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**Question 56**

Let  $(X, Y)$  have joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & \text{if } x = 0, 1, \dots, y; y = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Then which of the following statements is/are true?

- 1)  $E(X|Y = 4) = 2$
- 2) The moment generating function of  $Y$  is  $e^{2(e^v-1)}$  for all  $v \in R$
- 3)  $E(X) = 2$
- 4) The joint moment generating function of  $(X, Y)$  is  $e^{-2+(1+e^u)e^v}$  for all  $(u, v) \in R^2$

(GATE ST 2023)

**Solution:** Given

$$p_{XY}(x, y) = \frac{e^{-2}}{x!(y-x)!} \quad (2)$$

Then,

$$p_Y(y) = \sum_{x=0}^{\infty} p_{XY}(x, y) dx \quad (3)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-2}}{x!(y-x)!} \quad (4)$$

$$= \frac{e^{-2} 2^y}{y!} \quad (5)$$

$$(6)$$

$$1) E(X|Y = 4) = 2:$$

$$E(X|Y = 4) = \sum_x x \frac{p_{XY}(x, y)}{p_Y(y)} \quad (7)$$

$$= \sum_{x=0}^4 x \frac{p_{XY}(x, 4)}{p_Y(4)} \quad (8)$$

$$= \sum_{x=0}^4 x \frac{4!}{2^4 x!(4-x)!} \quad (9)$$

$$= \sum_{x=0}^4 x \frac{24}{16 x!(4-x)!} \quad (10)$$

$$= \sum_{x=0}^4 x \frac{3}{2 x!(4-x)!} \quad (11)$$

$$= \frac{3}{2} \left( 0 + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6} \right) \quad (12)$$

$$= \frac{3}{2} \times \frac{4}{3} \quad (13)$$

$$= 2 \quad (14)$$

Hence, the given statement is true.

- 2) The moment generating function of  $Y$  is  $e^{2(e^v-1)}$  for all  $v \in R$ :

The  $z$ -transform of  $Y$  is defined as:

$$M_Y(z) = E[z^{-Y}] \quad (15)$$

$$= \sum_{k=0}^{\infty} p_Y(k) z^{-k} \quad (16)$$

$$= \sum_{k=0}^{\infty} \frac{e^{-2} 2^k}{k!} z^{-k} \quad (17)$$

$$= e^{-2} \sum_{k=0}^{\infty} \frac{z^{-k} 2^k}{k!} \quad (18)$$

$$= e^{-2} \sum_{k=0}^{\infty} \frac{\left(\frac{2}{z}\right)^k}{k!} \quad (19)$$

$$= e^{-2+\frac{2}{z}} \quad (20)$$

For moment generating function,  $z$  is replaced

by  $e^{-z}$ , we get

$$M_Y(z) = e^{-2 + \frac{2}{e^{-z}}} \quad (21)$$

$$= e^{-2 + 2e^z} \quad (22)$$

$$= e^{2(e^z - 1)} \quad (23)$$

$$\Rightarrow M_Y(v) = e^{2(e^v - 1)} \quad (24)$$

Hence, the given statement is true.

3)  $E(X) = 2$ :

$$p_X(x) = \sum_{y=x}^{\infty} p_{XY}(x, y) dy \quad (25)$$

$$= \sum_{y=x}^{\infty} \frac{e^{-2}}{x!(y-x)!} \quad (26)$$

$$= \frac{e^{-2}}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} \quad (27)$$

$$= \frac{e^{-2}e}{x!} \quad (28)$$

$$= \frac{e^{-1}}{x!} \quad (29)$$

Moment generating function is defined as:

$$M_X(t) = E(e^{tx}) \quad (30)$$

$$= \sum_{k=0}^{\infty} p_X(k) e^{tk} \quad (31)$$

In  $z$ -transform we replace  $e^t$  with  $z^{-1}$ .

$$M_X(v) = E(v^{-x}) \quad (32)$$

$$= \sum_{k=0}^{\infty} p_X(k) v^{-k} \quad (33)$$

$$= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} v^{-k} \quad (34)$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{v^{-k}}{k!} \quad (35)$$

$$= e^{-1+v^{-1}} \quad (36)$$

$$\Rightarrow M_X(v^{-1}) = e^{-1+v} \quad (37)$$

Now,

$$E[X] = \frac{dM_X(v^{-1})}{dz} \Big|_{v=1} \quad (38)$$

$$= \frac{de^{-1+v}}{dv} \Big|_{v=1} \quad (39)$$

$$= e^{-1+v} \Big|_{v=1} \quad (40)$$

$$= e^0 \quad (41)$$

$$= 1 \quad (42)$$

Therefore,  $E(X) = 1$ .

Hence, the given statement is wrong.

4) The joint moment generating function of  $(X, Y)$  is  $e^{-2+(1+e^u)e^v}$  for all  $(u, v) \in R^2$ :

$$M_{XY}(z_1, z_2) = E(e^{z_1 X + z_2 Y}) \quad (43)$$

$$= \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} e^{z_1 x + z_2 y} \frac{e^{-2}}{x!(y-x)!} \quad (44)$$

$$= e^{-2} \sum_{x=0}^{\infty} \frac{e^{z_1 x}}{x!} \sum_{y=x}^{\infty} \frac{e^{z_2 y}}{(y-x)!} \quad (45)$$

$$= e^{-2} e^{e^{z_2}} \sum_{x=0}^{\infty} \frac{e^{(z_1+z_2)x}}{x!} \quad (46)$$

$$= e^{-2+e^{z_2}} e^{e^{z_1+z_2}} \quad (47)$$

$$= e^{-2+(1+e^{z_1})e^{z_2}} \quad (48)$$

$$\Rightarrow M_{XY}(u, v) = E(e^{ux+vy}) \quad (49)$$

$$= e^{-2+(1+e^u)e^v} \quad (50)$$

Hence, the given statement is true.

Therefore, 1,2,4 statements are true.