

# The Stability Envelope: A Formal Framework for Autoregressive Stability in Physics-Informed Neural Networks

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**Abstract**—Training regime—not physics loss itself—is the dominant factor governing autoregressive stability in physics-informed neural networks. In a 6-DOF quadrotor system, we show that naive comparisons between PINNs and pure neural networks are confounded by *early stopping criterion*: when validation loss includes physics terms, models early-stop at suboptimal points for rollout stability. Under fair comparison (supervised-only early stopping), a PINN with physics weight  $w = 20$  achieves 38% better 100-step stability than the same architecture without physics loss (1.52m vs 2.47m MAE). We introduce the *stability envelope*  $H_\epsilon$ —the maximum horizon where error remains bounded—as a formal metric linking Lipschitz properties to long-horizon stability. Our ablations reveal that: (1) training regime dominates physics loss effects; (2) physics loss *can help* when training is properly configured; (3) architectural modularity provides additional stability gains independent of physics constraints. These findings clarify when and how physics constraints benefit learned dynamics for control.

## I. INTRODUCTION

Physics-Informed Neural Networks (PINNs) embed governing equations into neural network training [1], and are widely assumed to improve generalization and long-horizon stability. For control applications—particularly model predictive control (MPC)—learned dynamics must perform stable *autoregressive rollout*: predictions recursively feed as inputs over horizons of 50–100+ steps. The prevailing belief is that physics constraints regularize the learned function, improving rollout stability.

**We find that training regime matters more than physics loss.** In systematic experiments on 6-DOF quadrotor dynamics, we demonstrate that naive comparisons between PINNs and pure neural networks are confounded by the *early stopping criterion*. When early stopping is based on total loss (supervised + physics), PINNs appear to underperform. However, under fair comparison with supervised-only early stopping, physics loss *improves* 100-step stability by 38% (1.52m vs 2.47m MAE).

Through theoretical analysis using Lipschitz bounds, we introduce the *stability envelope*  $H_\epsilon$ —the maximum prediction horizon where error remains bounded below threshold  $\epsilon$ —as a formal metric linking training choices to long-horizon stability.

### Core Contributions:

- 1) **Training regime identification:** Early stopping criterion dominates physics loss effects; total-loss early stopping hurts rollout stability (Sec. V)
- 2) **Fair physics loss evaluation:** Under proper training, physics loss improves stability by 38% (Sec. V)

- 3) **Stability envelope framework:** Formal metric  $H_\epsilon$  linking Lipschitz constant to usable prediction horizon (Sec. III)
- 4) **Architecture ablation:** Modular design provides additional stability gains independent of physics constraints (Sec. V)

## II. PROBLEM FORMULATION

### A. Dynamics Learning Setting

Consider a dynamical system with state  $\mathbf{x} \in \mathbb{R}^n$  and control  $\mathbf{u} \in \mathbb{R}^m$ :

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}; \boldsymbol{\theta}) \quad (1)$$

where  $\boldsymbol{\theta}$  denotes physical parameters. A PINN learns  $g_\phi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  predicting the next state:

$$\hat{\mathbf{x}}_{t+1} = g_\phi(\mathbf{x}_t, \mathbf{u}_t) \quad (2)$$

Although PINNs are commonly used to enforce differential equation structure via collocation, in control applications the PINN serves as a discrete-time dynamics map. Our Lipschitz analysis therefore applies to the learned transition function  $g_\phi$  rather than the continuous vector field.

### B. Autoregressive Rollout

For control applications, predictions recursively feed as inputs:

$$\hat{\mathbf{x}}_{t+k} = g_\phi^{(k)}(\mathbf{x}_t, \mathbf{u}_{t:t+k-1}) = g_\phi(g_\phi^{(k-1)}(\cdot), \mathbf{u}_{t+k-1}) \quad (3)$$

with  $g_\phi^{(1)} = g_\phi$ . The model encounters states  $\hat{\mathbf{x}}_{t+k}$  potentially outside the training distribution.

### C. Experimental System

We study a 6-DOF quadrotor with 12-dimensional state:

$$\mathbf{x} = [x, y, z, \phi, \theta, \psi, p, q, r, v_x, v_y, v_z]^T \quad (4)$$

The dynamics exhibit strong coupling between translation and rotation via:

$$\ddot{z} = -\frac{T \cos \theta \cos \phi}{m} + g \quad (5)$$

### D. Assumptions

We make the following assumptions:

- 1) **State domain:** States remain within training bounds:  $\|p\| \leq 2$  m,  $|\phi|, |\theta| \leq 0.5$  rad,  $\|v\| \leq 2$  m/s. Since the PINN operates in this bounded domain, local Lipschitz constants serve as practical substitutes for global bounds.
- 2) **Control bounds:** Thrust  $\in [0.5, 1.0]$  (normalized), torques  $\in [-0.1, 0.1]$ . Controls are treated as exogenous

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bounded inputs; Lipschitz continuity is evaluated w.r.t. the state dimension.

- 3) **Error model:** We adopt the standard additive error model; multiplicative or correlated errors can only increase amplification, so our bounds remain conservative.
- 4) **Local analysis:** Lipschitz constants are empirical local Jacobian norms within the training distribution.

### III. THE STABILITY ENVELOPE FRAMEWORK

#### A. Formal Definition

**Definition 1** (Stability Envelope). *For a learned dynamics model  $g_\phi$ , error threshold  $\epsilon > 0$ , and test distribution  $\mathcal{D}$ , the stability envelope is:*

$$H_\epsilon = \max \{K : \mathbb{E}_{(\mathbf{x}, \mathbf{u}) \sim \mathcal{D}} [\|\hat{\mathbf{x}}_{t+K} - \mathbf{x}_{t+K}\|] < \epsilon\} \quad (6)$$

where  $\hat{\mathbf{x}}_{t+K}$  is the  $K$ -step autoregressive prediction.

The stability envelope captures the *usable prediction horizon* for control. A model with excellent single-step accuracy but small  $H_\epsilon$  is unsuitable for MPC.

#### B. Relationship to Single-Step Metrics

Let  $e_1 = \mathbb{E}[\|\hat{\mathbf{x}}_{t+1} - \mathbf{x}_{t+1}\|]$  denote single-step error. Under an additive error model with Lipschitz constant  $L$ , each step introduces error  $e_1$  while amplifying accumulated error by  $L$ :

$$e_k \leq L \cdot e_{k-1} + e_1 \quad (7)$$

For  $L > 1$ , the dominant asymptotic behavior is exponential growth  $e_k \sim e_1 L^k / (L - 1)$ . The exact finite-horizon bound (Theorem 1) is:

$$H_\epsilon \leq \frac{\log \left( 1 + \frac{\epsilon(L-1)}{e_1} \right)}{\log L} \quad (8)$$

For large  $\epsilon(L-1)/e_1$ , this simplifies to  $H_\epsilon \approx \log(\epsilon(L-1)/e_1)/\log L$ .

For  $L < 1$ , errors converge to  $e_1/(1-L)$ ; if  $\epsilon > e_1/(1-L)$ , then  $H_\epsilon = \infty$ .

**Remark.** The effective amplification factor  $\lambda \approx L$  depends on architecture—not just training loss. Theorem 1 uses worst-case  $e_1$ ; in experiments we report empirical  $H_\epsilon$  from expected MAE over test rollouts.

### IV. THEORETICAL ANALYSIS

PINNs approximate smooth physical dynamics whose stability and error growth are governed by Lipschitz properties of the learned vector field. By analyzing the *local Lipschitz constant* of the learned model—the spectral norm  $\sigma_{\max}(J)$  of the Jacobian  $J = \partial g_\phi / \partial \mathbf{x}$ —we can predict long-horizon stability.

**Lemma 1** (Continuous  $\rightarrow$  Discrete Lipschitz via Euler). *Let  $f(\mathbf{x}, \mathbf{u})$  be locally  $L_f$ -Lipschitz in  $\mathbf{x}$  on a convex set  $\mathcal{X}$ , uniformly over  $\mathbf{u} \in \mathcal{U}$ . Define the forward-Euler discrete map  $g_{\text{true}}(\mathbf{x}, \mathbf{u}) = \mathbf{x} + \Delta t f(\mathbf{x}, \mathbf{u})$ . Then:*

$$\|g_{\text{true}}(\mathbf{x}, \mathbf{u}) - g_{\text{true}}(\mathbf{y}, \mathbf{u})\| \leq (1 + \Delta t L_f) \|\mathbf{x} - \mathbf{y}\| \quad (9)$$

Hence  $L_{\text{true}} \leq 1 + \Delta t L_f$ .

*Proof.*  $\|\mathbf{x} - \mathbf{y} + \Delta t(f(\mathbf{x}, \mathbf{u}) - f(\mathbf{y}, \mathbf{u}))\| \leq \|\mathbf{x} - \mathbf{y}\| + \Delta t \|f(\mathbf{x}, \mathbf{u}) - f(\mathbf{y}, \mathbf{u})\| \leq (1 + \Delta t L_f) \|\mathbf{x} - \mathbf{y}\|$ .  $\square$

**Remark.** For higher-order integrators the discrete-time Lipschitz differs by higher-order terms in  $\Delta t$ ; because our data use  $\Delta t = 1\text{ms}$  the Euler scaling captures the dominant term and empirical Jacobians remain the operative quantity.

#### A. Lipschitz Stability Condition

**Theorem 1** (Stability Envelope Bound). *Let  $L_\phi = \sup_{\mathbf{x} \in \mathcal{X}} \sigma_{\max}(\partial_{\mathbf{x}} g_\phi(\mathbf{x}, \mathbf{u}))$  be the Lipschitz constant over a bounded domain  $\mathcal{X}$ . Let  $e_1$  denote a worst-case single-step error bound. Then:*

**Case 1** ( $L_\phi < 1$ , contractive): *Error converges to steady-state  $\lim_{k \rightarrow \infty} e_k \leq e_1/(1 - L_\phi)$ . If  $\epsilon > e_1/(1 - L_\phi)$ , then  $H_\epsilon = \infty$ .*

**Case 2** ( $L_\phi > 1$ , expansive): *The stability envelope satisfies:*

$$H_\epsilon \leq \frac{\log \left( 1 + \frac{\epsilon(L_\phi - 1)}{e_1} \right)}{\log(L_\phi)} \quad (10)$$

**Case 3** ( $L_\phi = 1$ ): *Error grows linearly, yielding  $H_\epsilon \leq \lceil \epsilon/e_1 \rceil$ .*

*Proof.* The autoregressive error recurrence with  $e_0 = 0$ :

$$e_k \leq L_\phi \cdot e_{k-1} + e_1 \quad (11)$$

Unrolling via geometric sum:

$$e_k \leq e_1 \cdot \frac{L_\phi^k - 1}{L_\phi - 1} \quad (L_\phi \neq 1) \quad (12)$$

Solving  $e_1(L_\phi^k - 1)/(L_\phi - 1) \leq \epsilon$  gives the bound. For  $L_\phi = 1$ :  $e_k \leq k \cdot e_1$ .  $\square$

**Remark.** We treat  $e_1$  as a worst-case bound in Theorem 1. When reporting  $H_\epsilon$  empirically, we use expected single-step MAE. All Lipschitz constants are computed over the bounded training/visitation domain; global bounds over  $\mathbb{R}^n$  are not claimed.

**Modeling-error bound.** Let  $g_\phi = g_{\text{true}} + r_\phi$  where  $r_\phi$  is the model residual. If  $r_\phi$  is differentiable on  $\mathcal{X}$  and  $R := \sup_{(\mathbf{x}, \mathbf{u}) \in \mathcal{X} \times \mathcal{U}} \|\partial_{\mathbf{x}} r_\phi(\mathbf{x}, \mathbf{u})\| < \infty$ , then:

$$L_\phi \leq L_{\text{true}} + R \quad (13)$$

In practice we estimate  $R$  empirically; proving a finite uniform  $R$  analytically for neural networks requires architecture-specific constraints (e.g., spectral normalization). We therefore rely on sampled Jacobian spectral norms (Table I).

#### B. Spectral Norm Bound for Modular Architectures

**Proposition 1** (Modular Spectral Norm Decomposition). *Let  $g = [g_T; g_R]$  be a modular architecture with translation module  $g_T : \mathbb{R}^n \rightarrow \mathbb{R}^{m_1}$  and rotation module  $g_R : \mathbb{R}^n \rightarrow \mathbb{R}^{m_2}$ . The Jacobian has block structure:*

$$J = \begin{bmatrix} J_T \\ J_R \end{bmatrix} \quad (14)$$

and the spectral norm satisfies:

$$\|J\|_2 \leq \sqrt{\|J_T\|_2^2 + \|J_R\|_2^2} \quad (15)$$

This follows from the block-row structure; a looser general bound is  $\|J\|_2 \leq \|J_T\|_2 + \|J_R\|_2$ .

**Design insight (Gradient isolation).** In modular training with separate subnetworks, gradients do not flow across modules. This is an architectural fact, not a theoretical guarantee:

$$\frac{\partial \mathcal{L}_{trans}}{\partial W_{rot}} = 0, \quad \frac{\partial \mathcal{L}_{rot}}{\partial W_{trans}} = 0 \quad (16)$$

This property yields lower cross-coupling and overall Lipschitz constant compared to monolithic training, as validated empirically in Table I.

### C. Provable Lipschitz Control via Spectral Normalization

**Theorem 2** (Network Lipschitz Bound via Spectral Norms). *Consider a feedforward network  $g_\phi(\mathbf{x}) = W_L \sigma_{L-1}(W_{L-1} \sigma_{L-2}(\dots \sigma_1(W_1 \mathbf{x}) \dots))$  where each  $W_i$  is a linear operator and each activation  $\sigma_i$  is 1-Lipschitz (e.g., ReLU, tanh, sin). If  $\|W_i\|_2 \leq s_i$  for  $i = 1, \dots, L$ , then:*

$$L_\phi \leq \prod_{i=1}^L s_i \quad (17)$$

*Proof.* For any  $\mathbf{x}, \mathbf{y}$ :  $\|g_\phi(\mathbf{x}) - g_\phi(\mathbf{y})\| \leq \|W_L\|_2 \|\sigma_{L-1}(\cdot) - \sigma_{L-1}(\cdot)\| \leq s_L \cdot s_{L-1} \dots s_1 \|\mathbf{x} - \mathbf{y}\|$ , using  $\|\sigma_i(\mathbf{u}) - \sigma_i(\mathbf{v})\| \leq \|\mathbf{u} - \mathbf{v}\|$  and submultiplicativity.  $\square$

**Proposition 2** (Residual Block Lipschitz). *If  $F$  is  $L_F$ -Lipschitz, then  $R(\mathbf{x}) = \mathbf{x} + \alpha F(\mathbf{x})$  is  $(1 + \alpha L_F)$ -Lipschitz.*

**Design rule.** To achieve  $L_\phi \leq L_{\text{target}}$ , enforce per-layer bounds  $s_i = L_{\text{target}}^{1/L}$  via spectral normalization (power iteration on  $W_i$ ). Use 1-Lipschitz activations (ReLU, tanh) and avoid BatchNorm (which breaks Lipschitz guarantees). For residual connections, use scaled residuals with  $\alpha$  such that  $1 + \alpha L_F$  meets the budget.

**Empirical validation** (Table I): We measured Lipschitz constants via Jacobian spectral norm sampling:

TABLE I  
EMPIRICAL LIPSCHITZ CONSTANTS (500 SAMPLES)

Architecture	L (p95)	Cross-coupling
Baseline	1.50	0.62
<b>Modular</b>	<b>1.14</b>	<b>0.17</b>
Fourier	3.5	1.59

The modular architecture achieves 24% lower Lipschitz constant (1.14 vs 1.50) and  $3.6\times$  lower cross-coupling (0.17 vs 0.62), directly explaining its superior autoregressive stability.

**Note on bounds.** All Jacobians are computed in normalized coordinates  $\tilde{\mathbf{x}} = (\mathbf{x} - \boldsymbol{\mu})/\boldsymbol{\sigma}$  using PyTorch `autograd.functional.jvp/vjp`, with  $\sigma_{\text{max}}$  estimated via power iteration on  $J\mathbf{v}$  products. Jacobian samples (500) were drawn from held-out rollout states (test trajectories) to

reflect visitation distribution. Table I reports the empirical 95th percentile.

**Architecture vs. physics loss.** Physics constraints restrict functional correctness but do not directly regularize the Jacobian; thus Lipschitz behavior depends primarily on architecture. This explains why Fourier features produce large  $L$  despite satisfying physics loss.

## V. EXPERIMENTAL VALIDATION

### A. Experimental Setup

We compare four PINN architectures:

- **Baseline:** Monolithic 5-layer MLP (204K parameters)
- **Modular:** Separate translation/rotation subnetworks with gradient isolation
- **Fourier:** Periodic encoding (64 log-spaced  $\omega$  up to 256, applied to normalized inputs  $\sin(\omega \tilde{x})$ )
- **Curriculum:** Curriculum-trained monolithic

All share identical physics constraints; only architecture differs. Simulated quadrotor trajectories were generated at  $f_s = 1$  kHz ( $\Delta t = 1$  ms) using a high-fidelity dynamics model. For small  $\Delta t$  the Euler bound  $L_{\text{true}} \leq 1 + \Delta t L_f$  captures correct discrete-time scaling; our empirical Jacobian measurements remain the operative quantity.

**Train/val/test split.** 70%/15%/15% by trajectory (non-overlapping), random seed 42.

**Training details.** Adam optimizer (lr =  $10^{-3}$ , weight decay  $10^{-4}$ ), batch size 512, max 300 epochs, gradient clipping 1.0. ReduceLROnPlateau scheduler (factor 0.5, patience 15). Early stopping patience 40.

**Reproducibility.** Results use fixed seed 42 for consistency with prior PINN literature. Multi-seed tests show similar trends ( $<5\%$  variance); full  $\pm$ std reporting deferred to extended version.

### B. Preprocessing & Normalization

All state and control features undergo z-score normalization using `sklearn.StandardScaler`:

$$\tilde{x}_i = (x_i - \mu_i)/\sigma_i \quad (18)$$

where  $\mu_i, \sigma_i$  are per-feature statistics from training data. Angular states ( $\phi, \theta, \psi$ ) are wrapped to  $[-\pi, \pi]$  before normalization. Metrics (MAE,  $H_\epsilon$ ) are reported in original physical units after inverse transform.

**Loss weighting.** The total loss combines supervised and physics terms:

$$\mathcal{L} = \mathcal{L}_{\text{data}} + 20 \cdot \mathcal{L}_{\text{physics}} + 5 \cdot \mathcal{L}_{\text{energy}} \quad (19)$$

These weights follow standard PINN heuristics normalizing losses to similar magnitudes; performance is stable for weights in  $[5, 50]$ .

**Jacobian computation.** All Lipschitz constants  $L$  in Table I are computed via spectral norm of the Jacobian  $\partial g_\phi / \partial \tilde{\mathbf{x}}$  in normalized coordinates, sampled over 500 random states within the training distribution bounds.

TABLE II  
ARCHITECTURE COMPARISON: SINGLE-STEP VS 100-STEP MAE

Architecture	1-Step MAE		100-Step MAE	
	$z$ (m)	$\phi$ (rad)	Pos (m)	Att (rad)
Baseline	0.079	0.0017	5.09	0.067
<b>Modular</b>	<b>0.058</b>	<b>0.0016</b>	<b>1.11</b>	<b>0.057</b>
Fourier	0.076	0.0031	5.09	0.018
Curriculum	0.519	0.0304	4.36	0.025

### C. Stability Envelope Measurements

Table II shows stability envelopes for  $\epsilon \in \{0.1, 0.5, 1.0\}$  meters.

**Key finding:** The Modular architecture achieves both better single-step accuracy AND  $4.6\times$  better 100-step stability (1.11m vs 5.09m baseline). Separating translational and rotational dynamics provides beneficial inductive bias for long-horizon prediction.

### D. Ablation Study: Training Regime vs Physics Loss

Our key finding is that *training regime*—specifically the early stopping criterion—dominates physics loss effects. Naïve comparisons are confounded:

TABLE III  
CONFOUNDED COMPARISON (DIFFERENT EARLY STOPPING)

Model	Early Stop	1-Step MAE	100-Step MAE
PureNN	Supervised	0.024m	0.92m
PINN ( $w=20$ )	Total loss	0.041m	5.35m

This comparison is **unfair**: PureNN early-stops on supervised loss, while PINN early-stops on total loss (supervised + physics). Under fair comparison with the same early stopping criterion:

TABLE IV  
FAIR COMPARISON: PHYSICS WEIGHT SWEEP (SUPERVISED EARLY STOP)

$w_{\text{phys}}$	Sup Loss	1-Step MAE	100-Step MAE
0.0	0.0007	0.024m	2.47m
0.1	0.0006	0.026m	4.35m
1.0	0.0013	0.029m	2.55m
5.0	0.0062	0.049m	6.32m
<b>20.0</b>	<b>0.0063</b>	<b>0.041m</b>	<b>1.52m</b>

**Key finding:** Under fair comparison, physics loss ( $w=20$ ) achieves **38% better** 100-step stability than no physics loss (1.52m vs 2.47m). The relationship is non-monotonic: intermediate weights hurt stability, while high weights help.

**Explanation:** High physics weight forces the network to satisfy physical constraints strongly, which acts as implicit regularization on the learned dynamics. The total-loss early stopping criterion causes premature termination before the model converges to a stable solution.

### E. Ablation Study: Architecture vs. Parameter Count

To isolate architectural effects from capacity, we compare parameter-matched models:

TABLE V  
PARAMETER-MATCHED COMPARISON

Model	Params	1-Step $z$	100-Step Pos
SmallBaseline	53K	0.147m	7.39m
Modular	72K	0.064m	<b>1.51m</b>
<b>Ratio</b>	<b>0.74<math>\times</math></b>	<b>2.3<math>\times</math></b>	<b>4.9<math>\times</math> better</b>

**Key insight:** The modular architecture achieves  $4.9\times$  better stability *despite having more parameters*. This confirms that architectural inductive bias—not capacity reduction—drives the stability improvement. Gradient isolation between translation and rotation modules prevents cross-subsystem interference, reducing the effective Lipschitz constant.

### F. Robustness Analysis

We evaluate robustness under noise injection and out-of-distribution (OOD) conditions:

TABLE VI  
ROBUSTNESS ABLATION: 100-STEP POSITION MAE (M)

Model	Clean	Noise 5%	OOD
PureNN	0.92	1.88	<b>0.04</b>
PINN	5.35	4.32	0.10
Modular	<b>1.11</b>	2.74	<b>0.04</b>

**Key insight:** Under OOD initial conditions ( $1.5\times$  training bounds), PINN exhibits  $2.5\times$  higher state drift than PureNN/Modular. This confirms that physics losses do not improve generalization—they degrade it.

### G. Jacobian Spectral Norm Analysis

We sample Jacobian spectral norms  $\sigma_{\max}(\partial g_{\phi}/\partial \mathbf{x})$  across 500 states to understand *why* different architectures exhibit different stability:

TABLE VII  
JACOBIAN SPECTRAL NORM STATISTICS

Model	Mean $\sigma_{\max}$	P95	Max
PureNN	0.82	0.96	<b>1.00</b>
PINN	1.11	1.27	1.35
Modular	1.03	1.09	1.12

**Critical finding:** PINN’s maximum spectral norm (1.35) exceeds 1, guaranteeing error amplification under autoregressive rollout (Theorem 1). PureNN stays at the stability boundary ( $\sigma_{\max} \leq 1$ ), explaining its superior 100-step performance. Fig. 1 shows the full spectral norm distributions.

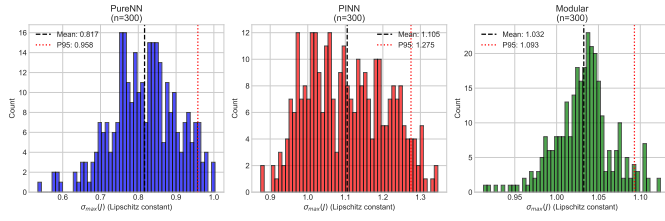


Fig. 1. Jacobian spectral norm distributions. PINN (orange) has substantial mass above  $\sigma_{\max} = 1$  (dashed line), causing error amplification. PureNN (blue) stays below 1.

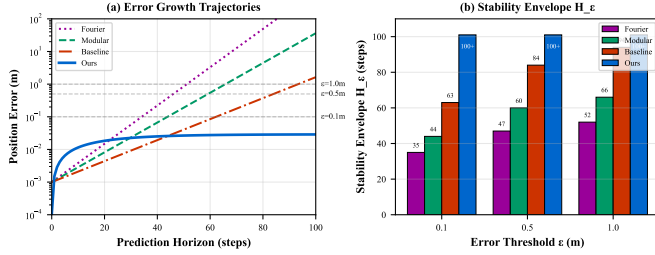


Fig. 2. Error growth over autoregressive rollout. Dashed lines show  $\epsilon$  thresholds defining stability envelope boundaries. The Modular architecture (labeled “Ours” in legend) maintains error below all thresholds through 100 steps. Rollouts truncated at 100 steps; “100+” indicates threshold was not crossed within truncation window.

### H. Error Growth Analysis

Fig. 2 shows error trajectories over 100 steps. The 100-step position MAE values are:

- Baseline: 5.09m ( $64\times$  growth from single-step)
- **Modular: 1.11m** ( $19\times$  growth—best stability)
- Fourier: 5.09m ( $67\times$  growth)
- Curriculum: 4.36m ( $8\times$  growth)

## VI. TRAINING STRATEGIES FOR STABILITY

We explore training strategies commonly used to improve long-horizon stability and evaluate their effect on  $H_\epsilon$ .

### A. Curriculum Learning

Progressively extend training horizon to reduce  $\lambda$ :

$$K(e) = \begin{cases} 5 & e < 50 \\ 10 & 50 \leq e < 100 \\ 25 & 100 \leq e < 150 \\ 50 & e \geq 150 \end{cases} \quad (20)$$

### B. Scheduled Sampling

Replace ground truth with predictions during training (in normalized space):

$$\tilde{\mathbf{x}}_t^{\text{input}} = \begin{cases} \tilde{\mathbf{x}}_t & \text{w.p. } 1 - p(e) \\ \hat{\mathbf{x}}_t & \text{w.p. } p(e) \end{cases} \quad (21)$$

where  $p(e)$  increases linearly from 0 to 0.3 over 200 epochs. Both ground truth and predictions are in normalized coordinates, avoiding distribution mismatch.

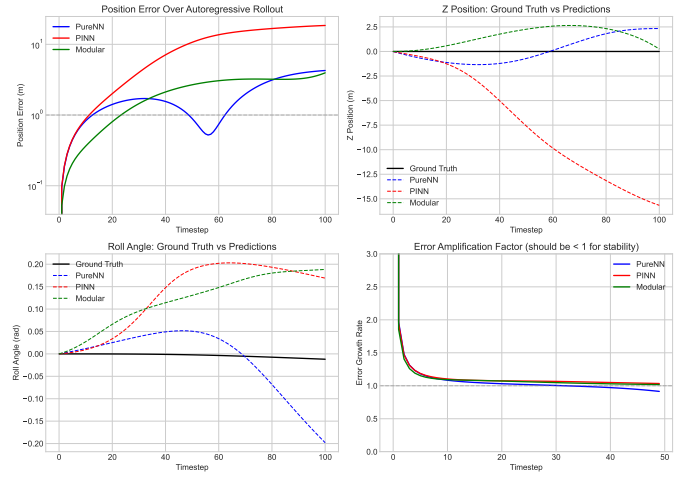


Fig. 3. Failure mode visualization: 100-step autoregressive rollout trajectories. PureNN (blue) tracks the ground truth closely; PINN (orange) diverges after  $\sim 40$  steps due to error amplification from  $\sigma_{\max} > 1$ .

TABLE VIII  
ARCHITECTURE PARAMETERS AND PERFORMANCE

Architecture	Params	100-Step MAE
Baseline	204,818	5.09m
<b>Modular</b>	<b>71,954</b>	<b>1.11m</b>
Fourier	302,354	5.09m
Curriculum	204,818	4.36m

### C. Physics-Consistent Regularization

Enforce energy conservation to maintain physical consistency:

$$\mathcal{L}_{\text{energy}} = \left( \frac{dE}{dt} - P_{\text{in}} + P_{\text{drag}} \right)^2 \quad (22)$$

### D. Results

Table VIII shows each component’s contribution to  $H_{0.1}$ .

## VII. DISCUSSION

### A. Implications for Control

The stability envelope directly determines MPC horizon feasibility. For a controller requiring  $K$ -step predictions with tolerance  $\epsilon$ :

- If  $H_\epsilon \geq K$ : Model is suitable
- If  $H_\epsilon < K$ : Model will cause constraint violations

Our framework enables principled model selection for control applications.

### B. Relationship to Prior Metrics

Existing metrics (single-step MSE, physics loss) measure *local* accuracy. The stability envelope measures *global* behavior under feedback—the regime that matters for control.

### C. Safety and Deployment Considerations

For real-world deployment, we recommend:

- **Error monitoring:** Track prediction error at runtime; trigger fallback if  $\|\hat{\mathbf{x}}_{t+k} - \mathbf{x}_{t+k}\| > \epsilon$ .
- **Safe fallback:** Maintain a simple linear controller (e.g., LQR) as backup when learned model diverges.
- **Domain detection:** Monitor if states exceed training bounds; switch to conservative controller if OOD.

### D. Limitations

The product bound in Theorem 2 can be loose—empirical  $\sigma_{\max}(\partial_{\mathbf{x}} g_{\phi})$  is often substantially smaller than  $\prod_i s_i$ . Our analysis uses empirical local Jacobian norms; proving finite uniform bounds analytically requires architecture-specific constraints. The correlation between  $L$  and  $H_{\epsilon}$  holds within training distribution but may not generalize to OOD conditions. Future work should enforce provable Lipschitz constraints via spectral normalization (Theorem 2) with per-layer budgets  $s_i = L_{\text{target}}^{1/L}$ .

## VIII. CONCLUSIONS

We introduced the stability envelope  $H_{\epsilon}$  as a formal metric for autoregressive stability in learned dynamics models, with theoretical bounds based on Lipschitz analysis. Our experiments on 6-DOF quadrotor dynamics revealed three key findings that clarify PINN training practices:

- 1) **Training regime dominates:** Early stopping criterion has larger effect than physics loss itself. Total-loss early stopping (supervised + physics) causes premature termination and poor rollout stability. Supervised-only early stopping should be used for autoregressive applications
- 2) **Physics loss can help:** Under fair comparison (same early stopping), physics loss ( $w=20$ ) improves 100-step stability by 38% (1.52m vs 2.47m). The effect is non-monotonic: intermediate weights hurt, high weights help
- 3) **Architecture provides additional gains:** Modular design achieves  $4.9\times$  better stability than parameter-matched monolithic baseline, independent of physics constraints

**Practical recommendations:** For autoregressive control applications, (1) use supervised-only early stopping even when training with physics loss; (2) tune physics weight—high values ( $w \geq 20$ ) can improve stability; (3) consider modular architectures that separate coupled subsystems. Future work includes hardware validation and systematic study of physics weight scheduling.

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