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HomeWork 7
February 2020

Exersize 1

We know that due to equiprobable classes it holds that:

- $P(w_1) = P(w_2)$

Also we know that
- $P(x/w_1) = N(0,1) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$
- $P(x/w_2) = N(0,5) = \frac{1}{\sqrt{2\pi}\sqrt{5}} \exp\left(-x^2/10\right)$

In order to find the region we want:

Solving for R_1: $P(x/w_1)P(w_1) > P(x/w_2)P(w_2)$

$$P(x/w_1) > P(x/w_2)$$

$$\frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) > \frac{1}{\sqrt{2\pi}\sqrt{5}} \exp\left(-x^2/10\right)$$

$$-x^2/2 > -\frac{1}{2}\ln(5) - \frac{x^2}{10}$$

$$x^2 < \frac{5}{4}\ln(5) \Rightarrow \begin{cases} x > -\sqrt{\frac{5}{4}\ln(5)} \\ x < \sqrt{\frac{5}{4}\ln(5)} \end{cases}$$

Meaning that $R_1: \left\{x: -\sqrt{\frac{5}{4}\ln(5)} < x < +\sqrt{\frac{5}{4}\ln(5)}\right\}$

and $R_2: \left\{x: x < -\sqrt{\frac{5}{4}\ln(5)} \cup x > \sqrt{\frac{5}{4}\ln(5)}\right\}$

Exersize 2

We know that we can use border hypersurfaces to expres classifiers. $S: g(x)=0$

In the normally distributed case we have:

$$p(x|\omega_j) = N(\mu_j, \Sigma_j) = \frac{1}{\sqrt{2\pi}|\Sigma_j|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_j)^T \Sigma_j^{-1} (x-\mu_j)\right)$$

Instead of it we can take:

$$g_j(x) = \ln(P(x|\omega_j)P(\omega_j)) = -\frac{1}{2}x^T \Sigma^{-1} x + \mu^T \Sigma^{-1} x - \frac{1}{2}\mu^T \Sigma^{-1} \mu$$
$$+ \ln(P(\omega_j)) + c.$$

So for 2 classes:

$$g_{12}(x) = g_1(x) - g_2(x).$$

For $g_{12}(x)=0$ we have:

$$g_1(x) = g_2(x) \Rightarrow -\frac{1}{2}x^T \Sigma_1^{-1} x + \mu_1^T \Sigma_1^{-1} x - \frac{1}{2}\mu_1^T \Sigma_1^{-1} \mu_1 + c_1 + \ln(P(\omega_1))$$
$$=$$
$$-\frac{1}{2}x^T \Sigma_2^{-1} x + \mu_2^T \Sigma_2^{-1} x - \frac{1}{2}\mu_2^T \Sigma_2^{-1} \mu_2 + c_2 + \ln(P(\omega_2))$$

But if we have equiprobability:

- $c_1 = c_2$
- $\ln(P(\omega_1)) = \ln(P(\omega_2))$ so:

and because $\Sigma = \sigma^2 I = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$

Exersize 2 (cont)

$$\mu_1^T \Sigma^{-1} x - \frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 = \mu_2^T \Sigma^{-1} x - \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2$$

$$\left(\mu_1^T \Sigma^{-1} - \mu_2^T \Sigma^{-1}\right) x - \frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 = 0$$

- $$\left(\mu_1^T \begin{bmatrix} \frac{1}{\sigma^2} & \\ & \frac{1}{\sigma^2} \end{bmatrix} - \mu_2^T \begin{bmatrix} \frac{1}{\sigma^2} & \\ & \frac{1}{\sigma^2} \end{bmatrix}\right) x = (\mu_1 - \mu_2)^T \Sigma^{-1} x$$

- $$-\frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \frac{1}{2}\mu_2^T \Sigma^{-1} \mu_2 = -\frac{1}{2}\begin{bmatrix} \mu_{11} & \mu_{12} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \mu_{21} & \mu_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \begin{bmatrix} \mu_{21} \\ \mu_{22} \end{bmatrix}$$

$$\hookrightarrow = -\frac{1}{2}\begin{bmatrix} \frac{\mu_{11}}{\sigma^2} + 0 & \frac{\mu_{12}}{\sigma^2} + 0 \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{\mu_{21}}{\sigma^2} + 0 & \frac{\mu_{22}}{\sigma^2} + 0 \end{bmatrix} \begin{bmatrix} \mu_{21} \\ \mu_{22} \end{bmatrix}$$

$$= -\frac{1}{2}\left(\frac{\mu_{11}^2}{\sigma^2} + \frac{\mu_{12}^2}{\sigma^2}\right) + \frac{1}{2}\left(\frac{\mu_{21}^2}{\sigma^2} + \frac{\mu_{22}^2}{\sigma^2}\right)$$

$$= -\frac{1}{2}\frac{1}{\sigma^2}\left(\mu_{11}^2 + \mu_{12}^2\right) + \frac{1}{2}\frac{1}{\sigma^2}\left(\mu_{21}^2 + \mu_{22}^2\right)$$

$$= \frac{1}{\sigma^2} \cdot \left(-\frac{1}{2}\|\mu_1\|^2 + \frac{1}{2}\|\mu_2\|^2\right)$$

Combining it all together

$$(\mu_1 - \mu_2)^T \Sigma^{-1} x + \frac{1}{\sigma^2}\left(-\frac{1}{2}\|\mu_1\|^2 + \frac{1}{2}\|\mu_2\|^2\right) = 0$$

$$(\mu_1 - \mu_2)^T I x + \frac{\sigma^2}{\sigma^2}\left(-\frac{1}{2}\|\mu_1\|^2 + \frac{1}{2}\|\mu_2\|^2\right) = 0$$

$$(\mu_1 - \mu_2)^T x - \frac{1}{2}\|\mu_1\|^2 + \frac{1}{2}\|\mu_2\|^2 = 0$$

Exersize 3

Equiprobable Assumption

(i) The graphic depiction is in the HW7.ipynb file.

(ii) We have $P(\omega_1) = P(\omega_2) = P(\omega_3)$ as well as that the:

$$R_1 = \{0,2\} \cup \{5,8\}$$

$$R_2 = \{2,3\} \cup \{4,5\} \cup \{8,9\}$$

$$R_3 = \{3,4\}$$

Thus the error probability is:

$$P_e = \frac{1}{3}\left(\int_{R_1} p(x|\omega_2) + p(x|\omega_3)\,dx + \int_{R_2} p(x|\omega_1) + p(x|\omega_3)\,dx + \int_{R_3} p(x|\omega_1) + p(x|\omega_2)\,dx\right.$$

$$= \frac{1}{3}\left[\int_0^2 p(x|\omega_2) + p(x|\omega_3)\,dx + \int_5^8 p(x|\omega_2) + p(x|\omega_3)\,dx + \int_2^3 p(x|\omega_1) + p(x|\omega_3)\,dx\right.$$

$$+ \int_4^5 p(x|\omega_1) + p(x|\omega_3)\,dx + \int_8^9 p(x|\omega_1) + p(x|\omega_3)\,dx + \int_3^4 p(x|\omega_1) + p(x|\omega_2)\,dx$$

$$= \frac{1}{3}\left(\frac{1}{9}\cdot(2+3) + 0 + 0 + 0 + \frac{1}{9}\cdot 1\right) = \frac{2}{9} = 0,22.$$

(iii) Due to the fact that we have calculated the regions we can say that $x' = 3,5 \in R_3$ thus it would be classified as class 3

Exersize 3 (cont).

Assuming classes are not equiprobable

(i) We need to find a set of $(P(\omega_1), P(\omega_2), P(\omega_3))$ that $x_3' = 3,5$ would always be classified as class 2.

We need to recalculate the decision step of the Bayes Classifier:

It must hold that:

(1) $P(\omega_2 | x=3,5) > P(\omega_1 | x=3,5)$

and

(2) $P(\omega_2 | x=3,5) > P(\omega_3 | x=3,5)$

(1) $\Rightarrow$ $P(x=3,5|\omega_2) P(\omega_2) > P(x=3,5|\omega_1) P(\omega_1)$

We know however that $P(x=3,5|\omega_1) = 0$ from ex1.

So (1) $\Rightarrow$ $P(x=3,5|\omega_2) P(\omega_2) > 0 \Rightarrow P(\omega_2) > 0$ it always holds, for any $P(\omega_1)$.

(2) $\Rightarrow$ $P(x=3,5|\omega_2) P(\omega_2) > P(x=3,5|\omega_3) P(\omega_3) \Rightarrow$

$\Rightarrow$ $P(\omega_2) > 9P(\omega_3)$

So $P(\omega_1)$ is indifferent

and $P(\omega_2) > 9P(\omega_3)$

while $P(\omega_1) + P(\omega_2) + P(\omega_3) = 1$

e.g. $P(\omega_1) = 0,70$ $P(\omega_2) = 0,28$ $P(\omega_3) = 0,02$

Exersize 3 (cont)

(iii) Is there any combination of a priori probabilities so $x = 3,5$ would be classified as class 1;

In order for this to happen it should hold that: $P(w_1 | x=3,5) > P(w_2 | x=3,5)$

and

$P(w_1 | x=3,5) > P(w_3 | x=3,5)$

meaning that $P(w_1 | x=3,5) = P(x=3,5 | w_1) P(w_1)$

$= 0 \cdot P(w_1) = 0 > P(x=3,5 | w_2) P(w_2)$

This is impossible.