Algebra: Maths Olympiad

G V V Sharma*

1. Find all the primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.
2. Solve the following equation for real x:
$$(x^2 + x - 2)^3 + (2x^3 - x - 1)^3 = 27(x^2 - 1)^3.$$
3. Let a,b,c be positive integers such that a divides b^2, b divides c^2, c divides a^2. Prove that abc divides $(a + b + c)^7$.
4. Let α and β be roots of the equation
$$x^2 + mx - 1 = 0, \tag{4.1}$$
when m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for n $\geq$ 0. Prove that for n $\geq$ 0.
   a) λ_n is an integer;and
   b) ged(λ_n, λ_{n+1})=1
5. Prove that the number of triples (A, B, C) when A, B, C are subsets of $\{1, 2, 3, ...n\}$ such that
$$A \cap B \cap C = \phi,$$
$$A \cap B = \phi,$$
$$B \cap C \neq 0 is 7^n - 2.6^n + 5^n.$$
6. Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that
   a) $y < x < 1$;and
   b) $x^2 + y^2 < 1$.
7. If x,y are integers,and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides xy - 12x + 15y.
8. If a, b, c are three real numbers such that $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.
9. Determine all triples (a, b, c) of positive integers such that a $\leq$ b $\leq$ c and
$$a + b + c + ab + bc + ac = abc + 1$$
10. If a, b, c are three real numbers such that $|a - b| \geq |c|$, $|b - c| \geq |a|$, $|c - a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.
11. Determine all triples (a, b, c) of positive integers such that a $\leq$ b $\leq$ c and
$$a + b + c + ab + bc + ac = abc + 1$$
12. Let a, b, c be the three natural numbers such that $a < b < c$ and ged(c-a, c-b) = 1. Suppose there exists an integer d such that a+d, b+d, c+d from the sides of a right angled triangle. Prove that there exist integers l, m such that c + d = $l^2 + m^2$.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

13. Find all pairs (a, b) of real numbers such that whenever α is a root of

$$x^2 + ax + b = 0, \tag{13.1}$$

α^2 - 2 is also a root of the equation.

14. Suppose a and b are the real numbers such that the roots of the cubic equation

$$ax^3 - x^2 + bx - 1 = 0 \tag{14.1}$$

are all positive real numbers. Prove that:

a) $0 < 3ab \leq 1$

b) $b \geq \sqrt{3}$.

15. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

16. In a book with page number from 1 to 100, some pages are not torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

17. Let

$$P_1(x) = ax^2 - bx - c,$$

$$P_2(x) = bx^2 - cx - a,$$

$$P_3(x) = cx^2 - ax - b$$

be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that a = b = c

18. Find the number of 4-digit numbers having non-zero digits and which are divisible by 4 but not 8.

19. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1 \tag{19.1}$$

20. Let a and b positive real numbers such that a + b = 1. Prove that

$$a^a b^b + a^b b^a \leq 1.$$

21. Let a and b be real numbers such that a $\neq$ 0. Prove that not all that roots of

$$ax^4 + bx^3 + x^2 + x + 1 = 0 \tag{21.1}$$

can be real.

22. Let

$$f(x) = x^3 + ax^2 + bx + c$$

and

$$g(x) = x^3 + bx^2 + cx + a,$$

where a, b, c are integers with c $\neq$ 0. Suppose that the following conditions hold:

a) f(1)=0;

b) the roots of g(x)=0 are the squares of the roots of f(x)=0

Find the value of $a^{2013} + b^{2013} + c^{2013}$.

23. Supppose that m and n are integers such that both the quadratic equations

$$x^2 + mx - n = 0 \tag{23.1}$$

and

$$x^2 - mx + n = 0 \tag{23.2}$$

have integer roots. Prove that n is divisible by 6.

24. Let

$$P_1(x) = x^2 + a_1 + b_1 \tag{24.1}$$

and

$$P_2(x) = x^2 + a_2 x + b_2 \tag{24.2}$$

be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist an integer m $\neq$ n such that P_1(m) = P_2(n), P_2(m) = P_1(n). Prove that $a_1 - a_2$ is even.

25. Find real numbers such that 3 < a < 4 and $a(a - 3\{a\})$ is an integer. {Here {a} denotes the fractional part of a. For example {1, 5} = 0.5; {-3, 4} = 0.6.}

26. Let a, b, c be positive real number such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$$

Prove that abc $\leq \frac{1}{8}$.

27. Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$$

Prove that abc $\leq \frac{1}{8}$.

28. Let a, b, c be three distinct positive real numbers such that abc = 1. Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

29. Let a, b, c be positive real number such that

$$\frac{ac}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$

Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}$$

30. Show that equation

a) $a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$

has no solutions in integers a, b.

31. Let

$$P(x) = x^2 + \frac{1}{2}x + b$$

and

$$Q(x) = x^2 + cx + d$$

be two polynomials with real coefficients such that P(x)Q(x) = Q(P(x)) for all real x. Find all the real roots of P(Q(x)) = 0.