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Question 56
Let (X, Y) have joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & if x = 0, 1, ..., y; y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases} \tag{1}$$

Then which of the following statements is/are true?

1) $E(X|Y = 4) = 2$
2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$
3) $E(X) = 2$
4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$

(GATE ST 2023)

Solution: Given

$$p_{XY}(x, y) = \frac{e^{-2}}{x!(y-x)!} \tag{2}$$

Then,

$$p_Y(y) = \sum_{x=0}^{\infty} p_{XY}(x, y)\, dx \tag{3}$$
$$= \sum_{x=0}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{4}$$
$$= \frac{e^{-2}2^y}{y!} \tag{5}$$
$$\tag{6}$$

1) $E(X|Y = 4) = 2$:

$$E(X|Y = 4) = \sum_x x\frac{p_{XY}(x, y)}{p_Y(y)} \tag{7}$$
$$= \sum_{x=0}^{4} x\frac{p_{XY}(x, 4)}{p_Y(4)} \tag{8}$$
$$= \sum_{x=0}^{4} x\frac{4!}{2^4 x!(4-x)!} \tag{9}$$
$$= \sum_{x=0}^{4} x\frac{24}{16x!(4-x)!} \tag{10}$$
$$= \sum_{x=0}^{4} x\frac{3}{2x!(4-x)!} \tag{11}$$
$$= \frac{3}{2}(0 + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6}) \tag{12}$$
$$= \frac{3}{2} \times \frac{4}{3} \tag{13}$$
$$= 2 \tag{14}$$

Hence, the given statement is true.

2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$:
The z-transform of Y is defined as:

$$M_Y(z) = E\left[z^{-y}\right] \tag{15}$$
$$= \sum_{k=0}^{\infty} p_Y(k)\, z^{-k} \tag{16}$$
$$= \sum_{k=0}^{\infty} \frac{e^{-2}2^k}{k!} z^{-k} \tag{17}$$
$$= e^{-2} \sum_{k=0}^{\infty} \frac{z^{-k}2^k}{k!} \tag{18}$$
$$= e^{-2} \sum_{k=0}^{\infty} \frac{(\frac{2}{z})^k}{k!} \tag{19}$$
$$= e^{-2+\frac{2}{z}} \tag{20}$$

For moment generating function, z is replaced by e^{-z}, we get

$$M_Y(z) = e^{-2+\frac{2}{e^{-z}}} \tag{21}$$
$$= e^{-2+2e^{z}} \tag{22}$$
$$= e^{2(e^{z}-1)} \tag{23}$$
$$\implies M_Y(v) = e^{2(e^{v}-1)} \tag{24}$$

Hence, the given statement is true.

3) $E(X) = 2$:

$$p_X(x) = \sum_{y=x}^{\infty} p_{XY}(x,y)\,dy \tag{25}$$
$$= \sum_{y=x}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{26}$$
$$= \frac{e^{-2}}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} \tag{27}$$
$$= \frac{e^{-2}e}{x!} \tag{28}$$
$$= \frac{e^{-1}}{x!} \tag{29}$$

Moment generating function is defined as:

$$M_X(t) = E\left(e^{tx}\right) \tag{30}$$
$$= \sum_{k=0}^{\infty} p_X(k)\,e^{tk} \tag{31}$$

In z-transform we replace e^{t} with z^{-1}.

$$M_X(v) = E\left(v^{-x}\right) \tag{32}$$
$$= \sum_{k=0}^{\infty} p_X(k)\,v^{-k} \tag{33}$$
$$= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} v^{-k} \tag{34}$$
$$= e^{-1} \sum_{k=0}^{\infty} \frac{v^{-k}}{k!} \tag{35}$$
$$= e^{-1+v^{-1}} \tag{36}$$
$$\implies M_X\left(v^{-1}\right) = e^{-1+v} \tag{37}$$

Now,

$$E[X] = \frac{dM_X(v^{-1})}{dz}\Big|_{v=1} \tag{38}$$
$$= \frac{de^{-1+v}}{dv}\Big|_{v=1} \tag{39}$$
$$= e^{-1+v}\Big|_{v=1} \tag{40}$$
$$= e^{0} \tag{41}$$
$$= 1 \tag{42}$$

Therefore, $E(X) = 1$.
Hence, the given statement is wrong.

4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^{u})e^{v}}$ for all $(u, v) \in R^2$:

$$M_{XY}(z_1, z_2) = E\left(e^{z_1 x + z_2 y}\right) \tag{43}$$
$$= \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} e^{z_1 x + z_2 y} \frac{e^{-2}}{x!(y-x)!} \tag{44}$$
$$= e^{-2} \sum_{x=0}^{\infty} \frac{e^{z_1 x}}{x!} \sum_{y=x}^{\infty} \frac{e^{z_2 y}}{(y-x)!} \tag{45}$$
$$= e^{-2} e^{e^{z_2}} \sum_{x=0}^{\infty} \frac{e^{(z_1+z_2)x}}{x!} \tag{46}$$
$$= e^{-2+e^{z_2}} e^{e^{z_1+z_2}} \tag{47}$$
$$= e^{-2+(1+e^{z_1})e^{z_2}} \tag{48}$$
$$\implies M_{XY}(u, v) = E\left(e^{ux+vy}\right) \tag{49}$$
$$= e^{-2+(1+e^{u})e^{v}} \tag{50}$$

Hence, the given statement is true.
Therefore, 1,2,4 statements are true.

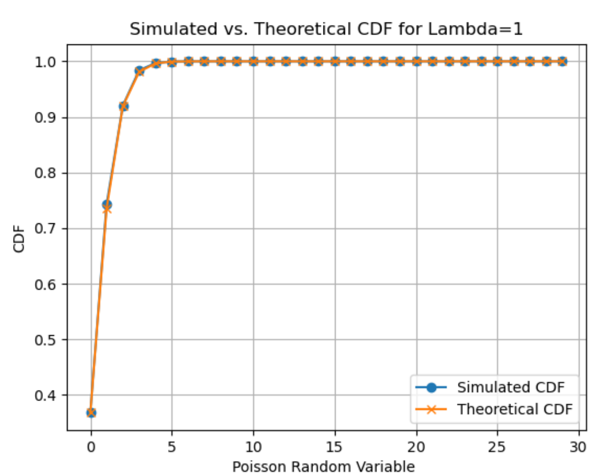

Fig. 1. CDF of X

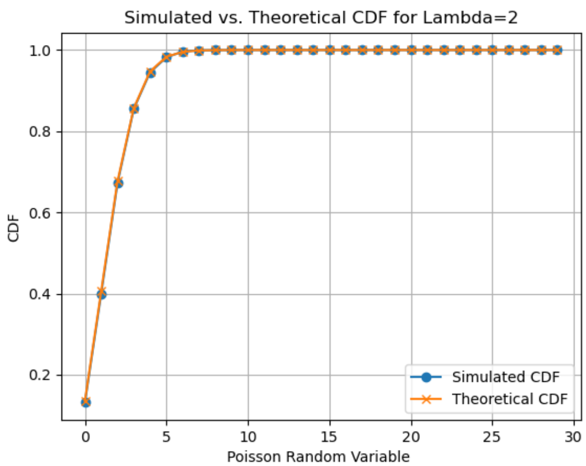

Fig. 2. CDF of Y

I. Simulation Steps for c

The provided C code simulates the generation of random numbers from a Poisson distribution using the cumulative distribution function (CDF) method. The simulation is performed for two different values of the Poisson parameter λ: $\lambda = 1$ (denoted as x) and $\lambda = 2$ (denoted as y).

A. Random Number Generation

Generate a random number from a uniform distribution between 0 and 1.
This function returns a random number between 0 and 1.

B. Poisson PMF Calculation

The Poisson probability mass function (PMF) is calculated using the CDF.
The function `poisson_pmf` takes a random number `u` and the Poisson parameter `lambda` as inputs and returns a Poisson-distributed random variable.
This function uses a loop to generate a Poisson-distributed random variable by comparing the generated uniform random number `u` with the cumulative distribution function.

C. Main Simulation

The main simulation is conducted in the `main` function.
It iterates over the two different values of λ: $\lambda = 1$ and $\lambda = 2$. For each λ, it performs 10,000 simulations and writes the results to separate text files.

II. Simulation Steps for python

The provided Python code simulates the cumulative distribution function (CDF) of a Poisson distribution for two different values of λ (lambda).

A. Simulating CDF

The `cdf_simulation` function simulates the CDF using the simulated data.
The CDF is calculated for each possible value of the Poisson random variable up to a chosen upper limit (in this case, 30).
The simulation involves counting the proportion of simulated data points that are less than or equal to each value.

B. Plotting and Saving CDF

The `plot_and_save_cdf` function takes the simulated CDF values, theoretical CDF values (calculated using `scipy.stats.poisson.cdf`), and the values of the Poisson random variable.
It then plots and saves the comparison between the simulated and theoretical CDFs.

C. Main Loop

The main loop iterates over two lambda values (1 and 2).
For each lambda value, it generates 10,000 simulated data points using `numpy.random.poisson`.
It then calculates the theoretical CDF using `scipy.stats.poisson.cdf`, simulates the CDF using the `cdf_simulation` function, and finally, plots and saves the comparison between the simulated and theoretical CDFs.