Geometry: Maths Olympiad

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1. Consider the cube ABCD$A^{'}B^{'}C^{'}D^{'}$ (ABCD and $A^{'}B^{'}C^{'}D^{'}$ are the upper and lower bases, respectively, and edges $AA^{'}, BB^{'}, CC^{'}, DD^{'}$are parallel). The point X moves at constant speed along the perimeter of the square ABCD in the direction ABCDA, and the point Y moves at the same rate along the perimeter of the square $B^{'}C^{'}$CB in the direction $B^{'}C^{'}CBB^{'}$. Points X and Y begin their motion at the same instant from the starting positions A and $B^{'}$, respectively. Determine and draw the locus of the midpoints of the segments XY.
2. On the circle K there are given three distinct points A, B, C. Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.
3. Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{r(r-2\rho)}$$

4. The tetrahedron SABC has the following property: there exist five spheres, each tangent to the edges SA, SB, SC, BCCA, AB, or to their extensions.
   a) Prove that the tetrahedron SABC is regular.
   b) Prove conversely that for every regular tetrahedron five such spheres exist.
5. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.
6. Let ABCD be a fixed convex quadrilateral with BC = DA and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy BE = DF. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.
7. Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

   Show that AP ≥ AI, and that equality holds if and only if P = I.
8. Let P be a regular 2006-gon. A diagonal of P is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P .The sides of P are also called good. Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P . Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.
9. Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P . Show that the sum of the areas assigned to the sides of P is at least twice the area of P.
10. Consider five points A, B, C, D and E such that ABCD is a parallelogram and BCED is a cyclic quadrilateral. Let l be a line passing through A. Suppose that l intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that EF = EG = EC. Prove that l is the bisector of angle DAB.

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11. In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.
12. An acute-angled triangle ABC has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at $A_1 and A_2$. Similarly, the circle passing through H with centre the midpoint of CA intersects the line CA at $B_1 and B_2$, and the circle passing through H with centre the midpoint of AB intersects the line AB at $C_1 and C_2$. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.
13. Let ABCD be a convex quadrilateral with $|BA| \neq |BC|$. Denote the in circles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD.Prove that the common external tangents of ω_1 and ω_2 intersect on ω.
14. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that OP = OQ.
15. Let ABC be a triangle with AB = AC. The angle bisectors of ∠ CAB and ∠ ABC meet the sides BC and CA at D and E, respectively. Let K be the in centre of triangle ADC.Suppose that ∠BEK = 45^o. Find all possible values of ∠ CAB.
16. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc BDC and F a point on the side BC such that

$$\angle BAF = \angle CAE < \frac{1}{2}\angle BAC.$$

    Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.
17. Let P be a point inside the triangle ABC. The lines AP , BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that SC = SP . Prove that MK = ML.
18. Let ABC be an acute triangle with circumcircle Γ. Let l be a tangent line to Γ, and let l_a, l_b and l_c be the lines obtained by reflecting l in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines l_a, l_b and l_c is tangent to the circle Γ.
19. Given triangle ABC the point J is the centre of the excircle opposite the vertex A.This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively.The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC.Prove that M is the midpoint of ST. (The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC,to the ray AB beyond B, and to the ray AC beyond C.)
20. Let ABC be a triangle with ∠BCA = 90^o, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that BK = BC. Similarly, let L be the point on the segment BX such that AL = AC. Let M be the point of intersection of AL and BK.Show that MK = ML.
21. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.
22. Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter

of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

23. Convex quadri lateral ABCD has $\angle$ABC = $\angle$CDA = 90^o. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^o, \angle THC - \angle DTC = 90^o$$

Prove that line BD is tangent to the circumcircle of triangle TSH.
24. Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle$PAB = $\angle$BCA and $\angle$CAQ = $\angle$ ABC. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.
25. Let ABC be an acute triangle with AB > AC.Let Γ be its circumcircle,H its orthocentre,and F the foot of the altitude from A.Let M be the midpoint of BC.Let Q be the point on Γ suchn that $\angle$HQA=90,and let K be the point on Γ such that $\angle$HKQ=90.Assume that the point A,B,C,K and Q are all different,and lie on Γ in this order. Prove that the circumcircle of triangle KQH and FKM are tangent to each other.
26. Triangle ABC has cicumcircle Ω and circumcenter O.A circle Γ with centre A intersects the segment BC at point D and E,such that B,D,E and C are all different and lie on the line BC in this order .Let F and G be the points of intersection of Γ and Ω,such that A,F,B,C and G lie on Ω in this order.Let K be the second point of intersection of the circumcircle of the triangle BDF and segment AB.Let L be the second point of intersection of circumcircle of triangle CGE and the segment CA. Suppose that lines FK and Gl are different and intersects at the point X.Prove that X lies on the line AO.
27. Trianlge BCF has a right angle at B.Let A be the point on line CF such that FA=FB and F lies between A and C.Point D is chosen such that DA=DC and AC is the bisector of $\angle$DAB.Point E is chosen such that EA=ED and AD is the bisector of $\angle$EAC.Let M be the midpoint of CF.Let X be the point such that AMXE is a parallelogram (where AM ||EM and AE || MX).Prove that lines BD,FX,and ME are concurrent.
28. Let R and S be different points on a circle Ω such that RS is not a diameter. Let l be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects l at two distinct points. Let A be the common point of Γ and l that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.
29. Let Γbe the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that AD = AE. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).
30. A convex quadrilateral ABCD satisfies ABCD = BCDA. Point X lies inside ABCD so that $\angle$XAB = $\angle$XCD and $\angle$XBC = $\angle$XDA.
Prove that $\angle$BXA + $\angle$DXC = 180^o.