

AI1103-Assignment 2

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Download all python codes from

<https://github.com/Sravanth-k27/AI1103/tree/main/Assignment-2/codes>

and latex-tikz codes from

<https://github.com/Sravanth-k27/AI1103/tree/main/Assignment-2/Assignment-2.tex>

QUESTION(GATE EC 55):

Let X_1 be an exponential random variable with mean 1 and X_2 a gamma random variable with mean 2 and variance 2. If X_1 and X_2 are independently distributed, then $P(X_1 < X_2)$ is equal to

SOLUTION(GATE EC 55):

- 1) Given that X_1 is an exponential random variable. Let the P.D.F of X_1 be

$$p_{X_1}(x_1) = \begin{cases} \lambda e^{-\lambda x_1} & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases} \quad (0.0.1)$$

C.D.F of x_1 is :

$$\begin{aligned} F_{X_1}(x_1) &= \int_{-\infty}^{x_1} p_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^0 p_{X_1}(x_1) dx_1 + \int_0^{x_1} p_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^0 0 \times dx_1 + \int_0^{x_1} \lambda e^{-\lambda x_1} dx_1 \\ &= 1 - e^{-\lambda x_1} \end{aligned} \quad (0.0.2)$$

$$\text{As mean} = \lambda \quad (0.0.3)$$

$$\text{Given that mean} = 1 \quad (0.0.4)$$

$$\text{so } \lambda = 1 \quad (0.0.5)$$

- 2) Given that X_2 is an gamma random variable. Let the P.D.F of X_2 be:

$$p_{X_2}(x_2) = \begin{cases} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} & x_2 \geq 0 \\ 0 & x_2 < 0 \end{cases} \quad (0.0.6)$$

$$\text{Since mean} = \frac{b}{a} = 2 \quad (0.0.7)$$

$$\text{Also, variance} = \frac{b}{a^2} = 2 \quad (0.0.8)$$

From (0.0.7) and (0.0.8)

$$b = 2, a = 1 \quad (0.0.9)$$

Since the total probability of X_2 is 1 so,

$$\int_{-\infty}^{\infty} p_{X_2}(x_2) dx_2 = 1 \quad (0.0.10)$$

$$\int_{-\infty}^0 p_{X_2}(x_2) dx_2 + \int_0^{\infty} p_{X_2}(x_2) dx_2 = 1 \quad (0.0.11)$$

$$\int_{-\infty}^0 0 \times dx_2 + \int_0^{\infty} \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 = 1 \quad (0.0.12)$$

$$\frac{a^b}{\Gamma(b)} \int_0^{\infty} x_2^{b-1} e^{-ax_2} dx_2 = 1 \quad (0.0.13)$$

$$\int_0^{\infty} x_2^{b-1} e^{-ax_2} dx_2 = \frac{\Gamma(b)}{a^b} \quad (0.0.14)$$

now substituting $a + \lambda$ for a in (0.0.14) gives

$$\int_0^{\infty} x_2^{b-1} e^{-(a+\lambda)x_2} dx_2 = \frac{\Gamma(b)}{(a+\lambda)^b} \quad (0.0.15)$$

Now we have to find $P(X_1 < X_2)$

- 3) Given that X_1 and X_2 are independent random variables, so

$$P(X_1 < X_2 | X_2) = F_{X_1}(X_2) = 1 - e^{-\lambda X_2} \quad (0.0.16)$$

Now,

$$P(X_1 < X_2) = \int_0^{\infty} F_{X_1}(X_2) \times p_{X_2}(x_2) dx_2 \quad (0.0.17)$$

from (0.0.6),(0.0.16)

$$P(X_1 < X_2) = \int_0^\infty (1 - e^{-\lambda X_2}) \times \frac{a^b x_2^{b-1} e^{-ax_2}}{\Gamma(b)} dx_2 \quad (0.0.18)$$

$$P(X_1 < X_2) = \frac{a^b}{\Gamma(b)} \int_0^\infty x_2^{b-1} (e^{-ax_2} - e^{-(a+\lambda)x_2}) dx_2 \quad (0.0.19)$$

from (0.0.14) and (0.0.15)

$$P(X_1 < X_2) = \frac{a^b}{\Gamma(b)} \left(\frac{\Gamma(b)}{a^b} - \frac{\Gamma(b)}{(a+\lambda)^b} \right) \quad (0.0.20)$$

$$P(X_1 < X_2) = 1 - \frac{a^b}{(a+\lambda)^b} \quad (0.0.21)$$

$$P(X_1 < X_2) = 1 - \left(\frac{a}{a+\lambda} \right)^b \quad (0.0.22)$$

from (0.0.5) and (0.0.9)

$$P(X_1 < X_2) = 1 - \left(\frac{1}{1+1} \right)^2 \quad (0.0.23)$$

$$P(X_1 < X_2) = 1 - \frac{1}{4} = \frac{3}{4} \quad (0.0.24)$$