Feasibility of equilibria in large ecosystems from a random matrix theory standpoint

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Disclaimer

A crash course on Large Random Matrices

Modelling and Understanding Ecological Networks

Hand waving

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You are going to attend a very **naive** speech on mathematical ecology by a **non-specialist**. Please, be merciful.

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- First result (1948) goes back to Nobel Laureate Eugene Wigner known as

  $\boxed{\text{Wigner's semi-circle law}}$

Example 1: Wigner Model

Matrix model

Let $\mathbf{X}_N = (X_{ij})$ symmetric $N \times N$

$$\mathbf{X}_N = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{12} & \ddots & & \\ \vdots & & \ddots & \\ X_{1N} & & & X_{NN} \end{pmatrix}$$

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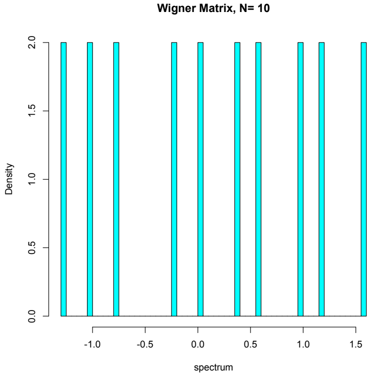

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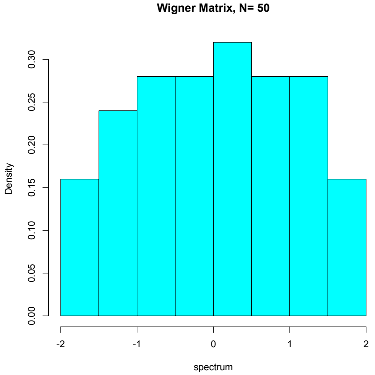

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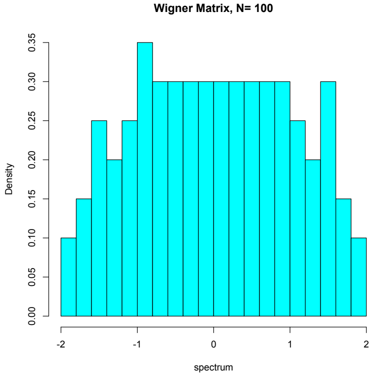

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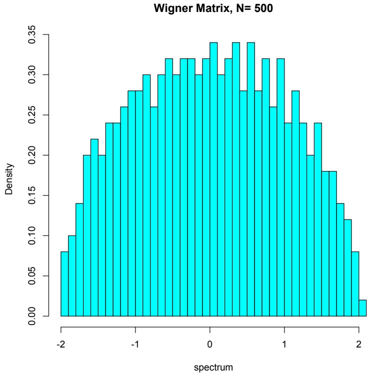

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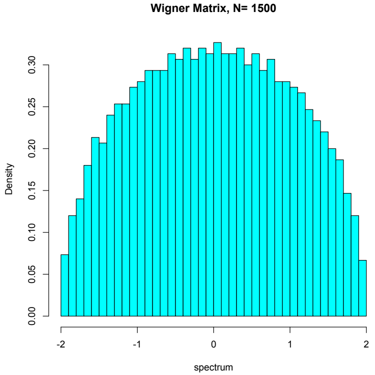

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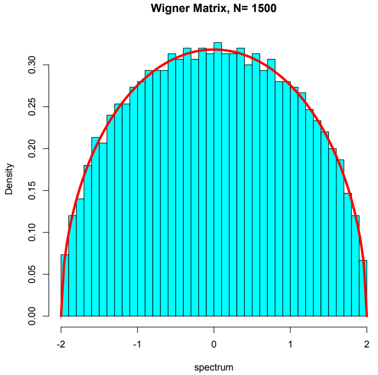

Figure: The semi-circular distribution (in red) with density $x \mapsto \frac{\sqrt{4-x^2}}{2\pi}$

Wigner's theorem (1948)

"The histogram of a Wigner matrix converges to the **semi-circular distribution**"

Example 2: Large Covariance Matrices

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Let $\mathbf{X}_N$ be a $N \times n$ rectangular matrix

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with dimensions of **same order**:

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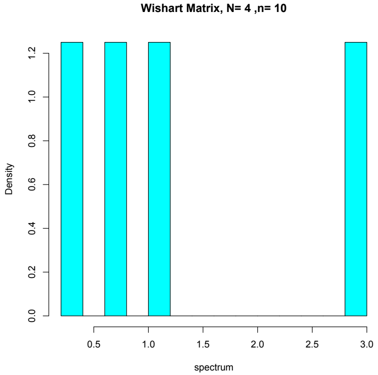

Figure: Spectrum's histogram - $\frac{N}{n} = 0.4$

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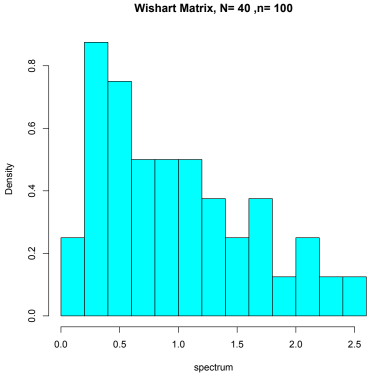

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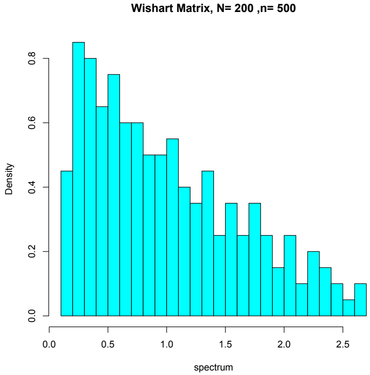

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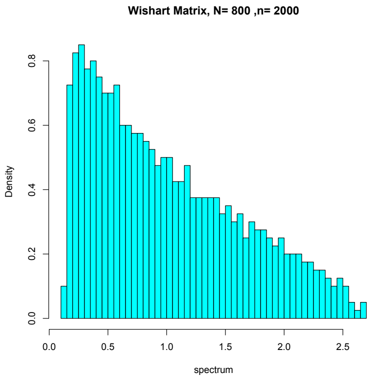

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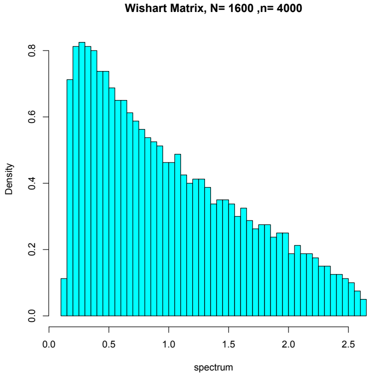

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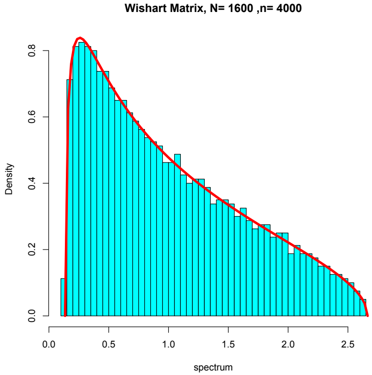

Figure: Marčenko-Pastur's distribution (in red)

Marčenko-Pastur's theorem (1967)

"The histogram of a **Large Covariance Matrix** converges to **Marčenko-Pastur distribution** with given parameter (here **0.4**)"

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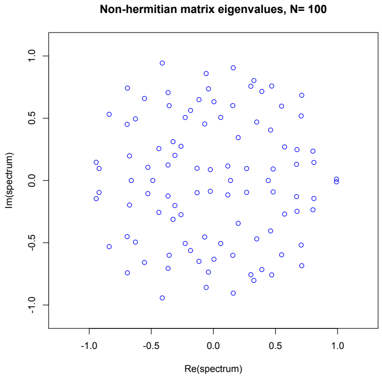

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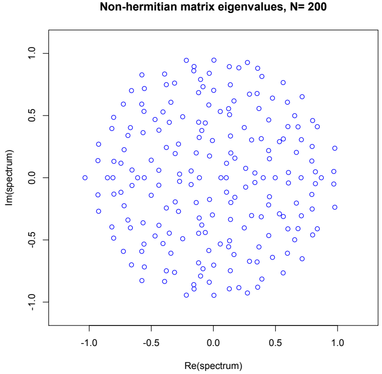

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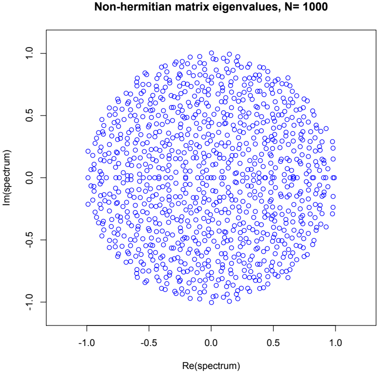

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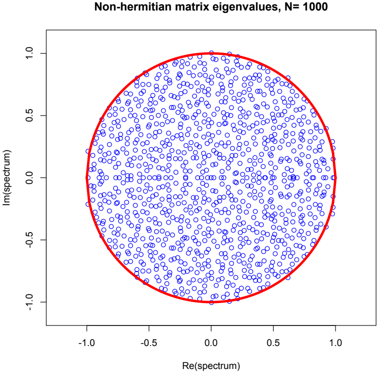

Figure: The circular law (in red)

Theorem: The Circular Law (Ginibre, Metha, Girko, Tao & Vu, etc.)

$\boxed{\text{The spectrum of } \mathbf{Y}_N \text{ converges to } \textbf{the uniform probability on the disc}}$

To go beyond: Many more results ..

- Matrices with a variance profile

$$Y_N = \frac{1}{\sqrt{N}} \left(a_{ij} X_{ij} \right) \quad \text{where} \quad (a_{ij}) \text{ are deterministic}$$

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- Spiked models, etc.

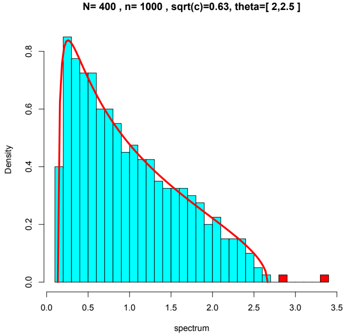

Figure: Spiked model: the largest eigenvalues separate from the others

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- δ is a parameter controlling the interaction $j \to i$ strength.

Interaction	**Value of δ**	**Comment**
strong	$\delta \in (0, 1/2)$	-
moderate	$\delta = 1/2$	RMT regime
weak	$\delta \in (1/2, 1)$	Perturbation theory

The Lotka-Volterra model II

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Equilibrium

- The equilibrium $\boldsymbol{a}^*$ is given by

$$\frac{da_i(t)}{dt} = 0 \quad \Longrightarrow \quad \boxed{\boldsymbol{a}^* = \left(\theta I_N - \frac{Z}{N^\delta}\right)^{-1} \boldsymbol{r}}$$

- The equilibrium is **feasible** if $\boxed{a_i^* > 0}$ for all i.

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- Given the jacobian $\mathcal{J}(\boldsymbol{a}^*)$, which is explicit for Lotka-Voltera systems

$$\boxed{\mathcal{J}(\boldsymbol{a}^*) = \mathrm{diag}(\boldsymbol{a}^*) \left(-\theta I_N + \frac{Z}{N^\delta}\right)}$$

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Conclusion

Static analysis (equilibrium and stability) of the Lotka-Volterra model can be addressed by means of RMT for large N

An intriguing argument for moderate interactions

Theorem (Dougoud et al.)

For moderate interactions,

- all the components a_i^*'s of the equilibrium vector

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As a consequence ..

$$\mathbb{P}\{a_i^* < 0 \text{ for some } i \in [n]\} \xrightarrow[N\to\infty]{} 1 \quad \Longleftrightarrow \quad \boxed{\textbf{No feasible equilibrium with proba 1!}}$$

Reference

- "The feasibility of equilibria in large ecosystems: A primary but neglected concept in the complexity-stability debate", *Dougoud, Vikenbosch, Rohr, Bersier, Mazza*, PLoS Comput. Biology, **2018**

Feasibility and the strenght of interactions

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- Consider the equilibrium vector

$$\boxed{\boldsymbol{a}^* = \left(\theta I_N - \frac{Z}{\boldsymbol{\kappa}_N}\right)^{-1} \mathbf{1}}$$

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$$\boldsymbol{\kappa}_N \gg \boldsymbol{\kappa}_N^* \quad (\text{Ex: } \boldsymbol{\kappa}_N = N) \implies \boxed{\textbf{All the equilibriums are feasible with probability 1}}$$

Simulations

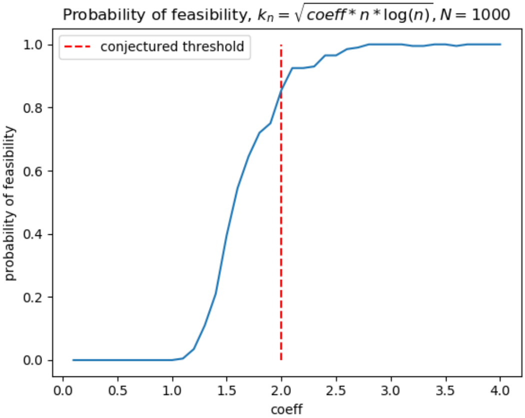

Figure: We plot the probability of feasibility as a function of the normalization parameter $\boldsymbol{\kappa}_N = \sqrt{\textbf{coeff} \times n \log(n)}$. As expected, we get an approximate threshold phenomenon around the critical value $\text{coeff} = 2$. We use uniforme random variables, centered and normalized for the interaction matrix, $N = 1000$. We compute the frequency over 200 simulations.

Disclaimer

A crash course on Large Random Matrices

Modelling and Understanding Ecological Networks

Hand waving

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2. General interaction matrix models

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random	Z_{ij} i.i.d. and $\mathbb{E}Z_{ij} = 0$	Circular law
structured	$Z_{ij} = 0$ for $(i,j) \in \mathcal{S}$	Sparse variance profiles / open
mutualistic	$\mathbb{E}Z_{ij} > 0$	open
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Figure: Various types of ecological networks

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For those of you interested in RMT

- GDR **MEGA** (**M**atrices **E**t **G**raphes **A**léatoires)