Question 41.2023

EE22BTECH11051

Question:

Suppose that X_1,X_2, ...,X_{10} are independen and identically distributed random vectors each having $N_2(\mu, \Sigma)$ distribution, where Σ is non-singular. If

$$\text{U} = \frac{1}{1 + (\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu)}, \tag{1}$$

where $\bar{X} = \frac{1}{10} \Sigma_{i=1}^{10} X_i$ then the value of $\log_e P(U \leq \frac{1}{2})$ equals

1) -5
2) -10
3) -2
4) -1

Solution:

It has been given that:

$$X_i \sim N_2(\mu, \Sigma) \tag{2}$$

$$\tag{3}$$

The distribution of $\bar{X}$ can be given as; The mean of $\bar{X}$ will be the average of the means of X1, X2, ..., X10, which is:

$$\mu_{\bar{X}} = \frac{\mu + \mu + \mu + ... + \mu}{10} = \frac{10\mu}{10} = \mu \tag{4}$$

And since the distributions X_1,X_2, ...,X_{10} are independent, the covariance between them is zero. Hence the new covariance is given as

$$\Sigma_{\bar{X}} = \frac{\Sigma}{10} \tag{5}$$

hence;

$$\bar{X} \sim N_2\left(\mu, \frac{\Sigma}{n}\right) \tag{6}$$

Now;

$$\frac{(\bar{X} - \mu)^2}{\left(\frac{\Sigma}{n}\right)} \sim \chi_2^2 \tag{7}$$

we can now write this as;

$$n(\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu) \sim \chi_2^2 \tag{8}$$

Let Y be;

$$Y = n(\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu) \tag{9}$$

Now;

$$\begin{aligned}
P\left(U \leq \frac{1}{2}\right) &= P\left(\frac{1}{1 + \frac{Y}{n}} \leq \frac{1}{2}\right) && (10)\\
&= P\left(\frac{Y}{n} \geq 1\right) && (11)\\
&= P(Y \geq 10) && (12)\\
&= \int_{10}^{\infty} \frac{1}{2} e^{-\frac{y}{2}}\, dy && (13)\\
&= e^{-5} && (14)
\end{aligned}$$

Hence;

$$\log_e P(U \leq \frac{1}{2}) = -5 \tag{15}$$