Exercise 1

Show that epimorphisms preserve neutral elements, i. e. if h is an epimorphism from a monoid $(M_1, \cdot, e_1)$ to a monoid $(M_2, \star, e_2)$ then $h(e_1) = e_2$.

1. Homomorphism: $\forall x, y \in M_1, h(x \cdot y) = h(x) \star h(y)$ and is surjective.
2. $\forall x \in M_1, h(x) = h(x \cdot e_1) = h(x) \star h(e_1)$,
3. $\forall x \in M_1, h(x) = h(e_1 \cdot x) = h(e_1) \star h(x)$,
4. So we've got:

$$h(x) \star h(e_1) = h(e_1) \star h(x) = h(x) \to h(e_1) = e_2$$

The result follows because $\forall y \in M_2$ is of the form $h(x)$ with $x \in M_1$.

Exercise 2

Give an example of a monoid that is not a group.

1. Monoid built from natural numbers and addition: $(\mathbb{N}, +)$,
   it's true that:

$$\forall x, y, z \in \mathbb{N}, x + (y + z) = (x + y) + z$$

and the neutral element is:

$$0 \to \forall x \in \mathbb{N}, x + 0 = x$$

2. Such a construction is not a Group, because it doesn't contain the inverse element, which is necessary for the Groups.

$$\neg(\forall x \in \mathbb{N}, x + x^{-1} = 0)$$

Exercise 3

Let φ be the Euler function, i.e

$$\varphi(n) = |\{k \in \mathbb{N} : GCD(n, k) = 1|\}$$

Show that $\Sigma_{k\in\{i\in\mathbb{N}:i|n\}}\varphi(n) = n$

1. Lets call this summation $F(n) = \varphi(d_1) + \varphi(d_2) + \ldots \varphi(d_r)$, where $d_i|n$
2. We know that if $p - prime$, then $F(p^k) = \varphi(1) + \varphi(p) + \varphi(p^2) + \ldots \varphi(p^k) = 1 + (p - 1) + (p^2 - p) + (p^3 - p^2) + \ldots + (p^k - p^{k-1}) = p^k$, because $\varphi(p^k) = p^{k-1}(p - 1)$.
3. Prime factorization of n: $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s}$, where $p_i^{k_i}$ and $p_j^{k_j}$ are relatively prime
4. $F(n) = F(p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots \cdot p_s^{k_s}) = F(p_1^{k_1}) \cdot \ldots \cdot F(p_s^{k_s}) = p_1^{k_1} \cdot \ldots \cdot p_s^{k_s} = n$

Exercise 4

Let X, Y, Z be sets and let $f : X \to Y$, $g : Y \to Z$ be functions.
Show that $F(g \circ f) = F(g) \circ F(f)$, where $F(f) : dom(f)* \to codomain(f)*$ and

$$(F(f))(x_1, x_2, ..., x_n) = (f(x_1), f(x_2), ..., f(x_n)).$$

What is the connection between F and the function `map`?

1. $F(g \circ f)(x_1, \ldots, x_n) = (g \circ f(x_1), \ldots, g \circ f(x_n)) = F(g)(f(x_1), \ldots, f(x_n)) = F(g) \circ F(f)(x_1, \ldots, x_n)$
2. Function F behaves the same as function `map`, it applies function f to each element of the argument list $(x_1, \ldots, x_n)$.