# COMPUTATIONAL FINANCE: 422

## *The Basic Theory of Interest*

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(Slides courtesy of Daniel Kuhn)

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# This Lecture

- The time value of money
  - Compounding
  - Present and future value
  - Net present value as a decision criterion
- The term structure of interest rates
  - Spot rates
  - Forward rates
  - Expectation dynamics

Further reading:

- D.G. Luenberger: *Investment Science*, Chapters 2,4

# Principal and Interest

Example: if you invest \$1.00 in a bank account that pays 8% interest per year, then at the end of 1 year you will have in your account \$1.08.

- Principal: amount invested ($W$).
- Interest: ‘rent’ paid on investment ($I$).
- Interest rate: interest per unit of currency invested ($r$).

$$\Rightarrow \quad I = W \times r$$

Account holdings:

- Initial wealth (today): $W_0 = W$;
- Terminal wealth (after one year): $W_1 = W(1 + r)$.

# Compound Interest I

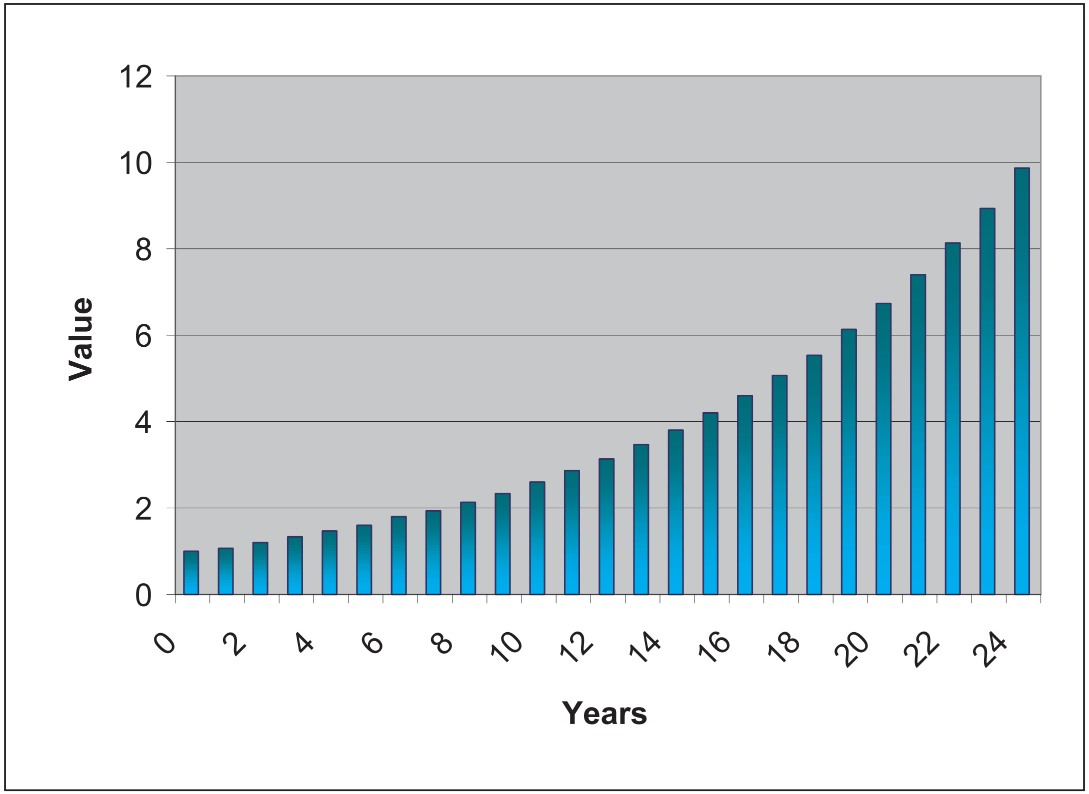
Consider a situation in which money is invested in a bank account over several periods. Assume that the interest rate in the $n$th year is $r_n$ for $n = 1, 2, 3, \ldots$. We obtain the following account holdings:

- today: $W_0 = W$;
- after 1 year: $W_1 = W(1 + r_1)$;
- after 2 years: $W_2 = W_1(1 + r_2) = W(1 + r_1)(1 + r_2)$;
- after $n$ years: $W_n = W_{n-1}(1 + r_n) = W \prod_{i=1}^{n}(1 + r_i)$.

If the interest rate is constant, i.e., $r_n = r$, then

$$W_n = W(1 + r)^n \quad \Rightarrow \quad r = \left(\frac{W_n}{W_0}\right)^{1/n} - 1 \,.$$

# Compound Interest II




The seven-ten rule:

- Money invested at 7% doubles in about 10 years;
- Money invested at 10% doubles in about 7 years (Figure).

# Compounding at Various Intervals

It is traditional to quote the interest rate on a yearly basis but then apply the appropriate proportion of that interest rate over each compounding period. Divide a year in $m$ equally spaced compounding periods.

- Nominal interest rate: $r$
- Length of a compounding period: $1/m$ [years]
- Interest rate for each of the $m$ periods: $r/m$
- Growth of the account over $k$ periods: $[1+r/m]^k$
- Growth of the account over 1 year: $[1+r/m]^m$
- The effective interest rate is the number $r_{\text{eff}}$ such that

$$1+r_{\text{eff}}=[1+r/m]^m\,.$$

# Continuous Compounding I

Increasing the number of compounding intervals per year infinitely leads to the idea of continuous compounding.

- Time measured in years: $t$
- Time measured in # compounding intervals: $k = tm$

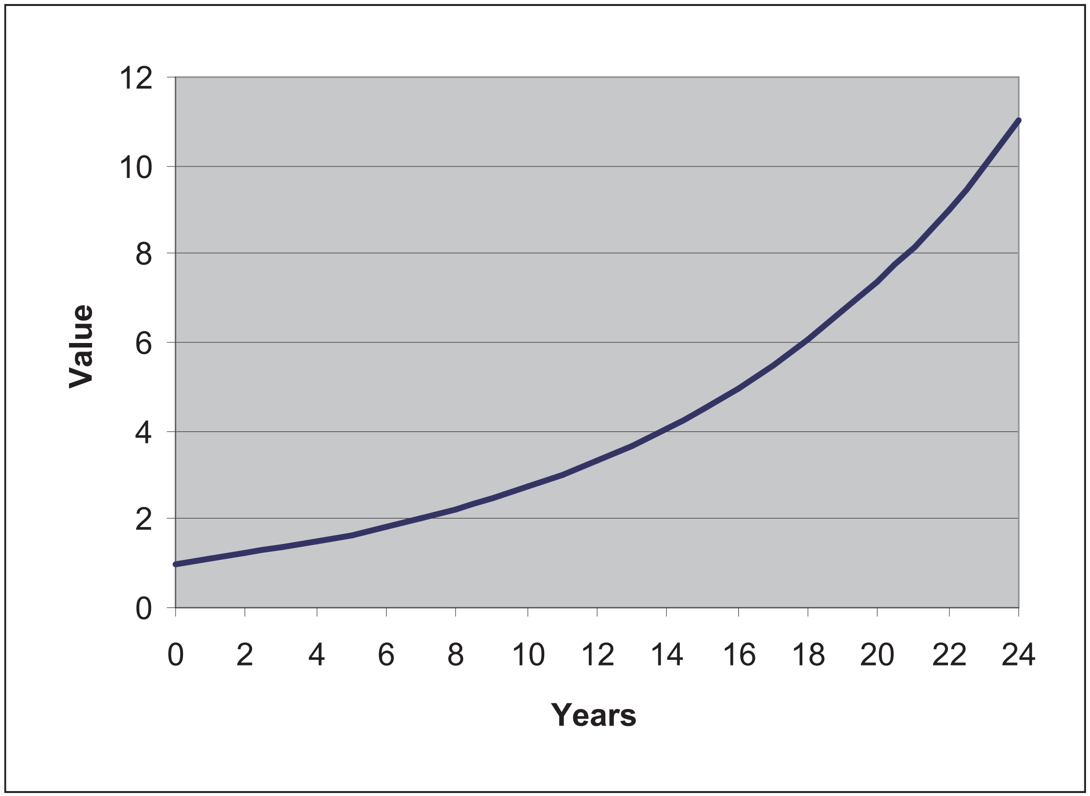
If $m$ is very large, then we can assume that $k \in \mathbb{N}$.

If $m$ tends to infinity, then the growth of an account with (nominal) interest rate $r$ over $t$ years becomes:

$$[1 + r/m]^k = [1 + r/m]^{mt} = ([1 + r/m]^m)^t \to e^{rt} \,.$$

The last expression corresponds to continuous compounding (in the limit $m \to \infty$) $\Rightarrow$ this leads to the familiar exponential growth curve.

# Continuous Compounding II




Under continuous compounding at 10% the value of $1

- doubles in about 7 years;
- grows by a factor of 8 in about 20 years.

# Debt

- A bank deposit grows over time due to interest compounding.
- If I borrow money from the bank at an interest rate $r$ and make no payments, then my debt increases over time according to the same formulas.

# Time Value of Money

- Money invested/borrowed today leads to increased value/debt in the future as a result of interest.
- The compounding formulas of the previous slides show how to calculate this future value.
- We can use the same formulas to determine the present value that should be assigned to money that is to be received at a later time.

# Present Value

Suppose that the annual interest rate $r$ is compounded $m$ times per year. The following are equivalent:

- receive an amount $A$ after $k$ compounding periods;
- receive an amount $d_k A$ today, where

$$d_k = \frac{1}{(1 + r/m)^k} < 1$$

denotes the discount factor corresponding to period $k$.

In fact, if we deposit $d_k A$ in a bank account today, then we receive $A$ after $k$ compounding periods.

$\Rightarrow$ $d_k A$ is the present value of $A$.

# The Ideal Bank

**Def.**: An ideal bank:

- applies the same interest rate to deposits and loans.
- has no service charges or transaction costs.
- has the same interest rate for any size of principal.

Interest rates for different transactions may be different:

- a 2-year certificate of deposit (CD) might offer a higher rate than a 1-year CD.

**Def.**: If an ideal bank has an interest value that is independent of the length of time for which it applies, it is called a constant ideal bank.

# Future and Present Value of Streams I

- Consider a cash flow stream $x_0, x_1, x_2, \ldots, x_n$.
- $x_k$ occurs at the end of period $k$.
- We can use a constant ideal bank to move all cash flows to the end of period $n$ or to the present time.

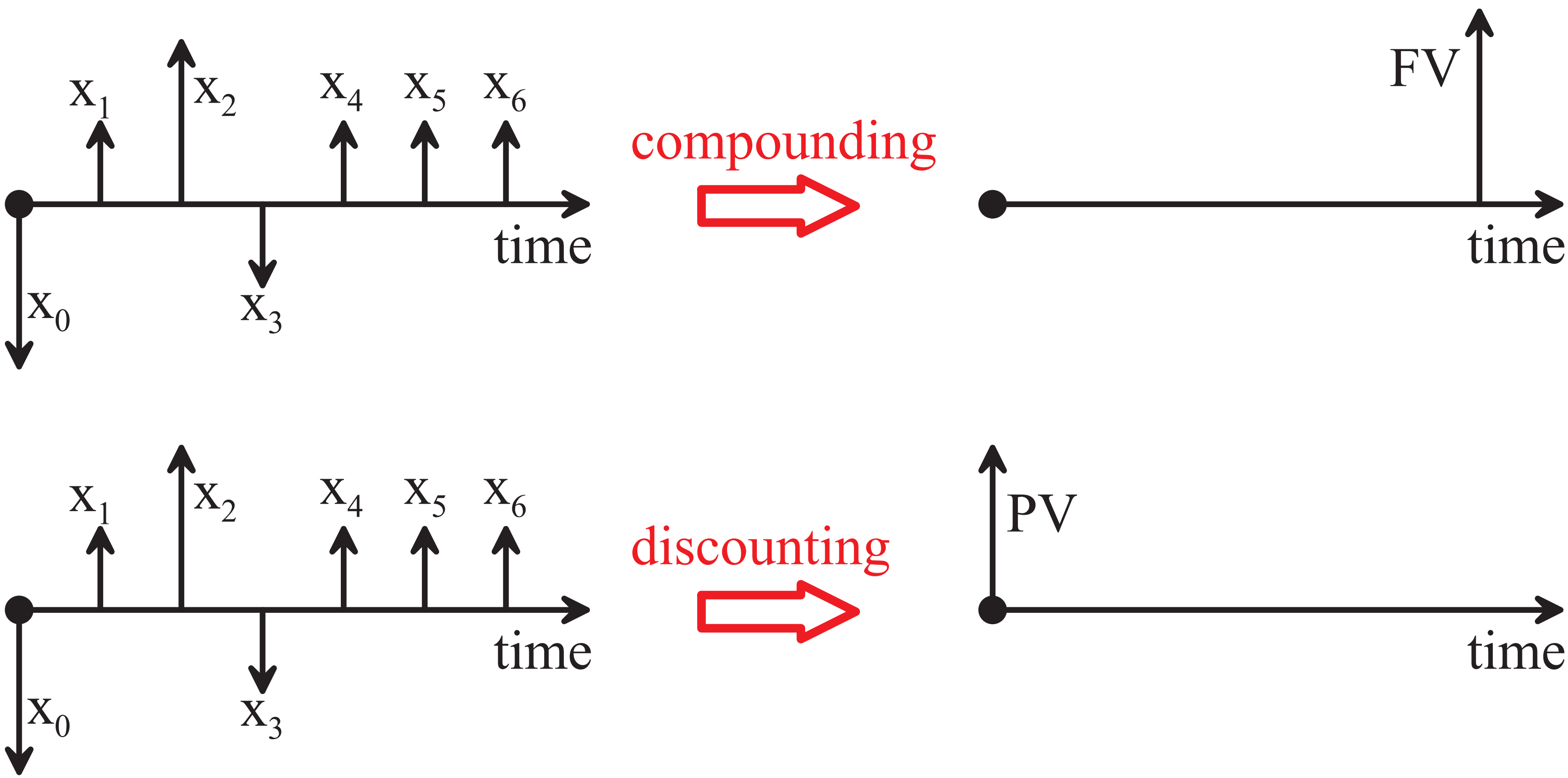
**Def.**: The future value of the stream is

$$\mathrm{FV} = \sum_{k=0}^{n} x_k (1 + r/m)^{n-k} \quad \leftarrow \quad \text{‘compounding’}$$

**Def.**: The present value of the stream is

$$\mathrm{PV} = \sum_{k=0}^{n} \frac{x_k}{(1 + r/m)^k} \quad \leftarrow \quad \text{‘discounting’}$$

# Future and Present Value of Streams II



# Present Value and an Ideal Bank

**Def.**: Two CF streams are equivalent if they can be transformed into each other by an ideal bank.

**Example**: A 10% bank can change

- $(1, 0, 0)$ to $(0, 0, 1.21)$ by receiving a deposit of $1 now and paying principal and interest of $1.21 in 2 years;
- $(0, 0, 1.21)$ to $(1, 0, 0)$ by issuing a loan for $1 now.

**Theorem**: The CF streams $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ are equivalent for a constant ideal bank with interest rate $r$ iff their PVs are equal.

$\Rightarrow$ Evaluate CF streams only on the basis of their PVs.

# Net Present Value

- Different choices can lead to different CF streams.
- PV can be used to rank these choices:
  the higher the PV, the more desirable the choice.
- Here, one must include all cash flows associated with an investment, both positive and negative.
- In this case, PV is termed net present value (NPV).

# When to Cut a Tree?

You want to plant trees in order to sell lumber:

- buy seedlings today: initial cost of $1$;
- two options as to when to harvest:
  (a) after 1 year: early moderate revenues of 2;
  (b) after 2 years: later but higher revenues of 3 (due to additional growth).

Net present values for $r = 10\%$:

(a) NPV $= -1 + 2/1.1 = 0.82$;

(b) NPV $= -1 + 3/(1.1)^2 = 1.48$.

$\Rightarrow$ it is best to cut later.

# When to Cut a Tree?

Assume that the proceeds of a harvest can be used to plant additional trees $\Rightarrow$ the business has several cycles.

Reconsider the two options:

(a) cut early: money is doubled every year;

(b) cut later: money is tripled every 2 years $\Rightarrow$ in average, money grows by a factor $\sqrt{3}$ per year.

$\Rightarrow$ it is best to cut early.

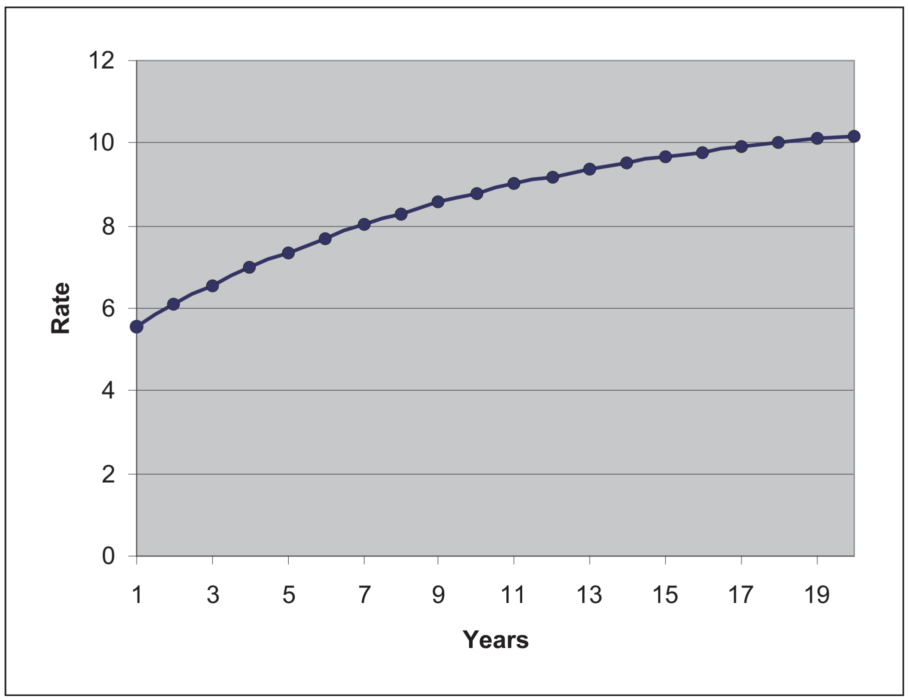
Repeatable activities must be compared over the same time horizon, e.g., 2 years in the tree cutting example:

$$\mathrm{NPV}(a) = -1 + 4/(1.1)^2 = 2.31 > \mathrm{NPV}(b) = -1 + 3/(1.1)^2 = 1.48$$

# The Term Structure of Interest Rates

In reality, there is a whole family of interest rates at any point in time — a different rate for each maturity time.

**Def.**: The spot rate $s_t$ is the annualized interest rate charged for money held form the present until time $t$.



# Compounding Conventions

Under different compounding conventions, the spot rate $s_t$ is defined as follows:

- yearly compounding: $s_t$ is defined such that

  $$(1 + s_t)^t$$

  is the growth factor of a deposit held for $t$ years ($t \in \mathbb{N}$);
- $m$ compounding periods/year: $s_t$ is defined such that

  $$(1 + s_t/m)^{mt}$$

  is the corresponding growth factor ($t \in \frac{1}{m}\mathbb{N}$);
- continuous compounding: $s_t$ is defined such that $e^{s_t t}$ is the corresponding growth factor ($t \in \mathbb{R}_+$).

# Properties of Spot Rate Curves

- Long commitments tend to offer higher interest rates than short commitments.

  $\Rightarrow$ Spot rate curves are normally upward sloped.

- The spot rate curve undulates around in time (like a branch in the wind).
- The spot rate curve is called
  - normally shaped: if it is increasing;
  - inverted: if it is decreasing.[a]
- The spot rate curve is smooth.

[a] The inverted shape occurs when short-term rates increase rapidly, and investors believe, that the rise is temporary.

# Discount Factors

For a given set of spot rates, we can define the corresponding **discount factors** $d_t$:

- **yearly compounding**:

$$d_t = \frac{1}{(1 + s_t)^t} \qquad t \in \mathbb{N}\,;$$

- **$m$ compounding periods/year**:

$$d_t = \frac{1}{(1 + s_t/m)^{mt}} \qquad t \in \frac{1}{m}\mathbb{N}\,;$$

- **continuous compounding**:

$$d_t = e^{-s_t t} \qquad t \in \mathbb{R}\,.$$

# Present Value

Given any CF stream $x_0, x_1, x_2, \ldots, x_n$, the present value relative to the prevailing spot rates is

$$\mathrm{PV} = x_0 + d_1 x_1 + d_2 x_2 + \cdots + d_n x_n \,.$$

Note that:

- $d_t$ acts like a price for cash received at time $t$;
- PV is the sum of 'price times quantity' for all cash components.

# Example: Price of a 10-year Bond

Consider an 8% bond maturing in 10 years:

- the bond pays $8 at the end of the years $1, 2, \ldots, 9$ and $108 at the end of year 10.
- the end-of-year discount factors for years $1, 2, \ldots, 10$ can be calculated from a given spot rate curve.
- We take the products of the cash flows with the corresponding discount factors and sum.

$\Rightarrow$ The value of the bond is $97.34.

| Year | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total PV |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Spot Rate | 5.571 | 6.088 | 6.555 | 6.978 | 7.361 | 7.707 | 8.020 | 8.304 | 8.561 | 8.793 | |
| Discount | 0.947 | 0.889 | 0.827 | 0.764 | 0.701 | 0.641 | 0.583 | 0.528 | 0.477 | 0.431 | |
| Cash Flow | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 108 | |
| PV | 7.58 | 7.11 | 6.61 | 6.11 | 5.61 | 5.12 | 4.66 | 4.23 | 3.82 | 46.50 | 97.34 |

# Forward Rates I

Forward rates are interest rates for money to be borrowed between two dates in the future, but under terms agreed upon today.

**Example**: Assume that you commit today to deposit \$1 in a bank account for 1 year, starting in 1 year from now. That loan will accrue interest at a prearranged rate $f$ (agreed upon now).

$f$ is the forward rate for money to be lent in this way.

$f$ can be determined from the current spot rates.

# Forward Rates II

Two possibilities to invest \$1 over a period of two years:[a]

1. Leave \$1 in a 2-year account.

   $\Rightarrow$ After 2 years you obtain $\$(1+s_2)^2$.

2. Place \$1 in a 1-year account and make arrangements that the proceeds $\$(1+s_1)$ will be lent for 1 year starting a year from now.

   $\Rightarrow$ After 2 years you obtain $\$(1+s_1)(1+f)$.

Comparison principle: $(1+s_2)^2 \overset{!}{=} (1+s_1)(1+f)$

$$\Rightarrow \quad f = \frac{(1+s_2)^2}{1+s_1} - 1$$

[a]Yearly compounding.

# Forward Rates III

**General forward rate definition**: The forward rate between times $t_1$ and $t_2$ ($t_1 < t_2$) is denoted by $f_{t_1,t_2}$. It is the interest rate charged for borrowing money at $t_1$ which is to be repaid (with interest) at $t_2$. $f_{t_1,t_2}$ is agreed on today ($t = 0$).

The forward rate $f_{i,j}$ satisfies (yearly compounding)

$$(1 + s_j)^j = (1 + s_i)^i (1 + f_{i,j})^{j-i} \quad \Rightarrow \quad f_{i,j} = \left[ \frac{(1 + s_j)^j}{(1 + s_i)^i} \right]^{1/(j-i)} - 1 \, .$$

- This is called the implied forward rate.
- It may be slightly different from the market forward rate due to market imperfections.

# Different Compounding Conventions

- Yearly compounding: $(1+s_j)^j = (1+s_i)^i(1+f_{i,j})^{j-i}$

$$\Rightarrow \quad f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{1/(j-i)} - 1$$

- $m$ periods/year: $(1+s_j/m)^j = (1+s_i/m)^i(1+f_{i,j}/m)^{j-i}$

$$\Rightarrow \quad f_{i,j} = m\left[\frac{(1+s_j/m)^j}{(1+s_i/m)^i}\right]^{1/(j-i)} - m$$

- Continuous compounding: $e^{s_{t_2}t_2} = e^{s_{t_1}t_1}e^{f_{t_1,t_2}(t_2-t_1)}$

$$\Rightarrow \quad f_{t_1,t_2} = \frac{s_{t_2}t_2 - s_{t_1}t_1}{t_2 - t_1}$$

# Spot Rate Forecasts I

The forward rate $f_{1,2}$ is

- the implied rate for money loaned for 1 year, a year from now;
- the market expectation of what the 1-year spot rate will be next year.

The same argument applies to all other rates, too.

$\Rightarrow$ The current spot rate curve $s_1, s_2, \ldots, s_n$ implies a set of forward rates $f_{1,2}, f_{1,3}, \ldots, f_{1,n}$, which define the expected spot rate curve $s'_1, s'_2, \ldots, s'_{n-1}$ for next year:

$$s'_{j-1} = f_{1,j} = \left[\frac{(1+s_j)^j}{1+s_1}\right]^{1/(j-1)} - 1 \quad \text{for } j = 2, 3, \ldots, n \qquad (1)$$

# Spot Rate Forecasts II

The entity of all future expected spot rate curves implied by an initial curve can be displayed as follows:

$$
\begin{array}{lllllll}
f_{0,1} & f_{0,2} & f_{0,3} & \cdots & f_{0,n-2} & f_{0,n-1} & f_{0,n} \\
f_{1,2} & f_{1,3} & f_{1,4} & \cdots & f_{1,n-1} & f_{1,n} & \\
f_{2,3} & f_{2,4} & f_{2,5} & \cdots & f_{2,n} & & \\
\vdots & \vdots & & & & & \\
f_{n-2,n-1} & f_{n-2,n} & & & & & \\
f_{n-1,n} & & & & & &
\end{array}
$$

The transformation (1) of the spot rate curve is termed expectation dynamics.

# Discount Factors

We denote by $d_{t_1,t_2}$ the discount factor to discount cash received at time $t_2$ back to time $t_1$ where $t_1 < t_2$.

- Yearly compounding:

$$d_{i,j} = \frac{1}{(1 + f_{i,j})^{j-i}}$$

- $m$ periods/year:

$$d_{i,j} = \frac{1}{(1 + f_{i,j}/m)^{j-i}}$$

- Continuous compounding:

$$d_{t_1,t_2} = e^{-f_{t_1,t_2}(t_2-t_1)}$$

# Running Present Value I

For any $i < j < k$ we have (yearly compounding)

$$d_{i,k} = d_{i,j} d_{j,k} \, .$$

The present value $\mathrm{PV}(0)$ of a CF stream $x_0, x_1, \ldots, x_n$ is

$$\begin{aligned} \mathrm{PV}(0) &= x_0 + d_1 x_1 + d_2 x_2 + \cdots + d_n x_n \\ &= x_0 + d_1 \left( x_1 + d_{1,2} x_2 + \cdots + d_{1,n} x_n \right) \\ &= x_0 + d_1 \mathrm{PV}(1) \, , \end{aligned}$$

where $\mathrm{PV}(1)$ is the present value of the stream $x_1, \ldots, x_n$ as viewed at time $1$. The values $d_{1,k}$, $k = 2, 3, \ldots, n$, are the discount factors 1 year from now under an assumption of expectation dynamics.

# Running Present Value II

Define now the time $k$ present value as

$$\mathrm{PV}(k) \;=\; x_k + d_{k,k+1}x_{k+1} + d_{k,k+2}x_{k+2} \cdots + d_{k,n}x_n\,.$$

The relations

$$d_{k,k+j} = d_{k,k+1}d_{k+1,k+j} \quad \text{for} \quad j = 1, 2, \ldots, n-j$$

imply that the present values $\mathrm{PV}(k)$ satisfy the recursion

$$\mathrm{PV}(k) = x_k + d_{k,k+1}\mathrm{PV}(k+1)\,.$$

$\Rightarrow$ $\mathrm{PV}(0)$ can be calculated by means of a backward recursion starting with $\mathrm{PV}(n) = x_n$.