Computational Finance
Answers

Tutorial 1

Exercise 1

1. Let x and y be two dependent random variables, and let α and β be real numbers. Prove that

$$\mathrm{var}(\alpha x + \beta y) = \alpha^2 \mathrm{var}(x) + 2\alpha\beta \mathrm{cov}(x, y) + \beta^2 \mathrm{var}(y).$$

Proof:

$$\begin{aligned}
\mathrm{var}(\alpha x + \beta y) &= E([\alpha x + \beta y - E(\alpha x + \beta y)]^2) \\
&= E([\alpha x + \beta y - \alpha E(x) - \beta E(y)]^2) \\
&= E([(\alpha x - \alpha E(x)) + (\beta y - \beta E(y))]^2) \\
&= E([\alpha^2 (x - E(x))^2 + 2\alpha\beta(x - E(x))(y - E(y)) \\
&\qquad + \beta^2 (y - E(y))^2]) \\
&= \alpha^2 E([x - E(x)]^2) + 2\alpha\beta E([(x - E(x))(y - E(y))]) \\
&\qquad + \beta^2 E([y - E(y)]^2) \\
&= \alpha^2 \mathrm{var}(x) + 2\alpha\beta \mathrm{cov}(x, y) + \beta^2 \mathrm{var}(y).
\end{aligned}$$

2. Suppose that there are two stocks. Let x and y denote the random values of the first and second stock, respectively, after one year. Furthermore, we know that $\mathrm{std}(x) = 0.20$, $\mathrm{std}(y) = 0.18$, and $\mathrm{cov}(x, y) = 0.01$. A *portfolio* is composed out of $\alpha = 2$ units of stock 1 and $\beta = 3$ units of stock 2. Calculate the variance of the *portfolio value* in one year, that is, $\mathrm{var}(\alpha x + \beta y)$.

Answer:

$$\begin{aligned}\text{var}(\alpha x + \beta y) &= \alpha^2 \text{var}(x) + 2\alpha\beta \text{cov}(x, y) + \beta^2 \text{var}(y) \\ &= 2^2(0.20)^2 + 0.12 + 3^2(0.18)^2 \\ &= 0.5716\end{aligned}$$

Exercise 2

Find the mean and the variance of a random variable described by the probability density function

$$p(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Answer:

The mean of the random variable is found using

$$\begin{aligned}E(x) &= \int_{-\infty}^{+\infty} xp(x)\mathrm{d}x = \int_0^1 xp(x)\mathrm{d}x + \int_1^2 xp(x)\mathrm{d}x \\ &= \int_0^1 x^2 \mathrm{d}x + \int_1^2 x(2-x)\mathrm{d}x = \int_0^1 x^2\mathrm{d}x + 2\int_1^2 x\mathrm{d}x - \int_1^2 x^2\mathrm{d}x \\ &= \left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^2}{2}\right]_1^2 - \left[\frac{x^3}{3}\right]_1^2 \\ &= \frac{1}{3} + 2\left(2 - \frac{1}{2}\right) - \left(\frac{8}{3} - \frac{1}{3}\right) = 1.\end{aligned}$$

In order to find the variance of the random variable we compute

$$\begin{aligned}E(x^2) &= \int_{-\infty}^{+\infty} x^2p(x)\mathrm{d}x = \int_0^1 x^2p(x)\mathrm{d}x + \int_1^2 x^2p(x)\mathrm{d}x \\ &= \int_0^1 x^3 \mathrm{d}x + \int_1^2 x^2(2-x)\mathrm{d}x = \int_0^1 x^3\mathrm{d}x + 2\int_1^2 x^2\mathrm{d}x - \int_1^2 x^3\mathrm{d}x \\ &= \left[\frac{x^4}{4}\right]_0^1 + 2\left[\frac{x^3}{3}\right]_1^2 - \left[\frac{x^4}{4}\right]_1^2 \\ &= \frac{1}{4} + 2\left(\frac{8}{3} - \frac{1}{3}\right) - \left(4 - \frac{1}{4}\right) = \frac{7}{6},\end{aligned}$$

and therefore the variance of the random variable is

$$\text{var}(x) = E(x^2) - (E(x))^2 = \frac{1}{6}.$$

Exercise 3

Write the second order Taylor series expansion of

1. $f(x) = e^x$, around $x = 1$.
2. $f(x) = e^{x^2}$, around $x = 1$.
3. $f(x_1, x_2) = e^{x_1 x_2}$, around $x_1 = x_2 = 0$.

Answer:

1. We have that $f'(x) = f''(x) = e^x$. Using the formula for the Taylor series expansion and keeping terms up to second order we have that

$$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= e + e(x-1) + \frac{1}{2}e(x-1)^2. \end{aligned}$$

2. We have that

$$\begin{aligned} f'(x) &= (x^2)'e^{x^2} = 2xe^{x^2} \\ f''(x) &= (f'(x))' = (2xf(x))' \\ &= (2x)'f(x) + 2xf'(x) \\ &= 2f(x) + 2x(2xe^{x^2}) = 2e^{x^2} + 4x^2e^{x^2}. \end{aligned}$$

Using the formula for the Taylor series expansion and keeping terms up to second order we have that

$$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 \\ &= e + 2e(x-1) + 3e(x-1)^2. \end{aligned}$$

3. We have that

$$
\begin{aligned}
f'(x_1, x_2) &= \begin{bmatrix} \frac{\partial f(x_1,x_2)}{\partial x_1} \\ \frac{\partial f(x_1,x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 e^{x_1 x_2} \\ x_1 e^{x_1 x_2} \end{bmatrix} \\
f''(x_1, x_2) &= \begin{bmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 x_2} \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_2 x_1} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \end{bmatrix} \\
&= \begin{bmatrix} x_2^2 e^{x_1 x_2} & e^{x_1 x_2} + x_1 x_2 e^{x_1 x_2} \\ e^{x_1 x_2} + x_1 x_2 e^{x_1 x_2} & x_1^2 e^{x_1 x_2} \end{bmatrix}
\end{aligned}
$$

Using the formula for the Taylor series expansion and keeping terms up to second order we have that

$$
\begin{aligned}
f(x_1, x_2) &= f(0,0) + f'(0,0)^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} f''(0,0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= 1 + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= 1 + x_1 x_2.
\end{aligned}
$$

Computational Finance
Solutions

Tutorial 2

Exercise 1

Let r_{eff} be the effective interest rate and r be the nominal interest rate. Therefore, their relation is given by

$$1 + r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m,$$

where m is the number of compounding periods and r is the interest rate.

1. For 3% compounded monthly, $r_{\text{eff}} = \left(1 + \frac{0.03}{12}\right)^{12} - 1 = 0.0304$.
2. For 18% compounded monthly, $r_{\text{eff}} = \left(1 + \frac{0.18}{12}\right)^{12} - 1 = 0.1956$.
3. For 18% compounded quarterly, $r_{\text{eff}} = \left(1 + \frac{0.18}{4}\right)^{4} - 1 = 0.1925$.

Exercise 2

We need to consider the present value of the two following payment streams,

1. Change the roof now, then every 20 years,

$$PV_1 = 20,000 \cdot \sum_{i=0}^{\infty} \frac{1}{(1+0.05)^{20i}} = 32,097.$$

2. Change the roof in 5 years and then every 20 years,

$$PV_2 = \frac{PV_1}{(1+0.05)^5} = 25,149.$$

Taking the difference of these two, we find the value of the roof to be

$$V = PV_1 - PV_2 = 6,948.$$

Exercise 3

1. Let the current spot rate curve be $s_1, s_2, \ldots, s_n$. Therefore, the expected spot rate curve $s_1^{'}, s_2^{'} \ldots, s_{n-1}^{'}$ for next year is given by

$$s_{j-1}^{'} = \left[\frac{(1+s_j)^j}{(1+s_1)}\right]^{\frac{1}{j-1}} - 1 \quad \text{for } j = 2, 3, \ldots, n.$$

Here $\boldsymbol{s} = \{0.05, 0.053, 0.056, 0.058, 0.06, 0.061\}$. Therefore,

$$s_1^{'} = \left[\frac{(1+s_2)^2}{(1+s_1)}\right]^{\frac{1}{2-1}} - 1 = 0.05600$$

$$s_2^{'} = \left[\frac{(1+s_3)^3}{(1+s_1)}\right]^{\frac{1}{3-1}} - 1 = 0.05901$$

$$s_3^{'} = \left[\frac{(1+s_4)^4}{(1+s_1)}\right]^{\frac{1}{4-1}} - 1 = 0.06068$$

$$s_4^{'} = \left[\frac{(1+s_5)^5}{(1+s_1)}\right]^{\frac{1}{5-1}} - 1 = 0.06251$$

$$s_5^{'} = \left[\frac{(1+s_6)^6}{(1+s_1)}\right]^{\frac{1}{6-1}} - 1 = 0.06321$$

Therefore, the spot rate curve for next year is

$$\boldsymbol{s}^{'} = \{0.05600, 0.05901, 0.06068, 0.06251, 0.06321\}.$$

2. The present value of an n-year bond with face value F and coupon rate c, is given by

$$PV = \sum_{k=1}^{n-1} d_k c F + d_n(1+c)F,$$

where d_i is the discount rate of the i^{th} year. For the two bonds under consideration, we have that

$$101 = \sum_{k=1}^{4} d_k(0.09 \cdot 100) + d_5(1.09 \cdot 100) = 9\sum_{k=1}^{4} d_k + 109 d_5,$$
$$93.2 = \sum_{k=1}^{4} d_k(0.07 \cdot 100) + d_5(1.07 \cdot 100) = 7\sum_{k=1}^{4} d_k + 107 d_5.$$

Multiplying the first equation by 7 and the second by 9 we obtain

$$707 = 63\sum_{k=1}^{4} d_k + 763 d_5,$$
$$838.8 = 63\sum_{k=1}^{4} d_k + 963 d_5.$$

Subtracting the last two equations gives

$$707 - 838.8 = (763 - 963) d_5,$$

from which we obtain that

$$d_5 = 0.659.$$

Therefore, the value of the zero coupon bond is given by

$$V = F d_5 = 65.9.$$

Exercise 4

Consider a divided stream that grows at a rate of g per period. Let $r > g$ be the discount rate per period. Then the present value of the dividend stream, starting one period from the present, is

$$\begin{aligned} V_0 &= \sum_{k=1}^{\infty} \frac{(1+g)^k D_0}{(1+r)^k} = D_0 \sum_{k=1}^{\infty} \left(\frac{1+g}{1+r}\right)^k \\ &= D_0 \left(\frac{1}{1 - \left(\frac{1+g}{1+r}\right)} - 1 \right) \\ &= \frac{(1+g) D_0}{r-g}, \end{aligned}$$

where D_0 is the current divided. The above expression is referred to as the discounted growth formula.

The total value of a share is given by

$$V_0 = \frac{1.10 \times \$1.37}{0.15 - 0.10} = \$30.14.$$

Computational Finance
Answers

Tutorial 3

Exercise 1

1.

$$P_A = \frac{100}{1.15} + \frac{100}{1.15^2} + \frac{1100}{1.15^3} = 885.84$$
$$P_B = 771.68$$
$$P_C = 657.52$$
$$P_D = 869.57$$

2.

$$D_A = \frac{1\frac{100}{1.15} + 2\frac{100}{1.15^2} + 3\frac{1100}{1.15^3}}{885.84} = 2.72$$
$$D_B = 2.84$$
$$D_C = 3$$
$$D_D = 1$$

3. C is most sensitive to a change in yield.

4. We must much the present value and the duration of the obligation. The equations to be solved are:

$$V_A + V_B + V_C + V_D = PV,$$
$$D_A V_A + D_B V_B + D_C V_C + D_D V_D = 2P_V,$$

where

$$PV = \frac{2000}{1.15^2}.$$

5. You can only use bond D in combination with bond C, because all other bonds have a duration larger than 2, and it is impossible to achieve a weighted average of 2. Hence we have

$$V_C + V_D = \frac{2000}{1.15^2} = 1512.29,$$

$$3V_C + V_D = 2 \cdot 1512.29 = 3024.58$$

Solving we obtain $V_C = 756.15$, $V_D = 756.15$.

Exercise 2

Let α, β be the outcomes of two die rolls. then $z = \alpha\beta$. By independence we know

$$\mathrm{E}[\alpha\beta] = \mathrm{E}[\alpha]\mathrm{E}[\beta] = 3.5^2 = 12.25$$

and

$$\mathrm{Var[z]} = \mathrm{E}[\alpha^2]\mathrm{E}[\beta^2] - (\mathrm{E}[\alpha]\mathrm{E}[\beta])^2 = \left(\frac{1+4+9+16+25+36}{6}\right)^2 - 12.25^2 = 79.97$$

Exercise 3

1. We have

$$P = \lim_{n\to\infty} \frac{F}{(1+\lambda/m)^n} + \frac{C}{\lambda}\left(1 - \frac{1}{(1+\lambda/m)^n}\right) = \frac{C}{\lambda}.$$

$$D_M = -\frac{1}{P}\frac{dP}{d\lambda} = -\frac{\frac{C}{\lambda^2}}{\frac{C}{\lambda}} = \frac{1}{\lambda}.$$

2. We have

$$D = \lim_{n\to\infty} \frac{1+\lambda/m}{\lambda} - \frac{1+\lambda/m + n(c/m - \lambda/m)}{c[(1+\lambda/m)^n - 1] + \lambda} = \frac{1+\lambda/m}{\lambda}$$

and therefore

$$D_M = \frac{D}{1+\lambda/m} = \frac{1}{\lambda}.$$

Computational Finance
Answers

Tutorial 4

Exercise 1
The portfolio's expected rate of return is

$$\bar{r} = w_1\bar{r}_1 + w_2\bar{r}_2 = 0.25 \cdot 0.1 + 0.75 \cdot 0.18 = 0.16.$$

In the lecture notes we can find the variance of the rate of return of a portfolio:

$$\sigma^2 = \sum_{i,j=1}^{n} w_i \sigma_{ij} w_j$$

for $n = 2$ this is

$$\sigma^2 = w_1\sigma_{11}w_1 + w_1\sigma_{12}w_2 + w_2\sigma_{21}w_1 + w_2\sigma_{22}w_2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2.$$

Using $\sigma_{12} = \rho\sigma_1\sigma_2 = 0.1 \cdot 0.15 \cdot 0.3 = 0.0045$, we have

$$\begin{aligned} \sigma^2 &= w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2 = \\ &= 0.25^2 \cdot 0.15^2 + 2 \cdot 0.25 \cdot 0.75 \cdot 0.0045 + 0.75^2 \cdot 0.3^2 = 0.053719 \end{aligned}$$

and $\sigma = 0.2318 \approx 23.2\%$.

Exercise 2

1. The investment portfolio consists of the concert and the insurance. Let u denote the units of insurances bought. Kate's total investment is therefore £$10^6 + 0.5 \cdot u$.

(a) There is a 50% chance that the concert will take place, in which case, she will get £$3 \cdot 10^6$ back at the end of the year.

(b) There is a 50% chance that the concert will not take place. She will then get £u back at the end of the year thanks to the insurance.

The investment's expected rate of return is therefore

$$\bar{r}(u) = \frac{0.5 \cdot 3 \cdot 10^6 + 0.5 \cdot u}{10^6 + 0.5 \cdot u} - 1.$$

2. Variance is always greater than or equal to 0. Thus, if we find u that reduces the variance to zero, we are done. If the variance is zero, then both possibilities (concert or no concert) have the same rate of return.

   (a) If the concert takes place, the rate of return of the investment is

   $$\frac{3 \cdot 10^6}{10^6 + 0.5 \cdot u} - 1$$

   (b) If the concert does not take place, the rate of return of the investment is

   $$\frac{u}{10^6 + 0.5 \cdot u} - 1$$

   .

   Therefore, if we set $u = 3 \cdot 10^6$ (the maximal allowed amount of insurance units), the rate of return is equal in both cases, and the variance of the rate of return of the investment is 0. The corresponding expected rate of return is

   $$\bar{r} = \frac{0.5 \cdot 3 \cdot 10^6 + 0.5 \cdot 3 \cdot 10^6}{10^6 + 0.5 \cdot 3 \cdot 10^6} - 1 = 0.20.$$

Exercise 3
We have to

$$\begin{array}{rc} \text{minimise} & \sigma^2(w_1, w_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ \text{subject to the constraint} & w_1 + w_2 = 1 \end{array}$$

Therefore, the Lagrangian function is given by

$$L(w_1, w_2, \lambda) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} - \lambda(w_1 + w_2 - 1).$$

At optimality, the derivatives with respect to w_1, w_2, and λ have to be equal to 0:

$$\begin{aligned} \frac{\partial L}{\partial w_1} &= 2w_1\sigma_1^2 + 2w_2\sigma_{12} - \lambda = 0 && (1) \\ \frac{\partial L}{\partial w_2} &= 2w_2\sigma_2^2 + 2w_1\sigma_{12} - \lambda = 0 && (2) \\ \frac{\partial L}{\partial \lambda} &= w_1 + w_2 - 1 = 0 && (3) \end{aligned}$$

By eliminating λ from (1) and (2) we obtain

$$\begin{aligned} & w_1\sigma_1^2 + w_2\sigma_{12} = w_2\sigma_2^2 + w_1\sigma_{12} \\ \Longleftrightarrow\ & w_1\sigma_1^2 + (1 - w_1)\sigma_{12} = (1 - w_1)\sigma_2^2 + w_1\sigma_{12} \\ \Longleftrightarrow\ & w_1\left(\sigma_1^2 - \sigma_{12} + \sigma_2^2 - \sigma_{12}\right) = \sigma_2^2 - \sigma_{12}, \end{aligned}$$

and therefore we find

$$\begin{aligned} w_1 &= \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}, \\ w_2 &= \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}. \end{aligned}$$

The mean rate of return then amounts to

$$\bar{r} = w_1\bar{r}_1 + w_2\bar{r}_2.$$

Alternative solution: we could express w_2 as a function of w_1, that is,

$$w_2 = 1 - w_1. \qquad (4)$$

Using (4), we can now express the portfolio variance as a function of w_1.

$$\sigma^2(w_1) = w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_{12}$$

We now proceed by solving $\frac{\mathrm{d}\sigma(w_1)}{\mathrm{d}w_1} = 0$ for w_1.

Exercise 4
We have

$$\begin{aligned}
&\sigma_i = \sigma && \forall i = 1, \dots, n \\
&w_i = \frac{1}{n} && \forall i = 1, \dots, n \\
&\sigma_{ij} = \rho\sigma_i\sigma_j = \rho\sigma^2 && \text{for } i \neq j.
\end{aligned}$$

The variance of the portfolio is

$$\begin{aligned}
\sigma_{\mathrm{p}}^2 =& \sum_{i,j} w_i w_j \sigma_{ij} = \frac{1}{n^2}\sum_{i,j}\sigma_{ij} = \frac{1}{n^2}\left(\sum_{i=j}\sigma_{ij} + \sum_{i\neq j}\sigma_{ij}\right) \\
=& \frac{1}{n^2}\left(\sum_{i}\sigma^2 + \sum_{i\neq j}\rho\sigma^2\right) = \frac{1}{n^2}\left(n\sigma^2 + (n^2-n)\rho\sigma^2\right) \\
=& \frac{(1-\rho)\sigma^2}{n} + \rho\sigma^2.
\end{aligned}$$

Therefore $\lim\limits_{n\to\infty}\sigma_{\mathrm{p}}^2 = \rho\sigma^2$.

Computational Finance
Answers

Tutorial 5

Exercise 1

1. Using the two fund theorem we know that the market portfolio is of the form

$$\alpha w + (1-\alpha)v = [0.8 - 0.2\alpha, 0.4\alpha - 0.2, 0.4 - 0.2\alpha]^T.$$

The market cannot contain assets in negative amounts, and therefore

$$0.5 \leq \alpha \leq 2.$$

The expected return of the market is

$$\bar{r}_M = 0.1 \cdot (0.8 - 0.2\alpha) + 0.2 \cdot (0.4\alpha - 0.2) + 0.1 \cdot (0.4 - 0.2\alpha) = 0.08 + 0.04\alpha.$$

Therefore

$$0.1 \leq \bar{r}_M \leq 0.16.$$

2. Since the expected return of the market must not be less than the return of the minimum-variance portfolio, we have

$$0.12 \leq \bar{r}_M \leq 0.16.$$

Exercise 2

1. $\bar{r} = 0.07 + \frac{0.23-0.07}{0.32}\sigma = 0.07 + 0.5\sigma$

2. (a) $\sigma = 0.64$
   (b) We solve
   $$0.07w + 0.23(1-w) = 0.39$$
   giving $w = -1$. Hence we need to borrow \$1000 at the risk free rate and invest \$2000 in the market.
3. $300 \cdot 1.07 + 700 \cdot 1.23 = 1182$

Exercise 3

1. $$\sigma_M^2 = \frac{1}{4}(\sigma_A^2 + 2\sigma_{A,B} + \sigma_B^2)$$
$$\sigma_{AM} = \frac{1}{2}(\sigma_A^2 + \sigma_{A,B}) \text{ hence } \beta_A = \frac{\sigma_A^2 + \sigma_{A,B}}{2\sigma_M^2}$$
$$\sigma_{BM} = \frac{1}{2}(\sigma_B^2 + \sigma_{A,B}) \text{ hence } \beta_B = \frac{\sigma_B^2 + \sigma_{A,B}}{2\sigma_M^2}$$
2. $$\bar{r}_A = 0.1 + \frac{5}{4}(0.18 - 0.1) = 0.2$$
$$\bar{r}_B = 0.1 + \frac{3}{4}(0.18 - 0.1) = 0.16$$

Exercise 4

The market consists of \$150 in shares of A and \$300 is shares of B. Hence the market return is $r_M = \frac{150}{450}r_A + \frac{300}{450}r_B = \frac{1}{3}r_A + \frac{2}{3}r_B$.

1. $$\bar{r}_M = \frac{1}{3}0.15 + \frac{2}{3}0.12 = 0.13$$
2. $$\sigma_M = \left[\frac{1}{9}0.15^2 + \frac{4}{9}\cdot\frac{1}{3}\cdot 0.15 \cdot 0.09 + \frac{4}{9}0.09^2\right]^{\frac{1}{2}} = 0.09$$

3.

$$\sigma_{AM} = \frac{1}{3}\sigma_A^2 + \frac{2}{3}\rho_{AB}\sigma_A\sigma_B = \frac{1}{3}0.15^2 + \frac{2}{3}\cdot\frac{1}{3}\cdot 0.15\cdot 0.09 = 0.0105$$

$$\beta_A = \frac{\sigma_{AM}}{\sigma_M^2} = 1.2963$$

4. Since Simpleland satisfies the CAPM exactly, stocks A,B plot on the security market line. Specifically, $\bar{r}_A - r_f = \beta_A(\bar{r}_M - r_f)$. Hence, $r_f = \frac{\bar{r}_A - \beta_A\bar{r}_M}{1-\beta_A} = 0.0625$.

Computational Finance
Answers

Tutorial 6

Exercise 1

a Expected Utilities

- For the risk free wealth variable x we have

$$E(U(x)) = E(U(5)) = U(5) = 5 - 0.04 \cdot 5^2 = 4.$$

- For the coin toss we have

$$E(U(x)) = \frac{1}{2} \cdot U(10) + \frac{1}{2} \cdot U(0) = \frac{1}{2} \cdot (10 - 0.04 \cdot 10^2) = 3.$$

The risk-free investment is to be preferred.

b Certainty Equivalent

The certainty equivalent is obtained by solving $E(U(x)) = U(C)$ for C. The risky investment has $E(U(x)) = 3$. We therefore have to solve the equation

$$3 = C - 0.04\ C^2,$$

which gives $C = 3.49$[1].

[1] The other solution to the quadratic equation $0.04\ C^2 - C + 3 = 0$ is $C \approx 21.5$. This is meaningless in our case.

Exercise 2
At the end of the year Jérôme earns his salary and, of course, his bonus. This can be $£80,000+0$ or $£80,000+10,000$, ..., each with probability $\frac{1}{7}$. The expected utility, displayed in thousands, is

$$E(U(x)) = \frac{1}{7} \cdot \left(\sqrt[4]{80} + \sqrt[4]{90} + \sqrt[4]{100} + \sqrt[4]{110} + \sqrt[4]{120} + \sqrt[4]{130} + \sqrt[4]{140} \right) = 3.23.$$

In order to find the certainty equivalent, we have to solve $E(U(x)) = U(C)$ for C. In this example, we have to solve the equation $\sqrt[4]{C} = 3.23$, which gives $C = 3.23^4 = 109$. Therefore, a basic salary of £109,000 with no extra bonus would be equivalent to the hedge-fund offer.[2]

Exercise 3
The derivatives of $V(x) = c+bU(x)$ are $V'(x) = bU'(x)$ and $V''(x) = bU''(x)$. Therefore, we find

$$a_V(x) = -\frac{V''(x)}{V'(x)} = -\frac{bU''(x)}{bU'(x)} = -\frac{U''(x)}{U'(x)} = a_U(x).$$

Exercise 4
For $U(x) = \ln(x)$ we have $U'(x) = \frac{1}{x}$ and $U''(x) = -\frac{1}{x^2}$. Therefore, we find

$$\mu(x) = -\frac{xU''(x)}{U'(x)} = -\frac{x(-\frac{1}{x^2})}{\frac{1}{x}} = 1.$$

For $U(x) = \gamma x^\gamma$, we have $U'(x) = \gamma^2 x^{\gamma-1}$ and $U''(x) = \gamma^2 \cdot (\gamma-1)x^{\gamma-2}$. Therefore, we find

$$\mu(x) = -\frac{xU''(x)}{U'(x)} = -\frac{x\gamma^2(\gamma-1)x^{\gamma-2}}{\gamma^2 x^{\gamma-1}} = 1-\gamma.$$

Exercise 5

a We use the following approximations

$$U(C) = U(\bar{x}) + U'(\bar{x})(C-\bar{x}), \tag{1}$$
$$E(U(x)) = U(\bar{x}) + \tfrac{1}{2}U''(\bar{x})\mathrm{var}(x). \tag{2}$$

[2] All calculations were performed with an accuracy of 3 significant digits.

The certainty equivalent C is defined via $U(C) = E(U(x))$. Using our approximations, we have

$$\begin{aligned}
U(\bar{x}) + U'(\bar{x})(C - \bar{x}) &\approx U(\bar{x}) + \frac{1}{2}U''(\bar{x})\mathrm{var}(x) \\
U'(\bar{x})C &\approx U'(\bar{x})\bar{x} + \frac{1}{2}U''(\bar{x})\mathrm{var}(x) \\
C &\approx \bar{x} + \frac{1}{2}\frac{U''(\bar{x})}{U'(\bar{x})}\mathrm{var}(x).
\end{aligned}$$

b In exercise 1, we have

$$\begin{aligned}
U(x) &= x - 0.04x^2, \\
U'(x) &= 1 - 0.08x, \\
U''(x) &= -0.08.
\end{aligned}$$

The payoff of the risky investment has expected value

$$\bar{x} = \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 = 5$$

and variance

$$\mathrm{var}(x) = \frac{1}{2} \cdot (10 - \bar{x})^2 + \frac{1}{2} \cdot (0 - \bar{x})^2 = 25.$$

Using the approximation of C (derived in 5(a)), we have

$$\begin{aligned}
C &\approx \bar{x} + \frac{1}{2}\frac{U''(\bar{x})}{U'(\bar{x})}\mathrm{var}(\bar{x}) = 5 + \frac{1}{2}\frac{-0.08}{1 - 0.08 \cdot 5}25 = \\
&= 5 - \frac{25}{2}\frac{8}{100 - 8 \cdot 5} = 5 - \frac{200}{120} = 5 - \frac{5}{3} = \frac{10}{3} = 3.33.
\end{aligned}$$

The exact certainty equivalent was 3.49.

Computational Finance
Answers

Tutorial 7

Exercise 1

[Residual rights] The investor must solve the portfolio optimization problem, as was the case in the notes, with the new information. There are three securities:

1. the original film venture
2. the risk-free asset
3. the residual rights.

The investor will purchase these securities in amounts θ_1, θ_2 and θ_3, respectively. Using the logarithmic utility function $U(x) = \ln x$, the problem is to select $\theta_1, \theta_2, \theta_3$ which solve the following maximization problem

$$\begin{array}{ll} \max & 0.3\ln(3\theta_1 + 1.2\theta_2 + 6\theta_3) + 0.4\ln(\theta_1 + 1.2\theta_2) + 0.3\ln(1.2\theta_2) \\ \text{s.t.} & \theta_1 + \theta_2 + \theta_3 = W \end{array}$$

The optimality conditions are

$$\frac{0.9}{3\theta_1 + 1.2\theta_2 + 6\theta_3} + \frac{0.4}{\theta_1 + 1.2\theta_2} = \lambda \tag{1}$$

$$\frac{0.36}{3\theta_1 + 1.2\theta_2 + 6\theta_3} + \frac{0.48}{\theta_1 + 1.2\theta_2} + \frac{0.36}{1.2\theta_2} = \lambda \tag{2}$$

$$\frac{1.8}{3\theta_1 + 1.2\theta_2 + 6\theta_3} = \lambda. \tag{3}$$

In addition, the budget constraint $\theta_1 + \theta_2 + \theta_3 = W$ must be satisfied. In order to solve these equations, we note from (3) that

$$\frac{0.9}{3\theta_1 + 1.2\theta_2 + 6\theta_3} = \frac{\lambda}{2}.$$

Substituting this expression into (1), we obtain

$$\frac{0.8}{\theta_1 + 1.2\theta_2} = \lambda \tag{4}$$

From the last equation and (3) we obtain

$$\begin{aligned}\frac{0.48}{\theta_1 + 1.2\theta_2} &= \frac{6}{10}\lambda, \\ \frac{0.36}{3\theta_1 + 1.2\theta_2 + 6\theta_3} &= \frac{2}{10}\lambda.\end{aligned}$$

Substituting these equations into (2), we find

$$\frac{2}{10}\lambda + \frac{6}{10}\lambda + \frac{0.36}{1.2\theta_2} = \lambda,$$

that is,

$$\lambda\theta_2 = \frac{3}{2}. \tag{5}$$

If we combine (5) and (4) we have that $0.8 = \lambda\theta_1 + 1.2\lambda\theta_2 = \lambda\theta_1 + 1.8$. Therefore,

$$\lambda\theta_1 = -1. \tag{6}$$

Similarly, by substituting (5) and (6) into (3) we get $1.8 = 3\lambda\theta_1 + 1.2\lambda\theta_2 + 6\lambda\theta_3 = -3 + 1.8 + 6\lambda\theta_3$, that is,

$$\lambda\theta_3 = \frac{1}{2}. \tag{7}$$

Multiplying the budget constraint by λ and using (5), (6) and (7), we find

$$\lambda W = \lambda\theta_1 + \lambda\theta_2 + \lambda\theta_3 = -1 + \frac{3}{2} + \frac{1}{2} = 1 \quad \Rightarrow \quad \lambda = \frac{1}{W}.$$

Substituting $\lambda = \frac{1}{W}$ into (5), (6) and (7) we thus obtain

$$\begin{aligned} w_1 &= -1.0W \\ w_2 &= 1.5W \\ w_3 &= 0.5W. \end{aligned}$$

In other words, the investor should short the ordinary film venture by an amount equal to his total wealth in order to invest in the other two alternatives.

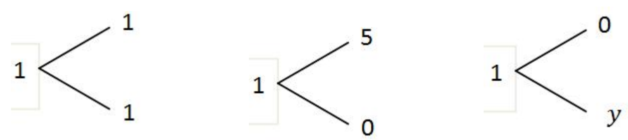
Exercise 2

Joe will choose the fraction w of his wealth M to bet on the horse so as to maximize his expected utility, that is, he solves

$$\max E[U(a)] = \frac{1}{4}\sqrt{M + 4wM} + \frac{3}{4}\sqrt{(1-w)M}.$$

1. The first order necessary conditions are
$$\frac{1}{2}\frac{M}{\sqrt{M+4wM}} - \frac{3}{8}\frac{M}{\sqrt{(1-w)M}} = 0,$$
which yields
$$w = \frac{7}{52} = 0.1346$$
Therefore Joe's maximizing choice is to bet 13.46% of his money and keep the rest in his pocket.

2. The three securities can be represented diagrammatically as:



Keep money in pocket   **Bet for No Arbitrage**   **Bet agains No Arbitrage**

3. The value of y that makes Joe's market arbitrage-free is obtained by expressing the last security (bet against no arbitrage) as linear combination of the other two securities (money in pocket, bet for no arbitrage). If θ_1 and θ_2 are the weights of the last two securities, then
$$\begin{array}{ccccc} \theta_1 \cdot 1 & + & \theta_2 \cdot 5 & = & 0 \\ \theta_1 \cdot 1 & + & \theta_2 \cdot 1 & = & 1 \\ \theta_1 \cdot 1 & + & \theta_2 \cdot 0 & = & y \end{array}$$
This is a system of three linear equations with unknowns θ_1, θ_2, y. From the last equation we find
$$y = \theta_1. \tag{8}$$
Subtracting the first from the second equation we obtain $4\theta_2 = 1$, or
$$\theta_2 = -\frac{1}{4}.$$
From the first equation we have that
$$\theta_1 = -5\theta_2 = \frac{5}{4}.$$
Substituting this expression into (8) we obtain
$$y = \frac{5}{4}.$$

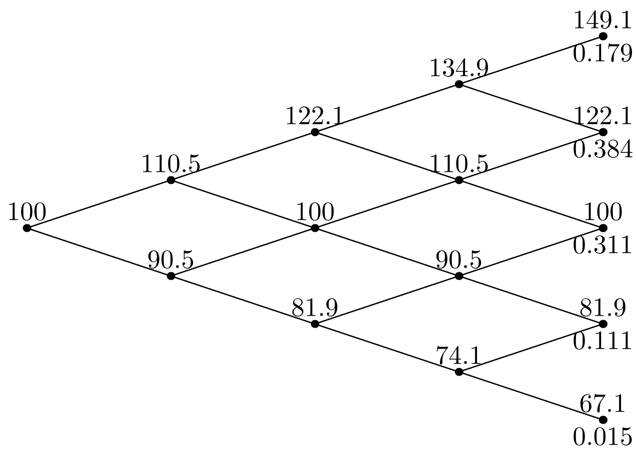
Computational Finance
Answers

Tutorial 8

Exercise 1 (**Stock lattice**)

The following parameters are given : $\Delta t = \frac{1}{4}$, $\nu = 0.12$, $\sigma = 0.20$ and $S(0) = 100$. Using the approximation formulae on slide 34, we obtain

$$
\begin{aligned}
u &\simeq e^{\sigma\sqrt{\Delta t}} = 1.105 \\
d &\simeq e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} = 0.905 \\
p &\simeq \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} = 0.65
\end{aligned}
$$

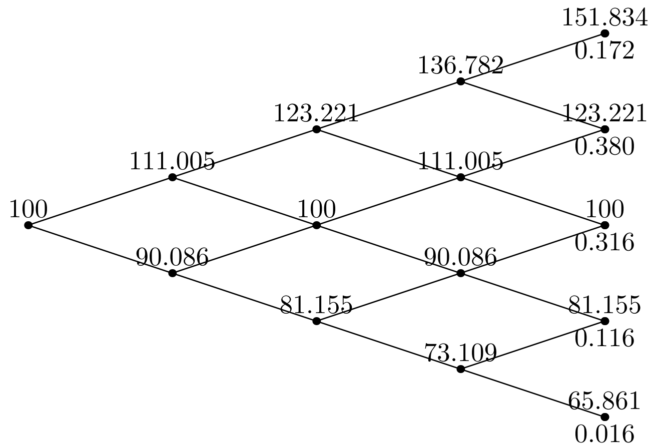


The binomial lattice of the stock prices for a period of 1 year is shown above. The numbers above the nodes are the stock prices and the numbers below the end nodes are the corresponding probabilites. Note that the probability of reaching the end node with stock price $S(0)u^k d^{n-k}$ is $\binom{n}{k} p^k(1-p)^{n-k}$.

On the other hand, using the exact formulae on slide 34 will give

$$
\begin{aligned}
p &= \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2/(\nu^2\Delta t)+1}} = 0.64367 \\
u &= e^{\sqrt{\sigma^2\Delta t + (\nu\Delta t)^2}} = 1.11005 \\
d &= e^{-\sqrt{\sigma^2\Delta t + (\nu\Delta t)^2}} = 0.90086
\end{aligned}
$$

The binomial lattice using exact formulae is shown below for comparison.



Exercise 2 (**Time scaling**)

Each movement in k corresponds to a month, and each movement in K corresponds to a year. Let k_K denote the first month of year K. Then

$$W(K) = \sum_{i=0}^{11} w(k_{K-1} + i).$$

So,

$$
\begin{aligned}
\mathrm{E}[W(K)] &= \mathrm{E}\left[\sum_{i=0}^{11} w(k_{K-1}+i)\right] = 12\nu \\
\mathrm{var}[W(K)] &= \mathrm{E}\left[\sum_{i=0}^{11} w(k_{K-1}+i)\right]^2 = 12\sigma^2.
\end{aligned}
$$

Exercise 3 (**Expectations**)

The formulae can be found on slides 21 and 22 in the lecture notes. Given $\nu = 0.12$, $\sigma = 0.40$, $t = 1$ and $S(0) = 1$, we have

$$\begin{aligned}
\mathrm{E}[\ln S(1)] &= \mathrm{E}[\ln S(0)] + \nu t = 0.12, \\
\mathrm{std}[\ln S(1)] &= \sigma\sqrt{t} = 0.40, \\
\mathrm{E}[S(1)] &= S(0)e^{(\nu+\frac{\sigma^2}{2})t} = e^{0.2} = 0.122, \\
\mathrm{std}[S(1)] &= S(0)e^{(\nu+\frac{\sigma^2}{2})t}(e^{\sigma^2 t} - 1)^{\frac{1}{2}} = e^{0.2}(e^{0.16} - 1)^{\frac{1}{2}} = 0.51.
\end{aligned}$$

Exercise 4 (**Lognormal random variables**)

If u is a lognormal variable, then $w = \ln u$ is a normal variable, that is $w \approx \mathcal{N}(\nu, \sigma^2)$. The expected value of u is

$$\begin{aligned}
\mathrm{E}[u] &= \mathrm{E}[e^w] \\
&= \int_{-\infty}^{+\infty} e^w \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\nu)^2}{2\sigma^2}} dw \\
&= \int_{-\infty}^{+\infty} \frac{e^w}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\nu-\sigma^2+\sigma^2)^2}{2\sigma^2}} dw \\
&= \int_{-\infty}^{+\infty} \frac{e^w}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\nu-\sigma^2)^2+2(w-\nu-\sigma^2)\sigma^2+\sigma^4}{2\sigma^2}} dw \\
&= \int_{-\infty}^{+\infty} \frac{e^w}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\nu-\sigma^2)^2}{2\sigma^2}} e^{-(w-\nu-\sigma^2)} e^{-\frac{\sigma^2}{2}} dw \\
&= e^{\nu+\frac{\sigma^2}{2}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\nu-\sigma^2)^2}{2\sigma^2}} dw \\
&= e^{\nu+\frac{\sigma^2}{2}}
\end{aligned}$$

where

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\nu-\sigma^2)^2}{2\sigma^2}} dw = 1$$

since it is the integral of the density function of a normal random variable that follows $\mathcal{N}(\nu + \sigma^2, \sigma^2)$.

Computational Finance
Answers

Tutorial 9

Exercise 1
[Monotonicity of the price of a call] Assume that $K_2 > K_1$, and suppose to the contrary that $C(K_2) > C(K_1)$. We can then buy call option 1 with strike price K_1 and sell call option 2 with strike price K_2. By assumption, we have an immediate positive reward. Furthermore, we can use option 1 to cover the obligations of option 2, since $\max\{0, S - K_1\} \geq \max\{0, S - K_2\}$ for all S. This is a type B arbitrage.

Exercise 2
[A Bull spread] For the bull spread we buy one call with strike price K_1 and sell another one with strike price $K_2 > K_1$. Therefore, the payoff of our spread is

$$C_{\text{spread}}(S) = \max\{0, S - K_1\} - \max\{0, S - K_2\},$$

or

$$C_{\text{spread}}(S) = \begin{cases} 0 & \text{for } S < K_1 \\ S - K_1 & \text{for } S \in [K_1, K_2] \\ K_2 - K_1 & \text{for } S > K_2 \end{cases}$$

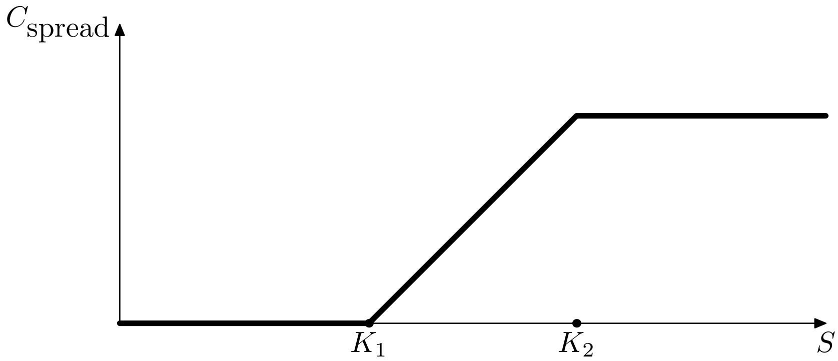
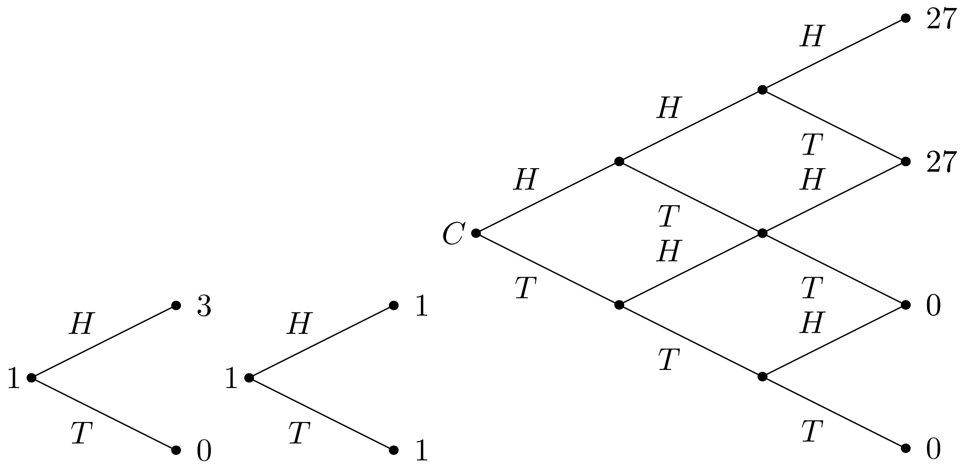


Figure 1: Payoff function of the bull spread.

We have to pay $C(K_1) - C(K_2)$ at the beginning, which is unfortunately positive, see Exercise 1.

Exercise 3
[A coin] The trees for the two basic propositions are shown in Figure 2(a) and 2(b) .



(a) Stochastic proposition (b) Deterministic proposition (c) New proposition

Figure 2: The two basic propositions and the new proposition.

The new proposition is visualised in Figure 2(c). Look at the top right subtree. It looks like 27 times the deterministic proposition.

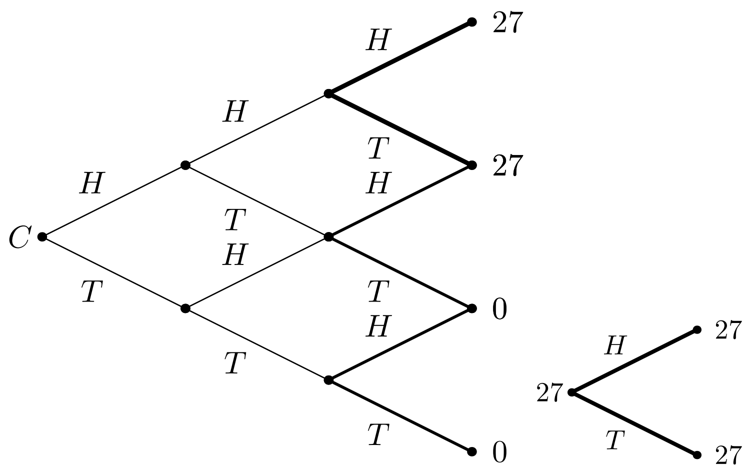


Figure 3: Step 1

Therefore, we can assign the value 27 to the root node of this subtree, and proceed to the next lower subtree that looks like 9 times the basic stochastic proposition. Moreover, the bottom right subtree looks like 0 times the deterministic one.

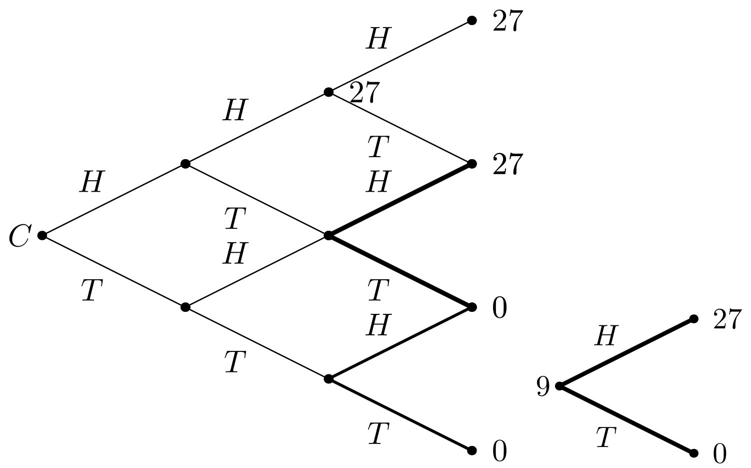


Figure 4: Step 2

The top middle subtree can be seen as a combination of 9 times the deterministic and 6 times the stochastic proposition, whose price is 15. The bottom middle subtree corresponds to 3 times the stochastic proposition.

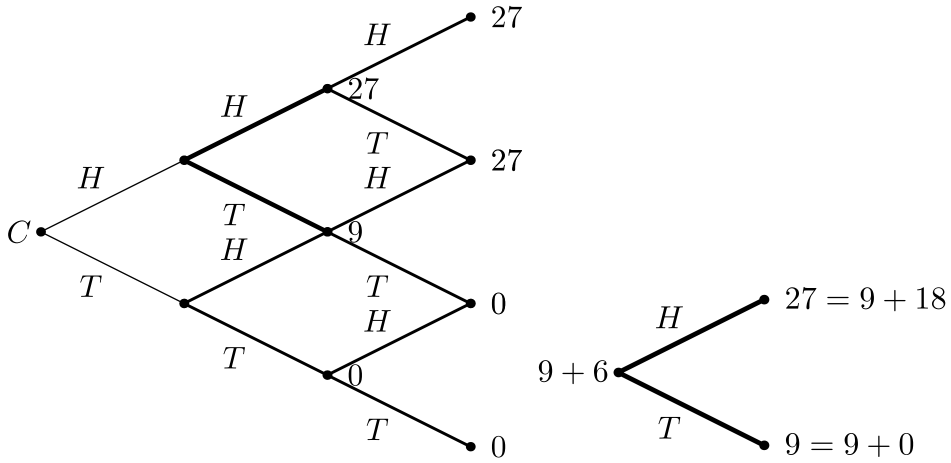


Figure 5: Step 3

The far left subtree can again be seen as a combination of the basic trees. Thus, we find the value of the proposition to be 7.

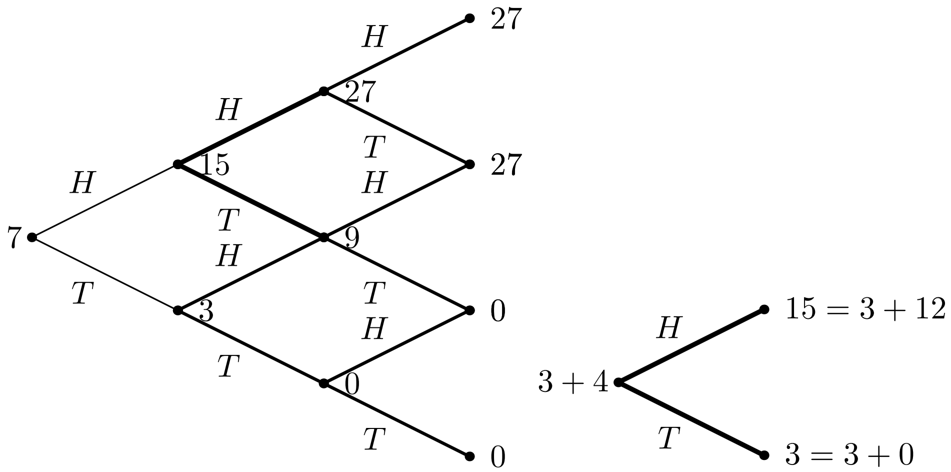


Figure 6: Step 4

Exercise 4
[A "happy call"] The payoff of the stock as a function of itself is S. The payoffs of the two calls with strike prices K and $2K$ are given by $\max\{0, S-K\}$ and $\max\{0, S-2K\}$, respectively. We want to express $\max\{0.5\cdot S, S-K\}$, as a linear combination of the basic payoff functions. First, we eliminate the part that is independent the strike price

$$\max\{0.5 \cdot S, S - K\} - 0.5 \cdot S = \max\{0, 0.5 \cdot S - K\} .$$

The term on the right side can be rewritten as $0.5 \cdot \max\{0, S - 2K\}$. This is equal to 0.5 times the payoff of a call with strike price $2K$. Thus, the "happy call" is equivalent to 0.5 times the stock and 0.5 times the call with strike price $2K$. By linearity of pricing, we have

$$C_H = 0.5 \cdot P + 0 \cdot C_1 + 2 \cdot C_2 .$$

Exercise 5
[Forward price formula] Indirect proof: assume $F > S/d$. We borrow an amount S and use this money to buy one unit of the commodity *now* at the current price S. Furthermore, we sell one forward contract, i.e., we commit to deliver the commodity at time T in return for receiving the forward price F. So far, we have not invested any money of our own.[1] Later, at time T, we have to pay back our debt with interests, which amounts to S/d. Moreover, we deliver the commodity and receive the amount F. Since $F > S/d$, we made a certain profit without investment. Since we exclude arbitrage opportunities, we conclude that $F \leq S/d$. The reverse argument shows that $F \geq S/d$. Thus, $F = S/d$.

[1] Recall that there are no initial payments associated with the forward contract.