Proving that the limit of a function does not exist,

Squirrelcoding*

Department of Mathematics, University of Softsquirrel

Dated: October 10, 2022

*Github: `Squirrelcoding`; Author

This document contains the proof that $\lim_{x \to 0} f(x)$ does not exist. This proof will be accomplished through the epsilon-delta definition of a limit. For the sake of this proof two things shall be assumed:

- $\epsilon < 1$
- $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

As a refresher the epsilon-delta definition of a limit for our $f(x)$ is defined as follows:

$$\forall \epsilon > 0 \quad \exists \delta \quad \text{such that}$$
$$0 < |x| < \delta \Longrightarrow |f(x) - L| < \epsilon < 1$$

Our limit has three possible cases:

- $\lim_{x \to 0} f(x) = 0$
- $\lim_{x \to 0} f(x) = 1$
- The limit does not exist.

In our first case we attempt to prove

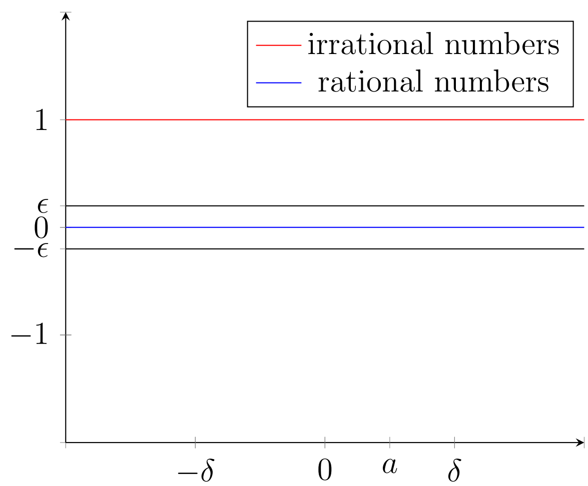
$$\lim_{x \to 0} f(x) = 0$$

Which according to the epsilon-delta definition of a limit states that

$$\forall \epsilon > 0 \quad \exists \delta \quad \text{such that}$$

$$0 < |x| < \delta \Longrightarrow |f(x)| < \epsilon < 1$$

If we have an irrational number a, we can divide it by a sufficiently large integer k such that $\frac{a}{k}$ satisfies the condition $0 < |x| < \delta$. Therefore it must satisfy the condition $|f(x) - L| < \epsilon$ which itself evaluates to $1 - 0 = 1$, since $L = 0$ and $f(\frac{a}{k}) = 1$ in this first case. Our result is *not* less than our epsilon. In the beginning of this proof we had assumed that $\epsilon < 1$. We can visualize this below. Our irrational number has ended up on the red line, when the numbers near 0 *should* be on the blue line.



Please note that although it appears that all numbers map to both lines, this is not the case. In this situation a is mapping onto the red line.

From this we can safely conclude

$$\lim_{x \to 0} f(x) \neq 0$$

Since we have found a number which does not satisfy the epsilon-delta definition for this limit.

A similar argument can be made for

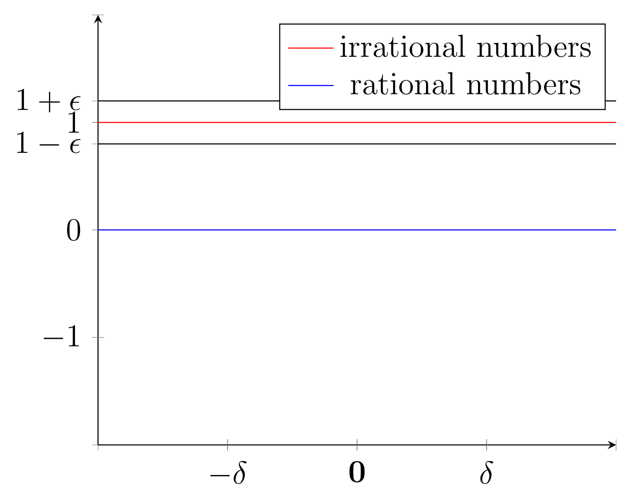
$$\lim_{x \to 0} f(x) = 1$$

We begin by re-stating the epsilon-delta definition for our function, this time L being equal to 1.

$$\forall \epsilon > 0 \quad \exists \delta \quad \text{such that}$$

$$0 < |x| < \delta \Longrightarrow |f(x) - 1| < \epsilon < 1$$

If we have a rational number n, we can divide it by a sufficiently large integer k such that $\frac{n}{k}$ satisfies $0 < |x| < \delta$. Because of this, the statement must satisfy $|f(x) - 1| < \epsilon$. If we plug in the numbers we get $|0 - 1| = |-1| = |1| < \epsilon$. But we had assumed $\epsilon < 1$ in the beginning of the proof. Therefore this statement is false. The situation can be visualized below.



Again take note that although it appears that all numbers map to both lines, this is not the case. In this situation $\frac{n}{k}$ is mapping onto the blue line.

$\frac{n}{k}$ lies on the blue line when it *should* lie on the red line. Therefore

$$\lim_{x \to 0} f(x) \neq 1$$

Because we have found yet another number which does not satisfy the epsilon-delta definition of the above limit.

In the previous two cases we have shown that it is possible for an irrational number a to exist within the interval $(-\delta, \delta)$. We have also shown that it is possible for a rational number $\frac{n}{\delta}$ to lie within the same interval. This is true for all numbers in the interval. From this we can conclude that an infinite amount of rational and irrational numbers exist within the interval $(-\delta, \delta)$. For every rational number in the interval, an even smaller irrational number exists, and vice versa. Because of this, it is impossible to find $\lim_{x \to 0} f(x)$ as the output of the function keeps changing. Therefore, by contradiction, we have proven that $\lim_{x \to 0} f(x)$ does not exist.