St. Xavier's College (Autonomous), Kolkata

Department of Statistics

MSc in Data Science

Semester 1

Paper 3

(Module I)

Linear Algebra

Some Results

1. **For 2 matrices A and B, Rank (AB) ≤ min {Rank(A), Rank(B)}**

Let C = $A^{m\times r}B^{r\times n}$

Rank(C) = Row rank (C) = Col rank (C)

Row Space(C) ⊆ Row Space (B) [Rows of C being linear combinations of the rows of B]

Hence, Rank(C) = Rank (AB) ≤ Row Rank (B) = Rank(B)---(i)

Col Space(C) ⊆ Row Space (A) [Cols of C being linear combinations of the rows of A]

Hence, Rank(C) = Rank (AB) ≤ Row Rank (A) = Rank(A)---(ii)

From (i) and (ii), Rank (AB) ≤ min {Rank(A), Rank(B)}

2. **(Rank-Nullity Theorem). Let A be an m×n. Then, dim(R(A)) + dim(N(A)) = n.**

Let dim(N(A)) = r ≤ n and

let B = $\{\underline{u}_1, \underline{u}_2, \dots \underline{u}_r\}$ be a basis of N(A).

Since B is a linearly independent set in R^n , extend it to get

C = $\{\underline{u}_1, \underline{u}_2, \dots \underline{u}_n\}, a$ basis of R^n.

Then, Col(A) = LS(AC) = LS($A\underline{u}_1, A\underline{u}_2, \dots, A\underline{u}_n$)

= LS(0, . . . , 0, A$\underline{u}_{r+1}, A\underline{u}_{r+2}, \dots A\underline{u}_n$)

= LS(A$\underline{u}_{r+1}, A\underline{u}_{r+2}, \dots A\underline{u}_n$).

So, D = {A$\underline{u}_{r+1}, A\underline{u}_{r+2}, \dots A\underline{u}_n$} spans Col(A).

We further need to show that D is linearly independent.

So, consider the homogeneous linear system given below in the unknowns $\alpha_1, \alpha_2, \dots, \alpha_{n-r}$

α_1A$\underline{u}_{r+1} + \alpha_2 A\underline{u}_{r+2} + \cdots + \alpha_{n-r} A\underline{u}_n = \underline{0}$

$\Leftrightarrow A(\alpha_1\underline{u}_{r+1} + \alpha_2\underline{u}_{r+2} + \cdots + \alpha_{n-r}\underline{u}_n)$= 0---(i)

Now, for (i) to have a non-null solution,

$\alpha_1\underline{u}_{r+1} + \alpha_2\underline{u}_{r+2} + \cdots + \alpha_{n-r}\underline{u}_n \in$ N(A) = LS(B).

Therefore, there exist scalars βi, 1 ≤ i ≤ r, such that $\sum_{i=1}^{n-r} \alpha_i\underline{u}_{r+i} = \sum_{j=1}^{r} \beta_j\underline{u}_j$.

Or equivalently, $\beta_1\underline{u}_1 + \cdots + \beta_r\underline{u}_r - \alpha_1\underline{u}_{r+1} - \cdots - \alpha_{n-r}\underline{u}_n = \underline{0}$ ---(ii)

Equation (ii) is a linear system in vectors from C with αi 's and βj 's as unknowns.

As C is a linearly independent set, the only solution of Equation (ii) is

αi = 0, for 1 ≤ i ≤ n − r and βj = 0, for 1 ≤ j ≤ r.

In other words, we have shown that the only solution of Equation (i) is the trivial solution.

Hence, {A$\underline{u}_{r+1}, A\underline{u}_{r+2}, \dots A\underline{u}_n$} is a basis of Col(A).

As we know that Row rank(A) = Col Rank(A), Dim R(A) = n-r

Thus, Dim R(A) + Dim N(A) = n

3. $\boldsymbol{R^{\perp}(A) = N(A)}$

To show that for each $\underline{x} \in$ N(A) and $\underline{u} \in$ R(A), $\underline{u}^T\underline{x}$ = 0.

As $\underline{u} \in$ R(A) there exists $\underline{y} \in R^m$ such that $\underline{u}^T = \underline{y}^T A$.

Further, x ∈ N(A) implies Ax = 0.

Thus, we see that $\underline{u}^T\underline{x} = \underline{y}^T A\underline{x} = \underline{y}^T\underline{0} = 0$

As this is true for any arbitrary $\underline{x}$ ∈ N(A) and $\underline{u}$ ∈ R(A), every vector in R(A) is orthogonal to every vector in N(A) and vice-versa.

(2) along with (3) implies $\boldsymbol{R}^{\perp}(\boldsymbol{A}) = \boldsymbol{N}(\boldsymbol{A})$