

St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MDTS 4113/SEM I

Linear Algebra

Notes

Linear Dependence and Independence of Vectors

- If one or more vectors in a set of vectors $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ can be written in terms of the other vectors in the set, then S is said to be a **dependent set of vectors**.

Alternatively,

A set of vectors $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ is said to be linearly dependent if there exists scalars c_2, \dots, c_n ,

not simultaneously zero, such that $c_1 \underline{a}_1 + c_2 \underline{a}_2 + \dots + c_n \underline{a}_n = \underline{0}$.

Claim: Both the definitions are equivalent.

Proof: WLG let \underline{a}_1 can be written in terms of the other vectors in the set.

Then, $\underline{a}_1 = c_2 \underline{a}_2 + \dots + c_n \underline{a}_n$,

→ $1 \cdot \underline{a}_1 - c_2 \underline{a}_2 - \dots - c_n \underline{a}_n = \underline{0}$

there exists scalars c_2, \dots, c_n , not simultaneously zero, such that $c_1 \underline{a}_1 + c_2 \underline{a}_2 + \dots + c_n \underline{a}_n = \underline{0}$

Let WLG $c_n \neq 0$

$$\underline{a}_1 = \frac{c_2}{c_n} \underline{a}_2 + \dots + \frac{c_n}{c_n} \underline{a}_n$$

- If no vector in a set of vectors $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ can be written in terms of the other vectors in the set, then S is said to be a **Linearly independent set of vectors**.

Alternatively,

A set of vectors $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ is said to be linearly independent if the

only scalars c_2, \dots, c_n for which $c_1 \underline{a}_1 + c_2 \underline{a}_2 + \dots + c_n \underline{a}_n = \underline{0}$ holds is

$$c_1 = c_2 = \dots = c_n = 0$$

Claim: Both the definitions are equivalent [check!]

Q. What can you say about a set containing a single vector \underline{a} ?

Independent if $\underline{a} \neq \underline{0}$

Dependent if $\underline{a} = \underline{0}$

Q. What can you say about a set of vectors containing a null vector ?

A null vector is not linearly independent of any other vector or a set of vectors as

$$\underline{0} = 0 \cdot \underline{a}_1 + 0 \cdot \underline{a}_2 + \dots + 0 \cdot \underline{a}_n$$

$$\text{Or, } 1 \cdot \underline{0} - 0 \cdot \underline{a}_1 - 0 \cdot \underline{a}_2 - \dots - 0 \cdot \underline{a}_n = \underline{0}$$

Thus, a null vector is dependent on any other vector and presence of null vector in a set makes the set dependent.

- A subset of a set of linearly independent vectors is linearly independent.
- A Super set of a set of linearly dependent vectors is linearly dependent.
- Any set of n vectors in R^m must be dependent if $n > m$.
- Consider the following homogenous system of n equations in m unknowns:

$$\begin{array}{l} \underline{A}\underline{\lambda} = \underline{0} \Rightarrow \begin{array}{l} a_{11}\lambda_1 + \dots + a_{1m}\lambda_m = 0, \\ a_{21}\lambda_1 + \dots + a_{2m}\lambda_m = 0, \\ \vdots \\ a_{n1}\lambda_1 + \dots + a_{nm}\lambda_m = 0. \end{array} \Rightarrow \lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2 + \dots + \lambda_m \underline{a}_m = \underline{0} \end{array}$$

$\underline{\lambda} = \underline{0}$ is always a solution.

Existence of non-zero solution depends on whether the column vectors $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ are dependent / independent.

- To check any set of vectors $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ for linear independence,
 - form the matrix A whose k columns are the given vectors.
 - Then solve the system $A\underline{\lambda} = 0$;

the vectors are dependent if and only if there is a solution other than $\underline{\lambda} = 0$.

- A is an m by k matrix. Suppose now that $k > m$.

Then it will be impossible for A to have k independent columns.

a homogeneous system $Ac = 0$ with more unknowns than equations

always has solutions $c \neq 0$.

Vector Spaces and Subspaces

Fields:

A field is a set together with some operations on the objects in that set which behave like ordinary addition, subtraction, multiplication, and division of numbers in the sense that they obey the nine rules of algebra listed below.

A set of real numbers F is called a field if:

1. Addition is commutative, $x+y = y+x$ for all x and y in F .
2. Addition is associative, $x + (y + z) = (x + y) + z$ for all x , y , and z in F .
3. There is a unique element 0 (zero) in F such that $x + 0 = x$, for every x in F .
4. To each x in F there corresponds a unique element $(-x)$ in F such that $x + (-x) = 0$.
5. Multiplication is commutative, $xy = yx$ for all x and y in F .
6. Multiplication is associative, $x(yz) = (xy)z$ for all x , y , and z in F .
7. There is a unique non-zero element 1 (one) in F such that $x1 = x$, for every x in F .
8. To each non-zero x in F there corresponds a unique element x^{-1} (or $1/x$) in F such that $xx^{-1} = 1$.
9. Multiplication distributes over addition; that is, $x(y + z) = xy + xz$, for all x , y , and z in F .

Vector Space

A set of vectors V , defined over a field F of scalars, is called a vector Space

if there exists a rule (or operation), called vector addition, which associates with each pair of vectors $\underline{v}, \underline{w} \in V$, a vector $(\underline{v} + \underline{w}) \in V$, in such a way that

- (a) addition is commutative, $\underline{v} + \underline{w} = \underline{w} + \underline{v}$;
- (b) addition is associative, $\underline{t} + (\underline{v} + \underline{w}) = (\underline{t} + \underline{v}) + \underline{w}$
- (c) there is a unique vector $\underline{0}$ in V , called the zero vector, such that $\underline{v} + \underline{0} = \underline{v}$ for all \underline{v} in V ;
- (d) for each vector \underline{v} in V there is a unique vector $-\underline{v}$ in V such that $\underline{v} - \underline{v} = \underline{0}$;

Also a rule (or operation), called multiplication by scalar, which associates with each scalar c in F and vector \underline{v} in V a vector $C \underline{v}$ in V , in such a way that

- (a) $1 \underline{v} = \underline{v}$ for every \underline{v} in V ;
- (b) $(c_1 c_2) \underline{v} = c_1(c_2 \underline{v})$;
- (c) $c(\underline{v} + \underline{w}) = c \underline{v} + c \underline{w}$
- (d) $(c_1 + c_2) \underline{v} = c_1 \underline{v} + c_2 \underline{v}$

Subspaces:

Let V be a vector space over the field F . A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V .

A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\underline{v}, \underline{w}$ in W and each scalar c in F , the vector $c\underline{v} + d\underline{w}$ belongs to W .

Some Results:

- **Let V be a vector space over the field F . The intersection of any collection of subspaces of V is a subspace of V .**

Proof. Let $\{W_\alpha\}$ be a collection of subspaces of V ,

let $W = \bigcap_\alpha W_\alpha$ be their intersection.

Note that W is defined as the set of all elements belonging to every W_α .

Since each W_α is a subspace, each contains the zero vector.

Thus the zero vector is in the intersection W , and W is non-empty.

Let α & β be vectors in W and let c be a scalar. By definition of W , both belong to each W_α , and because each W_α is a subspace, the vector $(c\alpha + \beta)$ is in every W_α .

Thus $(c\alpha + \beta)$ is again in W . Hence, W is a subspace of V .

- We have seen that for 2 subspaces W_1 & W_2 , $W_1 \cap W_2$ is always a subspace.
- $W_1 \cap W_2$ is the largest subspace contained in both W_1 & W_2

Q. What can you say about $\bigcup_\alpha W_\alpha$? $\rightarrow W_1 \cup W_2$ need not be a subspace.

Q. What about $S = W_1 + W_2 + \dots + W_k$? (**Linear Sum**)

$$S = \{ \underline{x} : \underline{x} = \sum_{i=1}^k \alpha_i \underline{s}_i \text{ s.t. } \alpha_i \in W_i \forall i \}$$

$W_1 + W_2$ is the smallest subspace containing both W_1 & W_2 .

Clearly, $W_1 + W_2 = W_2 + W_1$

Example: If S & T are any 2 distinct lines through the origin in \mathcal{R}^2 .

Both S & T are subspaces

$$S + T = \mathcal{R}^2$$

- **Let V be a vector space and $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k\}$ be a set of vectors $\in V$. Then the set of all possible linear combinations of these vectors is a vector subspace W .**

Proof: Let W be the set of all possible linear combinations of the vectors in S .

To prove W is a vector subspace, we need to show that $c\underline{x} + d\underline{y} \in W \forall c, d \in R$ and $\forall \underline{x}, \underline{y} \in W$

Consider any $\underline{x}, \underline{y} \in W$. Then $\underline{x} = c_1 \underline{a}_1 + c_2 \underline{a}_2 + \dots + c_k \underline{a}_k$ for some $c_i \in R$ and

$$\underline{y} = d_1 \underline{a}_1 + d_2 \underline{a}_2 + \dots + d_k \underline{a}_k \text{ for some } d_i \in R$$

$$\begin{aligned} \text{Now, } \underline{c}\underline{x} + \underline{d}\underline{y} &= \underline{c}(c_1 \underline{a}_1 + c_2 \underline{a}_2 + \dots + c_k \underline{a}_k) + \underline{d}(d_1 \underline{a}_1 + d_2 \underline{a}_2 + \dots + d_k \underline{a}_k) \\ &= (\underline{c}c_1 + \underline{d}d_1)\underline{a}_1 + (\underline{c}c_2 + \underline{d}d_2)\underline{a}_2 + \dots + (\underline{c}c_k + \underline{d}d_k)\underline{a}_k \in W \end{aligned}$$

- The set W of all linear combinations of vectors $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k$ is called the **span** of these vectors, and is written $\text{Span} \{ \underline{a}_1, \underline{a}_2, \dots, \underline{a}_k \}$
- Let W be a set of vectors. If $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k$ are vectors such that $W = \text{Span} \{ \underline{a}_1, \underline{a}_2, \dots, \underline{a}_k \}$ then we say $\{ \underline{a}_1, \underline{a}_2, \dots, \underline{a}_k \}$ is a generating set for W .
- The vectors $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k$ are the as generators or spanning set for W .
- The subspace W spanned by a set of vectors $\{ \underline{a}_1, \underline{a}_2, \dots, \underline{a}_k \}$ is the intersection of all subspaces of V which contain S .

- Rows of a matrix generate a subspace called the **row space** $R(A)$.

$$\bullet \quad R(A) = \{ \underline{x}: \underline{x}' = d_1 \underline{\beta}'_1 + d_2 \underline{\beta}'_2 + \dots + d_m \underline{\beta}'_m \} = (d_1 \ d_2 \ \dots \ d_m) \begin{bmatrix} \underline{\beta}'_1 \\ \underline{\beta}'_2 \\ \vdots \\ \underline{\beta}'_m \end{bmatrix}$$

$$= (d_1 \ d_2 \ \dots \ d_m) \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \underline{d}' A$$

Row space is a subspace of R^n

The row space of a matrix A is column space of A^T

Basis and Dimension

Basis for a Vector Space: A basis for a vector space V is a set of vectors which are independent and span the vector space V .

Every vector \underline{x} in the space V is a combination of the basis vectors, because they span the space.

- **Representation of \underline{x} is unique, because the basis vectors $\underline{v}_1, \dots, \underline{v}_n$ are independent:**

There is one and only one way to write \underline{x} as a combination of the basis vectors.

Proof: Suppose $\underline{x} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$ and also $\underline{x} = b_1\underline{v}_1 + \cdots + b_n\underline{v}_n$.

By subtraction $(a_1 - b_1)\underline{v}_1 + \cdots + (a_n - b_n)\underline{v}_n = \underline{0}$.

From the independence of the \underline{v}_i 's, each $(a_i - b_i) = 0$. Hence, $a_i = b_i \forall i$