Modeling Pendulum Motion

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Abstract

We describe here the kinematics and numerical solution of a single pendulum system. (Double pendulum simulation to be added).

1 Introduction

The pendulum is perhaps one of the most well known examples studied in classical mechanics. However, a clear exposition of analytical and numerical techniques to model the single and double pendulum systems are often difficult to find. This article attempts to present the analytical and numerical analysis of a pendulum system in an easy to understand format. This article supplements the Matlab and C with OpenGL source code which models the systems. The pendulum systems described here are idealized, in the sense that the rod and Bob are assumed to be massless and there is no resistance to the motion.

2 Single Pendulum

Let us start with a picture of a single pendulum system. The length of the pendulum is l. It makes

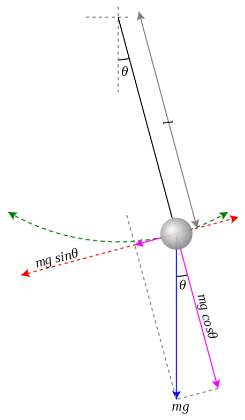


Figure 1: Source: Image by Krishnavedala from Wikipedia

an angle of θ from the equilibrium position. The direction of the bobs motion is along the green arc, while the direction of the instantaneous velocity vector is along the red line, tangent to the green arc. Both the angle and speed of the Bob are functions of time: $\theta(t)$ and $v(t) = \frac{d\theta}{dt}(t)$. By Newton's second law:

$$F = ma \implies -mg\sin(\theta) = ma \implies a = -g\sin(\theta) \tag{2.1}$$

It remains to express the acceleration a in terms of l and θ. Let us take s to be the arc length along the circular direction of motion. Then we have that:

$$\frac{s}{2\pi l} = \frac{\theta}{2\pi} \implies s = l\theta \implies v = \frac{ds}{dt} = l\frac{d\theta}{dt} \implies a = \frac{d^2 s}{dt^2} = l\frac{d^2\theta}{dt^2} \tag{2.2}$$

Substituting into (2.1), we arrive at the second order nonlinear ODE:

$$l\frac{d^2\theta}{dt^2} = -g\sin(\theta) \implies \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \tag{2.3}$$

The initial value problem for the single pendulum system is usually given as:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \quad ; \quad \theta(0) = \theta_0 \quad \text{and} \quad \frac{d\theta}{dt}(0) = v_0 \tag{2.4}$$

where the initial conditions give the initial displacement (angle) and speed of the bob.

2.1 Numerical Modeling

An analytical solution of (2.4) is difficult to obtain. However, it is easy to model numerically via numerical methods. We first rewrite the IVP as a first order system, by defining $p = \theta$ and $q = \theta'$:

$$\begin{aligned} \frac{dp}{dt} &= q \\ \frac{dq}{dt} &= -\frac{g}{l}\sin(p) \\ p(0) &= \theta_0 \quad \text{and} \quad q(0) = v_0 \end{aligned}$$

Next, we rewrite the system as a single first order ODE of a vector valued function. Letting:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \quad \text{and} \quad f(t) = \begin{bmatrix} q(t) \\ -\frac{g}{l}\sin(p(t)) \end{bmatrix} = \begin{bmatrix} u_2(t) \\ -\frac{g}{l}\sin(u_1(t)) \end{bmatrix}$$

We get:

$$\frac{d}{dt}u(t) = f(t) \quad ; \quad u(0) = \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} \theta_0 \\ v_0 \end{bmatrix} \tag{2.5}$$

We can solve (2.5) using the Runge-Kutta 4th order scheme using the following sequence of steps. We first discretize a time interval, say $t_i = 0$ to $t_f = 10$ using N steps, resulting in time step $h = \frac{t_i - t_f}{N}$. Then, we perform numerical integration of the system by looping the formulas:

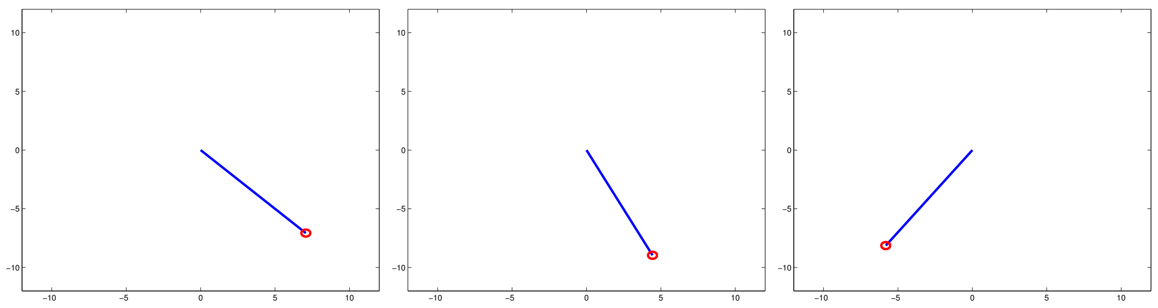
$$\begin{aligned} k_1 &= hf(t, u) \\ k_2 &= hf\left(t + \frac{h}{2}, u + \frac{k_1}{2}\right) \\ k_3 &= hf\left(t + \frac{h}{2}, u + \frac{k_2}{2}\right) \\ k_4 &= hf(t + h, u + k_3) \\ u &= u + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ t &= t + h \end{aligned}$$

Then at each step, $u(1) \approx \theta(t)$ and $u(2) \approx v(t)$. The RK4 scheme is implemented in the file *rk4sys_integrator.m*, which is called by *single_pendulum_driver.m*. The result of running the script is two arrays of values for θ and v.

What remains is to write a simple program to plot the motion of the pendulum. This is done by the script *plot_pendulum.m*. We now briefly describe how the plotting is done. Note that the motion is entirely two dimensional. For simplicity we assume the pendulum rod is attached to an anchor at the origin $(0,0)$, but can rotate freely around the anchor so that $\pi \leq \theta < \pi$. It then simply remains to translate the angle at each time t into a position vector $x(t)\hat{i} + y(t)\hat{j}$. There are four possibilities, which can be readily verified by making a small graph:

- If $0 \leq \theta < \frac{\pi}{2}$, then $x = l\cos(\frac{\pi}{2} - \theta)$ and $y = -l\sin(\frac{\pi}{2} - \theta)$.
- If $\frac{\pi}{2} \leq \theta < \pi$, then $x = l\cos(\theta - \frac{\pi}{2})$ and $y = l\sin(\theta - \frac{\pi}{2})$.
- If $-\frac{\pi}{2} \leq \theta < 0$, then $x = -l\cos(\frac{\pi}{2} - |\theta|)$ and $y = -l\sin(\frac{\pi}{2} - |\theta|)$.
- If $-\pi \leq \theta < -\frac{\pi}{2}$, then $x = -l\cos(|\theta| - \frac{\pi}{2})$ and $y = l\sin(|\theta| - \frac{\pi}{2})$.

The pendulum motion is very predictable. Starting with no initial velocity and from an angle of $\frac{\pi}{4}$ we get motion similar to below:



The OpenGL version of the simulation is written with the standard Free Glut library. The C code is implemented similar to the Matlab code, except it is easier to treat the two components of the function f separately in the RK4 loop code. Essentially, the following code can be used for the ODE integration:

```c
    theta = theta0;
    v = v0;
    for(i = 0; i<N; i++){
        k11 = h*v;
        k12 = -h*(g/l)*sin(theta);

        k21 = h*(v + k11/2);
        k22 = -h*(g/l)*sin(theta + k12/2);

        k31 = h*(v + k21/2);
        k32 = -h*(g/l)*sin(theta + k22/2);

        k41 = h*(v + k31);
        k42 =  -h*(g/l)*sin(theta + k32);

        theta = theta + (k11 + 2*k21 + 2*k31 + k41)/6;
        v = v + (k12 + 2*k22 + 2*k32 + k42)/6;
    }
```

Note that in the simple classical case of this problem f does not depend on t. The OpenGL graphics part is simple: we compute an array of angles and velocities and then convert angles into coordinates as before. We call the *display()* function each time we loop though the $\theta(t)$ array in order to plot a sequence of steps of the pendulum motion.