HOMEWORK - I

24-677 : Linear Control Systems

① Types of Systems:-

(input) $u \rightarrow$ [system] $\rightarrow y$ (output)

(i) $y(t) = 0$ for all t :-

$\Rightarrow$ Here, we know the general eqn; $\boxed{y(t) = H(u)}$

So, we need to find the output for 2 inputs say,

$u_1 \Rightarrow y_1(t) = H(u_1)$ & $\boxed{u_3 = \lambda u_1 + \beta u_2}$

$u_2 \Rightarrow y_2(t) = H(u_2)$

as given, $y(t) = 0$ then $y_1(t) = 0 = y_2(t)$

$u_3 \Rightarrow y_3(t) = H(u_3)$ So, $y_3(t) = H(u_3) = 0$

& $\boxed{y_3(t) = \lambda y_1(t) + \beta y_2(t)}$

$\Rightarrow 0 = \lambda(0) + \beta(0)$

$\Rightarrow 0 = 0$ So, the system is <u>Linear</u>

$\Rightarrow$ For time variance check, we need to assume $u_2(t) = u_1(t - t_0)$, where t_0 is a shift or delay and check for $y_2(t) = y_1(t - t_0)$

Here, $\left.\begin{array}{l} u_1 \Rightarrow y_1(t) = H(u_1) \\ u_2 \Rightarrow y_2(t) = H(u_2) \end{array}\right\} 0$

In this, the system always produces an output of 0, (constant)

Hence, This is <u>time-Invariant</u>

Scanned with CamScanner

(ii) $y(t) = u^3(t)$:-

$\Rightarrow$ Here, the inputs are;

$u_1 \Rightarrow y_1(t) = u_1^3(t)$

$u_2 \Rightarrow y_2(t) = u_2^3(t)$

$u_3 \Rightarrow y_3(t) = u_3^3(t)$ — ①

where, ~~$u_3 \neq \lambda u_1(t)$~~

$u_3(t) = \lambda u_1(t) + \beta u_2(t)$ & — ②

$y_3(t) = \lambda y_1(t) + \beta y_2(t)$ — ③

In ③, $y_3(t) = \lambda u_1^3(t) + \beta u_2^3(t)$

Now, $(u_3)^3_{(t)} = (\lambda u_1(t) + \beta u_2(t))^3$

Substitute $y_3(t)$ & $u_3^3(t)$, in eqn ①,

~~$\lambda y_1(t) + \beta y_2(t) = \lambda u_1(t$~~

$\lambda u_1(t) + \beta u_2(t) \neq (\lambda u_1(t) + \beta u_3(t))^3$

$\therefore$ This is a non-Linear

=> For Time invariance check, → delay/shift

Assume $u_2(t) = u_1(t-t_0)$

& check $y_2(t) = y_1(t-t_0)$

Inputs; $u_1 \Rightarrow y_1(t) = u_1^3(t)$

$u_2 \Rightarrow y_2(t) = u_2^3(t)$

$y_2(t) = u_2^3(t)$ → Substituting,

$y_2(t) = u_1^3(t-t_0)$ — ④

Now, $y_1(t) = u_1^3(t-t_0)$ — ⑤

④ = ⑤

Hence, it's time-Invariant

(iii) $y(t) = u(3t)$:

→ For Linearity, I/Ps,

$u_1 \Rightarrow \quad y_1(t) = u_1(3t)$

$u_2 \Rightarrow \quad y_2(t) = u_2(3t)$

$u_3 \Rightarrow \quad y_3(t) = u_3(3t) \quad -①$

where, $u_3(t) = \lambda u_1(t) + \beta u_2(t)$ & $-②$

$y_3(t) = \lambda y_1(t) + \beta y_2(t) \quad -③$

Take, $u_3(t) = \lambda u_1(t) + \beta u_2(t)$

$u_3(3t) = \lambda u_1(3t) + \beta u_2(3t)$

&, $y_3(t) = \lambda u_1(3t) + \beta u_2(3t)$

Substitute $u_3(3t)$ & $y_3(t)$ in ean ①

$\lambda u_1(3t) + \beta u_2(3t) = \lambda u_1(3t) + \beta u_2(3t)$

Hence, This is Linear

⇒ For Time invarience,

Assume = $u_2(t) = u_1(t - t_0)$ → delay ⇒ $u_2(3t) \Rightarrow u_1(3[t-t_0])$

Check $y_2(t) = y_1(t - t_0)$

of I/PS ⇒ $u_1 \Rightarrow y_1(t) = y\, u_1(3t)$

$u_2 \Rightarrow y_2(t) = u_2(3t)$

$y_2(t) = u_1[3(t-t_0)] \quad -④$

$y_1(t - t_0) = u_1[3(t-t_0)] \quad -⑤$

④=⑤ So, Time invarient

(iv) $y(t) = e^{-t} u(t-T)$:-

$\Rightarrow$ For Linearity, I/PS,

$u_1 \Rightarrow y_1(t) = e^{-t} u_1(t-T)$

$u_2 \Rightarrow y_2(t) = e^{-t} u_2(t-T)$

$u_3 \Rightarrow y_3(t) = e^{-t} u_3(t-T)$ — ①

& $u_3 = \lambda u_1 + \beta u_2$ — ②

$y_3 = \lambda y_1 + \beta y_2$ — ③

Take ②, $u_3(t) = \lambda u_1(t) + \beta u_2(t)$

$e^{-t} u_3(t-T) = e^{-t} u_1(t-T) + e^{-t} u_2(t-T)$

Take ③, $y_3(t) = e^{-t} \lambda u_1(t-T) + e^{-t} \beta u_2(t-T)$

Substitute the above eqns in ①

we get, ~~$y_3(t)$~~ =

① $e^{-t} u_1(t-T) + e^{-t} u_2(t-T) = e^{-t} \lambda u_1(t-T) + e^{-t} \beta u_2(t-T)$

$\therefore$ This is Linear system

$\Rightarrow$ For Time Invariance, Assume $u_2(t) = u_1(t-t_0)$ Assume delay as t_0

Hence, $u_2(t-T) = u_1(t-T-t_0)$

Check, $y_2(t) = y_1(t-t_0)$

I/PS $u_1 \Rightarrow y_1(t) = e^{-t} u_1(t-T)$

$y_2(t) = e^{-t} u_2(t-T)$

$y_2(t) = e^{-t} u_1(t-T-t_0)$ — ④

~~$y_1(t)$~~ $= e^{-t}$

$y_1(t-t_0) = e^{-(t-t_0)} u_1(t-T-t_0)$ — ⑤

④ $\neq$ ⑤

$\therefore$ The system is time-varying

(v)
$$y(t) = \begin{cases} 0 & t \leq 0 \\ u(t) & t > 0 \end{cases}$$

⇒ For Linearity, ~~if~~ $u_1(t) \Rightarrow y_1(t) = u_1(t)$

$u_2(t) \Rightarrow y_2(t) = u_2(t)$

$u_3(t) = y_3(t) = u_3(t)$

Now, ~~y~~ and $y_3(t) = \alpha y_1(t) + \beta y_2(t)$

$y_3(t) = \alpha u_1(t) + \beta u_2(t)$

} for $t > 0$

~~if~~ $y_3(t) = 0$ } for $t \leq 0$ (solved in part-I [I(i)])

Now, combined output, will not be a homogenous as it will have varied values for different sides of the axis, [The signs will be opposite]

∴ This is non-linear system

⇒ For Time invarience,

Since the system is defined in 2 different sides of the axis, which donot depend on value of 't'.

Hence the system is Time-Invarient

(2) State space representations:-

3 teams (each 2 drones)

transmitter (Ti) recieverr

$i \rightarrow P_i > 0\}$

$T_j \longrightarrow R_i \Rightarrow G_{ij}$

Signal Power at $(R_i) \Rightarrow S_i = G_{ii} p_i$

$\sigma > 0$

Noise + Interference at $R_i \Rightarrow q_i = \underbrace{\sigma^2}_{\text{noise}} + \underbrace{\sum_{j \neq i} G_{ij} P_j}_{\text{Interference}}$

given $j = i$ so, $\underbrace{\text{no diagnal elemets in the matrix}}_{0}$

$\to$ threshold value

$S_i = S_i / q_i \quad [S_i \geq \gamma]$

$P_i(k) \Rightarrow$ discrete signals
$\hookrightarrow 0, 1, 2$

$S_i(K) = S_i(K) / q_i(K) = \alpha\gamma$
$\hookrightarrow \alpha > 1$

$\Rightarrow P_i(K+1) = P_i(K) (\alpha\gamma / S_i(K))$

(i) $P_i(K+1) = P_i(K) \left[\frac{\alpha\gamma}{S_i(K)}\right]$

Now, $P(K+1) = AP(K) + B\sigma^2$?

$P(K+1) \Rightarrow P_i(K) = \left[\frac{\alpha\gamma}{S_i(K)}\right] \Rightarrow$ substitute $S_i(K)$

$P(K+1) \Rightarrow P_i(K) \left[\frac{\alpha\gamma}{G_{ii} p_i} \times \frac{\sigma^2 + \sum_{j \neq i} G_{ij} P_j(K))}{G_{ii} p_i(K)}\right]$

$$P_i(K+1) = P_i(K)\left[\frac{\alpha\gamma(\sigma^2 + \sum_{j\neq i} G_{ij} P_j(K))}{G_{ii} P_i(K)}\right] //$$

Now substitute $i=1,2,3$ .. (Give 3×3 matrix)

Let $i=1$,

$$P_1(K+1) = \cancel{P_1(K)}\left[\frac{\alpha\gamma(\sigma^2 + \sum G_{12}P_2(K) + G_{13}P_3(K))}{G_{11}\cancel{P_1(K)}}\right]$$

$$\Rightarrow \underbrace{\frac{\alpha\gamma(\sigma^2)}{G_{11}}}_{B} + \underbrace{\frac{\alpha\gamma(G_{12}P_2(K) + G_{13}P_3(K))}{G_{11}}}_{A}$$

Let $i=2$,

$$P_2(K+1) = \cancel{P_2(K)}\left[\frac{\alpha\gamma(\sigma^2 + G_{21}P_1(K) + G_{23}P_3(K))}{G_{22}\cancel{P_2(K)}}\right]$$

$$\Rightarrow \underbrace{\frac{\alpha\gamma(\sigma^2)}{G_{22}}}_{B} + \underbrace{\frac{\alpha\gamma(G_{21}P_1(K) + G_{21}P_3(K))}{G_{22}}}_{A}$$

Let $i=3$,

$$P_3(K+1) = \cancel{P_3(K)}\left[\frac{\alpha\gamma(\sigma^2 + G_{31}P_1(K) + G_{32}P_2(K))}{G_{33}\cancel{P_3(K)}}\right]$$

$$\Rightarrow \underbrace{\frac{\alpha\gamma(\sigma)^2}{G_{33}}}_{B} + \underbrace{\frac{\alpha\gamma(G_{31}P_1(K) + G_{32}P_2(K))}{G_{33}}}_{A}$$

matrix Form: $P(K+1) = A\,P_i(K) + B\,\sigma^2$

$$A = \begin{bmatrix} 0 & \frac{\alpha\gamma G_{12}}{G_{11}} & \frac{\alpha\gamma G_{13}}{G_{11}} \\ \frac{\alpha\gamma G_{21}}{G_{22}} & 0 & \alpha\gamma G_{23}/G_{22} \\ \frac{\alpha\gamma G_{31}}{G_{33}} & \frac{\alpha\gamma G_{32}}{G_{33}} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\alpha\gamma}{G_{11}} \\ \frac{\alpha\gamma}{G_{22}} \\ \frac{\alpha\gamma}{G_{33}} \end{bmatrix} //$$

(3) Linearization

$\Rightarrow$ eqn $\Rightarrow \ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x}_1 = x_2$ $\quad \dot{y}$

now, $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0.5x_1^3 - 2x_1 + (1+x_1)x_2 \end{bmatrix}$ $\quad f(x)$

$2x_1 - 0.5x_1^3 - (1+x_1)x_2$

To ~~find~~ the equilibrium points $\bar{x}$, $f(\bar{x}) = 0$

Given ~~equation~~ equilibrium pts are 0, $x_2 = 0$

$\Rightarrow 0.5x_1^3 - 2x_1 + (1+x_1)x_2 = 0$

$\Rightarrow$ As $x_2 = 0$

$\Rightarrow 0.5x_1^3 - 2x_1 + 0 = 0$

~~$\Rightarrow 0.5x_1^3 = 2x_1$~~ $\quad x_1(0.5x_1^2 - 2) = 0$

~~$\Rightarrow 0.5x_1^2 = 2$~~ $\quad x_1(2 - 0.5x_1^2) = 0$

~~$x_1^2 = \frac{2}{0.5}$~~ $\quad x_1 = 0$ or $+2$ or -2

3 equilibrium pts $\underline{0, -2, +2}$

$\bar{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\bar{x}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$; $\bar{x}_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

Now introduce a perturbation

$\delta x = x - \bar{x}$

$\dot{x} = \delta\dot{x}$

using Taylor series,

$\dot{x} \Rightarrow f(\bar{x}) + \frac{d f(\bar{x})}{dx} \delta x$

$$\delta x = \begin{bmatrix} 0 & 1 \\ 2-1.5x_1^2 - x_2 & -(1+x_1) \end{bmatrix} \delta x$$

~~case~~ Now substitute the pts;

$[0,0] \Rightarrow \delta x = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \delta x$ $\quad [2,0]\ \delta x = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \delta x$

$[-2,0] \Rightarrow \delta x = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \delta x$

(4) Given

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g\left(\frac{D}{x_1(t)+D}\right)^2 + \frac{\ln(u)}{m} \end{bmatrix} = f(x,u)$$

We know that equilibrium ~~po~~ of a system is achieved when $\dot{x}_1 = 0$ & $\dot{x}_2 = 0$

Given $\dot{x}_1 = x_2(t)$; so $x_2(t) = 0$

with ~~$\dot{x}_2 = 0$~~, $x_2(t) = 0$., $\frac{-gD^2}{(x_1(t)+D)^2} + \frac{1}{m}\ln(u) = 0$

$$\Rightarrow \frac{gD^2}{(x_1(t)+D)^2} = \frac{1}{m}\ln(u)$$

$\Rightarrow$ ~~$\frac{gD^2 \times m}{\ln(u)} = (x_1(t)+D)^2$~~ $\quad \frac{gD^2 m}{\ln(u)} = (x_1(t)+D)^2$

$\Rightarrow$ ~~$\frac{\sqrt{g}\,D}{x_1(t)+D} = \left(\frac{1}{m}\ln(u)\right)^2$~~ Sq root on both sides

$$\sqrt{\frac{gD^2 m}{\ln(u)}} = x_1(t) + D$$

$$x_1(t) = D\left[\sqrt{\frac{gm}{\ln(u)}} - 1\right]$$

Scanned with CamScanner

$$\bar{x} = \begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ D\left(\sqrt{\frac{gm}{en(u)}} - 1\right) \end{bmatrix}$$

Now, with pertubation,

$$\delta x = x - \bar{x}$$

so $x = \delta x + \bar{x}$

$$\dot{x} = \delta \dot{x}$$

$\delta \dot{x} =$

$$\begin{bmatrix} \delta \dot{x}_1(t) \\ \delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2g\left(\frac{D}{(x_1(t)+D)^3}\right) & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

<u>The points are</u> $(D, 0)$

(5) Given;

(i) $L = T - V = \frac{1}{2}\dot{r}^2\theta^2 - \frac{K}{r}$

The equation of motions are;

$$\ddot{r} = \dot{r}\dot{\theta}^2 - \frac{K}{r^2} + u_1 \quad \text{—} (1)$$

$$\ddot{\theta} = -2\frac{\dot{\theta}\dot{r}}{r} + \frac{u_2}{r} \quad \text{—} (2)$$

Given that $u_1 = u_2 = 0$,
$r(t) = p$
$\theta(t) = \cancel{(\ )}\omega t$

Here, $r = p$
$\theta = \omega t$
$\dot{\theta} = \omega$
$u_1 = 0$

Substituting in (1)

$$\ddot{r} = p\omega^2 - \frac{K}{r^2} + 0, \text{ Here } \cancel{\ } \dot{r} = \ddot{r} = 0 \text{ as } r \text{ is a constant}$$

$$\cancel{or}\ p\omega^2 - \frac{K}{p^2} + 0 = 0$$

$$p\omega^2 = \frac{K}{p^2}$$

$$\boxed{K = p^3\omega^2}$$

(ii) Let's define the states for this system;

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ which is } = \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

now,

$$\dot{x} = \begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - K/x_1^2 + u_1 \\ x_4 \\ \frac{-2x_4x_2}{x_1} + \frac{u_2}{x_1} \end{bmatrix}$$

$$\delta\dot{x} = \begin{bmatrix} 0 \\ 0 \\ \omega \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2K/x^3 & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 0 \\ \frac{2x_4x_2}{x_1^2} - \frac{u_1}{x^2} & \frac{-2x_4x_2}{x_1} & 0 & \frac{-2x_2}{x_1} \end{bmatrix} \delta x$$

$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/x_1 \end{bmatrix} \delta u$$