St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MSc in Data Science

Semester 1

Paper 3

(Module I)

Linear Algebra

Kronecker Product

Given an *m x n* matrix A and a *p x q* matrix B, their **Kronecker product** C = A tensor B, also called their matrix direct product, is an *mp x nq* matrix.

A= $\begin{bmatrix} a_{11} & a_{12} \ldots a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} \cdots a_{2(n-1)} & a_{2n} \\ & \vdots & \\ a_{m1} & a_{m2} \cdots a_{m(n-1)} & a_{mn} \end{bmatrix}$ and B= $\begin{bmatrix} b_{11} & b_{12} \ldots b_{1(q-1)} & b_{1q} \\ b_{21} & b_{22} \cdots b_{2(q-1)} & b_{2q} \\ & \vdots & \\ b_{p1} & b_{p2} \cdots b_{p(q-1)} & b_{pq} \end{bmatrix}$

C = A tensor B = $\begin{bmatrix} a_{11}B & a_{12}B \ldots a_{1(n-1)}B & a_{1n}B \\ a_{21}B & a_{22}B \cdots a_{2(n-1)}B & a_{2n}B \\ & \vdots & \\ a_{m1}B & a_{m2}B \cdots a_{m(n-1)}B & a_{mn}B \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{bmatrix}$$

Other names for the **Kronecker product** include **tensor product, direct product** or **left direct product.**

Example: A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

C = A$\otimes$ B = $\begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$ = $\begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$

In general, A$\otimes$ $B \neq B \otimes A$. [Solve Problem 3 in Assignment]

Applications: There are many other applications of the Kronecker product in signal processing, image processing, quantum computing and semidefinite programming.

The Kronecker product can be used to present linear equations in which the unknowns are matrices. Examples for such equations are: AX = B -----(1)

AX + XB = C ----(2)

AXB = C ----(3)

AX + Y B = C ----(4)

For any matrix A ∈ $M^{m\times n}$, the **vec –operator** is defined as
vec (A) = $(a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn})$, i.e. the entries of A are stacked columnwise forming a vector of length mn.

Equations (1) to (4) are equivalent to the following systems of equations:
AX = B -----(1) is equivalent to **(I $\otimes$ A)vec X = vec B** ----(5)
AX + XB = C ----(2) is equivalent to **(I $\otimes$ A) + (B^T $\otimes$ I) vec X = vec C** ----(6)
AXB = C ----(3) is equivalent to **(B^T $\otimes$ A)vec X = vec C** ----(7)
AX + Y B = C ----(4) is equivalent to **(I $\otimes$ A)vec X + (B T $\otimes$ I)vec Y = vec C** ----(8)

Results:

1. In general we can show that if $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m$ is an orthonormal basis in $R^m$and $\underline{u}_1$, $u_2, \ldots, \underline{u}_n$ is an orthonormal basis in R^n, then $\underline{v}_i \otimes u_j$ (i = 1,...,m; j = 1,...,n) is an orthonormal basis in the vector space R^{mn}. [Example: problem5 in Problem set 4]
2. The unit matrix can be represented with the help of an orthonormal basis in R^2 and the Kronecker product.
   Example: Let $\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then find $\underline{u}' \otimes \underline{u}$, $v' \otimes \underline{v}$ and verify that $\underline{u}' \otimes \underline{u}$ + $v' \otimes \underline{v}$.= I_2 Notice that the 2 × 2 unit matrix is also given by I_2= $\underline{u}$ $\underline{u}$'+ $\underline{v}$ $\underline{v}'$.
3. Let I_n be the n × n unit matrix and let I_n be the m × m unit matrix. Then $I_n \otimes I_m$ is the (nm) × (mn) unit matrix. Obviously, $I_n \otimes I_m = I_m \otimes I_n = I_{nm}$.
4. **Kronecker product involving Zero matrices:**
   Any Kronecker product that involves a zero matrix (i.e., a matrix whose entries are all zeros) gives a zero matrix as a result.

Let A_n be an arbitrary n × n matrix and let 0_m be the m × m zero matrix. Then $A_n \otimes 0_m = 0_{mxn}$.

5. **Kronecker product satisfies the associative law.**
   Let A be an $m \times n$ matrix, B be a $p \times q$ matrix and C be an $s \times t$ matrix.
   Then (A ⊗ B) ⊗ C ≡ A ⊗ (B ⊗ C). The matrix A ⊗ B ⊗ C has (*mps*) rows and (*nqt*) columns.
6. **Multiplication by a scalar:**
   It does not matter where we place multiplication with a scalar.
   Let A be an m × n matrix and B be a p × q matrix. Let c, d ∈ C. Then
   c⊗ A≡cA
   A⊗c ≡ cA
   (cA) ⊗ B ≡ c(A ⊗ B) ≡ A ⊗ (cB).
   (cA) ⊗ (dB) ≡ (cd)(A ⊗ B)

7. **Kronecker product obeys the distributive law.**
   Let A and B be $m \times n$ matrices and C and D be $p \times q$ matrices.
   Then we have (A + B) ⊗ (C + D) ≡ A ⊗ C + A ⊗ D + B ⊗ C + B ⊗ D.
8. Let r(A) be the rank of A and r(B) be the rank of B. Then r(A ⊗ B) = r(A)r(B).
   Example: Let A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then r(A) = 2 and r(B) = 1.
   Therefore r(A ⊗ B) = 2.
9. If A and B are diagonal matrices, then A ⊗ B is a diagonal matrix. Conversely, if A ⊗ B is a diagonal matrix and A ⊗ B ≠ 0, then A and B are diagonal matrices.
10. Let A and B be upper triangular matrices, then A ⊗ B is an upper triangular matrix.
11. The trace of the Kronecker product of two matrices is the product of the traces of the matrices.
    If A and B are square matrices, then the trace satisfies tr (A ⊗ B) = tr(A).tr(B)
12. Taking the transpose before carrying out the Kronecker product yields the same result as doing so afterwards.
    Let A be an m × n matrix and B be an p × q matrix. Then $(A \otimes B)^t \equiv A^t \otimes B^t$.
13. Let A be an invertible n × n matrix. Let B be an invertible m × m matrix. Then A ⊗ B is an invertible matrix with $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
14. Let A and B be positive definite matrices. Then A⊗B is a positive definite matrix. The same holds for positive semi-definite matrices.
15. An n × n matrix A is called nilpotent if $A^k = O_n$ for some positive integer k. If A and B be two n × n nilpotent matrices, then A ⊗ B is nilpotent.
16. **Mixed Products:** The product of two Kronecker products yields another Kronecker product.
    Assume that the matrix A is of order $m \times n$, B of order $p \times q$, C of order $n \times r$ and D of order $q \times s$. Then (A ⊗ B)(C ⊗ D)=(AC) ⊗ (BD).

17. Let A be an invertible m×m matrix. Let B be an invertible n × n matrix. Then A ⊗ B is invertible and det(A ⊗ B) = $(\det(A))^n$ $(\det(B))^m$. [det(A ⊗ B) is called the Zehfuss determinant of A and B].
    This implies that A ⊗ B is nonsingular if and only if both A and B are nonsingular.

Assignments:

1. Let A be a symmetric matrix of order n with A^2 = 0. Is it necessarily true that A = 0?
2. Let $\underline{u}$, $\underline{v}$ $\in R^n$, where $\underline{u}$, $\underline{v}$ are considered as column vectors. Show that $\underline{u}' \otimes \underline{v} = \underline{v}\underline{u}' = \underline{v} \otimes \underline{u}'$
3. Find the Kronecker product: a). A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$, b).). A = $\begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$, B =$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$
   and verify that A$\otimes B \neq B \otimes A$.
4. Let A and B be $n \times n$ matrices. Then (A ⊗ I_n)(I_n ⊗ B) = A ⊗ B.