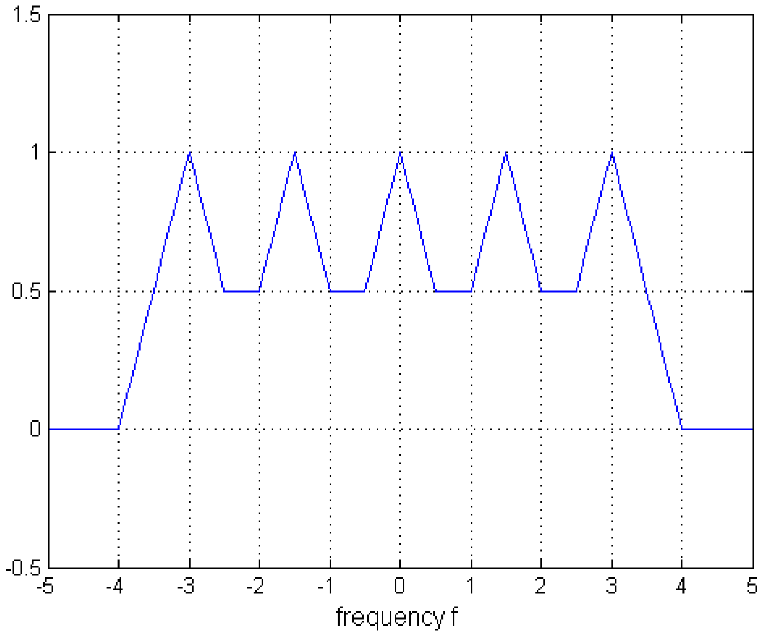

Figure 7.2: sampling with $f_s = 1,5$ *and* $B = 1$ (Aliasing)

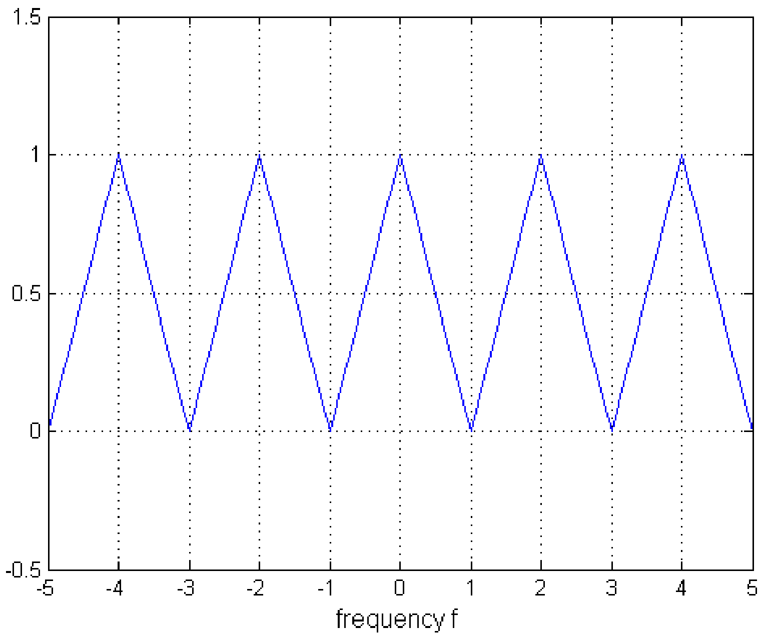

Figure 7.3: sampling with $f_s = 2$ *and* $B = 1$

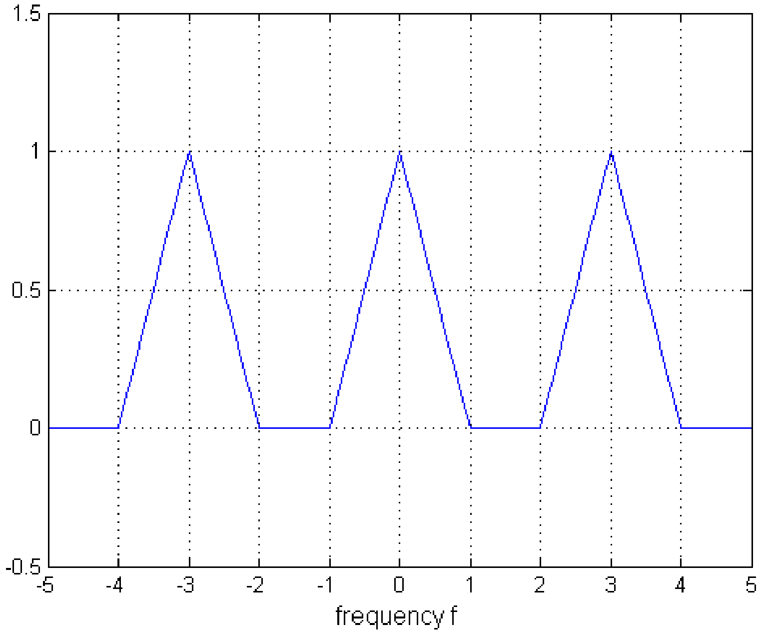

Figure 7.4: sampling with $f_s = 3$ *and* $B = 1$

repetitions was the true analog signal. Nevertheless it may also be used with advantage (and is done in modern communication systems) that a signal being in some interval $(k\frac{f_s}{2}, (k+1)\frac{f_s}{2})$ may be sampled with f_s and then after sampling one of the periodic replicas is picked. This can save the usage of an extra mixers. So sampling and mixing is then one process. To summarize the result this is the first part of the

Theorem 7.1
Shannon Sampling Theorem Part 1 *Suppose a signal x is band-limited with bandwidth B.*

1. *It can be sampled with a sample frequency without corruption (i.e. loss of information) if*
$$f_s > 2B \tag{7.1.9}$$
2. *The spectrum of the sampled signal is periodic with period f_s and given by*
$$X_T(f) := \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{7.1.10}$$

[1]

The practical implementation of a system to definitely make sure that a sampling

[1]Claude Shannon was a US American mathematician and engineer. During his work at AT&T Bell Labs he published this result back in 1948 [15]. Today it is known, that several others derived the same result, see e.g. [5]

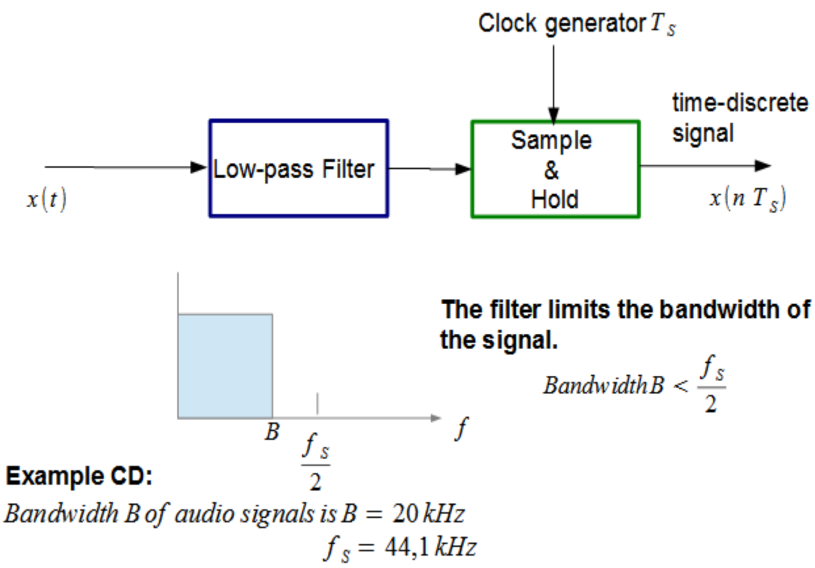

Figure 7.5: real sampling system with low-pass filter

process is free of aliasing would look as (see 7.5). First the signal is low-pass filtered to a bandwidth of less than $\frac{f_s}{2}$ before the Sample&Hold generates the time discrete signal. This low-pass filter is therefore often called Anti-aliasing Filter. The design of ADC (analog to digital converters is a special topic) needing much design experience. There are many more design parameters of interest(linearity, SNR, THD, spurious noise ....see e.g. [8])

7.2 Reconstruction of the Analog Signals from Samples

Up to this point the process of generating the time-discrete signal was considered. E.g. music is tuned into a digital signal and stored on a digital storage device like a CD. The question now is how to reconstruct the analog signal to make it audible again. To this end see figure 7.4 . The signal was sampled with an appropriate sampling frequency showing no spectral overlap in the periodic replica, i.e. no aliasing.Thus a low pass filter is needed to filter out all replicas see figure 7.6 (the sampling frequency is 3). This low pass filter has the property in the frequency domain:

$$LP(f) = \begin{cases} \frac{1}{fs} & -\frac{f_s}{2} < \frac{f_s}{2}, \\ 0 & else. \end{cases} \tag{7.2.11}$$

Thus the output then is:

$$X(f) = X_{f_s}(f)LP(f). \tag{7.2.12}$$

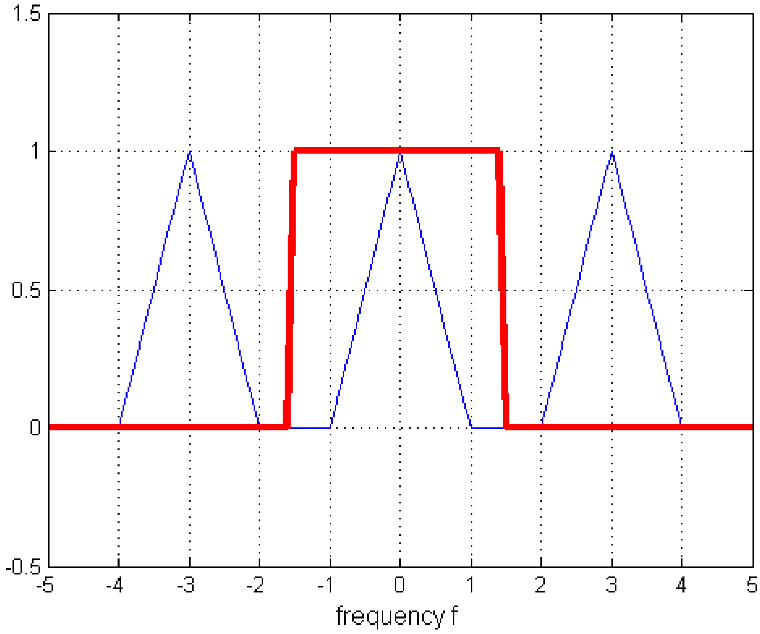

Figure 7.6: reconstruction of the original spectrum by low pass filtering.

Inserting (7.1.4) and noting the Fourier coefficients are the samples themselves one gets:

$$X(f) = LP(f) \sum_{k=-\infty}^{\infty} x_k e^{-j2\pi k \frac{f}{f_s}} \tag{7.2.13}$$

Now the inverse Fourier transform is applied to derive the signal in time domain:

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, \mathrm{d}f \\ &= \int_{-\infty}^{\infty} LP(f) \left(\sum_{k=-\infty}^{\infty} x_k e^{-j2\pi k \frac{f}{f_s}} \right) e^{j2\pi f t} \, \mathrm{d}f \\ &= \int_{-\infty}^{\infty} LP(f) \left(\sum_{k=-\infty}^{\infty} x_k e^{-j2\pi k \frac{f}{f_s}} \right) e^{j2\pi f t} \, \mathrm{d}f \\ &= \frac{1}{f_S} \sum_{k=-\infty}^{\infty} x_k \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} e^{-j2\pi (kT-t) f} \, \mathrm{d}f \end{aligned} \tag{7.2.14}$$

The integral is solved for the two cases $kT - t = 0 \neq 0$ or $kT - t \neq 0$:

$$\frac{1}{f_S} \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} e^{-j2\pi(kT-t)f} \,\mathrm{d}f = \begin{cases} 1, \quad (kT - t = 0), \\ \dfrac{e^{-j2\pi(kT-t)\frac{f_s}{2}} - e^{j2\pi(kT-t)\frac{f_s}{2}}}{-j2\pi(kT-t)f_s}, \quad (kT - t \neq 0) \end{cases}$$

$$= \begin{cases} 1, \quad (kT - t = 0), \\ \dfrac{\sin(\pi(kT-t)f_s)}{\pi(kT-t)f_s}, \quad (kT - t \neq 0). \end{cases} \tag{7.2.15}$$

Where in the last step use was made of Eulers formula. Defining the so called *sinc* function as

Definition 7.2
Sinc function

$$\mathrm{sinc}(x) = \begin{cases} 1 & , \quad x = 0 \\ \dfrac{sin\pi x}{\pi x} & , \quad x \end{cases} \tag{7.2.16}$$

The *sinc* function is a arbitrary often continuous differentiable function. It plays a major role in many areas of natural sciences. E.g. the diffraction of light going through a slit is also described by it.
Inserting (7.2.15) into (7.2.14) can be written with the *sinc* function as as

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \,\mathrm{sinc}(tf_s - k) \tag{7.2.17}$$

(7.2.17) is the convolution of the signal sample sequence x_k with the sinc function. It is also know as cardinal series and establishes the.

Theorem 7.2
Shannon Sampling Theorem Part 2
If an analog signal $x(t)$ is sampled with samples $x_k = x(kT)$ then it can be reconstructed from its samples with the cardinal series

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \,\mathrm{sinc}(tf_s - k) \tag{7.2.18}$$

Of course this series since it is an infinite sum can not be realized (which is equivalent to that an ideal low pass filter as used in the calculations is not feasible). In practical implementations only approximate low pass filters can be realised leading to an error. These are considerations also for A/D and D/A converters not covered here. Formula (7.2.17) is often referred to as an interpolation formula which is indeed true. [2]. This establishes the second part of the Shannon sampling theorem.

[2] This formula goes back to Joseph-Louis Lagrange (1736-1813) a famous Italian mathematician [5]

8 Discrete Fourier Transform

8.1 Definition of the DFT

In 6 it was shown that a sequence of samples of a discrete time signal has a spectral representation of:

$$X(f) := \sum_{k=-\infty}^{\infty} x_k e^{-jk2\pi\frac{f}{f_s}}. \tag{8.1.1}$$

and the signal is then

$$x_n = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X(f) e^{j2\pi\frac{f}{f_s}}\,\mathrm{d}f \tag{8.1.2}$$

For numerical calculations these formulas are not well suited. Therefore it is necessary to seek for some numerical (approximate) simplifications. In reality a signal will have a finite duration. N samples will be chosen from 0 to $N-1$. Furthermore the frequency $\frac{f}{f_s}$ is taken at the discrete points $\frac{n}{N}$ $n = 0, \cdots, N-1$

$$X_n := X(\frac{n}{N}) = \sum_{k=0}^{N-1} x_k e^{-jk2\pi\frac{n}{N}} \qquad n = 0, \cdots, N-1. \tag{8.1.3}$$

These is a set of $N-1$ equations, each of them are the product of a line of the matrix

$$F := \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{8.1.4}$$

, where

$$\omega = e^{-j2\pi/N} \tag{8.1.5}$$

with the row vector

$$\vec{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix} \tag{8.1.6}$$

Then with setting

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{pmatrix} \tag{8.1.7}$$

8.1.3 can be written as in matrix form

$$\vec{X} = F\vec{x} \tag{8.1.8}$$

The matrix F is called a transformation matrix. Scaled by $\frac{1}{\sqrt{N}}$ it becomes a unitary matrix,i.e. there holds that the conjugate transposed $= F^H$ is basically the inverse

$$FF^H = N. \tag{8.1.9}$$

This leads to a simple inverse transform (**inverse DFT**):

$$\vec{x} = \frac{1}{N} F^H \vec{X}. \tag{8.1.10}$$

These set of equations are numerically feasible to calculate on a computer. But it needs to be understood what the trade-off due to this rather simple modifications of the DTFT are.

8.2 Spectral analysis with the DFT

One thing that can be seen at once is: equation (8.1.10) gives a periodic sequence with period N. But it is not said that the input x_n is necessary periodic. Suppose the signal from which the samples were taken is a sine wave with some frequency f

$$x(t) = \sin(2\pi ft) \Rightarrow x_k = \sin(2\pi fkT) \tag{8.2.11}$$

Then a period of N would force that there holds:

$$\sin(2\pi fkT) = \sin(2\pi f(k+N)T). \tag{8.2.12}$$

Now the question is, if this periodic signal would be continuous or even differentiable. It is clear that any portion of a signal can be periodically repeated, but the difference may be if this new periodic construct is smooth or not (the properties of continuity and differentiability). This is obviously the case if for the frequency and the length of the sequence there holds:

$$fNT = M \in \mathbb{Z} \tag{8.2.13}$$

In fact the samples can be viewed as a multiplication of the whole signal sequence with a rect signal cutting out the relevant N samples, where

$$\mathrm{rect}_k = \begin{cases} 1, & 0 \leq k \leq N-1, \\ 0, & \text{else.} \end{cases} \tag{8.2.14}$$

So according to 6.0.3 the DTFT of rect is (note that this then is a geometric sum (9.3.6)) for $f \neq 0$:

$$
\begin{aligned}
\sum_{k=0}^{N-1} e^{-jk2\pi\frac{f}{f_s}} = & \\
&= \frac{1-e^{-j2\pi N\frac{f}{f_s}}}{1-e^{-j2\pi\frac{f}{f_s}}} \\
&= \frac{e^{-j\pi N\frac{f}{f_s}}\left(e^{j\pi N\frac{f}{f_s}}-e^{-j\pi N\frac{f}{f_s}}\right)}{e^{-j\pi\frac{f}{f_s}}\left(e^{j\pi\frac{f}{f_s}}-e^{-j\pi\frac{f}{f_s}}\right)} \\
&= \frac{\sin(\pi N\frac{f}{f_s})}{\sin(\pi\frac{f}{f_s})}e^{-j\pi(N-1)\frac{f}{f_s}}
\end{aligned}
\tag{8.2.15}
$$

For $f = 0$ one gets:

$$\sum_{k=0}^{N-1} e^{-jk2\pi\frac{0}{f_s}} = N; \tag{8.2.16}$$

This last expression is called the *Dirichlet Kernel*.[1]Sometimes also referred to as the *periodic sinc function*. Using the normalized angular frequency $\Omega = 2\pi\frac{f}{f_s}$ this simplifies to:

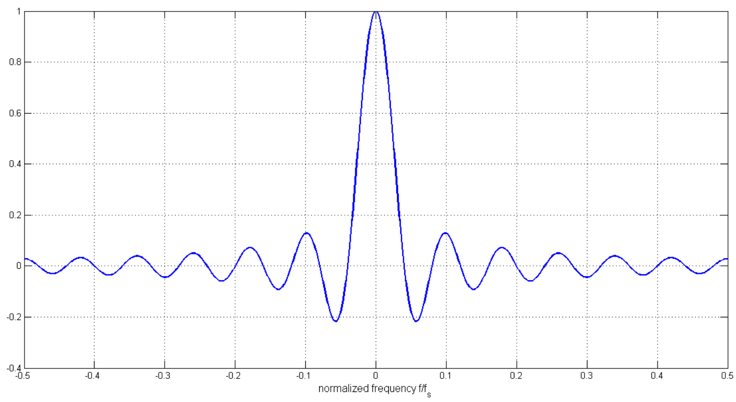

Figure 8.1: Dirichlet Kernel

$$D_N(\Omega) := \begin{cases} \frac{\sin(N\frac{\Omega}{2})}{\sin(\frac{\Omega}{2})}e^{-j(N-1)\frac{\Omega}{2}} & f \neq 0, \\ N & f = 0. \end{cases} \tag{8.2.17}$$

Thus the DFT will show the amplitude spectrum of the convolution of the signal spectrum $X(f)$ with the Dirichlet kernel and the amplitude response is:

$$|DFT(x_k)| = |X(f) * D_N(f)|. \tag{8.2.18}$$

[1]G.L. Dirichlet (1805-1859) was a German mathematician. He was successor of C.F. Gauss in Göttingen after he died. His main fields were number theory and Fourier analysis

If the signal frequency satisfies (8.2.13), then this means

$$D_N(f) = N \quad \text{for } fNT = M \in \mathbb{N} \tag{8.2.19}$$

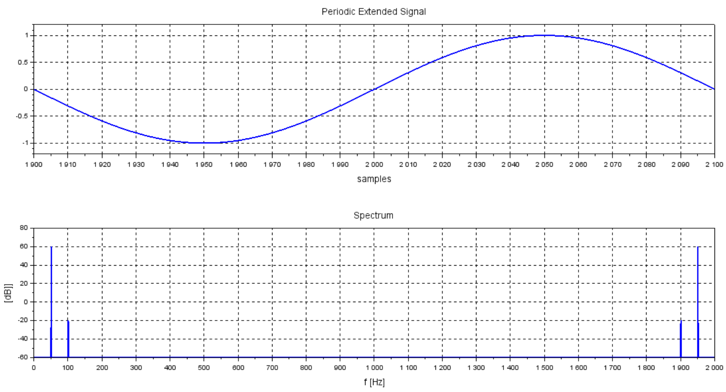

Figure 8.2: continuous periodic extension of sample sequence

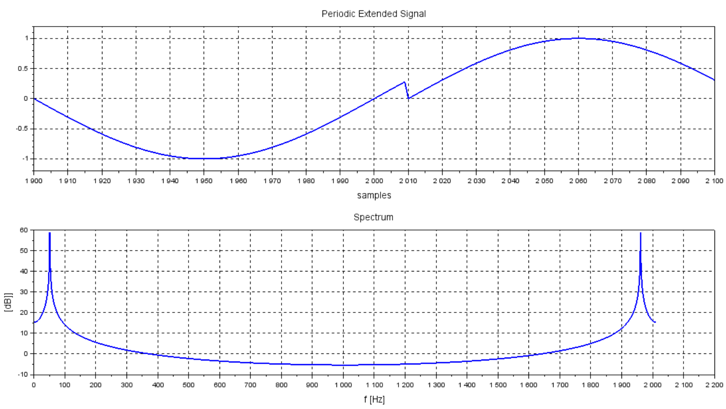

Figure 8.3: discontinuous periodic extension of sample sequence

8.3 Windows

As has been pointed out the discontinuity of the periodic repeated signal is the root cause of the spectrum calculated with the DFT showing the behaviour shown in 8.3. It is the result of the signal being convolved with the Dirichlet kernel. This is not completely surprising, since from Fourier series it is know that these do not converge towards the function value at discontinuities at finite jumps show the well known Gibbs phenomenon . The core idea is to instead using the rect sequence which causes the abrupt cut off (and thus the discontinuity) to apply an other sequence w_n $0 \leq n \leq N-1$ that forces the product

$$x_n w_n \quad (0 \leq n \leq N-1) \tag{8.3.20}$$

to be continuous and more over to be differentiable to an as high order as possible. Such sequences are called windows. An extensive discussion with a lot of deep analysis is found in[3]. See also [13].
In fact these windows have been studied in mathematics a long time before in the context to determine the convergence properties of Fourier series and there are know as kernels or convergence factors, see [2]. One of the first known windows is the so called Fejér-kernel given by:

$$w_k = \begin{cases} 1 - \frac{|k|}{N} & N \leq n \leq N-1 \\ 0 & else \end{cases} \tag{8.3.21}$$

[2] also known in signal analysis as the Bartlett window. Others to be mentioned are: Hanning window, Hamming window, Blackman- Harris window, Kaiser window, Tukey window. They are to be found in Matlab and Scilab. 8.4 shows the Hanning window as an example. As an example the previous discontinuous situ-

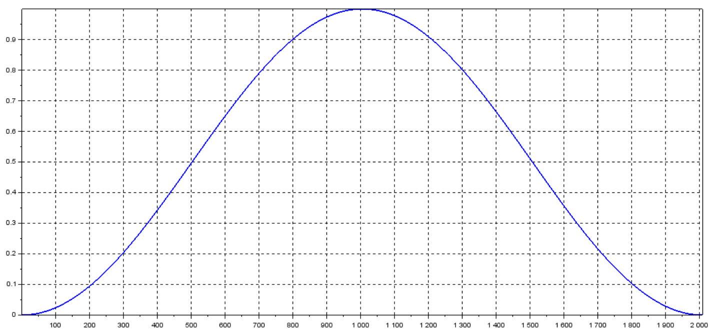

Figure 8.4: Hanning Window

ation has been taken with the Hanning window. As can be seen, the two signals can be separated again.

[2] Leopold Fejér (1880-1959) was a Hungarian mathematician who worked on Fourier series and their convergence behaviour. His results were fundamental to the understanding of Fourier series.

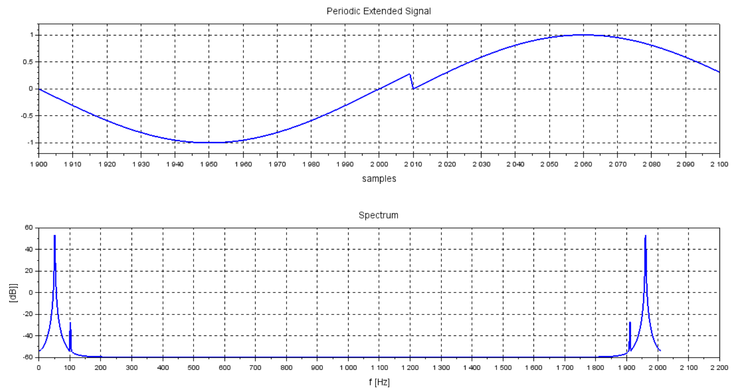

Figure 8.5: Spectrum derived with Hanning window

8.4 Fast Fourier Transform (FFT)

To evaluate the DFT several software programs have been developed (see e.g. [10]), [11][14]). They all make use of symmetries that can be seen in the DFT matrix and thus are very fast. Most semiconductor companies have building blocks in their hardware libraries that can be used in their chip design. These algorithms are called FFT (Fast Fourier Transform). Modern communication systems like WLAN, DVB-T,-S,-C (Digital Video Broadcasting) or DAB (Digital Audio Broadcasting) and others use the FFT in there Modulation and demodulation (OFDM (Orthogonal Frequency Division Multiplex) or in COFDM (Corrected OFDM). See[9]. Other applications are using it for implementing FIR and IIR filter by what is known as overlap- save and overlap-add filtering. [7],[14]. So from a theoretical point of view this doesn't give any additional insights. It should be mentioned that the FFT performs fastest, if $N = 2^L$, i.e. is a power of 2. A critical issue may be in applications that if N is a large number the FFT tends to introduce noise into the signals. This must be checked from case to case and highly depends on the word-length implemented for the FFT algorithm.

9 Appendix

9.1 Fourier Series

The Fourier series is given for a periodic function f with period T by:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T}; \tag{9.1.1}$$

with the Fourier coefficients being

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{j2\pi \frac{kx}{T}} \,\mathrm{d}x \qquad (k \in \mathbb{Z}). \tag{9.1.2}$$

The Fourier coefficients define what is know as spectrum. *Thus a periodic function / signal always has a discrete spectrum because it exists only for the distinct values in $\mathbb{Z}$.*

9.2 Fourier Transform

For a not necessary periodic function f the Fourier Transform is given by:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi ft} \,\mathrm{d}t \tag{9.2.3}$$

The inverse Fourier Transform is then given by:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi ft} \,\mathrm{d}f \tag{9.2.4}$$

One writes:

$$x(t) \quad \underset{\mathcal{L}}{\circ\!\!-\!\!\!-\!\!\bullet} \quad X(f) \tag{9.2.5}$$

9.3 Geometric Sum and Series

$$\sum_{k=0}^{N-1} q^k = \frac{1 - q^N}{1 - q} \tag{9.3.6}$$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q} \quad |q| < 1 \tag{9.3.7}$$

9.4 Notations

$\mathbb{N}$: set of natural numbers $\{1, 2, 3 \dots\}$
$\mathbb{N}_0$: $\mathbb{N} \cup 0$
$\mathbb{Z}$: set of whole numbers
$\mathbb{Q}$: set of rational numbers numbers
$\mathbb{R}$: set of real numbers
$\mathbb{C}$: set of complex numbers
$x \in \mathbf{A}$: x is element of the set$\mathbf{A}$
$x \notin \mathbf{A}$: x is not element of the set$\mathbf{A}$
$j = \sqrt{-1}$ imaginary unit
$z = a + jb \in \mathbf{C} \Rightarrow \overline{z} = a - jb$ conjugate complex number
iff: if and only if (the same as equivalent $\Leftrightarrow$)

9.5 Abbreviations

f denotes frequency [Hz]
T sampling period
$f_s = \frac{1}{T}$ sampling frequency
$\omega = 2\pi f$ angular frequency [rad x Hz]
$\Omega = 2\pi \frac{f}{f_s}$ normalized angular frequency

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