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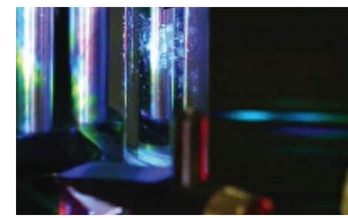

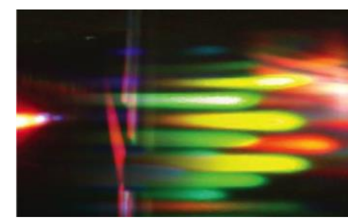

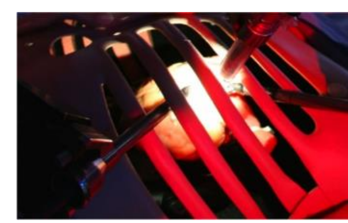

COMP70058 Computer Vision

Lecture 6 – Fourier Methods

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The Hamlyn Centre for Robotic Surgery

Contents

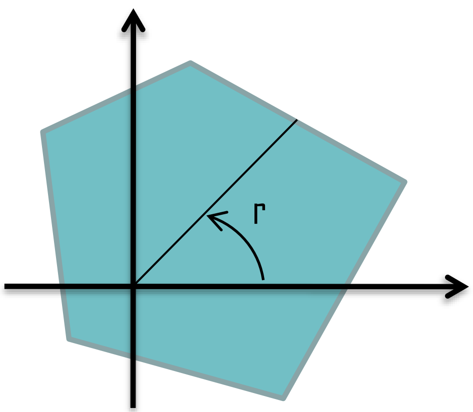
- Fourier Harmonics
- Basic Properties of Fourier Transform
- Correlation and Convolution
- 2D Fourier Transform

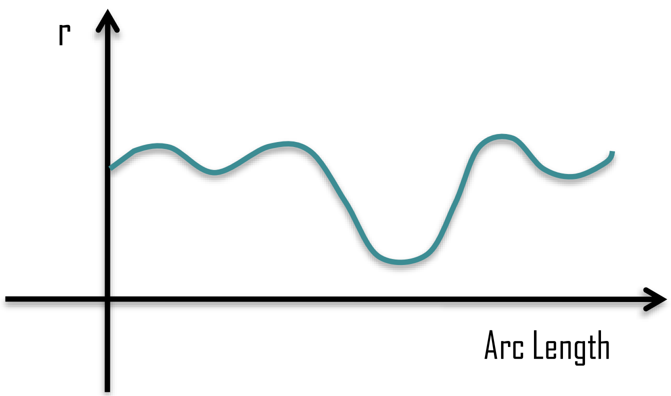
The Fourier Descriptor

- Can we classify objects by their features in the frequency domain rather than the spatial domain?

A simple analogy is with sound waves. A periodic **sound wave** can be easily described by a sinusoid of fundamental frequency, which determines the **pitch**, and a set of **overtones**, which are higher frequency sinusoids added to the fundamental which determine the timbre of the sound. All periodic functions can be broken into a set of sinusoidal waves.

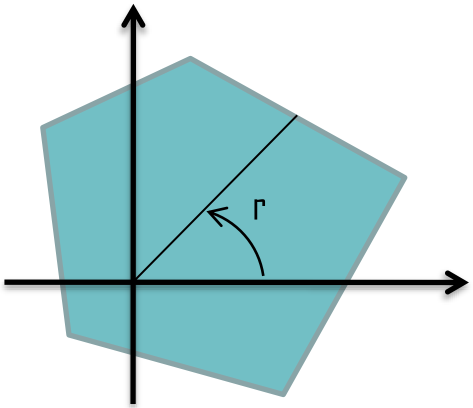
- For shapes, we can associate a periodic function such as:

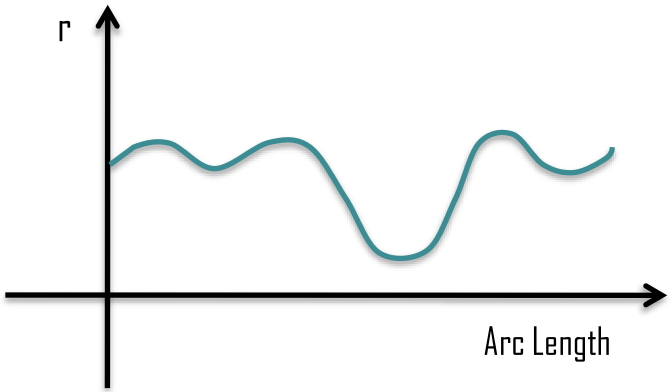




The Fourier Descriptor

- A start point on the boundary is defined again uniquely, in this case by taking the intersection of the boundary with a horizontal line through the centre. The boundary may then be described by plotting the **radial distance** of each boundary point from the centre against its distance along the boundary from the start point.
- The variation of magnitude with distance along the boundary is a periodic function. If the total length around the boundary is L the period is normalised to 2π by multiplying a length l by 2π/L.
- If we can find the frequency components of that periodic function, then these can be used to classify objects in a recognition algorithm.





If r is plotted against arc length for a closed contour we obtain a periodic function

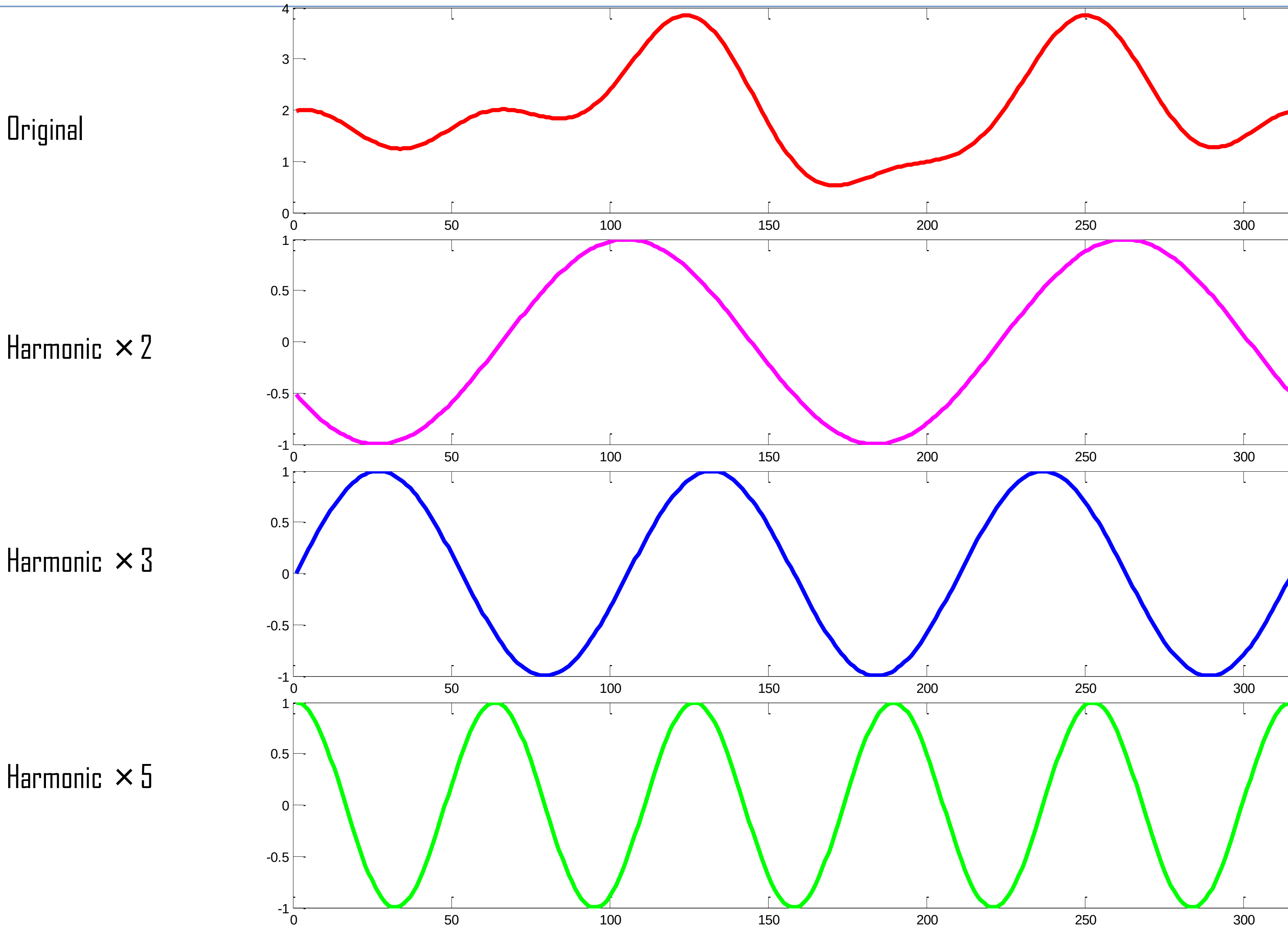
Joseph Fourier
1768 to 1830

Fellow of the Royal Society Elected 1823
Lunar features - Crater Fourier
Commemorated on the Eiffel Tower

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

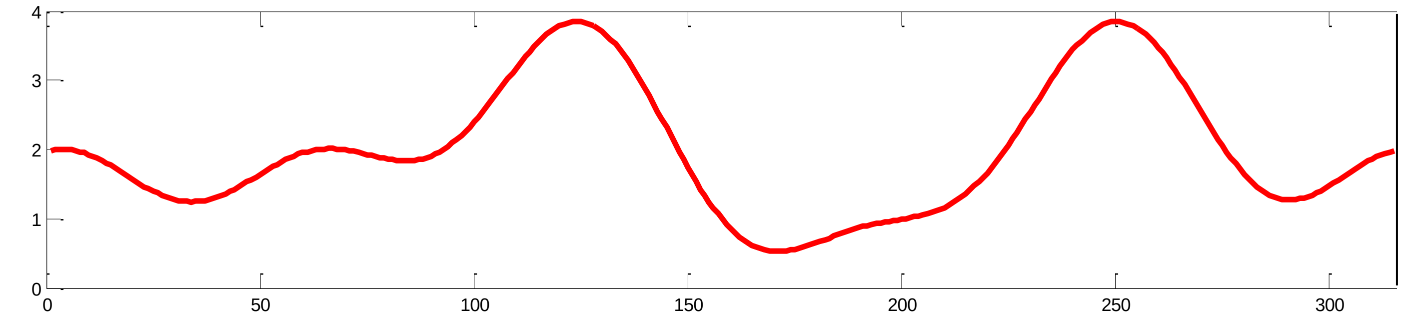
Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

$$\sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$$

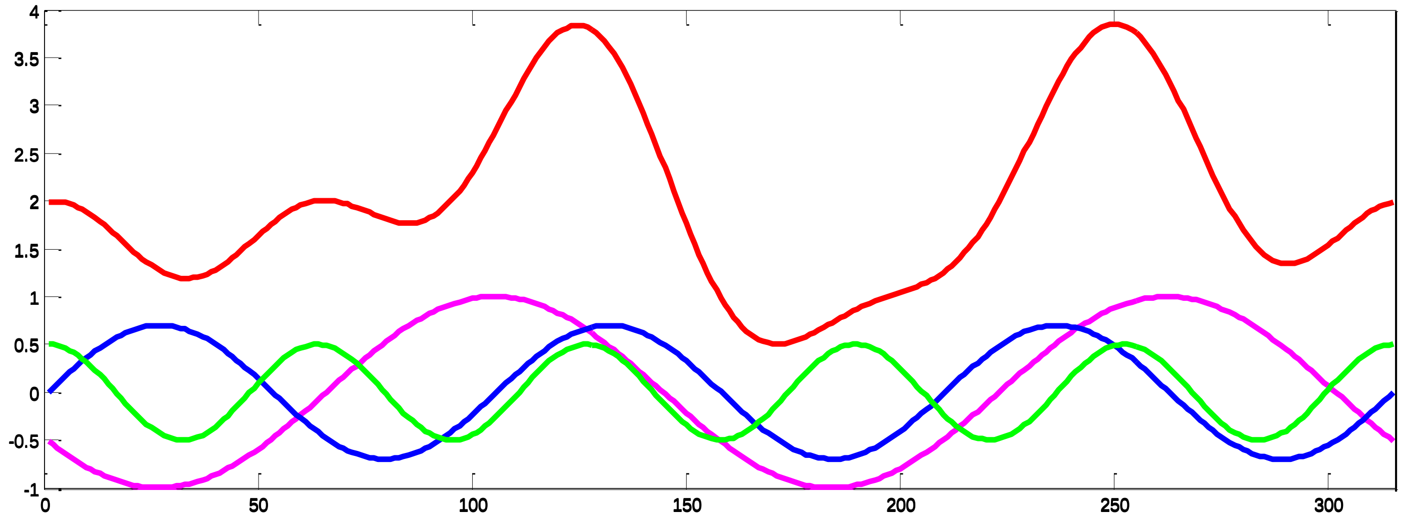


Original
Harmonic × 2
Harmonic × 3
Harmonic × 5

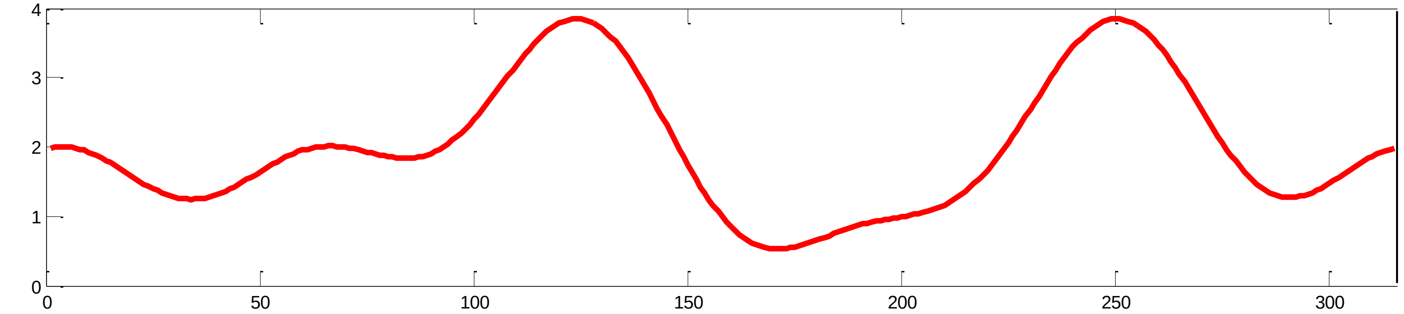
Original



Reconstructed



Original



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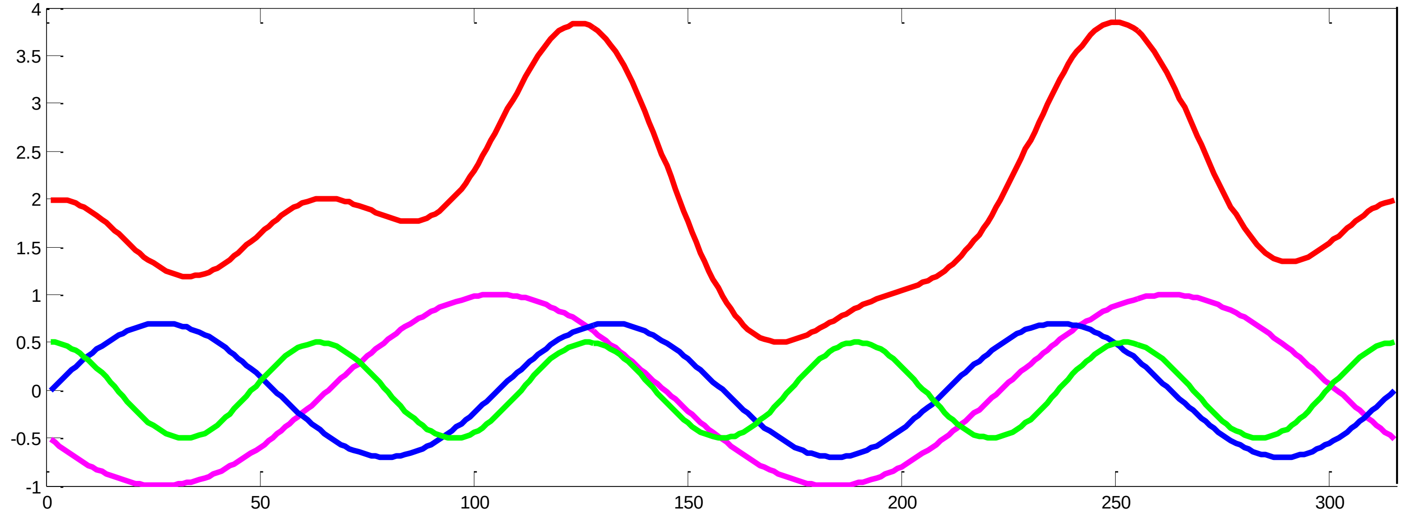
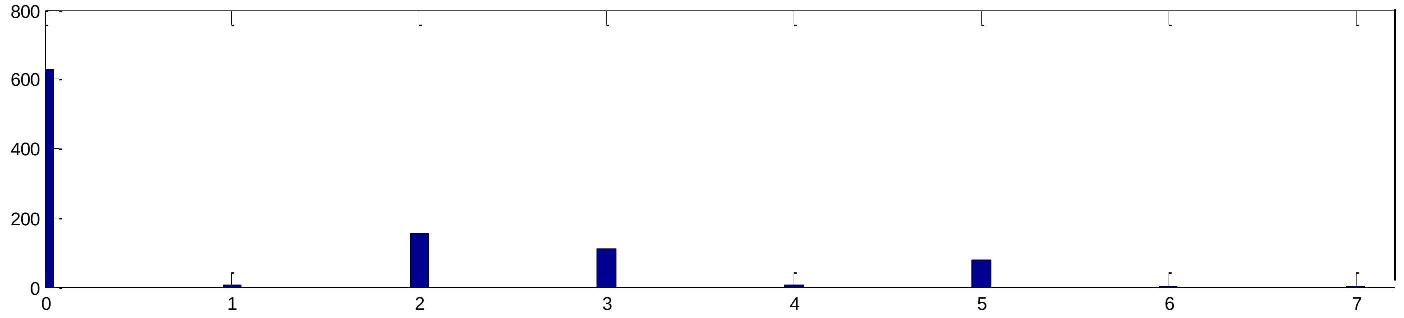


Spectrum



Fourier Methods – Mathematical Description

- A periodic function f(x+N)=f(x) can be approximated by an infinite sum of sinusoidal functions (cosine and sine), each with a frequency that is an integer multiple of 1/*N*.

$$f(x) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta + \ldots$$

where θ=2πx/N, and 2π/N is called the fundamental frequency

$$f(x) = \sum_{h=0}^{\infty} a_h \cos(2\pi h \frac{x}{N}) + b_h \sin(2\pi h \frac{x}{N})$$

The purpose of Fourier transform is to find an easy way to recover the values of a_0, a_1, b_1, a_2, b_2, … To do this, we can make use of the orthogonal properties of trigonometric functions.

Fourier Transform – Cosine Transform

- The discrete cosine transform can be represented as

$$F_c(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\cos(2\pi u\frac{x}{N})$$

- Because we represent the signal as

$$f(x) = \sum_{h=0}^{\infty} a_h \cos(2\pi h\frac{x}{N}) + b_h \sin(2\pi h\frac{x}{N})$$

- We have

$$F_c(u) = \frac{1}{2}a_u$$

Fourier Transform – Sine Transform

- Similarly we can define the discrete sine transform:

$$F_s(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\sin(2\pi u\frac{x}{N})$$

- Which for some integer u will evaluate to $b_u/2$

$$F_s(u) = \frac{1}{2}b_u$$

Fourier Methods – Orthogonal Functions

$$\int_0^{2\pi} \sin(p\theta)\sin(q\theta)d\theta = 0, \quad \text{if } p \neq q$$

$$\int_0^{2\pi} \sin(p\theta)\sin(q\theta)d\theta = \frac{1}{2}\int_0^{2\pi} \cos((p-q)\theta) - \cos((p+q)\theta)\,d\theta$$

$$= \frac{1}{2}\left(\frac{1}{p-q}\sin(p-q)\theta\Big|_0^{2\pi} - \frac{1}{p+q}\sin(p+q)\theta\Big|_0^{2\pi}\right) = 0, \quad \text{if } p \neq q$$

- What happens if p=q?

$$\int_0^{2\pi} \sin(p\theta)\sin(q\theta)d\theta = 0, \text{ and } \int_0^{2\pi} \cos(p\theta)\cos(q\theta)d\theta = 0, \text{ if } p \neq q$$

$$\int_0^{2\pi} \cos(p\theta)\sin(q\theta)d\theta = 0$$

Fourier Methods – Orthogonal Functions

- Since we are only interested in the discrete transform, which is the form in which we will compute it, we can approximate the above integrals by sums as follows:

$$\sum_{x=0}^{N-1} \sin(2\pi p \frac{x}{N}) \sin(2\pi q \frac{x}{N}) = 0, \text{ if } p \neq q$$

- Similar results can be derived for sin-cos, and cos-sin products.

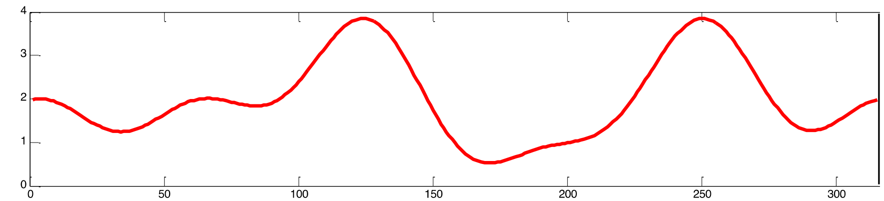
- When p=q, the sum will be (approximately) N/2, because

$$\int_0^N \sin^2\left(2\pi p \frac{x}{N}\right) dx = \int_0^N \cos^2\left(2\pi p \frac{x}{N}\right) dx = \frac{N}{2}$$

I leave this to you to prove, note: $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$ and $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

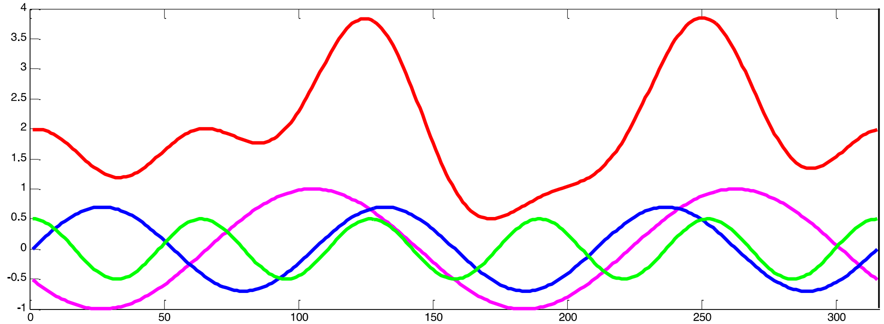
Fourier Methods - Summary

Original



$$f(x) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta +$$

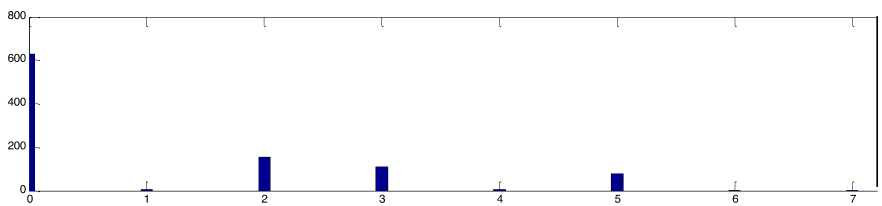
Reconstructed



$$F_c(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\cos(2\pi u \frac{x}{N}) \qquad a_u = 2F_c(u)$$

$$F_s(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\sin(2\pi u \frac{x}{N}) \qquad b_u = 2F_s(u)$$

Spectrum



$$\sqrt{F_s^2(u) + F_c^2(u)}$$

Fourier Series Forms

Sine-cosine form

$$f(x) = \sum_{h=0}^{\infty} a_h \cos(2\pi h \frac{x}{N}) + b_h \sin(2\pi h \frac{x}{N})$$

Amplitude-phase form

$$\mathrm{f(x)} = \frac{A_0}{2} + \sum_{h=1}^{N} \left(A_h \cos(2\pi h \frac{x}{N} - \varphi_h) \right)$$

Exponential form

$$\mathrm{f(x)} = \sum_{h=-N}^{N} c_h e^{i2\pi h\frac{x}{N}}$$

$$e^{i\theta} = \cos(\theta) + isin(\theta)$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Fourier Transform

- The Fourier Transform is a tool that converts a function into an alternative representation.
- The term *Fourier transform* refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time.
- The Fourier transform of a function is a complex-valued function of frequency, whose magnitude represents the amount of that frequency present in the original function, and whose argument is the phase offset of the basic sinusoid in that frequency.
- Linear operations performed in one domain (time or frequency) have corresponding operations in the other domain, which are sometimes easier to perform. For example, convolution in the time domain corresponds to multiplication in the frequency domain. This means that operations can be performed in the frequency domain and the result can be transformed to the time domain.

Fourier Transform – in complex form

- The Discrete Fourier Transform is defined as:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left(exp\left(-2\pi ju\frac{x}{N}\right)\right)$$

$$e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$$

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left(\cos(2\pi u\frac{x}{N}) - j\sin(2\pi u\frac{x}{N})\right)$$

$$F(u) = F_c(u) - jF_s(u)$$

The transform can be calculated directly from the above equations, but in practice an ingenious Fast Fourier algorithm, which makes use of the symmetries in the Cosine and Sine functions to reduce the number of multiplies, is used. Details of how it works can be found in any text on signal processing, and the code that implements it is widely published.

Fourier Transform – in complex form

- The Discrete Fourier Transform is defined as:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left(exp\left(-2\pi j u \frac{x}{N}\right)\right)$$

Amplitude

$$|F(u)| = \sqrt{F_{s}^{2}(u) + F_{c}^{2}(u)}$$

Phase

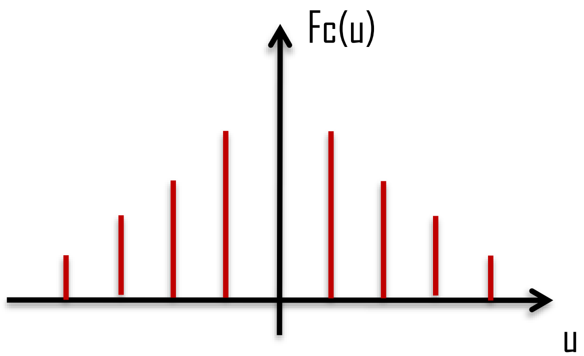
$$\varphi(u) = \tan^{-1}\frac{F_s(u)}{F_c(u)}$$

- The inverse transform can be represented as:

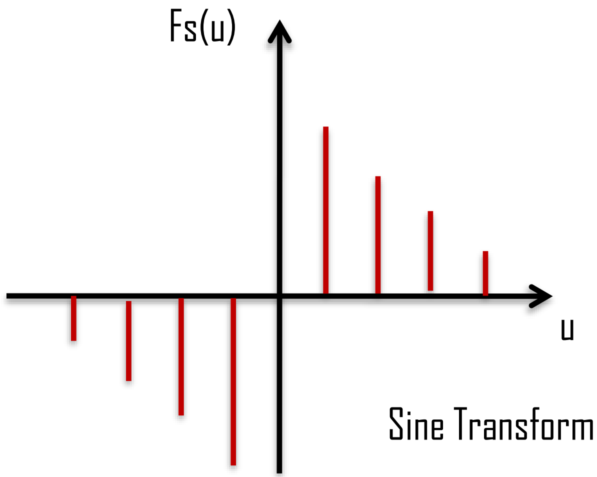
$$f(x) = \sum_{u=0}^{N-1} F(u)\left(exp\left(2\pi j u \frac{x}{N}\right)\right)$$

Symmetry Properties of the Fourier Transform

- Notice that because of the symmetry properties of the sine and cosine functions, the spectrum is well defined for negative values of u



Cosine Transform



Sine Transform

Fourier Transform – Basic Properties

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency $e^{-i2\pi u x}$

Fourier Transform – Basic Properties

Spatial Domain		Frequency Domain
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

$$
\begin{tikzcd}
g & = & f \arrow[d, "\text{FT}"] & * & h \arrow[d, "\text{FT}"] \\
G \arrow[u, "\text{IFT}"] & = & F & \times & H
\end{tikzcd}
$$

Fourier Transform – Basic Properties

$$g = f * h$$

$$G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u \tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u (x-\tau)} dx \right]$$

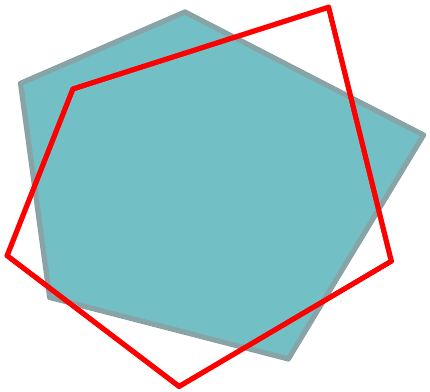
$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u \tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

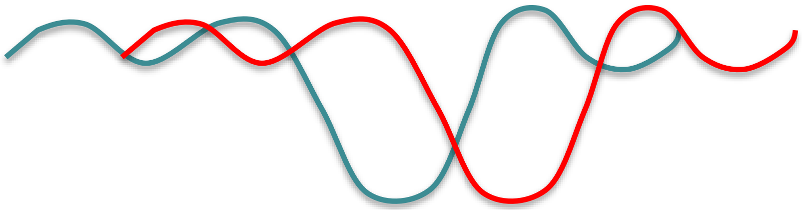
$$= F(u) H(u)$$

Fourier Transform – Basic Properties

- The advantage of using the Fourier transform is in its invariant properties
- Rotating the object causes a phase change to occur, and the same phase change is caused to all the components

$$f(x+a) \Rightarrow F(u)e^{2\pi jau}$$

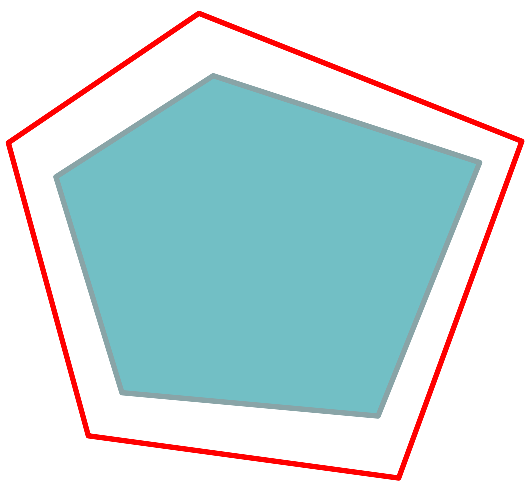


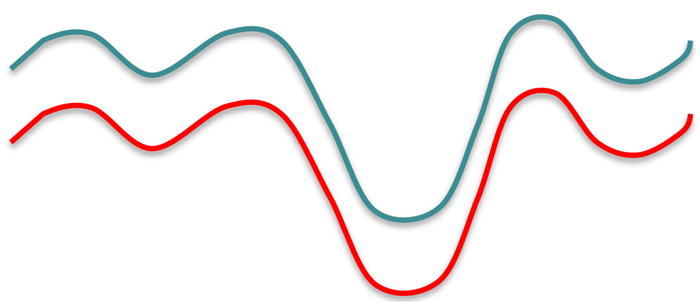


Fourier Transform – Basic Properties

- Scaling the object changes the magnitude of all the components by the same factor.
- If the magnitude of the spectrum is normalised, such that its maximum is equal to 1, then this normalised spectrum is invariant to object size.

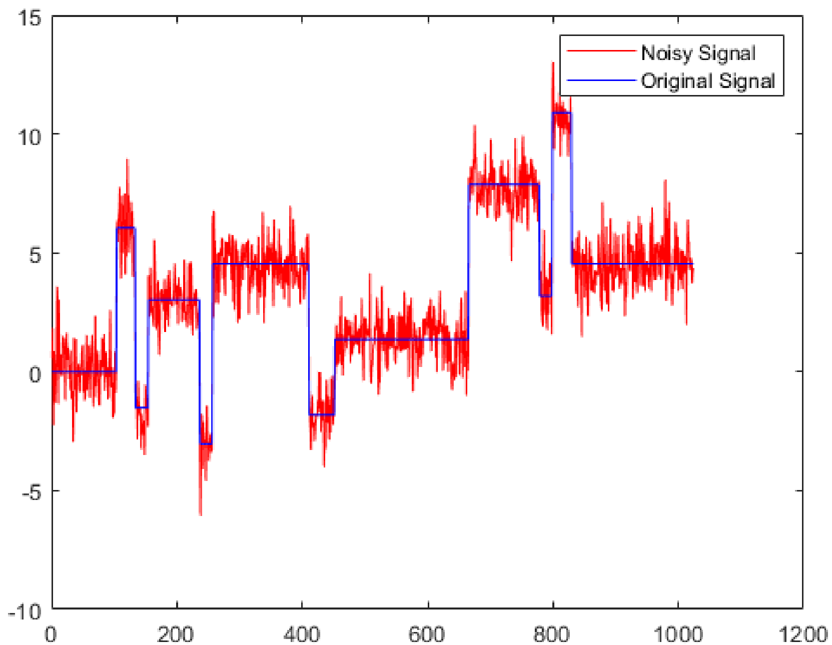
$$af(x) \Rightarrow aF(u)$$





Fourier Transform – Basic Properties

- Consider also the effect of noise and quantisation errors on the boundary. This will cause local variation of high frequency, and will not change the low frequencies.
- If the high frequency components of the spectrum are ignored, the rest of the spectrum is unaffected by noise.
- Thus, for object recognition, the Fourier descriptor offers many advantages over the template matching methods and it is faster to compute.



Fourier Transform – Correlation and Convolution

- A theorem which should be noted:

$$F(u)G(u) \Rightarrow \text{FT of the convolution of } f \text{ and } g$$

$$F(u)G(u)^* \Rightarrow \text{FT of the correlation of } f \text{ and } g$$

- where G(u)* is the complex conjugate of G(u)

If we can compute the transforms quickly, then we can also do template matching by simply carrying out pointwise multiplies of the spectra, and taking the inverse transform. However, this technique is rarely used, since the result for continuous space does not generalise to discrete space very well.

Fourier Transform for Template Matching

128 pixels

128 pixels

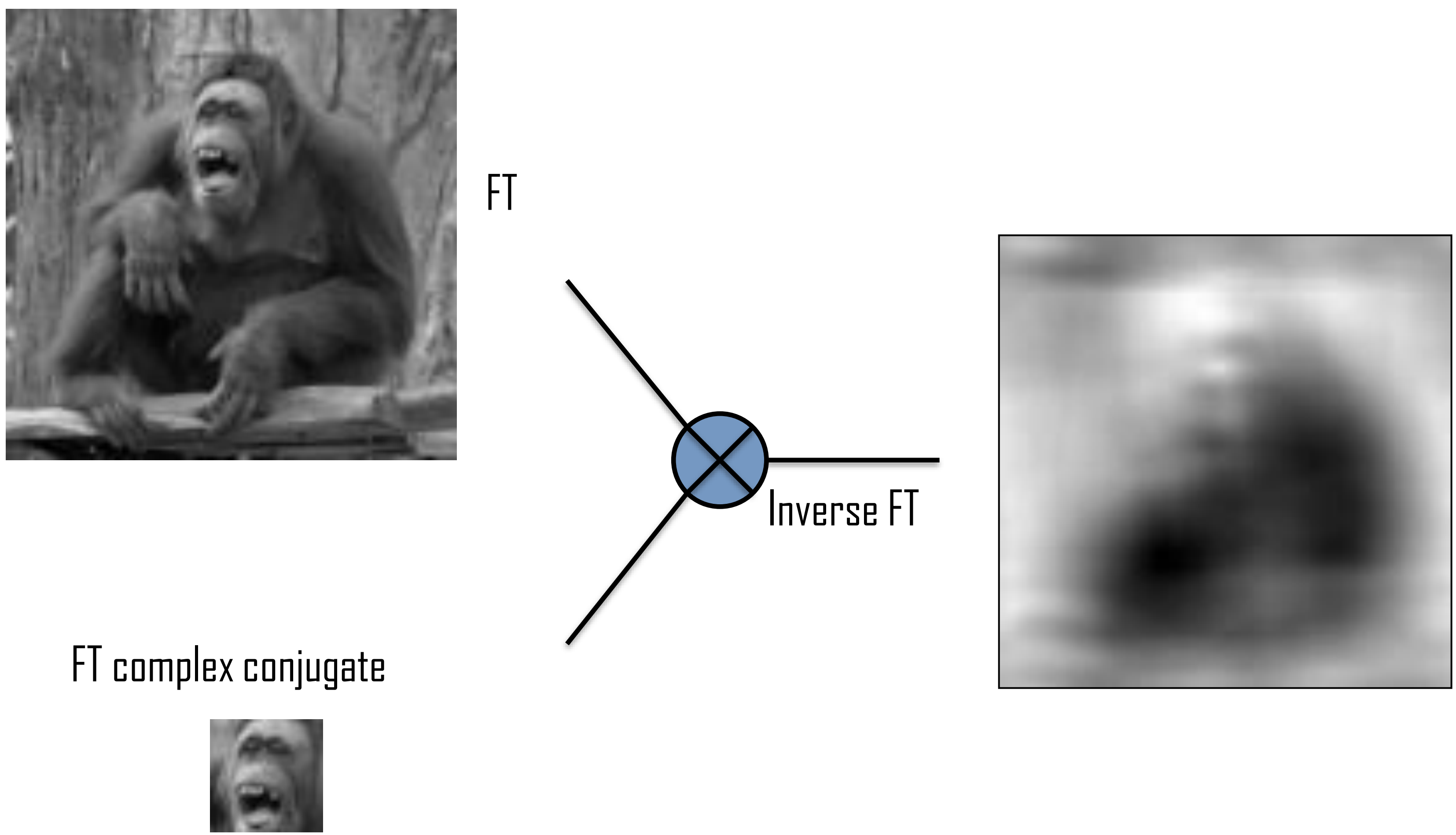
Image to be searched

32 pixels

32 pixels

Target

Fourier Transform for Template Matching



2D Fourier Transforms

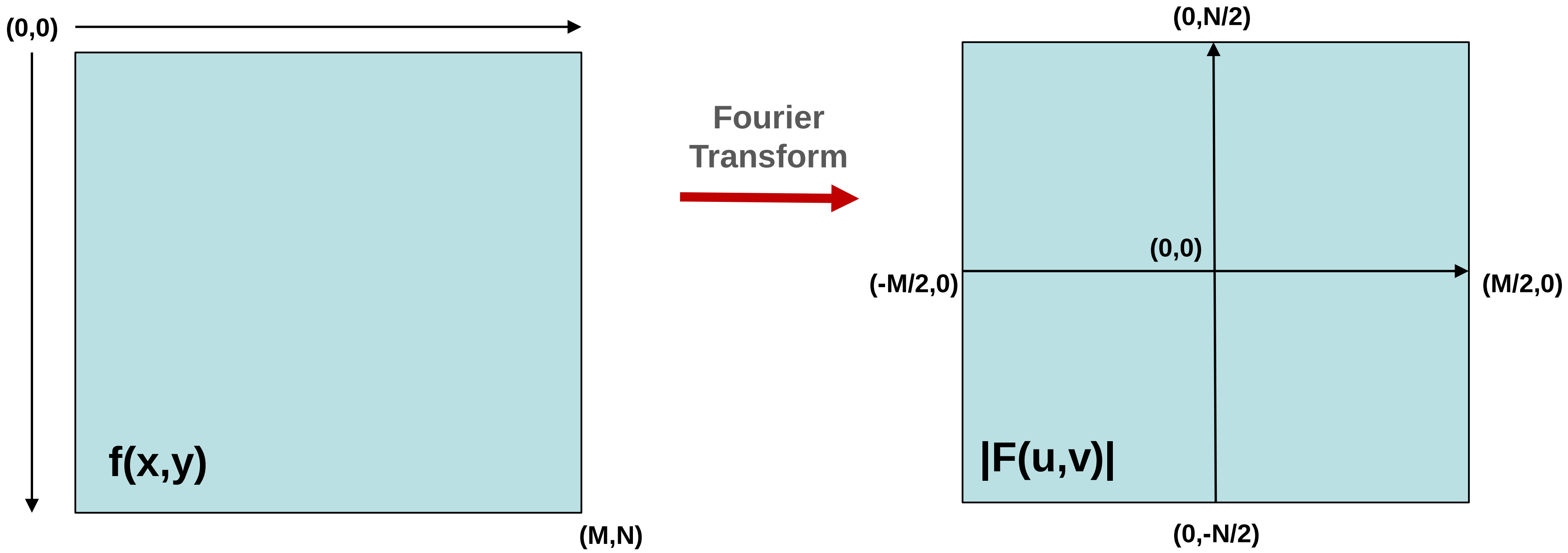
- We now look at the Fourier transform in two dimensions
- The equations are a simple extension of the one dimensional case, and the proof of the equations is, as before, based on the orthogonal properties of the sine and cosine functions

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp(-2\pi j u \frac{x}{M}) \exp(-2\pi j v \frac{y}{N})$$

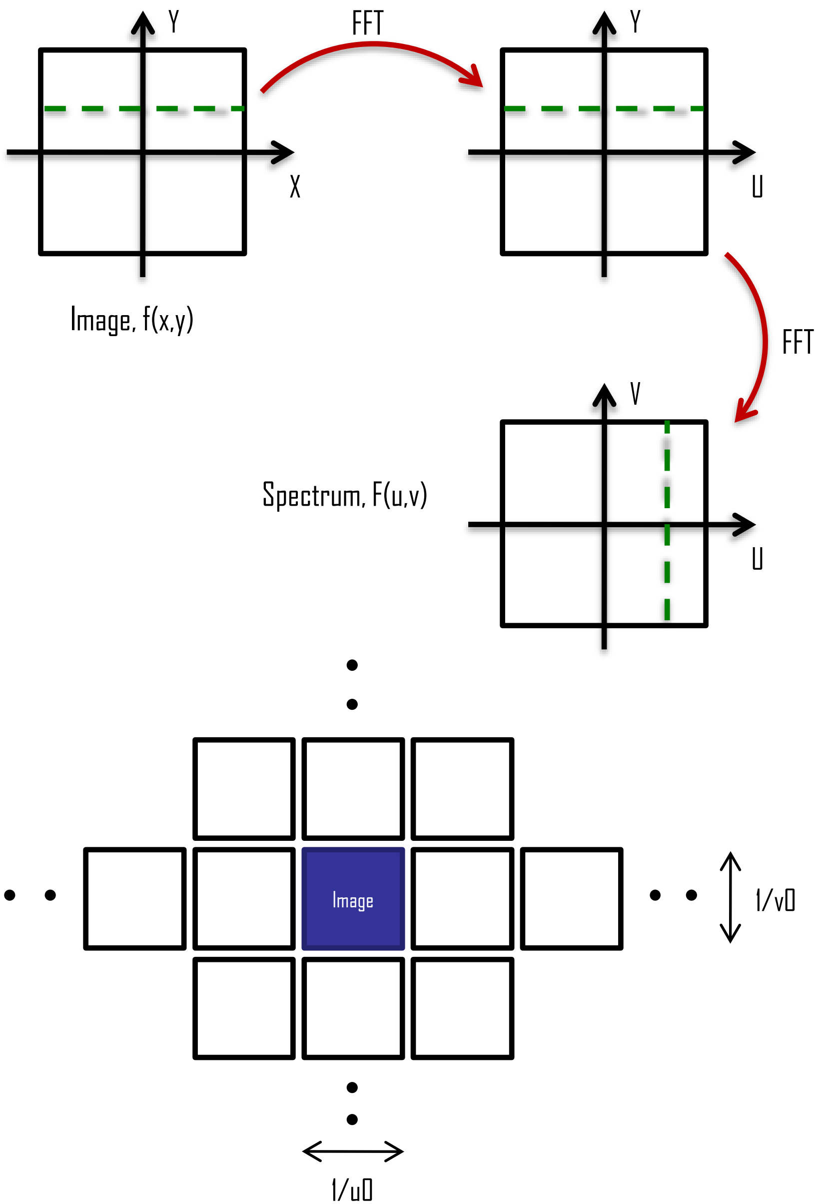
$$f(x,y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} F(u,v) \exp(2\pi j u \frac{x}{M}) \exp(2\pi j v \frac{y}{N})$$

2D Fourier Transforms

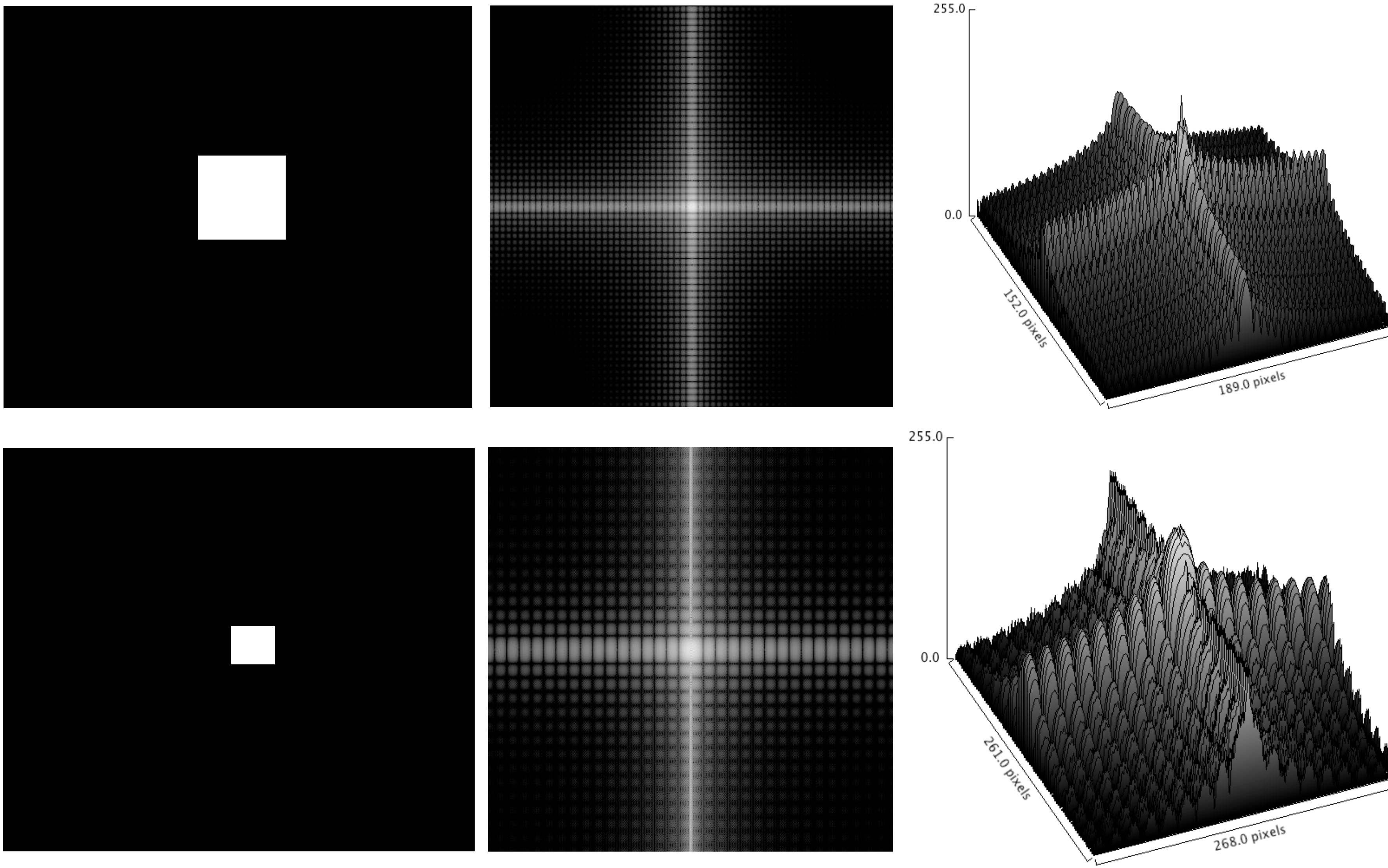
$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp(-2\pi ju\frac{x}{M})\exp(-2\pi jv\frac{y}{N})$$

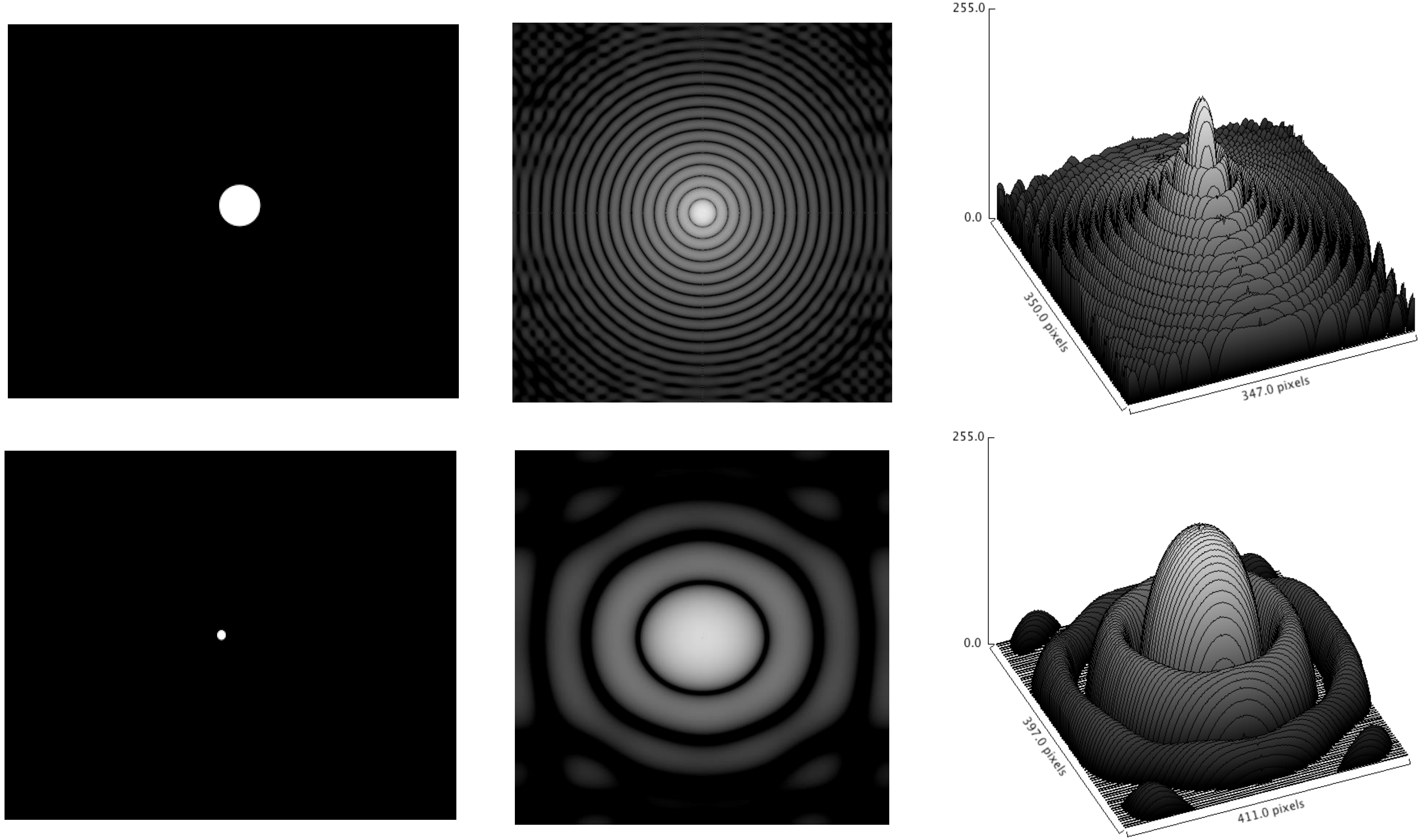


Separability of the 2D Fourier Transform



Again, a fast algorithm (Fast Fourier Transform - FFT) is available for computing this transform, providing that N and M are powers of 2. In fact, a two dimensional transform can be separated into a series of one dimensional transforms. In other words, we transform each horizontal line of the image individually to yield an intermediate form in which the horizontal axis is frequency u and the vertical axis is space y. We then transform individually each vertical line of this intermediate image to obtain each vertical line of the transformed image. Hence a two dimensional transform with an n by n image consists of 2n one dimensional transforms.


255.0
0.0
152.0 pixels
189.0 pixels
255.0
0.0
261.0 pixels
268.0 pixels

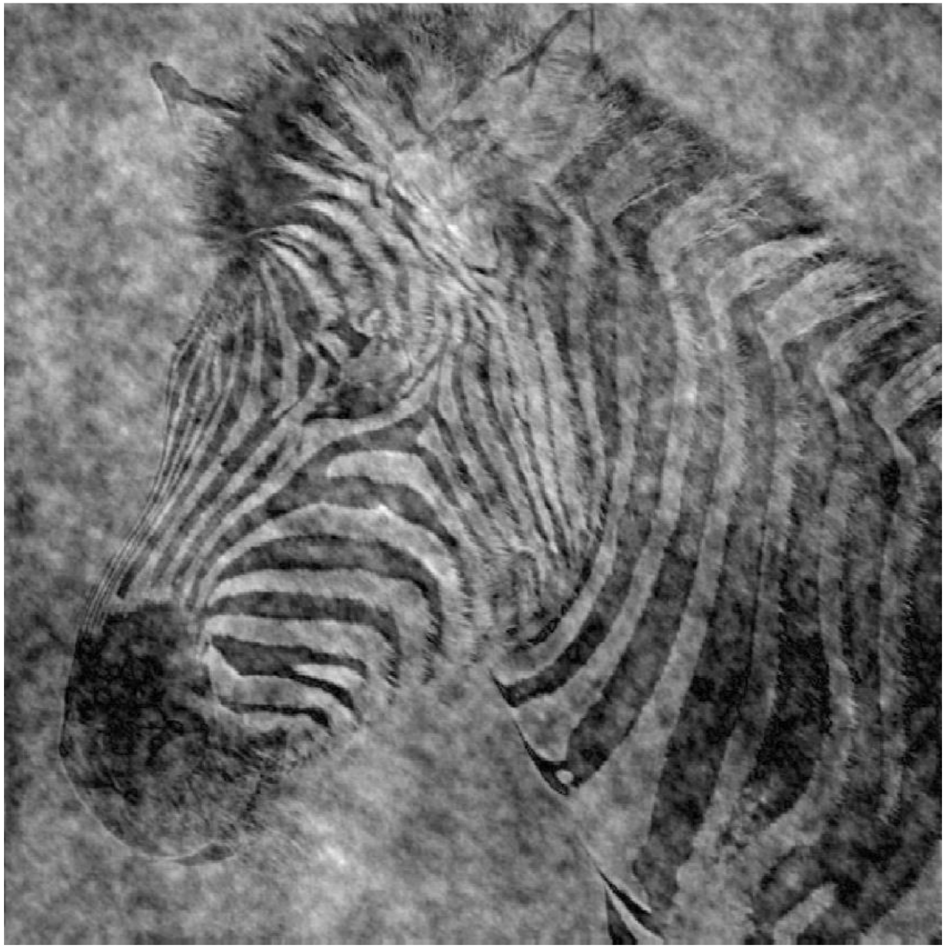

255.0
0.0
350.0 pixels
347.0 pixels
255.0
0.0
397.0 pixels
411.0 pixels

2D Fourier Transforms – Amplitude/Phase

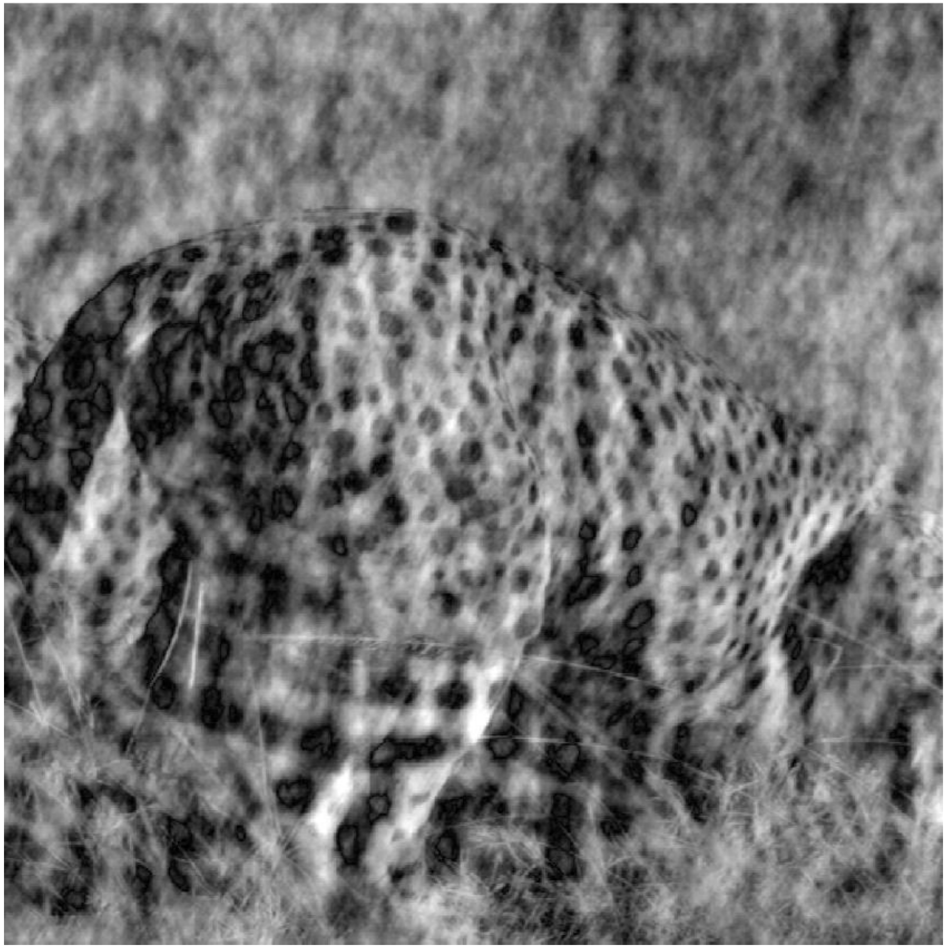
- In Fourier space, where is more of the information that we see in the visual world?
  - Amplitude
  - Phase

2D Fourier Transforms – Amplitude/Phase

Zebra phase, cheetah amplitude



Cheetah phase, zebra amplitude



Efros

2D Fourier Transforms – Amplitude/Phase

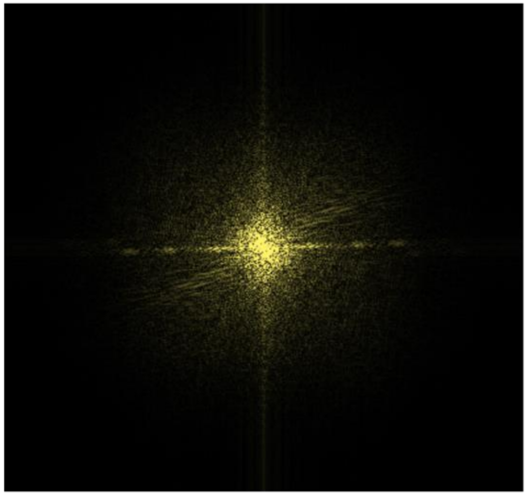
- In Fourier space, where is more of the information that we see in the visual world?
  - Amplitude
  - Phase

- The frequency amplitude of natural images are quite similar
  - Heavy in low frequencies, falling off in high frequencies

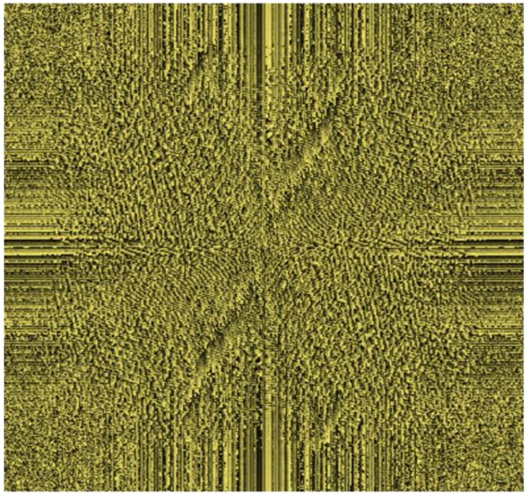
- Most information in the image is carried in the phase, not the amplitude
  - Not quite clear why

2D Fourier Transforms – Amplitude/Phase

Original

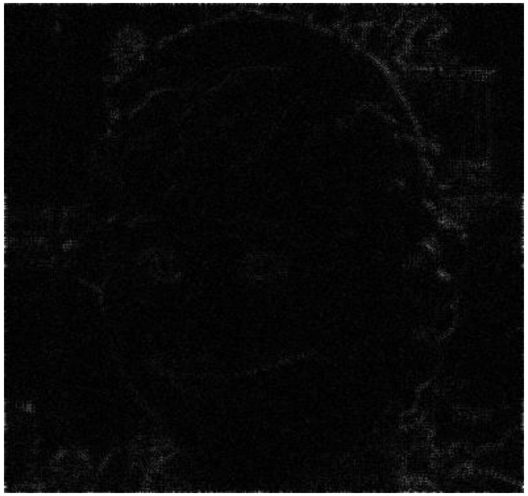


Magnitude



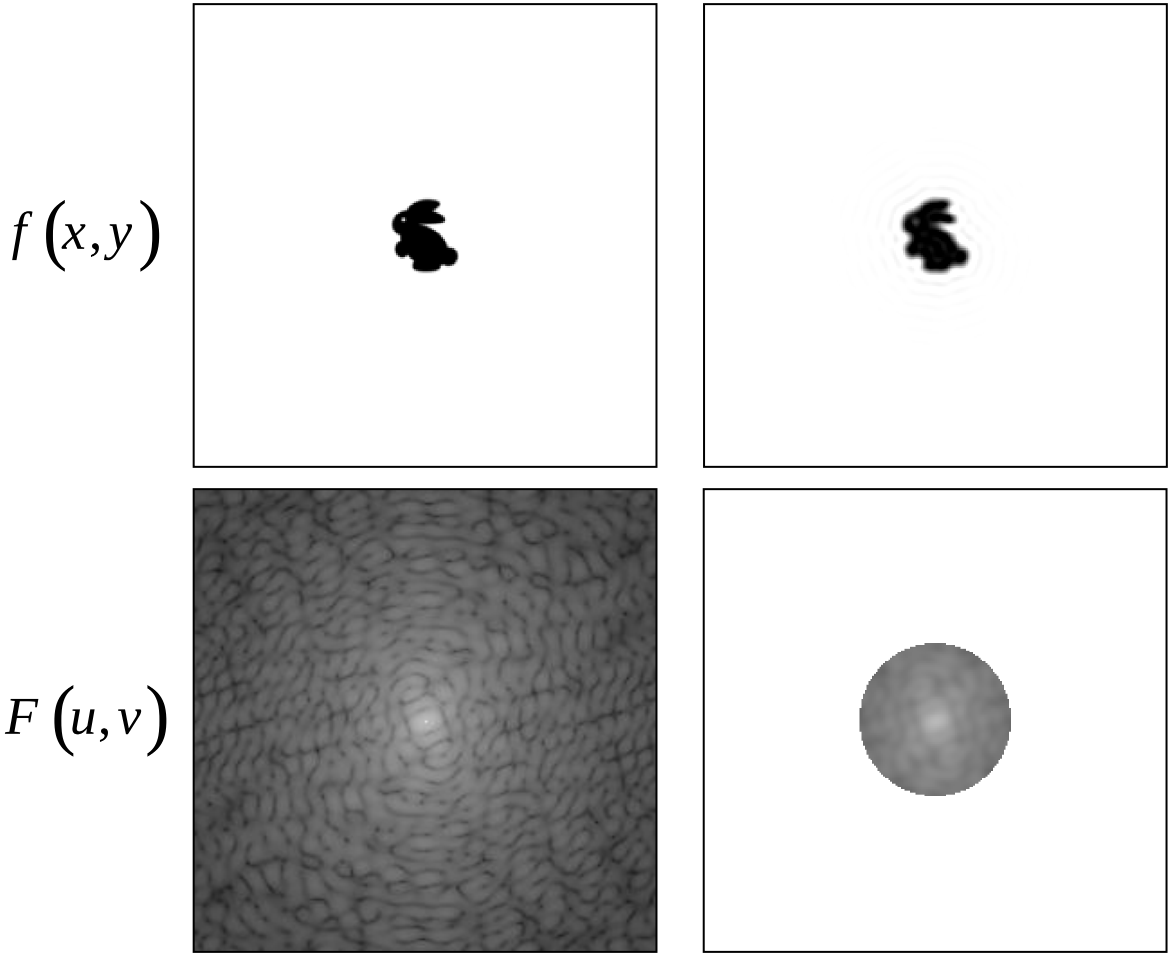
Phase

Reconstructed from magnitude information alone



Reconstructed from phase information alone

2D Fourier Transforms – A Pictorial Overview



Original Image

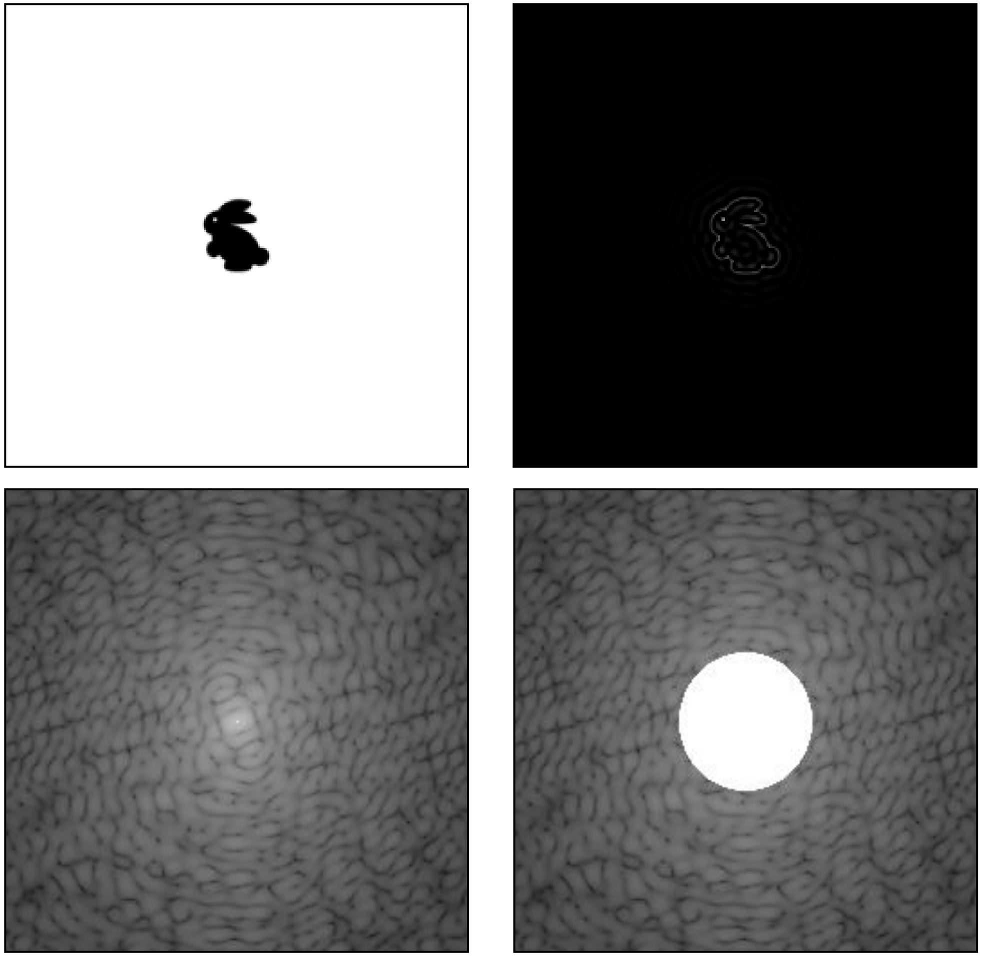
Low Pass Filtering

- Let the low frequencies pass and eliminate the high frequencies.
- It generates images with overall shading, but not much detail.

2D Fourier Transforms – A Pictorial Overview

$f(x,y)$

$F(u,v)$

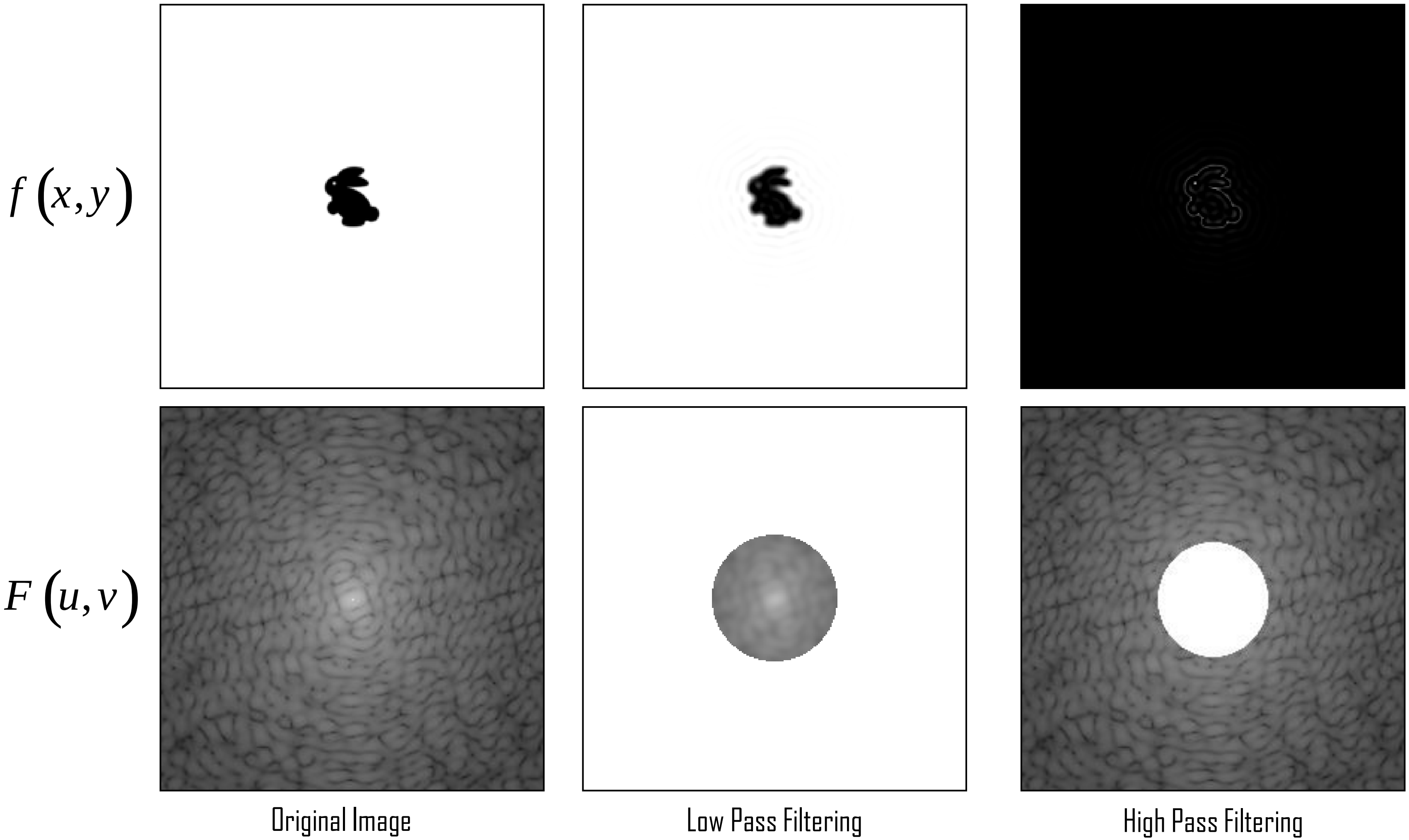


Original Image

High Pass Filtering

- Let the high frequencies pass (the details) and eliminate the low frequencies (the overall shape).
- This acts like an edge enhancer.

2D Fourier Transforms – A Pictorial Overview



Conclusions

- Fourier Harmonics
- Basic Properties of Fourier Transform
- Correlation and Convolution
- 2D Fourier Transform

Thank you!