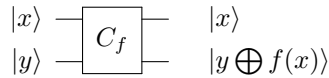# Quantum Algorithms, Spring 2022: Lecture 5 Scribe

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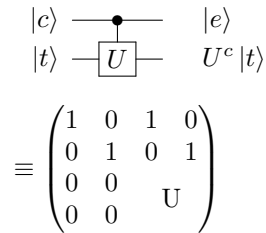
February 18, 2022

1 Recap

- Reversible circuits produce unwanted garbage bits that are dependent on the input and are entangle with the desired out bits, so we need : **Uncomputing!**

$$
\begin{array}{rcl}
|x\rangle & \boxed{C_f} & |x\rangle \\
|y\rangle & & |y \bigoplus f(x)\rangle
\end{array}
$$

- Quantum Circuits

  - Single Qubit Gates: $X, Y, Z, R_\phi, h, \ldots$
  - Two Qubit Gates: CNOT, any $C - U$ where U is a single qubit gate.

$$
\begin{array}{rcl}
|c\rangle & \bullet & |e\rangle \\
|t\rangle & \boxed{U} & U^c |t\rangle
\end{array}
$$

$$
\equiv \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & & \\ 0 & 0 & & U \end{pmatrix}
$$

$\forall$ U is any single qubit gate.

2 Universality of Quantum circuits

I will provide you some statements regarding the universality of quantum circuits without necessarily proving them.

- **Statement 1:** {CNOT, all single qubit gates} : universal for Quantum Computing

- **Statement 2:** The set of { CNOT, H, $R_{\pi/4}$ } : universal for Quantum Computing
  Any other quantum circuit can be well approximated using quantum circuits of only these gates.

2.1 Formalizing Statement 2

Let $G = \{CNOT, H, R_{\pi/4}\}$, then for any quantum circuit U, $\in$ a number t, such that

$$||U - U_t U_{t-1} \ldots U_1|| \leq \epsilon, where$$

$$
\text{each } U_j \in G
$$
$$
|| \quad || : \text{spectral norm}
$$
$$
||A|| = max_{\langle\psi||\psi\rangle=1}||A|\psi\rangle||
$$

- How large should 't' be? Clearly, it better not be too large.

- Luckily 't' isn't too large owing to crucial result by Solvay and Kitaev

3 Solovay Ketanov Theorem

- Any 't'-gate quantum circuit can be ϵ approximated using only $\mathcal{O}(t \, \text{polylog}(\frac{1}{\epsilon}))$ gates from G.

- **Proof:** Appendix of Neilsen and Chuang [?]

- There are also other universal gate sets: some are efficient than others.
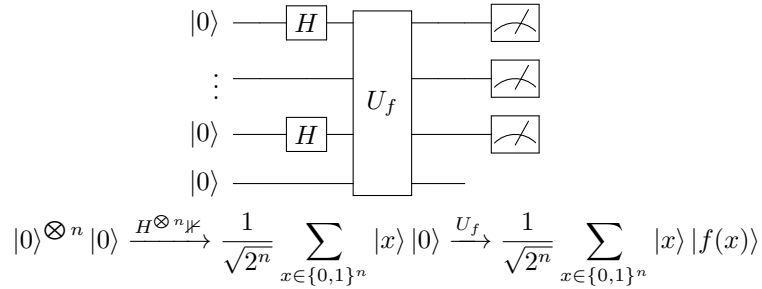
4 Quantum Parallelism

- Suppose we are interested in some function $f : \{0,1\}^n \to \{0,1\}$

$$
\begin{array}{ll}
|x\rangle & |x\rangle \\
& \boxed{U_f} \\
|y\rangle & |y \oplus f(x)\rangle
\end{array}
$$

So, if $f(x) = 0$, $\quad |x\rangle |y\rangle \xrightarrow{U_f} |x\rangle |y\rangle$

and if $f(x) = 1$, $\quad |x\rangle |y\rangle \xrightarrow{U_f} |x\rangle |\bar{y}\rangle$

$$
|0\rangle^{\otimes n} |0\rangle \xrightarrow{H^{\otimes n} \not| \not|} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |0\rangle \xrightarrow{U_f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle
$$

- By applying U_f only once, we are able to obtain a quantum state that contains in it all 2^n possible values of $f(x)$ in superposition!

- This in itself is not very useful. If we make projective measurement, we will observe some $|z\rangle |f(z)\rangle$ with probability $1/2^n$.

- Quantum parallelism is not enough to demonstrate the power of quantum computing.

- Quantum parallelism needs to be combined with interference, entanglement, to something better than classical computing.

5 Quantum Oracle : Phase Kickback Oracle

- From the above sections we know that for some function : $f : \{0,1\}^n \to \{0,1\}$ and

$$
\begin{array}{ll}
|x\rangle & |x\rangle \\
& \boxed{U_f} \\
|y\rangle & |y \oplus f(x)\rangle
\end{array}
$$

if $f(x) = 0$, $\quad |x\rangle |y\rangle \xrightarrow{U_f} |x\rangle |y\rangle$

and if $f(x) = 1$, $\quad |x\rangle |y\rangle \xrightarrow{U_f} |x\rangle |\bar{y}\rangle$

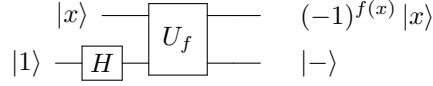If we substitute $|-\rangle$ for y we get :

$$
\text{if} \quad f(x) = 0, \quad |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \xrightarrow{U_f} |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]
$$

$$
\text{and if} \quad f(x) = 1, \quad |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \xrightarrow{U_f} - |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]
$$

- The phase get changed when $f(x) = 1$ (a kickback), hence we call this a phase kick back oracle with whose result we can guess $f(x)$! This can be rewritten as :

$$|x\rangle\,|-\rangle \xrightarrow{U_f} (-1)^{f(x)}\,|x\rangle\,|-\rangle$$

Rewriting the circuit for $y = |-\rangle$:

$$
\begin{array}{l}
|x\rangle \underline{\quad\boxed{}\quad} (-1)^{f(x)}\,|x\rangle \\
\quad\quad\boxed{U_f} \\
|1\rangle \underline{\boxed{H}\ } \underline{\quad} |-\rangle
\end{array}
$$

The second input and output lines can be dropped as they remain the same in another frequently used representation :

$$|x\rangle \underline{\boxed{U_f^{\pm}}} (-1)^{f(x)}\,|x\rangle$$

$$|x\rangle \xrightarrow{U_f^{\pm}} (-1)^{f(x)}\,|x\rangle$$

On passing $H^{\otimes n}\,|0^{\otimes n}\rangle$ into the phase kickback $U_f^{\pm}$ we get :
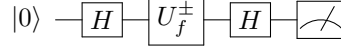
$$H^{\otimes n}\,|0^{\otimes n}\rangle \xrightarrow{U_f^{\pm}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \xrightarrow{U_f^{\pm}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}\,|x\rangle$$

The important thing to note here is that after passing through the oracle the amplitudes of the states have the information of $f(x)$

6 Deutsch Algorithm

Given a U_f for some boolean function $f : \{0,1\} \to \{0,1\}$ with the promise that either : $f(0) = f(1)$ or $f(0) \neq f(1)$, the task is to find the number of queries to U_f to determine which is the case.

- Classical Algorithm requires 2 queries by comparing outputs of inputs 0 and 1.

- Quantum Algorithm requires only 1 query! with the design :

$$|0\rangle \underline{\boxed{H}}\,\underline{\boxed{U_f^{\pm}}}\,\underline{\boxed{H}}\,\underline{\boxed{\diagup}}$$

$$H\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow{U_f^{\pm}} \frac{1}{\sqrt{2}}(-1^{f(0)}\,|0\rangle + -1^{f(1)}\,|1\rangle) \xrightarrow{H} \frac{(-1^{f(0)} + -1^{f(1)})\,|0\rangle + (-1^{f(0)} - 1^{f(1)})\,|1\rangle}{2}$$

we observe :

$$|0\rangle \text{ if } f(0) = f(1),\,and\,|1\rangle \text{ for } f(0) \neq f(1)$$

Therefore, only one query with input $|0\rangle$ is needed.

7 Physics Understanding of the Deutsch Problem

The physical setup of the Deutsch Algorithm is realised using Mach Zehnder Interferometer which consists of a beam splitter that creates an equal superposition of $|0\rangle$ and $|1\rangle$. The phase shifter adds a phase of 0 or π which passes through another beam splitter (acting as final H gate in Deutsch Algorithm) where the final states are recorded.
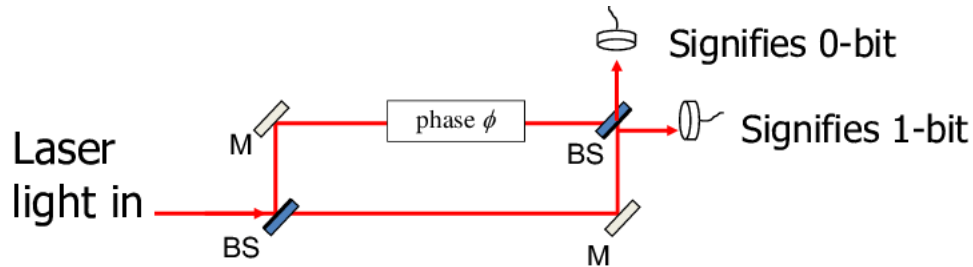

Figure 1: mach zehnder interferometer [2]