Sreeja Komakula - EE22BTECH11029

Question 56
Let (X, Y) have joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & if x = 0, 1, ..., y; y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases} \tag{1}$$

Then which of the following statements is/are true?

1) $E(X|Y = 4) = 2$
2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$
3) $E(X) = 2$
4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$

(GATE ST 2023)

Solution: Given

$$p_{XY}(x, y) = \frac{e^{-2}}{x!(y-x)!} \tag{2}$$

Then,

$$p_Y(y) = \sum_{x=0}^{\infty} p_{XY}(x, y)\, dx \tag{3}$$
$$= \sum_{x=0}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{4}$$
$$= \frac{e^{-2}2^y}{y!} \tag{5}$$
$$\tag{6}$$

1) $E(X|Y = 4) = 2$:

$$E(X|Y = 4) = \sum_{x} x \frac{p_{XY}(x, y)}{p_Y(y)} \tag{7}$$
$$= \sum_{x=0}^{4} x \frac{p_{XY}(x, 4)}{p_Y(4)} \tag{8}$$
$$= \sum_{x=0}^{4} x \frac{4!}{2^4 x!(4-x)!} \tag{9}$$
$$= \sum_{x=0}^{4} x \frac{24}{16x!(4-x)!} \tag{10}$$
$$= \sum_{x=0}^{4} x \frac{3}{2x!(4-x)!} \tag{11}$$
$$= \frac{3}{2}(0 + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6}) \tag{12}$$
$$= \frac{3}{2} \times \frac{4}{3} \tag{13}$$
$$= 2 \tag{14}$$

Hence, the given statement is true.

2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$:
The z-transform of Y is defined as:

$$M_Y(z) = E\left[z^{-y}\right] \tag{15}$$
$$= \sum_{k=0}^{\infty} p_Y(k) z^{-k} \tag{16}$$
$$= \sum_{k=0}^{\infty} \frac{e^{-2}2^k}{k!} z^{-k} \tag{17}$$
$$= e^{-2} \sum_{k=0}^{\infty} \frac{z^{-k}2^k}{k!} \tag{18}$$
$$= e^{-2} \sum_{k=0}^{\infty} \frac{(\frac{2}{z})^k}{k!} \tag{19}$$
$$= e^{-2+\frac{2}{z}} \tag{20}$$

For moment generating function, z is replaced by e^{-z}, we get

$$M_Y(z) = e^{-2+\frac{2}{e^{-z}}} \tag{21}$$
$$= e^{-2+2e^{z}} \tag{22}$$
$$= e^{2(e^{z}-1)} \tag{23}$$
$$\implies M_Y(v) = e^{2(e^{v}-1)} \tag{24}$$

Hence, the given statement is true.

3) $E(X) = 2$:

$$p_X(x) = \sum_{y=x}^{\infty} p_{XY}(x,y)\,dy \tag{25}$$
$$= \sum_{y=x}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{26}$$
$$= \frac{e^{-2}}{x!}\sum_{y=x}^{\infty} \frac{1}{(y-x)!} \tag{27}$$
$$= \frac{e^{-2}e}{x!} \tag{28}$$
$$= \frac{e^{-1}}{x!} \tag{29}$$

Moment generating function is defined as:

$$M_X(t) = E\left(e^{tx}\right) \tag{30}$$
$$= \sum_{k=0}^{\infty} p_X(k)\,e^{tk} \tag{31}$$

In z-transform we replace e^{t} with z^{-1}.

$$M_X(v) = E\left(v^{-x}\right) \tag{32}$$
$$= \sum_{k=0}^{\infty} p_X(k)\,v^{-k} \tag{33}$$
$$= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!}v^{-k} \tag{34}$$
$$= e^{-1}\sum_{k=0}^{\infty} \frac{v^{-k}}{k!} \tag{35}$$
$$= e^{-1+v^{-1}} \tag{36}$$
$$\implies M_X\left(v^{-1}\right) = e^{-1+v} \tag{37}$$

Now,

$$E[X] = \frac{dM_X(v^{-1})}{dz}\Big|_{v=1} \tag{38}$$
$$= \frac{de^{-1+v}}{dv}\Big|_{v=1} \tag{39}$$
$$= e^{-1+v}\Big|_{v=1} \tag{40}$$
$$= e^{0} \tag{41}$$
$$= 1 \tag{42}$$

Therefore, $E(X) = 1$.

Hence, the given statement is wrong.

4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^{u})e^{v}}$ for all $(u, v) \in R^2$:

$$M_{XY}(z_1,z_2) = E\left(e^{z_1x+z_2y}\right) \tag{43}$$
$$= \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} e^{z_1x+z_2y}\frac{e^{-2}}{x!(y-x)!} \tag{44}$$
$$= e^{-2}\sum_{x=0}^{\infty}\frac{e^{z_1x}}{x!}\sum_{y=x}^{\infty}\frac{e^{z_2y}}{(y-x)!} \tag{45}$$
$$= e^{-2}e^{e^{z_2}}\sum_{x=0}^{\infty}\frac{e^{(z_1+z_2)x}}{x!} \tag{46}$$
$$= e^{-2+e^{z_2}}e^{e^{z_1+z_2}} \tag{47}$$
$$= e^{-2+(1+e^{z_1})e^{z_2}} \tag{48}$$
$$\implies M_{XY}(u,v) = E\left(e^{ux+vy}\right) \tag{49}$$
$$= e^{-2+(1+e^{u})e^{v}} \tag{50}$$

Hence, the given statement is true.

Therefore, 1,2,4 statements are true.

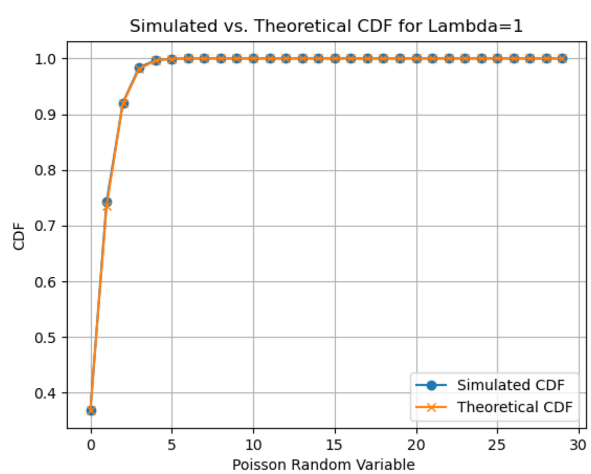

Fig. 1. CDF of X

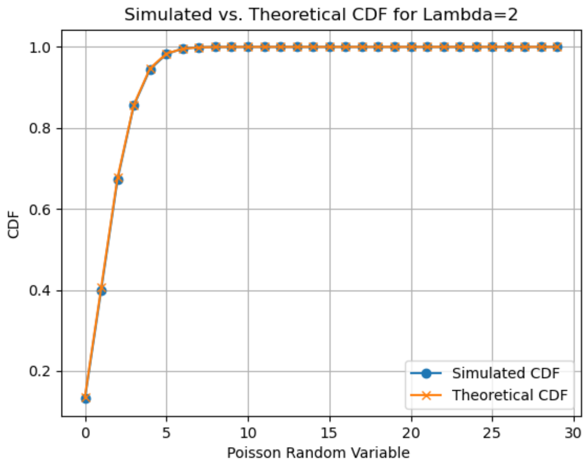
Simulated vs. Theoretical CDF for Lambda=2
CDF
1.0
0.8
0.6
0.4
0.2
0
5
10
15
20
25
30
Poisson Random Variable
Simulated CDF
Theoretical CDF

Fig. 2. CDF of Y