Homework 10: Hybrid Systems

24-760 Robot Dynamics & Analysis
Fall 2023

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Please turn in a PDF with the answers to the following questions.

Problem 1) Falling Block

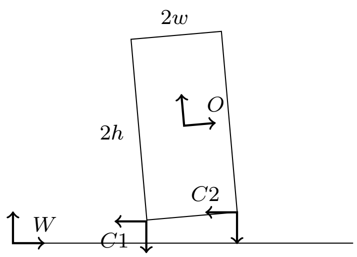

Figure 1: A block.

Consider a planar, rectangular block as shown above and considered in the last homework. The block has mass m, width $2w$, and height $2h$. Assume a tall block, where $h > 2w$. The state of the block in local coordinates is $q = [x, y, \theta]^T$ where each coordinate is expressed relative to the W frame. The gravity vector points in the $-y$ direction in the W frame, and there are no other applied wrenches or friction. There are two contact points on the bottom corners of the block, C_1 and C_2 that can make frictionless contact. Their position constraints are,

$$a_1(q) = y - h\cos(\theta) - w\sin(\theta)$$
$$a_2(q) = y - h\cos(\theta) + w\sin(\theta)$$

1.1) What are the possible contact modes, $\mathcal{J}$? Assume the block doesn't tip over (i.e. the only two possible contacts are at C_1 and C_2).

1.2) What is the domain, D_I, of each contact mode?

1.3) Write down the flow for the system, $\mathcal{F}$, i.e. the dynamics of the system in all possible contact modes. Please specify the matrices M, C, N, A, and Υ for the unconstrained contact mode $\{\}$ (i.e. neither C_1 or C_2 touching the ground), then for all other contact modes write down the updated version of any matrices that change. Feel free to use your solutions from HW9 as a start.

1.4) What are the feasible transitions between contact modes, $\tilde{\Gamma}$, based on the dynamics?

1.5) For each transition above, what are the corresponding guard conditions?

1.6) For each transition above, what is the corresponding reset map?

1.7) Now, let's drop the assumption that $h > 2w$, and for this problem only assume that $h > w$. Is the transition from $\{1\}$ to $\{1,2\}$ achievable? Is the transition from $\{2\}$ to $\{1,2\}$ achievable? That is, will it ever reach a state where it comes to rest?

ANSWERS :-

1.1 $J = \{ \{\}, \{c_1\}, \{c_2\}, \{c_1, c_2\} \}$

Here, the Block can be in one in four States: suspended in the air, tilted on contact point C_1, tilted on contact point C_2 or stable on both contact points C_1 & C_2

1.2 The term $A_i \dot{q}$ is the contact point's velocity, which is zero upon surface contact. Each contact mode's domain is determined accordingly;

$$D = \begin{cases} D\{\} = \{(q,\dot{q}) : a_1(q) \geq 0,\ a_2(q_2) \geq 0\} \\ D\{1\} = \{(q,\dot{q}) : a_1(q) = 0,\ A_1\dot{q} = 0,\ a_2(q) \geq 0\} \\ D\{2\} = \{(q,\dot{q}) : a_1(q) \geq 0,\ a_2(q) = 0,\ A_2\dot{q} = 0\} \\ D\{1,2\} = \{(q,\dot{q}) : a_1(q) = 0,\ A_1\dot{q} = 0,\ a_2(q) = 0,\ A_2\dot{q} = 0\} \end{cases}$$

(1.3) The unconstrained mode, signified by $\{\}$, indicates no contact forces on the system, with gravity as the sole force, eliminating the $A^T\lambda$ term and reducing Y to a zero matrix, allowing for matrix solution.

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m}{3}(h^2+w^2) \end{bmatrix},\quad N = \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad Y = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \tau = M\ddot{q} + C\dot{q} + N$$

Since M depends only on constants m, h, w the C matrix is zero. The N matrix, based on potential energy varies solely with the block's vertical position y.

In constrained modes, these matrices stay constant but an $A^T\lambda$ term is included to represent constrained forces

$$M\ddot{q} + C\dot{q} + N + A^T\lambda = \tau$$

$FC_1 \Rightarrow$ $A = Da_1$

$$= [0, 1, h\sin\theta \;\; -w\cos\theta]$$

$FC_2 \Rightarrow$ $A = Da_2$

$$= [0, 1, h\sin(\theta) + w\cos\theta]$$

$FC_1, FC_2 \Rightarrow$ $A = \begin{bmatrix} Da_1 \\ Da_2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & h\sin\theta - \omega\cos\theta \\ 0 & 1 & h\sin\theta + \omega\cos\theta \end{bmatrix}$$

(1.4) Feasible Transitions $\bar{T}$,

$$\Rightarrow \{ (\{\}, \{1\}), (\{\}, \{2\}), (\{1\}, \{2\}), (\{\}, \{1,2\}), (\{2\}, \{1\}) \}$$

Here, $(\{1\}, \{1,2\})$ & $(\{2\}, \{1,2\})$ won't come as $h > 2\omega$

(1.5)

$$G = \begin{cases} G_{\{\},1} = \{q, \dot{q} \in D_{\{\}} : a_1(q) = 0, A_1\dot{q} < 0\} \\ G_{\{\},2} = \{q, \dot{q} \in D_{\{\}} : a_2(q) = 0, A_2\dot{q} < 0\} \\ G_{\{\},\{1,2\}} = \{q, \dot{q} \in D_{\{\}} : a_1(q) = 0, A_1\dot{q} < 0, a_2(q) = 0, A_2\dot{q} < 0\} \\ G_{1,2} = \{q, \dot{q} \in D_1 : a_2(q) = 0, A_2\dot{q} < 0\} \\ G_{2,1} = \{q, \dot{q} \in D_2 : a_1(q) = 0, A_1\dot{q} < 0\} \end{cases}$$

1.6

$$R = \begin{cases} R_{\{1,2\},1}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_1 \\ R_{\{1,2\},2}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_2 \\ R_{\{1,2\},\{1,2\}}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_{\{1,2\}} \\ R_{1,2}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_2 \\ R_{2,1}(q, \dot{q}^-) = (q, \dot{q}^+) \in D_1 \end{cases} \Bigg\}$$

$$\dot{q}^+ = \dot{q}^- - A_J^{+T}(q)\, A_J(q)\, \dot{q}^-$$

1.7 Consider transition into $\{1,2\}$ and we assume that both constraints will be active,

$$A = \begin{bmatrix} 0 & 1 & h\sin\theta - w\cos\theta \\ 0 & 1 & h\sin\theta + w\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \dot{q}^+ \\ \hat{p} \end{bmatrix} = \begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix}$$

Now, $\dot{y} = w\dot{\theta}$

By substituting we get,

$$\begin{bmatrix} \dot{q}^{+} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \dot{x}^{-} \\ 0 \\ 0 \\ -\frac{m\dot{\theta}^{-}}{6w}(h^2 - 2w^2) \\ \frac{m\dot{\theta}^{-}}{6w}(h^2 - 4w^2) \end{bmatrix}$$

For contact at C_2, a negative $\dot{\theta}$ necessitates a negative impulse at C_2, indicating a positive force from the ground on the block.

The impulse at C_1 varies with block's dimensions (h, w).

$$h^2 - 2w^2 < 0$$

$$h < \sqrt{2}\, w$$

Now, transition from $\{2\}$ to $\{1, 2\}$

$$\dot{y} = -w\dot{\theta}$$

Substituting we get,

$$\begin{bmatrix} \dot{q}^{+} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \dot{x}^{-} \\ 0 \\ 0 \\ -\frac{m\dot{\theta}^{-}}{6w}(h^2+4w^2) \\ \frac{m\dot{\theta}^{-}}{6w}(h^2-2w^2) \end{bmatrix}$$

A positive $\dot{\theta}$ for contact at C_1 leads to negative impulse at C_1, which corresponds to a positive force from the ground. The impulse at C_2 is dependent on the block's h and w.

Here we find the same relation as well,

$$h < \sqrt{2}\, w$$

Thus, transitions from $\{1\}$ to $\{1,2\}$ and $\{2\}$ to $\{1,2\}$ are achievable if $h < \sqrt{2}\, w$.