A Comparative Study of Finite Difference Methods for Solving the Black-Scholes Partial Differential Equation for American Options Pricing

Josh Avery and Saad Rafiq
Department of Mathematics, College of Science and Engineering, Texas State University

Background

Pricing American Options is a complex problem due to the possibility of early exercise. Analytical solutions do not exist for American options, so numerical methods are required to determine their values accurately. Finite difference methods are a powerful numerical technique that can approximate solutions to partial differential equations such as the Black-Scholes PDE, which is used to price European options. To price American options, finite difference methods can be used to simulate future asset prices and determine the optimal exercise time. We aim to provide insights into the strengths and weaknesses of each method, specifically regarding run time, numerical convergence speed, and stability conditions.

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S}+(r-q)S\frac{\partial V}{\partial S}-rV=0 \quad (1)$$

Equation 1: The Black-Scholes Partial Differential Equation.

Image Credit: https://www.mainesbdc.org/

Hypothesis

We expected the Crank-Nicholson Method to have a balanced trade-off between runtime and mesh size. The Binomial Tree model was expected to have a costly trade-off between the mesh size and runtime but the most comprehensive application range (due to its stability conditions) over the other two methods. SERKv2 was expected to have a high convergence order but struggle with stability.

Rationale

The Binomial Tree and Crank-Nicolson methods are two popular finite difference methods for option pricing. These methods have been extensively studied in the literature and are widely used in practice. However, they have limitations, such as being computationally expensive and unstable for certain options.

Runge-Kutta schemes are established, powerful finite difference methods, and it has only recently been applied to option pricing effectively. Therefore, we wanted to investigate SERKv2's performance with the more established Binomial Tree and Crank-Nicolson methods.

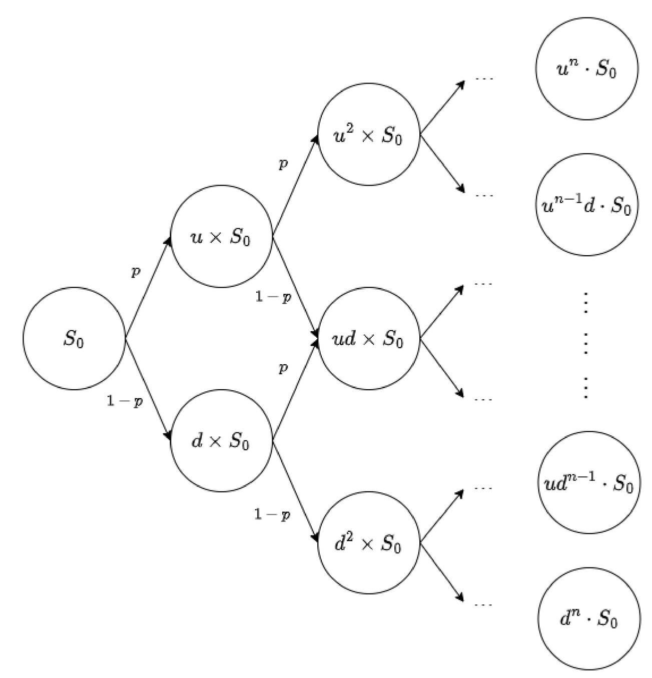
Methods

We present three finite-difference methods for option pricing:

- The Binomial Tree model,
- The Crank-Nicolson method, and
- The SERK2v2 method.

The Binomial Tree

This model is a simple, yet powerful tool that can be used to approximate the value of an option. It works by constructing a tree of possible future asset prices, then the option value is determined by backtracking through the tree according to equation 2:



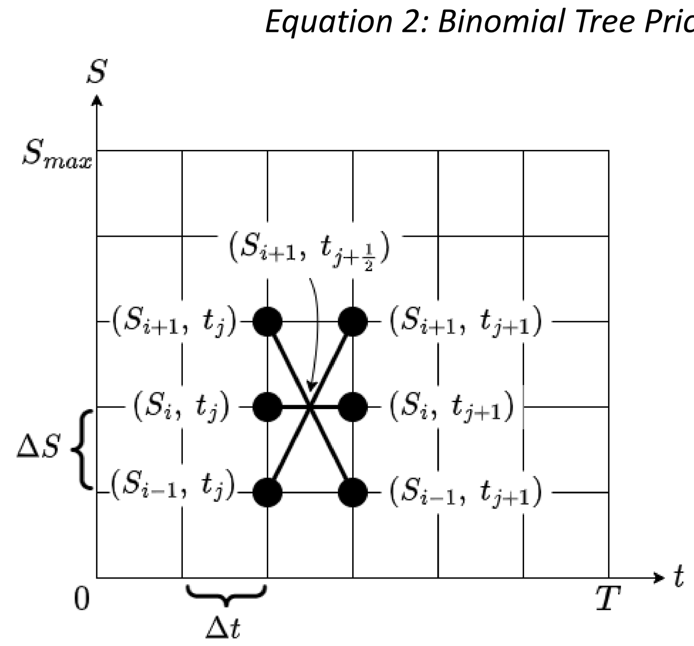
The Binomial Tree Model.

$$V_{t-\Delta t,i}=e^{-r\Delta t}\left(p\cdot V_{t,i}+(1-p)\cdot V_{t,i+1}\right) \quad (2)$$

Equation 2: Binomial Tree Pricing Formula.

Crank-Nicolson

The Crank-Nicolson method uses six points per approximation in its discretization. The method is based on a combination of explicit and implicit finite difference methods. We use an adaptation of the Crank-Nicolson model as discussed in Forsyth, Windcliff, & Vetzal (2004). The method uses an iterative approach by calculating the option's value based on the previous timestep's value.



The Crank-Nicolson Discretization.

$$V^{(j+1)}=\left(I_n+\frac{1}{2}A\Delta t\right)^{-1}\left(I_n-\frac{1}{2}A\Delta t\right)V^{(j)} \quad (3)$$

Equation 3: Timestep Iteration for the Crank-Nicolson Method.

Stabilized Explicit Runge-Kutta (SERKv2)

SERKv2, which is designed for solving parabolic PDEs like the heat equation, can be used to solve the Black-Scholes PDE after it has been transformed into a heat equation. The method first discretizes and systematizes the transformed equation, then solves these equations explicitly in a multi-stage approach. Our implementation is based on Martin-Vaquero, Khaliq, and Kleefeld (2014).

$$g_n^{(m)}=g_n^{(m)}+\Delta T\left[\begin{array}{c}\frac{1}{2}\sigma^2 S_n^2\frac{g_{n-1}^{(m)}-2g_n^{(m)}+g_{n+1}^{(m)}}{\Delta S^2}\\+\left(r-\frac{1}{2}\sigma^2\right)S_n\frac{g_{n+1}^{(m)}-g_{n-1}^{(m)}}{2\Delta S}-rg_n^{(m)}\end{array}\right] \quad (4)$$

Equation 4: SERKv2 stage evolution.

Results

Binomial Tree

	T	σ	K	r	q	S_0
GOOG	2	35.9	105	3.585%	0.0%	$104.70
AAPL	2	32.6	170	3.585	5.5%	$167.63

Table 1: Binomial Tree Parameters

Tree Height	Google Call	Apple Call	Google Put	Apple Put
20	0.0017359	0.0016396	0.0008454	0.0010092
100	0.02339	0.024604	0.0162228	0.0152802
500	0.5795052	0.5219235	0.4386829	0.402108
1,000	1.9802445	1.9216243	1.6476108	1.6310733
2,500	12.1569164	12.1644261	10.693373	10.4135767
5,000	47.5867264	44.4187062	43.4996731	42.2884514
10,000	179.086233	180.2901271	194.6959253	192.2900986
25,000	11130.097174	1218.819917	1179.468089	1123.494705
50,000	4709.238731	4397.227161	5117.154921	4945.517523

Table 2: Runtime (seconds) of the Bionomial Tree Methods Across Tree Heights

Crank-Nicolson

	T	σ	K	r	q	S_{max}	S_{min}
GOOG	2	35.9	105	3.585%	0.01%	500	0
AAPL	2	32.6	170	3.585%	5.5%	500	0

Table 3: Crank-Nicolson Parameters

		Stock Price Nodes (n_s)				
Times Nodes (n_t)		**1000**	**2000**	**3000**	**4000**	**5000**
Monthly	**24**	1.2450	4.9096	16.2104	36.0895	66.6167
Weekly	**104**	1.1965	5.1169	16.7891	37.1528	67.1918
Daily	**730**	1.4783	6.7505	24.4833	45.3177	84.1909
Hourly	**17,520**	7.6092	56.8880	130.5930	237.6383	360.4237

Table 4: Crank-Nicolson Runtimes (seconds) on Google Call Settings

		Stock Price Nodes (n_s)				
Times Nodes (n_t)		**1000**	**2000**	**3000**	**4000**	**5000**
Monthly	**24**	1.4172	5.2898	14.9086	35.1306	63.4802
Weekly	**104**	1.3650	5.3386	15.3934	36.1990	63.8750
Daily	**730**	1.7584	6.9092	19.2119	42.1976	74.3146
Hourly	**17,520**	7.5329	50.9348	118.3217	212.6308	337.5909

Table 5: Crank-Nicolson Runtime (seconds) on Google Call Settings

Stabilized Explicit Runge-Kutta (SERK2v2)

	T	stages	σ	K	r	M	N
GOOG	2	20	18.4076	140	3.585%	500	500
AAPL	2	20	13.6867	100	3.585%	500	500

Table 6: SERK2v2 Parameters

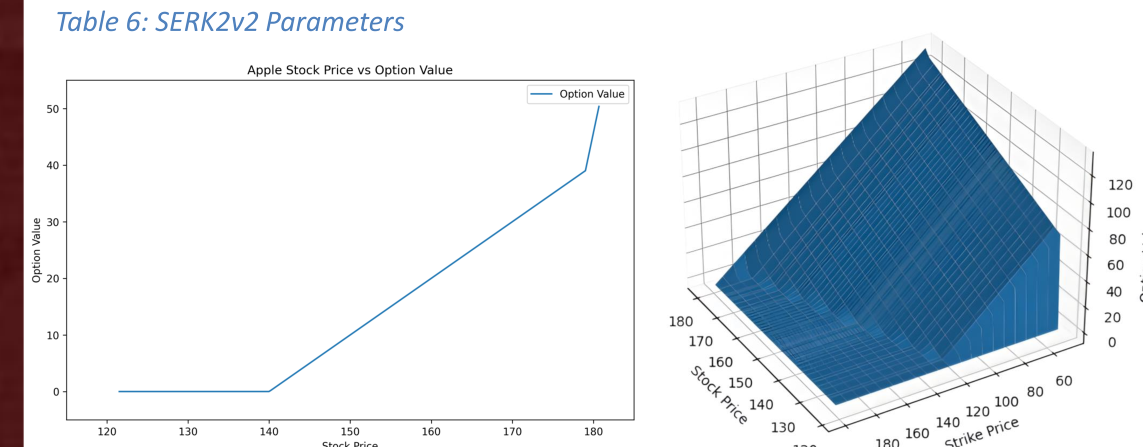


Figure 1: (Left) Apple Option Value vs Stock Price
(Right) Apple Option Value vs Strike Price vs Stock Price

Time Steps (M)	Stock Price Steps (N)	σ	Run Time (sec)
100	100	11.2777	0.1685
200	200	10.7519	0.2316
300	300	11.9730	0.3146
400	400	12.5641	0.4905
500	500	13.6867	0.5502

Table 7: SERK2v2 Runtime (seconds) on Apple Stock

Discussion and Future Work

The Binomial Tree model was much slower than initially anticipated. The method's stability performed as expected; however, the algorithm's runtime proved to be a barrier to the methods usage when using a fine timestep. Perhaps the Binomial Tree would perform well for volatile options with a shorter time horizon, as this would allow for a shorter tree height.

Googles Class-C stock does not pay dividends. However, in our experiments, we found that the Crank-Nicolson method was unstable for any value of q smaller than 0.01%. However, the Crank-Nicolson method performed much faster than the initial expectation of O(n4), however, the method is still slow for practicality. It is possible the method would perform optimally for options with larger time horizons that don't require cent precision or are not highly volatile

The SERK2v2 method showed to be efficient and optimal given the negligible runtime with varying N and M values. The method showed graphs that were displayed as expected.

In the future we plan to test the numerical approximation of the Black-Scholes PDE with different methods, as well as investigate optimization, lower-level language implementation (C++), and parallel processing.

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