Introduction

Marčenko-Pastur's theorem

Large covariance matrices

Spiked models

Statistical Test for Single-Source Detection
- The setup
- Asymptotic behaviour of the GLRT
- Fluctuations of the test statistics
- Power of the test
- The GLRT: Summary

Direction of Arrival Estimation

Applications to the MIMO channel

Conclusion

The hypothesis testing problem

Statistical Setup

let

$$\vec{\mathbf{y}}(k) = \begin{cases} \sigma \vec{\mathbf{w}}(k) & \text{under } H_0 \\ \vec{\mathbf{h}}\, s(k) + \sigma \vec{\mathbf{w}}(k) & \text{under } H_1 \end{cases} \qquad \text{for } k = 1:n$$

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Objective

Given n observations $(\vec{\mathbf{y}}(k), 1 \le k \le n)$, and the associated **sample covariance matrix**

$$\hat{\mathbf{R}}_n = \frac{1}{n} \mathbf{Y}_n \mathbf{Y}_n^* \quad \text{where} \quad \mathbf{Y}_n = [\vec{\mathbf{y}}(1), \cdots, \vec{\mathbf{y}}(n)] \quad \text{is } N \times n \ ,$$

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Neyman-Pearson procedure

Likelihood functions

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In case where σ^2 and $\vec{\mathbf{h}}$ are **known**, the **Likelihood Ratio Statistics**

$$\frac{p_1(\mathbf{Y}_N; \vec{\mathbf{h}}; \sigma^2)}{p_0(\mathbf{Y}_N; \sigma^2)}$$

provides a **uniformly most powerful test**:

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- the **maximum achievable power**

$$1 - \mathbb{P}(H_0 \mid H_1)$$

is guaranteed by Neyman-Pearson.

The GLRT

The Generalized Likelihood Ratio Test

In the case where $\vec{\mathbf{h}}$ and σ^2 are unknown, we use instead:

$$L_n = \frac{\sup_{\sigma^2, \vec{\mathbf{h}}} \; p_1(\mathbf{Y}_n, \sigma^2, \vec{\mathbf{h}})}{\sup_{\sigma^2} \; p_0(\mathbf{Y}_n, \sigma^2)}$$

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Expression of the GLRT

The GLRT statistics writes

$$L_n = \frac{\left(1 - \frac{1}{N}\right)^{(1-N)n}}{\left(\frac{\lambda_{\max}(\hat{\mathbf{R}}_n)}{\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}_n}\right)^n \left(1 - \frac{1}{N}\frac{\lambda_{\max}(\hat{\mathbf{R}}_n)}{\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}_n}\right)^{(N-1)n}}$$

and is a **deterministic function** of $\boxed{T_n = \frac{\lambda_{\max}(\hat{\mathbf{R}}_n)}{\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}_n}}$

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Under H_0

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Denote by

$$\boxed{\mathbf{snr} = \frac{\|\vec{\mathbf{h}}\|^2}{\sigma^2}}$$

the **Signal-to-Noise (SNR)** ratio.

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Remarks

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  Hence the rule of thumb

$$\boxed{\text{Detection occurs if } \mathbf{snr} \text{ higher than } \textbf{asymptotic data noise}.}$$

Simulations

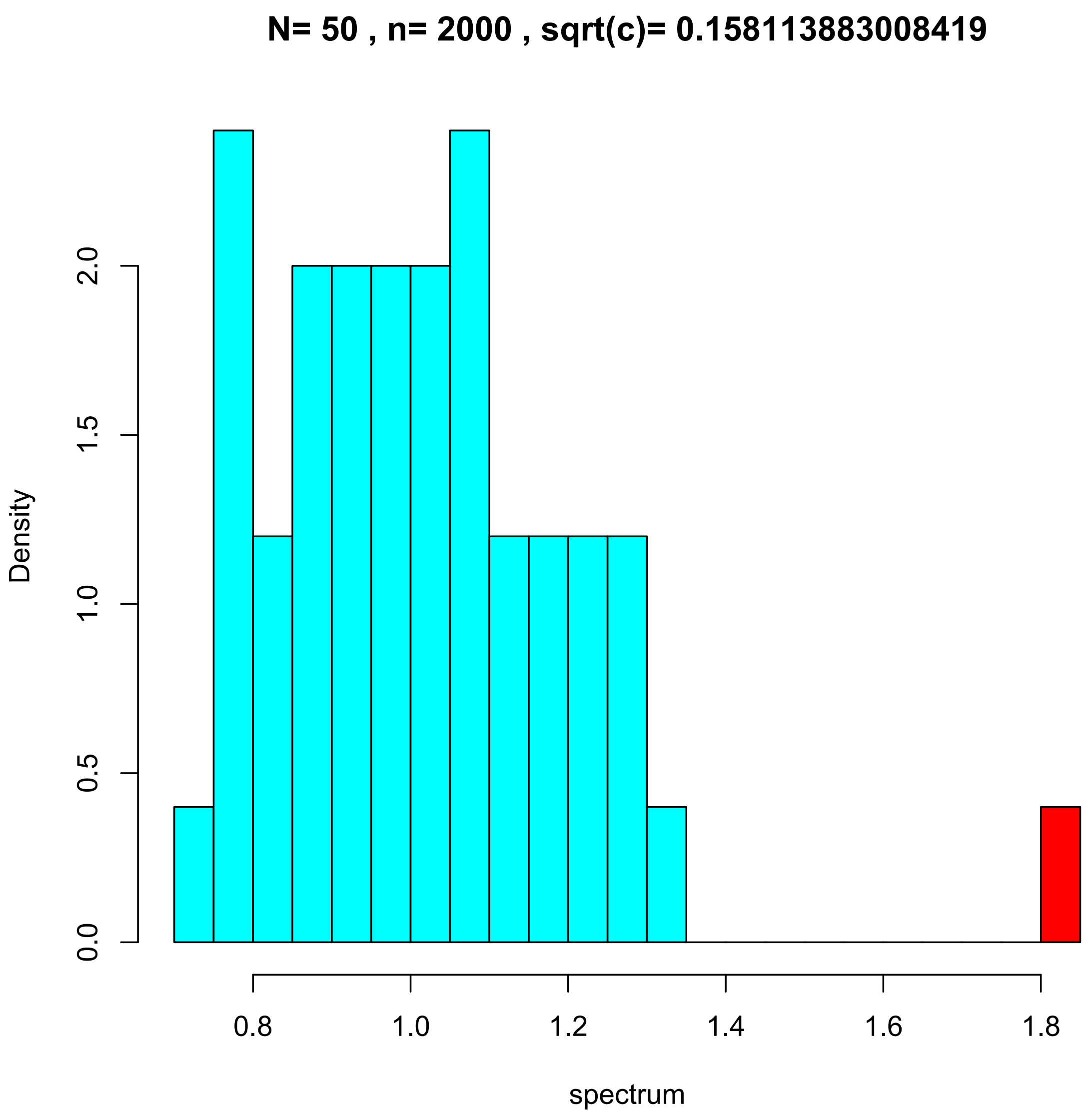

Figure : Influence of **asymptotic data noise** as $\sqrt{c}$ increases

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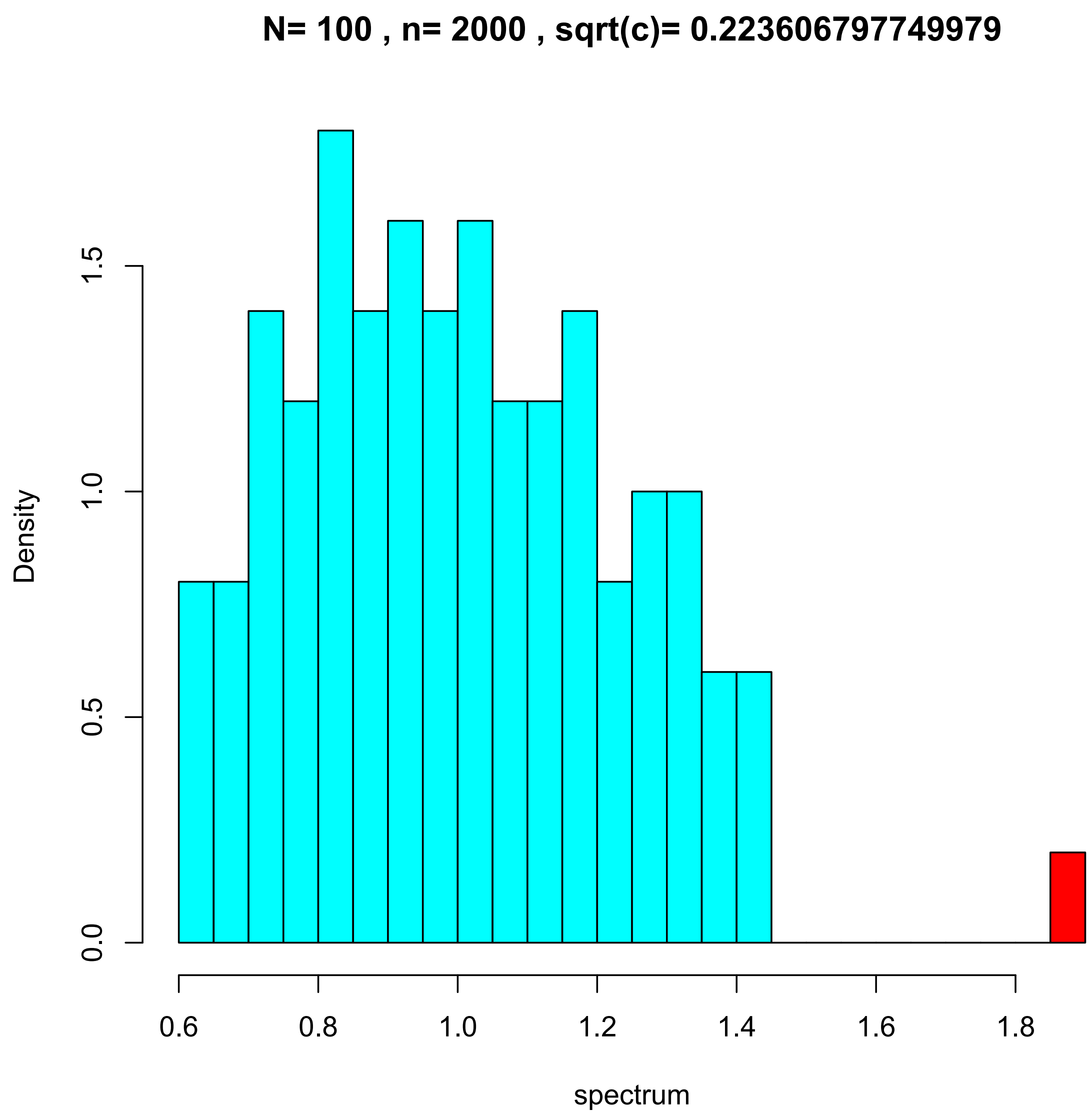

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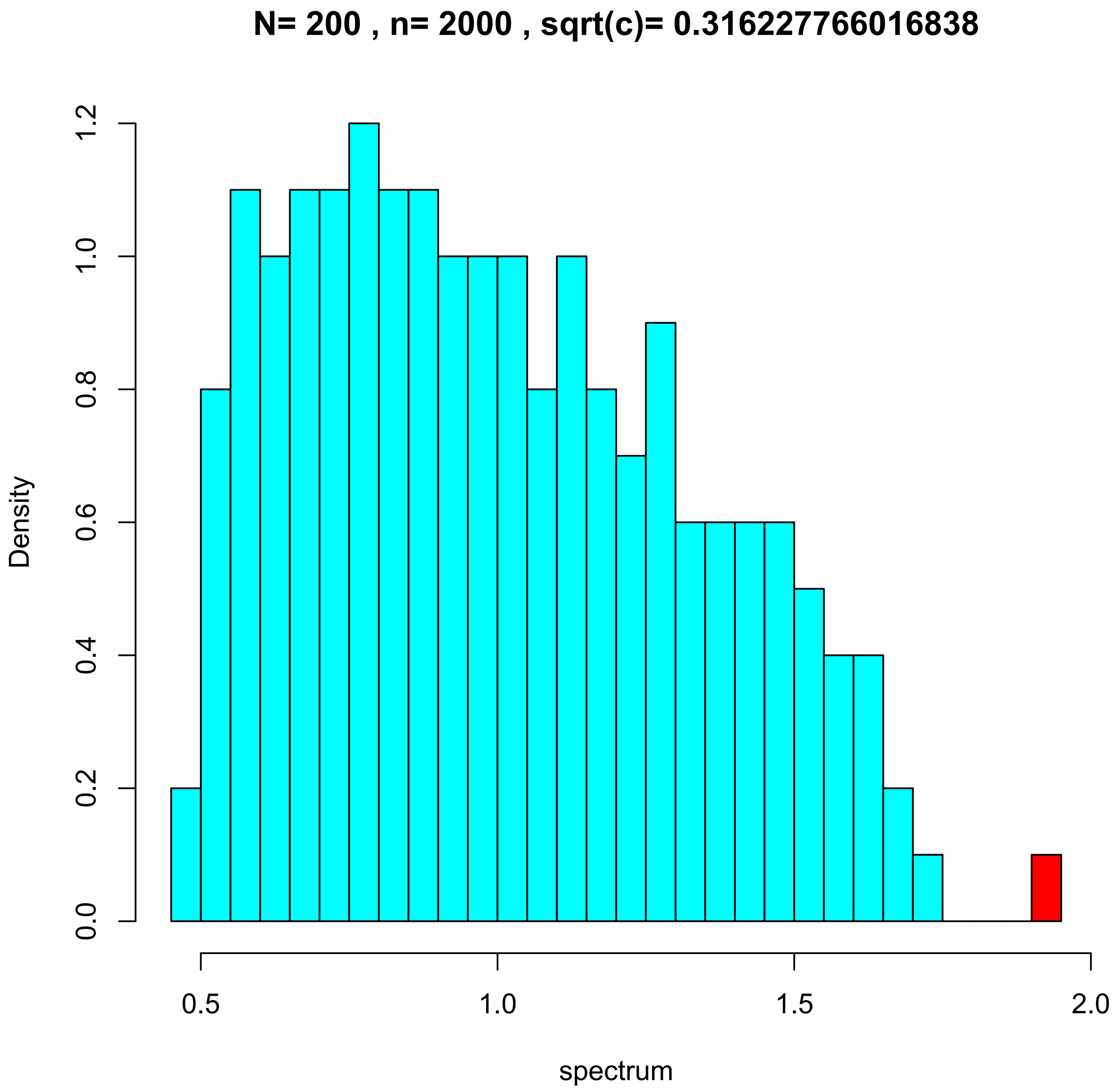

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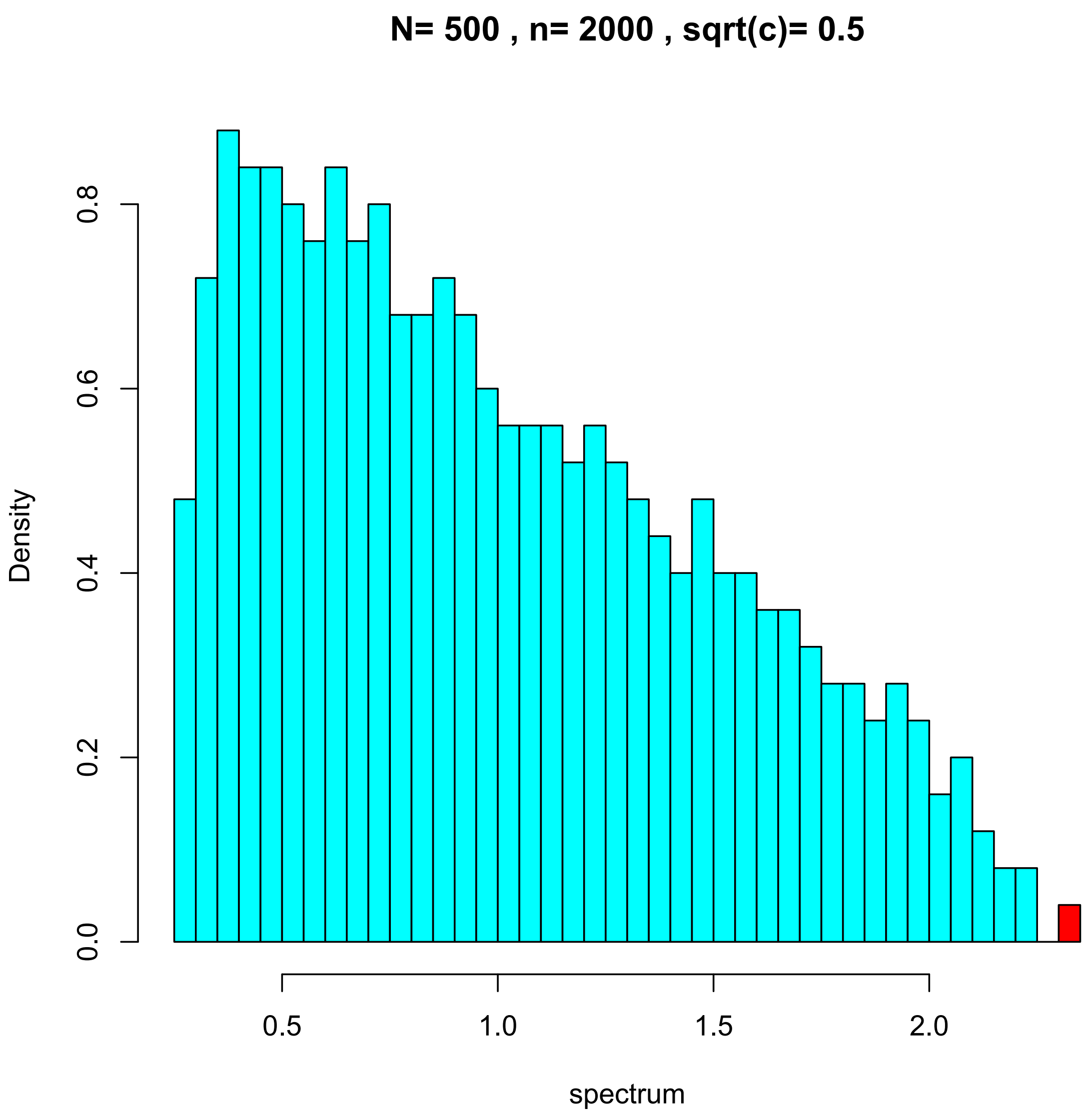

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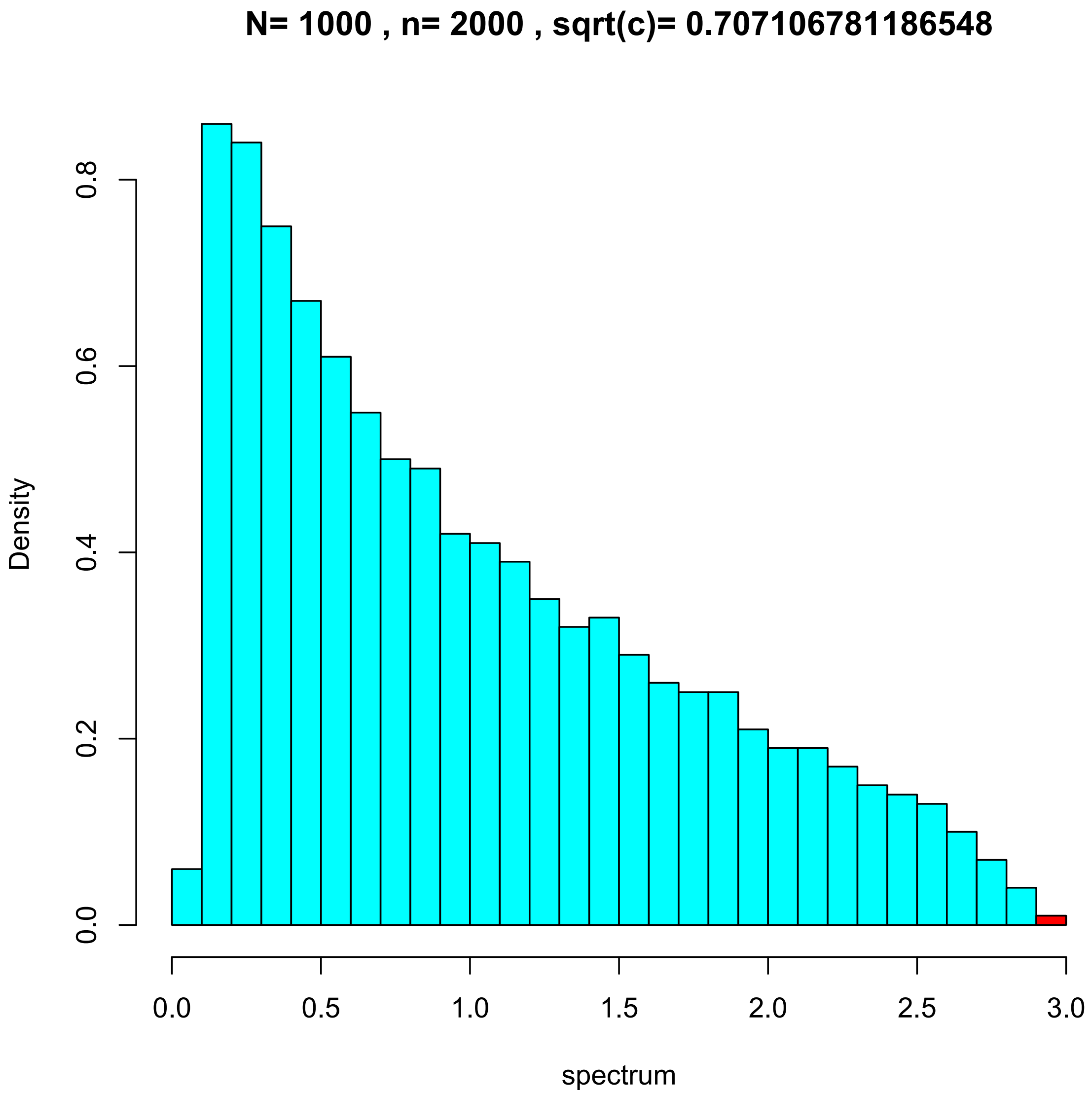

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- We are interested in the largest eigenvalue of the matrix model

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Spectrum of $\frac{1}{n}\mathbf{Y}_n\mathbf{Y}_n^*$ follows a spiked model with **rank-one perturbation**

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  Same limit as under H_0. The test statistics does not discriminate between the two hypotheses.

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- The exact distribution of the statistics

$$L_n = \frac{\lambda_{\max}(\hat{\mathbf{R}}_n)}{\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}_n}$$

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Fluctuations of $\lambda_{\max}(\hat{\mathbf{R}}_n)$: Tracy-Widom distribution **at rate $N^{2/3}$**

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Otherwise stated,

$$\boxed{\lambda_{\max}\left(\hat{\mathbf{R}}_n\right)=\sigma^2(1+\sqrt{c_n})^2+\frac{\Theta_N}{N^{2/3}}\boldsymbol{X}_{TW}+\varepsilon_n}$$

where $\boldsymbol{X}_{TW}$ is a random variable with Tracy-Widom distribution.

Details on Tracy-Widom distribution

Tracy-Widom distribution is defined by

- its cumulative distribution function

$$F_{TW}(x) = \exp\left\{-\int_x^\infty (u-x)^2 q^2(u)\,du\right\}$$

- where

$$q''(x) = xq(x) + 2q^3(x) \quad \text{and} \quad q(x) \sim \mathrm{Ai}(x) \text{ as } x \to \infty \ .$$

$x \mapsto \mathrm{Ai}(x)$ being the Airy function.

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Don't bother .. just download it

- For simulations, cf. `R Package 'RMTstat'`, by Johnstone et al.
- Also, Folkmar Bornemann (TU München) has developed fast `matlab` code

Tracy-Widom curve

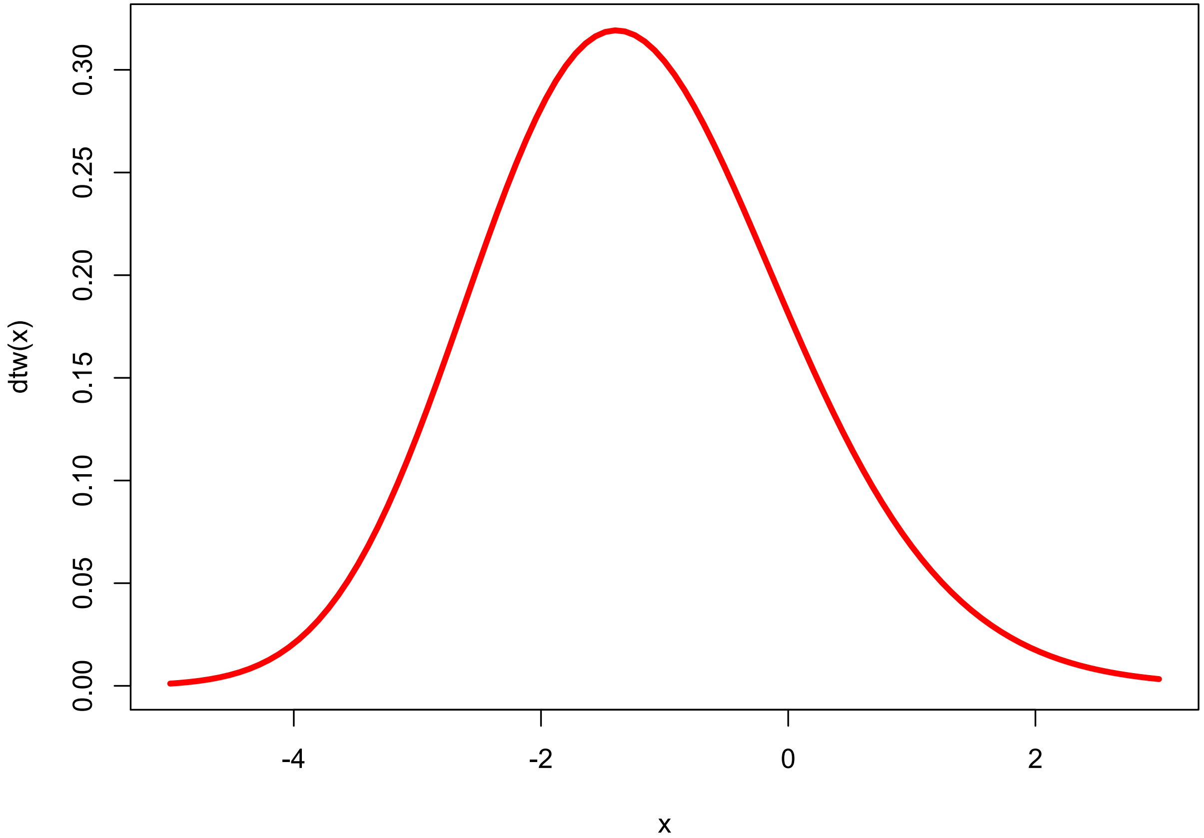

Figure : Tracy-Widom density

Tracy-Widom curve

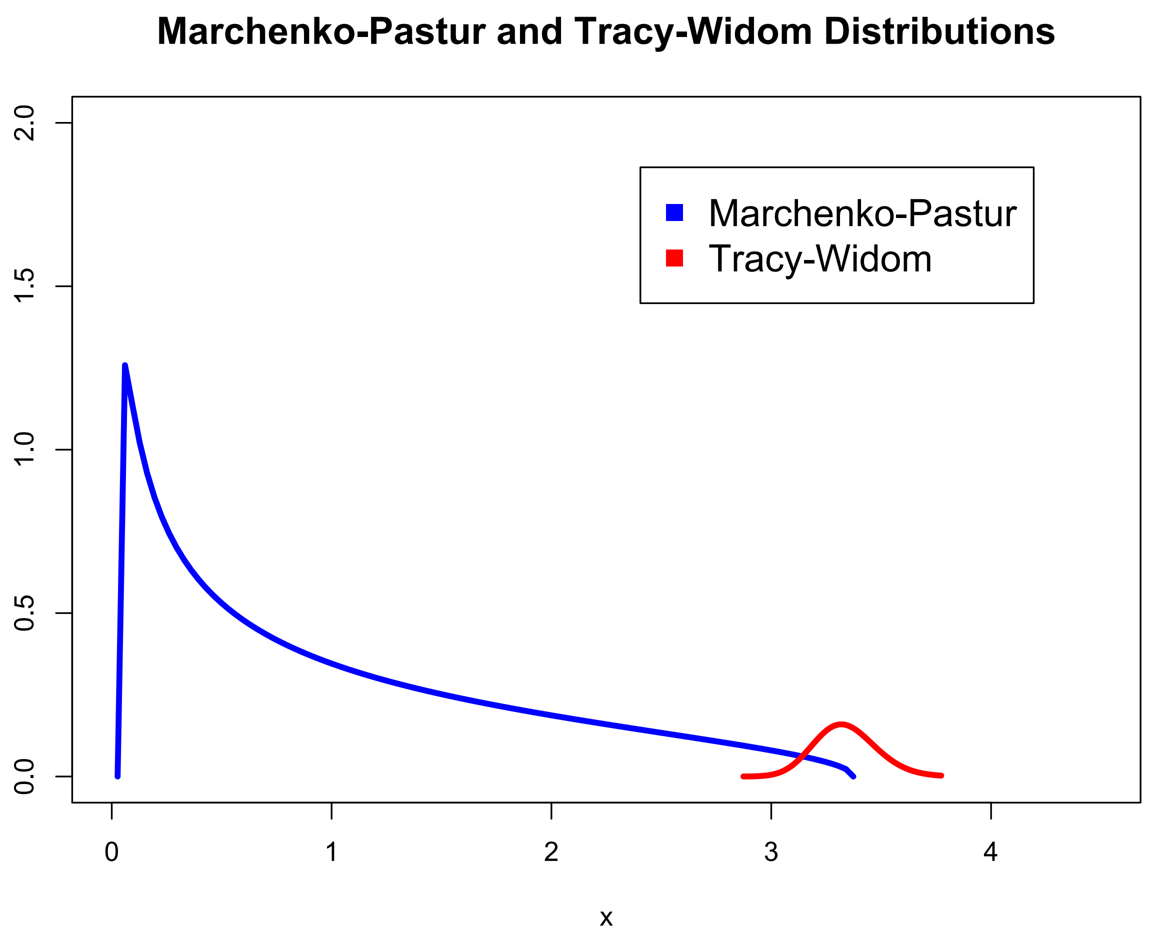

Figure : Fluctuations of the largest eigenvalue $\lambda_{\max}(\hat{\mathbf{R}}_n)$ under H_0

Fluctuations of the GLRT under H_0 - III

Fluctuations of $\frac{1}{N}\mathrm{Trace}\,(\hat{\mathbf{R}}_n)$: Gaussian distributions **at rate N**

$$N\left\{\frac{1}{N}\sum_{i=1}^{N}\lambda_i(\hat{\mathbf{R}}_n)-\sigma^2\right\}\xrightarrow[N,n\to\infty]{\mathcal{L}}\mathcal{N}(0,\Gamma)\ ,$$

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$$\boxed{\frac{1}{N}\mathrm{Trace}\,(\hat{\mathbf{R}}_n)=\frac{1}{N}\sum_{i=1}^{N}\lambda_i(\hat{\mathbf{R}}_n)=\sigma^2+\frac{\sqrt{\Gamma}}{N}\boldsymbol{Z}+\varepsilon_n}$$

where $\boldsymbol{Z}$ is a random variable with distribution $\mathcal{N}(0,1)$.

Fluctuations of the GLRT under H_0 - IV

Conclusion

- Fluctuations of $L_n = \frac{\lambda_{\max}(\hat{\mathbf{R}}_n)}{\frac{1}{N}\mathrm{Trace}\,\hat{\mathbf{R}}_n}$ are driven by $\lambda_{\max}(\hat{\mathbf{R}}_n)$:

$$\frac{N^{2/3}}{\widetilde{\Theta}_N}\left\{L_N - (1+\sqrt{c_n})^2\right\} \xrightarrow[N,n\to\infty]{\mathcal{L}} \mathbb{P}_{\mathrm{TW}} \quad \text{with} \quad \widetilde{\Theta}_N = (1+\sqrt{c_n})\left(\frac{1}{\sqrt{c_n}}+1\right)^{1/3}$$

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- In order to set the threshold $\boldsymbol{\alpha}$, we choose $\boldsymbol{t}_{\boldsymbol{\alpha}}^{\boldsymbol{n}}$ as

$$\boldsymbol{t}_{\boldsymbol{\alpha}}^{\boldsymbol{n}} = (1+\sqrt{c_n})^2 + \frac{\widetilde{\Theta}_N}{N^{2/3}}\boldsymbol{t}_{\boldsymbol{\alpha}}{}^{\textbf{Tracy-Widom}}$$

where $\boldsymbol{t}_{\boldsymbol{\alpha}}{}^{\textbf{Tracy-Widom}}$ is the corresponding quantile for a Tracy-Widom random variable:

$$\mathbb{P}\{\boldsymbol{X}_{TW} > \boldsymbol{t}_{\boldsymbol{\alpha}}{}^{\textbf{Tracy-Widom}}\} \le \alpha.$$

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Given a level of confidence $\alpha \in (0,1)$, the **type I error** defines the associate quantile $\boldsymbol{t_\alpha}$

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No optimality

Contrary to Neyman-Pearson procedure, there is **no theoretical guarantee** that the GLRT is a uniformily most powerful test.

- $\boxed{\text{It is therefore important to be able to compute the power of the GLRT}}$

Power of the GLRT II

- For fixed level of confidence α

$$\mathbb{P}_{H_0}\left(L_N > \boldsymbol{t_{\alpha}^{n}}\right) \leq \alpha \ ,$$

the type II error **exponentially decreases to 0**.

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- Indeed, we want to evaluate

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&= \quad \mathbb{P}_{H_1}\left(L_N \leq (1+\sqrt{c})^2 + \frac{\widetilde{\Theta}_N}{N^{2/3}} \boldsymbol{t_\alpha}^{\textbf{Tracy-Widom}}\right)\\
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- one can define its error exponent $\boldsymbol{\mathcal{E}}$ as:

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$$\boldsymbol{\mathcal{E}} = \lim_{N,n\to\infty} -\frac{1}{n}\log \mathbb{P}_{H_1}\left(L_N < \boldsymbol{t^n_\alpha}\right) .$$

- Hence, the type II error writes:

$$\mathbb{P}_{H_1}(L_N < t(\alpha)) \approx_{N,n\to\infty} e^{-n\boldsymbol{\mathcal{E}}}$$

Power of the GLRT IV: The error exponent

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$$\boxed{\boldsymbol{\mathcal{E}} = \frac{\lambda^+ - \lambda_{spk}^\infty}{1+\mathbf{snr}} - (1-c)\log\left(\frac{\lambda^+}{\lambda_{spk}^\infty} - 2c\left[F^+(\lambda^+) - F^+(\lambda_{spk}^\infty)\right]\right)}$$

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Elements of proof

- Proof essentially based on the large deviations of $\lambda_{\max}(\hat{\mathbf{R}}_n)$ under H_1

Power of the GLRT V

The error exponent curve

Instead of letting the type I error fixed, it is of interest to let it go exponentially to zero:

$$\mathbb{P}_{H_0}\left(L_N > t(\mathbf{a})\right) \approx_{N,n\to\infty} e^{-n\mathbf{a}}$$

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and to compute the corresponding type II error (or its error component $\boldsymbol{\mathcal{E}}(\mathbf{a})$)

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Interest

If we want to discriminate between two tests, we can compare their error expoenent curves.

Introduction

Marčenko-Pastur's theorem

Large covariance matrices

Spiked models

Statistical Test for Single-Source Detection
- The setup
- Asymptotic behaviour of the GLRT
- Fluctuations of the test statistics
- Power of the test
- The GLRT: Summary

Direction of Arrival Estimation

Applications to the MIMO channel

Conclusion

Summary

- Consider the following hypothesis

$$\vec{\mathbf{y}}(k) = \begin{cases} \sigma\vec{\mathbf{w}}(k) & \text{under } H_0 \\ \vec{\mathbf{h}}\, s(k) + \sigma\vec{\mathbf{w}}(k) & \text{under } H_1 \end{cases} \qquad \text{for } k = 1:n$$

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- The threshold can be asymptotically determined by Tracy-Widom quantiles.
- The type II error (equivalentlty power of the test) can be analyzed via the error exponent of the test

$$\boldsymbol{\mathcal{E}} = \lim_{N,n\to\infty} -\frac{1}{n} \log \mathbb{P}_{H_1}(L_N < \boldsymbol{t_\alpha}) \ ,$$

which relies on the study of large deviations of $\lambda_{\max}$ under H_1.