24 - 677
Fall 2023
Mid-term Exam
10/24/23
Time: 24 Hours

Name: SRECHARAN SELVAM

Andrew id: SSELVAM S

Print your initials on each page that has your answers

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use your equation sheet and calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- You are allowed to use course slides, homework solution sheets as references. You are allowed to search for the knowledge needed on the internet.
- You must conduct the exam independently. Discussion or seeking help from others, online or in-person, is prohibited.
- All answers need to be derived by hand to get points. You are allowed to use a calculator for basic calculation of scalars. You can use calculate/-computer programs to verify your answers but the effort does not account as credits.
- You can ask questions on campuswire but only to the TAs and instructors.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	10	
6	10	
7	20	
Total:	100	

Do not write in the table to the right.

S

1. Please state whether each of the following statement is **True** or **False**. Explanation is not required.
   (a) (3 points) The system $y(t) = \sin(t)u(3t)$ is linear.
   (b) (3 points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system. Assuming A^{-1} exists, the system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable.
   (c) (3 points) The following continuous time system is BIBO stable.
   $$\dot{x} = u, \quad y = x$$
   (d) (3 points) The following DT system is controllable if $a \neq 0$.
   $$x[k+1] = \underbrace{\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}}_{A} x[k] + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u[k]$$
   (e) (3 points) The system given in ($\boxed{d}$) is stabilizable when $a = 0$.

ANSWER:

(a) $y(t) = \sin(t)u(3t)$

for linearity, $H(\alpha u_1 + \beta u_2) = \alpha H(u_1) + \beta H(u_2)$

Now let's assume,

$y_1(t) = \sin(t)u_1(3t)$, $y_2(t) = \sin(t)u_2(3t)$

if $y_3(t) = y_1 + y_2$

$y_3(t) = \sin(t)u_3(3t)$

$y_3(t) = y_1(t) + y_2(t) \Rightarrow \sin u_1(3t) + \sin t\, u_2(3t)$

Thus, we get $u_3(3t) = u_1(3t) + u_2(3t)$

Hence, the system is linear and it is TRUE

S

(b) $\dot{x}(t) = Ax(t)$

eigen values of A will be < 0 as it is asymptotically stable (given)

Now, eigen values of $A^{-1} = \frac{1}{\lambda} < 0$

$\boxed{\dot{x}(t) = A^{-1}x(t) \text{ is TRUE}}$

(c) $\dot{x} = u$ & $y = x$

Now, $\dot{x} = [0]x + [1]u$

$y = [1]x$

$G(s) = [1]([s] - [0])^{-1}$

$\therefore s^{-1} = \frac{1}{s}$

As the poles are zero, This is not BIBO Stable (FALSE)

(d) $P = [B \ \ AB]$, $AB = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+a \\ 1 \end{bmatrix}$

$\downarrow$

$P = \begin{bmatrix} 1 & 1+a \\ 1 & 1 \end{bmatrix}$

This system is controllable when a is not equal to 0.

Hence, This is TRUE

S

(e) Given, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\det(\lambda I - A) = 0$, we get $\lambda = 1, 1$

So, now eigen vectors $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$J = M^{-1} A M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\left.\hat{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \text{rank}(\hat{B}) = 1 \right\}$ uncontrollable mode

Here $\lambda \leq 1$, as the eigen values are less than 1, The System is Stabilizable.

$\boxed{\text{Thus, the answer is TRUE}}$

ANSWERS;

(a) TRUE

(b) TRUE

(c) FALSE

(d) TRUE

(e) TRUE

S

2. Consider a model of fisheries management. State x_1 is the population level of a prey species, x_2 is the population level of a predator species, and x_3 is the effort expended by humans in fishing the predator species. The model is

$$\begin{aligned}\dot{x}_1 &= (r_1 - x_2)x_1 \\ \dot{x}_2 &= (r_2 - x_3)x_2 \\ \dot{x}_3 &= u \\ y &= x_2\end{aligned}$$

where u is the input, y is the measurement of the predator species, and $r_1 = 10$ and $r_2 = 25$

(a) (5 points) Find the equilibrium point if the prey species population is known to be $\bar{x}_1 = 20$.

(b) (5 points) Linearize the model using the equilibrium point from (a)

(c) (5 points) Find the transfer function of the linearized state model from (b)

ANSWER:

(a) Given, $\dot{x}_1 = (r_1 - x_2)x_1$

$\dot{x}_2 = (r_2 - x_3)x_2$

$\dot{x}_3 = u$

If $x_1 = \hat{x}_1 = 20$

Then, $\dot{x}_1 = 0$

$\dot{x}_2 = r_2$

$\dot{x}_3 = 0$

we know $\bar{x}_2 = r_1$ & $\bar{x}_3 = r_2$

So, at equilibrium point, $x_1 = 20$ & $\bar{u} = 0$

$x_2 = 10$

$x_3 = 25$

$$\boxed{\text{Answer} = [20 \quad 10 \quad 25] \ \& \ \bar{u} = 0}$$

S

(b)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} (r_1 - x_2) & -x_1 & 0 \\ 0 & (r_2 - r_3) & -x_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now, at equilibrium point,

$$\frac{\partial f}{dx} = \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) Here, we know

$$G(s) = C\,[sI - A]^{-1} B + D \qquad 0$$

$$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & x_1 & 0 \\ 0 & s & r_1 \\ 0 & 0 & s \end{bmatrix}$$

$$\Rightarrow |SI - A| = \begin{pmatrix} 1/s & -20/s^2 & 200/s^3 \\ 0 & 1/s & -10/s^2 \\ 0 & 0 & 1/s \end{pmatrix}$$

By substituting, we get,

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/s & -20/s^2 & 200/s^3 \\ 0 & 1/s & -10/s^2 \\ 0 & 0 & 1/s \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{So, } C[SI - A]^{-1} B + D = \frac{-10}{s^2}}$$

S

3. (15 points) For the following dynamical system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)$$

compute $x(0)$ when $u(t) = 0$ and $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

ANSWER:

$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ where $\det(AI - \lambda) = 0$

we get $\lambda_1 = 1$ & $\lambda_2 = 1$

So, from C-H Theorm,

$$f(\lambda) = e^{\lambda_1 t} = \beta_1 \lambda_1 + \beta_0$$

Now $\frac{df(\lambda)}{d\lambda} = t e^{\lambda t} = \beta_1$

So, for $\lambda = 1$, $f(\lambda) = f(1)$

$$\Rightarrow e^t = \beta_1 + \beta_0$$

$$\Rightarrow \beta_1 = te^t$$

Now, for $\lambda = 1$, $\frac{df(\lambda)}{d\lambda} = te^t = \beta_1$

$$\beta_0 = (1-t)e^t$$

S

So, $e^{At} = \beta_1 A + \beta_0 I$

$$e^{At} = \begin{bmatrix} te^t & 0 \\ 3e^t & te^t \end{bmatrix} + \begin{bmatrix} e^t - te^t & 0 \\ 0 & e^t - te^t \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & 0 \\ 3te^t & e^t \end{bmatrix}$$

Now, for $t = 2$

$$x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t_0 = 0$$

$$x(2) = \begin{bmatrix} e^2 & 0 \\ 6e^2 & e^2 \end{bmatrix} x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{x(0) = \begin{bmatrix} \frac{1}{e^2} \\ \frac{-6}{e^2} \end{bmatrix}}$$

S

4. Given an LTI system with state space representation

$$\dot{x}(t) = \overbrace{\begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 1 \\ \alpha \end{bmatrix}}^{B} u(t)$$
$$y(t) = \underbrace{\begin{bmatrix} 1 & \alpha \end{bmatrix}}_{C} x(t) + u(t)$$

where $\alpha \in \mathbb{R}$.

(a) (5 points) Find the range of α for which the system is exponentially stable.

(b) (10 points) For the supremum (least upper bound) of the range of α determined in (a), check whether the given system is BIBO stable.

ANSWER:

(a) Given, $A = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix}$, $\det(A - \lambda I) = 0$

$|A - \lambda I| = 0 \Rightarrow \begin{bmatrix} -1-\lambda & -\alpha \\ 0 & 1-\alpha-\lambda \end{bmatrix} = 0$

Thus, $\lambda_1 = -1$

$\lambda_2 = 1 - \alpha$

We know that for asymptotical stability

$Re(\lambda) < 0$

$\Rightarrow 1 - \alpha < 0$ & $\alpha - 1 > 0$

$\boxed{\text{Thus } \alpha > 1}$

S

(b) So, we get $\alpha = 1$

Now, we know $G(s) = C[sI - A]^{-1}B + D$

$$(sI - A) = \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s+1)} \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}$$

$$\Rightarrow C[sI - A]^{-1}B + D = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s+1 & \frac{-1}{s(s+1)} \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1$$

$$= \frac{1}{s+1} + \frac{s}{s(s+1)} + 1$$

$$= \frac{s + s + s(s+1)}{s(s+1)}$$

$$= \frac{s+3}{s+1}$$

As the pole $= -1$ $\}$ negative Real Part

$\boxed{\text{Thus, the system is BIBO stable}}$

S

5. Consider the following nonlinear system

$$\dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1$$
$$\dot{x}_2 = \frac{x_1}{1+x_1^2}$$

(a) (5 points) Using the candidate Lyapunov function $V(x) = x_1^2 + x_2^2$ and Lyapunov Direct method, first find the equilibrium point and then find the stability of the system at the equilibrium point

(b) (5 points) Linearize the system about the equilibrium point and find the stability of the linearized system using Lyapunov indirect method

ANSWER:

(a) we are given the system;

$$\dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1$$

$$\dot{x}_2 = \frac{x_1}{1+x_1^2}$$

we can find the equilibrium point when the derivatives are equal to zero.

For x_1,
$$0 = -\frac{x_1^2}{1+x_1^2} - 2x_1 \quad — ①$$

For x_2,
$$0 = \frac{x_1}{1+x_1^2} \quad — ②$$

S

From eqn (2), we can see that x_1 must be zero for the equilibrium point. Substituting this into eqn (1).

$\Rightarrow -0 + 2(0) = 0$

So, at the equilibrium point $(0,0)$, both derivatives are zero.

Now, Let's check the stability of this system

$$\Rightarrow V(x) = x_1^2 + x_2^2$$

$$\Rightarrow \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

We know, $\dot{x}_1 = \dfrac{-x_2}{1+x_1^2} - 2x_1$ and $\dot{x}_2 = \dfrac{x_1}{1+x_1^2}$

$$\Rightarrow \dot{V}(x) = -\frac{2x_1x_2}{1+x_1^2} - 4x_1^2 + \frac{2x_1x_2}{1+x_1^2}$$

$$\Rightarrow \dot{V}(x) = -4x_1^2 \leq 0$$

Therefore, $V(x)$ along with its partial derivatives are continuous and $V(x)$ is positive definite

S

Thus, $\dot{V}(x) = -4x_1^2 \leq 0$

$\Rightarrow$ $\dot{V}(x)$ is negative definite

$\Rightarrow$ This system is asymptotically stable. at the equibrium point.

(b) we already know that the equibrium point is $(0,0)$ where, $x_1=0$ & $x_2=0$

Let $f_1(x_1, x_2) = -\dfrac{x_2}{1+x_1^2} - 2x_1$

and, $f_2(x_1, x_2) = \dfrac{x_1}{1+x_1^2}$

So, $\left.\dfrac{\partial f_1}{\partial x_2}\right|_{x_1=0, x_2=0} = \left. -2 + \dfrac{x_2}{(1+x_1^2)^2} \cdot 2x_1 \right|_{x_1=0, x_2=0} = -2$

$\Rightarrow$ $\left.\dfrac{\partial f_1}{\partial x_2}\right|_{x_1=0, x_2=0} = \left.\dfrac{1}{1+x_1^2}\right|_{x_1=0, x_2=0} = -1$

And, $\dfrac{\partial f_2}{\partial x_1} = \left.\dfrac{(1+x_1^2) - x_1 \cdot 2x_1}{(1+x_1^2)^2}\right|_{x_1=0, x_2=0}$

S

$$\Rightarrow \left.\frac{1-x_1^2}{(1+x_1^2)^2}\right|_{x_1=0,\, x_2=0} = 1$$

$$\Rightarrow \frac{\partial f_2}{\partial x_2} = 0$$

Therefore, the linearised system at equilibrium point $x_1=0, x_2=0$ is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now, we need to check the stability for the linearised system

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$$

Now, $\det(\lambda I - A) = 0$

$$\Rightarrow \begin{vmatrix} \lambda+2 & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0$$

where $\lambda_1 = \lambda_2 = -1 \Big\}$ $re(\lambda) < 0$

$\hookrightarrow$ Locally asymptotically Stable

S

Now, for $e^{Jt} \rightarrow$ Jordan,

$$\operatorname{rank}(A+I) = \operatorname{rank}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = 1$$

$\Rightarrow$ The Jordan form of A is $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

For e^{Jt}, $\det(\lambda I - J) = 0$

$$\begin{vmatrix} \lambda+1 & -1 \\ 0 & \lambda+1 \end{vmatrix} = (\lambda+1)^2 = 0$$

we get, $\lambda_1 = \lambda_2 = -1$

$$e^{\lambda t} = \beta_1 \lambda + \beta_0 \qquad \Rightarrow \qquad e^{-t} = -\beta_1 + \beta_0 \qquad \Rightarrow \qquad \beta_1 = te^{-t}$$

$$te^{\lambda t} = \beta_1 \qquad\qquad \beta_1 = te^{-t} \qquad\qquad \beta_0 = (t+1)e^{-t}$$

So, $e^{Jt} = \beta_1 J + \beta_0 I$

$$\Rightarrow \begin{bmatrix} -te^{-t} & te^{-t} \\ 0 & -te^{-t} \end{bmatrix} + \begin{bmatrix} (t+1)e^{-t} & 0 \\ 0 & (t+1)e^{-t} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

, Thus for λ_1 & λ_2 we have $\mathrm{Re} = -1 < 0$

Therefore, the linearised system is Asymptotic stable and the original system is also asymptotic stable at the equilibrium points

S

S

6. (10 points) Find the minimal realization for

$$G(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}.$$

ANSWER:

Given, $G(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}$

Now, for $G(\infty)$, we get $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$G_{sp} = G_{(s)} - G_{(\infty)} = \begin{bmatrix} \frac{-1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix} \Rightarrow \frac{1}{s(s+1)} \begin{bmatrix} -s \\ 1 \end{bmatrix}$

Here, $\Delta s = s(s+1) = s^2 + s$, where $\alpha_1 = 1$

$\Rightarrow G_{sp} = \frac{1}{s(s+1)} \begin{bmatrix} -s \\ 1 \end{bmatrix}$

$\Rightarrow G_{sp} = \frac{1}{\Delta s} \left[\underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{N_1} s + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{N_2} \right]$

$N_1, N_2 \in R$

Thus we get,

$$A = [-1] \quad , B = [1]$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad , D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So, the state space of this system is,

$$\dot{x} = [-1] x + [1] u$$

$$y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

In order to check the minimal realization

For controllability P-matrix $= [1] = n$

↳ This is controllable

For Q-matrix $= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = n$

↳ This is observable

Thus, Relisation is both controllable & observable

So, it is a minimal realization

$$\boxed{\left.\begin{aligned} x &= [-1] x + [1] u \\ y &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \end{aligned}\right\} \text{is a minimal Realization}}$$

S

7. Suppose you are invited as a control engineering consultant to investigate a critical safety issue for an airplane company. You are provided with an approximate linear model of the lateral dynamics of the aircraft which has the state and control vectors

$$x = \begin{bmatrix} p & r & \beta & \phi \end{bmatrix}^T \quad \text{and} \quad u = \begin{bmatrix} \delta_a & \delta_r \end{bmatrix}^T$$

where p and r are incremental roll and yaw rates, β is an incremental sideslip angle, and ϕ is an incremental roll angle. The control inputs are the incremental changes in the aileron angle δ_a and in the rudder angle δ_r, respectively. In a consistent set of units, the linearized model is given as $\dot{x} = Ax + Bu$ with

eigen values

$$\mathbf{A} = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Answer the following questions with derivations. A simple yes or no without explanation will not get 0 credit.

(a) (5 points) Is the linearized aircraft model asymptotically stable? Is the linearized aircraft model stable i.s.L.?

(b) (5 points) Is the aircraft controllable with just δ_r? Is the aircraft controllable with both δ_r and δ_a?

(c) (5 points) Suppose a malfunction prevents manipulation of the rudder angle δ_r, is it possible to control the aircraft using only the aileron angle δ_a?

(d) (5 points) If you only have budget to measure one state, which one to measure (choose one from $\{p, r, \beta, \phi\}$) so that the whole system is observable?

AMSWER:

(a) Calculating eigenvalues of A,

$$A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \det(\lambda I - A)$$

$$\Rightarrow \det(\lambda I - A) = \lambda(\lambda+10)(\lambda^2+\lambda+1)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = -10, \lambda_3 = \frac{-1+\sqrt{3}i}{2}, \lambda_4 = \frac{-1-\sqrt{3}i}{2}$$

S

Here, as we have the linearised model of the aircraft system, and we have real parts ($\lambda_1 = 0$)

Since, there are positive and imaginary parts, the system is not asymptotically stable, But the system is stable i.s.L we have $\lambda_1 = 0$, So the system is stable i.s.L

(TRUE)

(b) For this we have to check the rank of the controllability matrix P,

$$P = [B, AB, A^2B, A^3B] = \begin{bmatrix} 0 & 0 & -1 & -11 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 11 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{rank}(P) = 4 = n$$

Therefore, the aircraft is controllable just using δ_r

Now, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ for δ_r, δ_a

$$P = [B \ AB \ A^2B \ A^3B]$$

S

$$P = \begin{bmatrix} 10 & 0 & -100 & 0 & 1000 & -1 & -10000 & 11 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 10 & 0 & -100 & 0 & 1000 & -1 \end{bmatrix}$$

$\text{Rank}(P) = 4 \; (n)$

Therefore, The aircraft is controllable using δ_r & δ_a

(c) If a malfunction prevents manipulation of the rudder angle δ_r, the aircraft is not controllable with only δ_a

$$B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow P = \left[B, \; AB, \; A^2B, \; A^3B \right] = \begin{bmatrix} 10 & -100 & 1000 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -100 & 1000 \end{bmatrix}$$

$\Rightarrow \text{rank}(P) = 2 < 4 \; (n)$

Thus, it is impossible to control the aircraft using only δ_a.

(d) To determine observability, we need to check observability matrix rank for each state variable.

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

for each state variables,

for $p = [1\ 0\ 0\ 0] \Rightarrow C_p$

for $r = [0\ 1\ 0\ 0] \Rightarrow C_r$

for $\beta = [0\ 0\ 1\ 0] \Rightarrow C_\beta$

for $\phi = [0\ 0\ 0\ 1] \Rightarrow C_\phi$

So, $Q_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \\ -1000 & -11 & -99 & 0 \end{bmatrix}$, $Q_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$Q_\beta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $Q_\phi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix}$

Here, it is evident the rank of Q_ϕ is $4 = n$

Thus, we can choose state variable 'ϕ'