Mid-term
ECE 271A
Electrical and Computer Engineering
University of California San Diego

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"0-1" loss

1. Note: In both **b)** and **c)** you are expected to give detailed explanations of your choices. Correct answers without a convincing explanation will receive little credit.

a) (5 points) Consider a classification problem with four Gaussian classes

$$P_{\mathbf{X}|Y}(\mathbf{x}|i) = \mathcal{G}(\mathbf{x}, \mu_i, \mathbf{\Sigma}_i),\ i \in \{1, \dots, 4\}.$$

Determine the Bayes decision rule, for the case in which the classes have equal probabilities

$$P_Y(i) = 1/4, \forall i$$

means

$$\mu_1 = (1,1)^T,\ \ \mu_2 = (1,-1)^T,\ \ \mu_3 = (-1,-1)^T,\ \ \mu_4 = (-1,1)^T,$$

and identity covariance.

b) (20 points) Figure 1 presents various classification boundaries for a 4-class problem. For each boundary, indicate if it could correspond to the BDR of the classification problem of a), by modifying some of the parameters. Indicate which parameters would have to be changed, and how. (Note: you don't have to give a numerical answer here, e.g. "the mean of class 3 would be $(3.33, 2.49)^T$". But you have to identify precisely how the new parameter values would obtained from the original ones, e.g. "for class 3 we would need to multiply $\mu_{3,1}$ by α and $\mu_{3,2}$ by $1-\alpha$".) If you think that multiple answers are valid pick the one that would require modifying less parameters.

c) (10 points) Consider a classification problem with two Gaussian classes

$$P_{\mathbf{X}|Y}(\mathbf{x}|i) = \mathcal{G}(\mathbf{x}, \mu_i, \Sigma),\ \ i \in \{0, 1\}$$

and equal class probabilities $P_Y(0) = P_Y(1) = 1/2$. Figure 2 presents four instances of this problem, showing the two Gaussians, represented by the mean point and an iso-contour, and a decision boundary, represented by a hyperplane. For each case, determine if the boundary is that of the BDR for this problem. You are allowed to make assumptions, e.g. "based on this plot, I assume that the boundary is orthogonal to the dashed line," which will be accepted if sensible. Explain your answer.

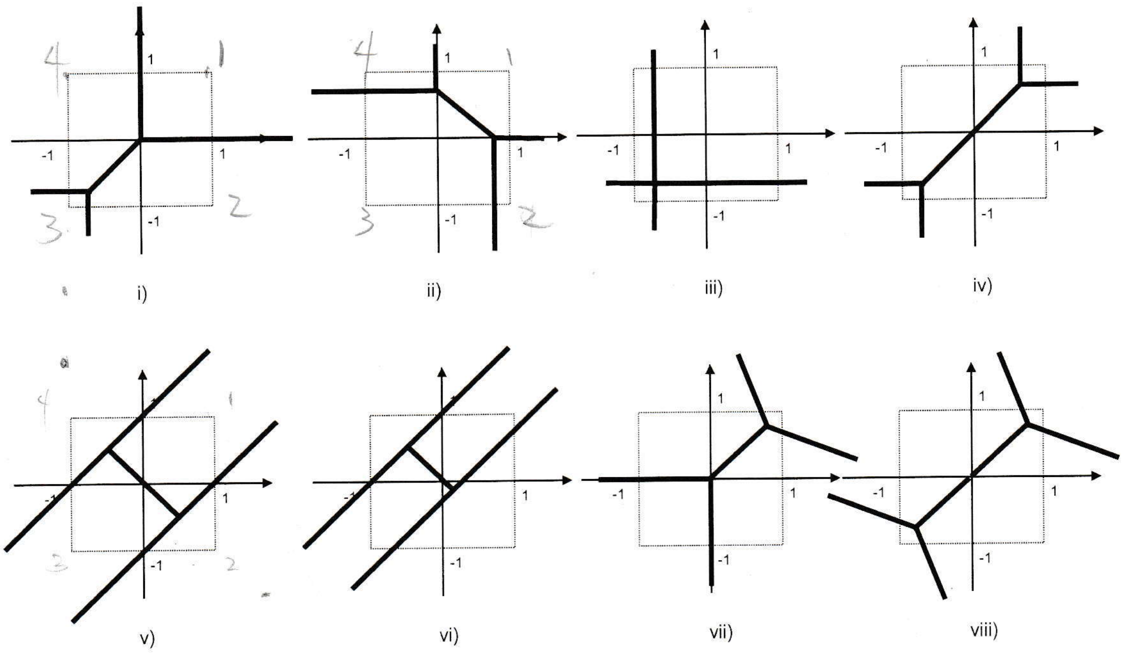

Figure 1: Classification boundaries.

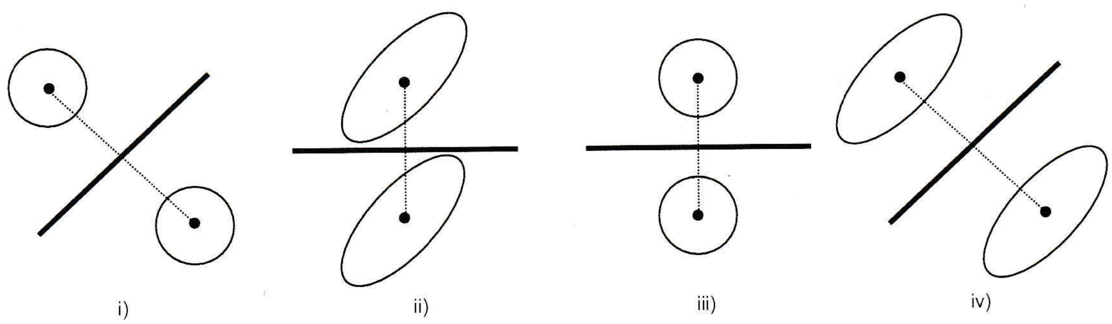

Figure 2: Binary classification problems.

2. Consider a sample of independent observations $\mathcal{D} = \{x_1, \ldots, x_T\}$ from a sequence of T Gaussian random variables $X_t \sim N(\mu_t, \sigma^2), t = 1, \ldots, T$, of variance σ^2 and mean

$$\mu_t = a + bt. \tag{1}$$

Note that observation x_t is drawn from random variable X_t.

In this problem we consider the case where $\mathcal{D}$ was (incorrectly) modeled as a sample from a collection of independent and identically distributed Gaussian random variables $X_t \sim (\mu, \sigma^2), t = 1, \ldots T$ of mean μ and variance σ^2. In all questions below

$$< \gamma >= \frac{1}{T} \sum_{t=1}^{T} \gamma_t \tag{2}$$

is the sample mean of a sample $\{\gamma_1, \ldots, \gamma_T\}$.

a) (5 points) What are the maximum likelihood estimators of μ and σ^2?

b) (20 points) Show that

$$\begin{aligned} E_{X_1,\ldots,X_T}[< x >] &= a + b < t > \\ E_{X_1,\ldots,X_T}[< x >^2] &= \frac{\sigma^2}{T} + (a + b < t >)^2. \end{aligned}$$

(**Note:** if you cannot answer this question, you can still use these formulas in the remaining questions.)

c) (20 points) What is the bias and variance, at time t, of the ML estimator of μ in **a)**?

d) (20 points) What is the bias of the ML estimator of σ^2 in **a)**?