St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MSc in Data Science

Semester 1

Paper 3

(Module I)

Linear Algebra

Determinant and Rank

Determinant is a scalar associated with a square matrix in a particular way.

Applications: computing inverse of a non-singular matrix, solving system of linear equations by Cramer's rule, the Jacobian used in transforming a multiple integral and many others.

The determinant of an nxn matrix A= $\{a_{ij}\}$ is given by

$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\,|M_{ij}|$ for any i=1,…,n [Expanding by the elements of a row]

$|A| = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\,|M_{ij}|$ for any j=1,…,n [Expanding by the elements of a column]

Where, $|M_{ij}|$ is the *minor* of a_{ij}: the determinant derived from A by crossing out the row and column corresponding to a_{ij}.

1. $C_{ij} = (-1)^{i+j}|M_{ij}|$: *Cofactor* of a_{ij} (signed minor).
2. Determinant of a transpose of a matrix equals the determinant of the matrix itself: $|A'| = |A|$
3. A is non-singular if $|A| \neq 0$. In that case, $|A^{-1}| = {}^{1}/_{|A|}$.
4. If 2 rows of A are same, $|A| = 0$.

5. Adding to one row (column) of a determinant any multiple of another row (column) does not affect the value of the determinant.
6. $|AB| = |A|\,|B|$ where A and B are square matrices of the same order.
7. The determinant of a lower triangular matrix is the product of its diagonal elements.

Note that 6 and 4 ⇒ Any determinant can be evaluated as the determinant of a lower triangular matrix.

8. $|AB| = |BA|$ as $|A||B| = |B||A|$
9. $|A^2| = |A|^2$,…., $|A^K| = |A|^K$
10. For orthogonal matrix A, $|A| = \pm 1$ (As A$A' = I \Rightarrow |A|^2 = 1$)
11. For idempotent matrix A, $|A| = 0$ *or* 1 (As $A^2 = A \Rightarrow |A|^2 = |A|$)
12. Similar matrices have the same determinant. Two matrices A and B are similar if ∃ a non-singular matrix P such that B = P^{-1}AP ⇒ $|B| = |P^{-1}||A||P|$= $|A|$.

13. Let A be a square matrix of order ≥ 2. Then Adjoint or Adjugate of the matrix A is defined as Adj(A) = $(C_{ij})^T$, where C_{ij}: *Cofactor* of a_{ij} (signed minor).
14. Properties of Adjoint matrix:

  i. A(adj A) = (adj A) A = A I, where I is the identity matrix of order n
  ii. For a zero matrix 0, adj(0) = 0
  iii. For an identity matrix I, adj(I) = I
  iv. If A is non-singular, $A^{-1} = \frac{1}{|A|} Adj(A)$.
  v. For any scalar k, adj(kA) = k^{n-1} adj(A)
  vi. adj(A^T) = (adj A)^T
  vii. det(adj A), i.e. |adj A| = (det A)$^{n-1}$
  viii. Suppose A and B are two matrices of order n, then adj(AB) = (adj B)(adj A)
  ix. For any non-negative integer p, adj(A^p) = (adj A)^p
  x. If A is invertible, then the above formula also holds for negative k.

15. Cramer's Rule: Let A be a non-singular matrix of order n. then the solution to the system of equations A$\underline{x}$=$\underline{b}$ is given by $|A_j| \big/ |A|$

    Where A_j denotes the matrix obtained from A by replacing jth column $\underline{a}_{*j}$ by $\underline{b}$.

16. If A and D are square matrices of possibly different orders, $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = \begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = |A|.|D|$.
17. Determinant of a partitioned matrix:
    i. If A and D are square and A is non-singular, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|.|D - CA^{-1}B|$.
    ii. If D is non-singular, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D|.|A - BD^{-1}C|$.
18. Let A, B, C and D be mxm matrices, then if A is non-singular and A commutes with C, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$.

19. The rank of a non-null matrix A is the largest integer k for which A has a non-vanishing minor of order k.