

# Adversarial Training against Systematic Uncertainty

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## I. INTRODUCTION

The discovery of new particles or physical phenomena in high energy physics experiments, like those at the Large Hadron Collider, requires the observation of statistically significant deviations from the predictions of the Standard Model, the current model of the known fundamental particles and their interactions. Typically, discovery requires satisfying the *5-sigma rule*, whereby the deviation must be at least five standard deviations from predictions, i.e. having a p-value of  $p \lesssim 3 \times 10^{-7}$ . The evaluation of the statistical significance must not only account for statistical uncertainties, but also systematic uncertainties that represent our incomplete knowledge of the theory of particle interactions or of the properties of the experimental detection apparatus. Systematic uncertainties can alter the expected rate of signal and background classes, e.g. their priors, as well as the distributions of features. As such, the signal class true-positive rate and the background class false-positive rate of a classifier can change with variations in the systematic uncertainty thereby decreasing the sensitivity of an analysis. In this paper, we present a new strategy for training a classifier, based on adversarial training [1], which aims to build a classifier whose decision function is invariant under variations due to systematic uncertainties. In this way, uncertainties on predictions from simulations will have a reduced impact when computing the significance of potential deviations after application of the discriminant model and thereby increase the sensitivity of the analysis to signals from potential new physics sources. [GL: Shall we be more explicit somewhere and put a formula underlining the effect of having a pivotal classifier on the resulting sensitivity? (versus a classifier whose decision function depends on the nuisance.)]

Given data  $X$  and associated labels  $Y$  taking values  $y \in \mathcal{Y} = \{s, b_{i=1\dots k}\}$  where  $s$  is the signal class and  $b_{i=1\dots k}$  are the background classes, then the probability density of the data can be written as a mixture

$$p(X) = \pi_s p(X|Y = s) + \sum_{i=1}^k \pi_{b_i} p(X|Y = b_i), \quad (1)$$

where  $\pi_y$  represents the prior for the signal or a given background, and  $p(X|Y)$  is the conditional distribution

of the data for the signal or a given background. Systematic uncertainties can affect this distribution in the following ways [GL: add references]:

- Uncertainties on the priors  $\pi_y$ . These arise from uncertainties on the rates we expect a given process to occur as predicted from theoretical calculations or from alternative measurements of the rate of a given class.
- Uncertainties in the simulators of the physical interactions, giving rise to uncertainties on the distributions  $p(X|Y)$ . These uncertainties are often determined by comparing the predictions of different simulations using differing models of the same physical process.
- Uncertainties on the detection apparatus and its effect on the measurement of particle properties, giving rise to uncertainties on the distributions  $p(X|Y)$ . For instance, this includes uncertainties on our knowledge of the mean and variance of the measured energy distribution of a particle.

In this paper, we will not address uncertainties of the first type mentioned above (uncertainties on the priors). Rather, we will focus on mitigating the effect of systematic uncertainties that alter the conditional distributions  $p(X|Y)$ .

Systematic uncertainties are parameterized by nuisance parameters in the statistical analysis of data [GL: add references]. Nuisance parameter often, though not always, have a suitable prior coming from alternative measurements in data. For instance, The average measured energy of an electron in simulation may be compared with those in data using a relatively pure sample of  $Z$  boson decays to electrons. The average measured electron energy can not be measured without uncertainty, and the measured uncertainty can be used in other analyses to estimate the uncertainty deriving from our imperfect knowledge of the electron's energy. A nuisance parameter is assigned to the electron energy uncertainty, which controls the variation of the electron energy in the statistical analysis of data. Nuisance parameters are typically, though not always, constrained by a suitable prior, e.g. a Gaussian distribution with a standard deviation

equal to the measured uncertainty. [GL: ... and accordingly, we want to build a classifier independent on the nuisance parameter, as a way to be independent of the corresponding systematics.]

[GL: Add paper outline.]

## II. PROBLEM STATEMENT

Let assume a probability space  $(\Omega, \mathcal{F}, P)$ , where  $\Omega$  is a sample space,  $\mathcal{F}$  is a set of events and  $P$  is a probability measure. Let consider the multivariate random variables  $X_z : \Omega \mapsto \mathbb{R}^p$  and  $Y : \Omega \mapsto \mathcal{Y}$ , where  $X_z$  denotes a dependence of the functional  $X$  on a nuisance parameter  $Z$  whose values  $z \in \mathcal{Z}$  define a parameterized family of its systematic uncertainties. That is,  $X_z$  and  $Y$  induce together a joint probability distribution  $p(X, Y|z)$ , where the conditional on  $z$  denotes  $X_z$ . For training, let further assume a finite set  $\{x_i, y_i, z_i\}_{i=1}^N$  of realizations  $X_{z_i}(\omega_i), Y(\omega_i)$ , for  $\omega_i \in \Omega$  and known values  $z_i$  of the nuisance parameter. Our goal is to learn a function  $f(\cdot; \theta_f) : \mathbb{R}^p \mapsto \mathcal{Y}$  of parameters  $\theta_f$  (e.g., a neural network-based classifier if  $\mathcal{Y}$  is a finite set of classes) and minimizing a loss  $\mathcal{L}_f(\theta_f)$  (e.g., the cross-entropy). In addition, we require that  $f(X_z; \theta_f)$  should be robust to the value  $z$  of the nuisance parameter – which remains unknown at test time. More specifically, we aim at building  $f$  such that in the ideal case

$$f(X_z(\omega); \theta_f) = f(X_{z'}(\omega); \theta_f) \quad (2)$$

for all samples  $\omega \in \Omega$  and all  $z, z'$  pairs of values of the nuisance parameter.

Since we do not have training tuples  $(X_z(\omega), X_{z'}(\omega))$  (for the same unknown  $\omega$ ), we propose instead to solve the closely related problem of finding a predictive function  $f$  such that

$$\begin{aligned} P(\{\omega | f(X_z(\omega); \theta_f) = y\}) \\ = P(\{\omega' | f(X_{z'}(\omega'); \theta_f) = y\}) \end{aligned} \quad (3)$$

for all  $y \in \mathcal{Y}$ . In words, we are looking for a predictive function  $f$  which is a pivotal quantity [2] with respect to the nuisance parameter. That is, such that the distribution of  $f(X_z; \theta_f)$  is invariant with respect to the value  $z$  of the nuisance. Note that a function  $f$  for which Eqn. 2 is true necessarily satisfies Eqn. 3. In general, the converse is however not true, since the sets of samples  $\{\omega | f(X_z(\omega); \theta_f) = y\}$  and  $\{\omega' | f(X_{z'}(\omega'); \theta_f) = y\}$  do not need to be the same for the equality to hold. In order to simplify notations, and as only Eqn. 3 is of direct interest in this work, we denote from here on the pivotal quantity criterion as

$$p(f(X; \theta_f)|z) = p(f(X; \theta_f)|z') \quad (4)$$

for all  $z, z' \in \mathcal{Z}$ .

## III. METHOD

Adversarial training was first proposed by [1] as a way to build a generative model capable of producing samples from random noise  $z \sim p_Z$ . More specifically, the authors pit a generative model  $g : \mathbb{R} \mapsto \mathbb{R}^p$  against an adversary classifier  $d : \mathbb{R}^p \mapsto \{0, 1\}$  whose antagonistic objective is to recognize real data  $X$  from generated data  $g(Z)$ . Both models  $g$  and  $d$  are trained simultaneously, in such a way that  $g$  learns to produce samples that are difficult to identify by  $d$ , while  $d$  incrementally adapts to changes in  $g$ . At the equilibrium,  $g$  models a distribution whose samples can be identified by  $d$  only by chance. That is, assuming enough capacity in  $d$  and  $g$ , the distribution  $p_g(Z)$  eventually converges towards the real distribution  $p_X$ .

In this work, we repurpose adversarial training as a means to constraint the predictive model  $f$  in order to satisfy Eqn. 4. As illustrated in Figure 1, we pit  $f$  against an adversary model  $r := p_{\theta_r}(z|f(X; \theta_f))$  of parameters  $\theta_r$  and associated loss  $\mathcal{L}_r(\theta_f, \theta_r)$ . This model takes as input realizations of  $f(X; \theta_f)$ , for the current value  $\theta_f$  of  $f$  parameters, and produces as output a function  $p_{\theta_r}(z|f(X; \theta_f))$  modeling the posterior probability density that  $z$  parameterizes the sample  $X$  observed through  $f(\cdot; \theta_f)$ . Intuitively, if  $p(f(X; \theta_f)|z)$  varies with  $z$ , then the corresponding correlation can be captured by  $r$ . By contrast, if  $p(f(X; \theta_f)|z)$  is invariant with  $z$ , as we require, then  $r$  should perform poorly and be close to random guessing. Training  $f$  such that it additionally minimizes the performance of  $r$  therefore acts as a regularization towards Eqn. 4.

If  $Z$  takes discrete values, then  $p_{\theta_r}$  can be represented e.g. as a probabilistic classifier  $\mathbb{R} \mapsto \mathbb{R}^{|\mathcal{Z}|}$  whose output  $j$  (for  $j = 1, \dots, |\mathcal{Z}|$ ) is the estimated probability mass  $p_{\theta_r}(z_j|f(X; \theta_f))$ . Similarly, if  $Z$  takes continuous values and if we assume some parametric distribution  $\mathcal{P}$  for  $Z|f(X; \theta_f)$  (e.g., a truncated gaussian over a bounded support), then  $p_{\theta_r}$  can be represented e.g. as network whose output  $j$  is the estimated value of the corresponding parameter  $\theta^j$  of that distribution (e.g., its mean and variance). As in [3], the estimated probability density  $p_{\theta_r}(z|f(X; \theta_f))$  can then be evaluated for any  $z \in \mathcal{Z}$ . As further explained in the next section, let us note that the adversary  $r$  may take any form, i.e. it does need to be a neural network, as long as it exposes a differentiable function  $p_{\theta_r}(z|f(X; \theta_f))$  of sufficient capacity to represent the true distribution within its bounded support.

As for generative adversarial networks, we propose to train  $f$  and  $r$  simultaneously, which we carry out by considering the value function

$$E(\theta_f, \theta_r) = \mathcal{L}_f(\theta_f) - \mathcal{L}_r(\theta_f, \theta_r) \quad (5)$$

that we optimize by finding the saddle point  $(\hat{\theta}_f, \hat{\theta}_r)$  such



FIG. 1. Architecture for the adversarial training of a binary classifier  $f$  against a continuous parameter  $Z$ .

that

$$\hat{\theta}_f = \arg \min_{\theta_f} E(\theta_f, \hat{\theta}_r), \quad (6)$$

$$\hat{\theta}_r = \arg \max_{\theta_r} E(\hat{\theta}_f, \theta_r). \quad (7)$$

Without loss of generality, the adversarial training procedure to obtain  $(\hat{\theta}_f, \hat{\theta}_r)$  is formally presented in Algorithm 1 in the case of a binary classifier  $f : \mathbb{R}^p \mapsto [0, 1]$  modeling  $p(Y = 1|X)$ . For reasons further explained in Section IV,  $\mathcal{L}_f$  and  $\mathcal{L}_r$  are respectively set to the expected value of the negative log-likelihood of  $Y|X$  under  $f$  and of  $Z|f(X; \theta_f)$  under  $r$ :

$$\mathcal{L}_f(\theta_f) = \mathbb{E}_{Y|X}[-\log p_{\theta_f}(Y|X)], \quad (8)$$

$$\mathcal{L}_r(\theta_f, \theta_r) = \mathbb{E}_{Z|f(X; \theta_f)}[-\log p_{\theta_r}(Z|f(X; \theta_f))], \quad (9)$$

The optimization algorithm consists in using stochastic gradient descent alternatively for solving Eqn. 6 and 7. [GL: Add a note to specify that quantities are assumed in expectation over their conditionings.]

#### IV. THEORETICAL RESULTS

In this section, we show that in the setting of Algorithm 1 where  $\mathcal{L}_f$  and  $\mathcal{L}_r$  are respectively set to expected value of the negative log-likelihood of  $Y|X$  under  $f$  and of  $Z|f(X; \theta_f)$  under  $r$ , the procedure converges to a classifier  $f$  which is a pivotal quantity in the sense of Eqn. 4.

In this setting, the nuisance parameter  $Z$  is considered as a random variable of bounded support, for which we require the uniform prior  $p(z)$  (for  $z \in \mathcal{Z}$ ). Importantly, classification of  $Y$  with respect to  $X$  is therefore considered in the context where  $Z$  is marginalized out, which means that the classifier minimizing  $\mathcal{L}_f$  is optimal with respect to  $Y|X$ , but not necessarily with  $Y|X, Z$ . Results hold for a nuisance parameter  $Z$  taking either categorical values or continuous values within a bounded support.

By abuse of notation,  $H(p_Z)$  denotes the differential entropy in this latter case. Finally, propositions below are derived in a non-parametric setting, by assuming that both  $f$  and  $r$  have enough capacity.

**Proposition 1.** *Let  $\hat{\theta}_r = \arg \max_{\theta_r} E(\theta_f, \theta_r)$  for fixed  $\theta_f$ . If  $p_{\hat{\theta}_r}(z|f(X; \theta_f)) = p(z)$  for all  $z \in \mathcal{Z}$ , then  $f$  is a pivotal quantity.*

*Proof.* The optimal parameters

$$\hat{\theta}_r = \arg \max_{\theta_r} E(\theta_f, \theta_r) = \arg \min_{\theta_r} \mathcal{L}_r(\theta_f, \theta_r)$$

are such that  $p_{\hat{\theta}_r}(z|f(X; \theta_f)) = p(z|f(X; \theta_f))$ . By assumption,  $p_{\hat{\theta}_r}(z|f(X; \theta_f)) = p(z)$ , and therefore  $p(z|f(X; \theta_f)) = p(z)$ . Using the Bayes' rule, we write

$$\begin{aligned} p(f(X; \theta_f)|z) &= \frac{p(z|f(X; \theta_f))p(f(X; \theta_f))}{p(z)} \\ &= p(f(X; \theta_f)), \end{aligned}$$

which holds for all  $z \in \mathcal{Z}$  and implies that  $f$  is a pivotal quantity.  $\square$

**Proposition 2.** *If there exists a saddle point  $(\hat{\theta}_f, \hat{\theta}_r)$  for Eqn. 6 and 7 such that  $E(\hat{\theta}_f, \hat{\theta}_r) = H(p_{Y|X}) - H(p_Z)$ , then  $f(\cdot; \hat{\theta}_f)$  is both an optimal classifier and a pivotal quantity.*

*Proof.* For fixed  $\theta_f$ , the adversary  $r$  is optimal at  $\hat{\theta}_r = \arg \max_{\theta_r} E(\theta_f, \theta_r) = \arg \min_{\theta_r} \mathcal{L}_r(\theta_f, \theta_r)$ , in which case  $p_{\hat{\theta}_r}(z|f(X; \theta_f)) = p(z|f(X; \theta_f))$  and  $\mathcal{L}_r$  reduces to the entropy  $H(p_{Z|f(X; \theta_f)})$  of the conditional distribution of the nuisance. The value function  $E$  can therefore be rewritten as

$$E'(\theta_f) = \mathcal{L}_f(\theta_f) - H(p_{Z|f(X; \theta_f)}).$$

In particular, we have the lower bound

$$H(p_{Y|X}) - H(p_Z) \leq \mathcal{L}_f(\theta_f) - H(p_{Z|f(X; \theta_f)})$$

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**Algorithm 1** Adversarial training of a classifier  $f$  against an adversary  $r$ .

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*Inputs:* training data  $\{x_i, y_i, z_i\}_{i=1}^N$ ;

*Outputs:*  $\hat{\theta}_f, \hat{\theta}_r$ ;

*Hyper-parameters:* Number  $T$  of training iterations, Number  $K$  of gradient steps to update  $r$ .

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1: for  $t = 1$  to  $T$  do
2:   for  $k = 1$  to  $K$  do ▷ Update  $r$ 
3:     Sample minibatch  $\{x_m, z_m\}_{m=1}^M$  of size  $M$ ;
4:     With  $\theta_f$  fixed, update  $r$  by ascending its stochastic gradient  $\nabla_{\theta_r} E(\theta_f, \theta_r) :=$ 

$$\nabla_{\theta_r} \sum_{m=1}^M \log p_{\theta_r}(z_m | f(x_m; \theta_f));$$

5:   end for
6:   Sample minibatch  $\{x_m, y_m, z_m\}_{m=1}^M$  of size  $M$ ; ▷ Update  $f$ 
7:   With  $\theta_r$  fixed, update  $f$  by descending its stochastic gradient  $\nabla_{\theta_f} E(\theta_f, \theta_r) :=$ 

$$\nabla_{\theta_f} \sum_{m=1}^M [-\log p_{\theta_f}(y_m | x_m) + \log p_{\theta_r}(z_m | f(x_m; \theta_f))],$$

   where  $p_{\theta_f}(y_m | x_m)$  denotes  $1(y_m = 0)(1 - f(x_m; \theta_f)) + 1(y_m = 1)f(x_m; \theta_f)$ ;
8: end for

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where the equality holds at  $\hat{\theta}_f = \arg \min_{\theta_f} E'(\theta_f)$  only when

- $\hat{\theta}_f$  minimizes the negative log-likelihood  $\mathcal{L}_f$  of  $X|Y$ , which happens when  $\hat{\theta}_f$  are the parameters of an optimal classifier and in which case  $\mathcal{L}_f$  reduces to its minimum value  $H(p_{Y|X})$ ,
- $\hat{\theta}_f$  maximizes the entropy  $H(p_{Z|f(X; \theta_f)})$ , which happens when all outcomes of  $Z|f(X; \theta_f)$  are equally likely and in which case  $p(z|f(X; \theta_f)) = p(z)$  for all  $z \in \mathcal{Z}$  since we require a uniform prior by construction. Note that in the continuous case, the supremum of the differential entropy over continuous distributions on the same bounded support is also realized by the uniform distribution over that support.

When the lower bound is active, we have  $p_{\theta_r}(z|f(X; \theta_f)) = p(z)$  because of the second condition, which induces that the optimal classifier  $f(\cdot; \hat{\theta}_f)$  is also a pivotal quantity as a consequence of Proposition 1.  $\square$

Proposition 2 suggests that if at each step of Algorithm 1 the adversary  $r$  is allowed to reach its optimum given  $f$  (e.g., by setting  $K$  sufficiently high) and if  $f$  is updated to improve  $\mathcal{L}_f(\theta_f) - H(p_{Z|f(X; \theta_f)})$ , then  $f$  should converge to a classifier which is both optimal and pivotal, provided such a classifier exists. On many problems of interest though, such a classifier may not exist because the nuisance parameter directly shapes the decision boundary, in which cases the lower bound  $H(p_{Y|X}) - H(p_Z) < \mathcal{L}_f(\theta_f) - H(p_{Z|f(X; \theta_f)})$  is strict:  $f$  can either be an optimal classifier or a pivotal quantity,

but not both simultaneously. In this situation, it is natural to rewrite the value function  $E$  as

$$E_\lambda(\theta_f, \theta_r) = \mathcal{L}_f(\theta_f) - \lambda \mathcal{L}_r(\theta_f, \theta_r), \quad (10)$$

where  $\lambda \geq 0$  is a hyper-parameter controlling the trade-off between the performance of  $f$  and its independence with respect to the nuisance parameter. Setting  $\lambda$  to a large value will preferably enforces  $f$  to be pivotal while setting  $\lambda$  close to 0 will rather constraint  $f$  to be optimal.

Interestingly, let us emphasize that these results hold using only the (1D) output of  $f$  (in the case of binary classification) as input to the adversary. We could similarly enforce an intermediate representation of the data to be pivotal, e.g. as in [4], but this is in fact not necessary.

## V. EXPERIMENTS

### A. Toy example

[GL: Define the architecture of  $r$  when  $Z$  is categorical ( $r$  is a standard classifier) or continuous ( $r$  is e.g a 2-output NN such that the distribution of  $Z|f(X)$  is modeled by a truncated Gaussian of known support). Cite Nix.]

### B. Physics example

## VI. RELATED WORK

To account for systematic uncertainties, experimentalists in high energy physics typically take as fixed a clas-

sifier  $f$  built from training data for a nominal value  $z_0$  of the nuisance parameter, and then propagate uncertainty [GL: add ref] by estimating  $p(f(x)|z)$  with a parameterized calibration procedure. Clearly, this classifier is however not optimal for  $z \neq z_0$ . In this setting, parameterized classifiers [5, 6] directly take (nuisance) parameters as additional input variables, hence ultimately providing the most statistically powerful approach for incorporating the effect of systematics on the underlying classification task. As argued in [7], such classifiers can however not be used on real data since the correct value  $z$  of the nuisance often remains unknown. This is typically not an issue in the context of parameter inference [5], where nuisance parameters are marginalized out, but otherwise often limits the range of their applications. In practice, parameterized classifiers are also computationally expensive to build and evaluate. In particular, calibrating their decision function, i.e. approximating  $p(f(x, z)|z)$  as a continuous function of  $z$ , remains an open challenge. By contrast, constraining  $f$  to be pivotal yields a classifier which may not be optimal with respect to  $Y|X, Z$ , as discussed in Section IV, but that can otherwise be used in a wider range of applications, since knowing the correct value  $z$  of the nuisance is not necessary. Similarly, calibration needs to be carried out only once, since the dependence on the nuisance is now built-in. [GL: Shall we also discuss the relation with [7] where point estimates of the nuisance are used as inputs to  $f$ ]

In machine learning, learning a pivotal quantity can

be related to the problem of domain adaptation [4, 8–12], where the goal is often stated as trying to learn a domain-invariant representation of the data. Likewise, our method also relates to the problem of enforcing fairness in classification [13, 14], which is stated as learning a classifier that is independent of some chosen attribute such as gender, color or age. For both families of methods, the problem can equivalently be stated as learning a classifier which is a pivotal quantity with respect to either the domain or the selected feature. In this context, [4, 14] are certainly among the closest to our work, in which domain invariance and fairness are enforced through an adversarial minimax setup composed of a classifier and an adversary discriminator. Following this line of work, our method can be regarded as a generalization that also supports the continuous case, which can be viewed as handling infinitely many domains, provided they can be continuously parameterized, or as enforcing fairness over continuous attributes.

[GL: Check related work of cited references to see if important related work are missing.]

## VII. CONCLUSIONS

## ACKNOWLEDGMENTS

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