Homework 5

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24-677 Special Topics: Linear Control Systems

Due: Oct 13, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Asymptotic stability and Lyapunov stability.* ***(10 points)***

For each of the systems given below, determine whether it is Lyapunov stable, whether it is asymptotic stable.

(a) **(5 points)**

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}_{A} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

(b) **(5 points)**

$$\dot{x} = \underbrace{\begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}}_{A} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

ANSWER:

(a) The given system is a DT linear system, where

$$A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} \Big\} \det(\lambda I - A)$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & 0 \\ 0.5 & \lambda-0.5 \end{vmatrix}$$

$$\Rightarrow (\lambda-1)(\lambda-0.5) = 0$$

So we get, $\lambda_1 = 1$ & $\lambda_2 = 0.5$

Now we can compute the Jordan form of A

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Now, $J^k = \begin{bmatrix} i^k & 0 \\ 0 & 0.5^k \end{bmatrix}$

for $\lambda_1 = 0.5$, we have $0.5 < 1$

for $\lambda_2 = 1$, we have $m = 0$

$\therefore$ This system is Lyapunov stable but not asymptotic stable.

(b) We know this given system is CT linear system where,

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

Now $\det(\lambda I - A) = \begin{bmatrix} \lambda+7 & 2 & -6 \\ 2 & \lambda+3 & 2 \\ 2 & 2 & \lambda-1 \end{bmatrix} = 0$

$$= (\lambda+1)(\lambda+5)(\lambda+3) = 0$$

we get, $\lambda_1 = -1$, $\lambda_2 = -3$, $\lambda_3 = -5$

Now we can compute Jordan form of A,

$$\left. J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \right\} e^{Jt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-5t} \end{bmatrix}$$

Therefore, This system is both Lyapunov stable and asymptotically stable.

Exercise 2. *Stabilizability* ***(20 points)***

Decompose the state equation $\xrightarrow{} A$ $\nearrow B$

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

$\hookrightarrow C$ $\quad D \to 0$

to a controllable form. Is the reduced state equation observable, stabilizable, detectable?

ANSWER:

The controllable matrix P,

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\} \text{rank}(P) = 2$$

Now, the bias matrix M as,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Now, $\hat{A} = M^{-1} A M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$\hat{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

For $\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$\hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For $\hat{C} = CM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$\hat{C} = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$$

The reduced state equation is;

$$\left.\begin{aligned} \dot{x}_c &= \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}}^{A} x_c + \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{B} u \\ y &= \underbrace{\begin{bmatrix} 3 & 3 \end{bmatrix}}_{C} x_c \end{aligned}\right\} \text{controllable}$$

Now, checking observability;

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \Big\} \; \text{rank}(Q) = 1 < 2\,(n)$$

Thus, The system is not observable.

$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$; $\lambda_1 = 0$, $\lambda_2 = -1$

eigen vectors ; $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ → not observable

↳ observable

Now for checking the stability of the system,
as we know that the reduced system is
always controllable, Hence this system is
stabilizable

Now for the detectability ?

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} , \lambda_1 = 0 \text{ \& } \lambda_2 = -1$$

The jordan form is, $J = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

Here we have $\lambda_2 = -1 < 0$

Therefore, the not observable mode is
Lyapunov stable

Thus, this is detectable.

$\therefore$ The Reduced System is not observable, detectable
and Stabilizable

Exercise 3. *Stability* ***(15 points)***

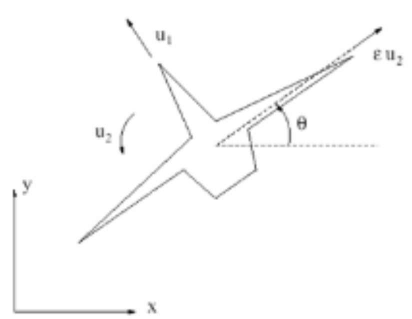


Figure 1: The VTOL. aircraft

The following is a planar model of a Vertical Take-off and Landing (VTOL) aircraft such as Lock-heed's F35 Joint Strike fighter around hover (cf. Figure 1):

$$m\ddot{x} = -u_1 \sin\theta + \epsilon u_2 \cos\theta$$
$$m\ddot{y} = u_1 \cos\theta + \epsilon u_2 \sin\theta - mg$$
$$J\ddot{\theta} = u_2$$

where x, y are the position of the center of mass of the aircraft in the vertical plane and θ is the roll angle of the aircraft. $u1$ and $u2$ are the thrust forces (control inputs). The thrust is generated by a powerful fan and is vectored into two forces $u1$ and $u2$. J is the moment of inertia, and ϵ is a small coupling constant. Determine the stability of the linearized model around the equilibrium solution

$$\tilde{x}(t), \tilde{y}(t), \tilde{\theta}(t) = 0, \tilde{u}_1(t) = mg; \tilde{u}_2(t) = 0.$$

The linearized model should be time invariant. The state $z = [\theta, \dot{x}, \dot{y}, \dot{\theta}]^T$, $u = [u_1, u_2]^T$

ANSWER:

we know the states variables are, $\dot{\theta}, \dot{x}, \dot{y}, \theta$

Given, $Z = \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$, Now for $\dot{Z}$

$$\dot{z} = f(\theta, x, y, \dot{\theta}) = \begin{array}{c} \begin{bmatrix} \dot{\theta} \\ \dfrac{-u_1 \sin\theta + \varepsilon u_2 \cos\theta}{m} \\ \dfrac{u_1 \cos\theta + \varepsilon u_2 \sin\theta - mg}{m} \\ \dfrac{u_2}{J} \end{bmatrix} \end{array} \begin{array}{l} \rightarrow f_1 \\ \rightarrow f_2 \\ \rightarrow f_3 \\ \rightarrow f_4 \end{array}$$

Now linearize the given system (z, u)

$$\dot{z} = \begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1(z,u)}{\partial z} \\ \dfrac{\partial f_2(z,u)}{\partial z} \\ \dfrac{\partial f_3(z,u)}{\partial z} \\ \dfrac{\partial f_4(z,u)}{\partial z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ \dfrac{-u_1 \cos\theta - \varepsilon u_2 \sin\theta}{m} & 0 & 0 & 0 \\ \dfrac{-u_1 \sin\theta + \varepsilon u_2 \cos\theta}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{J_1} z$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ \dfrac{-\sin\theta}{m} & \dfrac{\varepsilon \cos\theta}{m} \\ \dfrac{\cos\theta}{m} & \dfrac{\varepsilon \sin\theta}{m} \\ 0 & \dfrac{1}{J} \end{bmatrix}}_{J_2} u \quad \hookrightarrow \begin{bmatrix} \partial u_1 \\ \partial u_2 \end{bmatrix}$$

Now, $$J_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \dfrac{-u_1 \cos\theta - \varepsilon u_2 \sin\theta}{m} & 0 & 0 & 0 \\ \dfrac{-u_1 \sin\theta + \varepsilon u_2 \cos\theta}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we are given that,

$u_1(t) = mg, \quad u(t) = 0, \quad \theta(t) = 0, \quad x(t) = 0, \quad y(t) = 0$

So,
$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

now take J_2,

$$J_2 = \begin{bmatrix} 0 & 0 \\ \frac{-\sin\theta}{m} & \frac{\varepsilon\cos\theta}{m} \\ \frac{\cos\theta}{m} & \frac{\varepsilon\sin\theta}{m} \\ 0 & 1/J \end{bmatrix}$$

Substituting the given values we get,

$$J_2 = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon/m \\ 1/m & 0 \\ 0 & 1/J \end{bmatrix}$$

Now plugging in J_1 & J_2 matrices in the Linearized eqn:

$$\dot{z} = \begin{pmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \delta\theta \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon/m \\ 1/m & 0 \\ 0 & 1/J \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\Big\}$ $\det(A - \lambda) = \lambda^4 = 0$

λ's $= 0$

Jordan form $J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Therefore, the linearized model around the equilibrium solution is unstable.

Exercise 4. *Lyapunov's direct method* ***(10 points)***

An LTI system is described by the equations

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Use Lyapunov's direct method to determine the range of variable a for which the system is asymptotically stable. Consider the Lyapunov function,

$$V = x_1^2 + x_2^2$$

ANSWER:

Given Lyapunov function, $V = x_1^2 + x_2^2$

we know $V(x) > 0 \;\forall\, x \neq 0$

$\Rightarrow \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

where, $\dot{x}_1 = ax_1$ & $\dot{x}_2 = x_1 - x_2$

So, $\dot{V}(x) = 2ax_1^2 + 2x_1x_2 - 2x_2^2$

This system needs to be asymptotically stable

for all $x = [x_1 \; x_2]$, $\dot{V}(x) = 2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0$

$\Rightarrow a x_1^2 + x_1x_2 - x_2^2 < 0$ — (1)

$\Rightarrow a\left(\frac{x_1}{x_2}\right)^2 + \left(\frac{x_1}{x_2}\right) < 1$ — (2)

γ

$\Rightarrow$ In equation ②, $\quad ar^2 + ar - 1 < 0$

$$1 + 4a < 0$$

$$\boxed{\begin{array}{c} a < -0.25 \\ \therefore \text{Thus the system is asymptotically stable} \end{array}}$$

Exercise 5. *Stability of Non-Linear Systems* ***(20 points)***

Consider the following system:

$$\dot{x_1} = x_2 - x_1 x_2^2$$
$$\dot{x_2} = -x_1^3$$

(a) Is the approximated linearized system stable? Based on Lyapunov's Indirect method, is the system stable? [Hint: can you get a firm answer in this case?] **(5 points)**

(b) Based on Lyapunov's Direct method? **(5 points)** Consider the Lyapunov function:

$$V(x_1, x_2) = x_1^4 + 2x_2^2$$

(c) Plot the Phase Portrait plot of the original system and linearized system in a. **(5 points)**. Submit the code to Gradescope.

(d) Generate a 3D plot showing the variation of $\dot{V}$ with respect to x_1 and x_2. **(5 points)** [Hint: Use Axes3D python library]. Submit the code to Gradescope.

Note: For (c) and (d), include the code along with the plot in the pdf to be submitted. No need to submit .py file.

ANSWER:

(a) Linearising the model.

Let $\dot{x}_1$ and $\dot{x}_2$ be 0, So we get

$$x_2 - x_1 x_2^2 = 0 \quad \text{and} \quad -x_1^3 = 0$$

$$\dot{x}_1 = f_1(x_1, x_2) \Rightarrow -x_1 x_2^2 + x_2$$

$$\therefore \frac{\partial f_1}{\partial x_1} = -x_2^2 = 0 \quad ; \quad \frac{\partial f_1}{\partial x_2} = -2x_1x_2 + 1 = 1$$

$\Rightarrow \dot{x}_2 = f_2(x_1, x_2) = -x_1^3$

$\Rightarrow \dfrac{\partial f_2}{\partial x_1} = -3x_1^2 = 0 \; ; \; \dfrac{\partial f_2}{\partial x_2} = 0$

Therefore we get $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Now, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ & $\det(\lambda I - A) = 0$

Here we know λ_1 & λ_2 are zero.

Since both eigen values are zero, they are defective

Hence, Based on Lyapunov's indirect method, the linearized system is not stable.

(b) Given $V(x_1, x_2) = x_1^4 + 2x_2^2$

So, for all $x \neq 0$ and $x(0) = 0$ we have

$V(x_1, x_2) > 0$ [$V(x_1, x_2)$ is positive definite]

$$\Rightarrow \dot{V} = 4x_1^3\dot{x}_1 + 4x_2\dot{x}_2$$

Substitute $\dot{x}_1 = x_2 - x_1x_2^2$; $\dot{x}_2 = -x_1^3$

$$\Rightarrow \dot{V} = 4x_1^3(x_2 - x_1x_2^2) + 4x_2(-x_1^3)$$

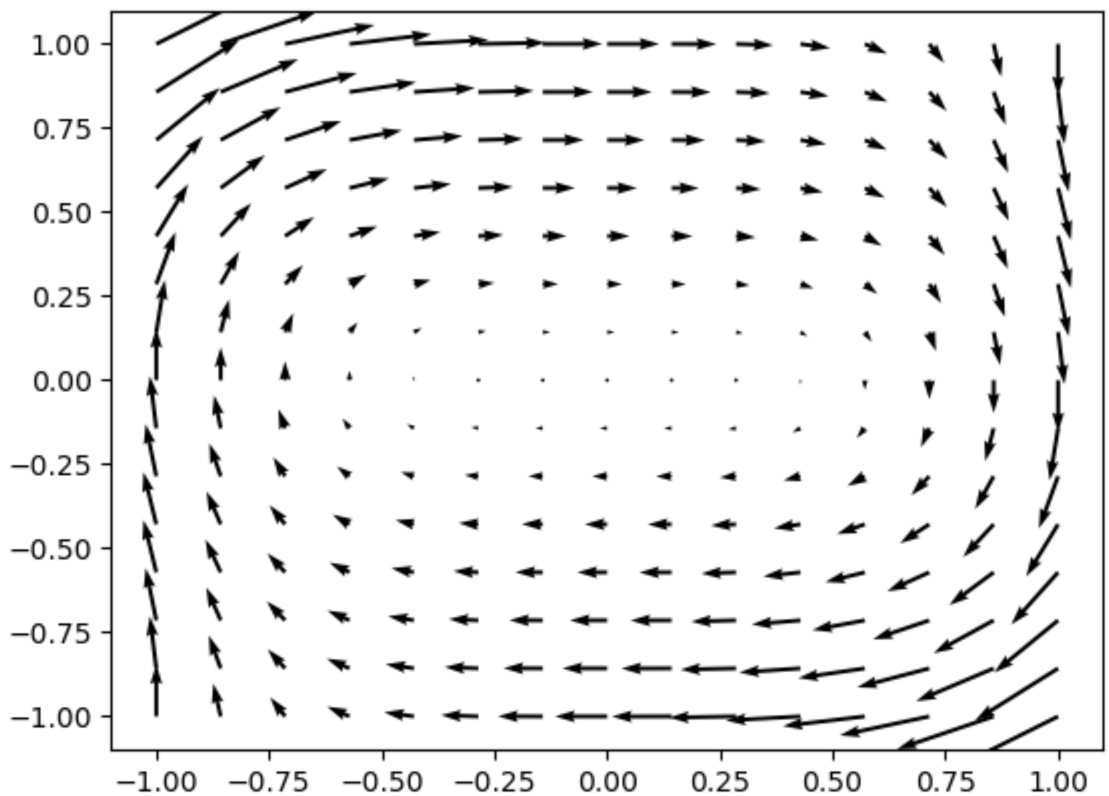
$$\Rightarrow \dot{V} = -4x_1^4x_2^2$$

Here, since $-4x_1^4x_2^2 < 0$, the energy of the system is reducing, so based on Lyapunov's direct method, the linear system is stable.

(c)

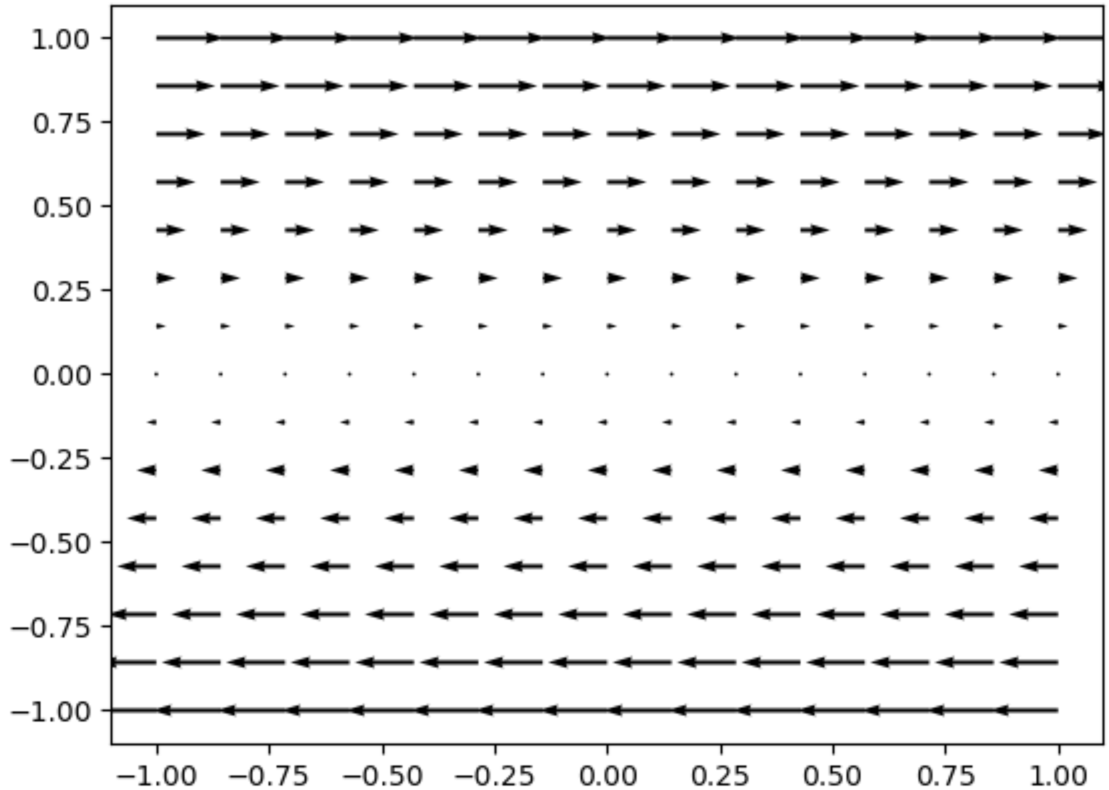
In [8]:

```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-1,1,15)
y = np.linspace(-1,1,15)
x1,x2 = np.meshgrid(x,y)
x1_dot = x2-x1*x2*x2
x2_dot = -x1*-x1*-x1
plt.figure()
plt.quiver(x1,x2,x1_dot,x2_dot)
plt.show()

x1_dot=x2
x2_dot=np.zeros(x1.shape)
plt.figure()
plt.quiver(x1,x2,x1_dot,x2_dot)
plt.show()
```



THE PHASE PORTRAIT PLOT OF THE ORIGINAL SYSTEM

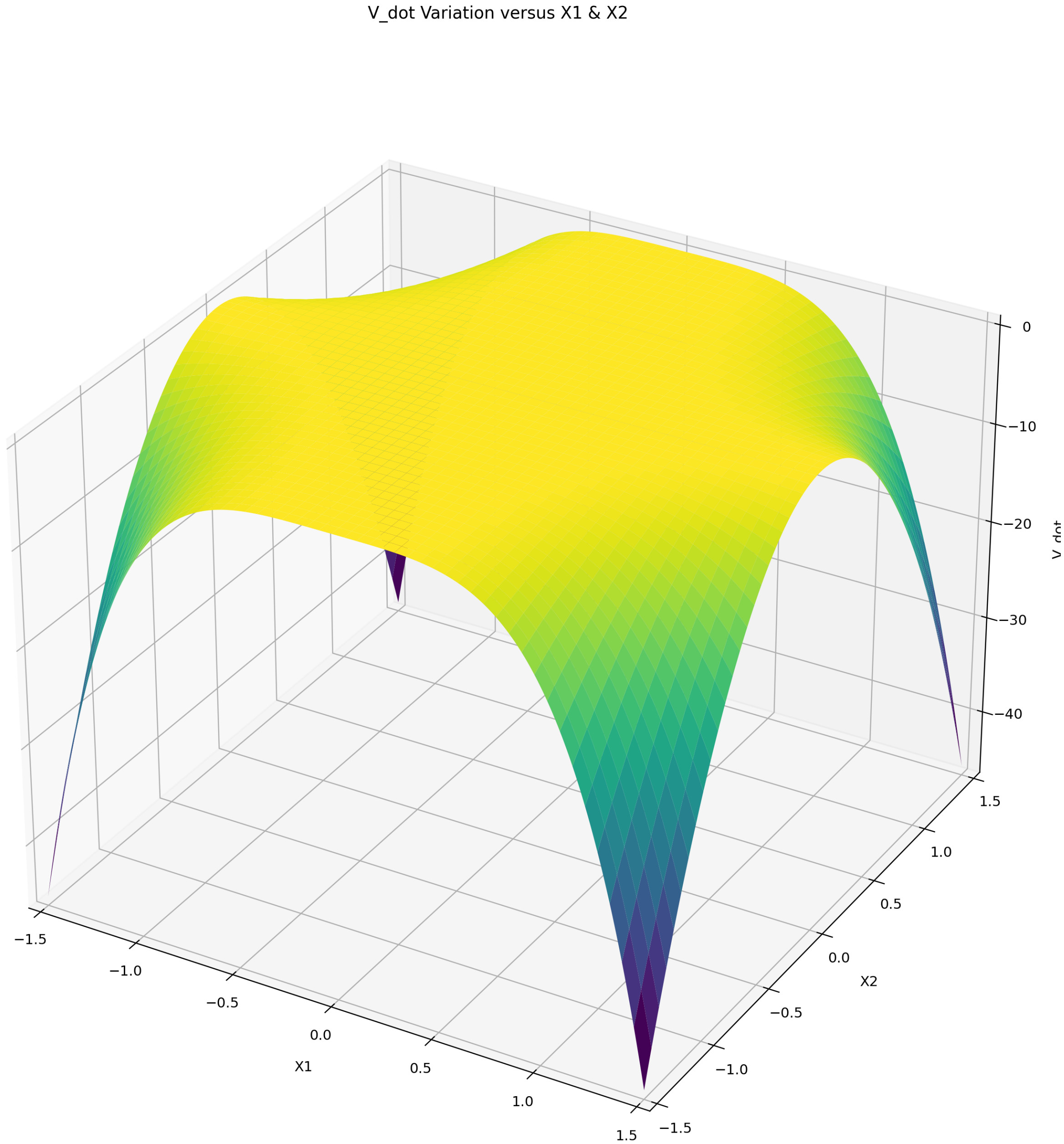
THE PHASE PORTRAIT PLOT OF THE LINEARIZED SYSTEM



(D)

In [19]:

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import cm

fig = plt.figure(figsize=(12,12),dpi=230)
ax = fig.add_subplot(111,projection='3d')
x = np.linspace(-1.5,1.5,400)
y = np.linspace(-1.5,1.5,400)
x1,x2 = np.meshgrid(x,y)
v_dot = -4*np.power(x1,4)*np.power(x2,2)
surf = ax.plot_surface(x1,x2,v_dot,cmap=cm.viridis,linewidth=0,antialiased=True)
ax.set_title("V_dot Variation versus X1 & X2",pad=40)
ax.set_xlabel('X1',labelpad=10)
ax.set_ylabel('X2',labelpad=10)
ax.set_zlabel('V_dot',labelpad=10)
ax.set_xlim(-1.5,1.5)
ax.set_ylim(-1.5,1.5)
plt.tight_layout()
plt.show()
```


V_dot Variation versus X1 & X2
0
−10
−20
−30
−40
V_dot
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
X1
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
X2

Exercise 6. *BIBO Stability* ***(10 points)***

For each of the systems given below, determine whether it is BIBO stable.

(a) **(5 points)**

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 5 & 5 \end{bmatrix} x(k)$$

(b) **(5 points)**

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

ANSWER:

(a) The given equation to generate the transfer function for the DT system

$$zI - A = \begin{bmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{bmatrix}$$

$$(zI - A)^{-1} = \frac{1}{(z-1)(z-0.5)} \begin{bmatrix} z-0.5 & 0 \\ -0.5 & z-1 \end{bmatrix}$$

where, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C = [5, 5]$, $D = [0]$

Hence,

$$G_D(s) = C(zI-A)^{-1}B + D$$

$$= \frac{1}{(z-1)(z-0.5)}\begin{bmatrix}5 & 5\end{bmatrix}\begin{bmatrix} z-0.5 & 0 \\ -0.5 & z-1 \end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix} = 0$$

= Transfer function = 0

Therefore, The given system is BIBO stable

(b) We are using the given equation to generate the transfer function for the CT system.

$$\Rightarrow \quad sI - A = \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}$$

$$\Rightarrow (sI-A)^{-1} = \frac{1}{s^3+9s^2+23s+15}\begin{bmatrix} s^2+2s-7 & -2s-10 & 6s+22 \\ 2s+2 & s^2+6s+5 & -2s+2 \\ -2s-10 & -2s-10 & s^2+10s+25 \end{bmatrix}$$

$$\Rightarrow \text{where, } B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \; C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \; D = 0$$

$$\Rightarrow G_C(s) = C(sI-A)^{-1}B + D$$

Substituting,

$$\Rightarrow G_c(s) = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{pmatrix} s^2+2s-7 & -2s-10 & 6s+22 \\ 2s+2 & s^2+6s+15 & -2s+2 \\ -2s-10 & -2s-10 & s^2+10s+25 \end{pmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow G_c(s) = \frac{1}{(s+1)(s+3)(s+5)} \begin{bmatrix} 0 & 0 \\ (s+1)(s+5) & 0 \end{bmatrix}$$

$$\Rightarrow G_c(s) = \frac{1}{s+3} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Here, we have poles at $s = -3$

Therefore, this system is BIBO stable.

Exercise 7. *BIBO Stability* ***(15 points)***

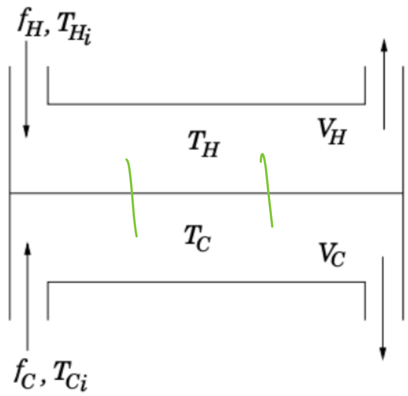


Figure 2: A simple heat exchanger

Consider a simplified model for a heat exchanger shown in Figure 2, in which f_C and f_H are the flows (assumed constant) of cold water and hot water, T_H and T_C represent the temperatures in the hot and cold compartments, respectively, T_{Hi} and T_{Ci} denote the temperature of the hot and cold inflow, respectively, and V_H and V_C are the volumes of hot and cold water. The temperatures in both compartments evolve according to:

$$V_C \frac{dT_C}{dt} = f_C(T_{Ci} - T_C) + \beta(T_H - T_C) \tag{1}$$

$$V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + \beta(T_C - T_H) \tag{2}$$

Let the inputs to the system be $u_1 = T_{Ci}$, $u_2 = T_{Hi}$, the outputs are $y1 = T_C$ and $y2 = T_H$, and assume that $f_C = f_H = 0.1(m^3/min)$, $\beta = 0.2(m^3/min)$ and $V_H = V_C = 1(m^3)$.

1. Write the state space and output equations for this system. **(5 points)**

2. In the absence of any input, determine $y_1(t)$ and $y_2(t)$. **(5 points)**

3. Is the system BIBO stable? Show why or why not. **(5 points)**

ANSWER:

① Let's choose state variables $x = \begin{bmatrix} T_C \\ T_H \end{bmatrix}$

and the input u is $= \begin{bmatrix} T_{Ci} \\ T_{Hi} \end{bmatrix}$

Now,
$$\dot{x} = \begin{bmatrix} \frac{-f_C+\beta}{V_C} & \frac{\beta}{V_C} \\ \frac{\beta}{V_C} & \frac{-f_H+\beta}{V_C} \end{bmatrix} x + \begin{bmatrix} f_C & 0 \\ 0 & f_H \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

Substitutions ; $f_C = 0.1$, $f_H = 0.1$, $\beta = 0.2$, $V_H = 1$, $V_C = 1$.

we get, $\Rightarrow$
$$\dot{x} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} x + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

② In absence of of any input, the solution of given system is integrated of the system.

$$x(t) = e^{\underbrace{\begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3-\lambda \end{bmatrix}}_{A}(t-t_0)} x_{t_0}.$$

Now, for finding eigen values,

$$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \Big\} \det(A-\lambda) = 0$$

$$\Rightarrow (\lambda+0.1)(\lambda+0.5) = 0$$

$\Rightarrow$ we get, $\lambda_1 = -0.1$ & $\lambda_2 = -0.5$

$$\Rightarrow f(\lambda) = g(\beta+\lambda) = \beta_{n-1}\lambda^{n-1} + \dots + \beta_1\lambda + \beta_0$$

By applying we get,

$$\Rightarrow \quad e^{\lambda_1 t} = \beta_1\lambda_1 + \beta_0 \ ; \quad e^{\lambda_2 t} = \beta_1\lambda_2 + \beta_0$$

$$\Rightarrow \quad e^{-0.1t} = -0.1\beta_1 + \beta_0 \ ; \quad e^{-0.5t} = -0.5\beta_1 + \beta_0$$

$$\Rightarrow \quad \beta_0 = 1.25e^{-0.1t} - 0.25e^{-0.5t} \ ; \quad \beta_1 = 2.5e^{-0.1t} - 2.5e^{-0.5t}$$

$$\Rightarrow f(A) = e^{At} = \beta_1 A + \beta_0 I = \begin{bmatrix} 0.5e^{-0.1t} + 0.5e^{-0.5t} & 0.5e^{-0.1t} - 0.5e^{-0.5t} \\ 0.5^{-0.1t} - 0.5^{-0.5t} & 0.5^{-0.1t} + 0.5^{-0.5t} \end{bmatrix}$$

Now for $y(t)$,

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$y(t) = Ce^{A(t-t_0)}x(t_0), \quad \text{where } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (0.5e^{-0.1t} + 0.5e^{-0.5t})x_1(0) + (0.5e^{-0.1t} - 0.5e^{-0.5t})x_2(0) \\ (0.5e^{-0.1t} - 0.5e^{-0.5t})x_1(0) + (0.5e^{-0.1t} + 0.5e^{-0.5t})x_2(0) \end{bmatrix}$$

③ We know, $SI - A = \begin{bmatrix} S+0.3 & -0.2 \\ -0.2 & S+0.3 \end{bmatrix}$

$$\Rightarrow (SI-A)^{-1} = \frac{1}{(S+0.1)(S+0.5)}\begin{bmatrix} S+0.3 & 0.2 \\ 0.2 & S+0.3 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = [0]$$

Here,

$$G_C(s) = C(SI-A)^{-1}B + D$$

$$\Rightarrow G_C(s) = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} S+0.3 & 0.2 \\ 0.2 & S+0.3 \end{bmatrix}\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}}{(S+0.1)(S+0.5)}$$

$$\Rightarrow G_c(s) = \frac{0.1}{(s+0.1)(s+0.5)} \begin{bmatrix} s+0.3 & 0.2 \\ 0.2 & s+0.3 \end{bmatrix}$$

Since the poles of $G_c(s)$ are negative values, Thus, the given system is BIBO stable.