St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MSc in Data Science

Semester 1

Paper 3

(Module I)

Linear Algebra

Introduction to Matrices

Definition: A rectangular array of numbers is called a matrix.

The horizontal arrays of a matrix are called its rows and the vertical arrays are called its columns.

A matrix having m rows and n columns is said to have the order m × n. A matrix A of order m × n can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

In a compact form the above matrix is represented by A = $[a_{ij}]$, 1 ≤ i ≤ m, 1 ≤ j ≤ n, or simply $[a_{ij}]_{m\times n}$

A matrix having only one column is called a column vector; and a matrix with only one row is called a row vector.

A = $[\underline{\boldsymbol{a}}_{*1} \quad \underline{\boldsymbol{a}}_{*2} \ \cdots \underline{\boldsymbol{a}}_{*(n-1)} \quad \underline{\boldsymbol{a}}_{*n}]$: (Row) Vector of column vectors

Or

A = $\begin{pmatrix} \underline{\boldsymbol{a}}'_{1*} \\ \underline{\boldsymbol{a}}'_{2*} \\ \vdots \\ \underline{\boldsymbol{a}}'_{m*} \end{pmatrix}$ **:** (Column) Vector of the row vectors

A 1xn matrix is called a row vector and an mx1 matrix is called a column vector.

Equality of two Matrices: Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ having the same order mxn are equal if $a_{ij} = b_{ij}$ for each i = 1, 2, . . . , m and j = 1, 2, . . . , n.

Different Types of Matrices

Square Matrix: An m x n matrix A is said to be a square matrix if m = n i.e. number of rows = number of columns.

Null Matrix or Zero Matrix

A matrix in which each entry is zero is called a zero-matrix, denoted by 0. For example $O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

Diagonal Matrix

A square matrix A = $[a_{ij}]_{n\times n}$, is said to be a diagonal matrix if a_{ij} = 0 for i ≠ j.
In other words, the non-zero entries appear only on the principal diagonal.
For example, the zero matrix O_n and $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$are a few diagonal matrices.

A diagonal matrix D of order n with the diagonal entries $(d_1, d_2, \ldots, d_n)$ is denoted by D = diag$(d_1, d_2, \ldots, d_n)$. If d_i = d for all i = 1, 2, . . . ,n then the diagonal matrix D is called a *scalar matrix*.

Identity Matrix

A square matrix A = $[a_{ij}]_{n\times n}$, with $a_{ij} = \begin{cases} 1 \; for \; i = j \\ 0 \; for \; i \neq j \end{cases}$ is called the identity matrix, denoted by I_n. For example, I_2 =$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and I_3 =$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Triangular Matrix

A square matrix in which all the elements below the principal diagonal are zero is called Upper Triangular matrix.
A square matrix in which all the elements above the principal diagonal are zero is called Lower Triangular matrix.

Given a square matrix A = $[a_{ij}]_{n\times n}$,
for upper triangular matrix, a_{ij} = 0, i > j
and for lower triangular matrix, a_{ij} = 0, i < j.

Operations on Matrices

let A = [a_{ij}] and B = [b_{ij}] be are two m×n matrices, then

i. Then for any element k ∈ R, kA = [ka_{ij}] is an mxn matrix. [**Multiplying a Scalar to a Matrix**]

Example: For example, if A = $\begin{bmatrix} 4 & 1 & 2 \\ 0 & -2 & 5 \end{bmatrix}$ and k = 5,

then 5A = $\begin{bmatrix} 20 & 5 & 10 \\ 0 & -10 & 25 \end{bmatrix}$.

ii. Then the sum A + B is defined to be the matrix D = [d_{ij}] with d_{ij} = a_{ij} + b_{ij}[**Addition of Matrices**]

Note that, we define the sum of two matrices only when the order of the two matrices are same.

iii. Let A = $\left[a_{ij}\right]_{m\times n}$,be an m × n matrix and C = $\left[c_{ij}\right]_{n\times r}$,be an n × r matrix.

The product AC is a matrix Z = $\left[z_{ij}\right]_{m\times r}$of order m × r,

with zij = $\sum_{k=1}^{n} a_{ik}c_{kj}$ = $a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}$

Observe that the product AC is defined if and only if the number of columns of A = the number of rows of C.

In a square matrix, A =$\left[a_{ij}\right]_{n\times n}$, of order n, the entries a_{11}, a_{22},...., a_{nn} are called the diagonal entries and form the principal diagonal of A.

Trace of a Matrix

The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A i.e. tr(A). Thus if A = $\left[a_{ij}\right]_{n\times n}$

then tr(A) = $\sum_{i=1}^{n} a_{ii}$ = a_{11} + a_{22} +......+ a_{nn}.

For two square matrices, A and B of the same order, (a) tr (A + B) = tr (A) + tr (B).

(b) tr (AB) = tr (BA), in case both AB and BA are defined.

- Let A, B and C be matrices of order m × n, and let k, ℓ ∈ R.

Then

i. A + B = B + A (commutative).
ii. (A + B) + C = A + (B + C) (associative).
iii. 0+A = A+0 [existence of additive identity]
iv. A+(-A) = (-A)+A = 0 [existence of additive inverse]
v. k(A+B) = kA+kB
vi. $(k + \ell)A = kA + \ell A$
vii. $k(\ell A) = (k\ell)A$.

- Suppose that the matrices A, B and C are so chosen that the matrix multiplications are defined.

i. Then (AB)C = A(BC). That is, the matrix multiplication is associative.
ii. For any k ∈ R, (kA)B = k(AB) = A(kB).
iii. Then A(B + C) = AB + AC. That is, multiplication distributes over addition.
iv. If A is an n × n matrix then $AI_n = I_nA = A$. [Multiplicative identity]
v. For any square matrix A of order n and D = D = $\mathrm{diag}(d_1, d_2, \ldots, d_n)$, we have
the first row of DA is d_1 times the first row of A;
for $1 \le i \le n$, the i th row of DA is d_i times the ith row of A.

A similar statement holds for the columns of A when A is multiplied on the right by D.

- Two square matrices A and B are said to commute if AB = BA.

In general, the matrix product is not commutative.
For example, consider the following two matrices A = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$and B = $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Check whether AB=BA.

Transpose of a Matrix

The matrix obtained from any given matrix A, by interchanging its rows and columns, is called the transpose of A and is denoted by A'.
If A = $[a_{ij}]_{m\times n}$, and A' = $[b_{ij}]_{n\times m}$, then $b_{ij} = a_{ij}$, ∀ i, j.

Properties of Transpose

(i) (A')' = A
(ii) (A + B)' = A' + B', A and B being conformable for addition
(iii) (αA)' = αA', α being scalar
(iv) (AB)' = B'A', A and B being conformable for multiplication

Some More Different types of Matrices

Symmetric Matrix and Skew-Symmetric Matrices
A matrix A over R is called symmetric if A^t = A and skew-symmetric if A^t = −A.

A matrix A is said to be orthogonal if $AA^t = A^tA = I$ [in case all the columns of A are mutually orthogonal]

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$, $C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}$

A is symmetric, B is skew-symmetric and C an orthogonal matrix.

Nilpotent Matrix

The matrices A for which a positive integer k exists such that A^k = 0 are called nilpotent matrices.
The least positive integer k for which A^k = 0 is called the order of nilpotency.
Example: 1. Let A = $[a_{ij}]_{n\times n}$ be an n×n matrix with aij =$\begin{cases} 1 \; if \; i = j+1 \\ 0 \; otherwise \end{cases}$.

Then A^n = 0 and $A^l \neq 0$ for 1 ≤ ℓ ≤ n − 1.
2. $B = \begin{pmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Find the order of nilpotency.

Idempotent Matrix
Let A =$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then A^2 = A.
The matrices that satisfy the condition that A^2 = A are called idempotent matrices
Submatrix of a Matrix
A matrix obtained by deleting some of the rows and/or columns of a matrix is said to be a submatrix of the given matrix.

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & -2 & 5 \end{bmatrix}.$$

Some of the submatrices of A are: [4], $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$, $\begin{bmatrix} 4 & 2 \\ 0 & 5 \end{bmatrix}$

The matrices $\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ are not submatrices of A.

Partitioning of Matrices:
Let A be an n × m matrix and B be an m × p matrix. Suppose r < m.

Then, we can decompose the matrices A and B as A = $[P^{nxr} \ Q^{nx(m-r)}]$ and B = $\begin{bmatrix} H^{rxp} \\ K^{(m-r)xp} \end{bmatrix}$; where P has order n × r and H has order r × p.

That is, the matrices P and Q are submatrices of A and P consists of the first r columns of A and Q consists of the last m − r columns of A.

Similarly, H and K are submatrices of B and H consists of the first r rows of B and K consists of the last m − r rows of B.

Result: Let A = $[a_{ij}\ ]$ = $[P^{nxr} \ Q^{nx(m-r)}]$ and B = $[b_{ij}]$ = $\begin{bmatrix} H^{rxp} \\ K^{(m-r)xp} \end{bmatrix}$ be defined as above. Then AB = P H + QK.

It may be possible to block the matrix in such a way that a few blocks are either identity matrices or zero matrices. In this case, it may be easy to handle the matrix product using the block form.

Example: 1. A=$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$

AB = $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} . \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} (e \quad f)$

<u>Assignment:</u>

1. Let A = $(a_1, a_2, \dots, a_n)$ and B=$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. Find AB and BA
2. Find examples for the following statements.

a). Suppose that the matrix product AB is defined. Then the product BA need not be defined.
b). Suppose that the matrices A and B are square matrices of order n. Then AB and BA may or may not be equal.

3. Show that for any square matrix A, S = $\frac{1}{2}(A + A^t)$ is symmetric, T = $\frac{1}{2}(A - A^t)$ is skew-symmetric, and A = S + T.
4. Show that the product of two lower triangular matrices is a lower triangular matrix. A similar statement holds for upper triangular matrices.
5. Show that the diagonal entries of a skew-symmetric matrix are zero.