

## Chapter 1

# Exploring Little Numbers

We manipulate numbers all the time and in this chapter we propose you a little journey into the way integers are mapped to their binary representations. We will open the box and take a language implementor perspective and happily explore how small integers are represented.

We will start with some simple reminders on math that are the basics of our digital world. Then we will have a look at how integers and in particular small integers are encoded. This is a typical simple knowledge that people can forget over time and our goal is simply to refresh it.

### 1.1 Power of 2 and Numbers

Let's start with some simple maths. In digital world, information is encoded as powers of 2. Nothing really new. In Smalltalk raising a number to a power is performed by sending the message `raisedTo:` to a number. Here are some examples. Figure 1.1 shows the powers of 2.

```
2 raisedTo: 0  
  → 1  
2 raisedTo: 2  
  → 4  
2 raisedTo: 8  
  → 256
```

Using a sequence of power of 2 we can encode numbers. For example, how can we encode the number 13? It cannot be higher than  $2^4$  because  $2^4 = 16$ . So it should be  $8 + 4 + 1$ ,  $2^3 + 2^2 + 2^0$ . Now when we order the powers of two as a sequence as shown in Figure 1.2, we see that 13 is encoded as 1101. So we can encode a number with a sequence of powers of 2. An element of this sequence is called a bit. 13 is encoded with 4 bits, corresponding to

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
32768	16384	8192	4096	2048	1024	512	256	128	64	32	16	8	4	2	1

Figure 1.1: Powers of 2 and their numerical equivalence.

$1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$ . This sequence of bits represents a binary notation for numbers. The most significant bit is located on the left of a bit sequence and the least on the right ( $2^0$  is the rightmost bit).

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
32768	16384	8192	4096	2048	1024	512	256	128	64	32	16	8	4	2	1
													1	1	1

Figure 1.2:  $13 = 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0$ .

## Binary notation

Smalltalk has a syntax for representing numbers in different bases. We write 2r1101 where 2 indicates the base or radix, here 2, and the rest the number expressed in this base. Note that we could also write 2r01101 or 2r0001101 since the leading zeroes do not change the number. This notation follows the convention that the least significant bit is the rightmost one.

```
2r1101
  → 13
13 printStringBase: 2
  → '1101'
Integer readFrom: '1101' base: 2
  → 13
```

Note that the last two messages `printStringBase:` and `readFrom:base:` do not handle well the internal encoding of negative numbers as we will see later. `-2 printStringBase: 2` returns `-10` but this is not the internal number representation known as two's complement. These messages just print/read the number in a given base.

The radix notation can be used to specify numbers in different bases. Obviously 15 written in decimal base (`10r15`) returns 15, while 15 in base 16 returns `16 + 5 = 21@` as illustrated by the following expressions.

```
10r15
  → 15
16r15
  → 21
```

## 1.2 Bit shifting is multiplying by 2 powers

Since integers are represented as sequences of bits, if we shift all the bits from a given amount we obtain another integer. Shifting bit is equivalent to perform a multiplication/division by two. Figure 1.3 illustrates this point. Smalltalk offers three messages to shift bits: `>> aPositiveInteger`, `<< aPositiveInteger` and `bitShift: anInteger`. `>>` divides the receiver, while `<<` multiply it by a power of two.

The following examples show how to use them.

```
2r000001000
  → 8
2r000001000 >> 1      "we divide by two"
  → 4
(2r000001000 >> 1) printStringBase: 2
  → '100'
2r000001000 << 1      "we multiply by two"
  → 16
```

The message `bitShift:` is equivalent to `>>` and `<<`, but it uses negative and positive integers to indicate the shift direction. A positive argument offers the same behavior than `<<`, multiplying the receiver by a power of 2. A negative is similar to `>>`.

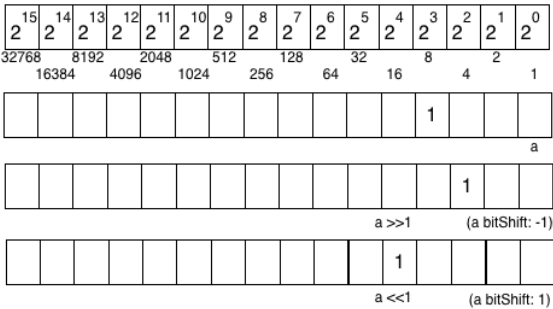


Figure 1.3: Multiplying and dividing by 2.

```
2r000001000
  → 8
2r000001000 bitShift: -1
  → 4
2r000001000 bitShift: 1
  → 16
```

Of course we can shift by more than one bit at a time.

```
2r000001000
  → 8
2r000001000 >> 2           "we divide by four"
  → 2
(2r000001000 >> 2) printStringBase: 2
  → '10'
2r000001000 << 2           "we multiply by four"
  → 32
```

The previous examples only show bit shifting numbers with one or two bits but there is no constraint at this level. The complete sequence of bits can be shifted as shown with 2r000001100 below and Figure 1.4.

```
(2 raisedTo: 8) + (2 raisedTo: 10)
  → 1280
2r010100000000
  → 1280
2r010100000000 >> 8
  → 2r0101
returns 5
```

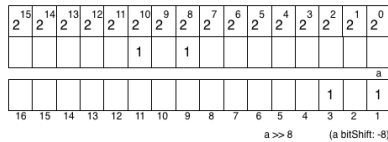


Figure 1.4: We move 8 times to the right. So from 1280 we get 5.

So far nothing really special. You should have learned that in any basic math lecture, but this is always good to walk on a hill before climbing a mountain.

## 1.3 Bit Manipulation and Access

Pharo offers common boolean operations for bit manipulation. Hence you can send the messages `bitAnd:`, `bitOr:`, and `bitXor:` to numbers. They will apply bit by bit the associated Boolean operation.

```
2r000001101 bitAnd: 2r01
  → 1
2r000001100 bitAnd: 2r01
  → 0
2r000001101 bitAnd: 2r1111
  → 1101
```

`bitAnd:` can then be used to select part of a number. For example, `bitAnd: 2r111` selects the three first bits.

```
2r000001101 bitAnd: 2r111
  → 5
2r000001101 bitAnd: 2r0
  → 0
2r0001001101 bitAnd: 2r1111
  → 13 "1101"
2r000001101 bitAnd: 2r111000
  → 8 "1000"
2r000101101 bitAnd: 2r111000
  → 40 "101000"
```

Using `bitAnd:` combined with a `bitShift:` we can select part of a number. Imagine that we encode three numbers on 10 bits: let's say one number encoded on 3 bits (a number between 0 and 7 — noted as XXX in ZZZYYYYXXX), one number encoded on 4 bits (0 to 15 — noted as YYYY in ZZZYYYYXXX), and finally the third one is encoded on 3 bits noted as ZZZ in ZZZYYYYXXX. The following expressions are handy to manipulate them. Accessing the second number cannot simply be done using only `bitShift:` because we will still have the third number present. We get `(2r1001111001 bitShift: -3)` returns 79, while we would like to get 15. The solution is either to use a `bitAnd:` to clear the third number and to do a `bitShift:` or to do a `bitShift:` and to clear the third number. The `bitAnd:` argument has to be adapted to select the right part of the encoding.

```
(2r1001111001 bitShift: -3)
  → 79
(2r1001111001 bitAnd: 2r0001111000)
  → 120
(2r1001111001 bitAnd: 2r0001111000) bitShift: -3
  → 15
(2r1001111001 bitShift: -3) bitAnd: 2r0001111
  → 15
```

**Bit Access.** Smalltalk lets you access bit information. The message `bitAt:` returns the value of the bit at a given position. It follows the Smalltalk convention that collections' indexes start with one.

```
2r000001101 bitAt: 1
  → 1

2r000001101 bitAt: 2
  → 0

2r000001101 bitAt: 3
  → 1

2r000001101 bitAt: 4
  → 1

2r000001101 bitAt: 5
  → 0
```

This is interesting to learn from the system itself. Here is the implementation of the method `bitAt:` on the `Integer` class.

```
Integer>>bitAt: anInteger
"Answer 1 if the bit at position anInteger is set to 1, 0 otherwise.
self is considered an infinite sequence of bits, so anInteger can be any strictly positive
integer.
Bit at position 1 is the least significant bit.
Negative numbers are in two-complements.

This is a naive implementation that can be refined in subclass for speed"

↑(self bitShift: 1 - anInteger) bitAnd: 1
```

We shift to the right from an integer minus one (hence `1 - anInteger`) and with a `bitAnd:` we know whether there is a one or zero in the location. Imagine that we have `2r000001101`, when we do `2r000001101 bitAt: 5` we will shift it from 4 and doing a `bitAnd: 1` with select that bits (*i.e.*, returns 1 if it was at 1 and zero otherwise, so its value). Doing a `bitAnd: 1` is equivalent to tell whether there is a 1 in the least significant bit.

Again, nothing really special but this was to refresh our memories. Now we will see how numbers are internally encoded in Pharo using two's complement. We will start by understanding the 10's complement and look at 2's complement.

## 1.4 Ten's complement of a number

To fully understand 2's complement it is interesting to see how it works with decimal numbers. There is no obvious usage for 10's complement but here the point we want to show is that a complement is the replacement of addition with subtraction (*i.e.*, adding the complement of A to B is equivalent to subtracting A from B).

The 10's complement of a positive decimal integer  $n$  is 10 to the power of  $k + 1$ , minus  $n$ , where  $k$  is the number of digits in the decimal representation of  $n$ .  $\text{Complement}_{10}(n) = 10^{k+1} - n$ . For example  $\text{Complement}_{10}(8) = 10^1 - 8$ ,  $\text{Complement}_{10}(1968) = 10^5 - 1968 = 8032$

It can be calculated in the following way:

1. replace each digit  $d$  of the number by  $9 - d$  and
2. add one to the resulting number.

**Examples.** The 10's complement of 1968 is  $9 - 1, 9 - 9, 9 - 6, 9 - 8 + 1$  *i.e.*,  $8031 + 1$  *i.e.*, 8032. Using the rule two we compute  $9 - 1, 9 - 9, 9 - 6, 10 - 8$  *i.e.*, 8032. So our 10's complement is 8032. Indeed  $1968 + 8032 = 10000 = 10^5$ . Therefore it correctly follows the definition above: 8032 is the result of  $10000 - 1968$ .

The 10's complement of 190680 is then  $9 - 1, 9 - 9, 9 - 0, 9 - 6, 9 - 8, 9 - 0 + 1$  *i.e.*, 809319 + 1 *i.e.*, 809320. Let's verify:  $190680 + 809320 = 1000000$ .

To compute the 10's complement of a number, it is enough to perform  $9-d$  for each digit and add one to the result.

Some books propose another equivalent way to compute the 10's complement. Computer scientists will probably prefer the first way since it is more regular and adding 1 is cheaper than making more tests.

1. All the zeros at the right-hand end of the number remain as zeros.
2. The rightmost non-zero digit  $d$  of the number is replaced by  $10 - d$ .
3. Each other digit  $d$  is replaced by  $9 - d$ .

### Subtraction at work

The key point of complement techniques is to convert subtractions into additions. Let's check that.

**Examples.** Suppose we want to perform the subtraction  $8 - 3 = 5$ . We will transform such subtraction into an addition using the 10's complement. The 10's complement of 3 is  $9 - 3 + 1 = 7$ . We add 7 to 8 and get 15. We drop the carry we obtained when from the addition and we obtain 5. In fact, the idea is that  $8 - 3 = 8 - (10 - 7) = 8 + 7 - 10 = 15 - 10 = 5$ .

Now let's compute  $98 - 60$ . The 10's complement of 60 is  $9 - 6, 9 - 0$  i.e., 39 + 1 i.e., 40.  $98 - 60 = 98 + 40 - 100 = 138 - 100 = 38$ . Therefore we could say that  $98 - 60 = 98 + 40 = (1)38$  and we drop the carry.

Now to perform  $-98 + 60$  we compute the 10's complement of 98, then the sum, then the 10's complement of the sum and negate. i.e.,  $-98 + 60$  becomes  $2 + 60 = 62$ , 62 10's complement is 38 therefore  $-98 + 60 = -38$ .

**Another look at it.** Replacing a number by its 10's complement is based on  $a - b = a - (10 - c)$  where  $10 = b + c$ . Imagine that we want to perform the following expression  $190680 - 109237$  which is equals to 81443. The 10's complement takes advantage of the fact that 109237 is also  $999999 - 890762$  or  $1000000 - 890763$  where 890763 is the 10's complement of 109237.

```
109237 = 999999 - 890762
109237 = 999999 - 890762 (+ 1 - 1)
109237 = 1000000 - 890762 - 1
```

Now the first subtraction is expressed as:

```
190680 - 109237
= 190680 - (1000000 - 890762 - 1)
= 190680 - 1000000 + 890762 + 1
= 190680 + 890762 + 1 - 1000000
= 1081443 - 1000000
= 81443
```

## 1.5 Negative Numbers

To know the value of a positive number it is simple: we just sum all the powers of 2 given by the binary representation as explained at the beginning of this Chapter. Now getting the value of a negative number is quite simple: we do the same except that we count the *sign bit* as negative, all the other ones as positive. The sign bit is the most significant bit i.e., the bit that represents the largest value (see Figure 1.5). For example, on 8 bit representation it will be the one associated with the weight  $2^7$ .

Let us illustrate that:  $-2$  is represented on 8 a bit encoding as: 1111 1110. To get the value out of the bit representation is simple we sum:  $-2^7 + 2^6 +$



most significant bit								
0	1	1	1	1	1	1	1	127
0	0	0	0	0	0	1	0	2
0	0	0	0	0	0	0	1	1
1	1	1	1	1	1	1	1	-1
1	1	1	1	1	1	1	0	-2
-2 <sup>7</sup>	2 <sup>6</sup>	2 <sup>5</sup>	2 <sup>4</sup>	2 <sup>3</sup>	2 <sup>2</sup>	2 <sup>1</sup>	0	-2
1	0	0	0	0	0	0	1	-127
1	0	0	0	0	0	0	0	-128

Figure 1.5: Negative numbers on 8 bits.

$2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 0 * 2^0$ , i.e.,  $-128 + 64 + 32 + 16 + 8 + 4 + 2$  and we get  $-2$ .

$-69$  is represented on 8 a bit encoding as: 1011 1011. To get the value out of the bit representation is simple we sum:  $-2^7 + 0 * 2^6 + 2^5 + 2^4 + 2^3 + 0 * 2^2 + 2^1 + 2^0$ , i.e.,  $-128 + 32 + 16 + 8 + 2 + 1$  and we get  $-69$ .

Following the same principle, check that the value of  $-1$  is the one described in Figure 1.5.

Let us count a bit: on a 8 bit representation we can then encode 0 to 255 positive integers or  $-128$  to  $64 + 32 + 16 + 8 + 4 + 2 + 1$  127. In fact we can encode from  $-1 * 2^7$  to  $2^7 - 1$ . More generally on  $N$  bits we can encode  $-1 * 2^{N-1}$  to  $2^{N-1} - 1$  integer values.

## 1.6 Two's complement of a number

Now we have all the pieces of the puzzle: we know how we can encode positive and negative numbers, we know how to use complement to turn subtraction into addition. Let us see how the two's complement is used to negate numbers and perform subtraction.

The two's complement is a common method to represent signed integers. The advantages are that addition and subtraction are implemented without having to check the sign of the operands and two's complement has only one representation for zero (avoiding negative zero). Adding numbers of different sign encoded using two's complement does not require any special processing: the sign of the result is determined automatically. The two's complement of a positive number represents the negative form of that number.

What the 10's complement shows us that it is computed by taking the difference of each digit with the largest number available in the base system, 9 in decimal base and adding one. Now in the case of binary, the largest number is one. Due to the fact that  $1 - 0 = 1$  and  $1 - 1 = 0$ , computing the *two's complement of number is exactly the same as flipping 1's to 0's and vice versa and adding 1*.

Let's look at the number 2. 2 is encoded on 8 bits as 00000010 and -2 as 11111110 as shown in Figure 1.6. 00000010 flipped is: 11111101 and we add one: so we get 11111110. The difference between -2 (11111110) and 126 (01111110) is given by the most significant bit which conveys the integer sign.

most significant bit								
0	1	1	1	1	1	1	1	127
0	1	1	1	1	1	1	0	126
0	0	0	0	0	0	1	0	2
0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	-1
1	1	1	1	1	1	1	0	-2
1	0	0	0	0	0	0	1	-127
1	0	0	0	0	0	0	0	-128

Figure 1.6: Overview of two's complement on 8 bits.

Try the following expressions in Pharo and experiment with them. We compute the direct inversion (bitwise NOT) and add one. Remember a bitwise NOT is turning any 1s into 0s and conversely. You can use the bitString message to obtain the bit representation of a number. If you check the length of the printed results, you will notice that they are 31 bits long instead of 32. This is because in Pharo and most Smalltalk implementations, small integers are specially encoded and this encoding requires one bit.

```
2 bitString
'0000000000000000000000000000010'
2 bitInvert bitString
'1111111111111111111111111111101'
(2 bitInvert + 1) bitString
'1111111111111111111111111111110'
-2 bitString
'1111111111111111111111111111110'
```

Note that the two's complement of a negative number is the corresponding positive value as shown by the following expressions: -2 two complement is 2. First we compute the direct inversion (bitwise NOT) and add one.

```
-2 bitString
'11111111111111111111111111111110'

-2 bitInvert bitString
'00000000000000000000000000000001'

(-2 bitInvert + 1) bitString
'00000000000000000000000000000010'

2 bitString
'00000000000000000000000000000010'
```

As you see negating a number and computing the two's complement gives the same binary representation.

```
(2r101 bitInvert + 1) bitString
returns '11111111111111111111111111111011'
2r101 negated bitString
returns '11111111111111111111111111111011'
```

There is one exception. On a given number of bits, let's say 8 bits as in Figure 1.6, we compute the negative of a number but computing its two's complement (flipping all the bits and adding 1), except for the most negative number. On a 8 bits representation, the most negative number is -128 (1000 0000), inverting it is (0111 1111), and adding one results in itself (1000 0000). We cannot encode 128 on 8 bits signed convention. Here the carry is "eaten" by the sign bit.

**Subtracting.** To subtract a number to another one, we just add the second number's two complement to the first one.

When we want to compute  $110110 - 101$ , we will compute the 2's complement and add it. So we will add 110110 and 111011, and get 110001. This is correct:  $54 - 5 = 49$ .

```
110110 - 101
 110110
+ 111011
-----
110001
```

Let us test this in Pharo

```
(2r110110 - 2r101) bitString
```



```
(SmallInteger maxVal negated - 1)
returns -1073741824
(SmallInteger maxVal negated - 1) class
returns SmallInteger
```

**Understanding some methods.** Now you should be able to understand the implementation of `SmallInteger>>bitInvert` method.

```
SmallInteger>>bitInvert
"Answer an Integer whose bits are the logical negation of the receiver's bits.
Numbers are interpreted as having 2's-complement representation."

↑ -1 - self.
```

```
2 bitString
returns '00000000000000000000000000000010'
2 bitInvert.
returns '11111111111111111111111111111101'

-1 bitString
returns '11111111111111111111111111111111'
2 negated (two complement)
returns '11111111111111111111111111111110'
```

## 1.7 SmallIntegers in Pharo

Smalltalk small integers use a two's complement arithmetic on 31 bits. A  $N$ -bit two's-complement numeral system can represent every integer in the range  $-1 * 2^{N-1}$  to  $2^{N-1} - 1$ . So for 31 bits Smalltalk systems small integers values are the range -1 073 741 824 to 1 073 741 823. Remember in Smalltalk integers are special objects and this marking requires one bit, therefore on 32 bits we have 31 bits for small signed integers. Of course since we also have automatic coercion this is not really a concern for the end programmer. Here we take a language implementation perspective. Let's check that a bit (this is the occasion to say it).

If you want to know the number of bits used to represent a `SmallInteger`, just evaluate:

```
SmallInteger maxVal highBit + 1
returns 31
```

`SmallInteger maxVal highBit` tells the highest bit which can be used to represent a positive `SmallInteger`, and  $+ 1$  accounts for the sign bit of the `SmallInteger` (0 for positive, 1 for negative).

Let us explore a bit.

```
2 raisedTo: 29
  returns 536870912

536870912 class
  returns SmallInteger

2 raisedTo: 30
  returns 1073741824

1073741824 class
  returns LargePositiveInteger

(1073741824 - 1) class
  returns SmallInteger

-1073741824 class
  returns SmallInteger

2 class maxVal
  returns 1073741823

-1 * (2 raisedTo: (31-1))
  returns -1073741824

(2 raisedTo: 30) - 1
  returns 1073741823

(2 raisedTo: 30) - 1 = SmallInteger maxVal
  returns true
```

## 1.8 Hexadecimal

We cannot finish this Chapter without talking about hexadecimal. In Smalltalk, the same syntax than for binary is used for hexadecimal. `16rF` indicates that F is encoded in 16 base.

We can get the hexadecimal equivalent of a number using the message `hex`. Using the message `printStringHex` we get the number printed in hexadecimal without the radix notation.

```
15 hex
  returns '16rF'

15 printStringHex
  returns 'F'
```

```
16rF printIt
returns 15
```

The following snippet lists some equivalence between a number and its hexadecimal equivalent.

```
{(1->'16r1'). (2->'16r2'). (3->'16r3'). (4->'16r4'). (5->'16r5'). (6->'16r6'). (7->'16r7').
(8->'16r8'). (9->'16r9'). (10->'16rA'). (11->'16rB'). (12->'16rC'). (13->'16rD'). (14
->'16rE'). (15->'16rF')}
```

When doing bit manipulation it is often shorter to use an hexadecimal notation over a binary one. Even if for bitAnd: the binary notation may be more readable

```
16rF printStringBase: 2
returns '1111'
2r00101001101 bitAnd: 2r1111
returns 2r1101
2r00101001101 bitAnd: 16rF
returns 2r1101
```

## Information subpart extraction

Often numbers are used to encode in a compact manner multiple information: we could imagine to use 1 bit for gender, 6 bits for age.... The resulting number would be meaningless but information extracted from it is meaningful. Now we are armed to select subpart of number to get such information.

```
2r1100010100000000
returns 50432
```

Imagine that only 4 bits interest us starting at the 8th bit. To get the encoded value, we shift out the 8 first ones and clear all the others after the next 4: so we bit shift and bit and as follows:

```
(2r1100010100000000 >> 8)
returns 2r11000101
returns 197

(2r1100010100000000 >> 8) bitAnd: 16rF
returns 2r0101
returns 5

2r11000101 bitAnd: 2r1111
returns 2r0101
returns 5
```

## 1.9 Conclusion

Smalltalk uses two's complement encoding for its internal number representation and supports bit manipulation of their internal representation. This is useful when we want to speed up algorithms using simple encoding. We have reviewed the following points:

**Alex** ► *I inserted the following* ◄

- Numerical values use complement to encode their negative value.
- Shifting a bit to the left multiple is equivalent to multiply it by 2, modulo the maximum value of its encoding size.
- On the opposite, shifting a bit to the right divides it by 2.
- Bits operations can be performed on any numerical values.
- Complement are useful to turn an addition into a subtraction, thus simplifying the operation.
- SmallInteger are coded on 31 bits on Pharo.