Numerical Methods for Stochastic Differential Equations.
Exercise 2. Coherence resonance.
Due date: October 23, 2023

Consider the Fitz Hugh-Nagumo model originally introduced to describe nerve pulses:

$$\epsilon\dot{x} = x - \frac{1}{3}x^3 - y \tag{1}$$
$$\dot{y} = x + a + D\xi(t)\,, \tag{2}$$

where $\xi(t)$ is a Gaussian white noise in Stratonovich sense, with zero mean and correlation $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$ and $\epsilon \ll 1$ is a small parameter introducing fast and slow time scales. Disregarding the noise term the deterministic system has a stable fixed point for $|a| > 1$ while $|a| < 1$ there is a stable limit cycle. In what follows consider $\epsilon = 0.01$ and $a = 1.05$.

1. *Program:* Implement a program to integrate this set of equations using the Heun method for white noise. Include the listing of the programs. In what follows use a time step $h = 0.01$ storing data every 10 time steps.

2. *Deterministic trajectories:* Integrate the dynamics for $D = 0$ starting from different initial conditions, plot the trajectories and discuss the results.

3. *Stochastic trajectories:* Integrate the dynamics for $D = 0.02$. Plot a few trajectories and discuss the differences with respect to the deterministic case.

4. *Trejectories for different D:* Plot a trajectory for $D = 0.02$, $D = 0.07$ and $D = 0.25$. Discuss the similarities and differences.

5. *Autocorrelation function:* Consider the autocorrelation function normalized to the variance

$$\tilde{C}(s) = \frac{\langle(\hat{y}(t)\hat{y}(t+s)\rangle_t}{\langle\hat{y}(t)\rangle_t} \tag{3}$$

   where $\hat{y}(t) \equiv y(t) - \langle y\rangle_t$ and $\langle\rangle_t$ stands for time averages over a long trajectory. Evaluate the autocorrelation function $\tilde{C}(s)$ for s in the interval $[0, 40]$ for values of s separated by an interval 0.1 (the same interval at which you are storing data) for $D = 0.02$, $D = 0.04$, $D = 0.07$, $D = 0.1$ and $D = 0.25$. Plot the autocorrelation functions that you have evaluated.

6. *Autocorrelation time:* From the autocorrelation function evaluated in the previous point evaluate the autorrelation time as:

$$\tau_c = \int_0^\infty \tilde{C}^2(s)ds \tag{4}$$

   Plot the values of the autocorrelation time that you have obtained as function of D and discuss the results.