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Question 56
Let (X, Y) have joint probability mass function

$$p_{XY}(x, y) = \begin{cases} \frac{e^{-2}}{x!(y-x)!} & if x = 0, 1, ..., y; y = 0, 1, 2, ... \\ 0 & otherwise. \end{cases} \tag{1}$$

Then which of the following statements is/are true?

1) $E(X|Y = 4) = 2$
2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$
3) $E(X) = 2$
4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$

(GATE ST 2023)

Solution: Given

$$p_{XY}(x, y) = \frac{e^{-2}}{x!(y-x)!} \tag{2}$$

Then,

$$p_Y(y) = \sum_{x=0}^{\infty} p_{XY}(x, y)\, dx \tag{3}$$
$$= \sum_{x=0}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{4}$$
$$= \frac{e^{-2}2^y}{y!} \tag{5}$$
$$\tag{6}$$

1) $E(X|Y = 4) = 2$:

$$E(X|Y = 4) = \sum_{x} x \frac{p_{XY}(x, y)}{p_Y(y)} \tag{7}$$
$$= \sum_{x=0}^{4} x \frac{p_{XY}(x, 4)}{p_Y(4)} \tag{8}$$
$$= \sum_{x=0}^{4} x \frac{4!}{2^4 x!(4-x)!} \tag{9}$$
$$= \sum_{x=0}^{4} x \frac{24}{16x!(4-x)!} \tag{10}$$
$$= \sum_{x=0}^{4} x \frac{3}{2x!(4-x)!} \tag{11}$$
$$= \frac{3}{2}(0 + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6}) \tag{12}$$
$$= \frac{3}{2} \times \frac{4}{3} \tag{13}$$
$$= 2 \tag{14}$$

Hence, the given statement is true.

2) The moment generating function of Y is $e^{2(e^v-1)}$ for all $v \in R$:
The z-transform of Y is defined as:

$$Y(z) = Z_Y(n) \tag{15}$$
$$= \sum_{n=0}^{\infty} y(n) z^{-n} \tag{16}$$
$$= \sum_{n=0}^{\infty} \frac{e^{-2}2^n}{n!} z^{-n} \tag{17}$$
$$= e^{-2} \sum_{n=0}^{\infty} \frac{z^{-n}2^n}{n!} \tag{18}$$
$$= e^{-2} \sum_{n=0}^{\infty} \frac{(\frac{2}{z})^n}{n!} \tag{19}$$
$$= e^{-2+\frac{2}{z}} \tag{20}$$

For moment generating function, z is replaced by e^{-z}, we get

$$M_Y(z) = e^{-2+\frac{2}{e^{-z}}} \tag{21}$$
$$= e^{-2+2e^z} \tag{22}$$
$$= e^{2(e^z-1)} \tag{23}$$
$$\implies M_Y(v) = e^{2(e^v-1)} \tag{24}$$

Hence, the given statement is true.

3) $E(X) = 2$:

$$p_X(x) = \sum_{y=x}^{\infty} p_{XY}(x,y)\,dy \tag{25}$$
$$= \sum_{y=x}^{\infty} \frac{e^{-2}}{x!(y-x)!} \tag{26}$$
$$= \frac{e^{-2}}{x!} \sum_{y=x}^{\infty} \frac{1}{(y-x)!} \tag{27}$$
$$= \frac{e^{-2}e}{x!} \tag{28}$$
$$= \frac{e^{-1}}{x!} \tag{29}$$

Now,

$$E(X) = \sum_{x=0}^{\infty} x p_X(x) \tag{30}$$
$$= \sum_{x=0}^{\infty} x \frac{e^{-1}}{x!} \tag{31}$$
$$= \sum_{x=0}^{\infty} \frac{e^{-1}}{(x-1)!} \tag{32}$$
$$= e^{-1} \sum_{x=0}^{\infty} \frac{1}{(x-1)!} \tag{33}$$
$$= e^{-1}e \tag{34}$$
$$= 1 \tag{35}$$

Therefore, $E(X) = 1$.
Hence, the given statement is wrong.

4) The joint moment generating function of (X, Y) is $e^{-2+(1+e^u)e^v}$ for all $(u, v) \in R^2$:

$$M_{XY}(z_1, z_2) = E\left(e^{z_1 x + z_2 y}\right) \tag{36}$$
$$= \sum_{x=0}^{\infty} \sum_{y=x}^{\infty} e^{z_1 x + z_2 y} \frac{e^{-2}}{x!(y-x)!} \tag{37}$$
$$= e^{-2} \sum_{x=0}^{\infty} \frac{e^{z_1 x}}{x!} \sum_{y=x}^{\infty} \frac{e^{z_2 y}}{(y-x)!} \tag{38}$$
$$= e^{-2} e^{e^{z_2}} \sum_{x=0}^{\infty} \frac{e^{(z_1+z_2)x}}{x!} \tag{39}$$
$$= e^{-2+e^{z_2}} e^{e^{z_1+z_2}} \tag{40}$$
$$= e^{-2+(1+e^{z_1})e^{z_2}} \tag{41}$$
$$\implies M_{XY}(u, v) = E\left(e^{ux+vy}\right) \tag{42}$$
$$= e^{-2+(1+e^u)e^v} \tag{43}$$

Hence, the given statement is true.
Therefore, 1,2,4 statements are true.