Institute for Cross-Disciplinary Physics and Complex

Exercise 1

Stochastic Differential Equations

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October 22, 2023

a) Firstly, it is easy to compute the fixed points of the deterministic differential equation:

$$\dot{x} = ax - bx^3 = 0 \quad \longrightarrow \quad x_{st}(a - bx_{st}^2) = 0 \quad \longrightarrow \quad x_0 = 0,\ x_\pm = \pm\sqrt{\frac{a}{b}}.$$

In our scenario, we set $a = 4$ and $b = 1$, resulting in $x_\pm = \pm 2$. The trajectories of the Stochastic Differential Equation (SDE) are illustrated in Figure 1. Initially, starting from the condition at $x = 0$, the absence of noise would yield a stationary point. However, the presence of noise leads the system to evolve towards the two other fixed points at $x_\pm = \pm 2$. Depending on the random fluctuations, there are instances when the trajectories converge to $x_+ = 2$, while in other cases, they approach $x_- = -2$.

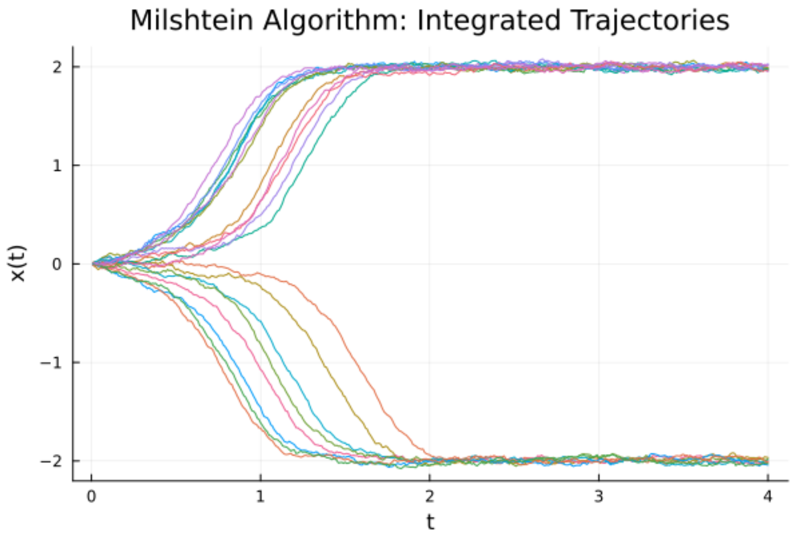

Figure 1: 20 trajectories for the SDE using Milshtein algorithm for $D = 0.01$, $h = 0.001$ storing data with $\Delta t = 0.01$.

b) The first, second, and fourth moments of the average over 1000 trajectories are shown in Figure 2. Regarding $\langle x(t)\rangle$, its value is quite close to zero, as one would expect since the average of noise over a sufficiently large number of trajectories should converge to zero. In this case, we observe that $\langle x(t)\rangle$ fluctuates around -0.02, primarily once it reaches the fixed point at time $t \approx 2$. This variation is attributed to random fluctuations, and the behavior will differ across various trajectories. Increasing the number of trajectories in the averaging process should bring the value even closer to zero.

For $\langle x^2(t)\rangle$, we can observe that it converges to approximately 4 once it reaches the fixed point at time $t \approx 2$. This is because we are calculating the average of $\langle x^2(t)\rangle$, and the square of the fixed points is $x_\pm^2 = 4$. Similar behavior is observed with $\langle x^4(t)\rangle$, which converges to approximately 16, reflecting $x_\pm^4 = 16$. The second and fourth moments also exhibit some fluctuations, although they are not readily discernible on the plot's scale.

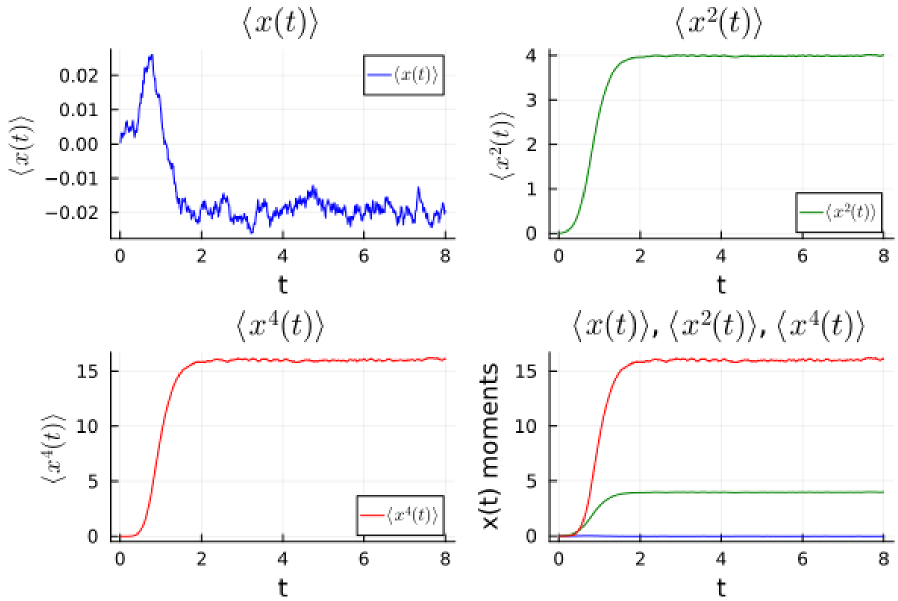

Figure 2: First, second and fourth moments of $x(t)$.

The variance of x^2 is shown in Figure 3. The peak corresponds to the moment when the trajectories transition from $x \approx 0$ to one of the other two fixed points. This peak occurs because, due to the noise, not all trajectories reach the fixed points simultaneously (as can be seen in Fig. 1), resulting in the observed peak.

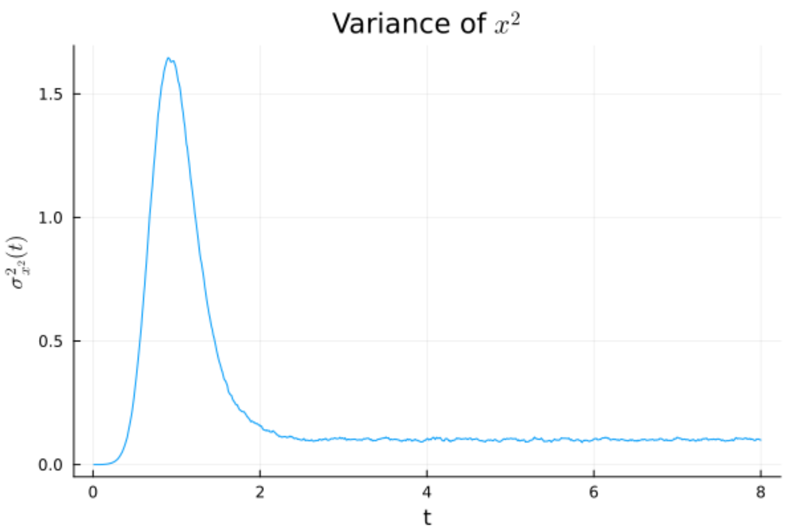

Figure 3: Variance of x^2 as a function of time.

c) Figure 4 presents nine distinct histograms for $x(t_i)$ at specific time instances, namely $t_i \in \{0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.5, 3.0, 4.0\}$. These plots offer a visual representation of how the distribution of $x(t_i)$ evolves over time. Initially, there is a peak around zero, exhibiting a Gaussian-like shape. As time progresses, this distribution gradually broadens, encompassing a wider range of positions (transitioning toward the fixed points $x\pm$). By the time we reach approximately $t \approx 1.5$, the position distribution becomes entirely concentrated around the fixed points $x_\pm = \pm 2$. Toward the end, two equally prominent peaks emerge with fluctuations around their maximum values. This phenomenon is attributed to the random noise, which introduces an equal probability of reaching either of the two fixed points.

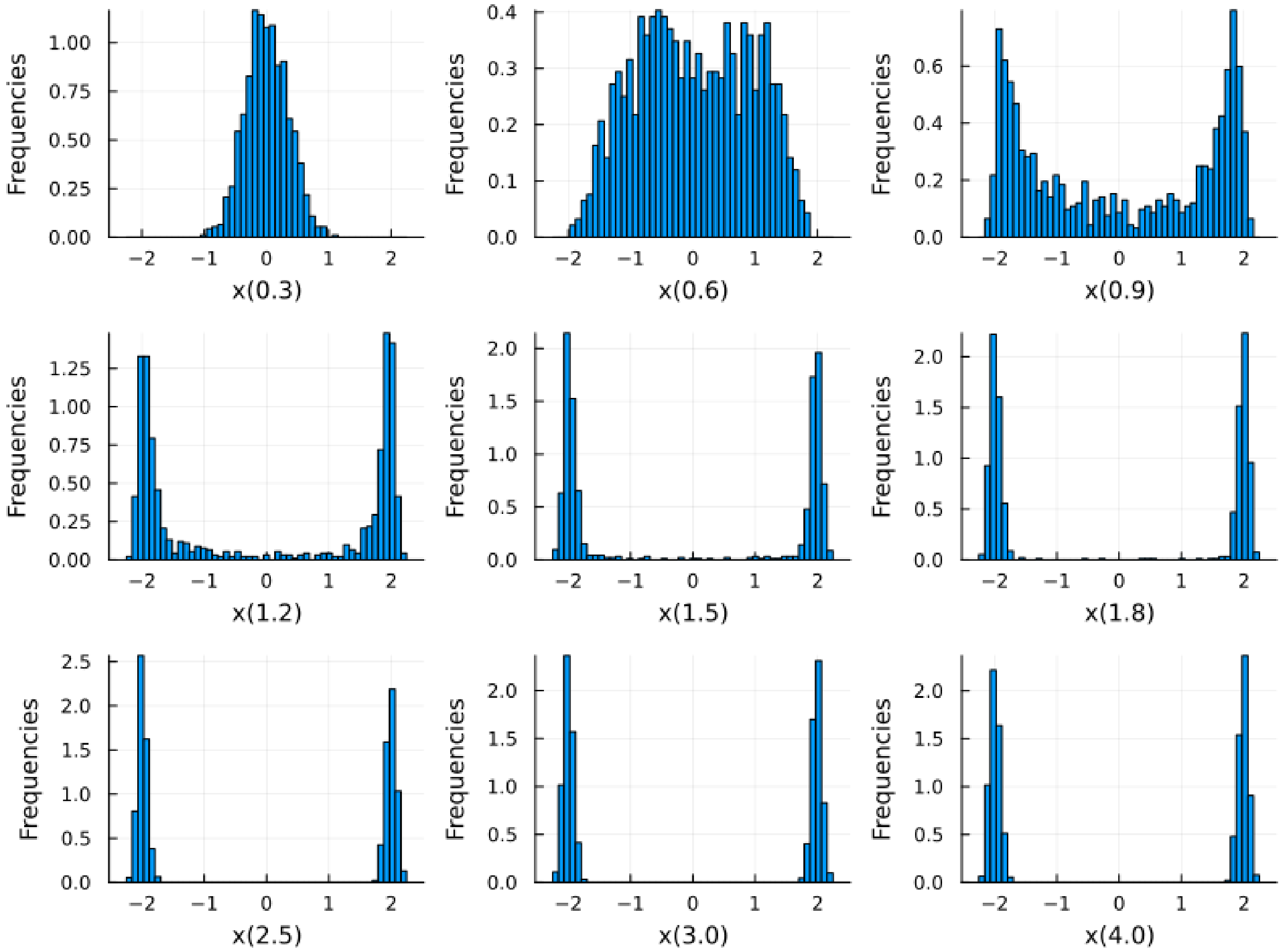

Figure 4: Histogram for the position $x(t_i)$ at concrete given time t_i

d) The histogram for the first passage time distribution at $|x(t)| = 0.5$ is illustrated in Fig. 5. This distribution exhibits a clear asymmetry, with a notable peak at lower values of the first passage time. This can be intuitively explained as follows: given that the initial condition corresponds to the unstable fixed point $x_0 = 0$ the presence of noise can easily cause the trajectory to fall into the system's dynamics, resulting in a shorter first passage time. However, there is still a small chance that noise may cause the trajectory to linger near the fixed point for an extended period, leading to a longer first passage time. While this scenario is less likely, it accounts for the long tail observed at larger time values in the distribution.

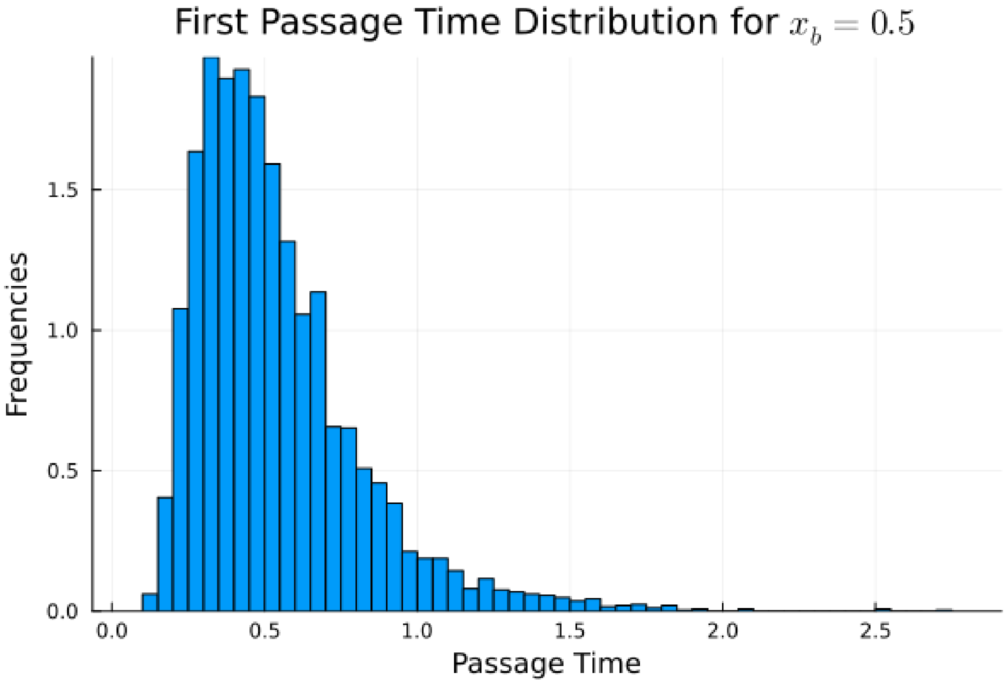

Figure 5: Histogram for the first passage time obtained using 5000 trajectories.

e) The mean first passage time (μ_τ), averaged over 5000 trajectories for various values of D, is shown in Fig. 6. We have also plotted the function $f(D) = \frac{-1}{2a} \ln D$. To facilitate a proper comparison between $f(D)$ and the experimental μ_τ, we additionally performed a linear fit to the mean first passage time data.

It is evident that the lines represented by μ_τ and $g(D)$ are seemingly parallel. Indeed, we can observe that the respective slopes are quite close, given that $\frac{-1}{2a} = -0.125$ for $a = 4$, and the slope of the fit is $m = -0.12536(73)$. The negative slope was expected since a larger noise level results in a quicker departure of the trajectory from the unstable fixed point x_0.

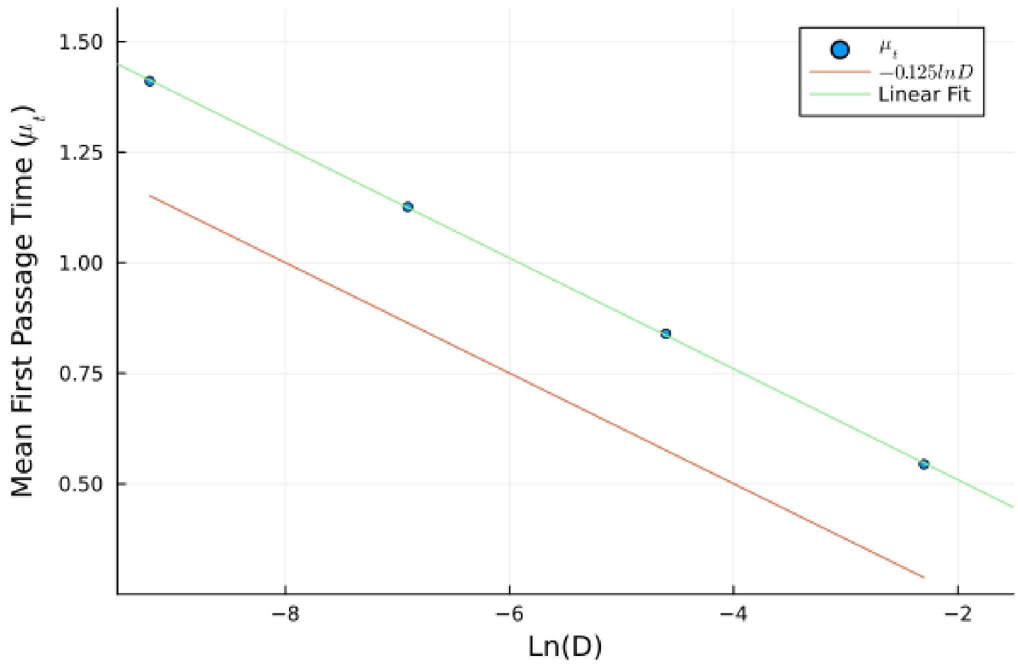

Figure 6: Plot for the first passage time as a function of LnD. The coefficients of the linear fit are $\mu_\tau = m\text{Ln}D + n$ with $m = -0.12536(73)$ and $n = 0.2584(47)$.

References

[1] Julia:Random Numbers