Practical Problem Set - 4 (DB)

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Question 1: Let S be the 2-dimensional subspace of R^3 spanned by the orthogonal vectors $v_1 = (1,2,1)$ and $v_2 = (1,-1,1)$. Write the vector $v = (-2,2,2)$ as the sum of a vector in S and a vector orthogonal to S.

```
v11 = c(1, 2, 1)
v12 = c(1, -1, 1)
v13 = c(-2, 2, 2)
v14 = v11*dot(v13, v11)/dot(v11, v11) + v12*dot(v13, v12)/dot(v12, v12)
v14

## [1] 0 2 0

v15 = v13 - v14
v15

## [1] -2  0  2
```

Here, the vector $(-2,2,2) = (0,2,0) + (-2,0,2)$; where $(0,2,0) \in S$ and $(-2,0,2) \perp S$.

Question 2: Let P be the subspace of R^3 specified by the equation $2x + y = 2z = 0$. Find the distance between P and the point $q = (3,2,1)$.

The basis for the system $2x + y = 2z = 0$ is $(-0.5,1,0)$. Hence, we need to calculate the orthogonal projection of the point $(3,2,1)$ on the line spanned by the vector $(-0.5,1,0)$.

```
v21 = c(-0.5, 1, 0)
v22 = c(3, 2, 1)
v22 - v21*dot(v21, v22)/dot(v21, v21)

## [1] 3.2 1.6 1.0

Norm(v22)

## [1] 3.741657
```

The required distance is 3.74 units.

Question 3: Let H be the 3-dimensional subspace of R^4 with basis $B = \{v_1 = (0,1,-1,0), v_2 = (0,1,0,1), v_3 = (1,-1,0,0)\}$.

```
v31 = c(0, 1, -1, 0)
v32 = c(0, 1, 0, 1)
v33 = c(1, -1, 0, 0)
```

```r
M31 = cbind(v31, v32, v33)
M31
```

```
##      v31 v32 v33
## [1,]   0   0   1
## [2,]   1   1  -1
## [3,]  -1   0   0
## [4,]   0   1   0
```

3.(i) Find the orthogonal basis for H.

```r
M32 = GramSchmidt(M31, normalize = F, verbose = F, tol =
sqrt(.Machine$double.eps))
M32
```

```
##      v31 v32        v33
## [1,]   0 0.0  1.0000000
## [2,]   1 0.5 -0.3333333
## [3,]  -1 0.5 -0.3333333
## [4,]   0 1.0  0.3333333
```

The columns of the matrix M_{32} above are an orthogonal basis of H.

3.(ii) Find the orthonormal basis for H.

```r
M33 =GramSchmidt(M31, normalize = T, verbose = F, tol =
sqrt(.Machine$double.eps))
M33
```

```
##            [,1]      [,2]       [,3]
## [1,]  0.0000000 0.0000000  0.8660254
## [2,]  0.7071068 0.4082483 -0.2886751
## [3,] -0.7071068 0.4082483 -0.2886751
## [4,]  0.0000000 0.8164966  0.2886751
```

The columns of the matrix M_{33} above are an orthonormal basis of H.

3.(iii) What are the components of the vector $x = (1,1,-1,1)$ relative to this orthonormal basis?

```r
v33 = c(1, 1, -1, 1)
c31 = dot(v33, M33[, 1])
c32 = dot(v33, M33[, 2])
c33 = dot(v33, M33[, 3])
v34 = c(c31, c32, c33)
v34
```

```
## [1] 1.4142136 0.8164966 1.1547005
```

```r
#check if v34 is in H
rref(cbind(M33, v33))
```

```
##                  v33
## [1,] 1 0 0 1.4142136
## [2,] 0 1 0 0.8164966
```

```
## [3,] 0 0 1 1.1547005
## [4,] 0 0 0 0.0000000
```

As we can see, v_{33} can be spanned by first 3 column vectors of the augmented matrix above.

The components of the vector $x = (1,1,-1,1)$ relative to this orthonormal basis represented by the columns of M_{33} are $1.414,\ 0.816,\ 1.155$ respectively.

```
#verification
v33

## [1]  1  1 -1  1

t(M33 %*% matrix(v34))

##      [,1] [,2] [,3] [,4]
## [1,]    1    1   -1    1
```

3.(iv) What happens if you attempt to find the components of the vector $y = (1,1,1,1)$ relative to the orthonormal basis?

```
v35 = c(1, 1, 1, 1)
c35 = dot(v35, M33[, 1])
c36 = dot(v35, M33[, 2])
c37 = dot(v35, M33[, 3])
v36 = c(c35, c36, c37)
v36

## [1] 0.0000000 1.6329932 0.5773503

#check if v35 is in H
rref(cbind(M33, v35))

##            v35
## [1,] 1 0 0   0
## [2,] 0 1 0   0
## [3,] 0 0 1   0
## [4,] 0 0 0   1
```

As we can see, v_{35} can not be spanned by first 3 column vectors of the augmented matrix above.

```
#verification
v35

## [1] 1 1 1 1

t(M33 %*% matrix(v36))

##      [,1] [,2] [,3] [,4]
## [1,]  0.5  0.5  0.5  1.5
```

As we can see, $v_{35} \neq (0.5,0.5,0.5,1.5)$. This is because the vector v_{35} is not in H, meaning no linear combination of basis vectors of H can give $(1,1,1,1)$.

Question 4: Find the Kronecker product of the following matrices and verify that $A \otimes B \neq B \otimes A$:

4.(i) $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$.

```
M41 = matrix(c(2, 3, 0, 1), nrow = 2, byrow = T)
M42 = matrix(c(0, -1, -1, 1), nrow = 2, byrow = T)
M41%x%M42

##      [,1] [,2] [,3] [,4]
## [1,]    0   -2    0   -3
## [2,]   -2    2   -3    3
## [3,]    0    0    0   -1
## [4,]    0    0   -1    1

M42%x%M41

##      [,1] [,2] [,3] [,4]
## [1,]    0    0   -2   -3
## [2,]    0    0    0   -1
## [3,]   -2   -3    2    3
## [4,]    0   -1    0    1
```

4.(ii) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 2 \\ 6 & 7 & 3 \end{bmatrix}$.

```
M43 = matrix(c(1, 2, 3, 4, 1, 0), nrow = 3, byrow = T)
M44 = matrix(c(0, 5, 2, 6, 7, 3), nrow = 2, byrow = T)
M43%x%M44

##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    0    5    2    0   10    4
## [2,]    6    7    3   12   14    6
## [3,]    0   15    6    0   20    8
## [4,]   18   21    9   24   28   12
## [5,]    0    5    2    0    0    0
## [6,]    6    7    3    0    0    0

M44%x%M43

##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    0    0    5   10    2    4
## [2,]    0    0   15   20    6    8
## [3,]    0    0    5    0    2    0
## [4,]    6   12    7   14    3    6
## [5,]   18   24   21   28    9   12
## [6,]    6    0    7    0    3    0
```

Question 5: Let $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find $u \otimes u, u \otimes v, v \otimes u$ and $v \otimes v$. Which space do they span?

```
V51 = c(1, 0)
V52 = c(0, 1)
#u⊗u
matrix(V51)%x%matrix(V51)

##      [,1]
## [1,]    1
## [2,]    0
## [3,]    0
## [4,]    0

#u⊗v
matrix(V51)%x%matrix(V52)

##      [,1]
## [1,]    0
## [2,]    1
## [3,]    0
## [4,]    0

#v⊗u
matrix(V52)%x%matrix(V51)

##      [,1]
## [1,]    0
## [2,]    0
## [3,]    1
## [4,]    0

#v⊗v
matrix(V52)%x%matrix(V52)

##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
## [4,]    1
```

These 4 vectors together span R^4.

Question 6: Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix}$. Find $rank(A), rank(B)$ and $rank(A \otimes B)$.

```
M61 = matrix(c(2, 0, 1, 3, 1, 0), nrow = 2, byrow = T)
M62 = matrix(c(1, 2, 2, 1, 2, 0), nrow = 2, byrow = T)
#rank(A)
R(M61)
```

```
## [1] 2

#rank(B)
R(M62)

## [1] 2

#rank(A ⊗B)
R(M61 %x% M62)

## [1] 4
```