St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MDTS 4113/SEM I

Module I

Linear Algebra

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Notes

Linear Dependence and Independence of Vectors

- If one or more vectors in a set of vectors S = $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$ can be written in terms of the other vectors in the set, then S is said to be a **<u>dependent set of vectors</u>**.

Alternatively,

A set of vectors S = $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$ is said to be linearly dependent if there exists scalars $c_2, \ldots, c_n$,

not simultaneously zero, such that $c_1\underline{a}_1 + c_2\,\underline{a}_2 + \ldots c_n\,\underline{a}_n = \underline{0}$.

Claim: Both the definitions are equivalent.

Proof: WLG let $\underline{a}_1$ can be written in terms of the other vectors in the set.

Then, $\underline{a}_1 = c_2\underline{a}_2 + \ldots c_n\underline{a}_n$,

$\Rightarrow$ $1.\underline{a}_1 - c_2\underline{a}_2 - \ldots - c_n\underline{a}_n = \underline{0}$

there exists scalars $c_2, \ldots, c_n$, *not simultaneously zero, such that* $c_1\underline{a}_1 + c_2\,\underline{a}_2 + \ldots c_n\,\underline{a}_n = \underline{0}$

Let WLG $c_n \neq 0$

$\underline{a}_1 = \frac{c_2}{c_1}\underline{a}_2 + \ldots \frac{c_n}{c_1}\underline{a}_n$

- If no vector in a set of vectors S = $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$ can be written in terms of the other vectors in the set, then S is said to be a **Linearly independent set of vectors.**

Alternatively,

A set of vectors S = $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_n\}$ is said to be linearly independent if the

only scalars $\boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ for which $\boldsymbol{c}_1\underline{\boldsymbol{a}}_1 + \boldsymbol{c}_2\,\underline{\boldsymbol{a}}_2 + \ldots \boldsymbol{c}_n\,\underline{\boldsymbol{a}}_n = \underline{\mathbf{0}}$ holds is

$\boldsymbol{c}_1 = \boldsymbol{c}_2 = \cdots = \boldsymbol{c}_n = 0$

Claim: Both the definitions are equivalent [check!]

Q. What can you say about a set containing a single vector $\underline{a}$?

Independent if $\underline{a} \neq \underline{0}$

Dependent if $\underline{a} = \underline{0}$

Q. What can you say about a set of vectors containing a null vector?

A null vector is not linearly independent of any other vector or a set of vectors as

$\underline{0} = 0.\underline{a}_1 + 0.\underline{a}_2 + \ldots + 0.\underline{a}_n$

Or, $1.\underline{0} \; - 0.\underline{a}_1 - 0.\underline{a}_2 - \ldots - 0.\underline{a}_n = \underline{0}$

Thus, a null vector is dependent on any other vector and presence of null vector in a set makes the set dependent.

- **A subset of a set of linearly independent vectors is linearly independent.**
- **A Super set of a set of linearly dependent vectors is linearly dependent**.
- Any set of n vectors in R^m must be dependent if n>m.
- Consider the following homogenous system of n equations in m unknowns:

$$A\underline{\lambda} = \underline{0} \Rightarrow \begin{aligned} a_{11}\lambda_1 + \cdots + a_{1m}\lambda_m &= 0, \\ a_{21}\lambda_1 + \cdots + a_{2m}\lambda_m &= 0, \\ &\vdots \\ a_{n1}\lambda_1 + \cdots + a_{nm}\lambda_m &= 0. \end{aligned} \Rightarrow \lambda_1\underline{a}_1 + \lambda_2\underline{a}_2 + \cdots + \lambda_m\underline{a}_m = \underline{0}$$

$\underline{\boldsymbol{\lambda}} = \underline{\mathbf{0}}$ is always a solution.

Existence of non-zero solution depends on whether the column vectors $\underline{\boldsymbol{a}}_1$, $\underline{\boldsymbol{a}}_2$,…, $\underline{\boldsymbol{a}}_n$ are dependent / independent.

- **To check any set of vectors $\underline{\boldsymbol{a}}_1$, $\underline{\boldsymbol{a}}_2$,…, $\underline{\boldsymbol{a}}_n$ for linear independence,**

→ form the matrix A whose k columns are the given vectors.

→ Then solve the system $A\underline{\lambda} = 0$;

the vectors are dependent if and only if there is a solution other than $\underline{\lambda} = 0$.

- **A is an m by k matrix. Suppose now that k > m.**

Then it will be impossible for A to have k independent columns.

a homogeneous system Ac = 0 with more unknowns than equations

always has solutions c ≠ 0.

Vector Spaces and Subspaces

<u>Fields</u>:

<u>A field is a set together with some operations on the objects in that set</u> which behave like ordinary addition, subtraction, multiplication, and division of numbers in the sense that they obey the nine rules of algebra listed below.

A set of real numbers F is called a field if:

1. Addition is commutative, x+y = y+x for all x and y in F.

2. Addition is associative, x + (y + z) = (x + y) + z for all x, y, and z in F.

3. There is a unique element 0 (zero) in F such that x + 0 = x, for every x in F.

4. To each x in F there corresponds a unique element (-x) in F such that x + (-x) = O.

5. Multiplication is commutative, xy = yx for all x and y in F.

6. Multiplication is associative, x(yz) = (xy)z for all x, y, and z in F.

7. There is a unique non-zero element 1 (one) in F such that x1= x, for every x in F.

8. To each nOll-zero x in F there corresponds a unique element X^{-1}(or 1/x) in F such that XX^{-1} = 1.

9. Multiplication distributes over addition; that is, x(y + z) = xy + xz, for all x, y, and z in F.

<u>Vector Space</u>

A set of vectors V, defined over a field F of scalars, is called a vector Space

if there exists a rule (or operation), called vector addition, which associates with each pair of vectors $\underline{v}$, $\underline{w}$ $\in$ V, a vector $(\underline{v}+\underline{w})\in$ V, in such a way that

(a) addition is commutative, $\underline{v}+\underline{w}=\underline{w}+\underline{v}$;

(b) addition is associative, $\underline{t}+(\underline{v}+\underline{w})=(\underline{t}+\underline{v})+\underline{w}$

(c) there is a unique vector $\underline{0}$ in V, called the zero vector, such that $\underline{v}+\underline{0}=\underline{v}$ for all $\underline{v}$ in V;

(d) for each vector $\underline{v}$ in V there is a unique vector - $\underline{v}$ in V such that $\underline{v}-\underline{v}=\underline{0}$;

Also a rule (or operation), called multiplication by scalar, which associates with each scalar c in F and vector $\underline{v}$ in V a vector C $\underline{v}$ in V, in such a way that

(a) 1 $\underline{v}$ = $\underline{v}$ for every $\underline{v}$ in V;
(b) $(c_1c_2)\,\underline{v} = c_1(c_2\underline{v})$;
(c) c$(\underline{v}+\underline{w})$= c $\underline{w}+c\,\underline{v}$
(d) $(c_1+c_2)\,\underline{v} = c_1\underline{v}+c_2\underline{v}$

Subspaces:

Let V be a vector space over the field F. A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V.

A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\underline{v}$, $\underline{w}$ in W and each scalar c in F, the vector C $\underline{v}$ +d $\underline{w}$ belongs to W.

Some Results:

- **Let V be a vector space over the field F. The intersection of any collection of subspaces of V is a subspace of V.**

  Proof. Let $\{W_\alpha\}$ be a collection of subspaces of V,
  let W= $\bigcap_\alpha W_\alpha$ be their intersection.
  Note that W is defined as the set of all elements belonging to every W_α.
  Since each W_α is a subspace, each contains the zero vector.
  Thus the zero vector is in the intersection W, and W is non-empty.
  Let α & β be vectors in W and let c be a scalar. By definition of W, both belong to each W_α, and because each W_α is a subspace, the vector (c$\alpha + \beta$) is in every W_α .
  Thus (c$\alpha + \beta$) is again in W. Hence, W is a subspace of V.

- We have seen that for 2 subspaces W_1 & W_2, $W_1 \cap W_2$ is always a subspace.
- $W_1 \cap W_2$ is the largest subspace contained in both W_1 & W_2

  Q. What can you say about $\bigcup_\alpha W_\alpha$? -→ $W_1 U W_2$ need not be a subspace.

  Q. What about **S** = W_1+ $W_2 + \cdots + W_k$**? (Linear Sum)**
  S =$\{\underline{x}$: $\underline{x}$=$\sum_{i=1}^{k} \alpha_i \ s.t \ \alpha_i \in \mathrm{W_i} \ \forall i\}$

  W_1+ W_2 is the smallest subspace containing both W_1 & W_2.
  Clearly, $\mathrm{W_1}$+ $\mathrm{W_2}$ = $\mathrm{W_2}$+ $\mathrm{W_1}$

  Example: If S & T are any 2 distinct lines through the origin in $\mathcal{R}^2$.

  Both S & T are subspaces
  S+T = $\mathcal{R}^2$

- **Let V be a vector space and S={$\underline{a}_1$, $\underline{a}_2$,…, $\underline{a}_k$} be a set of vectors ∈ V. Then the set of all possible linear combinations of these vectors is a vector subspace W.**

  Proof: Let W be the set of all possible linear combinations of the vectors in S.
  To prove W is a vector subspace, we need to show that $c\underline{x} + d\underline{y} \in \mathrm{W} \ \forall c, d \in$ R and $\forall \underline{x}, \ \underline{y} \in \mathrm{W}$

Consider any $\underline{x}$, $\underline{y} \in$ W. Then $\underline{x} = c_1\underline{a}_1 + c_2\,\underline{a}_2 + \ldots + c_k\,\underline{a}_k$ for some $c_i \in$ R and

$\underline{y} = d_1\underline{a}_1 + d_2\,\underline{a}_2 + \ldots + d_k\,\underline{a}_k$ for some $d_i \in$ R

Now, $\mathbf{c}\underline{\boldsymbol{x}} + \boldsymbol{d}\underline{\boldsymbol{y}} = \mathbf{c}(\boldsymbol{c}_1\underline{\boldsymbol{a}}_1 + \boldsymbol{c}_2\,\underline{\boldsymbol{a}}_2 + \ldots + \boldsymbol{c}_k\,\underline{\boldsymbol{a}}_k) + \mathbf{d}(\boldsymbol{d}_1\underline{\boldsymbol{a}}_1 + \boldsymbol{d}_2\,\underline{\boldsymbol{a}}_2 + \ldots + \boldsymbol{d}_k\,\underline{\boldsymbol{a}}_k)$

$= (\mathbf{c}\boldsymbol{c}_1 + \mathbf{d}\boldsymbol{d}_1)\underline{\boldsymbol{a}}_1 + (\mathbf{c}\boldsymbol{c}_2 + \mathbf{d}\boldsymbol{d}_2)\underline{\boldsymbol{a}}_2 + \cdots + (\mathbf{c}\boldsymbol{c}_k + \mathbf{d}\boldsymbol{d}_k)\underline{\boldsymbol{a}}_k \in \boldsymbol{W}$

- The set W of all linear combinations of vectors $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k$ is called the **span** of these vectors, and is written Span $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k\}$
- Let W be a set of vectors. If $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k$ are vectors such that W = Span $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k\}$ then we say $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k\}$ is a generating set for W.
- The vectors $\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k$ are the as generators or spanning set for W.
- The subspace W spanned by a set of vectors $\{\underline{\boldsymbol{a}}_1, \underline{\boldsymbol{a}}_2, \ldots, \underline{\boldsymbol{a}}_k\}$ is the intersection of all subspaces of V which contain S.

- Rows of a matrix generate a subspace called the **row space R(A).**

- **R (A)** = $\{\underline{x}: \underline{x}' = d_1\underline{\beta'}_1 + d_2\underline{\beta'}_2 + \cdots + d_m\underline{\beta'}_m\} = (\mathrm{d}_1\ \ \mathrm{d}_2\ \ \ldots.\mathrm{d}_\mathrm{m})\begin{bmatrix} \underline{\beta'}_1 \\ \underline{\beta'}_2 \\ \vdots \\ \underline{\beta'}_\mathrm{m} \end{bmatrix}$

$$= (d_1\ \ d_2\ \ldots.d_m)\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \underline{d'}\,\mathrm{A}$$

Row space is a subspace of R ^{n}

The row space of a matrix A is column space of A^T

Basis and Dimension

Basis for a Vector Space: A basis for a vector space V is a set of vectors which are independent and span the vector space V.

Every vector $\underline{x}$ in the space V is a combination of the basis vectors, because they span the space.

- **Representation of $\underline{x}$ is unique, because the basis vectors $\underline{v}_1, \ldots, \underline{v}_n$ are independent:**

There is one and only one way to write $\underline{x}$ as a combination of the basis vectors.

<u>Proof</u>: Suppose $\underline{x} = a_1\underline{v}_1 + \cdots + a_2\underline{v}_n$ and also $\underline{x} = b_1\underline{v}_1 + \cdots + b_2\underline{v}_n$.

By subtraction $(a_1 - b_1)\underline{v}_1 + \cdots + (a_n - b_n)\underline{v}_n = \underline{0}$.

From the independence of the $\underline{v}_i' s$, each $(a_i - b_i)$ = O. Hence, $a_i = b_i \ \forall \ i$