Proving $\lim_{x \to a} \sqrt{x} = \sqrt{a}$

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1 Introduction

I have spent two hours of my life on this proof. In this proof we are going to prove

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}$$

Throughout this proof I would like you to keep in mind that this has been written by a high schooler who has absolutely no idea what a mathematical proof should look like. This solution for this problem was inspired by question 37, chapter 2.4 of *Calculus: Early Transcendentals*. I have poured blood, sweat, and tears into figuring it out, exploding my brain multiple times in the process.

2 Scratchwork

The epsilon-delta definition for our limit goes as follows:

$$\forall \epsilon > 0 \quad \exists \delta \quad \text{such that}$$
$$0 < |x - a| < \delta \Longrightarrow |\sqrt{x} - \sqrt{a}| < \epsilon$$

Before we begin we will assume two things:

- $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}$
- a is a constant and shall be treated like any other number.

Let us begin by manipulating $|\sqrt{x} - \sqrt{a}| < \epsilon$

$$|\sqrt{x} - \sqrt{a}| < \epsilon \longrightarrow \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \epsilon \longrightarrow |x-a| \cdot \frac{1}{\sqrt{x}+\sqrt{a}} < \epsilon$$

In the middle of the manipulation we used our first assumption. But now we have arrived at a problem: *We cannot do anything to find ϵ in terms of δ*. If we attempt to divide the inequality by $\frac{1}{\sqrt{x}+\sqrt{a}}$ we will end up with an epsilon dependent on x.

Similar to other epsilon-delta proofs we must introduce an auxiliary problem. Our goal is to find a constant C such that

$$|\sqrt{x} - \sqrt{a}| = |x-a| \cdot \frac{1}{\sqrt{x}+\sqrt{a}} < \delta C = \epsilon$$

In order to find C we must first place an upper bound on δ. The upper bound we will choose is $\frac{a}{2}$. Note that the number we have chosen is rather arbitrary and we may choose any number in the interval $(0, a)$. The reason for selecting a number within this interval is clever, you will find out why soon. We are now left with $|x - a| < \frac{a}{2}$. Let us manipulate the expression further.

$$|x-a| < \frac{a}{2} \longrightarrow -\frac{a}{2} < x - a < \frac{a}{2}$$
$$\longrightarrow a - \frac{a}{2} < x < a + \frac{a}{2}$$
$$\longrightarrow \sqrt{a - \frac{a}{2}} < \sqrt{x} < \sqrt{a + \frac{a}{2}}$$

Let's step aside from the proof for a brief moment and present you with an inequality:

$$\text{if } a > b \text{ then}$$

$$\frac{1}{a+c} < \frac{1}{b+c}$$

You may have noticed that a very similar pattern has emerged in our inequalities.

$$\text{If } \sqrt{x} > \sqrt{a - \frac{a}{2}} \text{ then}$$

$$\frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}}$$

Here is where the decision to select a number from $(0, a)$ becomes apparent. Usually in an epsilon-delta proof we would have chosen 1 as the upper bound for our δ. However if we had done so then we wouldn't be able to find the limits for numbers $0 < n < 1$ since $\sqrt{n-1}$ isn't defined for the purposes of this proof.

Now that this inequality is true we can simply multiply both sides of the inequality by $|x - a|$.

$$|x-a| \cdot \frac{1}{\sqrt{x}+\sqrt{a}} < |x-a| \cdot \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}}$$

Considering $|x - a| < \delta$ the following inequality must also be true.

$$|x-a| \cdot \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}} < \delta \cdot \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}}$$

If we set $\delta = \min\{\frac{a}{2}, (\sqrt{a-\frac{a}{2}}+\sqrt{a})\epsilon\}$. Because of this if $\delta = \frac{a}{2}$ then $\frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}}$. The same statement holds true if $\delta = \sqrt{a-\frac{a}{2}}+\sqrt{a}$ because $\sqrt{a-\frac{a}{2}}+\sqrt{a} < \frac{a}{2}$ due to our bound. If this is still not clear let me present the following:

$$\text{If } 0 < b < c < a \text{ and } a \text{ is our upper bound then}$$

$$a - c > 0 \Longrightarrow a - b > 0$$

The exact same thing is occurring. a is our upper bound, $\frac{a}{2} = c$ and $\sqrt{a-\frac{a}{2}}+\sqrt{a} = b$. We are now done with our scratch work and may proceed to write a formal proof.

3 Writing a formal proof

The epsilon-delta definition of our limit states

$$\forall \epsilon > 0 \quad \exists \delta \quad \text{such that}$$

$$0 < |x-a| < \delta \implies |\sqrt{x} - \sqrt{a}| < \epsilon$$

Let $\epsilon > 0$. Choose $\delta = \min\{\frac{a}{2}, (\sqrt{a - \frac{a}{2}} + \sqrt{a})\epsilon\}$.

$$|x-a| \implies \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \implies |x-a| \cdot \frac{1}{\sqrt{x}+\sqrt{a}}$$

If $\delta = \frac{a}{2}$ then $\sqrt{a - \frac{a}{2}} < \sqrt{x}$. If $\delta = (\sqrt{a - \frac{a}{2}} + \sqrt{a})\epsilon$, $\sqrt{a - \frac{a}{2}} < \sqrt{x}$ still holds true because $(\sqrt{a - \frac{a}{2}} + \sqrt{a})\epsilon < \frac{a}{2}$ due to our bound. As a consequence of these inequalities and one of our assumptions, the following must be true.

$$\frac{1}{\sqrt{x}+\sqrt{a}} < \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}}$$

Multiplying both sides of the inequality by $|x-a|$ we get

$$|x-a| \cdot \frac{1}{\sqrt{x}+\sqrt{a}} < |x-a| \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}}$$

If we look back at the epsilon-delta definition of our limit we see that $|x-a| < \delta$. Therefore

$$|x-a| \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}} < \delta \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}}$$

Substituting δ by $(\sqrt{a - \frac{a}{2}} + \sqrt{a})\epsilon$ will have the following effect:

$$|x-a| \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}} < \delta \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}}$$

$$|x-a| \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}} < ((\sqrt{a - \frac{a}{2}} + \sqrt{a})\epsilon) \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}}$$

$$|x-a| \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}} < ((\cancel{\sqrt{a - \frac{a}{2}} + \sqrt{a}})\epsilon) \frac{1}{\cancel{\sqrt{a - \frac{a}{2}} + \sqrt{a}}}$$

$$|x-a| \cdot \frac{1}{\sqrt{a - \frac{a}{2}} + \sqrt{a}} < \epsilon$$

Tracing back our work we see the following

$$|x-a| = |x-a| \cdot \frac{1}{\sqrt{x}+\sqrt{a}}$$

$$< |x-a| \cdot \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}}$$

$$< \delta \cdot \frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}} = ((\sqrt{a-\frac{a}{2}}+\sqrt{a})\epsilon)\frac{1}{\sqrt{a-\frac{a}{2}}+\sqrt{a}} = \epsilon$$

Removing everything in between we have proven

$$|x-a| < \epsilon$$

By proving the epsilon-delta definition for our limit we have shown that

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}$$