Assignment 3 (DB) - Answers

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Question 1

Let $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. Find AB and BA.

Answer

Matrix A being an $(1 \times n)$ matrix and matrix B being a $(n \times 1)$ matrix, both the matrix operations AB and BA are valid.

$$AB = \sum_{i=1}^{n} a_i b_i$$

$$BA = \begin{pmatrix} b_1a_1 & b_1a_2 & \dots & b_1a_n \\ b_2a_1 & b_2a_2 & \dots & b_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ b_na_1 & b_na_2 & \dots & b_na_n \end{pmatrix}$$

Here, AB is a scalar quantity, whereas BA is a $(n \times n)$ matrix.

Question 2

Find examples for the following statements:

1. Suppose that the matrix product AB is defined. Then the product BA need not be defined.

2. Suppose that the matrices A and B are square matrices of order n. Then AB and BA may or may not be equal.

Answer

1. Say $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$. Then AB is defined as number of columns of A = number of rows of B, and will be of dimension (3×2). But, BA is not defined, since number of columns of $B \neq$ number of rows of A.

2. Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, whereas $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Question 3

Show that for any square matrix A, $S = \frac{1}{2}(A + A^T)$ is symmetric, $T = \frac{1}{2}(A - A^T)$ is skew-symmetric, and $A = S + T$.

Answer

Say A is a $(n \times n)$ matrix whose elements are $a_{(i,j)}$; $\ i, j \in \{1, 2, \ldots n\}$.
The $(i,j)^{th}$ element of $S = \frac{1}{2}(A + A^T)$ would be $s_{(i,j)} = \dfrac{1}{2}(a_{(i,j)} + a_{(j,i)}) = \dfrac{1}{2}(a_{(j,i)} + a_{(i,j)}) = s_{(j,i)}$; the $(j,i)^{th}$ element of S. Hence, S is symmetric by definition.
The $(i,j)^{th}$ element of $T = \frac{1}{2}(A - A^T)$ would be $t_{(i,j)} = \dfrac{1}{2}(a_{(i,j)} - a_{(j,i)}) = -\dfrac{1}{2}(a_{(j,i)} - a_{(i,j)}) = -t_{(j,i)}$; the negative of the $(j,i)^{th}$ element of T. Hence, S is skew-symmetric by definition.
For the last part, $S + T = \{s_{(i,j)} + t_{(i,j)}\} = \{\dfrac{1}{2}(a_{(i,j)} + a_{(j,i)}) + \dfrac{1}{2}(a_{(i,j)} - a_{(j,i)})\} = a_{(i,j)} = A$.

Question 4

Show that the product of two lower triangular matrices is a lower triangular matrix. A similar statement holds for upper triangular matrices.

Answer

Lets A and B be two $(n \times n)$ lower triangular matrices. We fix $i < j$. Then

$$[AB]_{(i,j)} = \sum_{k=1}^{n} A_{(i,k)}.B_{(k,j)}$$

Now, either $i < k$ or $k < j$ holds, because the negation $k \leq i < j \leq k$ is not possible. So either $A_{(i,k)} = 0$ or $B_{(k,j)} = 0$, since A and B both are lower triangular matrices. Therefore all the addends are zero; i.e. $[AB]_{(i,j)} = 0 \;\forall\; i, j \in \{1, 2, \ldots n\}$ and $i < j$. Hence, that the product of two lower triangular matrices is a lower triangular matrix.

Question 5

Show that the diagonal entries of a skew-symmetric matrix are zero.

Answer

A $(n{\times}n)$ matrix A is defined to be a skew-symmetric matrix if $\forall\; i, j \in \{1, 2, \ldots n\},\; A[i, j] = -A[j, i]$. For diagonal elements of A, $i = j$; which gives $A[i, i] = 0 \;\forall\; i \in \{1, 2, \ldots n\}$. Hence, the diagonal entries of a skew-symmetric matrix are zero.