Homework 3

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24-677 Special Topics: Linear Control Systems

Due: Sept 29, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Controllability and Observability* ***(10 points)***

Is the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad \text{— (1)}$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

A, B, C

Controllable?**(5 points)** Observable?**(5 points)** Provide your derivation.

ANSWER:

Need to check if the state equation given is controllable or observable

From eqn (1),

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix},$$

$$n = 3$$

Now to find eigen values of A;

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -3 & -3-\lambda \end{bmatrix} = 0$$

$$= -(\lambda+1)^3 = 0$$

$$\therefore \Rightarrow \boxed{\lambda_1 = \lambda_2 = \lambda_3 = -1}$$

To use PBH Test :-

$$\lambda I - A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

For controllability :-

$$P = (B, AB, A^2B)$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow$ The rank$(P) = 3$ which is $= n$

$$\boxed{\therefore \text{ The system is controllable}}$$

For observability :-

we know, $Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$; $C = [1 \quad 2 \quad 1]$

$$\Rightarrow CA = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \end{bmatrix}$$

$$\Rightarrow CA^2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Thus, $Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$Q \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ } Rank(Q) = 1 which is less than $n(3)$

$1 < 3(n)$

$\boxed{\therefore \text{ The system is not observable}}$

Exercise 2. *Jordan form test* ***(15 points)***

Is the Jordan-form state equation controllable **(7.5 points)** and observable? **(7.5 points)**

$$\underbrace{\dot{x} = \overbrace{\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}^{J} x + \overbrace{\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}^{\hat{B}} u}$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x \quad \hat{C}$$

ANSWER:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad \hat{B} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now, for $\lambda = 2$;

$$\hat{B}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \longrightarrow$$

$$\xrightarrow[R_2 - R_1]{R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \Bigg\}\ \text{The Rank}(\hat{B}^2) = 3$$

$$\hat{C}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{C_3 - C_1 - C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Bigg\}\ \text{The Rank of } \hat{C}^2 = 2$$

$$\boxed{\therefore \text{This is not observable}}$$

For $\lambda = 1$,

$$\hat{B}^1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Big\}\ \text{Rank}(\hat{B}^1) = 2$$

$$\hat{C}^1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \Bigg\}\ \text{Rank}(\hat{C}^1) = 2$$

Thus, Both $\hat{B}^1$ and $\hat{B}^2$ has Full row rank, So the given system is <u>controllable</u>

The $\hat{C}^2$ donot have full column rank, So the given system is <u>not observable</u>

Exercise 3. *Controllability* ***(10 points)***

Recall the Exercise 2 of Homework 2 from last week. Is that system controllable? **(5 points)** Why?
Now lets move the inlet pipe from tank 1 to tank 2, as shown in the figure. Is this system controllable now? **(5 points)** Why?

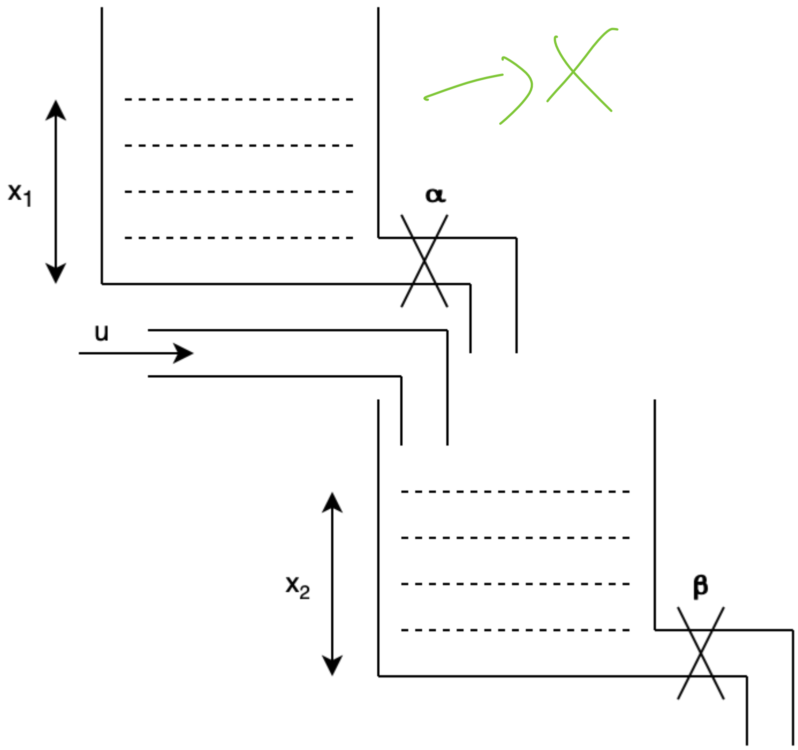

Figure 1: Revised Tank Problem

The system dynamics are

$$\frac{dx_1}{dt} = -\alpha x_1$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$$

ANSWER:

FROM HW-2 we got;

$$\dot{X} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\Rightarrow \dot{X} = \overset{A}{\begin{bmatrix} -0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}} X + \overset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} u$$

$$\Rightarrow A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \; n = 2$$

$$\Rightarrow P = (B, AB) = \begin{bmatrix} 1 & -0.1 \\ 0 & 0.1 \end{bmatrix} \Bigg\} \begin{array}{l} \text{Rank}(P) = 2 \\ \text{which is} = n \\ \rho(2) = 2(n) \end{array}$$

$$\boxed{\text{Thus, The system is controllable.}}$$

<u>For the New system:-</u>

$$\dot{X} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Substitute,

$$\dot{X} = \overset{A}{\begin{bmatrix} -0.1 & \beta \\ 0.1 & -0.2 \end{bmatrix}} X + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u$$

Now,

$$A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and } n = 2$$

$$\therefore P = [B, AB] = \begin{bmatrix} 0 & 0 \\ 1 & 0.2 \end{bmatrix}$$

The rank (P) = 1 which is < 2 (n)

P(1) < 2 (n)

$$\boxed{\text{Thus, The system is not controllable}}$$

Reason :- When we transfer the pipe from Tank 1 to Tank 2, the system loses its controllability. It is clear that with this arrangement, we have no means to regulate or influence the water level in Tank 1. This is why old system is controllable and the new system is not controllable.

Exercise 4. *Gauss Elimination and LU Decomposition* ***(20 points)***

1. Solve the following system of linear equations using Gauss Elimination Method

$$
\begin{aligned}
a)\ x + y + z &= 3 \\
x + 2y + 3z &= 0 \\
x + 3y + 2z &= 3
\end{aligned}
$$

$$
\begin{aligned}
b)\ x + 2y - z &= 1 \\
2x + 5y - z &= 3 \\
x + 3y + 2z &= 6
\end{aligned}
$$

$$
\begin{aligned}
c)\ x_1 + x_2 - x_3 + x_4 &= 1 \\
2x_1 + 3x_2 + x_3 &= 4 \\
3x_1 + 5x_2 + 3x_3 - x_4 &= 5
\end{aligned}
$$

2. Solve the following system of linear equations using LU Decomposition Method

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 &= 3 \\
3x_1 + 8x_2 + 14x_3 &= 13 \\
2x_1 + 6x_2 + 13x_3 &= 4
\end{aligned}
$$

Provide your derivation.

ANSWER:

1 — (a)

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}
$$

$$
\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right]
\xrightarrow[R_3 - R_1]{R_2 - R_1}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \end{array}\right] \longrightarrow
$$

$$\xrightarrow[R_3-2R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \Bigg\} \Rightarrow \begin{cases} x - z = 6 \\ y + 2z = -3 \\ -3z = 6 \end{cases} \text{solving}$$

we get,

$$\Rightarrow \boxed{\begin{aligned} x &= 4 \\ y &= 1 \\ z &= -2 \end{aligned}}$$

1-(b)

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 2 & 5 & -1 & 3 \\ 1 & 2 & -1 & 1 \\ 1 & 3 & 2 & 6 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R_1-2R_2} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 5 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow$$

$$\xrightarrow[R_2+R_3]{\frac{R_3}{2}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1-R_3} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R_2-R_1} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \Leftrightarrow R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}}$$

we get,

$$\boxed{\begin{aligned} x &= 5 \\ y &= -1 \\ z &= 2 \end{aligned}}$$

1-(C)

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{bmatrix} \xrightarrow[R_2-2R_1]{R_3-3R_1} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{bmatrix} \rightarrow$$

$$\xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ \underbrace{0 \quad 0 \quad 0 \quad 0 \quad -2} \end{bmatrix} \Rightarrow 0x_1+0x_2+0x_3+0x_4 \neq -2$$

This should be zero.

→ The equation can't be satisfied, Thus there can be $\boxed{\text{no solution}}$ (Solution donot exist)

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② The equation can be written as;

$$\overset{A}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}}$$

$$\boxed{Ax = B}$$

Assume product of L & U;

where, $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$

Now,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{31} & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}L_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Solving,

$\Rightarrow \quad u_{11} = 1, \quad u_{12} = 2, \quad u_{13} = 4$

$L_{21} u_{11} = 3 \Rightarrow L_{21} = 3$

$L_{21} U_{12} + u_{22} = 8 \Rightarrow u_{22} = 2$

$L_{21} U_{13} + u_{23} = 14 \Rightarrow u_{23} = 2$

$L_{31} u_{11} = 2 \Rightarrow L_{31} = 2$

$L_{31} u_{12} + L_{32} u_{22} = 6$

$\Rightarrow L_{32} = 1$

$L_{31} u_{13} + L_{32} U_{23} + u_{33} = 13 \Rightarrow u_{33} = 3$

Now, we get,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \text{ \& } \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

As we know, $\quad Ax = B$

$A = LU$

we get $\boxed{LUx = B}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$Ux = Y$

Assume $Ux = Y$, Now for $LY = B$

$$\Rightarrow \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \overset{Y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}}$$

By substituting

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_2 + 4x_3 &= 3 \\ 2x_2 + 2x_3 &= 4 \\ 3x_3 &= -6 \end{aligned}\right\}$$

solving we get

$$\boxed{\begin{aligned} x_1 &= 3 \\ x_2 &= 4 \\ x_3 &= -2 \end{aligned}}$$

Exercise 5. *SVD (15 points)*

Use SVD to compress the following image to 50%, 10%, and 5% of the original storage required. You will find the image in the Canvas homework folder. For this problem you need to upload code and attached the corresponding compressed images as well as the number of singular value you used for each level of compression. **Note**: you don't need to care about the actual file size after compression since PNG itself is a more complex data structure that contains the grayscale data and other metadata of the image. You only need to care about the storage required in theory, i.e., how many values you need to store the pixel information. For example, for a greyscale image of size $m \times n$, you need $m \times n$ values to store the information if without compression.

You are expected to **attach the images at each compression level in the writeup and also mention the no. of singular values** you used to obtain the compression at each level.

m

Formula

$$r = \frac{m \times n \times size\%}{m+n+1}$$

Figure 2: CMU_Grayscale.png

n

ANSWER:

(i) 50%

The no: of singular values used $= \boxed{\frac{675 \times 1200 \times 50\%}{675+1200+1} = 216}$

(ii) 10%

The no: of singular values used $= \boxed{\frac{675 \times 1200 \times 10\%}{675+1200+1} = 43}$

(iii) 5%

The no: of singular values used $= \boxed{\frac{675 \times 1200 \times 5\%}{675+1200+1} = 22}$

Exercise 6. *Design for Controllability and Observability* ***(20 points)***

Given the following Linear Time Invariant (LTI) system with a tunable parameter γ,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B}u$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. What values of γ makes the system controllable but not observable? **(10 points)**
2. What values of γ makes the system observable but not controllable? **(10 points)**

ANSWER: The state space equation for the LTI system;

$$A = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad ; \quad n = 2$$

Now, the controllability matrix is; (P)

$$P = [B, AB] = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$$

Now, the observability matrix is; (Q)

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \gamma-3 & -1 \end{bmatrix}$$

① : If we need to make this system controllable but not observable, then the rank(P) = 2 = n (full rank) and rank(Q) < 2 = n

$\hookrightarrow 2 - \gamma = 0$

$\therefore \gamma = 2$

Now let $\gamma = 2$, then $P = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$ } rank(P) = 2 = n(2)

Here, $Q = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ } rank(Q) = 1 < n(2)

So, when $\gamma = 2$, The above system is controllable but not observable

② : If we need to make this system observable, but not controllable then the rank(P) < 2 = n(2) and rank(Q) = 2 = n(2)

So, rank(P) < 2 when $\gamma = 0$ and rank(Q) = 2

$\Rightarrow 2 - \gamma \neq 0$

$\Rightarrow \gamma \neq 2$

Thus $\gamma = 0$,

Let γ be 0, then we have $P = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ & $Q = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix}$

we know Rank(P) = 1 < n(2) and rank(Q) = 2 = n(2)

So, when $\gamma = 0$, The above system is observable but not controllable.

Exercise 7. *State Space Representation, Controllability* ***(10 points)***

We have an LED strip with 5 red LEDs whose brightnesses we want to set. These LEDs are addressed as a queue: at each time step, we can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

1. Model the system as a discrete system with input $u(t)$ as the brightness command to the left-most LED. The state to be the brightness of the five LEDs. Output equals to the state. Write out the state equations in matrix form. **(5 points)**

$x(k)$

2. Check the system's controllability. Explain intuitively what the controllability means in this system. **(5 points)**

Note: you do NOT need to consider the 0-255 constraints on the input.

ANSWER:

1 : Let's assume the brightness of the left most LED is $x_1(K)$ and the brightness of the right most LED is $x_5(K)$

This system can be written in the $x(K+1) = Ax(K) + Bu$ as this is a linear system.

From left to Right we get;

$$x(K+1) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} x(K) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} u(K)$$

$$x(K) = \begin{bmatrix} x_1(K) \\ x_2(K) \\ x_3(K) \\ x_4(K) \\ x_5(K) \end{bmatrix}$$

(2): From Last we got,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ \& } B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, n = 5$$

Now, using Controllability test, (P)

$$P = [B, AB, A^2B, A^3B, A^4B]$$

where, $AB = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $A^2B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $A^3B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$A^4B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\Rightarrow$ combining we get $\Big\}$ $P = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$

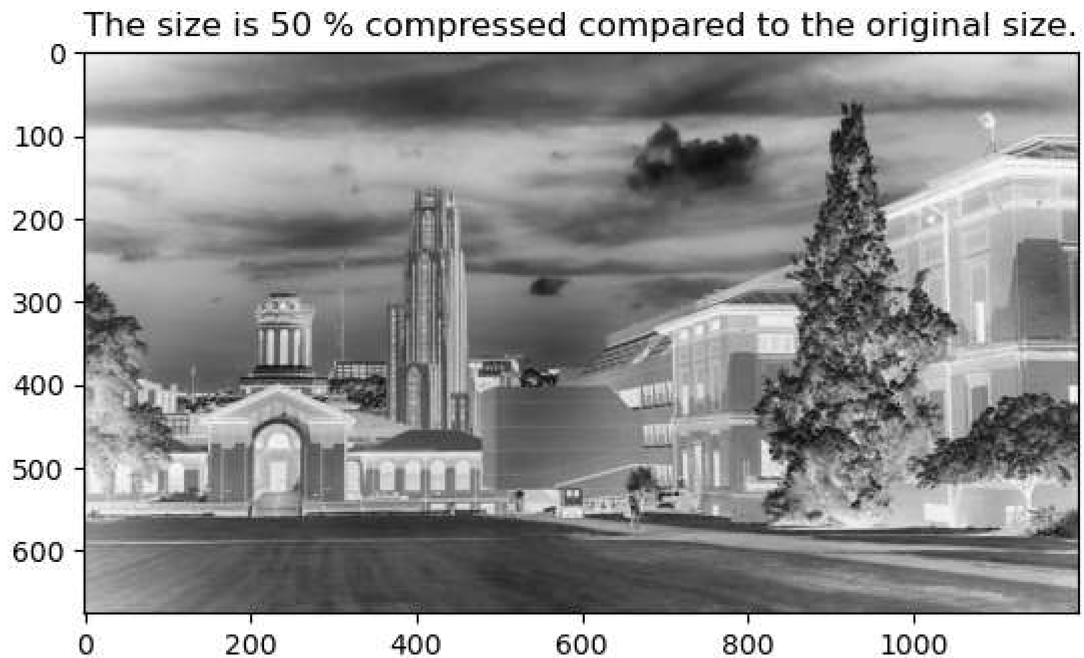
$\underbrace{\text{Rank}(P) = 5 = n(5)}_{\text{full rank}}$

$\boxed{\text{Thus, the system is Controllable}}$

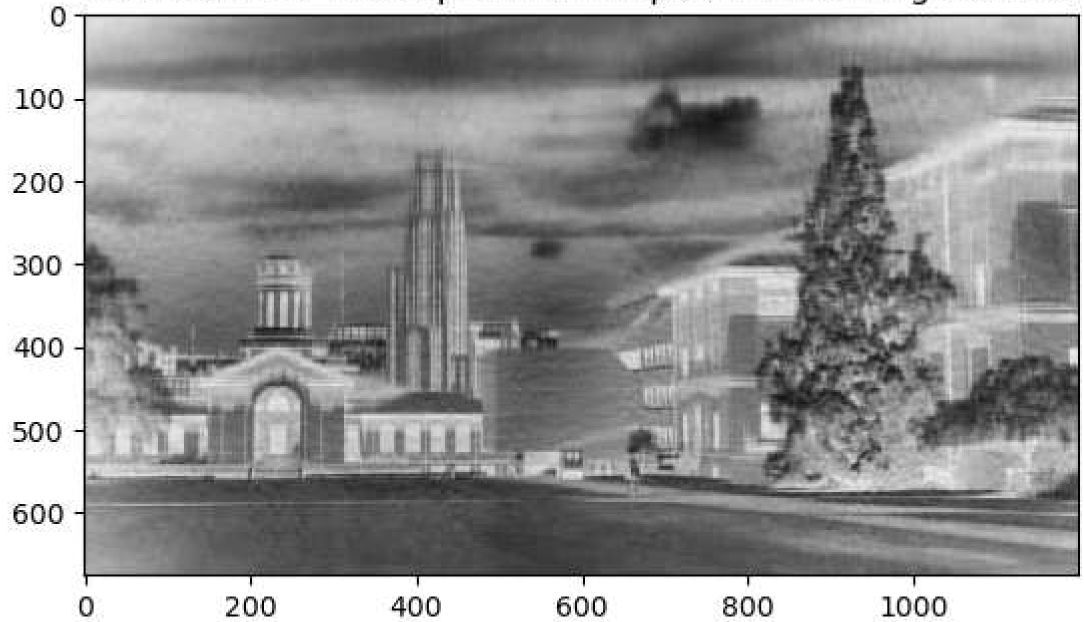
EXPLAINATION :- A system is defined as
controllable if it has the capacity to transition
from any given intial state to any final state
within a certain time frame. In the context of
our LED strip, being controllable signifies that
we have the flexiblity to showcase any
desired brightness levels on the strip. However,
it's important to note that achieving these
specific brightness levels might not be instaneous
and could require several time intervals to
fully realise.

In [18]:
```
import numpy as np
import matplotlib.pyplot as plt
from PIL import Image
import math

def reconstruct_matrix(U, s, Vt, r):
    m = len(U)
    n = len(Vt)
    C = np.zeros((m, n))

    i = 0
    while i <= r:
        col_U = U[:, i].reshape(m, 1)
        row_Vt = Vt[i].reshape(1, n)
        C += s[i] * np.dot(col_U, row_Vt)
        i += 1
    return C.astype('float32')

img_path = "C:\\Users\\srech\\Downloads\\CMU_Grayscale.png"
img_data = plt.imread(img_path)

lvl = 0.5                                # Compressing 50% of the original size
r = (int) (img_data.shape[0] * img_data.shape[1] * lvl / (img_data.shape[0] + img_data.shape[1]))
U, s, Vt = np.linalg.svd(img_data)
re_img = reconstruct_matrix(U, s, Vt, r)
plt.title("The size is 50 % compressed compared to the original size.")
plt.imshow(re_img, cmap=plt.cm.gray_r)
plt.show()

img_data = plt.imread(img_path)

lvl = 0.1                                # Compressing 10% of the original size
r = (int) (img_data.shape[0] * img_data.shape[1] * lvl / (img_data.shape[0] + img_data.shape[1]))
U, s, Vt = np.linalg.svd(img_data)
re_img = reconstruct_matrix(U, s, Vt, r)
plt.title("The size is 10 % compressed compared to the original size.")
plt.imshow(re_img, cmap=plt.cm.gray_r)
plt.show()

img_data = plt.imread(img_path)

lvl = 0.05                               # Compressing 5% of the original size
r = (int) (img_data.shape[0] * img_data.shape[1] * lvl / (img_data.shape[0] + img_data.shape[1]))
```

```
U, s, Vt = np.linalg.svd(img_data)
re_img = reconstruct_matrix(U, s, Vt, r)
plt.title("The size is 5 % compressed compared to the original size.")
plt.imshow(re_img, cmap=plt.cm.gray_r)
plt.show()
```



The size is 10 % compressed compared to the original size.



The size is 5 % compressed compared to the original size.