Introduction

Marčenko-Pastur's theorem

Large covariance matrices

Spiked models

Statistical Test for Single-Source Detection

Direction of Arrival Estimation

Position of the problem MUSIC Algorithm

Spiked model eigenvectors

Summary

Applications to the MIMO channel

Conclusion

 \blacktriangleright Consider the following $N\times n$ spiked model:

$$\tilde{\mathbf{X}}_N = (\mathbf{I}_N + \theta \vec{\mathbf{u}} \vec{\mathbf{u}}^*)^{1/2} \mathbf{X}_N \text{ with } ||\vec{\mathbf{u}}|| = 1,$$

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- if $\theta > \sqrt{c}$ then $\lambda_{\max} \to \sigma^2(1+\theta)(1+c/\theta)$, i.e. λ_{\max} separates from the bulk.

Spiked model eigenvectors II Preliminary observations

Preliminary observations

1. Let N finite, $n \to \infty$, then

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Theorem

Let $ec{a}_N$ be a deterministic vector with norm 1, then

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[N,n \to \infty]{a.s.} 0$$
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▶ Of course $\kappa(c) \to 1$ if $c \to 0$.

Reminder from complex analysis

▶ We need a simple result from complex analysis:

$$\frac{1}{2i\pi} \oint_{\mathcal{C}^-} \frac{dz}{z} = 1$$

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▶ If C^+ does not enclose λ , then the integral equals zero.

Proof II Our objective

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To express \vec{v}_{\max} with the help of the **resolvent** $\tilde{\mathbf{Q}}_n(z) = \left(\frac{1}{n}\tilde{\mathbf{X}}_N\tilde{\mathbf{X}}_N^* - z\mathbf{I}_N\right)^{-1}$

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By the spectral theorem,

$$\frac{1}{n}\tilde{\mathbf{X}}_{N}\tilde{\mathbf{X}}_{N}^{*} = \mathbf{O}_{N} \begin{pmatrix} \lambda_{\max} \\ & \ddots \\ & & \lambda_{N} \end{pmatrix} \mathbf{O}_{N}^{*}$$

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Recall that

• if $\theta > \sqrt{c}$, $\lambda_{\rm max}$ separates from the bulk and consider a contour \mathcal{C}^+ exclusively enclosing the eigenvalue $\lambda_{\rm max}$.

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We have

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N = \frac{1}{2i\pi} \oint_{\mathcal{C}^+} \vec{\boldsymbol{a}}_N^* \tilde{Q}_n(z) \vec{\boldsymbol{a}}_N \, dz$$

$$\begin{split} &\frac{1}{2i\pi} \oint_{\mathcal{C}^{+}} \vec{a}_{N}^{*} \tilde{Q}_{n}(z) \vec{a}_{N} \, dz \\ &= \frac{1}{2i\pi} \oint_{\mathcal{C}^{+}} \vec{a}_{N}^{*} [\vec{v}_{\text{max}} \, \boldsymbol{O}_{N-1}] \begin{pmatrix} \frac{1}{\lambda_{\text{max}} - z} & \\ & \ddots & \\ & \frac{1}{\lambda_{N} - z} \end{pmatrix} \begin{bmatrix} \vec{v}_{\text{max}}^{*} & \\ \boldsymbol{O}_{N-1}^{*} \end{bmatrix} \vec{a}_{N} \, dz \\ &= \vec{a}_{N}^{*} [\vec{v}_{\text{max}} \, \boldsymbol{O}_{N-1}] \begin{pmatrix} \frac{1}{2i\pi} \oint \frac{1}{\lambda_{\text{max}} - z} \, dz & \\ & \ddots & \\ & \frac{1}{2i\pi} \oint \frac{1}{\lambda_{N} - z} \, dz \end{pmatrix} \begin{bmatrix} \vec{v}_{\text{max}}^{*} \\ \boldsymbol{O}_{N-1}^{*} \end{bmatrix} \vec{a}_{N} \\ &= \vec{a}_{N}^{*} [\vec{v}_{\text{max}} \, \boldsymbol{O}_{N-1}] \begin{pmatrix} 1 & \\ & \ddots & \\ & & \\ \end{pmatrix} \begin{bmatrix} \vec{v}_{\text{max}}^{*} \\ \boldsymbol{O}_{N-1}^{*} \end{bmatrix} \vec{a}_{N} \end{split}$$

We have

$$ec{m{a}}_N^* ec{m{v}}_{ ext{max}} ec{m{c}}_{ ext{max}}^* ec{m{a}}_N = rac{1}{2 m{i} \pi} \oint_{\mathcal{C}^+} ec{m{a}}_N^* ilde{Q}_n(z) ec{m{a}}_N \, dz$$

Indeed.

 $= \ ec{m{a}}_N^* ec{m{v}}_{ ext{max}} ec{m{v}}_{ ext{max}}^* ec{m{a}}_N \ .$

$$\begin{split} &\frac{1}{2i\pi}\oint_{\mathcal{C}^+}\vec{a}_N^*\tilde{Q}_n(z)\vec{a}_N\,dz\\ &= &\frac{1}{2i\pi}\oint_{\mathcal{C}^+}\vec{a}_N^*[\vec{v}_{\max}\;\boldsymbol{O}_{N-1}]\left(\begin{array}{ccc} \frac{1}{\lambda_{\max}-z} & & \\ & \ddots & \\ & & \frac{1}{\lambda_N-z} \end{array}\right)\left[\begin{array}{c} \vec{v}_{\max}^* \\ \boldsymbol{O}_{N-1}^* \end{array}\right]\vec{a}_N\,dz\\ &= &\vec{a}_N^*[\vec{v}_{\max}\;\boldsymbol{O}_{N-1}]\left(\begin{array}{ccc} \frac{1}{2i\pi}\oint\frac{1}{\lambda_{\max}-z}\,dz & & \\ & \ddots & \\ & & \frac{1}{2i\pi}\oint\frac{1}{\lambda_N-z}\,dz \end{array}\right)\left[\begin{array}{c} \vec{v}_{\max}^* \\ \boldsymbol{O}_{N-1}^* \end{array}\right]\vec{a}_N\\ &= &\vec{a}_N^*[\vec{v}_{\max}\;\boldsymbol{O}_{N-1}]\left(\begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 0 \end{array}\right)\left[\begin{array}{c} \vec{v}_{\max}^* \\ \boldsymbol{O}_{N-1}^* \end{array}\right]\vec{a}_N \end{split}$$

Proof IV

Recall

$$ec{m{a}}_N^* ec{m{v}}_{ ext{max}} ec{m{v}}_{ ext{max}}^* ec{m{a}}_N = rac{1}{2 m{i} \pi} \oint_{\mathcal{C}^+} ec{m{a}}_N^* ilde{\mathbf{Q}}_n(z) ec{m{a}}_N \, dz$$

and temporarily forget about the integral.

Proof IV

Recall

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and temporarily forget about the integral.

Our objective now is:

to find a new formulation of $\vec{a}_N^* \tilde{Q}_n(z) \vec{a}_N$ and clearly separate the contribution from the perturbation (\vec{u} and θ) and the resolvent $\mathbf{Q}_n(z)$ from the non-pertubated model.

Proof IV

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Our objective now is:

to find a new formulation of $\vec{a}_N^* \tilde{Q}_n(z) \vec{a}_N$ and clearly separate the contribution from the perturbation (\vec{u} and θ) and the resolvent $\mathbf{Q}_n(z)$ from the non-pertubated model.

Introduce the notations

$$\mathbf{Z}_N = \frac{1}{n} \mathbf{X}_N \mathbf{X}_N^*$$
 and $\tilde{\mathbf{Z}}_N = \frac{1}{n} \tilde{\mathbf{X}}_N \tilde{\mathbf{X}}_N^*$

and recall the formula for the inverse of a rank-one perturbation:

$$(\mathbf{A} + \vec{\mathbf{u}}\vec{\mathbf{u}}^*)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\vec{\mathbf{u}}\vec{\mathbf{u}}^*\mathbf{A}^{-1}}{1 + \vec{\mathbf{u}}\mathbf{A}\vec{\mathbf{u}}^*},$$

In particular

$$\mathbf{\Pi}^{-1} = (\mathbf{I}_N + \theta \vec{\mathbf{u}} \vec{\mathbf{u}}^*)^{-1} = \mathbf{I}_N - \frac{\theta}{1+\theta} \vec{\mathbf{u}} \vec{\mathbf{u}}^*$$

$$\tilde{\mathbf{Q}}_n(z) = \left(\mathbf{\Pi}^{1/2} \tilde{\mathbf{Z}}_N \mathbf{\Pi}^{1/2} - z \mathbf{I}_N\right)^{-1}$$

$$\tilde{\mathbf{Q}}_n(z) = \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_N\mathbf{\Pi}^{1/2} - z\mathbf{I}_N\right)^{-1}$$
$$= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_N - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2}$$

$$\tilde{\mathbf{Q}}_{n}(z) = \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_{N}\mathbf{\Pi}^{1/2} - z\mathbf{I}_{N}\right)^{-1}$$

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$$= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_{N} - z(\mathbf{I}_{N} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^{*})^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2}$$

$$\begin{split} \tilde{\mathbf{Q}}_{n}(z) &= \left(\Pi^{1/2}\tilde{\mathbf{Z}}_{N}\Pi^{1/2} - z\mathbf{I}_{N}\right)^{-1} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\Pi^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z(\mathbf{I}_{N} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^{*})^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\left(\mathbf{I}_{N} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)\right)^{-1}\Pi^{-1/2} \end{split}$$

$$\begin{split} \tilde{\mathbf{Q}}_{n}(z) &= \left(\mathbf{\Pi}^{1/2}\tilde{\mathbf{Z}}_{N}\mathbf{\Pi}^{1/2} - z\mathbf{I}_{N}\right)^{-1} \\ &= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_{N} - z\mathbf{\Pi}^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\ &= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_{N} - z(\mathbf{I}_{N} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^{*})^{-1}\right)^{-1}\mathbf{\Pi}^{-1/2} \\ &= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_{N} - z\left(\mathbf{I}_{N} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)\right)^{-1}\mathbf{\Pi}^{-1/2} \\ &= \mathbf{\Pi}^{-1/2}\left(\mathbf{Z}_{N} - z\mathbf{I}_{N} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)^{-1}\mathbf{\Pi}^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \end{split}$$

$$\begin{split} \tilde{\mathbf{Q}}_{n}(z) &= \left(\Pi^{1/2}\tilde{\mathbf{Z}}_{N}\Pi^{1/2} - z\mathbf{I}_{N}\right)^{-1} \\ &= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z\Pi^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z(\mathbf{I}_{N} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^{*})^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z\left(\mathbf{I}_{N} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z\mathbf{I}_{N} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)^{-1}\Pi^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\ &= \Pi^{-1/2} \left(\mathbf{Q}_{n} - \frac{\mathbf{Q}_{n}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\mathbf{Q}_{n}}{1+\xi\vec{\mathbf{u}}^{*}\mathbf{Q}_{n}\vec{\mathbf{u}}}\right)\Pi^{-1/2} \end{split}$$

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= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z\Pi^{-1}\right)^{-1}\Pi^{-1/2} \\
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= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z\left(\mathbf{I}_{N} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)\right)^{-1}\Pi^{-1/2} \\
= \Pi^{-1/2} \left(\mathbf{Z}_{N} - z\mathbf{I}_{N} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)^{-1}\Pi^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\
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Hence

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

$$\begin{split} \tilde{\mathbf{Q}}_{n}(z) &= \left(\Pi^{1/2}\tilde{\mathbf{Z}}_{N}\Pi^{1/2} - z\mathbf{I}_{N}\right)^{-1} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\Pi^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z(\mathbf{I}_{N} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^{*})^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\left(\mathbf{I}_{N} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\mathbf{I}_{N} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)^{-1}\Pi^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\ &= \Pi^{-1/2}\left(\mathbf{Q}_{n} - \frac{\mathbf{Q}_{n}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\mathbf{Q}_{n}}{1+\xi\vec{\mathbf{u}}^{*}\mathbf{Q}_{n}\vec{\mathbf{u}}}\right)\Pi^{-1/2} \end{split}$$

Hence

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

Not so ugly!

$$\begin{split} \tilde{\mathbf{Q}}_{n}(z) &= \left(\Pi^{1/2}\tilde{\mathbf{Z}}_{N}\Pi^{1/2} - z\mathbf{I}_{N}\right)^{-1} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\Pi^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z(\mathbf{I}_{N} + \theta\vec{\mathbf{u}}\vec{\mathbf{u}}^{*})^{-1}\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\left(\mathbf{I}_{N} - \frac{\theta}{1+\theta}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)\right)^{-1}\Pi^{-1/2} \\ &= \Pi^{-1/2}\left(\mathbf{Z}_{N} - z\mathbf{I}_{N} + \xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\right)^{-1}\Pi^{-1/2} \quad \text{where } \xi = z\frac{\theta}{1+\theta} \\ &= \Pi^{-1/2}\left(\mathbf{Q}_{n} - \frac{\mathbf{Q}_{n}\xi\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\mathbf{Q}_{n}}{1+\xi\vec{\mathbf{u}}^{*}\mathbf{Q}_{n}\vec{\mathbf{u}}}\right)\Pi^{-1/2} \end{split}$$

Hence

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

Not so ugly! And we have separated the contribution of the perturbation from the non-perturbated model.

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

Recall

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and integrate the first term

$$rac{1}{2oldsymbol{i}\pi}\oint_{\mathcal{C}^+}oldsymbol{ec{a}}_N^*oldsymbol{\Pi}^{1/2}\mathbf{Q}_n(z)oldsymbol{\Pi}^{1/2}oldsymbol{ec{a}}_N=??$$

Recall

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$$rac{1}{2m{i}\pi}\oint_{\mathcal{C}^+}m{ec{a}}_N^*m{\Pi}^{1/2}m{Q}_n(z)m{\Pi}^{1/2}m{ec{a}}_N=m{0}$$

Why?

Recall

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and integrate the first term

$$\frac{1}{2i\pi} \oint_{\mathcal{C}^+} \vec{a}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{a}_N = 0$$

Why? Because

- 1. the contour only encloses $\lambda_{\max}(\tilde{\mathbf{Z}}_n)$ which is away from the bulk,
- 2. but all the eigenvalues of \mathbf{Z}_n are in the bulk. Hence:

$$\frac{1}{2i\pi} \oint_{\mathcal{C}^+} \frac{1}{\lambda_i(\mathbf{Z}_n) - z} \, dz = 0.$$

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Last step is to simplify the remaining expression by systematically use the **isotropic MP** law:

$$\vec{c}^* \mathbf{Q}_n(z) \vec{d} - \vec{c}^* \vec{d} \mathbf{g}_{MP}(z) \xrightarrow[N,n\to\infty]{a.s.} 0$$

Recall

$$\vec{\boldsymbol{a}}_N^* \tilde{\mathbf{Q}}_n(z) \vec{\boldsymbol{a}}_N = \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N - \xi \frac{\vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} \vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N}{1 + \xi \vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}}}$$

and integrate the first term

$$\frac{1}{2i\pi} \oint_{\mathcal{C}^+} \vec{\boldsymbol{a}}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n(z) \mathbf{\Pi}^{1/2} \vec{\boldsymbol{a}}_N = 0$$

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Last step is to simplify the remaining expression by systematically use the **Isotropic MP** law:

$$\begin{bmatrix}
\vec{c}^* \mathbf{Q}_n(z) \vec{d} - \vec{c}^* \vec{d} \mathbf{g}_{\check{\mathbf{M}}\mathbf{P}}(z) \xrightarrow[N,n\to\infty]{a.s.} 0
\end{bmatrix} \Rightarrow \begin{cases}
\vec{a}_N^* \mathbf{\Pi}^{1/2} \mathbf{Q}_n \vec{\mathbf{u}} & \approx \vec{a}_N^* \mathbf{\Pi}^{1/2} \vec{\mathbf{u}} \mathbf{g}_{\check{\mathbf{M}}\mathbf{P}}(z) \\
\vec{\mathbf{u}}^* \mathbf{Q}_n \mathbf{\Pi}^{1/2} \vec{a}_N & \approx \vec{\mathbf{u}}^* \mathbf{\Pi}^{1/2} \vec{a}_N \mathbf{g}_{\check{\mathbf{M}}\mathbf{P}}(z) \\
\vec{\mathbf{u}}^* \mathbf{Q}_n \vec{\mathbf{u}} & \approx \mathbf{g}_{\check{\mathbf{M}}\mathbf{P}}(z)
\end{cases}$$

After simplifications,

$$\vec{\boldsymbol{a}}_{N}^{*}\vec{\boldsymbol{v}}_{\text{max}}\vec{\boldsymbol{v}}_{\text{max}}^{*}\vec{\boldsymbol{a}}_{N} \approx -\frac{1}{2i\pi} \oint_{\mathcal{C}^{+}} |\vec{\boldsymbol{a}}_{N}^{*}\boldsymbol{\Pi}^{1/2}\vec{\mathbf{u}}|^{2} \frac{\mathbf{g}_{\text{MP}}^{2}(z)}{\xi^{-1} + \mathbf{g}_{\text{MP}}(z)} dz$$
$$= -\frac{\vec{\boldsymbol{a}}_{N}^{*}\vec{\mathbf{u}}\vec{\mathbf{u}}^{*}\vec{\boldsymbol{a}}_{N}}{1 + \theta} \oint_{\mathcal{C}^{+}} \frac{\mathbf{g}_{\text{MP}}^{2}(z)}{\xi^{-1} + \mathbf{g}_{\text{MP}}(z)} dz$$

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It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}^+} \frac{\mathbf{g}_{\text{MP}}^2(z)}{\xi^{-1} + \mathbf{g}_{\text{MP}}(z)} dz$$

by residue calculus (not that difficult).

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by residue calculus (not that difficult).

A minor miracle occurs: This factor admits a closed form formula!

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}^+} \frac{\mathbf{g}_{\tilde{\mathbf{M}}\mathbf{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\tilde{\mathbf{M}}\mathbf{P}}(z)} dz = \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1}$$

After simplifications,

$$\vec{\boldsymbol{a}}_{N}^{*} \vec{\boldsymbol{v}}_{\text{max}} \vec{\boldsymbol{v}}_{\text{max}}^{*} \vec{\boldsymbol{a}}_{N} \approx -\frac{1}{2i\pi} \oint_{\mathcal{C}^{+}} |\vec{\boldsymbol{a}}_{N}^{*} \mathbf{\Pi}^{1/2} \vec{\mathbf{u}}|^{2} \frac{\mathbf{g}_{\text{MP}}^{2}(z)}{\xi^{-1} + \mathbf{g}_{\text{MP}}(z)} dz$$
$$= -\frac{\vec{\boldsymbol{a}}_{N}^{*} \vec{\mathbf{u}} \vec{\mathbf{u}}^{*} \vec{\boldsymbol{a}}_{N}}{1 + \theta} \oint_{\mathcal{C}^{+}} \frac{\mathbf{g}_{\text{MP}}^{2}(z)}{\xi^{-1} + \mathbf{g}_{\text{MP}}(z)} dz$$

It remains to compute the correction factor

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}^+} \frac{\mathbf{g}_{\text{MP}}^2(z)}{\xi^{-1} + \mathbf{g}_{\text{MP}}(z)} dz$$

by residue calculus (not that difficult).

A minor miracle occurs: This factor admits a closed form formula!

$$-\frac{1}{1+\theta} \oint_{\mathcal{C}^+} \frac{\mathbf{g}_{\tilde{\mathbf{M}}\mathbf{P}}^2(z)}{\xi^{-1} + \mathbf{g}_{\tilde{\mathbf{M}}\mathbf{P}}(z)} dz = \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1}$$

Finally:

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[N,n \to \infty]{a.s.} 0$$
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Summary

1. Expression of $ec{v}_{\mathrm{max}}$ with the help of the resolvent

$$ec{m{a}}_N^* ec{m{v}}_{ ext{max}} ec{m{v}}_{ ext{max}}^* ec{m{a}}_N = rac{1}{2 m{i} \pi} \oint_{\mathcal{C}^+} ec{m{a}}_N^* ilde{Q}_n(z) ec{m{a}}_N \, dz$$

Summary

1. Expression of $\vec{v}_{\rm max}$ with the help of the resolvent

$$ec{oldsymbol{a}}_N^* ec{oldsymbol{v}}_{ ext{max}} ec{oldsymbol{d}}_N^* ec{oldsymbol{a}}_N = rac{1}{2 oldsymbol{i} \pi} \oint_{\mathcal{C}^+} ec{oldsymbol{a}}_N^* ilde{Q}_n(z) ec{oldsymbol{a}}_N \, dz$$

2. Convenient expression of $\vec{v}_{\rm max}$ where the contribution of the perturbation is separated from the resolvent of the non-perturbated model (MP)

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N \approx -\frac{\vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N}{1+\theta} \oint_{\mathcal{C}^+} \frac{\mathbf{g}_{\mathrm{MP}}^2(z)}{\xi^{-1} + \mathbf{g}_{\mathrm{MP}}(z)} dz$$

Summary

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3. Residue calculus to find the final form

$$\vec{\boldsymbol{a}}_N^* \vec{\boldsymbol{v}}_{\max} \vec{\boldsymbol{v}}_{\max}^* \vec{\boldsymbol{a}}_N - \left(1 - \frac{c}{\theta^2}\right) \left(1 + \frac{c}{\theta}\right)^{-1} \vec{\boldsymbol{a}}_N^* \vec{\mathbf{u}} \vec{\mathbf{u}}^* \vec{\boldsymbol{a}}_N \xrightarrow[N,n \to \infty]{a.s.} 0$$
.