Check for Definiteness

Change of Variables

It is possible to simplify a quadratic form $\underline{x}'A\underline{x}$ by change of variables $\underline{x}$ =R $\underline{y}$ or $\underline{y}=R^{-1}\underline{x}$

Where R is a non-singular matrix.

Why non-singular transformation??

One-to-one transformation.

Substitution of $\underline{x}$ =R $\underline{y}$ in $\underline{x}'A\underline{x}$ gives

F= $\underline{x}'A\underline{x}= (\mathrm{R}\underline{y})'A\,(\mathrm{Ry})= \underline{y}'R'\mathrm{AR}\,\underline{y} = \underline{y}'\mathrm{B}\,\underline{y}$

Note: if A is symmetric, B is also symmetric.

Congruence: A square matrix B is said to be congruent to a square matrix A if ∃ a non-singular matrix R such that B= R'AR.

Note: the matrix B of the quadratic form $\underline{y}'B\,\underline{y}$ obtained by the non-singular transformation of the variables x=Ry in the form $\underline{x}'A\underline{x}$ is congruent to A.

|A| is called the Discriminant of the quadratic form $\underline{x}'A\underline{x}$.

If B= $R'AR$ is congruent to A, then the discriminant of the Q.F $\underline{y}'B\,\underline{y}$ is

$$|B|=|R'||A||R|=|R|^2|A|$$

Note that: under a non-singular transformation of the variables $\underline{x}=R\,\underline{y}$, the discriminant of the new Q.F assumes a magnitude of $|R|^2$times that of the original one.

$|R|$ is called the modulus of the transformation

$\underline{x}=R\,\underline{y}$

- If $\underline{x}$ is allowed to vary over entire R^n, then the values taken by F= $\underline{x}'A\underline{x}$ is called the range of the quadratic form.

- Under a non-singular transformation of variables the range of a quadratic form remains unchanged.

Let F=$\underline{x}'A\underline{x}$ be the given Q.F

Transformation: $\underline{x}$ =R $\underline{y}$ such that $|R| \neq 0$ $\Rightarrow$ $\underline{y}=R^{-1}\underline{x}$

then the new Q.F= $\underline{y}'R'AR\,\underline{y}$ = $\underline{y}'B\,\underline{y}$

Note that for any $\underline{x}$ there is unique $\underline{y}$ and vice-versa such that F= $\underline{x}'A\underline{x}$= $\underline{y}'\mathrm{B}\,\underline{y}$

Hence, $\underline{x}'A\underline{x}$ and $\underline{y}'\mathrm{B}\,\underline{y}$ must have the same range.

If $|\mathrm{R}|=0$, this property doesn't hold.

- A positive definite form remains positive definite under a non-singular transformation of variables.

Since we know that the range of $\underline{y}'\mathrm{B}\,\underline{y}$ and $\underline{x}'A\underline{x}$ are same, it is only suff to show that $\underline{y} = \underline{0}$ is the only $\underline{y}$ for which $\underline{y}'\mathrm{B}\,\underline{y} = 0$.

Now, $\underline{x}'A\underline{x} = 0$ only if $\underline{x} = \underline{0}$ [$\underline{x}'A\underline{x}$ being pd]

However, $\underline{y} = R^{-1}\,\underline{x}$ and hence $\underline{y} = \underline{0}$ is the only value of $\underline{y}$ for which $\underline{x} = \underline{0}$.

Consider the following QFs: $x_1^2 + 2x_2^2 + 4x_3^2$ → PD

$-2x_1^2 - 2x_2^2$ → ND

$4x_1^2 - 3x_2^2$ → Indefinite

$3x_1^2 + 4x_1x_2 + 2x_2^2$??

$3x_1^2 + x_2^2 + 5x_3^2 + 4x_1x_2 + 2x_1x_3 + 6x_2x_3$??

Diagonalization of Quadratic Forms

F= $\underline{x}'A\underline{x}$

Consider a non-singular transf: $\underline{x}$ =R $\underline{y}$....(*)

Where the columns of the matrix R are an orthonormal set of eigen vectors for A.

R is therefore an orthogonal matrix and the transformation (*) is an orthogonal transf.

The new Q.F is $\underline{y}'R'AR\,\underline{y} = \underline{y}'D\,\underline{y}$

D= $(\lambda_j \delta_{ij})$: Diagonal matrix with eigen values of A as the diagonal elements.

$\therefore \underline{y}'D\,\underline{y} = \sum_{j=1}^{n} \lambda_j {y_j}^2$ $\Longrightarrow$ No cross product terms

A Q.F with only squares of the variables is said to be in diagonal form.

If we know the eigen values of A, we can immediately determine the form of the Q.F.

We can do this since

- The range of the form is unchanged under a non-singular transf
- A positive or negative definite Q.F remains positive or negative definite under a non-singular transf

- F= $\underline{x}'A\underline{x}$ is pd(nd) iff every eigen value of A is>0(<0)

- F= $\underline{x}'A\underline{x}$ is psd(nsd) iff every eigen value of A is$\geq$0($\leq$0) and atleast one of the eigen values vanishes.

- F= $\underline{x}'A\underline{x}$ is indefinite iff A has both positive and negative eigen values.

Diagonalization by completion of the square

F= $a_{11}{x^2}_1 + 2a_{12}x_1x_2 + a_{22}{x^2}_2$

If a_{11} or a_{22} is not zero, WLG let $a_{11} \neq 0$

F= $a_{11}[{x^2}_1+2\frac{a_{12}}{a_{11}}x_1x_2+(\frac{a_{12}}{a_{11}})^2{x^2}_2-(\frac{a_{12}}{a_{11}})^2{x^2}_2+\frac{a_{22}}{a_{11}}{x^2}_2]$

= $a_{11}[(x_1 + \frac{a_{12}}{a_{11}}\ x_2\ \ )^2+ [\frac{a_{22}}{a_{11}}-(\frac{a_{12}}{a_{11}})^2]\ {x^2}_2]$

Transformation: $y_1 = x_1 + \frac{a_{12}}{a_{11}}\ x_2$

$$y_2 = x_2$$

$\underline{y} = S\underline{x} = \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} \\ 0 & 1 \end{pmatrix} \underline{x}$

F= $a_{11}{y_1}^2 + [a_{22} - \frac{{a_{12}}^2}{a_{11}}]{y_2}^2$

This transformation of variables is non-singular since |S|=1.

The coefficients of ${y_1}^2$ &${y_2}^2$ are not in general eigen values.

Consider F= x'Ax = $\sum_{i,j} a_{ij}x_i x_j$ be a positive definite QF.

The terms involving $x_1$are

$a_{11}{x_1}^2$+ $2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n$-------(*)

Since the form is pd, it must be positive when $x_2 = x_3 = \cdots = x_n = 0 \;\&\; x_1 \neq 0$

Then F= $a_{11}{x_1}^2$ and $a_{11} must\ be\ positive$

$a_{11}{x_1}^2 + 2a_{12}x_1x_2 + \cdots + 2a_{1n}x_1x_n$-------(*)

(*) can be written as

$a_{11}({x_1}^2 + 2\sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} x_1x_k)$

$= a_{11}[{x_1}^2 + 2\sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} x_1x_k + \left(\sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} \; x_k \right)^2 \quad - (\sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} \; x_k)^2]$

$=a_{11}[(x_1 \; + \sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} x_k)^2 - (\sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} \; x_k)^2]$

Transformation:

$v_1 = (x_1 \ \ + \sum_{k=2}^{n} \frac{a_{1k}}{a_{11}} x_k)$

$v_2 = x_2, \ldots, v_n = x_n$

Or, $\underline{v}$= $S_1 \underline{x}$ where $S_1 \ = \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} \ldots & \frac{a_{1n}}{a_{11}} \\ 0 & 1 \ldots & 0 \\ & \vdots & \\ 0 & 0 \ldots & 1 \end{pmatrix}$

$|\ S_1\ |$=1

Thus, a non singular transformation of variables reduces F to

F= $a_{11}{v_1}^2 + \sum_{i,j=2}^{n} b_{ij}v_iv_j$ ----(i)

and a_{11}>0

This procedure is now repeated.

Since (i) is pd, $b_{22} > 0$

We complete the square for the variable v_2

Define another transformation of variables

$w_1 = v_1,\ w_2 = v_2 + \sum_{k=3}^{n} \frac{b_{2k}}{b_{22}} v_k, w_3 = v_3, \ldots\ w_n = v_n$

Or $\underline{w} = S_2\underline{v}$, $|S_2|$=1 and obtain the form

F= $a_{11}{w_1}^2$+ $b_{22}{w_2}^2$+$\sum_{i,j=3}^{n} c_{ij}w_iw_j$

With a_{11}, b_{22}>0

Repeating this process n-1 times

F= $a_{11}{y_1}^2$+ $b_{22}{y_2}^2$+ $c_{33}{y_3}^2 + \cdots z_{nn}{y_n}^2$

The non singular transformation y=Sx which gives this reduction is

S= $S_{n-1}S_{n-2} \ldots . S_2S_1$

- $x'Ax$ is pd iff A can be written as $B'B$ for some non-singular B.

<u>If part</u>: Consider $Q=x'Ax$

$\exists$ an orthogonal matrix S, made up of the eigen vectors of A such that $S'AS=\Lambda$

Since A is pd, all its eigen values are positive.

$A=S\Lambda S' = S\sqrt{\Lambda}\sqrt{\Lambda}S' = (\sqrt{\Lambda}S')'(\sqrt{\Lambda}S') = B'B$

Only if part: $A = B'B$

$Q = x'Ax$

$= x'B'Bx = (Bx)'(Bx) > 0$ when $x > 0$

If A is a pd matrix, all the following results are equivalent:

1. x’Ax > 0 ∀ x ≠0

   =0 iff x=0

2. All eigen values of a are positive.
3. All principal order minors of A are positive, i.e,

$$a_{11} > 0; \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$$

….,|A|>0

1) $\Rightarrow$ 2)

$Ax=\lambda x$ for some $x\neq 0$

$\Rightarrow$ $x'Ax = \lambda x'x = \lambda > 0$ [from 1]

[∵ A is a symmetric matrix, eigen vectors can be taken as orthonormal unit vectors]

2) $\Rightarrow$ 1)

A is symmetric matrix $\Rightarrow$ $\exists$ a full set of n orthonormal eigen vectors of A

Let these be $x_1, x_2, \ldots x_n$

They form a basis of R^n

$\therefore$ x= $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Ax = A($c_1x_1 + c_2x_2 + \cdots + c_nx_n$)

= $c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_3x_n$

$\therefore$ x'Ax = $(c_1x'_1 + c_2x'_2 + \cdots + c_nx'_n)\,(c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_3x_n)$

= $c^2{}_1\ \lambda_1 + c^2{}_2\lambda_2 \quad + \cdots + c^2{}_n\lambda_3$

Now λ_i > 0 ∀ i=1(1)n

So, x'Ax>0 ∀ x ≠0

Now, suff to show that 1) $\Rightarrow$ 3) and 3) $\Rightarrow$ 1)

A be an nxn matrix

Let A_1= a_{11}, A_2= $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,…, A_n= |A|

1) ➡ 3)

x'Ax >0 $\forall$ x $\neq$ 0

now, $(x_k{}'\ 0')\ A\binom{x_k}{0}$ =

$$(x_k{}'\ 0')\ \begin{pmatrix} A_k & B^{kx(n-k)} \\ C^{(n-k)xk} & D^{(n-k)x(n-k)} \end{pmatrix} \binom{x_k}{0}$$

= $x_k{}'\ A_k x_k$ > 0 $\forall$ $x_k \neq 0$

=0 iff x_k=0

So, A_k is pd and using 2), all eigen values of A_k are positive.

$\therefore$ $|A_k|$ = $\prod \lambda_i$ > 0 for every k

3) $\Rightarrow$ 1)

Since $a_{11} > 0$, we can perform a non-singular transformation

x=$R_1 v$,$|R_1|$=1 such that

$a_{11}{v_1}^2 + \sum_{i,j=2}^{n} b_{ij} v_i v_j$……(1)

If we set $x_i = v_i = 0\ (i = 3, \dots, n)$, (1) becomes a form in two variables whose discriminant is $a_{11}b_{22}$.

When the above variables are set to zero, the original form x'Ax reduces to a form in 2 variables whose discriminant is

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$>0

The discriminants are equal since the modulus of the transformation is unity.

Thus $a_{11}b_{22}$>0 & b_{22}>0 since $a_{11} > 0$

Another non-singular transformation of unit modulus reduces (1) to

$$a_{11}{w_1}^2 + b_{22}{w_2}^2 + \sum_{i,j=3}^{n} c_{ij}w_iw_j$$

Setting $w_i = v_i = x_i = 0(i = 4, \ldots n)$→ c_{33}>0

$$a_{11}b_{22}c_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0$$

And c_{33}>0 since a_{11}, b_{22}>0

This process is continued until we have $\sum_{i=1}^{n} d_i {y_i}^2$

d_i>0 $\forall i$

If x'Ax is nd, x'(-A)x is pd.

|-A|= $(-1)^n|A|$

A nsc for x'Ax to be nd or equivalently for x'(-A)x to be pd is

$a_{11} < 0$; $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$>0; $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0$

…., $(-1)^n|A|$>0