Homework 2: Motion

24-760 Robot Dynamics & Analysis
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Note: For homework submission, please submit the PDF of the written portion to "Homework 2" and a zipped folder of your Matlab code to "Homework 2 Programming" in Gradescope.

Problem 1) Lunar Motion

Consider a simplified model of the motion of the earth and the moon. Attach a stationary coordinate frame to the center of the earth (s), a frame with the same origin that rotates with the earth (e), and a frame to the moon (m). The axes of rotation are all aligned with each other and pointing in the $+z$ direction of each frame. Assume the moon's orbit around the earth is circular with radius l_m. The earth's radius is r_e and the moon's radius is r_m. The moon rotates about the earth at a rate of 1 revolution per 28 days, and about it's own axis at a rate of 1 revolution per 28 days. The earth rotates about it's own axis at a rate of 1 revolution per day. *Hint: Draw a figure to keep track of the different frames.*

1.1) Just consider the earth's rotation to start. At time t, assume the earth is rotated so that the earth's $+x$ axis is aligned with the stationary $-y$. What is R_{se}? What is g_{se}? Use this configuration for the other parts of this question.

1.2) For a point q on the surface of the earth, $q_e = [0, r_e, 0]^T$, calculate the location of this point in the stationary frame using a rigid body transformation.

1.3) What is the body velocity of the earth's rotation, V_{se}^b? What is the spatial velocity V_{se}^s?

1.4) Using that body and spatial velocity, what is the instantaneous velocity of the point q_e in the earth's frame, v_{qe}? What is the velocity in the stationary frame, v_{qs}?

1.5) Now consider the position of the moon relative to the earth. Assume at time t that the moon is located at $[l_m, 0, 0]^T$ in the stationary frame, with the moon's x axis pointing to the earth. Calculate g_{sm}, then calculate g_{em} based on g_{sm} and g_{se}.

1.6) What is the body velocity of the moon's motion, V_{sm}^b?

1.7) Calculate the spatial velocity, V_{sm}^s, using an adjoint operation. What is special about v_{sm}^s?

1.8) Based on the rotation of the earth and the orbit of the moon, how long is a lunar day on earth? That is, from the earth's perspective, how long does the moon take to come back over the same spot on the surface of the earth?

1. LUNAR MOTION

1.1 Given that;

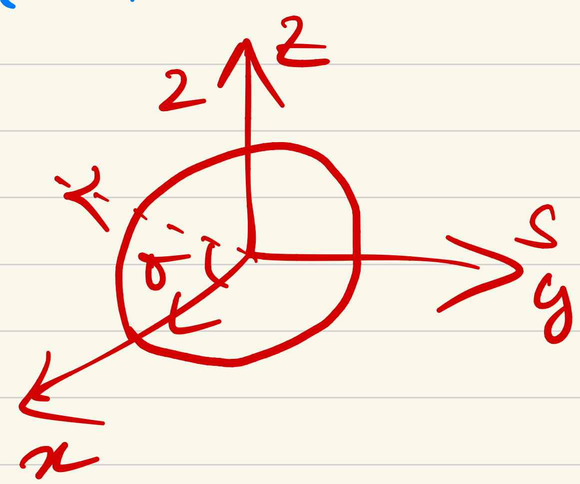
e = Rotates with the earth.

S = stationary.

m = Frame to moon.

⇒ In this configuration; Frame e is aligned with frame s, but it is rotated by an angle θ around the z-axis of frame s, since the earth's rotation axis is the z-axis of frame s.

⇒ The rotation angle 'θ' can be calculated as $\theta = -\omega t$, where ω is the angular velocity of the earth's rotation and 't' is the time.



Now Rotation matrix RSe is;

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

However, the frame is rotated by $\theta = 90^\circ$,

$$\boxed{\therefore R_{SE} = \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}}$$

Now, for g_{se}, which represents the acceleration due to gravity in Frame 'e'. Gravity points towards the center of the Earth and is represented as $(0, 0, -g)$.

However, the origins are same for both frames.

$$\therefore g = \left[\begin{array}{c:c} R_{se} & 0 \\ 0\ 0\ 0 & 1 \end{array}\right]$$

$$\boxed{\therefore \bar{g}_{se} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}}$$

1.2 Given pt; $q_e = \begin{bmatrix} 0 \\ r_c \\ 0 \end{bmatrix}$

we can use a rigid Body transformation.

Now, for $\bar{q}_s$

$\hookrightarrow \bar{g}_{se}\, \bar{q}_e$

Now,

$$\bar{q}_s = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4\times 4} \underbrace{\begin{bmatrix} 0 \\ r_e \\ 0 \\ 1 \end{bmatrix}}_{4\times 1} = \underbrace{\begin{bmatrix} r_e \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{4\times 1}$$

Now, $\boxed{q_s = \begin{bmatrix} r_e \\ 0 \\ 0 \end{bmatrix}}$ } In 'S'-coordinate frame

1.3 To calculate the body velocity of the earth's rotation (V_{se}^{b}) and the spatial velocity (V_{se}^{s}) of a point on the earth's surface due to its rotation, we will consider the rotation of the earth.

$$\bar{V}_{se}^{b} = \bar{g}_{se}^{-1} \times \bar{g}_{se}$$

$$\bar{V}_{se}^{b} = \begin{bmatrix} R^{-1} & P^{-1} \\ 000 & 1 \end{bmatrix} \begin{bmatrix} R & P \\ 000 & 0 \end{bmatrix} = \begin{bmatrix} R^{-1}R & P^{-1}P \\ 000 & 0 \end{bmatrix}$$

$$\boxed{\bar{V}_{se}^{b} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}}$$

1.4 Instantaneous Velocity of point qe in the earth's frame (V_{qe}):

$$V_{qe} = \omega \times r_e$$

And, $V_{qs} \Rightarrow$ Velocity of point qe in the stationary frame (V_{qs})

Now, $V_{qs} = \hat{V}_{se}^{s} \times q_s$

$\bar{V}_{qs}$

$$\Rightarrow \begin{bmatrix} 0 & -\theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} \begin{bmatrix} r_e \\ 0 \\ 0 \\ 1 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ \theta r_e = \theta \\ 0 \\ 0 \end{bmatrix}_{4\times 1}$$

$$\boxed{\bar{V}_{qs} = \begin{bmatrix} 0 \\ 2\pi r_e \\ 0 \\ 0 \end{bmatrix}}$$

Now, $V_{ae} = \bar{V}^{b}_{se}\, a_e = \begin{bmatrix} 0 & -\theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} \begin{bmatrix} 0 \\ r_e \\ 0 \\ 1 \end{bmatrix}_{4\times 1}$

$$\boxed{V_{ae} = \begin{bmatrix} -2\pi r_e \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4\times 1}}$$

1.5

$g_{sm} = [0, -g\_moon, 0]$ → acceleration due to gravity on moon's surface.

Gravitational acceleration of the earths relative to the moon in the stationary frame, based on g_{sm} & g_{se}

The origin of the moon is only translated along x-axis only.

$$\therefore \bar{P}_{sm} = \begin{bmatrix} l_m \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

when $\theta = 180^\circ$, (z-axis); $R_{sm} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $\left.\begin{bmatrix} R_{sm} & P_{sm} \\ 0\ 0\ 0 & 1 \end{bmatrix}\right\} \Rightarrow g_{sm}$

$$\Downarrow$$

$$\boxed{\begin{bmatrix} -1 & 0 & 0 & l_m \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4} = g_{sm}}$$

Now, for $g_{em} \Rightarrow g_{es} \times g_{sm}$

$$\begin{bmatrix} R^T & : & -P \\ 0\,0\,0 & : & 1 \end{bmatrix} \quad \Rightarrow g_{se}^{-1} \times g_{sm}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4} \begin{bmatrix} -1 & 0 & 0 & l_m \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

$$\therefore\ g_{sm} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & l_m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

1.6 Finding V^b_{sm}, we need to use

$$V^b_{sm} = g_{sm}^{-1} \times \dot{g}_{sm}$$

$$\Rightarrow \begin{bmatrix} R^{-1}_{sm} & -p \\ 000 & 1 \end{bmatrix}_{4\times 4} \begin{bmatrix} \dot{R}_{sm} & \dot{p} \\ 000 & 0 \end{bmatrix}_{4\times 4} = \begin{bmatrix} R^{-1}_{sm}\dot{R}_{sm} & -p\cdot\dot{p} \\ 000 & 1 \end{bmatrix}$$

Here, $\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -l_m \sin\theta\cdot\dot{\theta} \\ l_m \cos\theta\cdot\dot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -l_m\pi/14 \\ 0 \end{bmatrix}$

Now, $R^{-1}_{sm} \times \dot{R}_{sm} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} \theta & \dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

$$\Rightarrow \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

Now, $-R^{-1}_{sm} \cdot p$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -l_m \pi/14 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ l_m \pi/14 \\ 0 \end{bmatrix}$$

Now,

$$\hat{V}^b_{sm} = \begin{bmatrix} 0 & \overset{\pi/14}{-\theta} & 0 & 0 \\ \underset{\pi/14}{\theta} & 0 & 0 & l_m \pi/14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$$

$$V^b_{sm} = \begin{bmatrix} v^b_{sm} \\ \omega_{sm} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ l_m \pi/14 \\ 0 \\ 0 \\ 0 \\ -\pi/14 \end{bmatrix}_{6\times 1}$$

1.7 Calculate the moon's body velocity in its own rotating frame

$$V^s_{sm} = adj(g_{sm})\, V^b_{sm}$$

$$adj(g_{sm}) = \begin{bmatrix} R_{sm} & P.R_{sm} \\ 000 & R_{sm} \end{bmatrix}$$

where, $R_{sm} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -lm \\ 0 & lm & 0 \end{bmatrix}$

$$P\,R_{sm} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -lm \\ 0 & lm & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

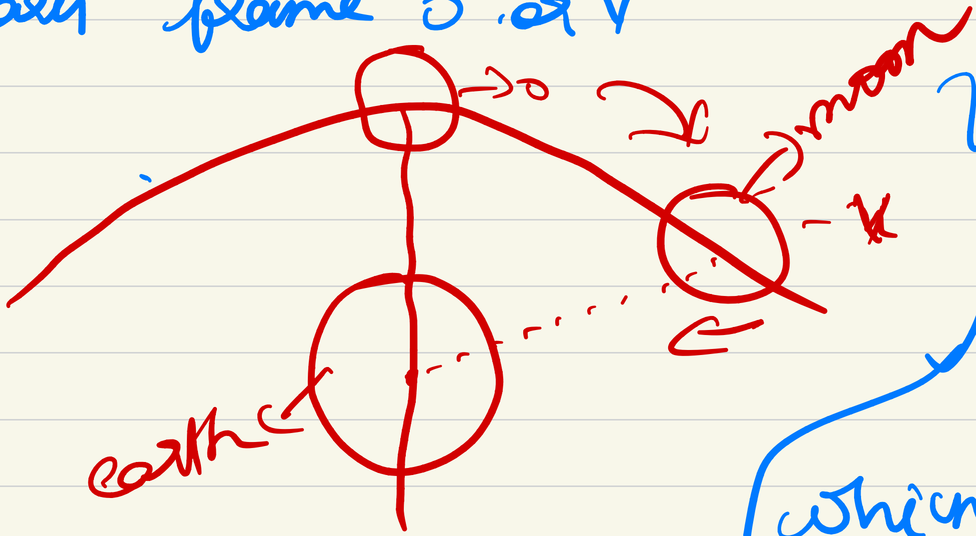
$$P.R_{sm} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -lm \\ 0 & -lm & 0 \end{bmatrix}$$

$$adj(g_{sm}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -lm \\ 0 & 0 & 1 & 0 & -lm & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6}$$

Now,

$$V^s_{sm} = \underset{6\times 6}{adj(g_{sm})} \times \underset{6\times 1}{\begin{bmatrix} 0 \\ \ln^{\pi/14} \\ 0 \\ 0 \\ 0 \\ -\pi/14 \end{bmatrix}} \Rightarrow \underset{6\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\pi/14 \end{bmatrix}}$$

Now, the speciality of V^s_{sm} is that the velocity of translation of moon w.r.t stationary frame 0 or V



There will be some side of the moon which always faces the earth's surface.

1.8 A lunar day, is the time it takes for the moon to return to same position in the sky relative to the earth & the sun. This is approx 29.5 days.

→ Length of a lunar day is longer than the moon's orbital period around the earth, which is about 27.3 days. This difference is due to combined motion of the earth and the moon in their orbits around the Sun.

→ So only considering rotation vectors

$$\omega_{rc} - \omega_{Sun} = \begin{bmatrix} 0 \\ 0 \\ \frac{27\pi}{14} \end{bmatrix}$$

→ Since the angle @ earth will be 2π,

$$\Rightarrow \text{Time} = \frac{\text{angle}}{\text{angular Velocity}} = \frac{2\pi}{\frac{27\pi}{14}} = \underline{\underline{1.036}}$$

$$\boxed{\therefore \text{Lunar Day} = 1.036 \text{ days}}$$