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- Consider the following $N \times n$ spiked model:

$$\begin{aligned}\tilde{\mathbf{X}}_N &= \left(\mathbf{I}_N + \theta \vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)^{1/2} \mathbf{X}_N \quad \text{with} \quad \|\vec{\mathbf{u}}\| = 1 \ , \\ &= \mathbf{\Pi}^{1/2} \mathbf{X}_N\end{aligned}$$

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- if $\boxed{\theta > \sqrt{c}}$ then $\lambda_{\max} \to \sigma^2(1+\theta)(1+c/\theta)$, i.e. $\lambda_{\max}$ **separates from the bulk.**

Spiked model eigenvectors II

Preliminary observations

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Theorem

Let $\vec{\boldsymbol{a}}_N$ be a deterministic vector with norm 1, then

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Proof I

Reminder from complex analysis

- We need a simple result from complex analysis:

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}^-}\frac{dz}{z} = 1$$

if $\mathcal{C}^-$ is a contour (take a circle of radius 1) enclosing counterclockwise 0.

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- If $\mathcal{C}^+$ does not enclose λ, then the integral **equals zero**.

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By the spectral theorem,

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and consider a contour $\mathcal{C}^+$ **exclusively enclosing** the eigenvalue $\lambda_{\max}$.

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&= \vec{a}_N^* \vec{v}_{\max} \vec{v}_{\max}^* \vec{a}_N \; .
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Proof IV

Recall

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N \, dz$$

and temporarily forget about the integral.

Proof IV

Recall

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N \, dz$$

and temporarily forget about the integral.

- Our objective now is:

to find a new formulation of $\vec{\boldsymbol{a}}_N^* \tilde{Q}_n(z) \vec{\boldsymbol{a}}_N$ and clearly separate the contribution from the perturbation ($\vec{\mathbf{u}}$ and θ) and the resolvent $\mathbf{Q}_n(z)$ from the non-pertubated model.

Proof IV

Recall

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N \, dz$$

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- Our objective now is:

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Introduce the notations

$$\mathbf{Z}_N = \frac{1}{n} \mathbf{X}_N \mathbf{X}_N^* \quad \text{and} \quad \tilde{\mathbf{Z}}_N = \frac{1}{n} \tilde{\mathbf{X}}_N \tilde{\mathbf{X}}_N^*$$

and recall the formula for the inverse of a rank-one perturbation:

$$(\mathbf{A} + \vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{A}^{-1}}{1 + \vec{\mathbf{u}} \mathbf{A} \vec{\mathbf{u}}^*},$$

In particular

$$\mathbf{\Pi}^{-1} = (\mathbf{I}_N + \theta \vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1} = \mathbf{I}_N - \frac{\theta}{1+\theta} \vec{\mathbf{u}}\vec{\mathbf{u}}^*$$

Proof V

$$\tilde{\mathbf{Q}}_n(z) \quad = \quad \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_N\mathbf{\Pi}^{1/2} - z\mathbf{I}_N\right)^{-1}$$

Proof V

$$
\begin{aligned}
\tilde{\mathbf{Q}}_n(z) &= \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_N\mathbf{\Pi}^{1/2} - z\mathbf{I}_N\right)^{-1} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2}
\end{aligned}
$$

Proof V

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&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z(\mathbf{I}_N + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2}
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&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z(\mathbf{I}_N + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z\left(\mathbf{I}_N - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2}
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\tilde{\mathbf{Q}}_n(z) &= \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_N\mathbf{\Pi}^{1/2} - z\mathbf{I}_N\right)^{-1} \\
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&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z(\mathbf{I}_N + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z\left(\mathbf{I}_N - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\
&= \mathbf{\Pi}^{-1/2}(\mathbf{Z}_N - z\mathbf{I}_N + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta}
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&= \boldsymbol{\Pi}^{-1/2}\left(\mathbf{Q}_n - \frac{\mathbf{Q}_n\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}\right)\boldsymbol{\Pi}^{-1/2}
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&= \boldsymbol{\Pi}^{-1/2}(\mathbf{Z}_N - z\mathbf{I}_N + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\boldsymbol{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
&= \boldsymbol{\Pi}^{-1/2}\left(\mathbf{Q}_n - \frac{\mathbf{Q}_n\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}\right)\boldsymbol{\Pi}^{-1/2}
\end{aligned}
$$

Hence

$$
\vec{\boldsymbol{a}}_N^*\tilde{\mathbf{Q}}_n(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^*\boldsymbol{\Pi}^{1/2}\mathbf{Q}_n(z)\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi\frac{\vec{\boldsymbol{a}}_N^*\boldsymbol{\Pi}^{1/2}\mathbf{Q}_n\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}
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Proof V

$$\begin{aligned}
\tilde{\mathbf{Q}}_n(z) &= \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_N\mathbf{\Pi}^{1/2} - z\mathbf{I}_N\right)^{-1} \\
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&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Q}_n - \frac{\mathbf{Q}_n\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}\right)\mathbf{\Pi}^{-1/2}
\end{aligned}$$

Hence

$$\vec{\boldsymbol{a}}_N^*\tilde{\mathbf{Q}}_n(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}_n(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi\frac{\vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}_n\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}$$

Not so ugly!

Proof V

$$\begin{aligned}
\tilde{\mathbf{Q}}_n(z) &= \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_N\mathbf{\Pi}^{1/2} - z\mathbf{I}_N\right)^{-1} \\
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&= \mathbf{\Pi}^{-1/2}(\mathbf{Z}_N - z\mathbf{I}_N + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
&= \mathbf{\Pi}^{-1/2}\left(\mathbf{Q}_n - \frac{\mathbf{Q}_n\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}\right)\mathbf{\Pi}^{-1/2}
\end{aligned}$$

Hence

$$\vec{\boldsymbol{a}}_N^*\tilde{\mathbf{Q}}_n(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}_n(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi\frac{\vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}_n\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{Q}_n\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}$$

Not so ugly! And we have separated the contribution of the perturbation from the non-perturbated model.

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \mathbf{Q}_n(z) \boldsymbol{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \boldsymbol{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2}\mathbf{Q}_n(z)\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2}\mathbf{Q}_n \vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{Q}_n \boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2}\mathbf{Q}_n(z)\boldsymbol{\Pi}^{1/2}\vec{\boldsymbol{a}}_N = ??$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N = \mathbf{0}$$

Why?

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}_n(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}_n \vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi\vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}_n(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N = 0$$

Why? Because

1. the contour only encloses $\lambda_{\max}(\tilde{\mathbf{Z}}_n)$ which is away from the bulk,
2. but all the eigenvalues of $\mathbf{Z}_n$ are **in the bulk**. Hence:

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \frac{1}{\lambda_i(\mathbf{Z}_n)-z}\,dz = 0.$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N = 0$$

Why? Because

1. the contour only encloses $\lambda_{\max}(\tilde{\mathbf{Z}}_n)$ which is away from the bulk,
2. but all the eigenvalues of $\mathbf{Z}_n$ are **in the bulk**. Hence:

$$\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \frac{1}{\lambda_i(\mathbf{Z}_n) - z} \, dz = 0.$$

Last step is to simplify the remaining expression by systematically use the **isotropic $\check{\mathrm{M}}$P law**:

$$\boxed{\vec{c}^* \mathbf{Q}_n(z) \vec{d} - \vec{c}^* \vec{d} \, \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) \xrightarrow[N,n\to\infty]{a.s.} 0}$$

Proof VI

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z)\vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}_n(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}_n \vec{\mathbf{u}}\vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N}{1+\xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2\boldsymbol{i}\pi}\oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2}\mathbf{Q}_n(z)\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N = 0$$

Why? Because

1. the contour only encloses $\lambda_{\max}(\tilde{\mathbf{Z}}_n)$ which is away from the bulk,
2. but all the eigenvalues of $\mathbf{Z}_n$ are **in the bulk**. Hence:

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Last step is to simplify the remaining expression by systematically use the **Isotropic M̌P law**:

$$\boxed{\vec{c}^*\mathbf{Q}_n(z)\vec{d} - \vec{c}^*\vec{d}\,\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) \xrightarrow[N,n\to\infty]{a.s.} 0} \Rightarrow \left\{ \begin{array}{lcl} \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\mathbf{Q}_n\vec{\mathbf{u}} & \approx & \vec{\boldsymbol{a}}_N^*\mathbf{\Pi}^{1/2}\vec{\mathbf{u}}\,\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) \\ \vec{\mathbf{u}}^*\mathbf{Q}_n\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N & \approx & \vec{\mathbf{u}}^*\mathbf{\Pi}^{1/2}\vec{\boldsymbol{a}}_N\,\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) \\ \vec{\mathbf{u}}^*\mathbf{Q}_n\vec{\mathbf{u}} & \approx & \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z) \end{array}\right.$$

Proof VII

After simplifications,

$$
\begin{aligned}
\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz \\
&= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz
\end{aligned}
$$

Proof VII

After simplifications,

$$\begin{aligned}\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz \\ &= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz\end{aligned}$$

It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz$$

by residue calculus (not that difficult).

Proof VII

After simplifications,

$$\begin{aligned}
\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz \\
&= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz
\end{aligned}$$

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$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz$$

by residue calculus (not that difficult).

A **minor miracle** occurs:

Proof VII

After simplifications,

$$\begin{aligned}
\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz \\
&= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz
\end{aligned}$$

It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz$$

by residue calculus (not that difficult).

A **minor miracle** occurs: **This factor admits a closed form formula!**

Proof VII

After simplifications,

$$\begin{aligned}
\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \boldsymbol{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz \\
&= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz
\end{aligned}$$

It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz$$

by residue calculus (not that difficult).

A **minor miracle** occurs: **This factor admits a closed form formula!**

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz = \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1}$$

Proof VII

After simplifications,

$$\begin{aligned}
\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N &\approx -\frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} |\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \vec{\mathbf{u}}|^2 \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz \\
&= -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz
\end{aligned}$$

It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz$$

by residue calculus (not that difficult).

A **minor miracle** occurs: **This factor admits a closed form formula!**

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{MP}}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{MP}}}(z)} \, dz = \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1}$$

Finally:

$$\boxed{\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[N,n\to\infty]{a.s.} 0 \, .}$$

Summary

1. Expression of $\vec{\boldsymbol{v}}_{\max}$ with the help of the resolvent

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2\boldsymbol{i}\pi} \oint_{\mathcal{C}+} \vec{\boldsymbol{a}}_N^* \tilde{Q}_n(z) \vec{\boldsymbol{a}}_N \, dz$$

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2. Convenient expression of $\vec{\boldsymbol{v}}_{\max}$ where the contribution of the perturbation is separated from the resolvent of the non-perturbated model (M̌P)

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N \quad \approx \quad -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}+} \frac{\mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\check{\mathrm{M}}\mathrm{P}}(z)} \, dz$$

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3. Residue calculus to find the final form

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[N,n\to\infty]{a.s.} 0 \; .$$