Institute for Cross-Disciplinary Physics and Complex

Exercise 2

Stochastic Differential Equations

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1) We've applied the Heun method to all equations, with the note that the noisy term for the x equation is zero.

2) In this part of the exercise, we've set the iterations parameters to $n_{wtr} = 100$ (representing the number of iterations for the writing loop), where each iteration comprises $n_{stp} = 100$ steps with a time increment of $h = 0.001$. Consequently, the total duration of the simulation amounts to $t_{total} = hn_{wtr}n_{stp} = 10$ time units. When we employ $D = 0$, we obtain the deterministic trajectories of the system. In Figures 1, 2, 3, and 4, we provide four illustrative examples of these trajectories. It's important to note that all of them ultimately converge towards the single stable fixed point of the system, considering the specific parameter values we've chosen ($a = 1.05$ and $\epsilon = 0.01$)

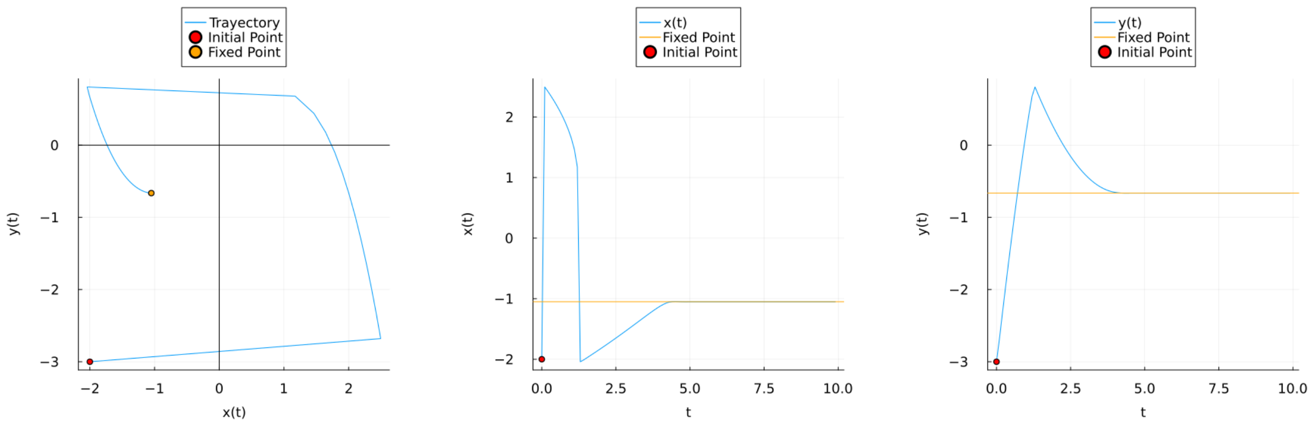

Figure 1: Example of a deterministic trajectory for $(x_0, y_0) = (-2, -3)$.

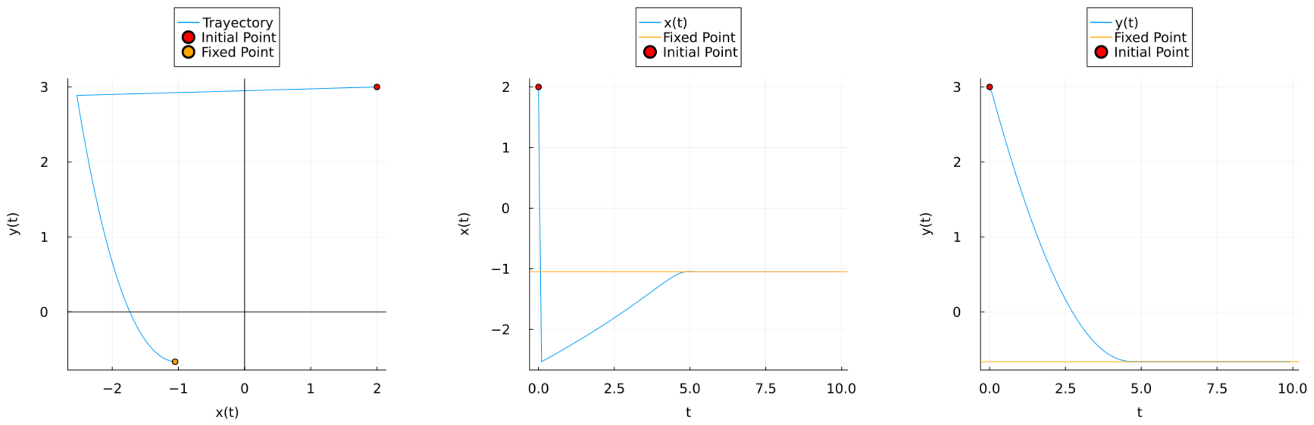

Figure 2: Example of a deterministic trajectory for $(x_0, y_0) = (2, 3)$.

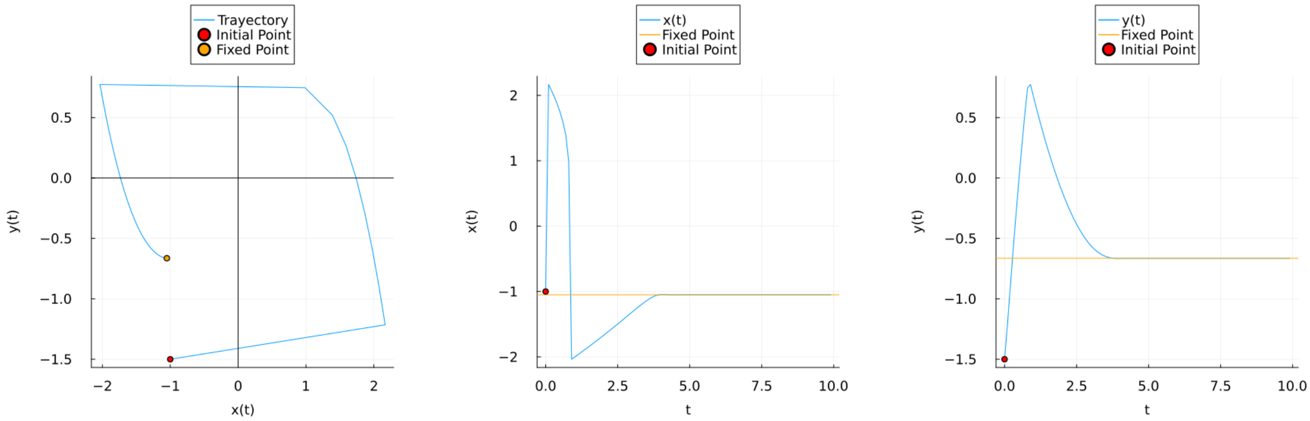

Figure 3: Example of a deterministic trajectory for $(x_0, y_0) = (-1, -1.5)$.

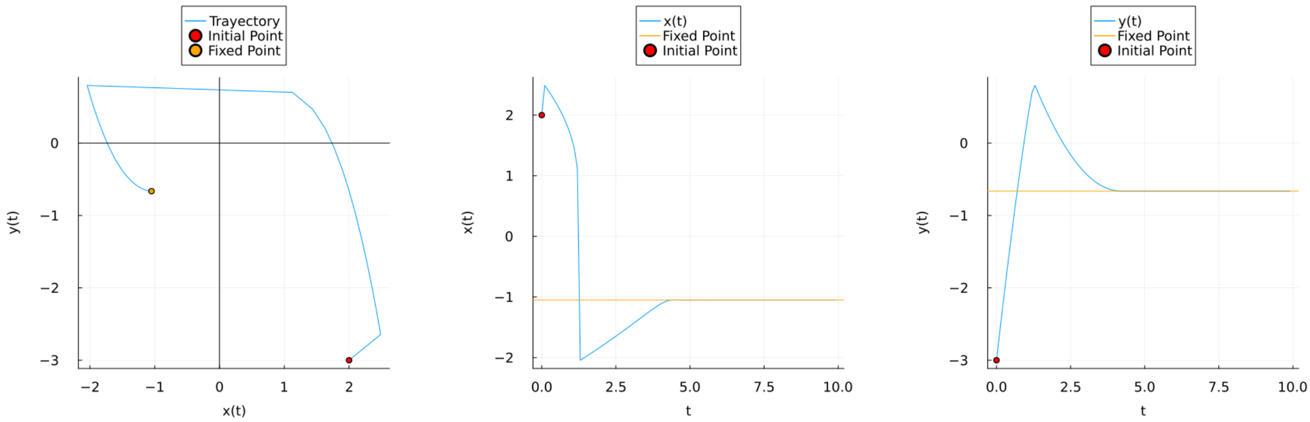

Figure 4: Example of a deterministic trajectory for $(x_0, y_0) = (2, 3)$.

3) For this part of the exercise, we will maintain the consistent values for $n_{wtr} = 100$, $n_{stp} = 100$, and $h = 0.001$. To compute the stochastic trajectories, we will adopt a value of $D = 0.02$. We have provided four illustrative examples of these trajectories in Figures 5, 6, 7, and 8. It's worth noting that these trajectories share the same initial conditions as their deterministic counterparts. Initially, they all converge towards the single stable fixed point, mirroring the deterministic behavior. However, due to the influence of stochastic noise, they do not remain indefinitely at this fixed point and instead exhibit cyclic behavior, periodically returning to the fixed point. Importantly, each of these cycles appears slightly distinct from the others, as the noise introduces unique variations to the trajectories on each cycle.

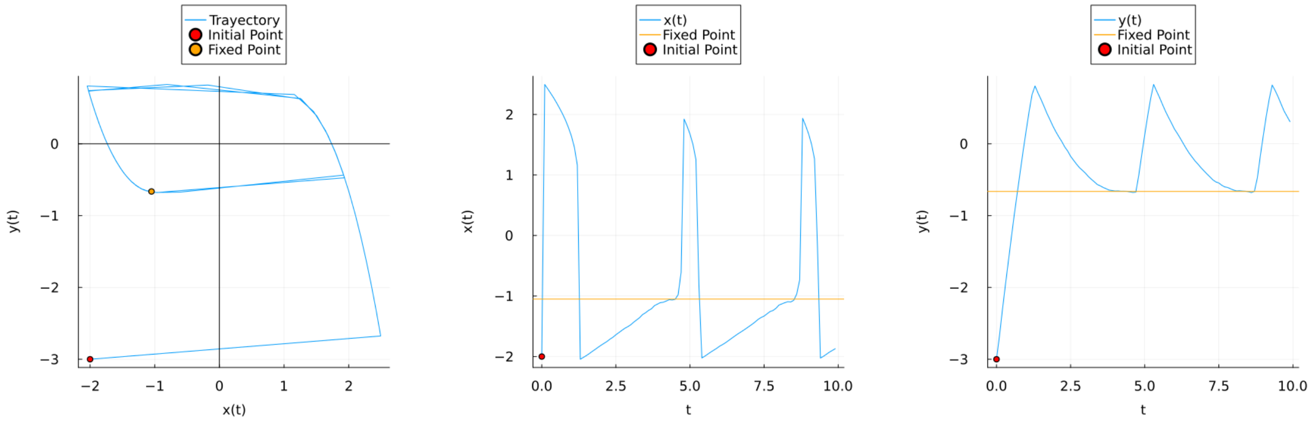

Figure 5: Example of stochastic trajectory for $(x_0, y_0) = (-2, -3)$.

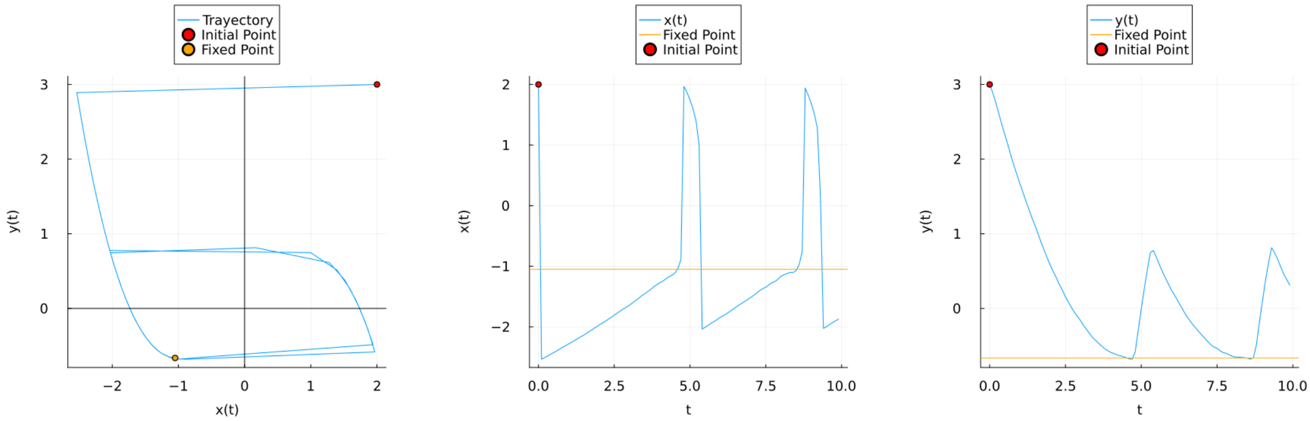

Figure 6: Example of a stochastic trajectory for $(x_0, y_0) = (2, 3)$.

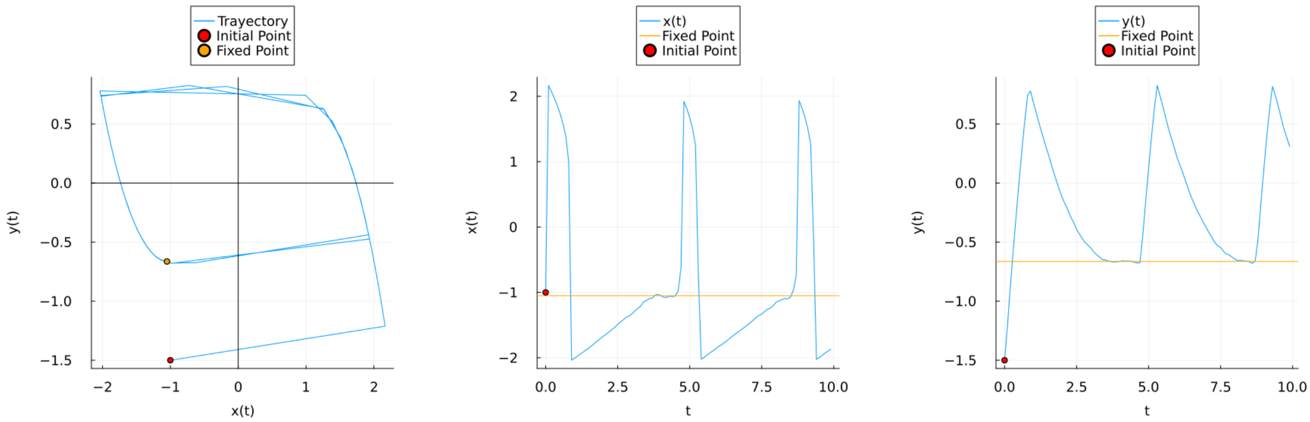

Figure 7: Example of a stochastic trajectory for $(x_0, y_0) = (-1, -1.5)$.

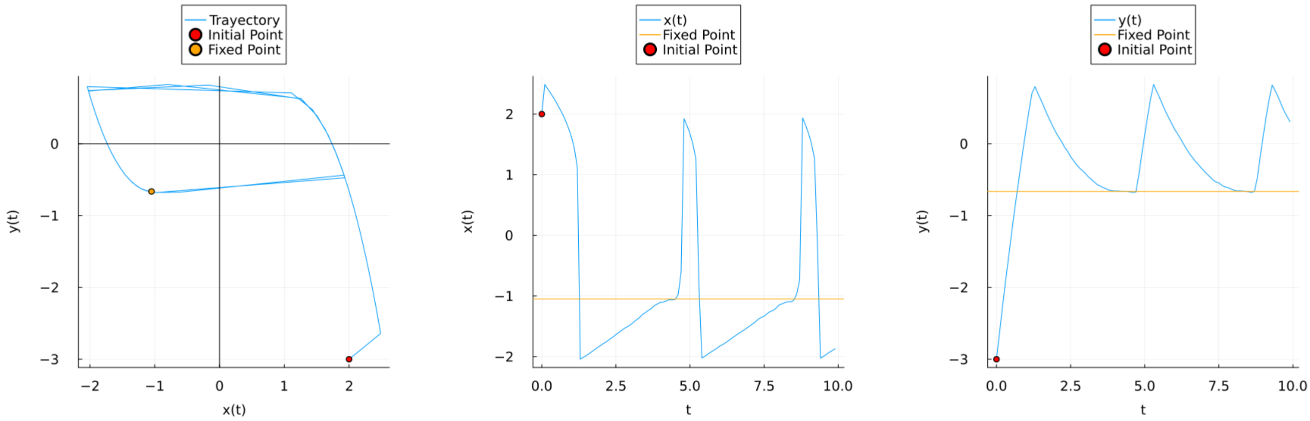

Figure 8: Example of a stochastic trajectory for $(x_0, y_0) = (2, 3)$.

4) In this particular segment of the exercise, we've adjusted $n_{wtr} = 400$, $n_{stp} = 25$, and $h = 0.001$. Although the total simulation time remains unchanged, these adjustments allow us to represent the trajectory more smoothly by using additional data points. In Figures 9, 10, and 11, we present four distinct examples of stochastic trajectories with an initial condition of $(x_0, y_0) = (-1, -1)$, while varying D within the set $\{0.02, 0.07, 0.25\}$. It's noteworthy that these trajectories exhibit the same cyclic behavior, periodically returning to the fixed point. However, it's crucial to emphasize that as the value of D increases, we observe larger fluctuations, leading to noticeable changes in the trajectory. Indeed, as evident in Figure 11, when $D = 0.25$, the noise significantly impacts the trajectory compared to the other cases.

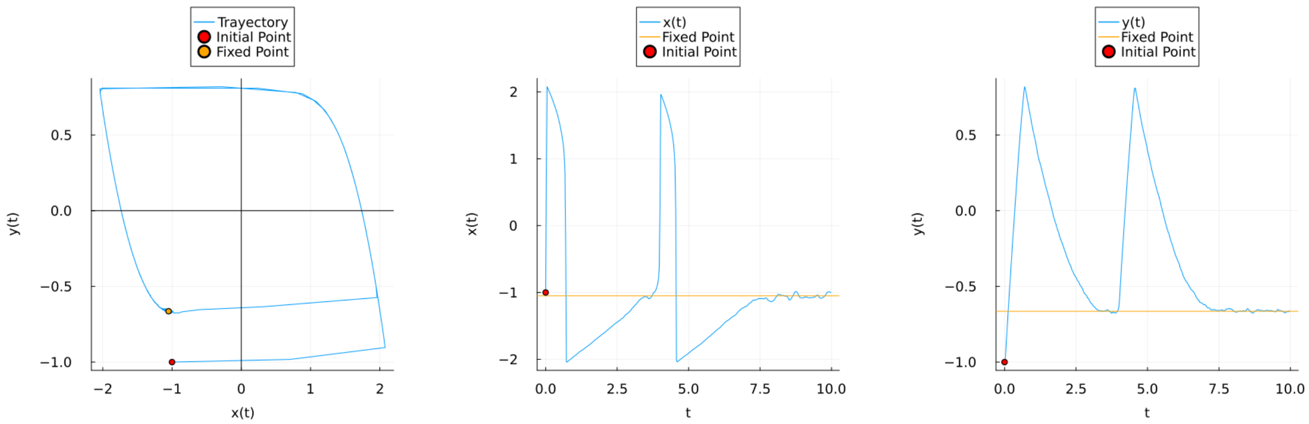

Figure 9: Example of a stochastic trajectory for $D = 0.02$ and $(x_0, y_0) = (-1, -1)$.

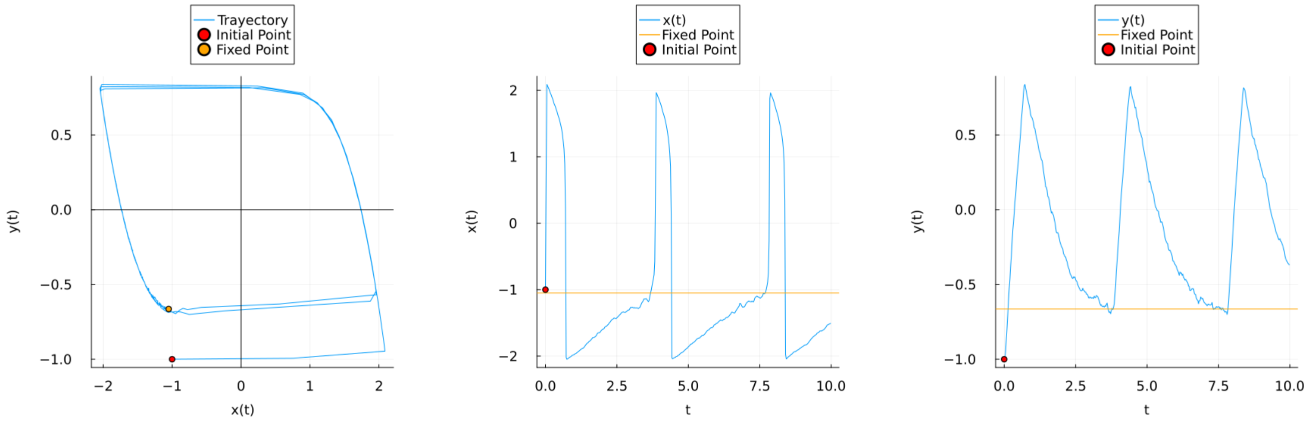

Figure 10: Example of a stochastic trajectory for $D = 0.07$ and $(x_0, y_0) = (-1, -1)$.

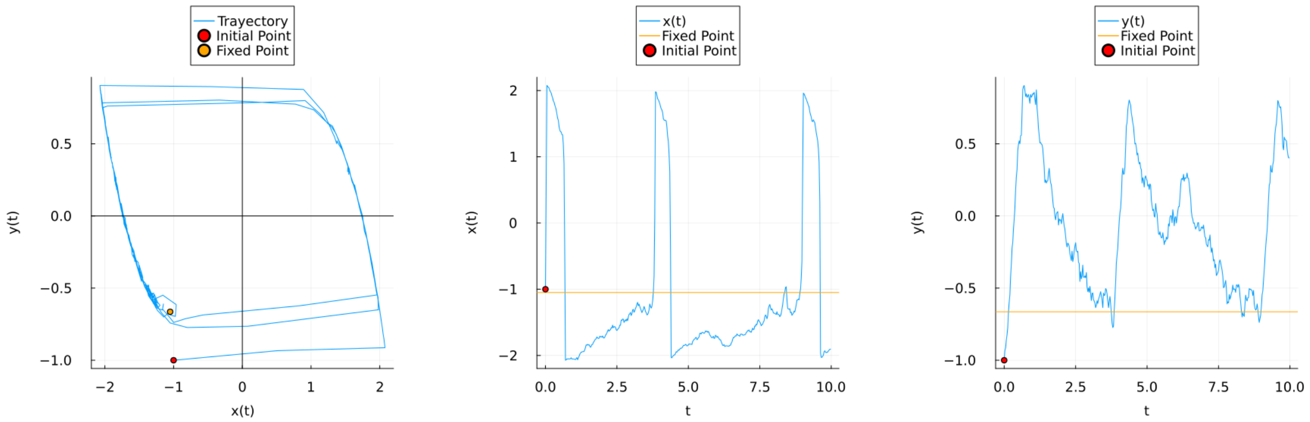

Figure 11: Example of a stochastic trajectory for $D = 0.25$ and $(x_0, y_0) = (-1, -1)$.

5) In this segment of the exercise, we've set $n_{wtr} = 2500$, $n_{stp} = 100$, and $h = 0.001$. As a result, the total simulation time increases, allowing for a greater number of points to calculate the autocorrelation. The results for different values of D, specifically 0.02, 0.04, 0.07, 0.1, 0.25, and 0.9, are displayed in Figure 12. It's evident that as the time shift increases, the autocorrelation rapidly fluctuates around zero. Moreover, we observe periodic behavior in the autocorrelation as s increases, which is anticipated as the noisy trajectory also exhibits periodicity.

For the highest values of D, we notice a reduction in the amplitude of the autocorrelation oscillations due to increased noise. However, this amplitude exhibits a slight increase when transitioning from $D = 0.02$ to $D = 0.1$. This can be explained as follows: As shown in Figure 9, there's a brief period during which the trajectory remains close to the fixed point due to the noise before departing and completing another cycle. This duration diminishes as the noise level rises (refer to Figure 10). The autocorrelation increases whenever this duration between cycles approaches zero. Nonetheless, if the noise intensity exceeds this threshold ($D \approx 0.1$), it can become overwhelming, leading to an almost negligible autocorrelation.

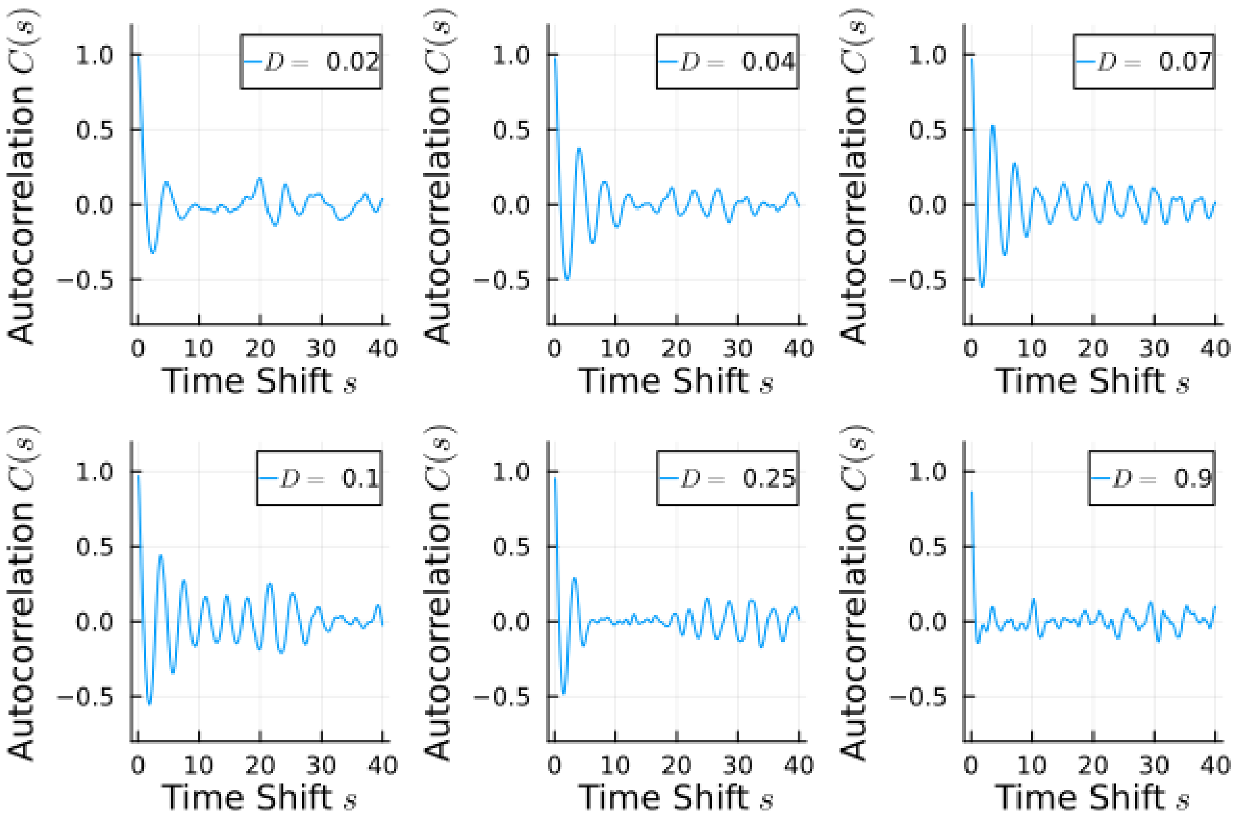

Figure 12: Autocorrelation as a function of the time shift for various values of D.

6) In this final part, we've utilized the autocorrelation values presented earlier. The autocorrelation time in relation to D is displayed in Figure 13. A peak is observed at $D = 0.01$. The autocorrelation time signifies the duration during which the system's behavior maintains self-correlations. When a peak in autocorrelation time emerges, it indicates a specific timescale in the system's behavior where correlations are most pronounced. As previously described, short interval exists during which the trajectory remains near the fixed point due to the influence of noise before resuming its cycle (see Figure 9). This interval gradually diminishes as the noise level intensifies (refer to Figure 10). The autocorrelation increases as this duration between cycles approaches zero, resulting in the peak observed in the autocorrelation time for that specific noise strength ($D = 0.01$).

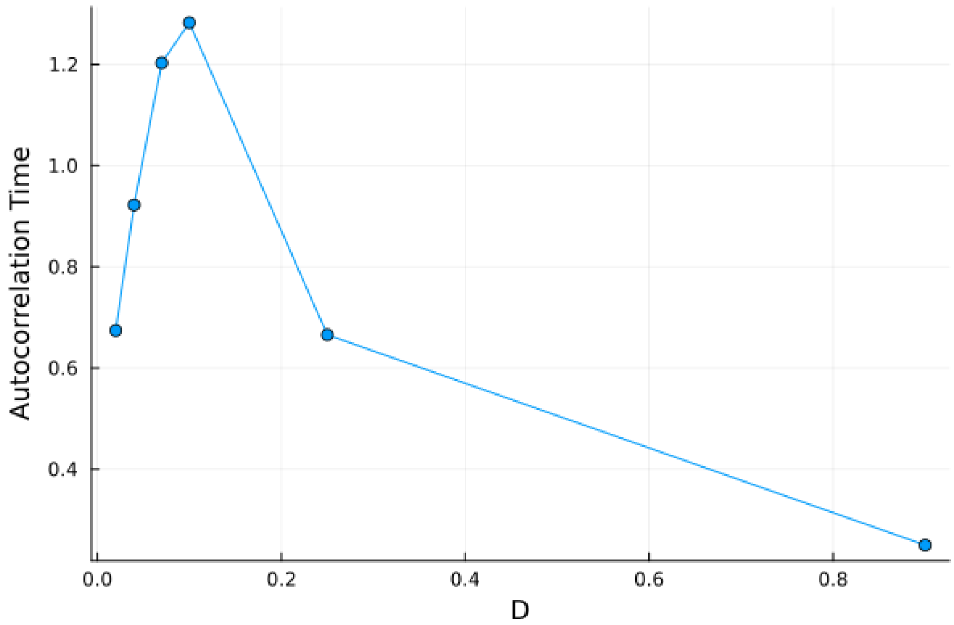

Figure 13: Autocorrelation time as a function of D.