Question 9.3.2

EE22BTECH11051 - Sreekar Cheela

There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Solution:

Gaussian Distribution

Parameter	Values	Description
n	10	Number of articles
p	0.05	Probability of being defective
Y	$0 \leq Y \leq 10$	Number of defective elements
$\mu = np$	0.5	mean
$\sigma = \sqrt{np(1-p)}$	0.475	standard deviation

1) Central limit theorm:

$$Y \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \tag{1}$$

$$\tag{2}$$

Due to continuity correction $\Pr(X = x)$ can be approximated using gaussian distribution as

$$p_Y(x) \approx \Pr(x - 0.05 < Y < x + 0.05) \tag{3}$$
$$\approx \Pr(Y < x + 0.05) - \Pr(Y < x - 0.05) \tag{4}$$
$$\approx F_Y(x + 0.05) - F_Y(x - 0.05) \tag{5}$$

Now, the CDF of Y can be found by;

$$F_Y(y) = \Pr(Y \leq y) \tag{6}$$
$$= p_Y(y) \tag{7}$$

We also know that;

$$Q(x) = \Pr(X > x), x > 0, X \sim N(0,1) \tag{8}$$
$$Q(-x) = \Pr(X > -x), x < 0, X \sim N(0,1) \tag{9}$$
$$= 1 - Q(x) \tag{10}$$

Hence, the CDF is given as:

$$F_Y(y) = \begin{cases} 1 - Q\left(\frac{y-\mu}{\sigma}\right), & \text{if } y > \mu \\ 1 - Q\left(\frac{y-\mu}{\sigma}\right) = Q\left(\frac{\mu-y}{\sigma}\right), & \text{if } y < \mu \end{cases} \tag{11}$$

Now, we get:

$$F_Y(1) = p_Y(1.05) \tag{12}$$
$$= 1 - Q\left(\frac{1.05 - 0.5}{\sqrt{0.05}}\right) \tag{13}$$
$$= 1 - Q\left(\frac{0.55}{0.2236}\right) \tag{14}$$
$$= 1 - Q(2.4596) \tag{15}$$
$$= 0.99304 \tag{16}$$

2) Binomial Distribution:

$$n = 10; p = \frac{1}{20} \tag{17}$$

Pmf of X for $0 \le k \le 10$ is

$$p_X(k) = {}^nC_k p^k (1-p)^{n-k} \tag{18}$$

Then the probability is given as:

$$p_X(0) + p_X(1) = {}^{10}C_0 \left(\frac{1}{20}\right)^0 \left(1 - \frac{1}{20}\right)^{10} + {}^{10}C_1 \left(\frac{1}{20}\right)^1 \left(1 - \frac{1}{20}\right)^9 \tag{19}$$

Hence we get;

$$p_X(0) + p_X(1) = 29\left(\frac{19^9}{20^{10}}\right) = 0.91386 \tag{20}$$

Hence we can say probability calculated through central limit theorem is very close to the one calculated through binomial distribution.

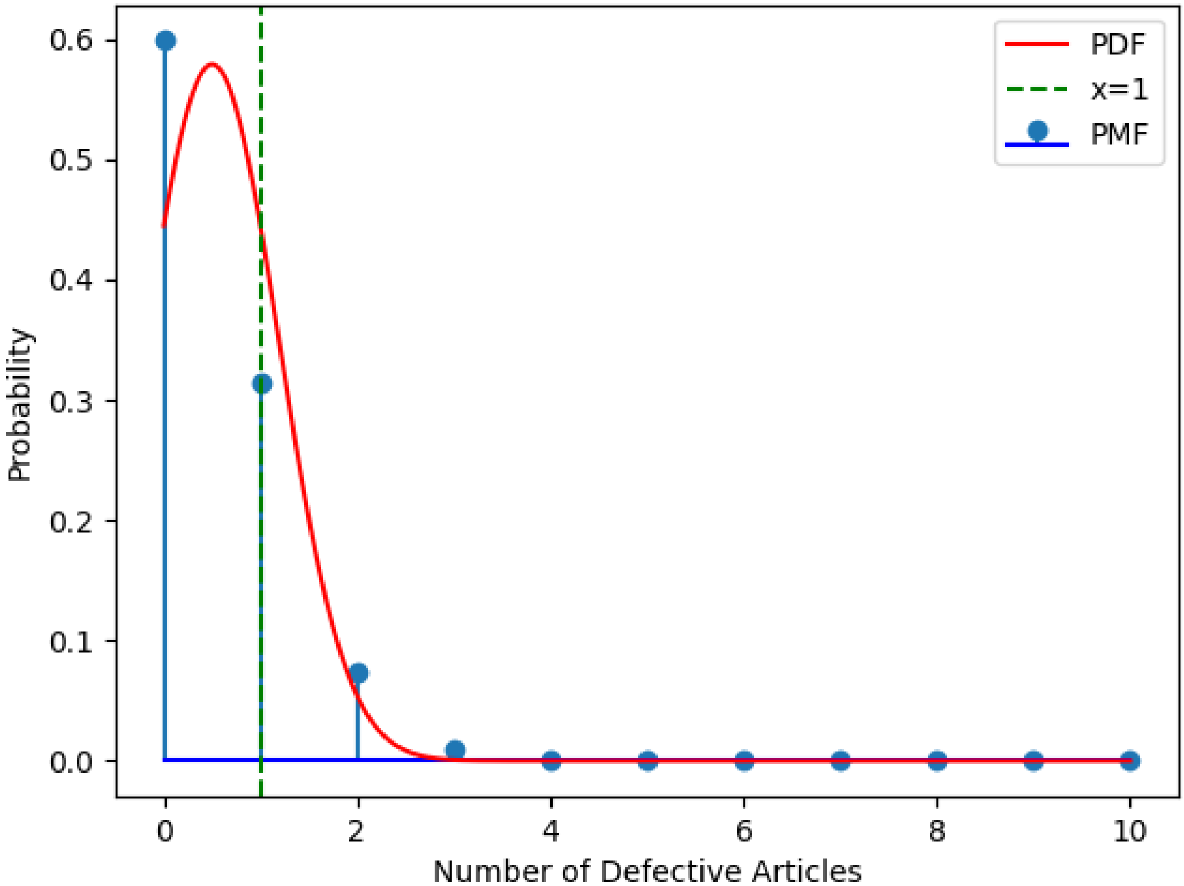

Fig. 1: Binomial vs Gaussian