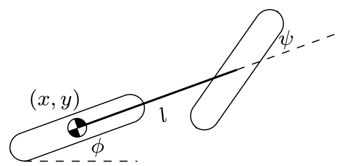
Homework 6: Mobile Robot Kinematics

24-760 Robot Dynamics & Analysis
Fall 2023

Name: Solutions SRECHARAN SELVAM

SSELVAM

Problem 1) Bicycle



You are building a bicycle. To model the system, consider only the horizontal plane kinematics. The wheels are radius $r = 0.3$m and spaced $l = 1$m apart. Assume the steering turns the front wheel about its center. The state of the system $q = [\psi, x, y, \phi]^T$ consist of the steering angle ψ, the position (x, y) of the rear wheel, and the orientation ϕ of the rear wheel. Consider the bicycle as a kinematic system with input u consisting of forward velocity of the rear wheel v and the rotational velocity of the steering wheel $\dot{\psi}$, so $u = [v, \dot{\psi}]^T$.

1.1) What are the kinematic constraints, $A\dot{q} = 0$, on the velocity of the bicycle states?

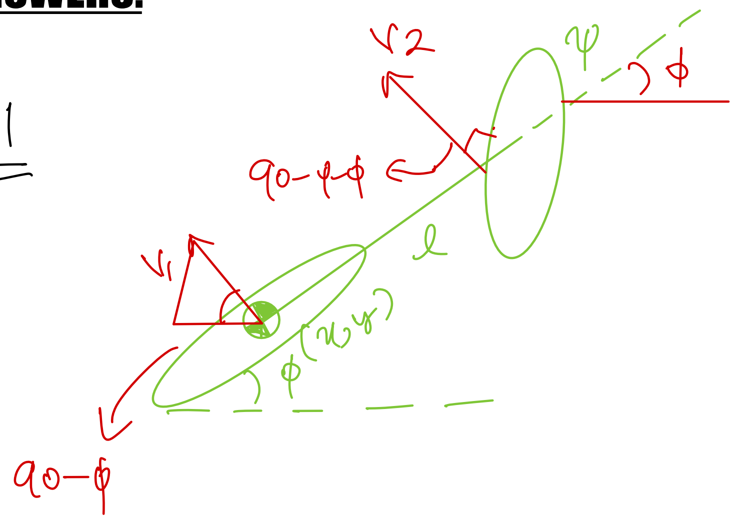
1.2) What are the kinematic freedoms, $\dot{q} = H(q)u$, i.e. what are the system velocities written in terms of the control input? *(Note that the wheels reading used G instead of H).*

1.3) Show that your kinematic constraints (represented by A) and kinematic freedoms (represented by H) are consistent with each other.

1.4) What is the turning radius of the bicycle for a given steering angle ψ? That is, what radius circle will the rear wheel trace out when the front wheel is at an angle of ψ?

ANSWERS:

1.1



There are 2 motions restrictions for this system. If we assume no wheel skids, then each wheel's side to side velocity is zero. The back wheel's speed is given by $[\dot{x}, \dot{y}]^T$ in the space frame. We'll name this frame as 'S'. The back wheel's frame B, the front wheel's frame F, and the positions of the back and front wheels ϕ and ψ. By transforming the back wheel's velocity from the space frame (YPS) to the B frame, we can understand it better.

$$\Rightarrow \quad -V_2 \cos(90-\psi-\phi) \Rightarrow V_2 \cos(90-(\psi+\phi))$$

$$\Rightarrow \quad V_2 \sin(90-\psi-\phi) \Rightarrow V_2 \sin(90-(\psi+\phi))$$

$$\Rightarrow \dot{x} = -v_2 \sin(\varphi + \phi)$$

$$\Rightarrow \dot{y} = v_2 \cos(\varphi + \phi)$$

$$V_{pb} = R_{sb}^{T} V_{ps} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} v \\ 0 \end{bmatrix}$$

The second constraint deal with the velocity of the front wheel in relation to the world which is represented as V_{qf},

$$V_{qb} = \omega_{sb}^{b} + q_{rb} + V_{sb}^{b} = \begin{bmatrix} 0 \\ \dot{\phi} l \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ \dot{\phi} L \end{bmatrix}$$

$$V_{qf} = R^{T}_{bf} V_{qb} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} v \\ \dot{\phi} l \end{bmatrix}$$

$$= \begin{bmatrix} v\cos\varphi + \dot{\phi} l \sin\varphi \\ -v\sin\varphi + \dot{\phi} l \cos\varphi \end{bmatrix} = \begin{bmatrix} v_f \\ 0 \end{bmatrix}$$

Here we can get,

$$A(q) = \begin{bmatrix} 0 & -\sin\phi & \cos\phi & 0 \\ 0 & -\cos\phi \sin\psi & -\sin\phi \sin\psi & l\cos\psi \end{bmatrix}$$

For Rotations,

$$R = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \dot{R} = \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\phi}$$

For positions,

$$\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}$$

Now,

$$V^b = \begin{bmatrix} R^T \dot{p} \\ (R^T \dot{R})^\vee \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \\ 0 \\ 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

For frame F,

$$R = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} \ell \\ 0 \\ 0 \end{bmatrix}$$

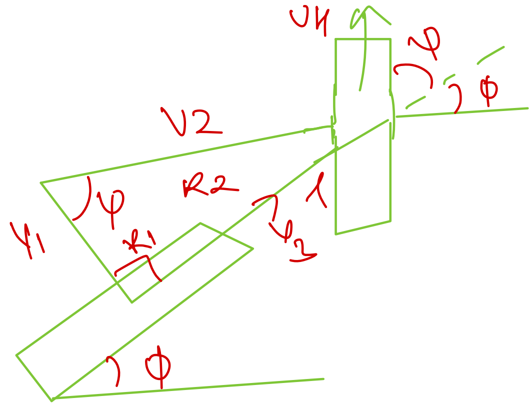
$Ad_{g^{-1}}$ $\qquad\qquad$ V^b

$$V^b = \begin{bmatrix} \cos\varphi & \sin\psi & 0 & 0 & 0 & \ell\sin\psi \\ -\sin\psi & \cos\psi & 0 & 0 & 0 & \ell\cos\varphi \\ 0 & 0 & 1 & 0 & L & 0 \\ 0 & 0 & 0 & \cos\psi & \sin\psi & 0 \\ 0 & 0 & 0 & -\sin\psi & \cos\varphi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}\cos\phi + \dot{y}\sin\phi \\ -\dot{x}\sin\phi + \dot{y}\cos\phi \\ 0 \\ 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$V^b = \begin{bmatrix} \dot{x}\cos(\phi+\psi) + \dot{y}\sin(\phi+\psi) + \ell\dot{\phi}\sin\psi \\ -\dot{x}\sin(\phi+\psi) + \dot{y}\cos(\phi+\psi) + \ell\dot{\phi}\cos\psi \\ 0 \\ 0 \\ 0 \\ \dot{\phi} + \dot{\psi} \end{bmatrix}$$

$$\boxed{A(q) = \begin{bmatrix} 0 & -\sin\phi & \cos\varphi & 0 \\ 0 & -\sin(\phi+\psi) & \cos(\phi+\psi) & \ell\cos\varphi \end{bmatrix}}$$

1.2



For rear wheel,

$$\dot{x} = v_3 \cos(\phi)$$

$$\dot{y} = v_3 \sin(\phi)$$

$$v = R_1 \dot{\theta}$$

$$\dot{\theta} = \frac{v}{R_1}$$

$$\tan\varphi = \frac{L}{R_1}$$

$$R_1 = \frac{L}{\tan\varphi}$$

$$\Rightarrow \quad \dot{\theta} = \frac{v \tan\varphi}{L}$$

$$\Rightarrow \begin{bmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos(\phi) & 0 \\ \sin(\phi) & 0 \\ \frac{\tan\varphi}{L} & 0 \end{bmatrix} [u]$$

$$\boxed{\Rightarrow \underbrace{\begin{bmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}}_{\dot{q}} = \underbrace{\begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \\ \frac{\tan\phi}{L} & 0 \end{bmatrix}}_{H_q(u)} \begin{bmatrix} v_1 \\ \dot{\varphi} \end{bmatrix}}$$

1.3

We know, $A\dot{q} = 0$

$$\Rightarrow A H(q) u = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -\sin\phi & \cos\phi & 0 \\ 0 & -\sin(\phi+\psi) & \cos(\phi+\psi) & l\cos\psi \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \cos\phi & 0 \\ \sin\phi & 0 \\ \frac{\tan(\phi)}{L} & 0 \end{bmatrix} [u] = 0$$

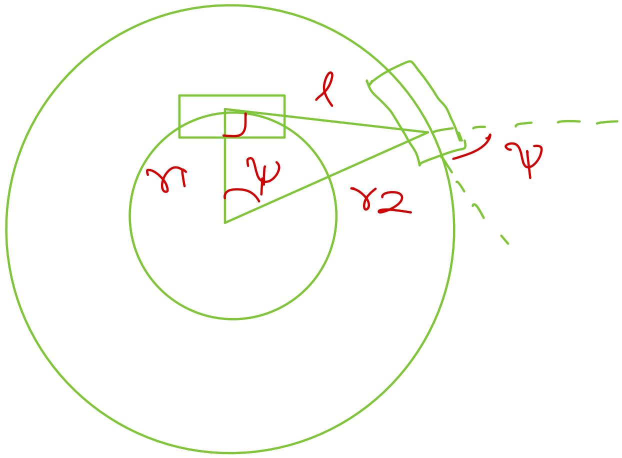
$$\Rightarrow \begin{bmatrix} -\sin\cancel{\phi}\cos\phi + \sin\cancel{\phi}\cos\phi + 0 & 0 \\ -\sin(\phi+\psi)\cos\phi + \sin(\phi)\cos(\phi+\psi) + \sin\psi & 0 \end{bmatrix} [u] = 0$$

$$u = \begin{bmatrix} v \\ \dot{\psi} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \dot{\psi} \end{bmatrix} = 0$$

$$\boxed{\text{Thus, } AH(q)u = 0}$$

1.4



The turning radius is determined using a right angle triangle formed by wheel centers and their perpendicular intersections. This triangle has a known angle 'ψ' and length 'l'.

Rear wheel : $\dfrac{l}{r_1} = \tan(\psi)$

$$\boxed{\Rightarrow r_1 = \frac{l}{\tan\psi}}$$