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Question 56
Let (X, Y) have joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & if x = 0, 1, ..., y; y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases} \tag{1}$$

Then which of the following statements is/are true?

1) $E(X|Y = 4) = 2$
2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$
3) $E(X) = 2$
4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$

(GATE ST 2023)

Solution: Given

$$p_{XY}(x, y) = \frac{e^{-2}}{x!(y-x)!} \tag{2}$$

Then,

$$\begin{aligned} p_Y(y) &= \sum_{x=0}^{\infty} p_{XY}(x, y)\, dx && (3) \\ &= \sum_{x=0}^{\infty} \frac{e^{-2}}{x!(y-x)!} && (4) \\ &= \frac{e^{-2}2^y}{y!} && (5) \\ & && (6) \end{aligned}$$

1) $E(X|Y = 4) = 2$:

$$\begin{aligned} E(X|Y = 4) &= \sum_{x} x \frac{p_{XY}(x, y)}{p_Y(y)} && (7) \\ &= \sum_{x=0}^{4} x \frac{p_{XY}(x, 4)}{p_Y(4)} && (8) \\ &= \sum_{x=0}^{4} x \frac{4!}{2^4 x!(4-x)!} && (9) \\ &= \sum_{x=0}^{4} x \frac{24}{16x!(4-x)!} && (10) \\ &= \sum_{x=0}^{4} x \frac{3}{2x!(4-x)!} && (11) \\ &= \frac{3}{2}(0 + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6}) && (12) \\ &= \frac{3}{2} \times \frac{4}{3} && (13) \\ &= 2 && (14) \end{aligned}$$

Hence, the given statement is true.

2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$:
The z-transform of Y is defined as:

$$\begin{aligned} M_Y(z) &= E\left[z^{-y}\right] && (15) \\ &= \sum_{k=0}^{\infty} p_Y(k)\, z^{-k} && (16) \\ &= \sum_{k=0}^{\infty} \frac{e^{-2}2^k}{k!} z^{-k} && (17) \\ &= e^{-2} \sum_{k=0}^{\infty} \frac{z^{-k}2^k}{k!} && (18) \\ &= e^{-2} \sum_{k=0}^{\infty} \frac{(\frac{2}{z})^k}{k!} && (19) \\ &= e^{-2+\frac{2}{z}} && (20) \end{aligned}$$

For moment generating function, z is replaced by e^{-z}, we get

$$M_Y(z) = e^{-2+\frac{2}{e^{-z}}} \tag{21}$$
$$= e^{-2+2e^{z}} \tag{22}$$
$$= e^{2(e^{z}-1)} \tag{23}$$
$$\implies M_Y(v) = e^{2(e^{v}-1)} \tag{24}$$

Hence, the given statement is true.

3) $E(X) = 2$:

$$p_X(x) = \sum_{y=x}^{\infty} p_{XY}(x,y)\,dy \tag{25}$$
$$= \sum_{y=x}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{26}$$
$$= \frac{e^{-2}}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} \tag{27}$$
$$= \frac{e^{-2}e}{x!} \tag{28}$$
$$= \frac{e^{-1}}{x!} \tag{29}$$

Moment generating function is defined as:

$$M_X(t) = E\left(e^{tx}\right) \tag{30}$$
$$= \sum_{k=0}^{\infty} p_X(k)\, e^{tk} \tag{31}$$

In z-transform we replace e^t with z^{-1}.

$$M_X(v) = E\left(v^{-x}\right) \tag{32}$$
$$= \sum_{k=0}^{\infty} p_X(k)\, v^{-k} \tag{33}$$
$$= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} v^{-k} \tag{34}$$
$$= e^{-1} \sum_{k=0}^{\infty} \frac{v^{-k}}{k!} \tag{35}$$
$$= e^{-1+v^{-1}} \tag{36}$$
$$\implies M_X\left(v^{-1}\right) = e^{-1+v} \tag{37}$$

Now,

$$E[X] = \frac{dM_X(v^{-1})}{dz}\Big|_{v=1} \tag{38}$$
$$= \frac{de^{-1+v}}{dv}\Big|_{v=1} \tag{39}$$
$$= e^{-1+v}\Big|_{v=1} \tag{40}$$
$$= e^{0} \tag{41}$$
$$= 1 \tag{42}$$

Therefore, $E(X) = 1$.

Hence, the given statement is wrong.

4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^{u})e^{v}}$ for all $(u, v) \in R^2$:

$$M_{XY}(z_1, z_2) = E\left(e^{z_1 x + z_2 y}\right) \tag{43}$$
$$= \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} e^{z_1 x + z_2 y} \frac{e^{-2}}{x!(y-x)!} \tag{44}$$
$$= e^{-2} \sum_{x=0}^{\infty} \frac{e^{z_1 x}}{x!} \sum_{y=x}^{\infty} \frac{e^{z_2 y}}{(y-x)!} \tag{45}$$
$$= e^{-2} e^{e^{z_2}} \sum_{x=0}^{\infty} \frac{e^{(z_1+z_2)x}}{x!} \tag{46}$$
$$= e^{-2+e^{z_2}} e^{e^{z_1+z_2}} \tag{47}$$
$$= e^{-2+(1+e^{z_1})e^{z_2}} \tag{48}$$
$$\implies M_{XY}(u, v) = E\left(e^{ux+vy}\right) \tag{49}$$
$$= e^{-2+(1+e^{u})e^{v}} \tag{50}$$

Hence, the given statement is true.

Therefore, 1,2,4 statements are true.

Simulation:

To make the simulation of the given question, generate a large set of random variables, say X and Y, with the probability:

$$P(X = k) = \frac{e^{-1}}{k!}\ for\ k = 0, 1, 2, 3, \ldots. \tag{51}$$
$$P(Y = k) = \frac{e^{-2}2^{y}}{k!}\ for\ k = 0, 1, 2, 3, \ldots. \tag{52}$$

In the simulation process, we use the concept of Inverse Transform Sampling. This involves generating a uniform random variable U from the range $[0, 1]$ and then inverting the generalized form of CDF, i.e. F_X, to obtain X and F_Y, to obtain Y. The inversion is done using the formula:

$$X = {F_X}^{-1}(U) \tag{53}$$
$$Y = {F_Y}^{-1}(U) \tag{54}$$

CDF of X is given as

$$F_X(x) = \sum_{i=0}^{x} p_X(i) \tag{55}$$

CDF of Y is given as

$$F_Y(y) = \sum_{i=0}^{y} p_Y(i) \tag{56}$$

Since, this distribution is discrete, we use following method to find the discrete random variable:

$$\sum_{i=0}^{k-1} p_X(i) \leq U < \sum_{i=0}^{k} p_X(i) \tag{57}$$

$$\sum_{i=0}^{k-1} p_Y(i) \leq U < \sum_{i=0}^{k} p_Y(i) \tag{58}$$

In simpler terms, you're using U to navigate through the distribution's cumulative probabilities to find the corresponding value of X and Y. Using this method, random number is compared to cumulative probabilities (CDF) for various values of the random variable k until the CDF exceeds it. This process continues until a match is found and k is returned as the generated random variable.

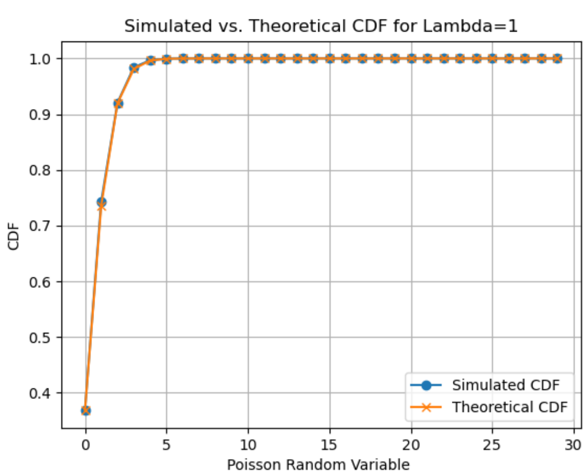

Fig. 1. CDF of X

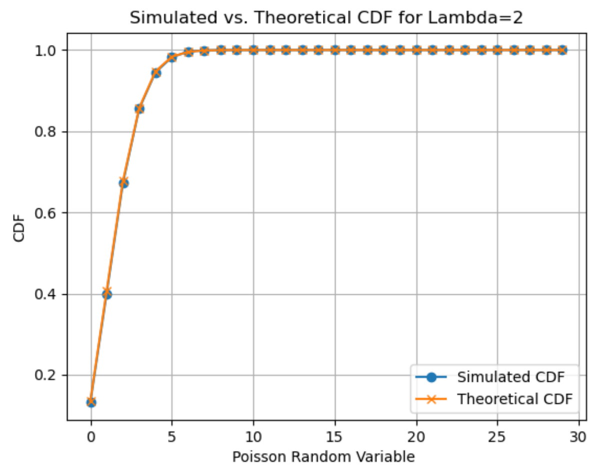

Fig. 2. CDF of Y