AI1103-Assignment 2

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Download all python codes from

https://github.com/Sravanth−k27/AI1103/tree/main/Assignment−2/codes

and latex-tikz codes from

https://github.com/Sravanth−k27/AI1103/tree/main/Assignment−2/Assignment−2.tex

Question(Gate EC 55):

Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed,then $P(X_1 < X_2)$ is equal to

Solution(Gate EC 55):

1) Given that X_1 is an exponential random variable. Let the P.D.F of X_1 be

$$P_{X_1}(x_1) = \begin{cases} \lambda e^{-\lambda x_1} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{0.0.1}$$

$$\text{As mean} = \lambda \tag{0.0.2}$$
$$\text{Given that mean} = 1 \tag{0.0.3}$$
$$\text{so } \lambda = 1 \tag{0.0.4}$$

The P.D.F of X_1 is:

$$P_{X_1}(x_1) = \begin{cases} e^{-x_1} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{0.0.5}$$

2) Given that X_2 is an gamma random variable.Let the P.D.F of X_2 be:

$$P_{X_2}(x_2) = \begin{cases} \frac{a^b x_2^{b-1} e^{-ax_2}}{(b-1)!} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{0.0.6}$$

$$\text{Since mean} = \frac{b}{a} = 2 \tag{0.0.7}$$
$$\text{Also,variance} = \frac{b}{a^2} = 2 \tag{0.0.8}$$

From (0.0.7) and (0.0.8)

$$b = 2, a = 1 \tag{0.0.9}$$

So, the P.D.F of X_2 is :

$$P_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{0.0.10}$$

Now we have to find $P(X_1 < X_2)$

3) Given that X_1 and X_2 are independent random variables

$$P_{(X_1|X_2)}(x_1|x_2) = P_{X_1}(x_1) \tag{0.0.11}$$

4) Consider the random variable $Z = X_1 - X_2$
Since

$$P(X_1 < X_2) \implies P(X_1 - X_2 < 0) \tag{0.0.12}$$

So, we have to find C.D.F of random variable Z for z $\leq$ 0 i.e $P(z \leq 0)$

C.D.F of random variable $Z = X_1 - X_2$ for $z \leq 0$:

$$F_Z(z) = P(X_1 - X_2 \leq z) \tag{0.0.13}$$
$$= \int_{-z}^{\infty} \int_{0}^{z+x_2} P_{X_1X_2}(x_1, x_2) dx_1 dx_2 \tag{0.0.14}$$
$$= \int_{-z}^{\infty} \int_{0}^{z+x_2} P_{(X_1|X_2)}(x_1|x_2) \times P_{X_2}(x_2) \tag{0.0.15}$$

from (0.0.11)

$$= \int_{-z}^{\infty} \int_{0}^{z+x_2} P_{X_1}(x_1) \times P_{X_2}(x_2) dx_1 dx_2 \tag{0.0.16}$$
$$= \int_{-z}^{\infty} P_{X_2}(x_2) \left(\int_{0}^{z+x_2} P_{X_1}(x_1) dx_1 \right) dx_2 \tag{0.0.17}$$
$$= \int_{-z}^{\infty} x_2 e^{-x_2} \left(\int_{0}^{z+x_2} e^{-x_1} dx_1 \right) dx_2 \tag{0.0.18}$$

Now for $F_Z(0)$

$$\begin{aligned}
F_Z(0) &= \int_0^\infty x_2 e^{-x_2} \left(\int_0^{x_2} e^{-x_1} dx_1 \right) dx_2 \\
&= \int_0^\infty x_2 e^{-x_2} \left(1 - e^{-x_2}\right) dx_2 \\
&= \int_0^\infty x_2 e^{-x_2} dx_2 - \int_0^\infty x_2 e^{-2x_2} dx_2 \\
&= 1 - \frac{1}{4} = \frac{3}{4}
\end{aligned} \tag{0.0.19}$$

So, $P(X_1 < X_2) = F_Z(0) = \frac{3}{4}$

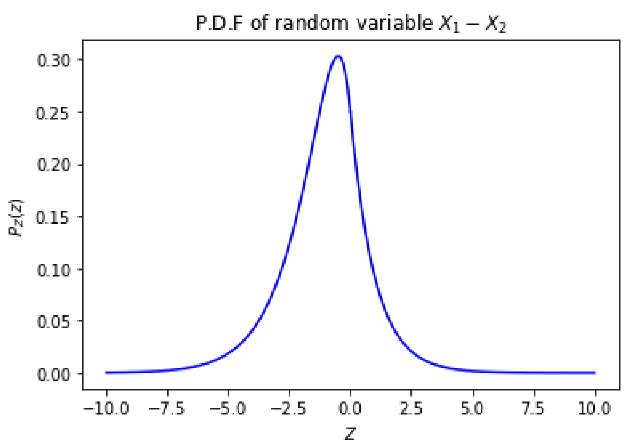

Fig. 4: P.D.F of Z

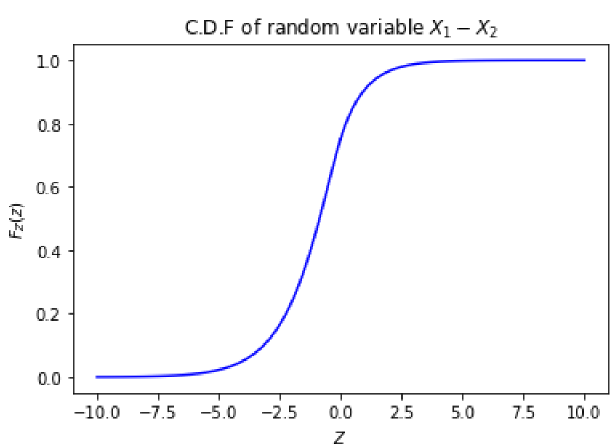

Fig. 4: C.D.F of Z