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Question 56
Let (X, Y) have joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & if x = 0, 1, ..., y; y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases} \tag{1}$$

Then which of the following statements is/are true?

1) $E(X|Y = 4) = 2$
2) The moment generating function of Y is $e^{2(e^{v}-1)}$ for all $v \in R$
3) $E(X) = 2$
4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^{u})e^{v}}$ for all $(u, v) \in R^2$

(GATE ST 2023)

Solution: Given

$$p_{XY}(x, y) = \frac{e^{-2}}{x!(y-x)!} \tag{2}$$

Then,

$$\begin{aligned} p_Y(y) &= \sum_{x=0}^{\infty} p_{XY}(x, y)\, dx && (3) \\ &= \sum_{x=0}^{\infty} \frac{e^{-2}}{x!(y-x)!} && (4) \\ &= \frac{e^{-2}2^{y}}{y!} && (5) \\ & && (6) \end{aligned}$$

We know that

$$\begin{aligned} p_{XY}(x, y) &= p_Y(y)\, p_{X|Y}(x, y) && (7) \\ \implies p_{X|Y}(x, y) &= \frac{p_{XY}(x, y)}{p_Y(y)} && (8) \\ &= \binom{y}{x}\frac{1}{2^{y}} && (9) \end{aligned}$$

1) $E(X|Y = 4) = 2$:

$$\begin{aligned} E(X|Y = 4) &= \sum_{x} x p_{X|Y}(x, 4) && (10) \\ &= \sum_{x=0}^{4} x p_{X|Y}(x, 4) && (11) \\ &= 0(p_{X|Y}(0, 4)) + 1(p_{X|Y}(1, 4)) + 2(p_{X|Y}(2, 4)) + 3(p_{X|Y}(3, 4)) + 4(p_{X|Y}(4, 4)) && (12) \\ &= \frac{1}{2^4}\binom{4}{1} + 2 \times \frac{1}{2^4}\binom{4}{2} + 3 \times \frac{1}{2^4}\binom{4}{3} + 4 \times \frac{1}{2^4}\binom{4}{4} && (13) \\ &= 32 \times \frac{1}{16} && (14) \\ &= 2 && (15) \end{aligned}$$

Hence, the given statement is true.

2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$:
Generally, the moment generating function for poisson distribution is given as:

$$M_Y(t) = e^{-\lambda(1-e^t)} \tag{16}$$
$$\implies M_Y(v) = e^{-2(1-e^v)} \tag{17}$$
$$= e^{2(e^v-1)} \tag{18}$$

Hence, the given statement is true.

3) $E(X) = 2$:

$$p_X(x) = \sum_{y=x}^{\infty} p_{XY}(x,y)\,dy \tag{19}$$
$$= \sum_{y=x}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{20}$$
$$= \frac{e^{-2}}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} \tag{21}$$
$$= \frac{e^{-2}e}{x!} \tag{22}$$
$$= \frac{e^{-1}}{x!} \tag{23}$$

Here $\lambda = 1$.
Therefore, $E(X) = 1$.
Hence, the given statement is wrong.

4) The joint moment generating function of (X,Y) is $e^{-2+(1+e^u)e^v}$ for all $(u,v) \in R^2$:

$$M_{XY}(s_1,s_2) = E\left(e^{s_1x+s_2y}\right) \tag{24}$$
$$= \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} e^{s_1x+s_2y}\frac{e^{-2}}{x!(y-x)!} \tag{25}$$
$$= e^{-2}\sum_{x=0}^{\infty}\frac{e^{s_1x}}{x!}\sum_{y=x}^{\infty}\frac{e^{s_2y}}{(y-x)!} \tag{26}$$
$$= e^{-2}e^{e^{s_2}}\sum_{x=0}^{\infty}\frac{e^{(s_1+s_2)x}}{x!} \tag{27}$$
$$= e^{-2+e^{s_2}}e^{e^{s_1+s_2}} \tag{28}$$
$$= e^{-2+(1+e^{s_1})e^{s_2}} \tag{29}$$
$$\implies M_{XY}(u,v) = E\left(e^{ux+vy}\right) \tag{30}$$
$$= e^{-2+(1+e^u)e^v} \tag{31}$$

Hence, the given statement is true.