

Question 41.2023

EE22BTECH11051

Question:

Suppose that X_1, X_2, \dots, X_{10} are independent and identically distributed random vectors each having $N_2(\mu, \Sigma)$ distribution, where Σ is non-singular. If

$$U = \frac{1}{1 + (\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu)}, \quad (1)$$

where $\bar{X} = \frac{1}{10} \sum_{i=1}^{10} X_i$ then the value of $\log_e \Pr(U \leq \frac{1}{2})$ equals

- 1) -5
- 2) -10
- 3) -2
- 4) -1

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Solution:

We are given a bivariate distribution;

$$X_i \sim N_2(\mu, \Sigma) \quad (2)$$

(3)

The distribution of \bar{X} can be given as:

The mean of \bar{X} will be the average of the means of X_1, X_2, \dots, X_{10} , which is:

$$\mu_{\bar{X}} = \frac{\mu + \mu + \mu + \dots + \mu}{10} = \frac{10\mu}{10} = \mu \quad (4)$$

And since the distributions X_1, X_2, \dots, X_{10} are independent, the covariance between them is zero. Hence we can find the new covariance as:

$$\Sigma_{\sum_{i=1}^{10} X_i} = \sum_{i=1}^{10} \Sigma_{X_i} = 10\Sigma \quad (5)$$

$$\Sigma_{\bar{X}} = \frac{1}{100} \Sigma_{\sum_{i=1}^{10} X_i} \quad (6)$$

$$\Sigma_{\bar{X}} = \frac{\Sigma}{10} \quad (7)$$

hence;

$$\bar{X} \sim N_2\left(\mu, \frac{\Sigma}{n}\right) \quad (8)$$

Now let us convert this normal distribution into χ^2_2 distribution; where χ^2_k distribution is given as;

$$f(x) = \frac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} \quad x \geq 0 \quad (9)$$

(10)

hence we get;

$$\frac{(\bar{X} - \mu)^T (\bar{X} - \mu)}{\left(\frac{\Sigma}{n}\right)} \sim \chi^2_2 \quad (11)$$

we can now write this as;

$$n(\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu) \sim \chi^2_2 \quad (12)$$

Let Y be;

$$Y = n(\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu) \quad (13)$$

which follows χ^2_2 distribution

Now;

$$\Pr\left(U \leq \frac{1}{2}\right) = \Pr\left(\frac{1}{1 + \frac{Y}{n}} \leq \frac{1}{2}\right) \quad (14)$$

$$= \Pr\left(\frac{Y}{n} \geq 1\right) \quad (15)$$

$$= \Pr(Y \geq 10) \quad (16)$$

Since $k = 2$ for Y, we get;

$$= \int_{10}^{\infty} \frac{1}{2} e^{-\frac{y}{2}} dy \quad (17)$$

$$= e^{-5} \quad (18)$$

Hence;

$$\log_e \Pr\left(U \leq \frac{1}{2}\right) = -5 \quad (19)$$

Simulation Steps:

- 1) Set the number of simulations
- 2) Run a loop for number of simulations specified
- 3) Generate random vectors from a bivariate normal distribution
- 4) Calculate the sample mean and set the covariance matrix
- 5) Using the given formula, calculate U for each simulation
- 6) Calculate the simulated probability and take the natural log of that value

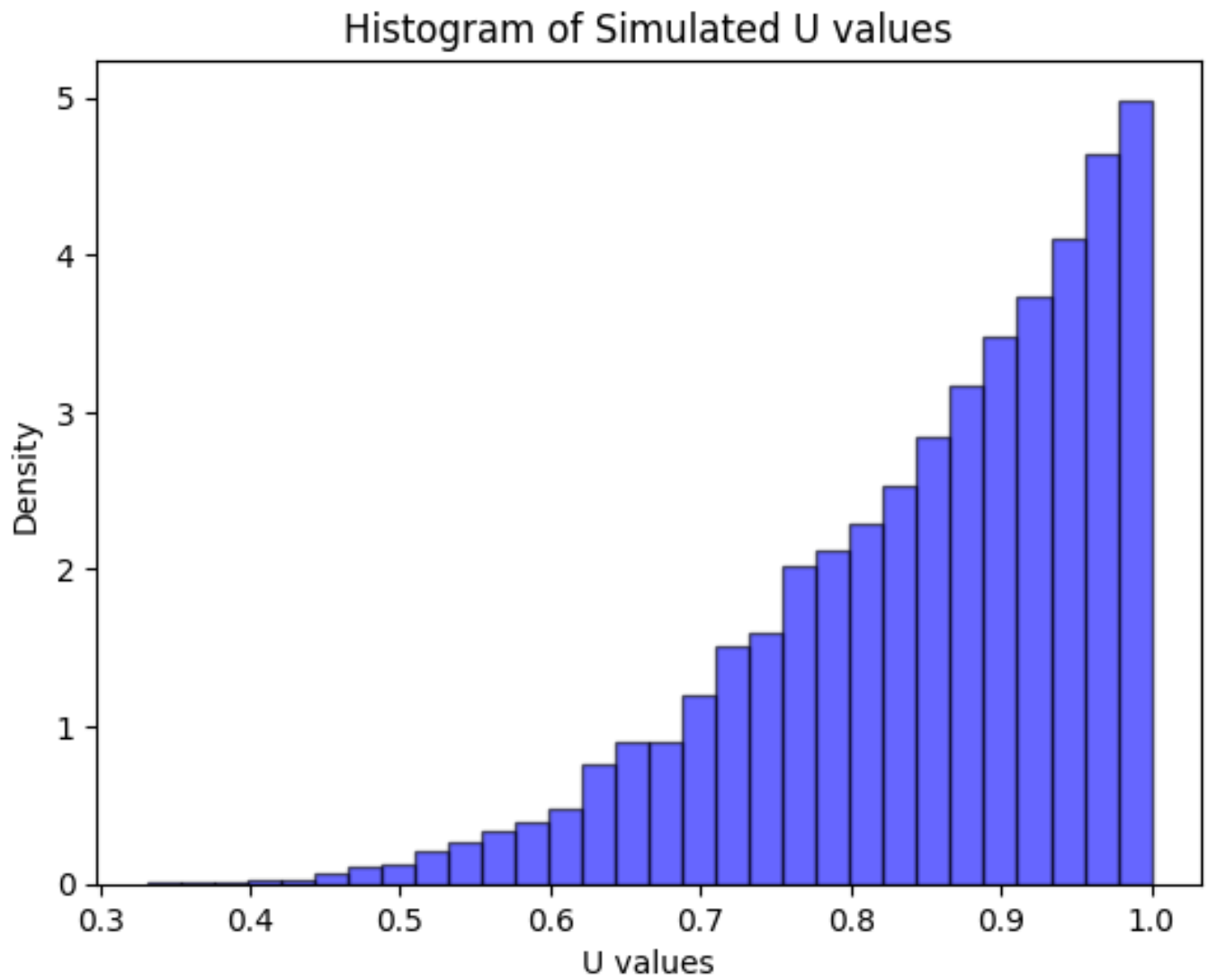


Fig. 1. Plot of $p_X(n)$. Simulations are close to the analysis.