Linear Algebra and its applications

Assignment

1. Find the equation of the parabola $y = A + Bx + Cx^2$ that passes through three points $(1,1)$, $(2,-1)$ and $(3,1)$ using Gaussian Elimination

Answer

$$y = A + Bx + Cx^2$$

at $(1,1)$, $1 = A + B + C \leftarrow (1)$

at $(2,-1)$, $-1 = A + 2B + 4C \leftarrow (2)$

at $(3,1)$, $1 = A + 3B + 9C \leftarrow (3)$

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$ | $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$2C = 4$ $\quad$ $B + 3C = -2$ $\quad$ $A + B + C = 1$

$\boxed{C = 2}$ $\quad$ $\boxed{B = -8}$ $\quad$ $\boxed{A = 7}$

$$y = 7 + (-8)x + 2x^2$$

2. Find LU Decomposition for the matrix

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

nswer $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - (-5)R_1$
$R_4 \to R_4 - 5R_1$

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$R_3 \to R_3 - (-2)R_2$
$R_4 \to R_4 - (-2)R_2$

$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$R_4 \to R_4 - 3R_3$

$$U = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ -5 & -2 & 3 & 1 \end{bmatrix}$$

3. Let $T: R^3 \to R^3$ defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

i) Find the matrix T relative to the standard basis of R^3

ii) Find the basis for 4 Fundamental subspaces of T

iii) Find the Eigen values and Eigen vectors of T.

iv) Decompose $T = QR$

swer i) $T(1,0,0)=(1,0,1)$, $T(0,1,0)=(2,1,1)$, $T(0,0,1)=(-1,1,-2)$

$$T=\begin{bmatrix}1&2&-1\\0&1&1\\1&1&-2\end{bmatrix} \Rightarrow \quad R_3\to R_3-R_1 \quad \begin{bmatrix}1&2&-1\\0&1&1\\0&-1&-1\end{bmatrix} \Rightarrow$$

$$R_3\to R_3+R_2 \quad \begin{bmatrix}1&2&-1\\0&1&1\\0&0&0\end{bmatrix}$$

ii) Column ① and ② $\Rightarrow$ Pivots and So, they are Linear independent

$C(A)=\{(1,0,1),(2,1,1)\}$ $\quad$ $C(A^T)=\{(1,2,-1),(0,1,1)\}$

$\boxed{\dim(C(A))=2}$ $\quad$ $\boxed{\dim(C(A^T))=2}$

For $N(A)$ and $N(A^T)$, T has to be in row reduced form

$$T=\begin{bmatrix}1&2&-1\\0&1&1\\0&0&0\end{bmatrix} \Rightarrow \quad R_1\to R_1-2R_2 \quad \begin{array}{ccc}x&y&z\end{array} \begin{bmatrix}1&0&-3\\0&1&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&-3\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

z is free variable

$x-3z=0$ $\quad$ $y+z=0$

$\boxed{x=3z}$ $\quad$ $\boxed{y=-z}$

$$N(A)=z\begin{bmatrix}3\\-1\\1\end{bmatrix} \quad N(A)=\{(3,-1,1)\} \quad \dim(N(A))=1$$

$$T=\left[\begin{array}{ccc|c}1&2&-1&b_1\\0&1&1&b_2\\1&1&2&b_3\end{array}\right] \Rightarrow \quad R_3\to R_3-R_1 \quad \left[\begin{array}{ccc|c}1&2&-1&b_1\\0&1&1&b_2\\0&-1&-1&b_3-b_1\end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + b_2 \end{array}\right]$$

For consistency

$-b_1 + b_2 + b_3 = 0$

$N(A^T) = \{(-1, 1, 1)\}$ $\quad \dim(N(A^T)) = 1$

iii) Eigen Values and Eigen Vectors

$$\begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$(1-\lambda)\left([1-\lambda][-2-\lambda] - 1\right) + 1\left(2 + (1-\lambda)\right) = 0$$

$$(1-\lambda)^2 (2+\lambda) = 2$$

$$(1-\lambda)(1-\lambda)(2+\lambda) - 2 = 0$$

$$\lambda^3 = 3\lambda$$

$$\boxed{\lambda = \sqrt{3}, -\sqrt{3}, 0}$$

Eigen vector for $\lambda = \sqrt{3}$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{1-\sqrt{3}} R_1$$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 1 - \frac{2}{1-\sqrt{3}} & -(2+\sqrt{3}) + \frac{1}{1-\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -0.732 & 2 & -1 \\ 0 & -0.732 & 1 \\ 0 & 3.732 & -5.098 \end{bmatrix}$$

$$\begin{bmatrix} -0.732 & 2 & -1 \\ 0 & -0.732 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Ux = 0$$

$$-0.732x + 2y - z = 0$$

$$-0.732y + z = 0$$

$$\underline{y = 1.366z}$$

$$-0.732x = -1.7322z$$

$$\boxed{x = 2.366z}$$

Eigen Vector $= z \begin{bmatrix} 2.3664 \\ 1.366 \\ 1 \end{bmatrix}$

For $\lambda = \sqrt{3}$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2.732 & 2 & -1 \\ 0 & 2.732 & 1 \\ 1 & 1 & -0.2679 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2.732} R_1 \quad = \begin{bmatrix} 2.732 & 2 & -1 \\ 0 & 2.732 & 1 \\ 0 & 0.2679 & 0.0981 \end{bmatrix}$$

$$\begin{bmatrix} 2.732 & 2 & -1 \\ 0 & 2.732 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\underline{z=1}$, $\quad 2.732y + z = 0$ $\quad\quad 2.732x + 2y = z$

$2.732y = -z$ $\quad\quad x = \frac{z-2y}{2.732}$

$\underline{y = -0.3660z}$ $\quad\quad \underline{x = 0.63396}$

$$x_2 = \begin{bmatrix} 0.63396 \\ -0.3660 \\ 1 \end{bmatrix} \quad \text{if } \lambda=0, \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ From $N(A)$ solution obtained

iv) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\quad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad q_2 = \frac{l_2}{\|l_2\|} = \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \end{bmatrix}$

$q_3 = \begin{bmatrix} -2/7 \\ 3/7 \\ -6/7 \end{bmatrix}$

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$$Q = \begin{bmatrix} 1/\sqrt{2} & 0.5/\sqrt{1.5} & -2/7 \\ 0 & 1/\sqrt{1.5} & 3/7 \\ 1/\sqrt{2} & -0.5/\sqrt{1.5} & -6/7 \end{bmatrix} \quad R = \begin{bmatrix} 1.4142 & 2.1213 & -2.121 \\ 0 & 1.2249 & 1.2249 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Fit a best straight line $y = C + Dx$ for the following data using Least square principles

x	-4	1	2	3
y	4	6	10	8

Answer Let $y = mx + c$ $\quad A \Rightarrow n \times 2$

$$\downarrow$$
$$n \times 1$$

$$\underset{A}{\begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} c \\ m \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}}$$

$$\hat{x} = (A^T A)^{-1} A^T . B \Rightarrow \hat{x} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} . A^T.B$$

$$A^T A = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \qquad A^T B = \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}^T \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix} = \begin{bmatrix} 28 \\ 23+6 \end{bmatrix}_{2\times 1}$$

$$\hat{x} = (A^T A)^{-1} A^T B$$

$$= \begin{bmatrix} 15/58 & -1/58 \\ -1/58 & 1/29 \end{bmatrix} \qquad A^T B = \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 193/29 \\ 20/29 \end{bmatrix} \qquad m = \frac{20}{29}, \quad c = \frac{193}{29}$$

5. Find the projection matrices P and Q onto the plane $x_1 + x_2 + 3x_3 + 4x_5 = 0$ and its orthogonal complements

$$V = \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$\downarrow$ Pivot

$$x_1 = -x_2 + (-3)x_3 - 4x_5$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5\times 3}$$

$$P = A(A^T A)^{-1} . A^T$$

$$P = \begin{bmatrix} 25/27 & -1/27 & -1/9 & 0 & -4/27 \\ -1/27 & 26/27 & -1/9 & 0 & -4/27 \\ -1/9 & -1/9 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4/27 & -4/27 & -4/9 & 0 & 11/27 \end{bmatrix}$$

$$P + Q = I$$

$$Q = I - P = \begin{bmatrix} 1/27 & 1/27 & 1/9 & 0 & 4/27 \\ 1/27 & 1/27 & 1/9 & 0 & 4/27 \\ 1/9 & 1/9 & 1/3 & 0 & 4/9 \\ 0 & 0 & 0 & 1 & 0 \\ 4/27 & 4/27 & 4/9 & 0 & 16/27 \end{bmatrix}$$

6. For which range of number 'a', the matrix A is positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

Which 3x3 matrix (symmetric) B produces these function $f = x^T A x$?

where $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3)$

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Answer. $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

$|a| > 0 \Rightarrow a > 0$

$\begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} > 0$

$a^2 - 4 > 0$

$a^2 > 4$

$a > \pm 2$

$(a-2)(a+2) > 0$

$(a-2) > 0$

$a > 2$

$a+2 > 0$

$a > -2$

$|A| > 0$

$a(a^2-4) - 2(2a-4) + 2(4-2a) > 0$

$a(a-2)(a+2) - 4(a-2) + 4(2-a) > 0$

$a^3 - 12a + 16 > 0$

$2-a > 0$

$a < 2$

$a \in (2, \infty)$

$f = x^T A x$

$f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 + (-2)(x_2x_3)$

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

7. Find the SVD of A, $U \Sigma V^T$ where

$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$A^TA = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$

$|A - \lambda I| = 0$

$(81-\lambda)(9-\lambda) - 27^2 = 0$

$729 - 81\lambda - 9\lambda + \lambda^2 - 27 = 0$

λ_1, λ_2 of A^TA

$\lambda_1 = 90$

$\begin{bmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{bmatrix} = 0$

$\sqrt{\lambda_1} = 3\sqrt{10}$ $\quad \lambda_2 = 0$

$\Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\lambda_1 \text{ vector } x_1 = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$+9x + 27y = 0$$

$$27x + 81y = 0$$

$$y = \begin{bmatrix} -27/9 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

if $A = 0$, $x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

For v, $\lambda_1 = 90$ and $\lambda_2 = 0$, $V_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$V_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \qquad V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = V^T$$

Finding U. Eigen Values $\Rightarrow$ 90, 0, 0

$$U_1 = \frac{A.V_1}{\sigma_1} = \frac{1}{3\sqrt{10}} A.V = \begin{bmatrix} -0.266 \\ 0.533 \\ 0.533 \end{bmatrix}$$

$\sigma_2 = 0$, $(AA^T - 0.I)x = 0$

$$AA^T = x$$

$$\begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finding Null Space

$R_2 \rightarrow R_2 + 2R_1$

$$\begin{bmatrix} 10 & -20 & -20 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$10x - 20y - 20z = 0$$

$$x - 2y - 2z = 0$$

$$x = 2y + 2z$$

$$x = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

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For $\lambda=0$, $u = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}-0.266 & 0.8944 & a\\ 0.533 & 0.4472 & b\\ 0.533 & 0 & c\end{bmatrix}$

U is orthogonal to $[U_3 \perp U_2]$

$-0.266a + 0.533b + 0.533c = 0$

$\underline{a = 0.4}$, $\underline{b = 0.8}$, $\underline{c = 1}$

$V_3 = \frac{1}{\sqrt{1.8}}\begin{bmatrix}0.4\\0.8\\1\end{bmatrix}$

$$A = \underbrace{\begin{bmatrix}1/3 & 2/\sqrt{5} & 0.4/\sqrt{1.8}\\ -2/3 & 1/\sqrt{5} & 0.8/\sqrt{1.8}\\ -2/3 & 0 & 1/\sqrt{1.8}\end{bmatrix}}_{u}\underbrace{\begin{bmatrix}\sqrt{90} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}-3/\sqrt{10} & 1/\sqrt{10}\\ 1/\sqrt{10} & 3/\sqrt{10}\end{bmatrix}}_{V^T}$$