Algebra: Maths Olympiad

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1. Determine all real numbers x which satisfy the inequality:
$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}$$
2. Solve the equation
$$\cos^2 x + \cos^2 2x + \cos^2 3x = 1$$
3. Let x, y, z be three positive reals such that xyz $\geq$ 1. Prove that
$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$
4. Let a_1, a_2,....... be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, ..., a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence a_1, a_2,.......
5. Determine all positive integers relatively prime to all the terms of the infinite sequence
$$a_n = 2^n + 3^n + 6^n - 1, n \geq 1.$$
6. Real numbers $a_1, a_2, ..., a_n$ are given. For each i $(1 \leq i \leq n)$ define
$$d_i = max\{a_j : 1 \leq j \leq i\} - min\{a_j : i \leq j \leq n\}$$
and let
$$d = max\{d_i : 1 \leq i \leq n\}.$$
   a) Prove that, for any real numbers $x_1 \leq x_2 \leq .... \leq x_n$, $\max\{|x_i - a_i| : 1 \leq i \leq n\} \geq \frac{d}{2}$.
   b) Show that there are real numbers $x_1 \leq x_2 \leq .... \leq x_n$such that equality holds in.
7. Let a and b be positive integers. Show that if 4ab-1 divides $(4a^2 - 1)^2$, then a = b.
8. Let n be a positive integer. Consider
$$S = \{(x, y, z) : x, y, z \in \{0, 1, ..., n\}, x + y + z > 0\}$$
as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include (0, 0, 0).
9. a) Prove that
$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$
for all real numbers x, y, z, each different from 1, and satisfying xyz = 1.
   b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying xyz = 1.
10. Prove that there exist infinitely many positive integers n such that $n^2 + 1$ has a prime divisor which is greater than $2n + \sqrt{2n}$.

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11. Let n and k be positive integers with k ≥ n and k - n an even number. Let 2n lamps labelled 1, 2,....., 2n be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps n + 1 through 2n are all off. Let M be the number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps n + 1 through 2n are all off, but where none of the lamps n + 1 through 2n is ever switched on. Determine the ratio $\frac{N}{M}$.
12. Let $a_1, a_2, ..., a_n$be distinct positive integers and let M be a set of n - 1 positive integers not containing $s = a_1 + a_2 + ... + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, ..., a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.
13. Let $a_1, a_2, a_3, ...$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = max\{a_k + a_{n-k} | 1 \leq k \leq n-1\}$$

for all $n > s$. Prove that there exist positive integers l and N, with $l \leq s$ and such that $a_n = a_l + a_{n-l}$ for all $n \leq N$.
14. Given any set A = $\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with 1 ≤ i ≤ j ≤ 4 for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.
15. Find all positive integers n for which there exist non-negative integers $a_1, a_2, ..., a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + ... + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + ... + \frac{n}{3^{a_n}} = 1$$

16. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = (1 + \frac{1}{m_1})(1 + \frac{1}{m_2})(1 + \frac{1}{m_k}).$$

17. Let Q > 0 be the set of positive rational numbers. Let f : Q>0 → R be a function satisfying the following three conditions:
    a) for all x, y ∈ Q>0, we have f(x)f(y) ≥ f(xy);
    b) for all x, y ∈ Q>0, we have f(x + y) ≥ f(x) + f(y);
    c) there exists a rational number a > 1 such that f(a) = a. Prove that f(x) = x for all x ∈ Q > 0.
18. Let $a_0 < a_1 < a_2 <$ be an infinite sequence of positive integers. Prove that there exists a unique integer n ≤ 1 such that

$$a_n < \frac{a_0 + a_1 + ... + a_n}{n} \leq a_{n+1}$$

19. Determine all triple (a,b,c) of positive integers such that each of the numbers.
    ab - c, bc - a , ca - b is a power of 2.
    (A power of 2 is an integer of the form 2^n,where n is a non-negative integer)
20. The sequence a_1,a_2,....of integers satisfies the following conditions:
    a) $1 \leq a_j \leq 2015$ for all j ≥ 1;
    b) k + a_k ≠ l + a_l for all 1 ≤ k < l.

Prove that there exist two positive integers b and N such that

$$\sum_{j=m+1}^{n} (a_j - b) \leq (1007)^2$$

for all integers m and n satisfying n > m ≥ N.

21. A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements.Let P(n) = $n^2 + n + 1$ what is the least possible value of the positive integer b such that there exists a non-negative integer a for which the set
$\{P(a+1), P(a+2), ..., P(a+b)\}$
is fragrant?

22. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2,. . . .$ by:
a_{n+1} =

$$\begin{cases} \sqrt{a_n} if \sqrt{a_n} \ \ is \ \ an \ \ integer, \\ a_{n+3} \ \ otherwise \end{cases}$$

for all each n ≥ 0
Determine all values of a_0 for which there is a number A such that a_n = A for infinitely many values of n.

23. An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers a_0, a_1, . . . , a_n such that, for each (x, y) in S, we have:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + ... + a_{n-1} x y^{n-1} + a_n y^n = 1.$$

24. Find all integers n ≥ 3 for which there exist real numbers $a_1, a_2, . . . , a_n + 2$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and $a_i a_{i+1} + 1 = a_{i+2}$ for i = 1, 2, . . . , n.

25. Let $a_1, a_2, . . .$ be an infinite sequence of positive integers. Suppose that there is an integer N > 1 such that, for each n ≥ N, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that$a_m = a_{m+1}$ for all m ≥ M.