Probabilistic Graphical Models

C. Sai Sreenivas
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HW0

1. Undirected Graphs:

$$p(x_1, x_2, \ldots x_N) = \frac{1}{Z} \prod_{c \in C} \psi_c(x_c)$$

↳ clique

ψ_c: compatability function

Z: Normalization factor

C: Set of maximal clique of graph

Sum-Product Algorithm:

$$\mu_s(x_s) = K \cdot \psi_s(x_s) \cdot \prod_{t \in N(s)} M^{*}_{ts}(x_s)$$

$$M_{ts}(x_s) = \sum_{x_{V_t}} \psi_{st}(x_s, x_t) \cdot p(x_{V_t}; T_t)$$

Consider a tree $G = (V, E)$ the joint distribution of the RVs on the vertices can be factorized as:

$$p(x_1, x_2, \ldots x_n) = \frac{1}{Z} \prod_{u \in V} \psi_u(x_u) \cdot \prod_{(s,t) \in E} \psi_{st}(x_s, x_t)$$

$$\mu_s(x_s) = \sum_{x_1, \ldots x_{s-1}, x_{s+1}, \ldots x_n} p(x_1, \ldots x_n)$$

Now,

$$V = \{s\} \cup \{ \bigcup_{t \in N(s)} V_t \}$$

$$E = \{ \bigcup_{t \in N(s)} (s,t) \} \cup \{ \bigcup_{t \in N} E_t \}$$

$$p(x_1, \ldots x_n) = \frac{1}{Z} \prod_{u \in V} \psi_u(x_u) \cdot \prod_{(s,t) \in E} \psi_{st}(x_s, x_t)$$

$$= \frac{1}{Z} \cdot \psi_s(x_s) \cdot \prod_{u \in V \setminus s} \psi_u(x_u) \cdot \prod_{(s,t) \in E} \psi_{st}(x_s, x_t)$$

$$= \frac{1}{Z} \cdot \psi_s(x_s) \cdot \prod_{\substack{u \in \cup V_t \\ t \in N(s)}} \psi_u(x_u) \cdot \prod_{\substack{(l,m) \in \cup E_t \\ t \in N(s)}} \psi_{lm}(x_l, x_m) \cdot \prod_{(s,t),\, t \in N(s)} \psi_{st}(x_s, x_t)$$

Now

$$\mu_s(x_s) = \sum_{x_1, \dots x_{s-1}, x_{s+1} \dots x_n} P(x_1, \dots x_s, \dots x_n)$$

$$= \sum_{x_1, \dots x_{s-1}, x_{s+1}, \dots x_n} \frac{1}{Z} \cdot \psi_s(x_s) \cdot \prod_{u \in V \setminus s} \psi_u(x_u) \cdot \prod_{(l,m) \in \cup E_t} \psi_{lm}(x_l, x_m) \cdot \prod_{t \in N(s)} \psi_{st}(x_s, x_t)$$

Recall

$$P(x_{V_t}; T_t) = \prod_{u \in V_t} \psi_u(x_u) \cdot \prod_{(v,w) \in E_t} \psi_{vw}(x_v, x_w)$$

$$\mu_s(x_s) = K \cdot \psi_s(x_s) \cdot \sum_{x_1 \dots x_{s-1} \dots x_n} \prod_{t \in N(s)} P(x_{V_t}; t) \cdot \prod_{t \in N(s)} \psi_{st}(x_s, x_t)$$

$$\Rightarrow \mu_s(x_s) = K \cdot \psi_s(x_s) \cdot \sum_{x_1} \sum_{x_2} \dots \sum_{x_n} \prod_{t \in N(s)} \psi_{st}(x_s, x_t) \cdot P(x_{V_t}, T_t)$$

$$= K \cdot \psi_s(x_s) \cdot \prod_{t \in N(s)} M_{ts}(x_s)$$

$$\downarrow$$

$$\sum_{x_{V_t}} \psi_{st}(x_s, x_t) \cdot P(x_{V_t}; T_t)$$

Reference diagram: