Sreeja Komakula - EE22BTECH11029

Question 56
Let (X, Y) have joint probability mass function

$$p_{XY}(x,y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & if x = 0,1,....,y; y = 0,1,2,.... \\ 0 & otherwise. \end{cases} \tag{1}$$

Then which of the following statements is/are true?

1) $E(X|Y=4) = 2$
2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$
3) $E(X) = 2$
4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$

(GATE ST 2023)

Solution: Poisson distribution is defined as follows:

$$p_Y(y) = \frac{e^{-\lambda}\lambda^y}{y!} \text{ where } E(Y) = \lambda \tag{2}$$

Now, the joint probability function for $0 < x < y$ can be written as:

$$\begin{aligned} p_{XY}(x,y) &= \frac{e^{-2}}{x!(y-x)!} && (3)\\ &= \frac{e^{-2}2^y y!}{y!x!(y-x)!2^y} && (4)\\ &= \frac{e^{-2}2^y}{y!}\frac{y!}{x!(y-x)!2^y} && (5)\\ &= p_Y(y)\binom{y}{x}\frac{1}{2^y} && (6)\\ &= p_Y(y)\,p_{X|Y}(x,y) && (7)\\ \implies p_{X|Y}(x,y) &= \binom{y}{x}\frac{1}{2^y} && (8) \end{aligned}$$

1) $E(X|Y=4) = 2$:

$$\begin{aligned} E(X|Y=4) &= \sum_x x p_{X|Y}(x,4) && (9)\\ &= \sum_{x=0}^{4} x p_{X|Y}(x,4) && (10)\\ &= 0(p_{X|Y}(0,4)) + 1(p_{X|Y}(1,4)) + 2(p_{X|Y}(2,4)) && (11)\\ &= \frac{1}{2^4}\binom{4}{1} + 2\times\frac{1}{2^4}\binom{4}{2} + 3\times\frac{1}{2^4}\binom{4}{3} + 4\times\frac{1}{2^4}\binom{4}{4} && (12)\\ &= 32\times\frac{1}{16} && (13)\\ &= 2 && (14) \end{aligned}$$

Hence, the given statement is true.

2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$:
Generally, the moment generating function for poisson distribution is given as:

$$\begin{aligned} M_Y(t) &= e^{-\lambda(1-e^t)} && (15)\\ \implies M_Y(v) &= e^{-2(1-e^v)} && (16)\\ &= e^{2(e^v-1)} && (17) \end{aligned}$$

Hence, the given statement is true.

3) $E(X) = 2$:
The distribution of X is now the poisson distribution with mean:

$$\begin{aligned} \mu &= \frac{\lambda}{2} = \frac{2}{2} = 1 && (18)\\ \implies E(X) &= 1 && (19) \end{aligned}$$

Hence, the given statement is wrong.

4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$:

$$M_{XY}(u, v) = E(e^{ux+vy}) \tag{20}$$

$$= \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} e^{ux+vy} \frac{e^{-2}}{x!(y-x)!} \tag{21}$$

$$= e^{-}2 \sum_{x=0}^{\infty} \frac{e^{ux}}{x!} \sum_{y=x}^{\infty} \frac{e^{vy}}{(y-x)!} \tag{22}$$

$$= e^{-}2e^{e^v} \sum_{x=0}^{\infty} \frac{e^{(u+v)x}}{x!} \tag{23}$$

$$= e^{-2+e^v} e^{(u+v)x} \tag{24}$$

$$= e^{-2+(1+e^u)e^v} \tag{25}$$

Hence, the given statement is true.