EE5601: Representation Learning

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(1) Given $X \in \mathbb{R}^{d \times N}$ where each is assumed to be a zero mean.

let $Y = PX$ be the linear transformation and P be the optimal linear transformation matrix.

$P \in \mathbb{R}^{d \times d}$.

$$\text{Covariance}(Y) = \text{Cov}(Y) = \frac{1}{N} \cdot Y \cdot Y^T$$

$$\Rightarrow \text{Cov}(Y) = \frac{1}{N} \cdot Y \cdot Y^T$$

$$= \frac{1}{N} \cdot (PX) \cdot (PX)^T$$

$$= \frac{1}{N} \cdot (P \cdot X) \cdot X^T \cdot P^T = \frac{1}{N} P \cdot X \cdot X^T \cdot P^T$$

$$\left[\text{Covariance}(X) = \text{Cov}(X) = \frac{1}{N} \cdot X \cdot X^T \right]$$

$$\Rightarrow \text{Cov}(Y) = P \cdot \text{Cov}(X) \cdot P^T \quad - (1)$$

Now,

$Cov(X)$ is symmetric so we can represent it uniquely as $Cov(X) = E.D.E^T$ — ②

Eigen Vectors (Orthogonal Matrix)   Diagonal Matrix with Eigen Values

Substitute ② in ①

$\Rightarrow Cov(Y) = P.E.D.E^T.P^T$

For $Cov(Y) = D$ i.e a diagonal matrix

P should be E^T $\Rightarrow P.E = E^T.P^T = I$ [From ②]

So, $P = E^T$

$\therefore$ The optimal linear decorrelating transformation for X

is $\boxed{Y = E^T.X}$

(2) $$L(X;\theta) = \sum_{n=1}^{N} \log\left[\sum_{k=1}^{K} \pi_k \mathcal{N}(x_n; \mu_k, \Sigma_k)\right]$$

posterior probability : $$\gamma(z_k) = \frac{\pi_k \cdot \mathcal{N}(x; \mu_k; \Sigma_k)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(x_n; \mu_k, \Sigma_k)}$$

$\underline{\underline{\mu_k:}}$

$$\frac{\partial L(X;\theta)}{\partial \mu_k} = 0$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\partial}{\partial \mu_k}\left[\log\left[\sum_{k=1}^{K} \pi_k \mathcal{N}(x_n; \mu_k, \Sigma_k)\right]\right] = 0$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\frac{\partial[\pi_k \cdot \mathcal{N}(x_n; \mu_k, \Sigma_k)]}{\partial \mu_k}}{\sum_{k=1}^{K} \pi_k \mathcal{N}(x_n; \mu_k; \Sigma_k)} = 0 \quad - (1)$$

$$\rightsquigarrow \mathcal{N}(x_n; \mu_k, \Sigma_k) = \frac{1}{\sqrt{(2\pi)^d |\Sigma_k|}} \cdot e^{-\frac{1}{2}(x-\mu_k)^T \cdot \Sigma_k^{-1}(x-\mu_k)}$$

$$\frac{\partial}{\partial \mu_k} \mathcal{N}(x_n; \mu_k, \Sigma_k) = \mathcal{N}(x_n; \mu_k, \Sigma_k) \cdot \frac{\partial}{\partial \mu_k}\left[-\frac{1}{2}(x_n-\mu_k)^T \cdot \Sigma_k^{-1} \cdot (x_n-\mu_k)\right]$$

$$-\frac{1}{2}(x-\mu_k)^T \cdot \Sigma_k^{-1} \cdot (x-\mu_k) = -\frac{1}{2}\sum_{i=1}^{d} \frac{(x_i-\mu_{ki})^2}{\sigma_{ki}^2}$$

$$\Rightarrow \frac{\partial}{\partial \mu_k} \mathcal{N}(x_n; \mu_k, \Sigma_k) = \mathcal{N}(x_n; \mu_k, \Sigma_k) \cdot \sum_{i=1}^{d} \frac{(x_i-\mu_{ki})}{\sigma_{ki}^2}$$

$$= \mathcal{N}(x_n; \mu_k, \Sigma_k) \cdot \Sigma_k^{-1} \cdot (x_n-\mu_k)$$

Substitute in ①

$$\Rightarrow \sum_{n=1}^{N} \underbrace{\frac{\pi_k \mathcal{N}(x_n; \mu_k, \Sigma_k)}{\sum_{k=1}^{K} \pi_k \cdot \mathcal{N}(x_n; \mu_k, \Sigma_k)}}_{\gamma(z_{nk})} \Sigma_k^{-1} \cdot (x_n-\mu_k) = 0$$

$$\Rightarrow \sum_{n=1}^{N} \gamma(z_{nk}) \cdot \Sigma_k^{-1} \cdot (x_n-\mu_k) = 0$$

$$\Rightarrow \sum_{n=1}^{N} \gamma(z_{nk}) \cdot x_n = \sum_{n=1}^{N} \gamma(z_{nk}) \cdot \mu_k$$

$$\Rightarrow \boxed{\mu_k = \frac{\sum_{n=1}^{N} \gamma(z_{nk}) \cdot x_n}{\sum_{n=1}^{N} \gamma(z_{nk})}}$$

Σ_K :

$$\frac{\partial}{\partial \Sigma_K} L(X; \theta) = 0$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\partial}{\partial \Sigma_K} \left[\log \left[\sum_{K=1}^{K} \pi_K . \mathcal{N}(x_n ; \mu_K, \Sigma_K) \right] \right] = 0$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\frac{\partial}{\partial \Sigma_K} \left(\pi_K . \mathcal{N}(x_n ; \mu_K, \Sigma_K) \right)}{\sum_{K=1}^{K} \pi_K . \mathcal{N}(x_n ; \mu_K, \Sigma_K)} = 0 \quad - (1)$$

Here we take a particular dimension (or) 1 dim-ension and generalise it to d-dimensional

$$\Rightarrow \mathcal{N}(x_n ; \mu_K, \sigma_K) = \frac{1}{\sqrt{(2\pi)^d} \, \sigma_K} . e^{-\frac{1}{2}(x-\mu) . \frac{1}{\sigma^2} . (x-\mu)}$$

$$\frac{\partial}{\partial \sigma_K} \mathcal{N}(x_n ; \mu_K, \sigma_K) = \mathcal{N}(x_n ; \mu_K, \sigma_K) \left[\frac{-1}{\sigma_K} + \frac{(x_n - \mu_K)^2}{\sigma_K^3} \right] - (2)$$

Substitute (2) in (1) :

$$\Rightarrow \sum_{n=1}^{N} \frac{\pi_K . \mathcal{N}(x_n ; \mu_K, \sigma_K)}{\sum_{K=1}^{K} \pi_K . \mathcal{N}(x_n ; \mu_K, \sigma_K)} \left[\frac{-1}{\sigma_K} + \frac{(x_n - \mu_K)^2}{\sigma_K^3} \right] = 0$$

$\gamma(z_{nk})$

$$\Rightarrow \sum_{n=1}^{N} \gamma(z_{nk}) \cdot \left[-\sigma_k^2 + (x_n - \mu_k)^2 \right] = 0$$

$$\Rightarrow \sigma_k^2 = \frac{\sum_{n=1}^{N} \gamma(z_{nk}) \cdot (x_n - \mu_k)^2}{\sum_{n=1}^{N} \gamma(z_{nk})}$$

So by generalising it to a matrix space we can write it as:

$$\therefore \boxed{\Sigma_k = \frac{\sum_{n=1}^{N} \gamma(z_{nk}) \cdot (x_n - \mu_k) \cdot (x_n - \mu_k)^T}{\sum_{n=1}^{N} \gamma(z_{nk})}}$$

$\underline{\underline{\pi_k :}}$

While maximising $L(X,\theta)$ w.r.t π_k we must consider the constraint $\sum_{k=1}^{K} \pi_k = 1$

$$\Rightarrow \cancel{L(X,\theta)} \text{ Maximise } \left[L(X,\theta) + \lambda \left(\sum_{k=1}^{K} \pi_k - 1 \right) \right] = M$$

$$\Rightarrow \frac{\partial M}{\partial \pi_k} = 0$$

$$\Rightarrow \frac{\partial L(X,\theta)}{\partial \pi_k} + \lambda = 0$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\frac{\partial}{\partial \pi_k}\left[\pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)\right]}{\sum_{k=1}^{K} \pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)} + \lambda = 0$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\cancel{\pi_k}\, \mathcal{N}(x_n ; \mu_k, \Sigma_k)}{\sum_{k=1}^{K} \pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)} + \lambda = 0 \quad - (1)$$

Multiply π_k on both sides and sum over K

$$\Rightarrow \sum_{n=1}^{N} \frac{\sum_{k=1}^{K} \pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)}{\sum_{k=1}^{K} \pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)} + \lambda \sum_{k=1}^{K} \pi_k = 0$$

$$\Rightarrow \sum_{n=1}^{N} 1 + \lambda \underbrace{\sum_{k=1}^{K} \pi_k}_{1} = 0 \Rightarrow \lambda = -N \quad - (2)$$

Substitute (2) in (1)

$$\Rightarrow \sum_{n=1}^{N} \frac{\mathcal{N}(x_n ; \mu_k, \Sigma_k)}{\sum_{k=1}^{K} \pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)} = N$$

Multiply with π_k on both sides

$$\Rightarrow \sum_{n=1}^{N} \underbrace{\frac{\pi_k \mathcal{N}(x_n ; \mu_k, \Sigma_k)}{\sum_{k=1}^{K} \pi_k . \mathcal{N}(x_n ; \mu_k, \Sigma_k)}}_{\gamma(n_k)} = N . \pi_k$$

$$\Rightarrow \therefore \boxed{\pi_k = \frac{\sum_{n=1}^{N} \gamma(n_k)}{N}}$$