Institute for Cross-Disciplinary Physics and Complex

Exercise 3

Stochastic Differential Equations

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a) In the main.jl file provided, I have implemented the Heun method to integrate the stochastic differential equation (SDE) with an Ornstein-Uhlenbeck noise $\xi_{OU}(t)$ of zero mean and $\langle \xi_{OU}(t)\xi_{OU}(t')\rangle = (1/2\tau)e^{-|t-t'|/\tau}$. The integration of the SDE is done by the **heunOrnsteinUhlenbeck** function that is define at the beginning of the main.jl file.

b) We will be using $g(x) = 1$, $D = 0.01$ and $a = 2$ for calculating 10 trajectories in $t \in [0, 5]$ for the initial condition $x_0 = 0$ with $t_0 = 0$. The values of D and a will be constant for all the exercise unless otherwise is said. In order to to that, we have set the iterations parameters to $n_{wtr} = 50$ (representing the number of iterations for the writing loop), where each iteration comprises $n_{stp} = 10$ steps with a time increment of $h = 0.01$. Consequently, the total duration of the simulation amounts to $t_{total} = hn_{wtr}n_{stp} = 5$ time units.

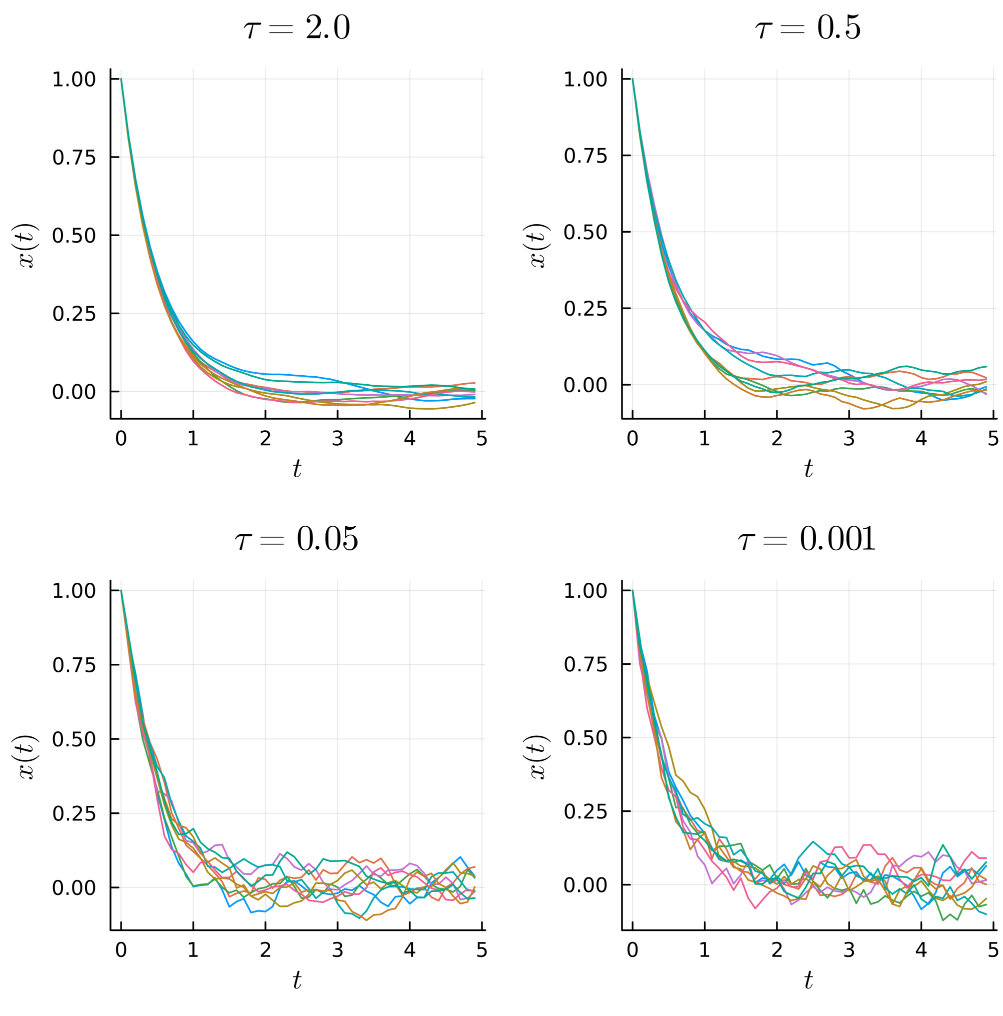

Figure 1: 10 Stochastic trajectory for $x_0 = 0$ and $t_0 = 0$ for different values of τ.

The results for $\tau \in [2, 0.5, 0.05, 0.001]$ are shown in Fig. 1. It can be clearly seen that fluctuations are stronger for lower values of τ. This behaviour was expected since lower values of τ means that the correlation decays faster leading to faster fluctuations. Note that we are not losing a significant amount of precision by using these low values of τ since $h/\tau = 0.01/0.001 = 10 < 18.4$ following the criteria that we discussed during the lectures.

c) Setting the parameters to the same value as for the previous calculation but fixing $\tau = 0.5$, we will calculate the first moment $\langle x(t) \rangle$ for $t \in [0, 5]$ averaging over $n = 10,\ 100,\ 1000$ trajectories. Results are shown in Fig. 2. From the plots, we can see that the fluctuations are smaller as we increased the number of trajectories for averaging. This was expected since the effect of the noise tends to cancel out with different trajectories.

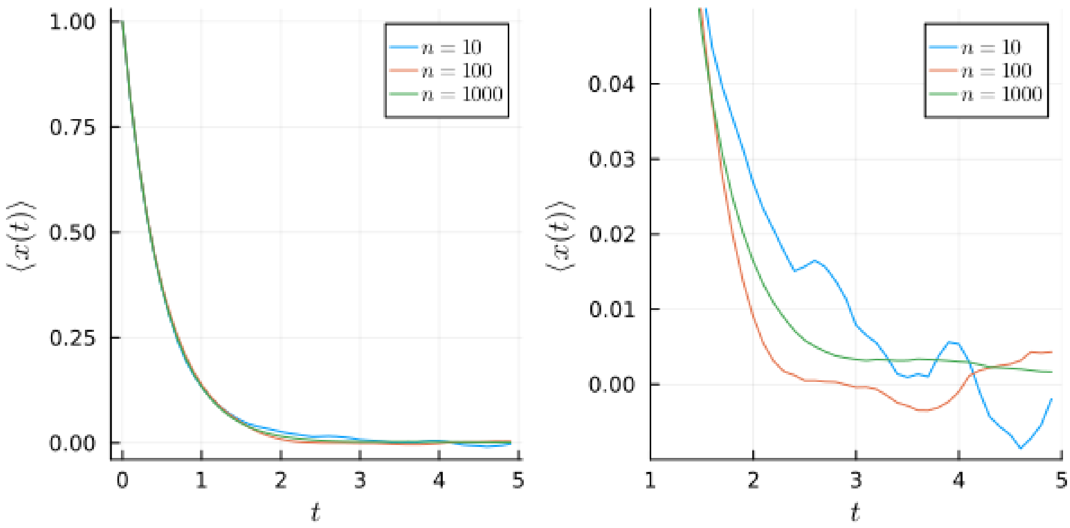

Figure 2: First moment $\langle x(t) \rangle$ obtained by averaging averaging over $n = 10,\ 100,\ 1000$ trajectories.

d) Setting the parameters to the same value as for the previous calculation, we will calculate the second moment $\langle x^2(t) \rangle$ for $t \in [0, 5]$ averaging over $n = 10,\ 100,\ 1000$ trajectories. We will also plot the deterministic trayectory that can be easily obtained as follows:

$$\dot{x}_{det} = -a x_{det} \quad \Longrightarrow \quad x_{det}(t) = x(0) e^{-at} \tag{1}$$

Results are shown in Fig. 3. From the plots, we can see that the fluctuations are smaller as we increased the number of trajectories for averaging mirroring the behaviour of $\langle x(t) \rangle$ as n increases. In fact, it is straight forward to see that the deterministic trajectory $x^2_{det}(t)$ could be a good approximation for the stochastic result of $\langle x^2(t) \rangle$. However, for larger values of t, discrepancies on the order of 10^{-3} emerge due to fluctuations. For $x^2(t)$, fluctuations do not cancel each other; negative fluctuations transform into positive ones, leading to a marginally positive average instead of converging to zero.

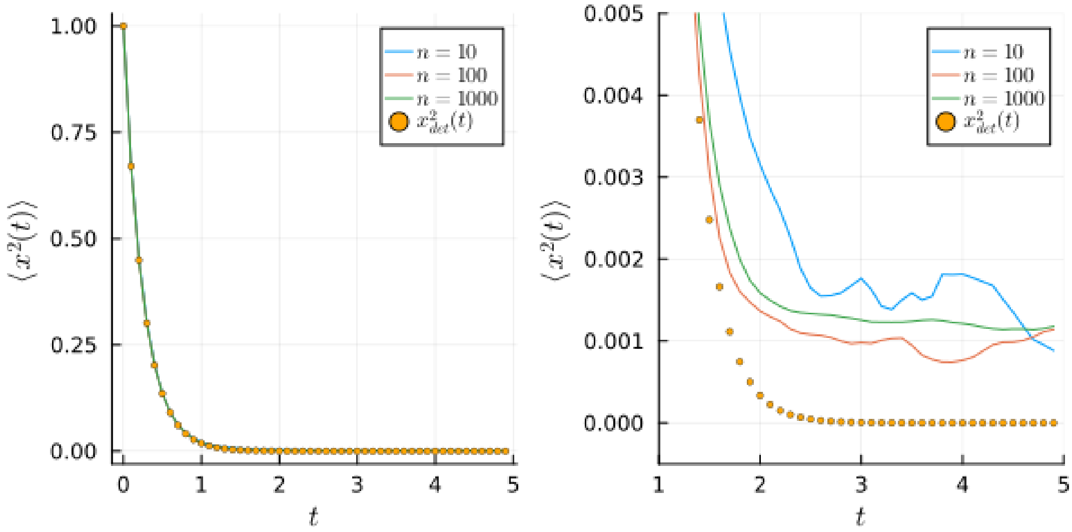

Figure 3: Second moment $\langle x^2(t) \rangle$ obtained by averaging averaging over $n = 10,\ 100,\ 1000$ trajectories and the deterministic trajectory $x^2_{det}(t)$.

e) In this part of the exercise we want to calculate the correlation function $C(t,s) = \langle x(t)x(t+s) \rangle$ averaging over 1000 trajectories. Using $x(t=0) = 0$, $h = 0.01$, $\tau = 0.5$, $n_{wrt} = 250$, $n_{stp} = 5$ we will compute $C_n(t,s) \equiv C(t,s)/C(t,0)$ for $s \in [0,5]$ and $\Delta s = 0.05$ time units.

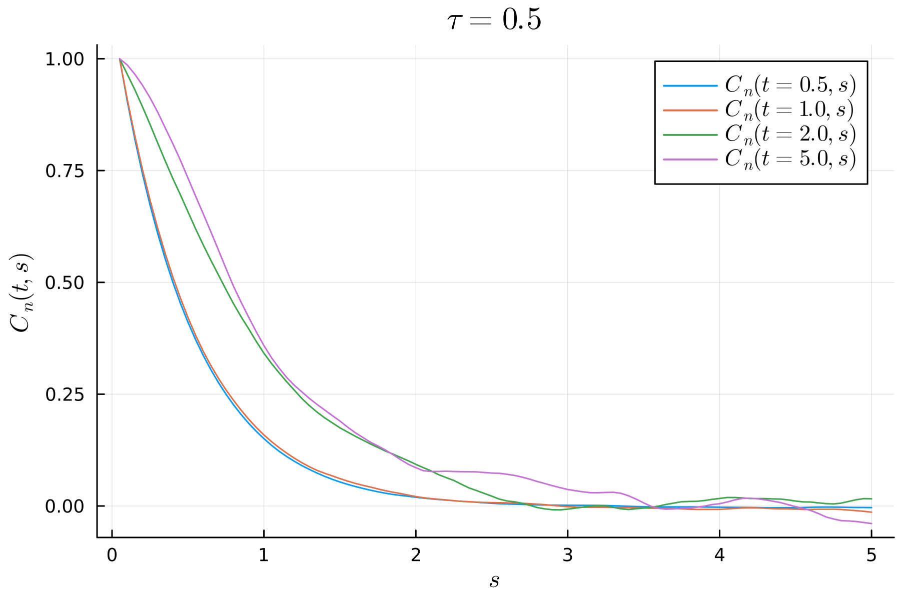

Figure 4: Normalized correlation function $C_n(t,s)$ for different values of time t.

Results are shown in Fig. 4 where we can see that $C_n(t,s)$ reach the stationary regime for $t \geq 2$ approximately. Calculating the correlation time for $t = 5$ we obtain $s_5 = 1.1$ time units. For $t = 2$, the result is $s_2 = 1.0$ which is quite similar to s_5. These correlation times are obtained by searching for the value s_t that makes $C(t, s_t) \approx e^{-1}$.

f) We will compute $C_n(t, s)$ using the same parameters as for the previous exercise but using $\tau = 0.05$ and $\tau = 2.0$. The results are shown in Fig. 5. We can see that for larger values of τ, the system reaches the stationary regime longer times. On the other hand, for $\tau = 0.05$, the system is already in the stationary regime for $t = 0.5$ even though fluctuations grows stronger as t increases due to the effect of the noise for $x(t \to \infty)$. This behaviour was expected since lower values of τ implies lower correlation in the noise term. The correlation time for $t = 5.0$ is $s_5(\tau = 0.05) = 0.5$ and $s_5(\tau = 2.0) = 2.7$.

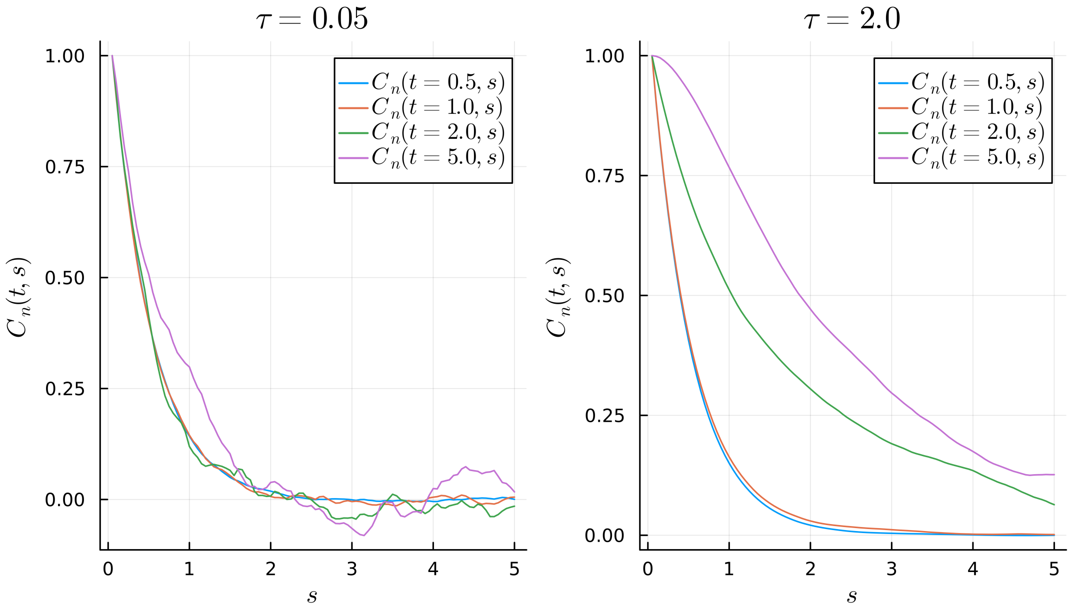

Figure 5: Normalized correlation function $C_n(t, s)$ for different values of time t and $\tau = 0.05, 2.0$.

g) In this final part of the exercise, we will use $g(x) = x$ which means that the noise will be multiplicative instead of additive. We will plot 10 stochastic trajectories using $x(t = 0) = 0$, $h = 0.01$, $\tau = 0.5$, $n_{wrt} = 500$, $n_{stp} = 1$. Results are shown in Fig. 6. Comparing these trajectories to the corresponding case in Fig. 1, we can see that fluctuations are significantly smaller for the case of the multiplicative noise. In fact, all the stochastic trajectories for the case of the multiplicative noise are very close to each other. This behaviour was expected since the multiplicative noise gets smaller at each time step leading to small different in the trajectory at earlier times.

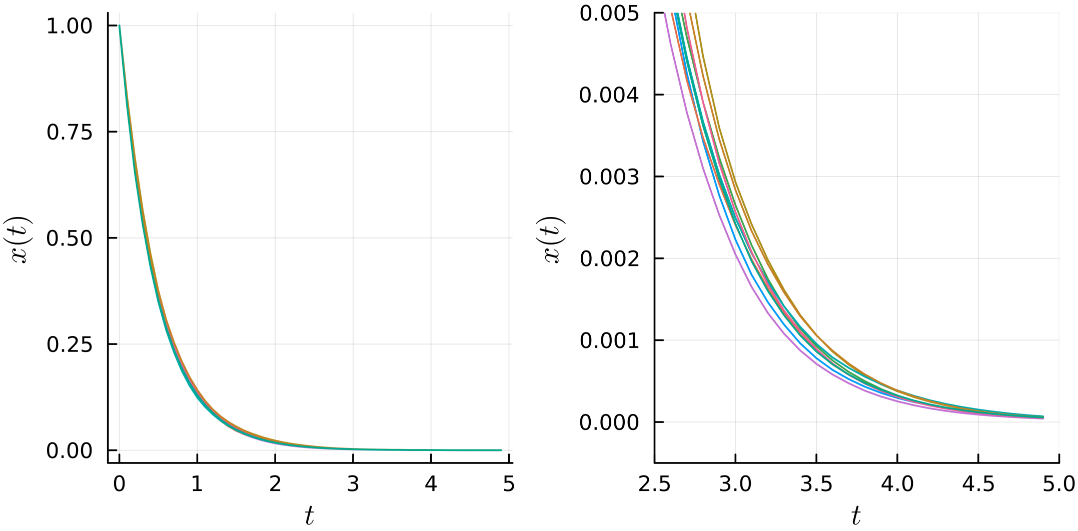

Figure 6: 10 stochastic trajectories for $g(x) = x$ using $\tau = 0.5$.