Question 41.2023

EE22BTECH11051

Question:
Suppose that X_1,X_2, ...,X_{10} are independen and identically distributed random vectors each having $N_2(\mu, \Sigma)$ distribution, where Σ is non-singular. If

$$U = \frac{1}{1 + (\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu)}, \tag{1}$$

where $\bar{X} = \frac{1}{10}\Sigma_{i=1}^{10} X_i$ then the value of $\log_e P(U \leq \frac{1}{2})$ equals

1) -5
2) -10
3) -2
4) -1

Solution:
We are given a bivariate distribution;

$$X_i \sim N_2(\mu, \Sigma) \tag{2}$$

$$\tag{3}$$

The distribution of $\bar{X}$ can be given as:

The mean of $\bar{X}$ will be the average of the means of X1, X2, ..., X10, which is:

$$\mu_{\bar{X}} = \frac{\mu + \mu + \mu + ... + \mu}{10} = \frac{10\mu}{10} = \mu \tag{4}$$

And since the distributions X_1,X_2, ...,X_{10} are independent, the covariance between them is zero. Hence the new covariance is given as

$$\Sigma_{\bar{X}} = \frac{\Sigma}{10} \tag{5}$$

hence;

$$\bar{X} \sim N_2\left(\mu, \frac{\Sigma}{n}\right) \tag{6}$$

Now let us convert this normal distribution into χ_2^2 distrribution;

$$\frac{(\bar{X} - \mu)^2}{\left(\frac{\Sigma}{n}\right)} \sim \chi_2^2 \tag{7}$$

we can now write this as;

$$n(\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu) \sim \chi_2^2 \tag{8}$$

Let Y be;

$$Y = n(\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu) \tag{9}$$

which follows χ_2^2 distrribution
Now;

$$P\left(U \leq \frac{1}{2}\right) = P\left(\frac{1}{1 + \frac{Y}{n}} \leq \frac{1}{2}\right) \tag{10}$$

$$= P\left(\frac{Y}{n} \geq 1\right) \tag{11}$$

$$= P(Y \geq 10) \tag{12}$$

χ_k^2 distribution is given as;

$$f(x) = \frac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} \quad x \geq 0 \tag{13}$$

$$\tag{14}$$

Since $k = 2$, we get;

$$= \int_{10}^{\infty} \frac{1}{2} e^{-\frac{y}{2}} \, dy \tag{15}$$

$$= e^{-5} \tag{16}$$

Hence;

$$\log_e P(U \leq \frac{1}{2}) = -5 \tag{17}$$