Question 41.2023

EE22BTECH11051

Question:

Suppose that X_1,X_2, ...,X_{10} are independen and identically distributed random vectors each having $N_2(\mu, \Sigma)$ distribution, where Σ is non-singular. If

$$U = \frac{1}{1 + (\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu)}, \tag{1}$$

where $\bar{X} = \frac{1}{10}\Sigma_{i=1}^{10} X_i$ then the value of $\log_e \Pr\left(U \leq \frac{1}{2}\right)$ equals

1) -5
2) -10
3) -2
4) -1

GATE ST 2023

Solution:

We are given a bivariate distribution;

$$X_i \sim N_2(\mu, \Sigma) \tag{2}$$

$$\tag{3}$$

The distribution of $\bar{X}$ can be given as:

The mean of $\bar{X}$ will be the average of the means of X1, X2, ..., X10, which is:

$$\mu_{\bar{X}} = \frac{\mu + \mu + \mu + ... + \mu}{10} = \frac{10\mu}{10} = \mu \tag{4}$$

And since the distributions X_1,X_2, ...,X_{10} are independent, the covariance between them is zero. Hence we can find the new covariance as:

$$\text{Cov}(\bar{X}) = E[(\bar{X} - \mu)(\bar{X} - \mu)^T] \tag{5}$$

$$= E[(\frac{1}{10}\Sigma_{i=1}^{10} X_i - \mu)(\frac{1}{10}\Sigma_{j=1}^{10} X_j - \mu)^T] \tag{6}$$

$$= E[\frac{1}{100}\Sigma_{i=1}^{10}\Sigma_{j=1}^{10}(X_i - \mu)(X_j - \mu)^T] \tag{7}$$

$$= \frac{1}{100}\Sigma_{i=1}^{10}\Sigma_{j=1}^{10} E[(X_i - \mu)(X_j - \mu)^T] \tag{8}$$

Since X_1,X_2, ...,X_{10} are independen and identically distributed random vectors, we get the covariance term $E[(X_i - \mu)(X_j - \mu)^T]$ to be Σ hence;

$$\mathrm{Cov}(\bar{X}) = \frac{1}{100} * 10 * \Sigma \tag{9}$$

$$\bar{X} \sim N_2\left(\mu, \frac{\Sigma}{n}\right) \tag{10}$$

Now let us convert this normal distribution into χ_2^2 distrribution; where χ_k^2 distribution is given as;

$$f(x) = \frac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} \quad x \geq 0 \tag{11}$$

$$\tag{12}$$

hence we get;

$$\frac{(\bar{X} - \mu)^T(\bar{X} - \mu)}{\left(\frac{\Sigma}{n}\right)} \sim \chi_2^2 \tag{13}$$

we can now write this as;

$$n(\bar{X} - \mu)^T\Sigma^{-1}(\bar{X} - \mu) \sim \chi_2^2 \tag{14}$$

Let Y be;

$$Y = n(\bar{X} - \mu)^T\Sigma^{-1}(\bar{X} - \mu) \tag{15}$$

which follows χ_2^2 distrribution
Now;

$$\Pr\left(U \leq \frac{1}{2}\right) = \Pr\left(\frac{1}{1 + \frac{Y}{n}} \leq \frac{1}{2}\right) \tag{16}$$

$$= \Pr\left(\frac{Y}{n} \geq 1\right) \tag{17}$$

$$= \Pr\left(Y \geq 10\right) \tag{18}$$

Since $k = 2$ for Y, we get;

$$= \int_{10}^{\infty} \frac{1}{2} e^{-\frac{y}{2}}\, dy \tag{19}$$

$$= e^{-5} \tag{20}$$

Hence;

$$\log_e \Pr\left(U \leq \frac{1}{2}\right) = -5 \tag{21}$$

Simulation Steps:

1) Set the number of simulations
2) Run a loop for number of simulations specified
3) Generate random vectors from a bivariate normal distribution
4) Calculate the sample mean and set the covariance matrix
5) Using the given formula, calculate U for each simulation
6) Calculate the simulated probability adn take the natural log of that value

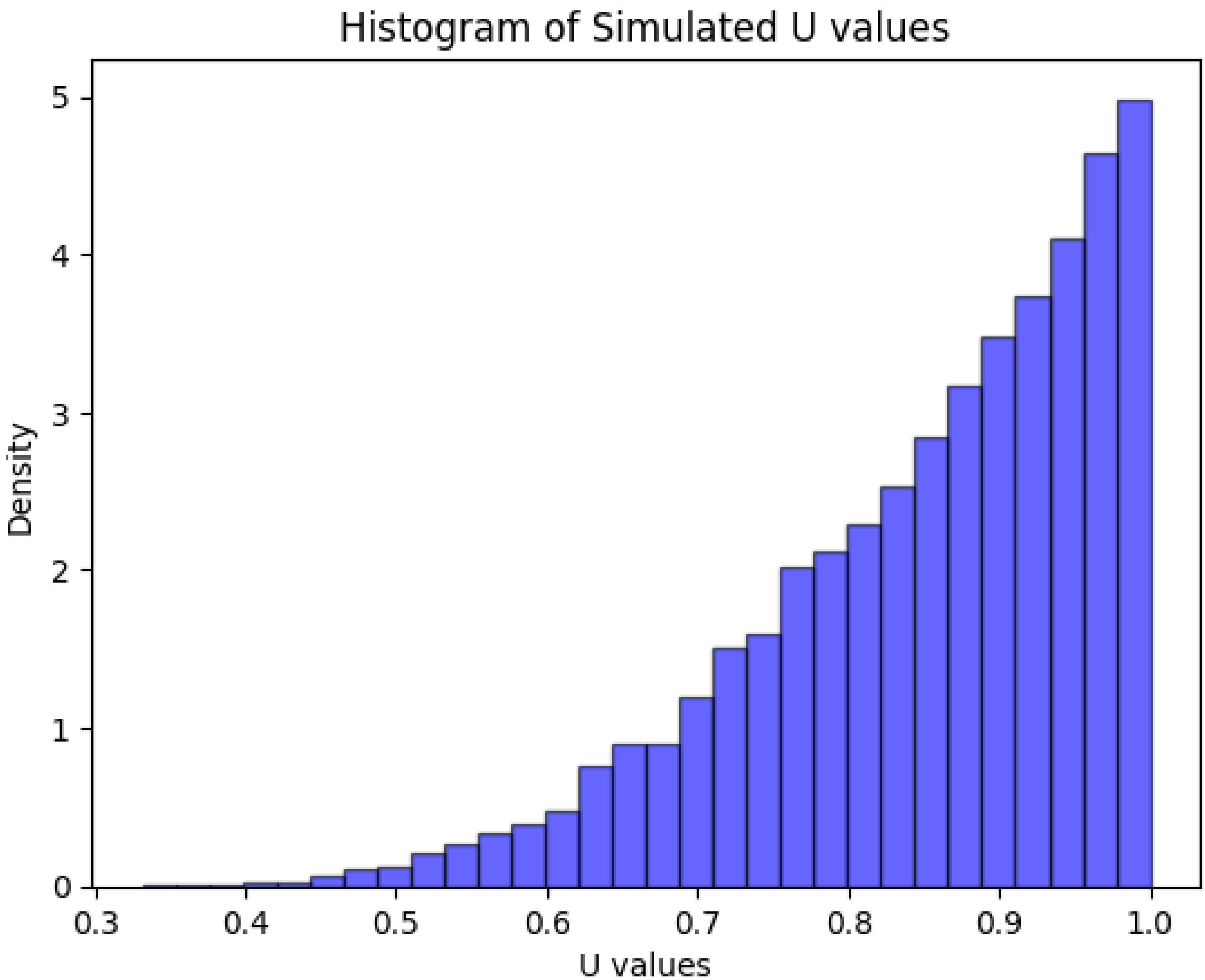

Fig. 1. Plot of $p_X(n)$.Simulations are close to the analysis.