Homework 7: Lagrangian Dynamics

24-760 Robot Dynamics & Analysis
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For this homework, please compose everything in a single Matlab script, except helper functions. In the main script, you need to include all reasoning (either type it or insert a picture of your hand written result), calculation, and required output (with the same output variable names given in problem statements). Please fill in all the `TODO` sections and clearly label sections based on which part they are for. Please use the precise variable names that we define in the template and do not overwrite them in later sections. If you used any helper functions, please put them together with the main script in a zip file named as andrewID_24760_HW7.zip, where andrewID is your Andrew ID.

Please make sure to use the predefined symbolic variables in the code template, especially the differential state, for example, we defined `q1` and its first and second derivative `dq1` and `ddq1` there. You should be able to complete the homework without defining any new symbolic variables.

Hint: Look at the Matlab functions `diff`, `gradient`, and `jacobian`.

Problem 1) Unconstrained Lagrangian

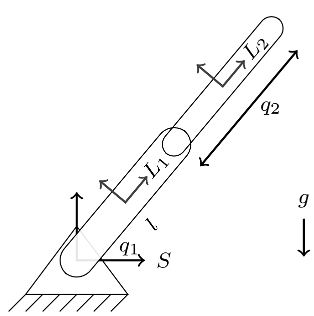

Figure 1: Two link robot. For each frame, the label indicates the x-axis.

Consider the dynamics of the two link robot shown in Figure 2, with one rotational joint and one prismatic joint in the plane. Each link is a rod of length l and mass m with uniform mass distribution and center of mass (COM) at frame L_i. The second joint, q_2, extends the second link from length 0 (fully retracted) to length l (fully extended).

For this question we will consider the generalized coordinates, $q_g = [q_1, q_2]^T$, with no constraints. In the next question we will consider maximal coordinates with constraints. We use subscript g and m to distinguish them (i.e. L_g and L_m). *(HINT: It may be beneficial to create reusable functions that perform Lagrange's equation to differentiate L to get the EOM, or that calculates C from M, etc).*

1.1) What is the kinematic energy $T_g(q_1, q_2, \dot{q}_1, \dot{q}_2)$, potential energy $V_g(q_1, q_2)$, and Lagrangian in generalized coordinates, $L_g(q_1, q_2, \dot{q}_1, \dot{q}_2)$?

Please compute and save them in the symbolic variables T_g, V_g, and L_g respectively in the script.

1.2) Assuming that joints 1 and 2 have motor torque and force τ and F, compute the applied forces to each generalized coordinate Υ_g and the dynamic equations of motion by using the Lagrange equations to differentiate the Lagrangian in generalized coordinates.

Please compute and save them in the symbolic variables Y_g and EOM_g1 in the script. EOM_g1 should be a 2 by 1 symbolic matrix that is equal to [0; 0]. It is obtained by subtracting Υ_g on both sides of the equations.

1.3) Re-compute the dynamic equations of motion by directly computing the M_g, C_g, N_g, and Υ_g matrices in the manipulator equation,

$$M_g(q_g)\ddot{q}_g + C_g(q_g, \dot{q}_g)\dot{q}_g + N_g(q_g, \dot{q}_g) = \Upsilon_g$$

Check that you get the same answer as Problem 1.2.

Please compute and save them in the symbolic variables M_g, C_g, N_g, and EOM_g2 in the script. EOM_g2 should be a 2 by 1 symbolic matrix that is equal to [0; 0] and should be the same as EOM_g1 from Problem 1.2.

Problem 2) Constrained Lagrangian

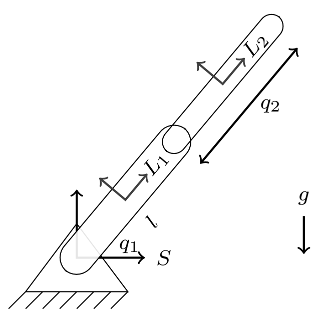

Figure 2: Two link robot. For each frame, the label indicates the x-axis.

In the last question we considered the generalized coordinates, $q_g = [q_1, q_2]^T$, with no constraints. In this question we consider maximal coordinates with constraints.

Maximal coordinates represent the full position and orientation of each link. For each link, use local coordinates for the link frame at the COM, namely L_1 is at (x_1, y_1, ϕ_1) and L_2 is at (x_2, y_2, ϕ_2) with ϕ_i measured counter-clockwise from the S frame. The new combined state is $q_m = [x_1, y_1, \phi_1, x_2, y_2, \phi_2]^T$.

2.1) What position and velocity constraints are there on the system, $a_m(q_m)$ and $A_m\dot{q}_m$?

Please compute and save them in the symbolic variables a_m, A_m respectively in the script.

2.2) What is the kinematic energy $T_m(q_m, \dot{q}_m)$, potential energy $V_m(q_m)$, and Lagrangian in maximal coordinates, $L_m(q_m, \dot{q}_m)$?

Please compute and save them in the symbolic variables `T_m`, `V_m`, and `L_m` respectively in the script.

2.3) The actuator effort is harder to represent in maximal coordinates. For each link, consider the force or torque applied to a frame at the end of the link (with equal and opposite signs for the joint effort between the links). Assuming that joints 1 and 2 have motor torque and force τ and F, calculate Υ_m, the resulting applied force on the two links in maximal coordinates.

Please compute and save it in the symbolic variable `Y_m` in the script.

2.4) Compute the dynamic equations of motion by using the constrained Lagrange equations to differentiate the Lagrangian in maximal coordinates.

Please compute and save it in the symbolic variable `EOM_m1` in the script. `EOM_m1` should be a 6 by 1 symbolic matrix that is equal to [0; 0; 0; 0; 0; 0]. It is obtained by subtracting Υ_m on both sides of the equations.

2.5) Re-compute the dynamic equations of motion by directly computing the M_m, C_m, N_m, and Υ_m matrices in the constrained manipulator equation,

$$M_m(q_m)\ddot{q}_m + C_m(q_m, \dot{q}_m)\dot{q}_m + N_m(q_m, \dot{q}_m) + A^T(q_m)\lambda = \Upsilon_m$$

Check that you get the same answer as Problem 2.4.

Please compute and save them in the symbolic variables `M_m`, `C_m`, `N_m`, and `EOM_m2` in the script. `EOM_m2` should be a 6 by 1 symbolic matrix that is equal to [0; 0; 0; 0; 0; 0] and should be the same as `EOM_m1` from Problem 2.4.

2.6) What are the constraint forces?

Please compute and save it in the symbolic variable `lambdaVec` in the script.

2.7) (Optional) Finally, show that the dynamic equations in maximal coordinates (Problem 2.4 or 2.5) are equivalent to the dynamic equations in generalized coordinates (Problem 1.2 or 1.3) by using the constraint equations and the change of basis between q_g and q_m so that $q_m = h(q_g)$ and $\dot{q}_m = H\dot{q}_g$. The converted EOM in generalized coordinates from maximal coordinates should yield the same results as Problem 1.2 or 1.3.

Please compute and save them in the symbolic variables `h`, `H`, and `EOM_g3` respectively in the script.

PROBLEM 1 :- Unconstrained Lagrangian

$\boxed{1.1}$ To compute the lagrangian in generalized coordinates, we need to express the kinetic and potential energies in terms of these coordinates. The kinetic energy is determined by the com's linear velocity and each link's angular velocity in these coordinates.

$$T_q = \frac{1}{2} m v_{s L1}^{b\,2} + \frac{1}{2} m v_{sL2}^{b2} + \frac{1}{2} I_0 \dot{q}_1^2 + \frac{1}{2} I_0 \dot{q}_1^2$$

$$\Rightarrow T_q = \frac{1}{2} m \left(\frac{l}{2}\dot{q}_1\right)^2 + \frac{m l^2 \dot{q}_1^2}{12} + \frac{1}{2} m \left[\left(\left(\frac{l}{2} + q_2\right)\dot{q}_1\right)^2 + \dot{q}_2^2\right]$$

$$\Rightarrow T_q = \frac{m}{2}\left[\dot{q}_2 + \dot{q}_1\left(\frac{l}{2} + q_2\right)^2\right) + \frac{5}{54} m l^2 \dot{q}_1^2$$

Now for the Potential Energy.

$$V_q = mg\left(\frac{l}{2} + q_2\right)\sin q_1 + mg\frac{l}{2}\sin q_1$$

$$V_q = mg(l + q_2)\sin q_1$$

To get lagrangian, $L_q = T_q - V_q$

$$L_g = \frac{5}{24} m l^2 \dot{q}_1^2 + \frac{m}{2}\left[\dot{q}_2^2 + \dot{q}_1^2\left(\frac{l}{2} + q_2\right)^2\right] - mg \sin q_1 (l + q_2)$$

1.2 we know the lagrangian eqns;

$$\frac{d}{dt}\left(\frac{\partial L_g}{\partial \dot{q}_i}\right) - \frac{\partial L_g}{\partial q_i} = \tau$$

Now for q_1,

$$\frac{\partial L_g}{\partial \dot{q}_1} = \frac{5}{12} m l^2 \dot{q}_1 + m\dot{q}_1\left[\frac{l}{2} + q_2\right]^2$$

$$\frac{\partial L_g}{\partial q_1} = -mg \cos q_1 . (l + q_2)$$

For q_2,

$$\frac{\partial L_g}{\partial \dot{q}_2} = m\dot{q}_2$$

$$\frac{\partial L_g}{\partial q_2} = m\dot{q}_1^2\left[\frac{l}{2} + q_2\right] - mg \sin q_1$$

we know $\tau = \begin{bmatrix} \tau_1 \\ F_2 \end{bmatrix}$

$\tau_1 \rightarrow$ on Joint-1

$F_2 \rightarrow$ on Joint-2

Now, By substituting we get,

$$\Rightarrow 2m\dot{q}_1\dot{q}_2\left(\frac{l}{2}+q_2\right) + mg\cos q_1(l+q_2) + m\ddot{q}_1\left[\frac{5}{12}l^2 + \left(\frac{l}{2}+q_2\right)^2\right] = \tau_1$$

$$\Rightarrow mg\sin q_1 + m\ddot{q}_2 - m\dot{q}_1^2\left[\frac{l}{2}+q_2\right] = F_2$$

1.3

The M matrix is $= \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$

$\hookrightarrow m_1 = m_2$

To calculate M_q, $M_q = \sum_i J_{sli}^{bT} M_i J_{sli}^b$ — (1)

where, $J_{sl_1}^b = \begin{bmatrix} 0 & 0 \\ l/2 & 0 \\ 1 & 0 \end{bmatrix}$ & $J_{sl_2}^b = \begin{bmatrix} 0 & 1 \\ l/2+q_2 & 0 \\ 1 & 0 \end{bmatrix}$

Now substitute and combine in eqn (1);

$$M_q = \begin{bmatrix} \frac{5ml^2}{12} + m\left(\frac{l}{2}+q_2\right)^2 & 0 \\ 0 & m \end{bmatrix}$$

Now, for coriolis matrix,

$$C_q = \begin{bmatrix} \dot{q}_2 m\left(\frac{l}{2}+q_2\right) & \dot{q}_1 m\left(l/2+q_2\right) \\ -\dot{q}_1 m\left(l/2+q_2\right) & 0 \end{bmatrix}$$

The non-linear term present is;

$$N_q = \frac{\partial V}{\partial q} = \begin{bmatrix} mg\cos(q_1)\cdot(l+q_2) \\ mg\sin q_1 \end{bmatrix}$$

PROBLEM 2 :- Constrained Lagrangian

2.1 For a system with 2 DOF and 3 maximal coordinates per link, we have 6 coordinates but need to account for four constraints. These constraints, represented by $a_m(q_m) \in \mathbb{R}^4$, are functions of the maximal coordinates that become zero when the constraints due to the system's joints are satisfied. This helps us identify and represent the constrained motions within the system.

$$a_m = \begin{bmatrix} x_1 - \frac{L\cos(\phi_1)}{2} \\ y_1 - \frac{L\sin(\phi_1)}{2} \\ y_2\cos(\phi_2) - x_2\sin(\phi_2) \\ \phi_2 - \phi_1 \end{bmatrix}$$

Now we can find A_m,

$$A_m = \begin{bmatrix} 1 & 0 & \frac{l\sin\phi_1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{l\cos\phi_1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\phi_2 & \cos\phi_2 & -x_2\cos\phi_2 - y_2\sin\phi_2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{2.2}$ maximal coordinates facilitate the formulation of Lagrangians, as they allow for direct expression of kinetic and potential energies without additional computations. The system's Lagrangian is given by;

$$L_m = \frac{m l^2 [\dot{\phi}_1^2 + \dot{\phi}_2^2]}{24} + \frac{m[\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2]}{2} - mg(y_1 + y_2)$$

$\boxed{2.3}$ The system experiences actuator forces from a motor generating torque between the global and link 1, and a linear actuator links 1 & 2. The Applied torque τ only affects coordinate ϕ_1 as torques don't influence translational motion.
Inversely, F influences x_1, x_2, y_1, y_2, as the actuator impacts both links.

So,
$$\boxed{\gamma_m = \begin{bmatrix} -F\cos\phi_1 \\ -F\sin\phi_1 \\ \tau \\ F\cos\phi_1 \\ F\sin\phi_1 \\ 0 \end{bmatrix}}$$

$\boxed{2.4}$

EOM $\Rightarrow$

$$m\ddot{x}_1 + \lambda_1 = -F\cos\phi_1$$
$$m\ddot{y}_1 + mg + \lambda_2 = -F\sin\phi_1$$
$$\frac{1}{12}ml^2\ddot{\phi}_1 + \frac{l}{2}\left(\sin\phi_1\lambda_1 - \cos\phi_1\lambda_2\right) - \lambda_4 = \tau$$

$$m\ddot{x}_2 - \sin\phi_1\lambda_3 = F\cos\phi_1$$
$$m\ddot{y}_2 + mg + \cos\phi_1\lambda_3 = F\sin\phi_1$$

$$\frac{1}{12} m l^2 \ddot{\phi}_2 - x_2 \cos\phi_2 \lambda_3 - y_2 \sin\phi_2 \lambda_3 + \lambda_4 = 0$$

$\boxed{2.5}$ To calculate each term in the manipulator equation, we can use different Body Jacobians for each link. These Jacobians, $J^b_{sl_i}$, convert local coordinates $\dot{q}_m$ to body velocities $V^b_{sl_i} = J^b_{sl_i} \dot{q}_m$. $V^b_{sl_i}$ is expressed in the link's frame, requiring $J^b_{sl_i}$ to be a rotation matrix transforming vectors from the global frame to link frame.

$$J^b_{sl_1} = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 & 0 & 0 & 0 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \&$$

$$J^b_{sl_2} = \begin{bmatrix} 0 & 0 & 0 & \cos\phi_2 & \sin\phi_2 & 0 \\ 0 & 0 & 0 & -\sin\phi_2 & \cos\phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, we can calculate M_m, C_m and N_m

$$M_m = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{ml^2}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{ml^2}{12} \end{bmatrix} ,$$

$$C_m = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$N_m = \begin{bmatrix} 0 \\ mg \\ 0 \\ 0 \\ mg \\ 0 \end{bmatrix}$$

2.6

$$\lambda = \left(A_m M_m^{-1} A_m\right)^{-1}\left(A_m M_m^{-1}\left(r_m - C_m \dot{q}_m - N_m\right) + \dot{A}_m \dot{q}_m\right)$$

where,

$$\dot{A}_m = \frac{\partial A_m}{\partial x_1}\dot{x}_1 + \frac{\partial A_m}{\partial y_1}\dot{y}_1 + \frac{\partial A_m}{\partial \phi_1}\dot{\phi}_1 + \frac{\partial A_m}{\partial x_2}\dot{x}_2 + \frac{\partial A_m}{\partial y_2}\dot{y}_2 + \frac{\partial A_m}{\partial \phi_2}\dot{\phi}_2$$

(MATLAB)

2.7

(optional)