Cohomology of Groups

MAT 4104/511B

1 Group Extensions and $H^2(G, A)$

Let

$$E: \qquad 0 \longrightarrow A \xrightarrow{i} M \xrightarrow{\pi} G \longrightarrow 1 \tag{1}$$

be an extension of an abelian group A by a group G. We'll identify A with $i(A)$ and treat A as a subgroup of M. The group operation of A will be written additively and that of the group M will be written multiplicatively (so $1 \in M$ becomes $0 \in A$ and so on). Recall that, given such an extension, we can define a G-action on A, making it a G-module.

Let $\{u(x)\}_{x \in G}$ be a set of representatives of the elements of G in M (in other words, u is a **section** of π, i.e., it is a map from G to M satisfying $\pi \circ u(x) = x$ for all $x \in G$). Given two elements $x, y \in G$, we can find a unique element $f(x, y) \in A$ such that

$$u(x)u(y) = f(x, y)u(xy). \tag{2}$$

Proposition 1.1. *The function $f : G \times G \to A$ defined by Equation (2) is a 2-cocycle.*

Proof. Let $x, y, z \in G$. We have

$$\begin{aligned} u(x)(u(y)u(z)) &= u(x)(f(y, z)u(yz)) \\ &= (u(x)f(y, z)u(x)^{-1})(u(x)u(yz)) \\ &= (x \cdot f(y, z))f(x, yz)u(xyz) \end{aligned}$$

and

$$\begin{aligned} (u(x)u(y))u(z) &= f(x, y)u(xy)u(z) \\ &= f(x, y)f(xy, z)u(xyz). \end{aligned}$$

But group multiplication is associative, so

$$\begin{aligned} & & u(x)(u(y)u(z)) &= (u(x)u(y))u(z) \\ &\Longrightarrow & (x \cdot f(y, z))f(x, yz)u(xyz) &= f(x, y)f(xy, z)u(xyz) \\ &\Longrightarrow & (x \cdot f(y, z))f(x, yz) &= f(x, y)f(xy, z). \end{aligned}$$

Since the last equation only involves elements of the abelian group A, we may write it additively:

$$x \cdot f(y, z) + f(x, yz) = f(x, y) + f(xy, z),$$

that is,

$$x \cdot f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0.$$

This is precisely the 2-cocycle condition. $\square$

The function f is called the **factor set** for the extension E (associated to the map u). Now let $\{v(x)\}_{x\in G}$ be another set of representatives of the elements of G in M. Let $g : G \times G \to A$ be the 2-cocycle corresponding to this choice. We claim that the factor sets f and g differ by a 2-coboundary.

Proposition 1.2. *With notations as above,* $f + B^2(G, A) = g + B^2(G, A)$.

Proof. Let $x \in G$. Then $\pi(u(x)) = x = \pi(v(x))$, so $v(x)u(x)^{-1} \in \ker \pi = A$. Thus $v(x) = a(x)u(x)$, where $a(x) \in A$. Notice that

$$\begin{aligned}
g(x, y)v(xy) &= v(x)v(y) \\
&= a(x)u(x)a(y)u(y) \\
&= a(x)u(x)a(y)u(x)^{-1}u(x)u(y) \\
&= a(x)(x \cdot a(y))f(x, y)u(xy) \\
&= a(x)(x \cdot a(y))f(x, y)a(xy)^{-1}v(xy).
\end{aligned}$$

Cancelling out $v(xy)$, we obtain

$$g(x, y) = a(x) + x \cdot a(y) + f(x, y) - a(xy).$$

That is,

$$g(x, y) - f(x, y) = x \cdot a(y) - a(xy) + a(x) = d_1(a)(x, y).$$

Thus $g - f \in B^2(G, A)$, as required. $\square$

Hence given an extension E, there is a well-defined cohomology class in $H^2(G, A)$ determined by the factor set associated to some section of π. Next take an equivalent extension, say E'. This means we have the following commutative diagram:

$$\begin{tikzcd}
E: & 0 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & M \arrow[r, "\pi"] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, equal] & 1 \\
E': & 0 \arrow[r] & A \arrow[r, hook] & M' \arrow[r, "\pi'"] & G \arrow[r] & 1
\end{tikzcd}.$$

Notice that the set $\{v(x) := \alpha(u(x))\}_{x\in G}$ forms a set of representatives of the elements of G in M'. Indeed, $\pi'(\alpha(u(x))) = \pi(u(x)) = x$ by the commutativity of the diagram. Let's calculate the cocycle corresponding to v:

$$\begin{aligned}
v(x)v(y) &= \alpha(u(x))\alpha(u(y)) \\
&= \alpha(u(x)u(y)) && (\because \alpha \text{ is a group homomorphism}) \\
&= \alpha(f(x, y)u(xy)) \\
&= \alpha(f(x, y))\alpha(u(xy)) \\
&= f(x, y)v(xy). && (\text{commutative diagram})
\end{aligned}$$

Hence the cohomology class associated to the extension E' is again $f + B^2(G, A)$. Summarising,

Proposition 1.3. *Equivalent extensions define the same cohomology class in* $H^2(G, A)$.

To make life simpler, we now introduce a useful class of cocycles.

Definition 1.4. A 2-cocycle f is said to be **normalised** if $f(g, 1) = 0 = f(1, g)$ for all $g \in G$.

By restricting the representatives of the elements of G to satisfy $u(1) = 1$, we can make the corresponding cocycle normalised.

Lemma 1.5. *Every 2-cocycle is cohomologous to a normalised cocycle.*

Proof. Let f be a 2-cocycle. Consider the constant function f_1 on G whose value is $f(1,1)$. The 2-cocycle $f - d_1 f_1$ lies in the same cohomology class as f. Furthermore,

$$\begin{aligned} f(g,1) - d_1 f_1(g,1) &= f(g,1) - g \cdot f_1(1) + f_1(g) - f_1(1) \\ &= f(g,1) - g \cdot f(1,1) + f(1,1) - f(1,1) \\ &= f(g,1) - g \cdot f(1,1) \\ &= f(g,1) - g \cdot f(1,1) + f(g,1) - f(g,1) \\ &= -d_2 f(g,1,1) = 0. \qquad (\because f \in Z^2(G,A)) \end{aligned}$$

Similarly, we have $f(1,g) - d_1 f_1(1,g) = 0$. Thus $f - d_1 f_1$ is a normalised cocycle and the lemma follows. $\square$

Let A be a fixed G-module. Let $\mathrm{Ext}(G,A)$ denote the set of all equivalence classes of extensions of A by G giving rise to the given action of G on A.

Theorem 1.6. *There is a one-one correspondence between elements of* $\mathrm{Ext}(G,A)$ *and those of* $H^2(G,A)$.

Proof. Define a map $\theta : \mathrm{Ext}(G,A) \to H^2(G,A)$ by $\theta([E]) = f + B^2(G,A)$, where f is a normalised factor set associated to the equivalence class $[E]$. We claim that θ is a bijection.

Suppose $\theta([E]) = \theta([E'])$. We need to prove that $[E] = [E']$, i.e., there is a group homomorphism α making the following diagram commute:

$$\begin{tikzcd} E: & 0 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & M \arrow[r, "\pi"] \arrow[d, dashed, "\exists\alpha"] & G \arrow[r] \arrow[d, equal] & 1 \\ E': & 0 \arrow[r] & A \arrow[r, hook] & M' \arrow[r, "\pi'"] & G \arrow[r] & 1 \end{tikzcd} .$$

Let f be the normalised factor set for E associated to a section u and g be the normalised factor set for E' associated to a section v. By assumption $f + B^2(G,A) = g + B^2(G,A)$, so we can find a $\varphi \in C^1(G,A)$ such that $f - g = d_1(\varphi)$. Expand and rearrange $(f-g)(x,y) = d_1(\varphi)(x,y)$ to obtain

$$f(x,y) + \varphi(xy) = g(x,y) + x\varphi(y) + \varphi(x). \quad (x,y \in G) \tag{3}$$

Observe that every element of M can be uniquely written as $au(x)$ for $a \in A, x \in G$. (Indeed, if $m \in M$, we have $\pi(u(\pi(m))) = \pi(m)$, so $m[u(\pi(m))]^{-1} \in \ker \pi = A$, thus $m = au(\pi(m))$ for some $a \in A$.) Similarly, every element of M' can be uniquely written as $av(x)$ for $a \in A, x \in G$. Define $\alpha : M \to M'$ by

$$\alpha(au(x)) = a\varphi(x)v(x). \qquad (x \in G, a \in A)$$

Note that, for $a, b \in A, x, y \in G$, we have

$$\begin{aligned} \alpha(au(x)bu(y)) &= \alpha(au(x)bu(x)^{-1}u(x)u(y)) \\ &= \alpha(a(x \cdot b)u(x)u(y)) \\ &= \alpha(\underbrace{a(x \cdot b)f(x,y)}_{\in A} u(xy)) \\ &= a(x \cdot b)f(x,y)\varphi(xy)v(xy). \end{aligned}$$

Since $a(x\cdot b)f(x,y)\varphi(xy) \in A$, write it additively and use (3) to obtain

$$\begin{aligned} a + xb + f(x,y) + \varphi(xy) &= a + xb + g(x,y) + x\varphi(y) + \varphi(x) \\ &= a + \varphi(x) + xb + x\varphi(y) + g(x,y). \quad (A \text{ is abelian}) \end{aligned} \tag{4}$$

Continuing our calculation, we have

$$\begin{aligned} \alpha(au(x)bu(y)) &= a(x\cdot b)f(x,y)\varphi(xy)\nu(xy) \\ &= a\varphi(x)(x\cdot b)(x\cdot\varphi(y))g(x,y)\nu(xy) && \text{(from (4))} \\ &= a\varphi(x)\nu(x)b\nu(x)^{-1}\nu(x)\varphi(y)\nu(x)^{-1}\nu(x)\nu(y) \\ &= a\varphi(x)\nu(x)b\varphi(y)\nu(y) \\ &= \alpha(au(x))\alpha(bu(y)). \end{aligned}$$

This proves that α is a group homomorphism. Since f and g are normalised, it follows from (3) that $\varphi(1) = 0$ (take $x = y = 1$). So we have

$$\alpha(a) = \alpha(au(1)) = a\varphi(1)\nu(1) = a\varphi(1) = a + 0 = a \quad (a \in A)$$

and

$$\pi'(\alpha(au(x))) = \pi'(a\varphi(x)\nu(x)) = \underbrace{\pi'(a\varphi(x))}_{=1}\pi'(\nu(x)) = x = \pi(au(x)), \quad (a \in A, x \in G)$$

proving the commutativity of the diagram. Thus E and E' are equivalent extensions, i.e., $[E] = [E']$. This shows that θ is injective.

Next, we claim that θ is surjective. Let $f + B^2(G,A) \in H^2(G,A)$. Because of Lemma 1.5, we can safely assume that f is a normalised cocycle. Consider the set

$$M = \{(a,x) \mid a \in A, x \in G\}$$

and define a binary operation on M via

$$(a,x)(b,y) = (a + x\cdot b + f(x,y), xy).$$

It is easy to see that the above operation satisfy the group axioms. For instance, $(0,1)$ is the identity element:

$$\begin{aligned} (0,1)(a,x) &= (0 + 1\cdot a + f(1,x), x) \\ &= (a,x) = (a,x)(0,1), \end{aligned}$$

and the inverse of (a,x) is $(-x^{-1}a - x^{-1}f(x,x^{-1}), x^{-1})$. Using the 2-cocycle condition, it can be checked that the multiplication is associative. Thus, M is a group.

Now consider the group homomorphisms $i : A \to M$ defined by $a \mapsto (a,1)$ and $\pi : M \to G$ defined by $(a,x) \mapsto x$. The map i is injective and π is surjective with kernel $i(A)$. So we get the following exact sequence

$$E: \qquad 0 \longrightarrow A \xrightarrow{\ i\ } M \xrightarrow{\ \pi\ } G \longrightarrow 1.$$

For $x \in G$, let $u(x) = (0,x)$. It is clear that u is a section of π. Let's **not** identify A with $i(A)$. Then the induced action of G on A is given by

$$x \star a = i^{-1}(u(x)i(a)u(x)^{-1}). \quad (x \in G, a \in A)$$

Note that

$$\begin{aligned}u(x)i(a)u(x)^{-1} &= (0,x)(a,1)(-x^{-1}f(x,x^{-1}),x^{-1})\\ &= (x\cdot a,1) = i(x\cdot a).\end{aligned}$$

Thus $x \star a = x \cdot a$, i.e., the extension E induces the given G-module structure on A, so $[E] \in \mathrm{Ext}(G, A)$. Furthermore,

$$\begin{aligned}u(x)u(y) &= (0,x)(0,y)\\ &= (f(x,y),xy)\\ &= (f(x,y),1)(0,xy)\\ &= i(f(x,y))u(xy).\end{aligned}$$

Therefore the above choice of representative gives the 2-cocycle f, hence $\theta([E]) = f + B^2(G, A)$. Thus θ is surjective. ☐

A short exact sequence

$$E: \qquad 0 \longrightarrow A \xrightarrow{i} M \xrightarrow{\pi} G \longrightarrow 1$$

is equivalent to the short exact sequence

$$E': \qquad 0 \longrightarrow A \xrightarrow{i} A \rtimes M \xrightarrow{\pi} G \longrightarrow 1$$

if and only if there exists a group homomorphism $\phi : G \to M$ such that $\pi \circ \phi = \mathrm{id}$. The short exact sequence E is said to **split** if it satisfies the any of the above conditions.

If E is a split short exact sequence, then

$$\phi(x)\phi(y) = f(x,y)\phi(xy) = f(x,y)\phi(x)\phi(y),$$

hence $f(x,y) = 0$.

Remark 1.7. Under the bijection in Theorem 1.6, split extensions correspond to the trivial cohomology class.

2 Group Extensions and $H^1(G, A)$

Let A be a G-module and

$$\mathcal{E}: 1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

be an extension of A by G inducing the given G-action on A. We'll identify A with its image in E (hence A is an abelian normal subgroup of E), and G with E/A. Furthermore, we write every group operation multiplicatively. Let

$$\mathrm{Aut}_{A,G}(E) := \{\phi \in \mathrm{Aut}(E) \mid \phi(a) = a \text{ for all } a \in A \text{ and } \phi(x) \equiv x \bmod A \text{ for all } x \in E\}.$$

In other words, $\mathrm{Aut}_{A,G}(E)$ is the set of all automorphisms of E such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}: & 1 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
\mathcal{E}: & 1 \arrow[r] & A \arrow[r, hook] & E \arrow[r, "p"] & G \arrow[r] & 1
\end{tikzcd}$$

Next, we define a normal subgroup of $\mathrm{Aut}_{A,G}(E)$ by

$$\mathrm{Inn}_A(E) := \{\phi_a \in \mathrm{Aut}(E) \mid a \in A\},$$

where ϕ_a is defined by $x \mapsto axa^{-1}$ for all $x \in E$.

Theorem 2.1. *With notations as above, we have*

$$H^1(G, A) \cong \frac{\mathrm{Aut}_{A,G}(E)}{\mathrm{Inn}_A(E)}.$$

Proof. Define $\theta : \mathrm{Aut}_{A,G}(E) \to Z^1(G, A)$ by $\theta(\phi) = \phi'$, where ϕ' is given by

$$\phi'(xA) = \phi(x)x^{-1}. \quad (x \in E)$$

First of all, we verify that $\phi(x)x^{-1} \in A$. To that end, we compute $p(\phi(x)x^{-1})$:

$$p(\phi(x)x^{-1}) = p\phi(x)p(x)^{-1} = p(x)p(x)^{-1} = 1,$$

thus $\phi(x)x^{-1} \in \ker p = A$.

Secondly, we verify that the map ϕ' is well-defined 1-cocycle. To see this, consider the map $\phi' : E \to A$ given by $x \mapsto \phi(x)x^{-1}$ (just the ususal abuse of notation). For $x, y \in E$, we have

$$\begin{aligned}
\phi'(xy) &= \phi(xy)(xy)^{-1} \\
&= \phi(x)\phi(y)y^{-1}x^{-1} \\
&= \phi(x)x^{-1}x\phi(y)y^{-1}x^{-1} \\
&= (\phi(x)x^{-1})x(\phi(y)y^{-1})x^{-1} \\
&= \phi'(x)(x \cdot \phi'(y)).
\end{aligned}$$

Furthermore, for $x \in E$ and $a \in A$,

$$\begin{aligned}
\phi'(xa) &= \phi(xa)a^{-1}x^{-1} \\
&= \phi(x)\phi(a)a^{-1}x^{-1} \\
&= \phi(x)aa^{-1}x^{-1} \\
&= \phi(x)x^{-1} = \phi'(x).
\end{aligned}$$

Thus ϕ' factors through E/A. This shows that ϕ' is a well-defined 1-cocycle.

We now claim that θ is a group homomorphism. Let $\phi, \psi \in \mathrm{Aut}_{A,G}(E)$. Note that

$$\begin{aligned}
\theta(\phi\psi)(xA) &= \phi\psi(x)x^{-1} \\
&= \phi(\psi(x))\psi(x)\psi(x)^{-1}x^{-1} \\
&= \phi'(\psi(x)A)\psi'(xA) \\
&= \phi'(xA)\psi'(xA).
\end{aligned}$$

Thus, θ is a morphism of groups. (Here we write the group operation on $Z^1(G, A)$ multiplicatively.)

Next, we prove that θ is an isomorphism. Injectivity is easy. Indeed, suppose that $\theta(\phi) = 1$. Then $\phi'(xA) = \phi(x)x^{-1} = 1$. Thus $\phi(x) = x$ for all $x \in E$, that is, ϕ is the identity on E.

Now let $f \in Z^1(G, A)$. Define $\phi : E \to E$ via $\phi(x) = f(xA)x$. The map ϕ is a group homomorphism:

$$\begin{aligned}\phi(xy) &= f(xyA)xy \\ &= f(xAyA)xy \\ &= f(xA)(x \cdot f(yA))xy \\ &= f(xA)xf(yA)x^{-1}xy \\ &= f(xA)xf(yA)y \\ &= \phi(x)\phi(y).\end{aligned}$$

Moreover, $\phi(a) = f(aA)a = a$ and $p\phi(x) = p(f(xA)x) = p(x)$. The short five lemma implies that ϕ is an automorphism of E and hence $\phi \in \mathrm{Aut}_{A,G}(E)$. It follows from the definitions that $\theta(\phi) = f$. Hence θ is surjective as well.

Finally, we claim that $\theta(\mathrm{Inn}_A(E)) = B^1(G, A)$. Let $\phi_a \in \mathrm{Inn}_A(E)$. Then

$$\theta(\phi_a)(xA) = \phi_a(x)x^{-1} = axa^{-1}x^{-1} = a(x \cdot a)^{-1},$$

so $\theta(\phi_a) \in B^1(G, A)$. Conversely, if $f \in B^1(G, A)$, then define $\phi : E \to E$ by $\phi(x) = f(xA)x$. Since f is a coboundary, we have $f(xA)x = a(x \cdot a)^{-1}x = axa^{-1}$. Thus, $\phi \in \mathrm{Inn}_A(E)$ with $\theta(\phi) = f$. Thus the map

$$\tilde{\theta} : \frac{\mathrm{Aut}_{A,G}(E)}{\mathrm{Inn}_A(E)} \to \frac{Z^1(G, A)}{B^1(G, A)}$$

defined by setting $\tilde{\theta}([\phi]) = [\theta(\phi)]$ is an isomorphism. Hence $H^1(G, A) \cong \mathrm{Aut}_{A,G}(E)/\mathrm{Inn}_A(E)$, as required. $\square$

3 An Exact Sequence

Theorem 3.1. *Let G be a group and N a normal subgroup. Let A be a G-module. Then the sequence*

$$0 \longrightarrow H^1(G/N, A^N) \xrightarrow{\mathrm{Inf}} H^1(G, A) \xrightarrow{\mathrm{Res}} H^1(N, A)$$

is exact.

Proof. Let $f : G/N \to A^N$ be a 1-cocycle. By definition, $\mathrm{Inf}(f)(g) = f(gN)$ for all $g \in G$). Notice that

$$\mathrm{Inf}(f)(gn) = \mathrm{Inf}(f)(g) \tag{5}$$

for all $n \in N$ and $g \in G$. Suppose that $\mathrm{Inf}(f) = 0$ in $H^1(G, A)$. This means $\mathrm{Inf}(f) \in B^1(G, A)$, so there exists an element $a \in A$ such that

$$\mathrm{Inf}(f)(g) = f(gN) = ga - a$$

for all $g \in G$. Then by (5), we have $gna - a = ga - a$ for all $n \in N$, which implies $na = a$ for all $n \in N$, so $a \in A^N$. Thus $f \in B^1(G/N, A^N)$. This proves the injectivity of Inf. Also, note that $\mathrm{Res} \circ \mathrm{Inf}(f)(n) = f(nN) = 0$ for all $n \in N$. So $\mathrm{im}(\mathrm{Inf}) \subseteq \ker(\mathrm{Res})$.

Finally, we claim that $\mathrm{im}(\mathrm{Inf}) \supseteq \ker(\mathrm{Res})$. To that end, let f be a 1-cocycle contained in the kernel of Res. Then there exists $a \in A$ such that $f(n) = na - a$ for all $n \in N$. Define $\ell : G \to A$ by

$$\ell(x) = f(x) - xa + a. \quad (x \in G)$$

Clearly, ℓ is a 1-cocycle as it is a sum of two cocycles. Moreover,

$$\ell(n) = f(n) - na - a = 0$$

for all $n \in N$. Thus,

$$\ell(xn) = x\ell(n) + \ell(x) = \ell(x) \tag{6}$$

for all $n \in N$ and $x \in G$, so there is a well-defined 1-cocycle $\ell' : G/N \to A$ given by

$$\ell'(xN) := \ell(x)$$

for all $x \in G$. Furthermore,

$$n\ell(x) = \ell(nx) - \ell(n) = \ell(nx) = \ell(x\underbrace{x^{-1}nx}_{\in N}) \overset{(6)}{=} \ell(x)$$

for all $n \in N$ and $x \in G$, so the image of ℓ lies in A^N. Therefore $\ell' \in Z^1(G/N, A^N)$. Since $\mathrm{Inf}([\ell']) = [\ell] = [f]$, the claim follows. $\square$

Definition 3.2. A G-module of the form $M_1^G(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$, where A is an abelian group, is said to be a coinduced G-module (or coinduced for G).

Lemma 3.3. *Let G be a group and N a normal subgroup. If B is a coinduced G-module, then B is coinduced for N as well. Moreover, B^N is coinduced for G/N.*

Proof. Suppose that $B = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$. Since $\mathbb{Z}G$ is a $\mathbb{Z}N$-free module,

$$\mathbb{Z}G = \oplus_i \mathbb{Z}N = \mathbb{Z}N \otimes_{\mathbb{Z}} M,$$

where M is the abelian group $\oplus_i \mathbb{Z}$. Thus,

$$B = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}N \otimes_{\mathbb{Z}} M, A) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}N, \mathrm{Hom}_{\mathbb{Z}}(M, A)),$$

where the last isomorphism is the adjoint associativity. Hence, B is coinduced for N.

Notice that $f : G \to A$ belongs to $B^N = (\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A))^N$ if and only if $n \cdot f = f$ for all $n \in N$ if and only if $f(xn) = f(x)$ for all $n \in N$ and $g \in G$. The last condition allows us to define a map $\tilde{f} : G/N \to A$ by setting $\tilde{f}(xN) = f(x)$. Conversely, any map from G/N to A corresponds to an element of B^N; if $\tilde{f}$ is a map from G/N to A, define $f : G \to A$ via $f(x) = \tilde{f}(xN)$. One easily verifies that $f \in B^N$. Hence we may identify B^N with $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/N], A)$. Thus B^N is coinduced for G/N, as required. $\square$

Theorem 3.4. *Let G be a group and N a normal subgroup. Let A be a G-module. Let $q \geqslant 1$, and suppose that $H^i(N, A) = 0$ for all $1 \leqslant i \leqslant q-1$. Then the sequence*

$$0 \longrightarrow H^q(G/N, A^N) \xrightarrow{\mathrm{Inf}} H^q(G, A) \xrightarrow{\mathrm{Res}} H^q(N, A)$$

is exact.

We present Serre's proof here.

Proof. We do induction on q. Theorem 3.1 proves the $q = 1$ case. Suppose $q \geqslant 2$. Let $B = M_1^G(A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$. Since A injects into B, we have the following exact sequence:

$$0 \to A \to B \to C \to 0, \tag{7}$$

where C is the cokernel of the map $A \to B$. As $H^1(N, A) = 0$, we also have the exact sequence

$$0 \to A^N \to B^N \to C^N \to 0. \tag{8}$$

By Lemma 3.3, we know that B is coinduced for N and B^N is coinduced for G/N.

Now consider the following diagram

$$\begin{tikzcd}
 & H^{q-1}(G/N, B^N) \arrow[r] \arrow[d] & H^{q-1}(G, B) \arrow[r] \arrow[d] & H^{q-1}(N, B) \arrow[d] \\
0 \arrow[r] & H^{q-1}(G/N, C^N) \arrow[r] \arrow[d, "\delta"] & H^{q-1}(G, C) \arrow[r] \arrow[d, "\delta"] & H^{q-1}(N, C) \arrow[d, "\delta"] \\
0 \arrow[r] & H^q(G/N, A^N) \arrow[r] \arrow[d] & H^q(G, A) \arrow[r] \arrow[d] & H^q(N, A) \arrow[d] \\
 & H^q(G/N, B^N) \arrow[r] & H^q(G, B) \arrow[r] & H^q(N, B)
\end{tikzcd}$$

This diagram is commutative. The vertical arrows are the long exact sequences associated to the short exact sequences (7) and (8). Since B is coinduced for G and N and B^N is coinduced for G/N, the top and bottom rows are zeroes and hence δ is an isomorphism. Thus we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{q-1}(G/N, C^N) \arrow[r, "\mathrm{Inf}"] \arrow[d, "\delta"] & H^{q-1}(G, C) \arrow[r, "\mathrm{Res}"] \arrow[d, "\delta"] & H^{q-1}(N, C) \arrow[d, "\delta"] \\
0 \arrow[r] & H^q(G/N, A^N) \arrow[r, "\mathrm{Inf}"] & H^q(G, A) \arrow[r, "\mathrm{Res}"] & H^q(N, A)
\end{tikzcd}$$

Since B is coinduced for N, by a corollary of Shapiro's lemma, we have

$$H^i(N, C) \cong H^{i+1}(N, A) = 0$$

for all $1 \leqslant i \leqslant q-2$. Then by induction assumption, the top row is exact. Since δ is an isomorphism, the bottom row is exact as well. □