Practice Questions

Sreerag T.K

Roll no: 1811158

Vectors and Matrices

$$X = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad z = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

1. Inner dot product of y and z

$$= y^T z$$
$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$= 2 + 9$$
$$= 11$$

2. $$Xy = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2+12 \\ 1+9 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

3. $$X^{-1} = \frac{\mathrm{Adj}(X)}{|X|}$$

$$= \frac{\mathrm{Adj}\left(\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\right)}{\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}}$$

$$= \frac{\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}}{(6-4)}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

4. X is having linearly independent two columns so the rank of X is '2'.

Calculus

1. $y = x^3 + x - 5$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + x - 5)$$

$$= 3x^2 + 1$$

2. $f(x_1, x_2) = x_1 \sin(x_2) e^{-x_1}$

$$\Rightarrow \nabla f(x) = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \end{pmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1} f \\ \frac{\partial}{\partial x_2} f \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1}((x_1 \sin x_2) e^{-x_1}) \\ \frac{\partial}{\partial x_2}(x_1 (\sin x_2) e^{-x_1}) \end{bmatrix}$$

$$= \begin{bmatrix} \sin x_2 \left(e^{-x_1} - x_1\right) \\ x_1 e^{-x_1} \cos x_2 \end{bmatrix}$$

Probability and statistics

$$S = \{1, 1, 0, 1, 0\}$$

1. Sample mean $= \dfrac{1+1+0+1+0}{5}$

$$= 3/5$$

2. Sampl variance $= \dfrac{\sum (x_i - \bar{x})^2}{N}$

$$= \frac{\left(1-\frac{3}{5}\right)^2 + \left(1-\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(1-\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right)^2}{5}$$

$$= \frac{\cancel{\quad} \left(\frac{2}{5}\right)^2 \times 3 + \left(\frac{3}{5}\right)^2 \times 2}{5}$$

$$= \left(\frac{12}{25} + \frac{18}{25}\right)\frac{1}{5}$$

$$= \frac{30}{25} \times \frac{1}{5}$$

$$= \frac{6}{25}$$

3. Probability of 5 = product of probability of each trials

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{2^5}$$

$$= \frac{1}{32}$$

4. Let $p(x=1) = k$

Then $p(x=0) = 1-k$

Then Probability of S,

$$P(S) = K \times K \times (1-K) \times K \times (1-K)$$
$$= K^3 \times (1-K)^2$$
$$= K^3 (1 - 2K + K^2)$$
$$= K^3 - 2K^4 + K^5$$

P(S) is maximum when,

$$\frac{d}{dK}(P(S)) = 0$$

(same condition applies to minima also)

$$\Rightarrow \frac{d}{dK}(K^3 - 2K^4 + K^5) = 0$$
$$3K^2 - 8K^3 + 5K^4 = 0$$
$$\Rightarrow 3 - 8K + 5K^2 = 0$$
$$\Rightarrow 5K^2 - 8K + 3 = 0$$
$$\Rightarrow K = \frac{8 \pm \sqrt{64 - 15 \times 4}}{10}$$
$$= \frac{8 \pm \sqrt{64 - 60}}{10}$$

$= \frac{8 \pm 2}{10}$

$= \frac{6}{10}$ & $\frac{10}{10}$

$\Rightarrow \frac{3}{5}$ or 1

Now we have to check which value among these represents maxima ~~which~~.

W.K.T for maxima,

$$\frac{d^2 P(S)}{dk^2} < 0$$

Now let's check this condition.

For $k = 3/5$

$$\frac{d^2 P(S)}{dk^2} = \frac{d}{dk}\left(\frac{d}{dk}P(S)\right)$$

$$= \frac{d}{dk}\left(3k^2 - 8k^3 + 5k^4\right)$$

$$= 6k - 24k^2 + 20k^3$$

at $k = 3/5$

$$\frac{d^2 (P(S))}{dk^2} = 6 \times \frac{3}{5} - 24 \times \frac{9}{25} + \cancel{20}^{4} \times \frac{27}{25 \times \cancel{5}}$$

$$= -0.72 < 0$$

$\Rightarrow$ best value of probability

$$P(x=1) = \frac{3}{5} \quad \text{(for maxima)}$$

5

		y		
		a	b	c
z	T	0.2	0.1	0.2
	F	0.05	0.15	0.3

- $P(z=T \text{ AND } y=b) = 0.1$

- $$P(z=T / y=b) = \frac{P(z=T \text{ AND } y=b)}{P(y=b)}$$

$$= \frac{0.1}{0.1+0.15}$$

$$= \frac{0.1}{0.25}$$

$$= \frac{10}{25}$$

$$= \frac{2}{5}$$

Big - O Notation

1. $f(n) = \ln(n)$, $g(n) = \lg(n)$.

Both are true since both are related to each other by a multiplicative constant.

2. $f(n) = 3^n$, $g(n) = n^{100}$

Only $g(n) = O(f(n))$ is true.

Since $f(n) \gg g(n)$ for large n

3. $f(n) = 3^n$, $g(n) = 2^n$

Only $g(n) = O(f(n))$ is true.

Since $f(n) \gg g(n)$ for large n.

4. $f(n) = 1000n^2 + 2000n + 4000$
$g(n) = 3n^3 + 1$
Only $f(n) = O(g(n))$ is true
since $g(n) \gg f(n)$ for large n

Probability and Random Variables

a) $P(A \cup B) = P(A \cap (B \cap A^c)) \Rightarrow$ False

bcz $P(A \cap (B \cap A^c)) = P(A \cap (B - A \cap B))$

$\neq P(A \cup B)$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow$ True

c) $P(A) = P(A \cap B) + P(A^c \cap B) \Rightarrow$ False

~~d) P(A~~

d) $P(A_1 \cap A_2 \cap A_3) = P(A_3 | A_2 \cap A_1)$
$P(A_2 | A_1) P(A_1)$

$\Rightarrow$ True

Discrete and Continuous Distributions

a) Multivariable Gaussian

b) Bernoulli

c) Uniform

d) Binomial

e) $p^x (1-p)^{1-x}$

f) $\frac{1}{b-a}$ when $a \le x \le b$; 0 otherwise

g) $\binom{n}{x} p^x (1-p)^{n-x}$

h) $\frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu)\right)$

Mean, Variance and Entropy

$$Var(X) = E[(X - EX)^2]$$

We need to prove that

$$Var(X) = E(X^2) - E(X)^2$$

Start from RHS

$$\Rightarrow E[(X - EX)^2] = E(X^2 - 2X\,E(X) - (E(X))^2)$$

$$= E(X^2) - E(2X\,E(X)) - E(E(x)^2)$$

$$= E(X^2) - 2E(X)E(X) - E(E(X)^2)$$

$$= E(X^2) - 2E(X^2) + (E(X))^2$$

$= E(x^2) - E(x)^2$

$\Rightarrow Var(X) = E(X^2) - E(X)^2$

Hence proved.

b) Mean $= p$ variance $= p(1-p)$

entropy $= -(1-p)\log(1-p) - p\log p$

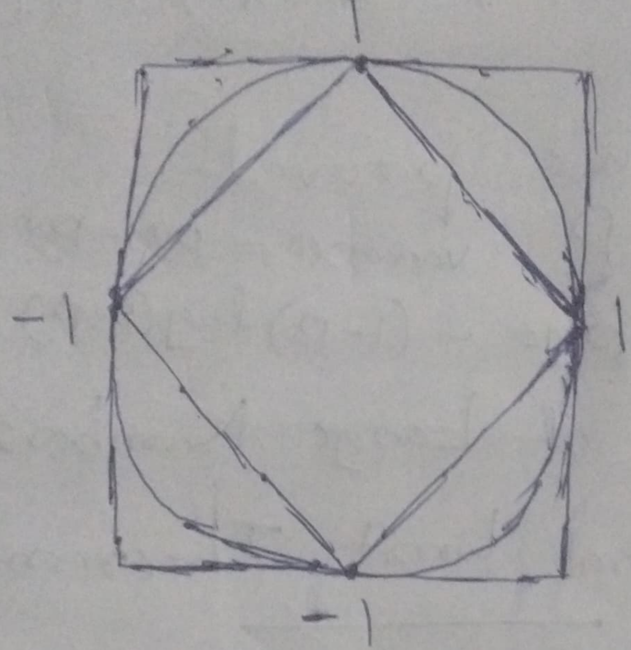
Law of Large Numbers and Central Limit Theorem

---

a) If it is a fair die, ~~probabl~~ probability of obtaining 3 is $\frac{1}{6}$. So by law of large Numbers Number of times 3 will show up in 6000 trials $= 6000 \times \frac{1}{6} = \underline{\underline{1000}}$

b) By central limit theorem when $n \to \infty$ LHS will tend to RHS

Linear Algebra



a) ~~inside reg~~ The whole inside region of the circle.

b) The whole inside region of the inner square.

c) The whole inside region of the outer square.

Geometry

a) line is $w^T x + b$.

Consider two ~~lines~~ points in the line $w^T x + b$ $(x_1 \& x_2)$

$$\Rightarrow \quad w^T x_1 + b = 0 \quad -(1)$$
$$w^T x_2 + b = 0 \quad -(2)$$

And $x_1 - x_2$ will be lying in the line $w^T x + b$

So if w is $\perp$r to $x_1 - x_2$, it will $\perp$r to the line $w^T x + b$.

So we need to prove that inner product of of w & $x_1 - x_2$ is zero.

Inner product of w & $x_1 - x_2$

$$= w^T (x_1 - x_2)$$

Now (1) - (2)

$$\Rightarrow w^T (x_1 - x_2) = 0$$

$\Rightarrow$ Their inner product is zero $\Rightarrow w \perp$ to $w^T x + b$

b) It can be found by projecting x into w. So we are taking inner product of both and dividing it by $|w|$ to make w a unit vector.

$$\Rightarrow \left| \frac{w^T x}{|w|} \right| \quad \text{---} (1)$$

now the eqn of line is

$$w^T x + b = 0$$

$$\Rightarrow w^T x = -b \quad \text{---} (2)$$

Applying eqn (2) in eqn (1)

$$\Rightarrow \left| \frac{-b}{||w||} \right| = \frac{b}{||w||}$$

Hence proved.