Numerical Methods for Stochastic Differential Equations.
Exercise 1. Decay from an unstable steady state.
Due date: October 23, 2023

Consider $\dot{x}(t) = ax - bx^3 + \sqrt{D}\xi(t)$ where $\xi(t)$ is a Gaussian white noise of zero mean and correlation $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. Take $a = 4$, $b = 1$, $D = 0.01$, $x(0) = 0$ as initial condition and use $h = 0.001$ as integration time step.

a) *Trajectories:* Integrate numerically 20 trajectories until time $t = 4$ storing data every $\Delta t = 0.01$ and plot them. Discuss the results.

b) *Transient anomalous fluctuations:* Evaluate $\langle x(t)\rangle$, $\langle x(t)^2\rangle$ and $\langle x(t)^4\rangle$, where $\langle ... \rangle$ stands for averages over trajectories, sampling at intervals $\Delta t = 0.01$. Use 1000 trajectories for the averages. Plot the results and also the variance of x^2, namely $\sigma^2_{x^2}(t) = \langle x(t)^4\rangle - \langle x(t)^2\rangle^2$ and show that it goes through a maximum at an intermediate time. Discuss the results.

c) *Probability density function:* Integrate numerically 5000 trajectories and evaluate an histogram for the distribution of x at times $t = 0.6$, $t = 0.9$, $t = 1.2$, $t = 1.5$, $t = 1.8$, $t = 3$ and $t = 4$. Normalize the histogram such that the total area is 1 and so that it corresponds to the probability distribution of x at these times and plot the results. Discuss the changes in shape.

d) *First passage time distribution:* Integrate the equations numerically to generate 5000 trajectories until the trajectory reaches $|x(t)| = x_b = 0.5$, recording the time at which the trajectory reaches that value. Plot a histogram of the distribution of the times normalized such that the total area is 1, this is known as the first passage time distribution. Is it symmetrical? Discuss the results.

e) *Mean first passage time:* Generate 5000 trajectories as before evaluating the average of the time at which trajectories reach $|x(t)| = x_b = 0.5$ (mean first passage time) for $D = 0.1$, $D = 0.01$, $D = 0.001$, and $D = 0.0001$. Plot the results as function of $\ln D$. Compare with $(-1/2a)\ln D$.

f) Include the listing of the programs.