Discrete: Maths Olympiad

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1. Find the smallest natural number n which has the following properties:
   a) Its decimal representation has 6 as the last digit.
   b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.
2. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$of the contestants. Moreover, no contestant solved all the 6 problems.Show that there are at least 2 contestants who solved exactly 5 problems each.
3. Determine the least real number M such that the inequality
$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \le M(a^2+b^2+c^2)^2$$
holds for all real numbers a, b and c.
4. Determine all pairs (x, y) of integers such that $1+2^x+2^{2x+1}=y^2$.
5. Let P (x) be a polynomial of degree n > 1 with integer coefficients and let k be a positive integer. Consider the polynomial Q(x) = P (P (. . . P (P (x)) . . .)), where P occurs k times. Prove that there are at most n integers t such that Q(t) = t.
6. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.
7. Find all functions f : (0, ∞) → (0,∞) (so, f is a function from the positive real numbers to the positive real numbers) such that
$$\frac{(f(w))^2+(f(x))^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$
for all positive real numbers w, x, y, z, satisfying wx = yz.
8. Suppose that $s_1, s_2, s_3, ...$ is a strictly increasing sequence of positive integers such that the sub sequences $s_{s_1}, s_{s_2}, s_{s_3}, ...$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, ...$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, ...$is itself an arithmetic progression.
9. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths a, f(b) and f(b + f(a)-1).
   (A triangle is non-degenerate if its vertices are not collinear.)
10. Determine all functions f : $R \to R$ such that the equality
$$f(\lfloor x\rfloor y = f(x)\lfloor f(y)\rfloor$$
holds for all x, y ∈R. (Here $\lfloor z\rfloor$ denotes the greatest integer less than or equal to z.
11. Let N be the set of positive integers. Determine all functions g : N → N such that (g(m) +n)(m+ g(n)) is a perfect square for all m, n ∈ N.

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12. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:
    Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}.
    Type 2: Choose a nonempty box B_k with $1 \leq$ k ≤ 4. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.
13. Given any set A = $\{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i \leq j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.
14. Let S be a finite set of at least two points in the plane. Assume that no three points of S are collinear. A windmill is a process that starts with a line l going through a single point P $\in$ S. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of S. This process continues indefinitely. Show that we can choose a point P in S and a line l going through P such that the resulting windmill uses each point of S as a pivot infinitely many times.
15. Let f : R $\rightarrow$ R be a real valued function defined on the set of real numbers that satisfies

    $$f(x + y) \leq y f(x) + f(f(x))$$

    for all real numbers x and y. Prove that f(x) = 0 for all x $\leq$ 0.
16. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference f(m)-f(n) is divisible by f(m-n). Prove that,for all integers m and n with $f(m) \leq f(n)$, the number f(n) is divisible by f(m).
17. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, ..., 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such away that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan,until all of the weights have been placed.Determine the number of ways in which this can be done.
18. The liars guessing game is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.At the start of the game A chooses integers x and N with 1 $\leq$x $\leq$ N. Player A keeps x secret,and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with yes or no, but is allowed to lie as many times as she wants; the only restriction is that, among any k + 1 consecutive answers, at least one answer must be truthful.After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:
    a) If n $\leq 2^k$, then B can guarantee a win.
    b) For all sufficiently large k, there exists an integer $n \leq 1.99^k$ such that B cannot guarantee a win.
19. Find all functions f : Z $\rightarrow$ Z such that, for all integers a, b, c that satisfy a + b + c = 0,the following equality holds:

    $$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

    (Here Z denotes the set of integers.)
20. A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian

configuration if the following two conditions are satisfied:

a) no line passes through any point of the configuration;
b) no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

21. Let n ≥ 3 be an integer, and consider a circle with n + 1 equally spaced points marked on it. Consider all labellings of these points with the numbers 0, 1, . . . , n such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels a < b < c < d with a + d = b + c, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c. Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that x + y ≤ n and gcd(x, y) = 1. Prove that M = N + 1.
22. Let n ≥ 2 be an integer. Consider an n × n chessboard consisting of n^2 unit squares.A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks,there is a k × k square which does not contain a rook on any of its k^2 unit squares.
23. For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$.Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.
24. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.
25. We say that a finite set S of the point in the plane is balanced if,for any two different points A and B in S,there is a point C in S such that AC=BC.We say that S is centre-free if for any three different point A,B,C in S,there is no point P in S such that PA=PB=PC.
    a) Show that for all integers n ≥ 3,there exist a balanced set consisting of n points.
    b) Determine all integers n ≥ 3 for which there exists a balanced centre-free set consisting of n points.
26. Let R be the set of Real numbers.Determine all functions f:R → R satisfying the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

27. Find the positive integer n for which each cell od an n × n table can be filled with one of the letter I,M and O in such a way that:
    a) In each row and each column,one third of the entires are I,one third are M and one third are O;and
    b) In any diagonal,if the number of entries on the diagonal is a multiple of three,then one third of the entries are I,one third are M and one third are O.

    NOTE: The row and column of an n × n table are each labelled 1 to n in a natural order.Thus each cell corresponding to a pair of positive integers(i, j) with 1 ≤ i,j ≤ n.For n > 1,the table has 4n-2 diagonlas of two types.A diagonal of first type consists of all cells(i,j) for which i+j is a constant,and a diagonal of the second type consists of all cells (i, j) for which i-j is a constant.
28. Let P=$A_1A_2...A_k$ be a convex polygon in the plane.The vertices $A_1, A_2..., A_k$ have integral co-ordinates and lie on a circle.Let S be the area of P.An odd positive integer n is given such that the squares of the side lengths of P are integers divisible by n.Prove that 2S is an integer divisible by n.
29. The equation

$$(x - 1)(x - 2)....(x - 2016) = (x - 1)(x - 2)....(x - 2016)$$

is written on the board,with 2016 linear factors on each side.What is the least possible value of k for which it is possible to erase exactly k of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solution?

30. There are n ≥ 2 line segments in the Plane such that every two segments,cross and no three segments meet at a point.Geoff has to choose an endpoint of each segment and place a frog on it,facing the other endpoint.Then he will clap his hand n-1 times.Every time he clap each frog will immediately jump forward to the next intersection point on its segment.Frogs never change the direction of their jumps.Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.
   a) Prove that Geoff can always fulfil his wish of n is odd.
   b) Prove that Geoff can never fulfil his wish of n is even.
31. Let R be the set of real numbers. Determine all functions f : R → R such that, for all real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

32. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbits starting point, A_0, and the hunters starting point, B_0, are the same. After n-1 rounds of the game,the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.
   a) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.
   b) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.
   c) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 109 rounds she can ensure that the distance between her and the rabbit is at most 100?
33. An integer N ≥ 2 is given. A collection of N(N+1) soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove N(N-1) players from this row leaving a new row of 2N players in which the following N conditions hold:
   a) no one stands between the two tallest players,
   b) no one stands between the third and fourth tallest players,
   c) (N) no one stands between the two shortest players.

   Show that this is always possible.
34. An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9 \end{array}$$

   Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to 1 + 2 +.......+ 2018?
35. A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20. Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance

between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone. Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.