AI1103-Assignment 2

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Download all python codes from

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https://github.com/Sravanth−k27/AI1103/tree/main/
    Assignment−2/codes
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and latex-tikz codes from

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https://github.com/Sravanth−k27/AI1103/tree/main/
    Assignment−2/Assignment−2.tex
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QUESTION(GATE EC 55):

Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed,then $P(X_1 < X_2)$ is equal to

SOLUTION(GATE EC 55):

1) Given that X_1 is an exponential random variable. Let the P.D.F of X_1 be

$$p_{X_1}(x_1) = \begin{cases} \lambda e^{-\lambda x_1} & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases} \tag{0.0.1}$$

C.D.F of x_1 is :

$$\begin{aligned} F_{X_1}(x_1) &= \int_{-\infty}^{x_1} p_{X_1}(x_1)dx_1 \\ &= \int_{-\infty}^{0} p_{X_1}(x_1)dx_1 + \int_{0}^{x_1} p_{X_1}(x_1)dx_1 \\ &= \int_{-\infty}^{0} 0 \times dx_1 + \int_{0}^{x_1} \lambda e^{-\lambda x_1}dx_1 \\ &= 1 - e^{-\lambda x_1} \end{aligned} \tag{0.0.2}$$

$$\text{As mean} = \lambda \tag{0.0.3}$$

$$\text{Given that mean} = 1 \tag{0.0.4}$$

$$\text{so } \lambda = 1 \tag{0.0.5}$$

2) Given that X_2 is an gamma random variable.Let the P.D.F of X_2 be:

$$p_{X_2}(x_2) = \begin{cases} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} & x_2 \geq 0 \\ 0 & x_2 < 0 \end{cases} \tag{0.0.6}$$

$$\text{Since mean} = \frac{b}{a} = 2 \tag{0.0.7}$$

$$\text{Also,variance} = \frac{b}{a^2} = 2 \tag{0.0.8}$$

From (0.0.7) and (0.0.8)

$$b = 2, a = 1 \tag{0.0.9}$$

Since the total probability of X_2 is 1 so,

$$\int_{-\infty}^{\infty} p_{X_2}(x_2)dx_2 = 1 \tag{0.0.10}$$

$$\int_{-\infty}^{0} p_{X_2}(x_2)dx_2 + \int_{0}^{\infty} p_{X_2}(x_2)dx_2 = 1 \tag{0.0.11}$$

$$\int_{-\infty}^{0} 0 \times dx_2 + \int_{0}^{\infty} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)}dx_2 = 1 \tag{0.0.12}$$

$$\frac{a^b}{\Gamma(b)} \int_{0}^{\infty} x_2^{b-1} e^{-ax_2}dx_2 = 1 \tag{0.0.13}$$

$$\int_{0}^{\infty} x_2^{b-1} e^{-ax_2}dx_2 = \frac{\Gamma(b)}{a^b} \tag{0.0.14}$$

now substituting $a + \lambda$ for a in (0.0.14) gives

$$\int_{0}^{\infty} x_2^{b-1} e^{-(a+\lambda)x_2}dx_2 = \frac{\Gamma(b)}{(a+\lambda)^b} \tag{0.0.15}$$

Now we have to find $P(X_1 < X_2)$

3) Given that X_1 and X_2 are independent random variables,so

$$P(X_1 < X_2|X_2) = F_{X_1}(X_2) = 1 - e^{-\lambda X_2} \tag{0.0.16}$$

Now,

$$P(X_1 < X_2) = \int_{0}^{\infty} F_{X_1}(X_2) \times p_{X_2}(x_2)dx_2 \tag{0.0.17}$$

from (0.0.6),(0.0.16)

$$P(X_1 < X_2) = \int_0^\infty (1 - e^{-\lambda X_2}) \times \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 \tag{0.0.18}$$

$$P(X_1 - X_2) = \frac{a^b}{\Gamma(b)} \int_0^\infty x_2^{b-1} (e^{-ax_2} - e^{-(a+\lambda)x_2}) dx_2 \tag{0.0.19}$$

from (0.0.14) and (0.0.15)

$$P(X_1 - X_2) = \frac{a^b}{\Gamma(b)} \left(\frac{\Gamma(b)}{a^b} - \frac{\Gamma(b)}{(a+\lambda)^b} \right) \tag{0.0.20}$$

$$P(X_1 - X_2) = 1 - \frac{a^b}{(a+\lambda)^b} \tag{0.0.21}$$

$$P(X_1 - X_2) = 1 - \left(\frac{a}{a+\lambda} \right)^b \tag{0.0.22}$$

from (0.0.5) and (0.0.9)

$$P(X_1 - X_2) = 1 - \left(\frac{1}{1+1} \right)^2 \tag{0.0.23}$$

$$P(X_1 - X_2) = 1 - \frac{1}{4} = \frac{3}{4} \tag{0.0.24}$$