Modelling transient heat transfer in the formation of ice sheets

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1 Introduction

The phenomenon of transient heat transfer is an unsteady state one, knowledge of which is crucial particularly with regard to start up and shut down of different chemical reactors **[3]** as well as energy changes in systems exposed to different temperature driving forces **[1]**.

The aim of this investigation was to model the effects that transient heat transfer has on ice formation in both fresh and sea water with the aid of reasonable assumptions as far as key parameters are concerned. The effects of conduction and convection were assumed to be the main contributors to heat transfer in this investigation with the main driving force for ice formation being the latent heat of fusion λ released upon the liquid water to ice phase transition **[1,2]**. Model I examines the fresh water case and model II examines the sea water scenario.

In the remainder of this report, the methods used in devising both models are explained, the results of the model calculations are discussed and, where appropriate, conclusions are made.

2 Methods and discussion

2.1 Model I

In modelling the formation of ice from fresh water, a control volume analysis was employed **[1]**. The formation of ice is caused by the absorption of large amounts of λ from the liquid water at the ice-water interface **[1]**.

It was assumed that, in the formation of ice, latent heat effects are much greater than sensible heat ones and that parameter values such as thermal conductivity k remain constant throughout the freezing process **[1,3]**. It was also assumed that the water beneath the ice-water interface remains still, such that it cools down to a near freezing temperature T. If not, then due to constant mixing, more heat will actually be available **[3]**.

The overall heat transfer coefficient U from the air-ice to ice-water interface can be obtained using equation 1 since, in slab geometry the heat flux is independent of ice thickness z formed **[1]**.

In equation 1 k_{ice} is the thermal conductivity of ice at 0 °C and h_{air} is the ambient air heat transfer coefficient.

$$U = \frac{h_{air}k_{ice}}{k_{ice}+zh_{air}} \tag{1}$$

If the ambient air $\boldsymbol{T}$ is -25 °C and the water $\boldsymbol{T}$ initially at 0 °C then, at instantaneous steady state when balancing the heat transfer rate with the change in total enthalpy related to the amount of ice, equation 2 gives the expression that can be used to estimate the time taken to form 1 m of ice where, $\boldsymbol{\rho_{ice}}$ is the density of ice at 0 °C and the upper integrand limit on the right hand side $\boldsymbol{\xi}$ is the ratio between $\boldsymbol{\rho_{ice}}$ and $\boldsymbol{\rho_{water}}$ at 0 °C. This ratio $\boldsymbol{\xi}$ accounts for the fact that ice expands on freezing **[1]**.

$$\int_0^t \frac{25}{\lambda\rho_{ice}}dt = \int_0^{\xi} \frac{1}{U}dz \tag{2}$$

Utilising the key parameter values assumed to remain constant in table 1 and that $\boldsymbol{h_{air}}$ = 100 kg s^{-3} K^{-1}, equation 2 yields that the estimated $\boldsymbol{t}$ taken to form 1 m thickness of ice is 2 428 000 s or 28.1 days.

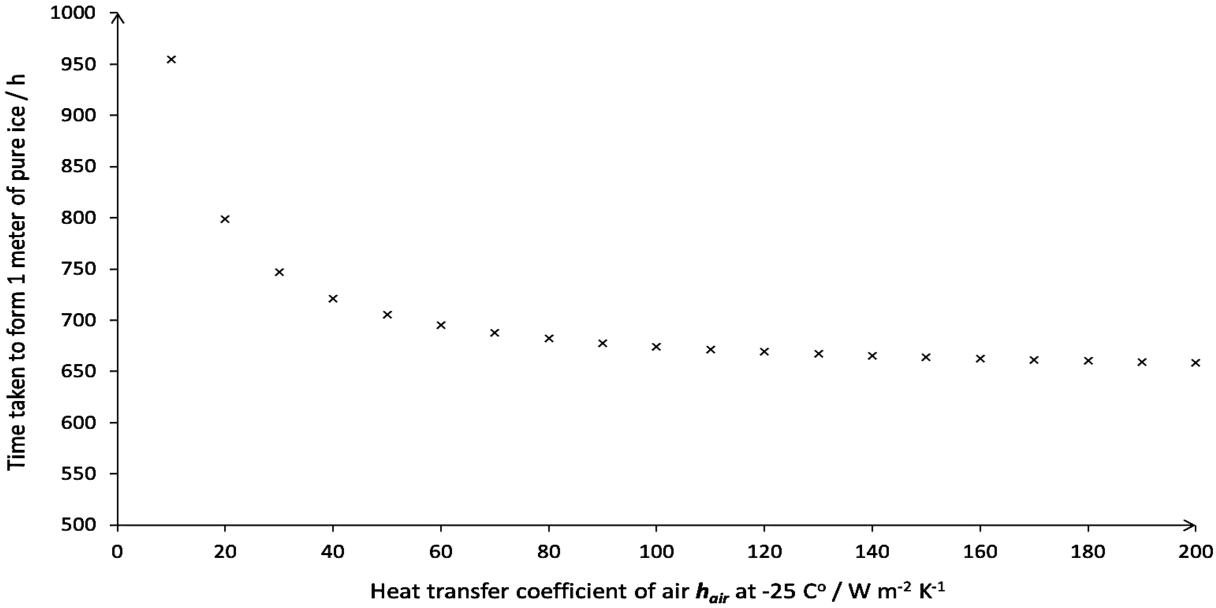
Table 1: Table listing all the parameters employed in modelling the formation of ice from a pool of pure water. The source of each parameter value employed can be found in References, section 4.

$\boldsymbol{\rho_{ice}}$ **at 0 °C [4] / kg m^{-3}**	$\boldsymbol{\rho_{water}}$ **at 0 °C [4] / kg m^{-3}**	$\boldsymbol{\lambda}$ **[4] / kJ kg^{-1}**	$\boldsymbol{k_{ice}}$ **at 0 °C [4] / kg m s^{-3} K^{-1}**	$\boldsymbol{\xi}$ **[4]**
916.2	999.8	334	2.22	0.92

The value of $\boldsymbol{h_{air}}$ may vary depending on wind conditions such as wind speed and actual ambient air $\boldsymbol{T}$ **[1,3]**. As a result, the actual $\boldsymbol{t}$ taken to form 1 m of ice may vary.

Figure 1 depicts the variation in $\boldsymbol{t}$ taken to form 1 m of ice with $\boldsymbol{h_{air}}$ varying from 10-200 kg s^{-3} K^{-1}. Figure 1 indicates that, at values of $\boldsymbol{h_{air}}$ less than 70 kg s^{-3} K^{-1}, the sensitivity of $\boldsymbol{t}$ with respect to $\boldsymbol{h_{air}}$ increases dramatically. For 70 < $\boldsymbol{h_{air}}$ < 190 kg s^{-3} K^{-1}, the % difference in $\boldsymbol{t}$ relative to its value at $\boldsymbol{h_{air}}$ = 100 kg s^{-3} K^{-1} is less than 3 %. The sensitivity is therefore lowest at this range. It's still low at $\boldsymbol{h_{air}}$ > 190 kg s^{-3} K^{-1} but skyrockets at $\boldsymbol{h_{air}}$ ~ 10 kg s^{-3} K^{-1} with a relevant % difference in $\boldsymbol{t}$ relative to $\boldsymbol{t}$ at $\boldsymbol{h_{air}}$ = 100 kg s^{-3} K^{-1} of 41.6 %.

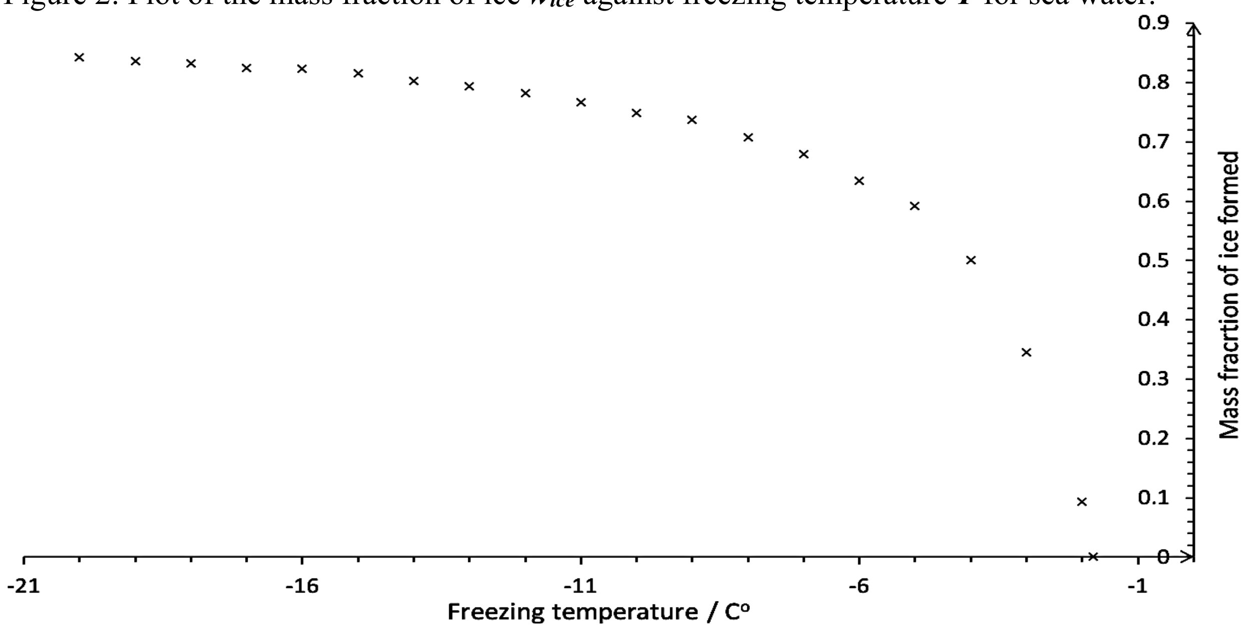
Figure 1: Plot showing the variation in $\boldsymbol{t}$ taken to form 1 m of ice from pure water for $10 < \boldsymbol{h_{air}} < 200$ kg s^{-3} K^{-1}.



2.2 Model II

With regard to the formation of sea ice, which is a mixture of ice crystals and saline solution **[2]**, it was firstly assumed that sea water contains 3.5 wt% NaCl and that it commences the freezing process at $\boldsymbol{T}$ = -1.8 °C **[2]**. A graph of the effect that different concentrations of NaCl have on the freezing $\boldsymbol{T}$ of NaCl solutions was used to obtain the mass fraction of ice $\boldsymbol{w_{ice}}$ formed at different $\boldsymbol{T}$ for $-1.8 < \boldsymbol{T} < -21.1$ °C **[2].** The lever rule was employed in the two-phase region between NaCl solution and ice to estimate $\boldsymbol{w_{ice}}$ and plot the graph depicted in figure 2 **[2,3]**.

Figure 2: Plot of the mass fraction of ice $\boldsymbol{w_{ice}}$ against freezing temperature $\boldsymbol{T}$ for sea water.



In addition, it was assumed that sea ice forms at $\boldsymbol{T}$ = -6.8 °C. The $\boldsymbol{w_{ice}}$ formed at this $\boldsymbol{T}$ was estimated at a value of 0.67 using the lever rule, as also tabulated in table 3. Furthermore, it was assumed that $\boldsymbol{\lambda}$ for the freezing of sea water is equivalent to that of pure water **[2]**.

Table 2: Table listing parameters employed in calculating volume fractions $\boldsymbol{\varphi_i}$ and mass fractions $\boldsymbol{w_i}$ of the ice and aqueous NaCl solution formed at -6.8 °C. The source of each parameter value employed can be found in References, section 4.

ρ_{ice} **at -6.8 °C [4] / kg m^{-3}**	$\rho_{sea\ water}$ **at -1.8 °C [4] / kg m^{-3}**	ρ_{soln} **of NaCl (aq.) at -6.8 °C [4] / kg m^{-3}**
918.0	1023.2	1075.3

The volume fractions of ice and NaCl solution, $\boldsymbol{\varphi_{ice}}$ and $\boldsymbol{\varphi_{soln}}$ respectively, were estimated using the parameter values tabulated in table 2 and the fact that the corresponding weight fraction of the NaCl solution $\boldsymbol{w_{soln}}$ at $\boldsymbol{T}$ = -1.8 °C is 0.33. The values for $\boldsymbol{\varphi_{ice}}$ and $\boldsymbol{\varphi_{soln}}$ were evaluated at 0.704 and 0.296 respectively using mass balances relative to 1 kg of starting sea water solution **[2]**.

Table 3: Table listing the calculated parameter values of volume fractions of ice and NaCl solution at -6.8 °C, $\boldsymbol{\varphi_{ice}}$ and $\boldsymbol{\varphi_{soln}}$ respectively along with the mass fraction of ice formed per kg of sea water $\boldsymbol{w_{ice}}$.

W_{ice} **formed at -6.8 °C**	φ_{ice} **at -6.8 °C**	φ_{soln} **of NaCl at -6.8 °C**
0.67	0.704	0.296

The thermal conductivity of the sea ice formed $\boldsymbol{k}$ was assumed to remain constant with changing sea ice $\boldsymbol{T}$ **[2]**. Additionally, $\boldsymbol{\rho}$ of the sea ice formed was assumed uniform **[2]**. It was also assumed that equation 3 relating the thermal conductivities of the ice and NaCl solution at $\boldsymbol{T}$ = -6.8 °C, $\boldsymbol{k_{ice}}$ and $\boldsymbol{k_{soln}}$ respectively, holds. Parameter values tabulated in table 4 were used in calculating $\boldsymbol{k}$ for sea ice at a value of 1.76 kg m s^{-3} K^{-1}.

$$k = k_{ice}\varphi_{ice} + k_{soln}\varphi_{soln} \tag{3}$$

Table 4: Table listing parameters employed in calculating the thermal conductivity $\boldsymbol{k}$ of sea ice at -6.8 °C. The table also lists the calculated value of $\boldsymbol{k}$ for sea ice. The source of each parameter value employed can be found in References, section 4.

k_{soln} **of NaCl at -6.8 °C [4,5] / kg m s^{-3} K^{-1}**	k_{ice} **at -6.8 °C [4] / kg m s^{-3} K^{-1}**	k **sea ice at -6.8 °C / kg m s^{-3} K^{-1}**
0.56	2.27	1.76

In terms of evaluating the heat transfer coefficient for natural convection from the sea to the ice front $\boldsymbol{h_{sea}}$, it was assumed that the density difference between the NaCl solution and the sea water at the sea ice-sea water interface $\boldsymbol{\Delta\rho}$ is 10 kg m^{-3} **[2]**. Moreover, the correlation in equation 4 between Nusselt number ***Nu*** and Rayleigh number ***Ra*** for free natural convection was assumed for the sea ice-sea water interface and equation 5 was employed to estimate $\boldsymbol{h_{sea}}$ at a value of 66.8 kg s^{-3} K^{-1} **[2]**. The parameter values tabulated in table 5 were used in calculating the value of $\boldsymbol{h_{sea}}$.

$$Nu = 0.0164Ra^{\frac{1}{3}} \tag{4}$$

$$\frac{h_{sea}}{k_{sea\,water}} = 0.0164\left(\frac{g\rho_{sea\,water}\Delta\rho}{\mu_{sea\,water}}\frac{C_p}{k_{sea\,water}}\right)^{\frac{1}{3}} \tag{5}$$

Table 5: Table listing parameters employed in calculating the heat transfer coefficient for natural convection from the sea to the ice front $\boldsymbol{h_{sea}}$ **[2]**. The source of each parameter value employed can be found in References, section 4.

$\mu_{sea\,water}$ at -1.8 °C [4] / 10^{-3} N m^{-2} s	k sea water at -1.8 °C [4,6] / kg m s^{-3} K^{-1}	$\Delta\rho$ / kg m^{-3}	g [4] / m s^{-2}	C_p [6] / J kg^{-1} K^{-1}
1.88	0.563	10	9.8145	3990

In estimating the rate of freezing of sea water, instantaneous steady state was assumed **[2]**. The heat conducted away from the freezing front $\boldsymbol{q_{cond}}$ was equated to the sum of the heat released by the rate of ice formation $\boldsymbol{F_{ice}}$ and that provided by natural convection using equation 6. In equation 6, the temperature driving force for the heat provided by natural convection is 5 °C as it is the difference between ***T*** of the sea water underneath of -1.8 °C and ***T*** of the ice formation front of -6.8 °C. Negligible variation in ***T*** across the sea ice thickness formed was assumed **[2,3]**.

$$q_{cond} = F_{ice}\lambda + 5h_{sea} = \frac{18.2kh_{air}}{k+h_{air}z} \tag{6}$$

The limiting thickness of the ice sheet formed $\boldsymbol{L_{max}}$ was estimated by setting the ice formation driving force $\boldsymbol{F_{ice}}$ in equation 6 to zero **[2]**. Equation 7 was derived and utilised when calculating the $\boldsymbol{L_{max}}$. The value of ice thickness ***z*** obtained in equation 7 was rescaled by multiplying it by ***v*** which is the ratio between the apparent sea ice ***ρ*** at ***T*** = -6.8 °C and the ***ρ*** of sea water at ***T*** = -6.8 °C tabulated in table 6. This ratio ***v*** accounts for the sea water expanding on freezing **[2,3]**. The apparent $\boldsymbol{\rho}$ of sea ice at $\boldsymbol{T}$ = -6.8 °C was calculated at 964.6 kg m^{-3} using $\boldsymbol{\varphi_{ice}}$ and $\boldsymbol{\varphi_{soln}}$ from table 3 and a mass balance based on 1 kg of starting sea water **[2]**. $\boldsymbol{L_{max}}$ was therefore estimated at 0.083 m.

$$z = \frac{k}{h_{air}}\left(\frac{18.2h_{air}}{5h_{sea}} - 1\right) \tag{7}$$

Table 6: Table listing the calculated parameter values of $\boldsymbol{h_{sea}}$ and limiting ice thickness $\boldsymbol{L_{max}}$ along with the ratio between sea water density at -1.8 °C and apparent sea ice density at -6.8 °C $\boldsymbol{v}$ employed in the calculation of $\boldsymbol{L_{max}}$.

$\boldsymbol{h_{sea}}$ / kg s^{-3} K^{-1}	$\boldsymbol{L_{max}}$ / m	$\boldsymbol{v}$ [4]
66.8	0.083	1.11

In estimating $\boldsymbol{t}$ taken to form 0.95$\boldsymbol{L_{max}}$, equation 8 was employed as $\boldsymbol{F_{ice}}$ varies with time. $\boldsymbol{F_{ice}}$ was equated to $\boldsymbol{\rho_{ice}}$ at $\boldsymbol{T}$ = -6.8 °C multiplied by d$\boldsymbol{z}$/d$\boldsymbol{t}$. In the first order ODE derived in equation 8, the left-hand side term accounts for the overall heat transfer coefficient $\boldsymbol{U}$ from the ice formation front to the air multiplied by the overall temperature driving force between the air at $\boldsymbol{T}$ = -25 °C and the ice formation front at $\boldsymbol{T}$ = -6.8 °C **[2]**. By solving the first order ODE analytically with an upper limit of 0.95$\boldsymbol{L_{max}}$/$\boldsymbol{v}$ or 0.074 m and initial conditions of $\boldsymbol{z}$ = 0 at $\boldsymbol{t}$ = 0, the time $\boldsymbol{t}$ taken was estimated at 195 000 s or 2.3 days, which is not that significantly long.

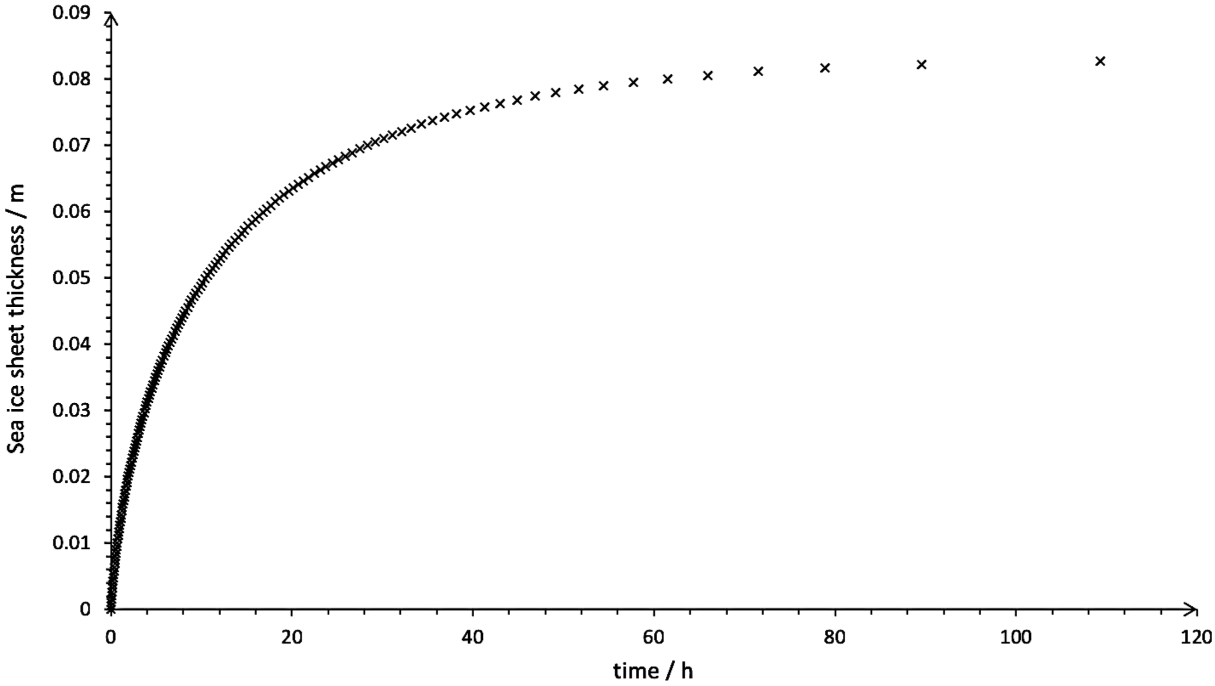
The graph depicted in figure 3 was plotted by use of equation 8 and it can indeed be observed that the thickness of ice formed tends to a limiting value of $\boldsymbol{L_{max}}$. The rate in ice thickness formation decreases with time, also depicted by the decreasing gradient with increasing $\boldsymbol{t}$ in figure 3 **[2]**.

$$\frac{18.2kh_{air}}{k+h_{air}z} = \lambda\rho_{ice}\frac{dz}{dt} + 5h_{sea} \tag{8}$$

In terms of estimating the difference in densities between the ice formation front and sea water underneath $\boldsymbol{\Delta\rho}$, an iterative scheme would be necessary whereby equations 6 and 8 could be used to balance the relevant terms in order to achieve the left-hand side terms being equal to the right-hand side terms at different $\boldsymbol{h_{sea}}$ values. The estimate would be refined accordingly in each successive turn **[2]**.

On the other hand, it would also be possible to devise a scheme by which to estimate $\boldsymbol{\Delta\rho}$ by using calculated volume fractions of relevant phases involved $\boldsymbol{\varphi_i}$ and mass balances based on starting sea water mass as well as using knowledge of the mass diffusivity of NaCl at relevant temperatures **[3]**.

Figure 3: Plot of the evolution of ice sheet thickness being formed in the sea with the passage of time.



3 Conclusions

The effects that transient heat transfer can have on the formation of ice in both fresh and sea water were effectively and quantitatively modelled based on a multitude of reasonable assumptions.

It can be concluded that, the estimated time taken to form 1 m thick ice in fresh water of 28.1 days is much longer than the estimated time taken to form $0.95\boldsymbol{L_{max}}$ thick sea ice of 2.3 days. On the other hand, sea ice can only form up to a limiting thickness $\boldsymbol{L_{max}}$ of approximately 0.083 m which is about 1/12th of a meter.

Furthermore, it can be conclusively said that at low values of $\boldsymbol{h_{air}}$ of approximately less than 70 kg s^{-3} K^{-1}, the sensitivity of the time taken to form 1 m thick ice in fresh water with respect to the parameter $\boldsymbol{h_{air}}$ increases dramatically.

Moreover, as time increases, the rate of ice growth indeed decreases for both fresh and sea water ice formation.

Knowledge of mass transport phenomena with regard to NaCl mass diffusivity in brine of varying NaCl concentrations at different temperatures would greatly aid in better modelling the effects of transient heat transfer in sea ice formation since mass transfer and heat transfer are coupled physical processes **[3]**.

4 References

[1] D.I. Wilson (2019) CET Lecture notes – Heat and Mass Transfer Fundamentals, pages 2-10 to 1-14.

[2] Chemical Engineering Tripos Part I, Exercise 4 – Transient Heat Transfer, LT 2019, Department of Chemical Engineering and Biotechnology, University of Cambridge, pages 1-2, 4 figure 1.

[3] F.P. Incorpera, D. P. De Witt, T. L. Bergman, A. S. Lavine (2007) Fundamentals of Heat and Mass Transfer. Edition: 6th. Wiley, NY.

[4] Engineeringtoolbox.com. (2019). *Engineering ToolBox*. [online] Available at: http://www.engineeringtoolbox.com/, properties of pure water at different temperatures, properties of pure ice at different temperatures, properties of brine at different NaCl w.t % and temperatures.

[5] A. Melinder, In: 'Thermophysical Properties of Aqueous Solutions Used as Secondary Working Fluids', Department of Energy Technology, Royal Institute of Technology, KTH Stockholm, Sweden, 2007, Doctoral Thesis, Report No 07/60, pages 16-18.

[6] M. H. Sharqawy, J. H. Lienhard, S. M. Zubair, In: 'Thermophysical properties of Seawater: a review of existing correlations and data', desalination and Water Treatment, www.deswater.com, 2010 Desalination publications. Doi no. 10.5004/dwt.2010.1079.

5 Nomenclature

$\boldsymbol{\lambda}$	specific latent heat of fusion of water	(J kg^{-1})
$\boldsymbol{\mu}$	viscosity of sea water at -1.8 ^{o}C	(N m^{-2} s)
$\boldsymbol{\nu}$	ratio between sea water density at -6.8 ^{o}C and apparent sea ice density at -6.8 ^{o}C	(-)
$\boldsymbol{\xi}$	ratio between density of pure ice at 0 ^{o}C and pure water at 0 ^{o}C	(-)

ρ_i	mass density of relevant species or phase denoted by subscript i	(kg m^{-3})
$\Delta\rho$	density difference between NaCl solution at -6.8 °C and sea water at -1.8 °C at sea ice-sea water interface	(kg m^{-3})
φ_i	volume fraction of relevant species or phase denoted by subscript i	(-)
C_p	specific heat capacity of sea water at -1.8 °C	(J kg^{-1} K^{-1})
F_{ice}	rate of ice formation	(kg m^{-2} s^{-1})
g	acceleration due to gravity	(m s^{-2})
h_i	heat transfer coefficient of relevant species or phase denoted by subscript i	(kg s^{-3} K^{-1})
k_i	thermal conductivity of relevant species or phase denoted by subscript i	(kg m s^{-3} K^{-1})
L_{max}	limiting thickness of ice sheet formed in the sea	(m)
m_i	mass of relevant species or phase denoted by subscript i	(kg)
Nu	Nusselt number	(-)
q_{cond}	heat conducted away from freezing front through the sea ice	(J m^{-2} s^{-1})
Ra	Rayleigh number	(-)
t	timescale for ice layer thickness formation	(s)
T	temperature	(°C)
U	overall heat transfer coefficient	(kg s^{-3} K^{-1})
V_i	volume of relevant species or phase denoted by subscript i	(m^3)
w_i	weight fraction of relevant species or phase denoted by subscript i	(-)

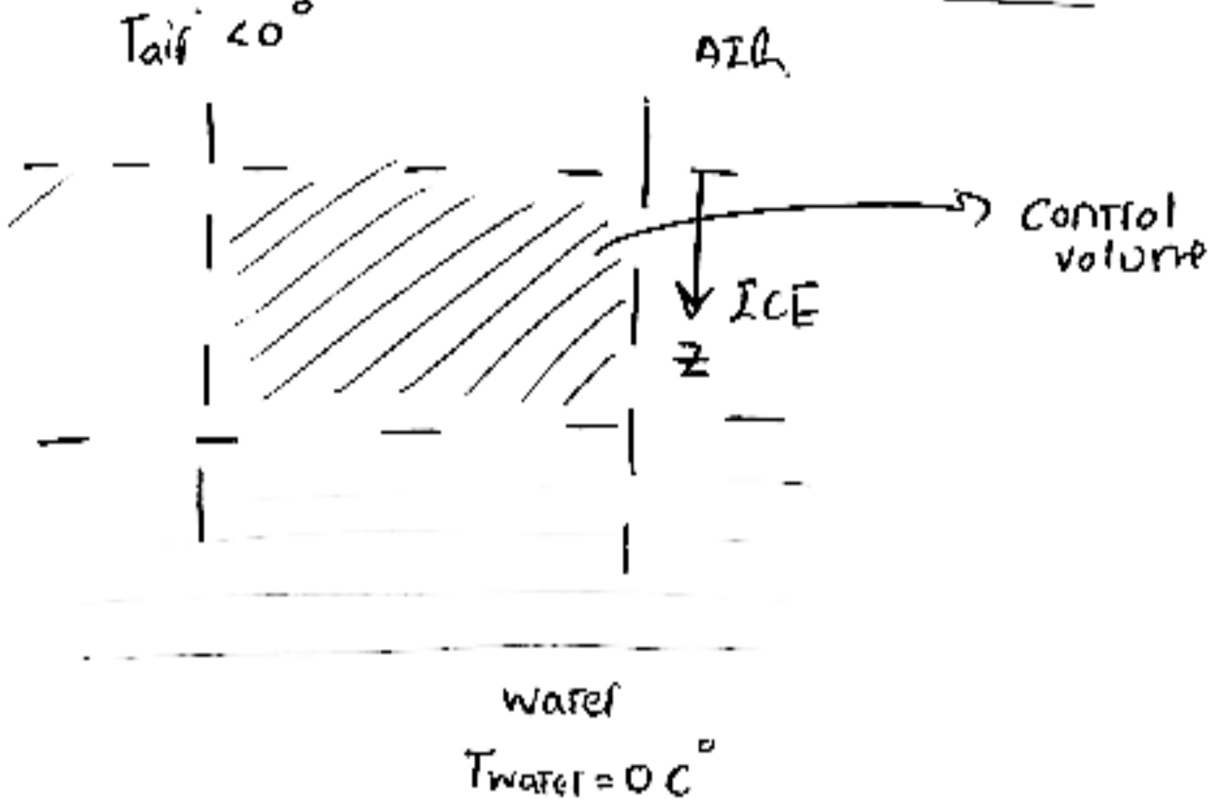
z	ice layer thickness	(m)

6 Appendix

Appendix 1.A: Sample derivations and calculation details for the ice formation models:

SAMPLE DERIVATIONS & CALCULATION DETAILS:

- Model(I) parts 1 and 2.
  - Formation of ice from large pool of pure water:



→ Enthalpy balance on C.V :

$$q = U(0 - T_{amb}) = \frac{d}{dt}(\rho_{Ice} \cdot \lambda \cdot z)$$

λ is the specific latent heat of fusion of water.

$$\therefore U(0 - T_{amb}) = \frac{dz}{dt}(\rho_{Ice} \cdot \lambda)$$

→ For a wide slab which is how the ice formed is treated :

Assuming ρ_{Ice} & λ are independent of z & t

$$\frac{1}{U} = \frac{1}{h_{air}} + \frac{z}{k_{ICE}}$$

$$h_{slab} = \frac{k_{slab\ material}}{z}$$

$$\therefore U = \frac{k_{Ice} + h_{air} \cdot z}{h_{air} \cdot k_{Ice}}$$

U is the overall heat transfer coefficient

• Calculating estimated t taken for 1m of ice to form when $h_{air} = 100\ Wm^{-2}K^{-1}$:

• Parameter values :

• $k_{Ice}(0C°) = 2.22\ Wm^{-1}K^{-1}$ • $T_{air} = -25\ C°$
• $\lambda = 334\ kJkg^{-1}$ • $\rho_{water}(0C°) = 999.8\ kgm^{-3}$
• $h_{air} = 100\ Wm^{-2}K^{-1}$ • $\rho_{pure\ ice}(0C°) = 916.2\ kgm^{-3}$

→ ∴ Using Model (I) which is :

$$\frac{h_{air} \cdot k_{Ice}}{k_{Ice} + 2h_{air}} \cdot (0 - T_{air}) = \rho_{Ice} \cdot \lambda \cdot \frac{dz}{dt}$$

It follows that :

$$\int_0^t \frac{h_{air} \cdot u_{Ice}}{\rho_{Ice} \cdot \lambda} \cdot (0-(-25))\, dt = \int_0^{1 \cdot \frac{916.2}{999.8}} (u_{Ice} + z h_{air})\, dz$$

• the upper limit on RHS is derived from a mass balance on initial C.V, assuming area @ interface hasn't changed.

i.e. $999.8 \cdot z \cdot A = 916.2 \cdot A \cdot 1$

$$\therefore z = \frac{916.2}{999.8} \cdot 1$$

$$\therefore \frac{h_{air}\, u_{Ice} \cdot (25)}{\rho_{Ice} \cdot \lambda} \cdot t = uz + \frac{1}{2} z^2 h_{air}$$

substituting all relevant parameters values & $z = \frac{916.2}{999.8}$, the time taken is estimated to be. 28.1 days.

• Sensitivity analysis on the value of t. with respect to hair :

→ when carrying out the analysis, the following was derived :

$$t = \frac{\left(uz + \frac{1}{2} z^2 h_{air}\right) \rho_{Ice}\, \lambda}{h_{air}\, u_{Ice} \cdot 25}$$

$$\therefore \left|\frac{\partial t}{t}\right| = \left| -h_{air}^{-2} \cdot \frac{\partial h_{air}}{\left(\frac{1}{h_{air}} + \frac{1}{2u_{Ice}}\right)} \right|$$

$$\therefore \frac{\Delta t}{t} = \frac{u_{Ice} \Delta h_{air}}{\left(u_{Ice} h_{air} + \frac{h_{air}^2}{2}\right)}$$

→ <u>Accept 1% discrepancy:</u>

$$0.01 = \frac{\Delta h_{air}}{\left(h_{air} + \frac{h_{air}^2}{2u_{Ice}}\right)}$$

$$\therefore 0.01 = \frac{\Delta h_{air}}{h_{air}}\left(1 - \frac{h_{air}}{2u_{Ice}} + O\left(h_{air}^2\right)\right)$$

Let $h_{air} = 100$ & $u_{Ice} = 222$.

$\therefore$ Accepted Δh_{air} is ± 23.52. From the value of $100\ Wm^{-2}K^{-1}$

- $70 < h_{air} < 190$.

  If $h_{air} < 70$ or $\underbrace{h_{air} > 190}_{\text{less significant}}$

  $\therefore$ discrepancy in value is $> 1\%$

- Model (II) parts. 3 and 4

→ • Formation of sea Ice :

• For plot of W_{Ice} w.t.% vs. Freezing Temperature, the lever rule was employed on graph provided for $(-1.8 > T > -21.1)C^{\circ}$

• Calculating mass of Ice formed per kg of sea water Frozen at $-6.8C^{\circ}$ and φ_{Ice} and φ_{soln} :

→ From plot :

W_{Ice} @ $-6.8C^{\circ} = 0.67$

W_{soln} @ $-6.8C^{\circ} = 0.33$

Applying overall mass balance with respect to 1kg of starting sea water :

• Relevant parameters :

- ρ_{Ice} @ $-6.8C^{\circ} = 918.004\ kg\,m^{-3}$
- ρ_{soln} @ $-6.8C^{\circ} = 1075.275\ kg\,m^{-3}$

$$\therefore\ V_{soln} = \frac{0.33}{1075275} \cdot 1$$

$$V_{Ice} = \frac{0.67}{918.004} \cdot 1$$

$$\therefore\ \varphi_{soln} = \frac{\frac{0.33}{1075.275}}{\frac{0.33}{1075.275} + \frac{0.67}{918.004}} = 0.296$$

$$\varphi_{Ice} = \frac{\frac{0.67}{918.004}}{\frac{0.67}{918.004} + \frac{0.33}{1075.275}} = 0.704$$

→ For thermal conductivity of sea ice formed k :

Relevant parameters :

$$\boxed{\begin{array}{l} k_{soln} = 0.577\ Wm^{-1}K^{-1} \\ k_{Ice} = 2.268\ Wm^{-1}K^{-1} \end{array}}$$

$$\therefore \quad k = 0.577 \cdot 0.296 + 0.704 \cdot 2.268 = 1.76\ Wm^{-1}K^{-1}$$

→ Estimating h_{sea} :

$$Nu = 0.0164\, Ra^{1/3}$$

$$\frac{h_{sea}}{k_{sea}} = 0.0164 \cdot \left(\frac{g \rho_{sea} \Delta\rho}{\mu_{sea}^2} \cdot Pr_{sea} \right)^{1/3}$$

• Relevant parameter values :

• $\mu_{sea} = 1.88 \cdot 10^{-3}\ Pa\,s$	• $k_{sea} = 0.563\ Wm^{-1}K^{-1}$
• $\rho_{sea} = 1023.2\ kgm^{-3}$	• $Cp_{sea} = 3987\ JKg^{-1}$
$Pr_{sea} = \frac{\mu_{sea} Cp_{sea}}{k_{sea}} \simeq 13.34$	• $\Delta\rho = 10\ kgm^{-3}$

$$\therefore \quad h_{sea} = 0.563 \cdot 0.0164 \cdot \left(\frac{9.8145 \cdot 1023.2 \cdot 10}{(1.88 \cdot 10^{-3})^2} \cdot Pr_{sea} \right)^{1/3}$$

$$\therefore\ h_{sea} = 66.78\ Wm^{-2}K^{-1}$$

Calculating the rate @ which sea ice freezes in part 5.

→ Estimating the limiting ice thickness, L_{max}:

- The equation used was:

$$q_{cond} = F_{ice} \cdot \lambda + h_{sea}\left[T_{front} - T_{sea}\right]$$

where $\left[T_{front} - T_{sea}\right] = 5C°$

For L_{max}, the driving force for ice formation F_{ice} in $kg\,m^{-2}s^{-1}$ was set to zero ∴

$$q_{cond} = h_{sea} \cdot 5.$$

- By extracting an overall heat transfer coefficient from the air-sea ice interface to the sea ice front for q_{cond}, the following expression was obtained:

$$\frac{k \cdot h_{air}}{k + h_{air} \cdot z} \cdot (-6.8 + 25) = h_{sea} \cdot 5$$

where k is the thermal conductivity of the sea ice formed.

- Relevant parameter values:

• $h_{air} = 100\ Wm^{-2}K^{-1}$	• $T_{sea\,ice} - T_{air} = 18.2$
• $k = 1.762\ Wm^{-1}K^{-1}$	• $T_{sea} - T_{front} = -5.$
• $h_{sea} = 66.78\ Wm^{-2}K^{-1}$	

z was therefore evaluated by: $\frac{1.762}{100}\left(\frac{100}{66.78} \cdot \frac{18.2}{5} - 1\right) = z$

at a value of: 0.0784 m

• The apparent sea ice density formed was computed

using : $\frac{1\ kg}{V_{soln} + V_{ice}} = \rho_{sea\ ice\ apparent}$

$\therefore \rho_{sea\ ice\ apparent} = \frac{1}{\frac{0.33}{1075.275} + \frac{0.67}{918.004}} = 964.56\ kg\,m^{-3}$

$\therefore$ L_{max} was obtained using a mass balance of :

$$L_{max} = \frac{1023.2\ kg\,m^{-3}}{964.56\ kg\,m^{-3}} \cdot 0.0784\ m = \boxed{0.0832\ m}$$

where $1023.2\ kg\,m^{-3}$ is the density of the sea water at $-1.8C°$.

→ Estimating the time taken to form ice sheet of thickness $0.95\ L_{max}$:

• $0.95\ L_{max} = 0.95 \cdot 0.0832\ m$

uppel limit on relevant integral though $= 0.95 \cdot 0.0784m = 0.0745m$

• Using $q_{cond} = F_{ice} \cdot \Lambda + h_{sea} \cdot 5$

where : • $q_{cond} = \frac{k \cdot h_{air}}{k + h_{air} \cdot Z}$ (18.2)

• $F_{ice} = \rho_{ice} \frac{dz}{dt}$

• Λ is the specific latent heat of water. at $334\ kJ\,kg^{-1}$

• The following ODE was set-up:

$$(18.2)\frac{\kappa \cdot h_{air}}{\kappa + h_{air} \cdot z} = \rho_{ice} \cdot \Delta \cdot \frac{dz}{dt} + h_{sea} \cdot 5$$

• This was re-arranged:

$$\frac{(18.2) \cdot \kappa h_{air}}{\rho_{ice} \cdot \Delta (\kappa + h_{air} \cdot z)} = \frac{dz}{dt} + \frac{h_{sea} \cdot 5}{\rho_{ice} \cdot \Delta}$$

• the following constants were assigned:

$$C_1 = \frac{18.2 \cdot \kappa \cdot h_{air}}{\rho_{ice} \cdot \Delta}, \quad C_2 = \frac{h_{sea} \cdot 5}{\rho_{ice} \cdot \Delta}, \quad C_3 = h_{air}$$

and $C_4 = \kappa$.

$$\therefore \quad \frac{C_1}{C_4 + C_3 z} = \frac{dz}{dt} + C_2$$

$$\therefore \int_0^t dt = \int_0^{0.0745} \frac{1}{\frac{C_1}{z \cdot C_3 + C_4} - C_2} \, dz$$

$$\therefore \int_0^t dt = \int_0^{0.0745} \frac{z \cdot C_3 + C_4}{C_1 - C_2 \cdot C_4 - C_2 C_3 z} \, dz$$

- The integral was computed analytically and using the initial condition that at $t=0$, $z=0$ the following expression for time was derived in general:

$$t = -88169.5 \cdot \ln(0.0784012 - z) - 918274 \cdot z - 224472.145$$

- The time taken to form $0.95 L_{max}$ was computed at 195747 seconds or 2.27 days.

- A plot of $z \cdot \left(\frac{1023.2}{964.56}\right)$ was formed.

  with the density ratio included assuming a constant expansion factor with dt for sea ice formed from sea water.

SPREADSHEET NUMERICAL WORKINGS FOR LIMITING SEA ICE THICKNESS L_{max} AND TIME TAKEN TO FORM $0.95L_{max}$:

PARAMETER VALUES				
h_{air}	ρ_{water} (0 degrees)	ρ_{ice} (0 degrees)	λ	
100	999.8	916.2	334*10^3	
k_{ice} (0 degrees)	$\rho_{apparent\ sea\ ice}$ (-6.8 degrees)	ρ_{soln} (-1.8 degrees)	ρ_{soln} (-6.8 degrees)	
2.22	964.5596202	1023.2	1075.3	
k_{ice} (-6.8 degrees)	φ_{soln}	φ_{ice}	k_{soln} (-6.8 degrees)	
2.268	0.296	0.704	0.557	

$\mu_{sea\ water}$ (-1.8 degrees)	C_p	$k_{sea\ ice}$ (-6.8 degrees)		
1.88*10^-3	3987	1488.2		
h_{sea}				
66.78				
OBTAINIG lmax				
	Zmax		**expression 1**	**expression 2**
	0.0783376		0.0176	4.451
	ACTUAL ZMAX			
	0.083100133			

OBTAINING z vs t			
hair	h_{sea}	$k_{sea\ ice}$ **(-6.8 degrees)**	**λ**
100	66.78	1.761544	334000
ΔT1	**ΔT2**	**ρ ice -1.8**	$\rho_{apparent\ sea\ ice}$ **(-6.8 degrees)**
18.2	5	918.004	964.55962
C1	**C2**	**C3**	**C4**
1.0456E-05	1.089E-06	100	1.761544
Upper limit	**C1-C2*C4**	**C2*C3**	**ANS**
0.07448155	8.5379E-06	0.0001089	z= 0.07900966 m
			t=54.3741667 h
			t=2.26559028 days
constant of integration			
-224472.14			

t	**z**	**ACTUAL SEA ICE thickness**	
0	0	0	
0.02915565	0.0005	0.0005304	
0.05932032	0.001	0.00106079	
0.09050704	0.0015	0.00159119	
0.12272914	0.002	0.00212159	
0.15600021	0.0025	0.00265199	
0.19033412	0.003	0.00318238	
0.22574501	0.0035	0.00371278	
0.26224731	0.004	0.00424318	
0.29985575	0.0045	0.00477358	
0.33858533	0.005	0.00530397	
0.37845138	0.0055	0.00583437	
0.41946955	0.006	0.00636477	
0.46165581	0.0065	0.00689517	
0.50502646	0.007	0.00742556	
0.54959814	0.0075	0.00795596	
0.59538785	0.008	0.00848636	
0.64241297	0.0085	0.00901676	
0.69069121	0.009	0.00954715	
0.74024072	0.0095	0.01007755	
0.79108	0.01	0.01060795	
0.84322798	0.0105	0.01113835	
0.896704	0.011	0.01166874	
0.95152784	0.0115	0.01219914	
1.00771973	0.012	0.01272954	
1.06530035	0.0125	0.01325994	
1.12429084	0.013	0.01379033	
1.18471285	0.0135	0.01432073	
1.24658853	0.014	0.01485113	
1.30994052	0.0145	0.01538153	
1.37479204	0.015	0.01591192	
1.44116681	0.0155	0.01644232	
1.50908916	0.016	0.01697272	
1.57858398	0.0165	0.01750312	
1.64967679	0.017	0.01803351	
1.72239371	0.0175	0.01856391	
1.79676152	0.018	0.01909431	
1.87280768	0.0185	0.01962471	

1.9505603	0.019	0.0201551	
2.03004826	0.0195	0.0206855	
2.11130112	0.02	0.0212159	
2.19434925	0.0205	0.0217463	
2.27922378	0.021	0.02227669	
2.36595668	0.0215	0.02280709	
2.45458074	0.022	0.02333749	
2.54512965	0.0225	0.02386789	
2.637638	0.023	0.02439828	
2.73214131	0.0235	0.02492868	
2.82867609	0.024	0.02545908	
2.92727985	0.0245	0.02598948	
3.02799116	0.025	0.02651987	
3.13084968	0.0255	0.02705027	
3.23589618	0.026	0.02758067	
3.34317262	0.0265	0.02811107	
3.45272217	0.027	0.02864146	
3.56458928	0.0275	0.02917186	
3.6788197	0.028	0.02970226	
3.79546055	0.0285	0.03023266	
3.91456039	0.029	0.03076305	
4.03616924	0.0295	0.03129345	
4.16033868	0.03	0.03182385	
4.28712189	0.0305	0.03235425	
4.41657371	0.031	0.03288464	
4.54875076	0.0315	0.03341504	
4.68371144	0.032	0.03394544	
4.82151608	0.0325	0.03447584	
4.96222697	0.033	0.03500623	
5.10590848	0.0335	0.03553663	
5.25262715	0.034	0.03606703	
5.40245177	0.0345	0.03659743	
5.55545348	0.035	0.03712782	
5.71170594	0.0355	0.03765822	
5.87128534	0.036	0.03818862	
6.03427062	0.0365	0.03871902	
6.20074357	0.037	0.03924941	
6.37078893	0.0375	0.03977981	
6.54449458	0.038	0.04031021	
6.7219517	0.0385	0.04084061	
6.90325489	0.039	0.041371	
7.0885024	0.0395	0.0419014	

7.27779629	0.04	0.0424318	
7.47124261	0.0405	0.0429622	
7.66895166	0.041	0.04349259	
7.87103819	0.0415	0.04402299	
8.07762163	0.042	0.04455339	
8.28882638	0.0425	0.04508379	
8.50478208	0.043	0.04561418	
8.72562389	0.0435	0.04614458	
8.95149283	0.044	0.04667498	
9.1825361	0.0445	0.04720538	
9.41890748	0.045	0.04773577	
9.6607677	0.0455	0.04826617	
9.90828487	0.046	0.04879657	
10.161635	0.0465	0.04932697	
10.4210022	0.047	0.04985736	
10.6865799	0.0475	0.05038776	
10.9585706	0.048	0.05091816	
11.237187	0.0485	0.05144856	
11.5226526	0.049	0.05197895	
11.8152024	0.0495	0.05250935	
12.1150836	0.05	0.05303975	
12.4225567	0.0505	0.05357015	
12.7378962	0.051	0.05410054	
13.061392	0.0515	0.05463094	
13.3933501	0.052	0.05516134	
13.7340941	0.0525	0.05569174	
14.0839665	0.053	0.05622213	
14.4433303	0.0535	0.05675253	
14.8125707	0.054	0.05728293	
15.1920965	0.0545	0.05781333	
15.5823428	0.055	0.05834372	
15.9837725	0.0555	0.05887412	
16.3968796	0.056	0.05940452	
16.8221912	0.0565	0.05993492	
17.2602712	0.057	0.06046531	
17.7117232	0.0575	0.06099571	
18.1771948	0.058	0.06152611	
18.657382	0.0585	0.06205651	
19.1530336	0.059	0.0625869	
19.6649572	0.0595	0.0631173	
20.1940256	0.06	0.0636477	
20.7411833	0.0605	0.0641781	

21.3074555	0.061	0.06470849	
21.8939568	0.0615	0.06523889	
22.5019023	0.062	0.06576929	
23.1326201	0.0625	0.06629969	
23.7875655	0.063	0.06683008	
24.4683381	0.0635	0.06736048	
25.176701	0.064	0.06789088	
25.9146046	0.0645	0.06842128	
26.6842135	0.065	0.06895167	
27.4879395	0.0655	0.06948207	
28.3284802	0.066	0.07001247	
29.2088667	0.0665	0.07054287	
30.1325202	0.067	0.07107326	
31.1033228	0.0675	0.07160366	
32.1257034	0.068	0.07213406	
33.2047459	0.0685	0.07266446	
34.346325	0.069	0.07319485	
35.5572794	0.0695	0.07372525	
36.8456344	0.07	0.07425565	
38.2208942	0.0705	0.07478605	
39.6944285	0.071	0.07531644	
41.2799954	0.0715	0.07584684	
42.9944612	0.072	0.07637724	
44.858813	0.0725	0.07690764	
46.8996218	0.073	0.07743803	
49.1512172	0.0735	0.07796843	
51.6590371	0.074	0.07849883	
54.4850067	0.0745	0.07902923	
57.7166258	0.075	0.07955962	
61.4833354	0.0755	0.08009002	
65.9885126	0.076	0.08062042	
71.5793385	0.0765	0.08115082	
78.9255433	0.077	0.08168121	
89.6075145	0.0775	0.08221161	
109.300165	0.078	0.08274201	