

St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MDTS 4113/SEM I

Module I

Linear Algebra

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Notes on Orthogonal Projections

Let V be an inner product space and W be a finite-dimensional subspace of V .

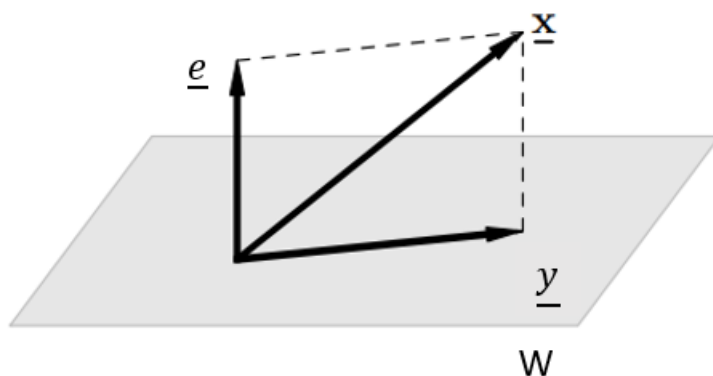
Then any vector $\underline{x} \in V$ is uniquely represented as $\underline{x} = \underline{y} + \underline{e}$, where $\underline{y} \in W$ and $\underline{e} \perp W$.

The component \underline{y} is the orthogonal projection of the vector \underline{x} onto the subspace W . The distance from \underline{x} to the subspace W is $\|\underline{e}\|$.

If $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ is an orthogonal basis for W then

$$\underline{y} = \frac{\langle \underline{x}, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 + \frac{\langle \underline{x}, \underline{v}_2 \rangle}{\langle \underline{v}_2, \underline{v}_2 \rangle} \underline{v}_2 + \dots + \frac{\langle \underline{x}, \underline{v}_n \rangle}{\langle \underline{v}_n, \underline{v}_n \rangle} \underline{v}_n$$

Projection of \underline{x} on \underline{v}_1 Projection of \underline{x} on \underline{v}_2 Projection of \underline{x} on \underline{v}_n



- The orthogonal projection of any vector \underline{x} onto \underline{v}_i : $Proj_{\underline{v}_i}(\underline{x}) = (\underline{x} \cdot \underline{v}_i) \underline{v}_i$, $i = 1, 2, \dots, n$.
- The orthogonal projection of any vector \underline{x} onto W : $Proj_W(\underline{x}) = \sum_{i=1}^n (\underline{x} \cdot \underline{v}_i) \underline{v}_i = \sum_{i=1}^n Proj_{\underline{v}_i}(\underline{x})$
- $\underline{x} = Proj_W(\underline{x}) + \underline{e}$, $\underline{e} \perp W$
- Define a matrix $P = [\underline{v}_1 \ \underline{v}_2 \ \dots \ \underline{v}_n]$ then, $Proj_W(\underline{x}) = PP^T \underline{x}$

Best Approximation:

Let W be a subspace.

- Then, for any vector \underline{y} , $Proj_W(\underline{y}) \in W$ is the best approximation to \underline{y} by vectors in W .
- More precisely, $Proj_W(\underline{y})$ is the closest point in W to \underline{y} ,

$$\text{i.e., } \|\underline{y} - Proj_W(\underline{y})\| \leq \|\underline{y} - \underline{w}\|, \text{ for any } \underline{w} \in W$$

- This closest distance is defined as the distance from \underline{y} to W :

$$\text{dist}(\underline{y}, W) = \min \{ \|\underline{y} - \underline{w}\| : \underline{w} \in W \} = \|\underline{y} - Proj_W(\underline{y})\|$$

Method of Least Squares observations:

$\underline{y}(\text{response})$

Consider a Linear Model: $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$

To estimate the parameter $\underline{\beta}$.

The error is $E = \|\underline{y} - X\underline{\beta}\|$: distance of \underline{y} to the point $X\underline{\beta}$ in the column space of X .

So, searching for the least square solution $\underline{\beta}$ minimizing E , is same as locating the point $p = X\underline{\beta}$, that is closer to \underline{y} than any other point in the column space.

