St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MDTS 4113/SEM I

Module I

Linear Algebra

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Gaussian Elimination

Consider a system of linear equations Ax=b

Forward Elimination Phase:

1. Search the first column of [A|b] from the top to the bottom for the first non-zero entry, and then if necessary, the second column (the case where all the coefficients corresponding to the first variable are zero), and then the third column, and so on. The entry thus found is called the current pivot.

2. Interchange, if necessary, the row containing the current pivot with the first row.

3. Keeping the row containing the pivot (that is, the first row) untouched, subtract appropriate multiples of the first row from all the other rows to obtain all zeroes below the current pivot in its column.

4. Repeat the preceding steps on the submatrix consisting of all those elements which are below and to the right of the current pivot.

5. Stop when no further pivot can be found.

The m × n coefficient matrix A of the linear system Ax = b is thus reduced to an (m × n) row ***echelon matrix U*** and the augmented matrix [A|b] is reduced to: [U|c] =

$$\left[\begin{array}{cccccccccccc|c}
0 & \dots & p_1 & * & * & * & * & * & * & * & \dots & * & c_1 \\
0 & \dots & 0 & \dots & p_2 & * & * & * & * & * & \dots & * & c_2 \\
0 & \dots & 0 & 0 & 0 & \dots & p_3 & * & * & * & \dots & * & c_3 \\
\vdots & & \vdots & \vdots & \vdots & 0 & 0 & \dots & \vdots & \vdots & & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & p_r & * & \dots & * & c_r \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_{r+1} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_m
\end{array}\right]$$

The entries denoted by $*$ and the ci 's are real numbers; they may or may not be zero. The pi 's denote the pivots; they are non-zero.

Gaussian elimination reduces the original system Ax=b to an upper triangular system Ux=c.

X is solved from Ux=c by back substitution.

If the system is reduced further to [E|x], where ***E is row reduced echelon matrix***, it is called ***Gauss-Jordan elimination***.

$$[E|c] = \left[\begin{array}{cccccccccccc|c} 0 & \dots & 1 & * & 0 & * & 0 & * & 0 & * & \dots & * & c_1 \\ 0 & \dots & 0 & \dots & 1 & * & 0 & * & 0 & * & \dots & * & c_2 \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & * & 0 & * & \dots & * & c_3 \\ \vdots & & \vdots & \vdots & \vdots & 0 & 0 & \dots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & \dots & * & c_r \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_{r+1} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_m \end{array}\right]$$

The system Ex=c can also be solved by back substitution.

Ex1: Solve the following system of linear equations by using the Gauss elimination method:

$$5x_1 + 6x_2 = 7$$
$$3x_1 + 4x_2 = 5$$

Solution: The system of linear equations has the following augmented matrix

$$\left(\begin{array}{cc|c} 5 & 6 & 7 \\ 3 & 4 & 5 \end{array}\right) \xrightarrow{\frac{1}{5}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 3 & 4 & 5 \end{array}\right) \xrightarrow{R_2 - 3R_1 \to R_2} \left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 0 & \frac{2}{5} & \frac{4}{5} \end{array}\right)$$

$$\xrightarrow{\frac{5}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & \frac{6}{5} & \frac{7}{5} \\ 0 & 1 & 2 \end{array}\right)$$

The last matrix is in row - echelon form. The corresponding reduced system is:

$$x_1 + \frac{6}{5}x_2 = \frac{7}{5} \quad \dots (1)$$

$$x_2 = 2 \quad \dots (2)$$

Substitute the value of x_2 in equation (1) ,we get the solution.

Ex2.

$$\begin{aligned} x - 2y + z &= 5 \\ 2x - 5y + 4z &= -3 \\ x - 4y + 6z &= 10. \end{aligned}$$

The augmented matrix for this system is the 3×4 matrix $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 2 & -5 & 4 & -3 \\ 1 & -4 & 6 & 10 \end{array}\right)$

First pivot

$R_2 \leftarrow R_2 + (-2)\, R_1$ $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & -2 & 5 & 5 \end{array}\right)$

2nd pivot

$R_3 \leftarrow R_3 + (-2)\, R_2$ $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & 0 & 1 & 31 \end{array}\right)$

The final matrix represents the linear system:

$$\begin{aligned} x - 2y + z &= 5 \\ -y + 2z &= -13 \\ z &= 31 \end{aligned}$$

by **backward substitution**

z = 31;

y = 13 + 2z = 75;

x = 5 + 2y − z = 124

- We can continue Gaussian elimination to simplify the augmented matrix further.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \end{array}\right)$$

- Simple augmented matrix gives a quick solution. This is called the **Gauss-Jordan Process**.
- Here, we ensure that all the pivots are equal to 1 and moreover all the other entries in the column containing the pivot are 0.
- In other words, we have 0's not only below but also above the pivot.

Gaussian Elimination Method consists of reducing the augmented matrix to a simpler matrix from which solutions can be easily found. This reduction is by means of elementary row operations.

Basic observation: Operations of three types on these equations do not alter the solutions- [**proof to be given later**]

1. Interchanging two equations.

2. Multiplying all the terms of an equation by a nonzero scalar.

3. Adding to one equation a multiple of another equation.

Now the above three operations on the equations in the linear system correspond to the following operations on the rows of the augmented matrix:

(i) interchanging two rows, $[R_{ij}]$

(ii) multiply a row by a nonzero scalar, $[c\ R_i]$

(iii) adding a multiple of one row to another. $[R_i + c\ R_j]$

These are called ***elementary row operations.***

Assignment:

Solve the following system of linear equations by using the Gauss elimination method:

1.

$$\begin{aligned} 4y + 2z &= 1 \\ 2x + 3y + 5z &= 0 \\ 3x + y + z &= 11 \end{aligned}$$

2.

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$