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Numerical Analysis II
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Project Proposal: A Comparative Study of Numerical Methods for Solving the Black-Scholes Partial Differential Equation for American Options Pricing

Introduction

Financial options are derivatives whose values are based on the prices of underlying assets. European options and American options are two types of financial options with different characteristics. European options can only be exercised at the expiration date, while American options can be exercised at any time before the expiration date. This difference has significant implications for pricing and hedging strategies.

The Black-Scholes model is a widely used mathematical model for pricing financial options. The Black-Scholes model assumes that the underlying asset follows a geometric Brownian motion and that the market is efficient. Black and Scholes showed in their seminal paper (Black & Scholes, 1973) that the pricing distribution of the underlying asset follows a lognormal distribution. Under these assumptions, the value of a European call option can be determined by solving the Black-Scholes partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where V is the value of the option, S is the price of the underlying asset, t is time, σ is the volatility of the underlying asset, and r is the risk-free interest rate.

However, the Black-Scholes model assumes that the option can only be exercised at expiration, which is not the case for American options. The problem of pricing American options requires finding the optimal exercise time, which is the time at which the option holder should exercise the option to maximize their profit. This problem is more complicated than pricing European options because the option holder can exercise the option at any time before expiration, which creates the possibility of early exercise.

The optimal exercise problem for American options can be formulated as an optimal stopping problem, where the option holder must decide when to stop waiting and exercise the option. An American option's value is modeled as the maximum of the value gained from immediate exercise or waiting until a later time to exercise. The challenge in solving this problem is that the optimal exercise time depends on the future price of the underlying asset, which is uncertain.

Numerical methods such as finite difference methods, spectral methods, and Monte Carlo methods have been used to approximate solutions to the Black-Scholes PDE for American options. These methods allow for the incorporation of the possibility of early exercise into the pricing model by simulating the future prices of the underlying asset and determining the optimal exercise time at each time step. The choice of numerical method depends on the characteristics of the problem, such as the complexity of the underlying asset price dynamics and the desired accuracy of the solution.

In our project, we will explore the use of numerical methods to solve the problem of pricing American options, focusing on the optimal exercise problem. We will compare and contrast the performance of finite difference methods and lattice methods in approximating the solution to the Black-Scholes PDE for American options with various boundary conditions. Our goal is to provide insights into each method's strengths and weaknesses and identify the circumstances under which each method is most appropriate while recording which experiments yield the most valuable exercises.

Methods

We chose to use finite difference methods and a lattice method to approximate solutions to the Black-Scholes PDE.

To approximate using finite difference methods, we will use both an explicit Runge-Kutta method as formulated by (Martin-Vaquero, Khaliq, & Kleefeld, 2014), and the Crank-Nicolson method as formulated in their paper (Crank & Nicolson, 1946). Runge-Kutta is a time-stepping technique that approximates the solution of the PDE at each time step based on its previous value, while Crank-Nicolson is a time-stepping technique that takes into account the midpoint between two-time steps to calculate the solution. We will implement both methods in MATLAB or Python and compare their accuracy, efficiency, and computational complexity.

In addition to finite difference methods, we will also use the binomial tree method to approximate the Black-Scholes equation, as proposed by (Cox, Ross, & Rubinstein, 1979). The binomial tree method is a discrete-time model that divides the time horizon of the option into smaller time steps and models the evolution of the underlying asset price over time as a binomial tree. It is one of the most widely used lattice methods to price American options. We will implement this method in MATLAB or Python and compare its accuracy, efficiency, and computational complexity with the finite difference methods.

By comparing the results obtained from different numerical methods, we aim to gain insights into their relative strengths and limitations for option pricing in the financial markets. We will evaluate the performance of each method by comparing its results with the closed-form solution of the Black-Scholes equation, when available, and by conducting sensitivity analyses on different input parameters, such as volatility and interest rates.

Hypothesis

The Crank-Nicholson Method is expected to have the highest accuracy, while the Runge-Kutta method is expected to have similar accuracy but be slower. The Binomial Tree model is expected to be the fastest but less accurate than the other methods.

Rationale

We chose to use the Crank-Nicolson method and the Runge-Kutta method, both of which are finite difference methods, for their ability to provide accurate and stable solutions with fewer time steps. The Crank-Nicolson method is a second-order accurate method that provides better accuracy than other time-stepping techniques and is unconditionally stable. The Runge-Kutta method is a higher-order accurate method that can provide more accurate solutions but may be more computationally expensive.

In addition to finite difference methods, we also chose to use the binomial tree method for its simplicity and intuitiveness in modeling the evolution of the underlying asset price over time as a binomial tree. The binomial tree method is a discrete-time model that can provide accurate solutions for options with a finite maturity.

While we believe that the methods we have chosen are well-suited to the problem of approximating the Black-Scholes equation, there are also other methods that could be explored in future work, such as Monte Carlo and Spectral methods as proposed by (Bouchard, Chau, Manai, & Sid-Ali, 2019) and (Song, Zhang, & Tian, 2014) respectively. We plan to investigate the use of these methods in future work. By continuing to investigate and compare different numerical methods, we can gain a deeper understanding of the behavior of financial instruments and improve our ability to analyze and price them effectively.

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