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本文档仅供学习交流，严禁用于商业用途.

1. (i) 解: $\forall A,B\in \mathfrak{g}$, $k\in\mathbb{K}$, 有: $tr(A+B)=tr(A)+tr(B)=0$, $A+B\in\mathfrak{g}$;
$tr(kA)=k\,tr(A)=0$, $kA\in\mathfrak{g}$.

$\therefore$ $\mathfrak{g}$ 是 $M_3(\mathbb{K})$ 线性子空间.

$\mathfrak{g}$ 的维数为 8, 基为 $X_1=E_{12}$, $X_2=E_{13}$, $X_3=E_{21}$, $X_4=E_{23}$, $X_5=E_{31}$, $X_6=E_{32}$, $X_7=E_{11}-E_{33}$, $X_8=E_{22}-E_{33}$:

对 $a_1,\cdots,a_8\in\mathbb{K}$, $\sum_{i=1}^{8}a_iX_i=\begin{pmatrix}a_7&a_1&a_2\\a_3&a_8&a_4\\a_5&a_6&-a_7-a_8\end{pmatrix}\in\mathfrak{g}$, 因此 $X_i\in\mathfrak{g}$.

且由 $0=\sum_{i=1}^{8}a_iX_i$ 可推出 $a_1=\cdots=a_8=0$. 因此 $X_1,\cdots,X_8$ 线性无关;

对 $\forall A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\in\mathfrak{g}$,

也有 $a_{12}X_1+a_{13}X_2+a_{21}X_3+a_{23}X_4+a_{31}X_5+a_{32}X_6+a_{11}X_7+a_{22}X_8=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&-a_{11}-a_{22}\end{pmatrix}=A$

(由 $tr A=0$ 知 $a_{33}=-a_{11}-a_{22}$)

$\therefore X_1,\cdots,X_8$ 是 $\mathfrak{g}$ 一组基.

(ii) 解: $\forall A,B\in\mathfrak{k}$, $k\in\mathbb{K}$, $(A+B)^T=A^T+B^T=-A-B=-(A+B)$.
$(kA)^T=kA^T=k(-A)=-kA$

$\therefore A+B\in\mathfrak{k}$, $kA\in\mathfrak{k}$ $\therefore\mathfrak{k}$ 是子空间.

基 $Y_1=E_{12}-E_{21}$, $Y_2=E_{13}-E_{31}$, $Y_3=E_{23}-E_{32}$

$\forall A,B\in\mathfrak{a}$, $k\in\mathbb{K}$, 设 $A=\begin{pmatrix}a_1&&\\&a_2&\\&&a_3\end{pmatrix}$, $B=\begin{pmatrix}b_1&&\\&b_2&\\&&b_3\end{pmatrix}$

则 $A+B=\begin{pmatrix}a_1+b_1&&\\&a_2+b_2&\\&&a_3+b_3\end{pmatrix}$, $kA=\begin{pmatrix}ka_1&&\\&ka_2&\\&&ka_3\end{pmatrix}$ 对角阵.

且 $tr(A+B)=0$, $tr(kA)=0$ $\therefore\mathfrak{a}$ 是子空间. 基为 $Z_1=E_{11}-E_{33}$ $Z_2=E_{22}-E_{33}$

$\forall A,B\in\mathfrak{n}$, $k\in\mathbb{K}$. $A=\begin{pmatrix}0&a_{12}&a_{13}\\&0&a_{23}\\&&0\end{pmatrix}$ $B=\begin{pmatrix}0&b_{12}&b_{13}\\&0&b_{23}\\&&0\end{pmatrix}$

$A+B=\begin{pmatrix}0&a_{12}+b_{12}&a_{13}+b_{13}\\&0&a_{23}+b_{23}\\&&0\end{pmatrix}\in\mathfrak{n}$, $kA=\begin{pmatrix}0&ka_{12}&ka_{13}\\&0&ka_{23}\\&&0\end{pmatrix}\in\mathfrak{n}$

$\therefore\mathfrak{n}$ 是子空间. 基为 $W_1=E_{12}$ $W_2=E_{13}$ $W_3=E_{23}$

(iii) 设 $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ $B=\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}$

则当 $A,B\in\mathfrak{g}$ 时, $tr(AB-BA)=tr(AB)-tr(BA)=\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}b_{ji}-\sum_{j=1}^{3}\sum_{i=1}^{3}b_{ji}a_{ij}=0$. $\therefore AB-BA\in\mathfrak{g}$

当 $A,B\in k$ 时，$(AB-BA)^T=(AB)^T-(BA)^T=B^TA^T-A^TB^T$

$=(-B)(-A)-(-A)(-B)=BA-AB=-(AB-BA)$

$\therefore AB-BA\in k$

当 $A,B\in a$ 时. $AB=\begin{pmatrix} a_{11}b_{11} & & \\ & a_{22}b_{22} & \\ & & a_{33}b_{33}\end{pmatrix}=BA$

$\therefore AB-BA=O_{3\times3}\in a$

当 $A,B\in n$ 时, $AB=\begin{pmatrix} 0 & 0 & a_{12}b_{23} \\ & 0 & 0 \\ & & 0\end{pmatrix}\in n$

同理 $BA\in n$ $\therefore AB-BA\in n$.

(iv) 证明：先证 $k+l+n$ 是直和.

对 $A\in k$, $B\in l$, $C\in n$. 若 $A+B+C=0$,

则由 B,C 均为上三角阵 知 $A=-B-C$ 也是上三角阵.

$\therefore A(i,j)=0$ $i>j$. 由 $A^T=-A$ 知: $A(j,i)=-A(i,j)$.

$\therefore$ 当 $i<j$ 时, $A(i,j)=-A(j,i)=0$; 当 $i=j$ 时 $A(i,i)=-A(i,i)$, $A(i,i)=0$.

$\therefore A=0$, $B+C=0$.

由 C 对角线上元素均为0 知: $B=-C$ 对角线上元素均为0.

由 B 对角阵: $B=0$. $\therefore C=0$

$\therefore k+l+n$ 是直和.

又 $\because \forall A\in g$ $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$

$=\begin{pmatrix} 0 & -a_{21} & -a_{31} \\ a_{21} & 0 & -a_{32} \\ a_{31} & a_{32} & 0\end{pmatrix}+\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33}\end{pmatrix}+$

$\begin{pmatrix} 0 & a_{12}+a_{21} & a_{13}+a_{31} \\ 0 & 0 & a_{23}+a_{32} \\ 0 & 0 & 0\end{pmatrix}\in k+a+n$

$\therefore g\subseteq k+a+n$

又 $\because k,n$ 中元素对角线均为0 $\therefore k,n\subseteq g$ $\therefore k+a+n\subseteq g$

$\therefore g=k\oplus a\oplus n$.

(v) 由(iii). f_A 是 g 到 g 映射.

又 $\because \forall X,\tilde{X}\in g$, $f_A(X+\tilde{X})=A(X+\tilde{X})-(X+\tilde{X})A$

$=AX-XA+A\tilde{X}-\tilde{X}A$

$=f_A(X)+f_A(\tilde{X})$,

$\forall X\in g$, $k\in K$, $f_A(kX)=A(kX)-(kX)A=kAX-kXA=k(AX-XA)$

$=kf_A(X)$

$\therefore f_A$ 是 g 上线性变换.

(vi) (ii)给出基为 $z_1 = \begin{pmatrix}1&&\\&0&\\&&-1\end{pmatrix}$, $z_2 = \begin{pmatrix}0&&\\&1&\\&&-1\end{pmatrix}$.

则当 $A = z_1$ 时 $f_A(E_{12}) = E_{12}$ $\quad f_A(E_{13}) = 2E_{13}$ $\quad f_A(E_{21}) = -E_{21}$

$f_A(E_{23}) = E_{23}$ $\quad f_A(E_{31}) = -2E_{31}$ $\quad f_A(E_{32}) = -E_{32}$

$f_A(E_{11} - E_{33}) = 0$ $\quad f_A(E_{22} - E_{33}) = 0$

f_{z_1} 在基 $X_1, \cdots, X_8$ 下矩阵 $\begin{pmatrix}1&&&&&&&\\&2&&&&&&\\&&-1&&&&&\\&&&1&&&&\\&&&&-2&&&\\&&&&&-1&&\\&&&&&&0&\\&&&&&&&0\end{pmatrix} = P_1$

当 $A = z_2$ 时 $f_A(E_{12}) = -E_{12}$ $\quad f_A(E_{13}) = E_{13}$ $\quad f_A(E_{21}) = E_{21}$

$f_A(E_{23}) = 2E_{23}$ $\quad f_A(E_{31}) = -E_{31}$ $\quad f_A(E_{32}) = -2E_{32}$

$f_A(E_{11} - E_{33}) = 0$ $\quad f_A(E_{22} - E_{33}) = 0$

f_{z_2} 在 $X_1, \cdots, X_8$ 下矩阵 $\begin{pmatrix}-1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&2&&&&\\&&&&-1&&&\\&&&&&-2&&\\&&&&&&0&\\&&&&&&&0\end{pmatrix} = P_2$.

(vii) 对 $\forall A \in \mathfrak{a}$ 设 $A = \begin{pmatrix}x&&\\&y&\\&&z\end{pmatrix}$ 则 $A = x z_1 + y z_2$

$\therefore f_A(X) = AX - XA = (xz_1 + yz_2)X - X(xz_1 + yz_2)$

$= x f_{z_1}(X) + y f_{z_2}(X)$ $\quad \therefore f_A = x f_{z_1} + y f_{z_2}$

$\therefore f_A$ 在基 $X_1, \cdots, X_8$ 下矩阵为 $xP_1 + yP_2$, 是对角阵.

$\therefore f_A$ 可对角化. (由 $i_1 i_2 i_3$ 为 $(1,2,3)$ 排列知 j_k 存在唯一)

(viii) 设 $j_k \in \{1,2,3\}$ 使 $i_{j_k} = k$. 则 $P_{i_1 i_2 i_3} = (e_{i_1}\ e_{i_2}\ e_{i_3}) = \begin{pmatrix}e_{j_1}'\\ e_{j_2}'\\ e_{j_3}'\end{pmatrix}$.

第k行仅 j_k 列为1, 其余列为0.

任取 $A \in \mathfrak{a}$. 设 $A = \begin{pmatrix}x&&\\&y&\\&&z\end{pmatrix}$

注意到 $\begin{pmatrix}e_{j_1}'\\ e_{j_2}'\\ e_{j_3}'\end{pmatrix}(e_{j_1}\ e_{j_2}\ e_{j_3}) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I$.

$\therefore P_{i_1 i_2 i_3}^{-1} = (e_{j_1}\ e_{j_2}\ e_{j_3})$

$\therefore B = P_{i_1 i_2 i_3}^{-1} A P_{i_1 i_2 i_3} = (e_{j_1}\ e_{j_2}\ e_{j_3})\begin{pmatrix}x&&\\&y&\\&&z\end{pmatrix}\begin{pmatrix}e_{j_1}'\\ e_{j_2}'\\ e_{j_3}'\end{pmatrix}$

$= xE_{j_1 j_1} + yE_{j_2 j_2} + zE_{j_3 j_3}$.

$\therefore B$ 是对角阵, 且 $\operatorname{tr}(B) = x + y + z = \operatorname{tr}(A) = 0$.

$\therefore B \in \mathfrak{a}$.

由于我熟悉 $P_{i_1 i_2 i_3}^{-1}$ 是什么样子, 因此把它从列表示改为行表示是我的第一想法.
但直接求B更简单:
设 $A = \begin{pmatrix}x_1&&\\&x_2&\\&&x_3\end{pmatrix}$.

则 $AP_{i_1 i_2 i_3}$
$= (Ae_{i_1}, Ae_{i_2}, Ae_{i_3})$
$= (x_{i_1}e_{i_1}, x_{i_2}e_{i_2}, x_{i_3}e_{i_3})$
$= P_{i_1 i_2 i_3}\begin{pmatrix}x_{i_1}&&\\&x_{i_2}&\\&&x_{i_3}\end{pmatrix}$

$B = \begin{pmatrix}x_{i_1}&&\\&x_{i_2}&\\&&x_{i_3}\end{pmatrix}$

后面与此相同.

2. 证明：若 A, B 为 V 中线性变换 f 在基 $(\alpha_1, \cdots, \alpha_n)$ 下和 $(\beta_1, \cdots, \beta_n)$ 下矩阵，

且有 $(\beta_1, \cdots, \beta_n) = (\alpha_1, \cdots, \alpha_n) P$，$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \in K^{n\times n}$ 可逆，

则 $f(\beta_i) = f\left(\sum_{j=1}^n \alpha_j p_{ji}\right) = \sum_{j=1}^n p_{ji} f(\alpha_j)$

$$= \sum_{j=1}^n p_{ji}\left(\sum_{k=1}^n A(k,j)\alpha_k\right)$$

$$= \sum_{k=1}^n \left(\sum_{j=1}^n A(k,j) p_{ji}\right)\alpha_k$$

另一方面 $f(\beta_i) = \sum_{j=1}^n B(j,i)\beta_j$

$$= \sum_{j=1}^n B(j,i)\left(\sum_{k=1}^n p_{kj}\alpha_k\right)$$

$$= \sum_{k=1}^n \left(\sum_{j=1}^n p_{kj} B(j,i)\right)\alpha_k.$$

$\therefore \sum_{j=1}^n A(k,j) p_{ji} = \sum_{j=1}^n p_{kj} B(j,i)$，$k = 1, \cdots, n$，

即 $AP = PB$，$B = P^{-1}AP$，A 与 B 相似.

若 A 与 B 相似. 取 $V = K^n$，$f: V \to V$，$\eta \mapsto A\eta$

则 f 是线性变换：$f(\eta_1 + \eta_2) = A(\eta_1 + \eta_2) = f(\eta_1 + \eta_2)$，$\eta_1, \eta_2 \in V$

$f(k\eta) = A(k\eta) = kA\eta = kf(\eta)$，$\eta \in V$，$k \in K$.

又 $\because f(\varepsilon_i) = A\varepsilon_i$.

$\therefore (f(\varepsilon_1), \cdots, f(\varepsilon_n)) = (A\varepsilon_1, \cdots, A\varepsilon_n) = A = (\varepsilon_1, \cdots, \varepsilon_n) A$.

构造矩阵为 A 的线性映射的最基本办法. $\longrightarrow$ $\therefore f$ 在基 $(\varepsilon_1, \cdots, \varepsilon_n)$ 下矩阵为 A.

设 $B = P^{-1}AP$，$P \in K^{n\times n}$ 可逆. 设 $P = (\eta_1, \cdots, \eta_n)$.

则由 $\operatorname{rank} P = n$ 知 $\eta_1, \cdots, \eta_n$ 是 K^n 的基.

$$(f(\eta_1), \cdots, f(\eta_n)) = (A\eta_1, \cdots, A\eta_n) = A(\eta_1, \cdots, \eta_n)$$

$$= AP = (\eta_1, \cdots, \eta_n) P^{-1}AP$$

$$= (\eta_1, \cdots, \eta_n) B.$$

$\therefore f$ 在基 $(\eta_1, \cdots, \eta_n)$ 下矩阵为 B.

由此即得 f 在 V 两组基下矩阵分别为 A 和 B.

3. 证明：由 $(A^TA)^T = A^T(A^T)^T = A^TA$ 知 A^TA 实对称.

又 $\because \forall \alpha \in \mathbb{R}^n$，$\alpha^T(A^TA)\alpha = (A\alpha)^T A\alpha \geq 0$

$\therefore A^TA$ 是半正定矩阵.

又 $\because$ 当 $A^TA\alpha = 0$ 时，$0 = \alpha^T(A^TA)\alpha = (A\alpha)^T A\alpha$. $\therefore A\alpha = 0$.

且当 $A\alpha = 0$ 时 $A^TA\alpha = A^T 0 = 0$. $\therefore A^TA\alpha = 0$ 与 $A\alpha = 0$ 同解.

$\therefore \operatorname{rank}(A^TA) = r(A) = r$.

设A^TA有两两不同特征值 $\mu_1, \cdots, \mu_l, \mu_0=0$,
每个特征值对应的特征子空间V_i有标准正交基 $\alpha_{i1}, \cdots, \alpha_{is_i}$
由 A^TA 可对角化知 $n=\sum_{i=0}^{l} \dim V_i = \sum_{i=0}^{l} s_i$.
又$\because$ $\alpha_{i1}, \cdots, \alpha_{is_i}$ 两两正交; 对$0 \le i_1, i_2 \le l$, $1 \le j_1 \le s_{i_1}$, $1 \le j_2 \le s_{i_2}$
有当 $i_1 = i_2$ 时 $\alpha_{i_1j_1}$ 与 $\alpha_{i_1j_2}$ 正交;
当 $j_1 \ne j_2$ 时, $\alpha_{i_1j_1}^T A^TA \alpha_{i_2j_2} = \alpha_{i_1j_1}^T(\mu_{i_2}\alpha_{i_2j_2})$,
$\alpha_{i_1j_1}^T A^TA \alpha_{i_2j_2} = (A^TA\alpha_{i_1j_1})^T \alpha_{i_2j_2} = (\mu_{i_1}\alpha_{i_1j_1})^T \alpha_{i_2j_2}$.
$\therefore (\mu_{i_2}-\mu_{i_1}) \alpha_{i_1j_1}^T \alpha_{i_2j_2} = 0$. $\alpha_{i_1j_1}^T \alpha_{i_2j_2} = 0$.
$\therefore \alpha_{i_1j_1}$ 与 $\alpha_{i_2j_2}$ 正交.
$\therefore$ 取$(\alpha_1, \cdots, \alpha_n) = (\alpha_{11}, \cdots, \alpha_{1s_1}, \cdots, \alpha_{l1}, \cdots, \alpha_{ls_l}, \alpha_{01}, \cdots, \alpha_{0s_0})$
则有 $\alpha_1, \cdots, \alpha_n$ 两两正交
且 $A^TA\alpha_i = \mu_{j_i}\alpha_i$, 其中$j_i$使得$s_1+\cdots+s_{j_i-1} < i \le s_1+\cdots+s_{j_i}$
又$\because$ $s_1+\cdots+s_l = n - s_0 = n - \dim V_0 = r$.
$\therefore \mu_{j_i} = 0$, $i>r$.
记 $\mu_{j_i} = \lambda_i$, 则 $\alpha_1, \cdots, \alpha_n$ 满足(i).

这里事实上重复了教材上实对称阵正交对角化的证明.
设 A^TA 的 n 个特征值 $\lambda_1 \ge \cdots \ge \lambda_n$.
由 A^TA 半正定: $\lambda_n \ge 0$.
由 $\operatorname{rank}(A^TA) = r$ 知 $\lambda_{r+1} = \cdots = \lambda_n = 0$.
取正交阵 $P = (\alpha_1, \cdots, \alpha_n)$
使 $P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.
则有 $\alpha_1, \cdots, \alpha_n$ 是$\mathbb{R}^n$的标准正交基,
$A^TA\alpha_i = \lambda_i\alpha_i$,
且对 $r<j$ 有 $\lambda_j = 0$;
即满足(i).

又$\because$ $(A\alpha_{i_1j_1})^T A\alpha_{i_2j_2} = \alpha_{i_1j_1}^T A^TA\alpha_{i_2j_2} = \alpha_{i_1j_1}^T \mu_{j_2}\alpha_{i_2j_2}$
$= \mu_{j_2}\alpha_{i_1j_1}^T\alpha_{i_2j_2} = 0$,
$0 \le i_1, i_2 \le l$, $1 \le j_1 \le s_{i_1}$, $1 \le j_2 \le s_{i_2}$.
$\therefore$ $A\alpha_{i_1j_1}$ 两两正交. 即 $A\alpha_1, \cdots, A\alpha_r$ 两两正交.
$\therefore \alpha_1, \cdots, \alpha_n$ 满足 (i) (ii), 即为所求.

4. 证明: $\dim(V_1 \cap V_2) + \dim((V_1+V_2)\cap V_3)$
$= \dim(V_1 \cap V_2) - \dim(V_1+V_2+V_3) + \dim(V_1+V_2) + \dim(V_3)$
$= \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1+V_2+V_3)$
$\dim(V_2 \cap V_3) + \dim((V_2+V_3)\cap V_1)$
$= \dim(V_2) + \dim(V_3) - \dim(V_2+V_3) + \dim(V_2+V_3) + \dim(V_1) - \dim(V_1+V_2+V_3)$
$= \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1+V_2+V_3)$
$\therefore$ 左=右, 原等式成立.

5. 解: $(v = \begin{pmatrix} a \\ b \\ c \end{pmatrix})$ $f(v) = v^T \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix} v$.

$\because f(v) - 2v^Tv = v^T\left(\begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix} - 2I\right)v = v^T\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}v.$

而 $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ 有特征值 0, 2(二重), 因此A半正定.

$\therefore f(v) - 2v^Tv \geq 0, \forall v \in \mathbb{R}^3$. ∴当v非0时 $\frac{f(v)}{v^Tv}$ 总大于等于2.

又∵A有特征值0的特征向量 $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = v_1$. $\therefore f(v_1) - 2v_1^Tv_1 = 0$,

$\frac{f(v_1)}{v_1^Tv_1} = 2$, 即 $\frac{f(v)}{v^Tv}$ 可取到2.

$\therefore \frac{f(v)}{v^Tv}$ 的最小值为2.