Conics Assignment

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October 2022

Problem Statement **-If two tangents drawn from a point P to the parabola y^2=4x are at right angles, then the locus of P is**

Solution

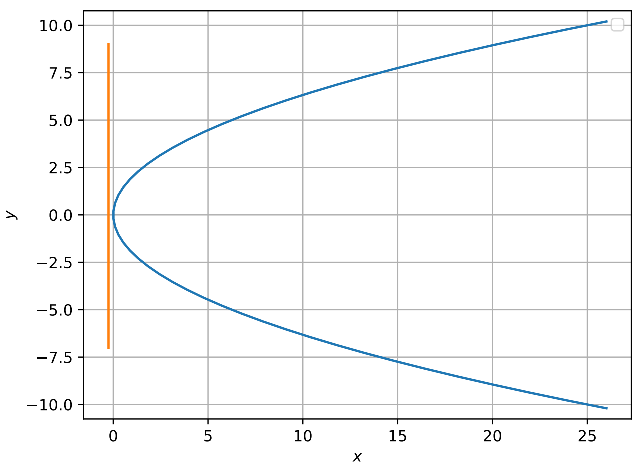

Figure 1:

Construction

Symbol	value	Description
V	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Vertex of parabola
F	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	focus of parabola
n	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	normal of directrix

Table 1:

Proof:

The given equation of parabola $y^2 = 4x$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where:

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

let vector point **P** as **h** where:

$$\mathbf{h} = \begin{pmatrix} k \\ l \end{pmatrix}$$

Given two tangents from point **h** make right angles with the parabola.The normal vectors of the tangents from a point **h** to the conic are given by

$$\mathbf{n} = \frac{\mathbf{e1}}{\mathbf{e1}^T\mathbf{h}} + \mu_i \mathbf{R}\mathbf{h} \tag{5}$$

where:

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{e1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

By substituting

$$\mathbf{n} = \begin{pmatrix} \frac{1}{k} - \mu_i l \\ \mu_i k \end{pmatrix} \tag{6}$$

where:

$$\mu_\mathbf{i} = \frac{-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}) \pm \sqrt{(\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}))^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f)\mathbf{m}^T\mathbf{V}\mathbf{m}}}{\mathbf{m}^T\mathbf{V}\mathbf{m}}$$

for μ_i substitute:

$$\mathbf{m} = \mathbf{R}\mathbf{h} = \begin{pmatrix} -l \\ k \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\mathbf{q} = \frac{\mathbf{e1}}{\mathbf{e1}^T\mathbf{h}} = \begin{pmatrix} \frac{1}{k} \\ 0 \end{pmatrix}$$

$$\mathbf{V} = \mathbf{V}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

By substituting

$$\mu_{\mathbf{1}} = \frac{1}{kl} \quad \mu_{\mathbf{2}} = \tfrac{1}{l}(\tfrac{1}{k}+4)$$

By substituting μ values in (6)

$$\mathbf{n_1} = \begin{pmatrix} 0 \\ \frac{1}{l} \end{pmatrix} \tag{7}$$

$$\mathbf{n_2} = \begin{pmatrix} -4 \\ \frac{1}{l}(\frac{1}{k}+4) \end{pmatrix} \tag{8}$$

Since the two tangents are perpendicular

$$\mathbf{n_1^T n_2} = 0$$

$$\begin{pmatrix} 0 & \frac{1}{l} \end{pmatrix} \begin{pmatrix} -4 \\ \frac{1}{l}(\frac{1}{k}+4) \end{pmatrix} = 0$$

$$k = \frac{-1}{4} \tag{9}$$

By substituting k value in (8)

$$\mathbf{n_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{10}$$

Since the tangent2 pass through the point h

$$\mathbf{n_2}^T \mathbf{h} = 0$$

$$\begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = 1 \tag{11}$$