Circle Assignment

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***Problem Statement* - In Figure 1. A,B,C are the three points with centre O such that ∠BOC=30° and ∠AOB=60°.If D is a point on the circle other than the arc ABC,find ∠ADC**

Solution

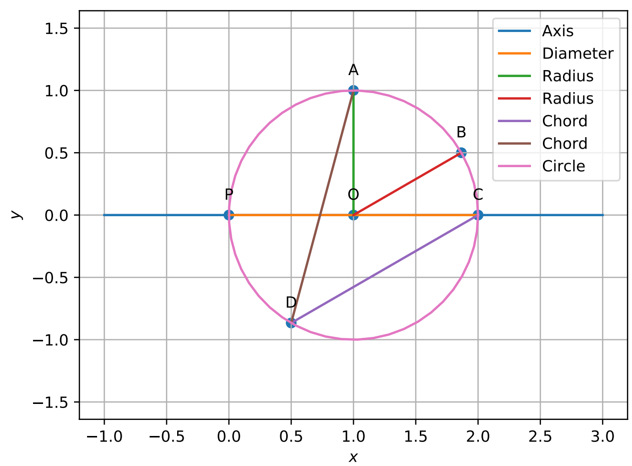

Figure 1:

Construction

The input parameters are the lengths

Symbol	value	Description
O		centre
∠BOC	30°	Angle between vectors B and C
∠AOB	60°	Angle between vectors A and B
∠ADC	??	Angle between vectors A and C

Table 1:

Assumptions

Let P be a point on the circle such that by expandig OC upto P we get diameter POC.

To find ∠ADC let the circle be unit circle and diameter POC on x axis.

Take three points C,A,D and α,β,γ be three angles made by the points C,A,D with respect to diameter POC.

From the Figure 2:

$$\alpha = \angle \mathbf{POC} = 180^\circ, \beta = \angle \mathbf{POA} = 90^\circ, \gamma = \angle \mathbf{POD} \tag{1}$$

Proof:

From assumptions the vector points C,A,D be

$$\mathbf{C} = \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix}, \mathbf{A} = \begin{pmatrix} cos\beta \\ sin\beta \end{pmatrix}, \mathbf{D} = \begin{pmatrix} cos\gamma \\ sin\gamma \end{pmatrix} \tag{2}$$

Let AC be the chord that subtends angles at the center O and at point D. The cosine of the angle subtended at point D is given by

$$cos(\angle ADC) = \frac{(A-D)^T(C-D)}{\|A-D\|\,\|C-D\|} \tag{3}$$

Where

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} cos\beta - cos\gamma \\ sin\beta - sin\gamma \end{pmatrix}, \mathbf{C} - \mathbf{D} = \begin{pmatrix} cos\alpha - cos\gamma \\ sin\alpha - sin\gamma \end{pmatrix} \tag{4}$$

$$(A-D)^T(C-D) = \begin{pmatrix} cos\beta - cos\gamma sin\beta - sin\gamma \end{pmatrix} \begin{pmatrix} cos\alpha - cos\gamma \\ sin\alpha - sin\gamma \end{pmatrix}$$

$$= (\cos\alpha - \cos\gamma)(\cos\beta - \cos\gamma) + (\sin\alpha - \sin\beta)(\sin\beta - \sin\gamma)$$

$$= -2\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha+\gamma}{2}\cdot(-2)\sin\frac{\beta-\gamma}{2}\sin\frac{\beta+\gamma}{2} + 2\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}\cdot 2\cos\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}$$

$$= 4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}(\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta+\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta+\gamma}{2})$$

$$= 4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\cos\left(\frac{\alpha+\gamma}{2} - \frac{\beta+\gamma}{2}\right)$$

$$= 4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\cos\frac{\alpha-\beta}{2} \tag{5}$$

$$\|A-D\|^2\,\|C-D\|^2 = ((\cos\alpha-\cos\gamma)^2+(\sin\alpha-\sin\gamma)^2)$$
$$((\cos\beta-\cos\gamma)^2+(\sin\beta-\sin\gamma)^2)$$

$$= (2-2\cos\alpha\cos\gamma-2\sin\alpha\sin\gamma)(2-$$
$$2\cos\beta\cos\gamma-2\sin\beta\sin\gamma)$$

$$= 4(1-\cos(\alpha-\gamma))(1-\cos(\beta-\gamma))$$
$$= 4\cdot 2\sin^2\frac{\alpha-\gamma}{2}\cdot 2\sin^2\frac{\beta-\gamma}{2}$$

$$= 16\sin^2\frac{\alpha-\gamma}{2}\sin^2\frac{\beta-\gamma}{2}$$

$$\|A-D\|\,\|C-D\| = 4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2} \tag{6}$$

Substituting (5) and (6) in (3),

$$cos(\angle ADC) = \frac{4sin\frac{\alpha-\gamma}{2}sin\frac{\beta-\gamma}{2}cos\frac{\alpha-\beta}{2}}{4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}}$$

$$cos(\angle ADC) = cos\frac{\alpha-\beta}{2} \tag{7}$$

By substituting α and β values in (7)

$$\angle ADC = \frac{\alpha-\beta}{2} = \frac{(180^\circ-90^\circ)}{2} = 45^\circ$$