Exersize 1

We have
- $p(x) = \sum_{j=1}^{m} P_j \, P(x|j)$ (1)
- $\sum_{j=1}^{m} P_j = 1$ (2)
- $\int_{-\infty}^{+\infty} p(x|j) = 1$ (3)

→ $P_1, P_2, \dots, P_j$ the a-priori probabilities

→ We need to find $[P_1, P_2, \dots, P_m] = \operatorname{argmax} \sum_{n=1}^{N} \sum_{j=1}^{M} P(j|x) \ln P_j$ subject to $\sum_{j=1}^{m} P_j = 1$. We need to prove that

$$P_j = \frac{1}{N} \sum_{i=1}^{N} P(j|x_i), \; j = 1, \dots, m.$$

Solution:

We have to solve an optimization problem under constraint. For this reason we will formulate the constraint into a regularized term and incorporate it in the "modified" Lagrangian function.

For this reason we formulate our Lagrangian Loss under constraint. Let $P_j \in [P_1, P_2, \dots P_m]$ for random j

$$L(P_1, P_2, \dots P_m) = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{M} \left(P(j|x_i) \ln P_j \right)}_{\text{Main Objective}} + \underbrace{\lambda \left(\sum_{j=1}^{M} P_j - 1 \right)}_{\text{Constraint}}$$

For P_j:

$$\frac{\partial L}{\partial P_j} = \frac{\partial}{\partial P_j} \left[\sum_{i=1}^{N} P(j|x_i) \ln P_j \right] + \frac{\partial}{\partial P_j} \left[\lambda P_j \right]$$

given that $P(j|x_i)$ is fixed-scalar in this step.

So:

$$\frac{\partial L}{\partial P_j} = \frac{1}{P_j} \sum_{i=1}^{N} P(j|x_i) + \lambda$$

We want $\frac{\partial L}{\partial P_j} = 0$: $\Rightarrow P_j = -\frac{1}{\lambda} \sum_{i=1}^{N} P(j|x_i)$ for $j \in [1, m]$

Also we need to solve under $\sum_{j=1}^{N} P_j = 1$:

So $\sum_{j=1}^{M} P_j = 1 \Rightarrow \sum_{j=1}^{M} \left(-\frac{1}{\lambda} \sum_{i=1}^{N} P(j|x_i) \right) = 1 \Rightarrow \sum_{j=1}^{M} \sum_{i=1}^{N} P(j|x_i) = -\lambda$

Substituting above: (Given that $\sum_{j=1}^{m} P_j = 1$ and $\sum_{j=1}^{m} P(j|x_i) = 1$)

$$P_j = \frac{1}{\sum_{i=1}^{N} \sum_{j=1}^{M} P(j|x_i)} \sum_{i=1}^{N} P(j|x_i) = \frac{1}{\sum_{i=1}^{N} 1} \sum_{i=1}^{N} P(j|x_i) = \frac{1}{N} \sum_{i=1}^{N} P(j|x_i)$$