Growth in a Restricted Solid-on-Solid Model

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Extensive simulations of growth in a stochastic ballistic deposition model on a $(d-1)$-dimensional substrate with a constraint on neighboring interface heights are described. The interface width obeys scaling even for small systems and grows as t^{β} with $\beta=1/(d+1)$. Generalizations to include irrelevant effects such as noise reduction are discussed as are possible reasons for the discrepancies in earlier results.

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Growth of a solid from the vapor has been receiving much attention recently. Although many real growth processes are controlled by diffusion processes in the vapor leading to fractal aggregates as in diffusion-limited aggregation (DLA)[1] or to dendritic growth,[2] the simpler processes of ballistic deposition[3,4] and Eden growth[5] are still not completely understood except in two dimensions despite their apparent simplicity. These latter processes form a compact cluster and the interesting physics is in the structure of the surface and in the scaling relations describing correlations in the surface.

The growth of an interface may be described by[6]

$$\partial h/\partial t = \nu \nabla^2 h + \lambda(\nabla h)^2 + \eta(\mathbf{x},t) , \qquad (1)$$

where $h(\mathbf{x},t)$ is the local deviation from the mean of the height of the interface from the $(d-1)$-dimensional substrate and η is a Gaussian random variable of width D describing local variations in deposition rate. A simple transformation[6] to $\mathbf{v}=-\nabla h$ converts this into the Burgers's equation[7] which has also been studied intensively.[8] The characteristics of such growth models are the predictions that height-height correlation functions scale as[6]

$$\lim_{t_0\to\infty} \langle [h(\mathbf{r},t+t_0)-h(0,t_0)]^2\rangle \sim r^{2\chi} f(rt^{-1/z}) , \qquad (2)$$

and that, in a finite system of size L, the width $\sigma(t)$ of the interface starting from a flat substrate at $t=0$ scales as[3]

$$\sigma^2(t) \equiv \langle h^2(\mathbf{r},t)\rangle \sim L^{2\chi} f(Lt^{-1/z}) , \qquad (3)$$

where the exponents χ, z obey $\chi+z=2$ (Refs. 6, 9, and 10) for temporally uncorrelated noise in the deposition rate. This identity follows from invariance under an infinitesimal tilt of the interface and from Galilean invariance of the corresponding Burgers's equation. The scaling form of the correlation function of (2) follows from renormalization-group arguments and the interface width of (3) from the assumptions of scaling and a single time-dependent length scale $\xi(t)\sim t^{1/z}$ for growth starting from an initially flat interface.

Most of the recent efforts on growth models of this type have focused on verifying the scaling form of Eqs. (2) and (3) and finding values of the exponents χ and $\beta \equiv \chi/z$ when the interface width $\sigma(t,L)\sim L^{\chi}$, $t \gg L^{\chi}$, and $\sim t^{\beta}$, $t \ll L^{z}$. In one dimension, which is the trivial problem of random deposition on independent sites, the system size L plays no role and the only relevant exponent is $\beta=\frac{1}{2}$. In two dimensions, there are various soluble models[9,11] which yield the values $\chi=\frac{1}{2}$ and $\beta=\frac{1}{3}$ as does the renormalization-group analysis[6] of Eq. (1). Numerical simulations of various ballistic deposition models[10,12] and also of the Eden surface[13,14] agree with these values in $d=2$.

For $d \geq 3$, no analytic results exist except (i) when $\lambda^2 D/\nu^3$ is sufficiently small, the interface is "ideal" with $z=2$,[6,15] and (ii) a "proof" that the $d=2$ values are superuniversal.[16] Extensive numerical work on the Eden model,[13,14] on ballistic deposition models,[9,17] and on directed polymers[18] give exponents which do not agree with each other nor with the superuniversality conjecture.[6,16]

In this Letter, we report on extensive numerical simulations on a particular growth model similar to the single step model of Ref. 9, undertaken to try to understand why the earlier simulations disagreed with each other. The growth algorithm is to randomly select a site on a cubic $(d-1)$-dimensional lattice and to permit growth by letting the height of the interface $h_i \to h_i+1$ provided the restricted solid-on-solid restriction on neighboring heights $|\Delta h| = 0,1,\ldots,N$ is obeyed at all stages. The initial configuration is a flat surface ($h_i=0$) and the algorithm ensures a compact d-dimensional cluster with no vacancies or overhangs and the slopes of hills and valleys is at most N. Most of our simulations were performed for $N=1$ with periodic boundary conditions but the results are, within the limits of our simulations, independent of N. Our simulations agree with the identity $\chi+z=2$ to better than 1%. Taking this identity to be exact,[6,9] we concentrated on the easiest exponent to measure which is $\beta=\chi/z$. We carried out simulations in $d=1$ and 2 to test our algorithm and obtained $\beta(1) = (0.50\pm 5)\times 10^{-4}$ and $\beta(2)=0.332\pm 0.005$ which agrees well with known results. In higher dimensions,

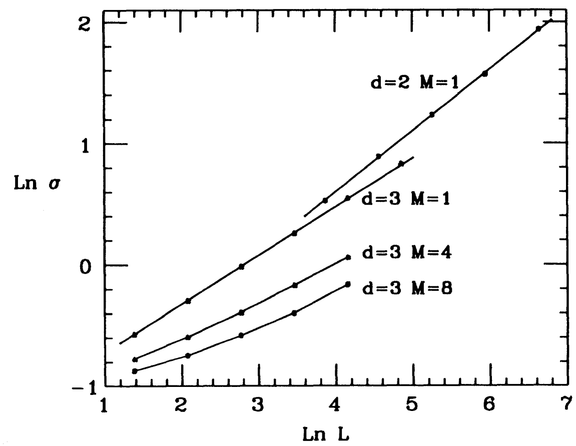

FIG. 1. Interface width σ as a function of system size L. M is the noise reduction parameter.

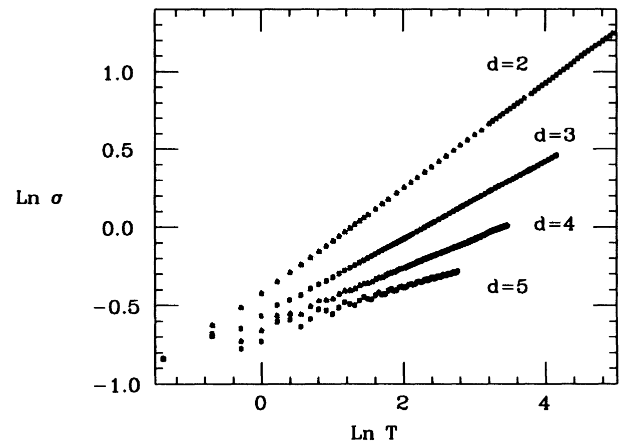

FIG. 2. Raw data of $\sigma(t)$ for the largest systems. Note the finite-size effects especially for $d=5$ ($L=32$).

$\beta(3)=0.250\pm0.005$, $\beta(4)=0.20\pm0.01$, and $\beta(5)\sim\frac{1}{6}$. These results lead us to the conjecture that the general form is

$$\beta(d)=1/(d+1),\quad \chi(d)=2/(d+2)\,,$$
$$z(d)=2(d+1)/(d+2)\,, \tag{4}$$

which is the main result of this Letter. Note that β and χ are larger than earlier estimates and disagree with all previous conjectures.[6,9,14,16,19] Also, as $d\to\infty$ we recover the result $z=2$ in agreement with results on a Cayley tree.[20]

In the rest of this Letter, we summarize our simulations and discuss why we believe the results from this model are more reliable than earlier work. The other major advantage of the model for $N=1$ is that even relatively small systems yield good scaling behavior. The exponent χ was determined by the usual procedure of starting from a flat surface in a system of linear size L, growing until the width of the interface is saturated by finite-site effects when $\sigma\sim L^{\chi}$, and averaging over very long times. This is necessary because in a system of size L, there are hills and valleys up to size $\sim L$ which persist for a time $\sim L^{z}$. Because of this, our runs to determine χ were restricted to relatively small systems—$L\lesssim 768$ for $d=2$ and $L\lesssim 128$ for $d=3$. Even these small systems gave excellent agreement with the known result $\chi=\frac{1}{2}$ in $d=2$ and yielded $\chi=0.40\pm0.01$ in $d=3$ as shown in Fig. 1. We made no attempt to find χ for $d>3$ because the computer time would be prohibitive.

To determine β, the exponent governing the rate of growth of the interface width, we used two methods. The first was a straightforward simulation of $\sigma(L,t)$ in the short-time regime. As a measure of t, we used the average height $\bar{h}$ which is completely linear in t. The general form of the curves are shown in Fig. 2 for the largest system used, $L=6144$ ($d=2$), 512 ($d=3$), 64 ($d=4$), and 32 ($d=5$). As expected, at very short times $\bar{h}<1$, random deposition effects dominate since the constraint on the heights $|\Delta h|=0,1$ has little effect. For $1<\bar{h}<0.02L^{z}$ the curves become straight lines with no noticeable curvature whose extent increases with L, but for large $\bar{h}$ a downward curvature sets in due to finite-size effects, eventually saturating at L^{χ} for $\bar{h}\gg L^{z}$. Also, fluctuations increase with $\bar{h}$ and to average these out a large number of runs were required. In $d=3$, a straight segment of the curve can be identified for $L\gtrsim 64$ whose extent increases with L with slope $\beta=0.25\pm0.005$ independent of L. Our estimates of the errors come from the largest system used ($L=512$). The independently measured exponents χ and β, using $z=\chi/\beta$, agree well with the identity $\chi+z=2$. Also, β and χ are independent of lattice structure as checked by measuring $\sigma(t)$ on a triangular substrate.

As a check on our results, we also measured the quan-

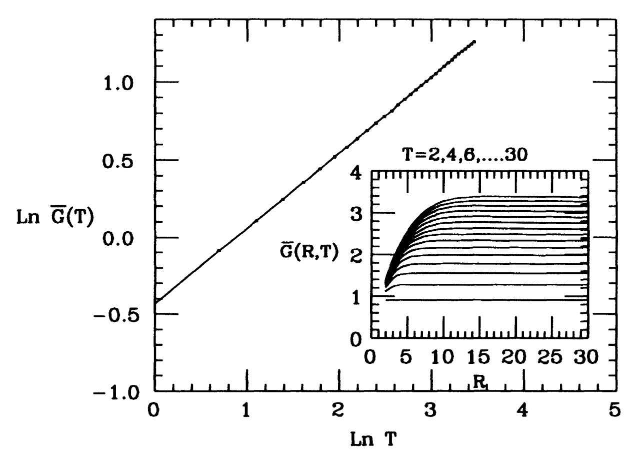

FIG. 3. $\bar{G}(t)$ for $d=3$ ($L=128$). Inset: Sequence of $\bar{G}(R,t)$ at fixed t.

tity $\bar{G}(r,t) \equiv \langle [h(r,t)-h(0,t)]^2 \rangle$ in $d=2,3$. At fixed $t \ll L^z$, $\bar{G}(r,t)$ saturates for $r=O(L)$ as in Fig. 3. To obtain β, $\bar{G}(r,t)$ is averaged over r for $L/4<r<L/2$ at fixed t to reduce fluctuations and obeys $\bar{G}(t) \sim t^{2\beta}$ with the same values of β as found from $\sigma(t)$. This method yields accurate values of β for smaller systems than the previous one. We also simulated $G(t) \equiv \langle [h(r,t+T) - h(r,T)]^2 \rangle$ in the saturated regime $T>L^z$. This was done because there is a good theoretical basis[6] for expecting $G(t) \sim t^{2\beta}$, whereas for $\sigma^2(t)$ and $\bar{G}(r,t)$ the power-law growth is a plausible assumption.[3] These simulations are very time consuming because of the very long-lived fluctuations and an enormous amount of averaging is needed to obtain good statistics. As expected, the results are the same as before. Finally, we checked the finite-size scaling form of Eq. (3) and the fit is shown in Fig. 4 for $d=3$.

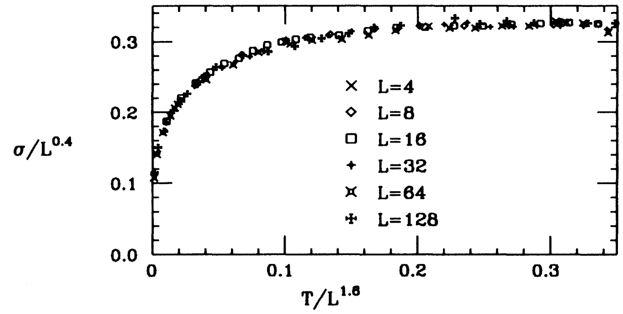

FIG. 4. Data collapse for $\sigma(L,t)$ in $d=3$ with $z=1.6$, $\chi=0.4$.

We now turn to the reasons why we believe that the results obtained from this model should be more reliable than those of earlier but bigger simulations. We make the very drastic but plausible assumption[6] that this model, ballistic deposition, and Eden growth are in the same universality class. When the constraint is relaxed to $|\Delta h| \leq N$, the region dominated by random deposition effects increases and is followed by a very slow crossover to asymptotic behavior. Finite-size effects appear at the same point as with $N=1$ so that the region of true power-law growth is rather small and the exponent β estimated from such simulations is not reliable. For the largest systems, we obtain quite good agreement with the earlier results for $N<4$, but if points with small $\bar{h}$ are used, β_{eff} increases slightly with N. The conclusion is that N is irrelevant but may have serious consequences for simulations. We expect that Eden growth[13,14] suffers from this problem since there is no constraint on Δh. The single step model[9,12] has a similar drawback because $\Delta h \neq 0$ and growth involves $h \rightarrow h+2$. We expect that asymptotic behavior will set in at a much larger value of $\bar{h}$ than in our model, as observed. Both χ and β are lower than our values which is probably due to the large intrinsic width of the interface.

We also investigated the effects of a noise reduction algorithm[14,21] in which a counter at a site is increased by one if a successful hit at that site occurs. When the counter reaches a value M, growth takes place and the counter is reset to zero. In this model, this was found to drastically reduce the quality of the scaling and the asymptotic regime as in Fig. 1. The best we could do with the data was to evaluate effective exponents χ and β and, although the identity $\chi+z=2$ was still approximately obeyed, both χ and β were reduced from their values in the original model, especially for $d=3$, but are close to earlier estimates.[14] A speculative explanation is that noise reduction is equivalent to decreasing the parameter $\lambda^2 D/\nu^3$. Since this is marginally relevant in $d=3$,[6] the crossover from the "ideal interface" behavior $\beta=\chi=0$ will be very slow resulting in a reduction in the effective exponents. In Eden growth, where there is no constraint, the effective exponents will be a compromise between the four competing effects—random deposition, small noise, asymptotic behavior, and finite size. The end result is a β_{eff} which increases with $\bar{h}$ to some maximum value and then decreases due to finite-size effects, which is precisely what is observed.[14]

We can summarize the results of our investigation by Eqs. (3) and (4). If other growth models are in the same universality class, the results from these must be viewed with suspicion because of random deposition and finite-size effects. In particular, noise reduction algorithms are a disaster for this model because they seem to be equivalent to increasing diffusion or surface tension and one needs very large systems to see asymptotic behavior. We suspect that similar effects occur in other models. We cannot say anything conclusive on the question of universality because of computer limitations and the lack of analytical tools. Some attempts[4,6,12,22] have been made to answer this, but they all suffer from the types of problems discussed earlier. The unmodified $N=1$ version of our model seems to be fairly free of these defects and to yield reliable results for relatively small systems. Since the exponents seem to be given by simple rational functions of d, one is led to think that it should be possible to analytically derive them.

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