Question 41.2023

EE22BTECH11051

Question:

Suppose that X_1,X_2, ...,X_{10} are independen and identically distributed random vectors each having $N_2(\mu, \Sigma)$ distribution, where Σ is non-singular. If

$$\text{U} = \frac{1}{1 + (\bar{X} - \mu)^T \Sigma^{-1} (\bar{X} - \mu)}, \tag{1}$$

where $\bar{X} = \frac{1}{10} \Sigma_{i=1}^{10} X_i$ then the value of $\log_e \Pr\left(U \leq \frac{1}{2}\right)$ equals

1) -5
2) -10
3) -2
4) -1

Solution:

We are given a bivariate distribution;

$$X_i \sim N_2(\mu, \Sigma) \tag{2}$$

$$\tag{3}$$

The distribution of $\bar{X}$ can be given as:

The mean of $\bar{X}$ will be the average of the means of X1, X2, ..., X10, which is:

$$\mu_{\bar{X}} = \frac{\mu + \mu + \mu + ... + \mu}{10} = \frac{10\mu}{10} = \mu \tag{4}$$

And since the distributions X_1,X_2, ...,X_{10} are independent, the covariance between them is zero. Hence the new covariance is given as

$$\Sigma_{\bar{X}} = \frac{\Sigma}{10} \tag{5}$$

hence;

$$\bar{X} \sim N_2\left(\mu, \frac{\Sigma}{n}\right) \tag{6}$$

Now let us convert this normal distribution into χ^2_2 distrribution; where χ^2_k distribution is given as;

$$f(x) = \frac{x^{\frac{k}{2}-1}}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} \quad x \geq 0 \tag{7}$$

$$\tag{8}$$

hence we get;

$$\frac{(\bar{X}-\mu)^2}{\left(\frac{\Sigma}{n}\right)} \sim \chi_2^2 \tag{9}$$

we can now write this as;

$$n(\bar{X}-\mu)^T\Sigma^{-1}(\bar{X}-\mu) \sim \chi_2^2 \tag{10}$$

Let Y be;

$$Y = n(\bar{X}-\mu)^T\Sigma^{-1}(\bar{X}-\mu) \tag{11}$$

which follows χ_2^2 distrribution
Now;

$$\Pr\left(U \leq \frac{1}{2}\right) = \Pr\left(\frac{1}{1+\frac{Y}{n}} \leq \frac{1}{2}\right) \tag{12}$$

$$= \Pr\left(\frac{Y}{n} \geq 1\right) \tag{13}$$

$$= \Pr\left(Y \geq 10\right) \tag{14}$$

Since $k = 2$ for Y, we get;

$$= \int_{10}^{\infty} \frac{1}{2} e^{-\frac{y}{2}}\, dy \tag{15}$$

$$= e^{-5} \tag{16}$$

Hence;

$$\log_e \Pr\left(U \leq \frac{1}{2}\right) = -5 \tag{17}$$