satisfied throughout the region. Show that the modulus of any analytic function is subharmonic.

7. Let u, v be any two real functions of (x,y). Using Green's theorem, obtain the first and second *Green's identities*, respectively,

$$\int_C u \frac{\partial v}{\partial n}\, ds = \int_A (u_x v_x + u_y v_y)\, dA + \int_A u(\Delta v)\, dA$$

$$\int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = \int_A [u(\Delta v) - v(\Delta u)]\, dA$$

Here C is a closed contour of area A; the symbol Δ represents the laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\partial/\partial n$ represents the outward normal derivative. Use these identities to show (*a*) that $\int_C \partial u/\partial n\, ds = 0$ for any function harmonic over the region inside C and (*b*) that a differentiable real function v is subharmonic if and only if $\Delta v \geqq 0$. [*Hint for* (*b*): Set $u = 1$, and let C be a circle.]

8. Show that the solution of a Dirichlet problem may be characterized as that function u which satisfies the boundary condition and also the condition that

$$\int_A (u_x^2 + u_y^2)\, dA$$

be a minimum. Interpret this condition geometrically.

2-6 *Taylor Series*

The fact that an analytic function has continuous derivatives of all orders suggests the existence of power-series representations for such a function. Let $f(z)$ be analytic in a region R, and let z_0 be a point inside this region. With center z_0, construct any circle C lying wholly in R. We shall show that within this circle $f(z)$ may be expressed as the sum of the convergent power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots a_n(z - z_0)^n + \cdots \tag{2-27}$$

in which the constants a_n are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \tag{2-28}$$

The proof is a straightforward application of Cauchy's integral formula. Let ζ be a point on the circle C; then, for any point z inside C,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \left(1 - \frac{z - z_0}{\zeta - z_0} \right)^{-1} d\zeta$$

Since $|z - z_0| < |\zeta - z_0|$, the series

$$\left(1 - \frac{z - z_0}{\zeta - z_0}\right)^{-1} = 1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots \tag{2-29}$$

is uniformly convergent for all points ζ on C, so that we may integrate term by term to give

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z_0} + \frac{z - z_0}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^3} + \cdots \\ &= f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \\ &\qquad \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots \end{aligned}$$

by use of Eq. (2-14). The expansion converges wherever the series (2-29) does, i.e., within C. Note that the coefficients depend only on the derivatives of $f(z)$ at z_0. By analogy with the Taylor-Maclaurin power-series expansion for real functions, an expansion of the form (2-27) is called a *Taylor series.*

We have previously seen (Exercise 8 of Sec. 2-2 or Exercise 9 of Sec. 2-3) that any power series represents an analytic function inside its circle of convergence; we have now proved the converse result that any analytic function can be expanded in a convergent power series.

Some Examples

Equation (2-28) provides a direct method for calculating the coefficients of the power-series expansion of a given function. For example, the expansion of e^z around the point $i\pi$ is obtained by setting

$$a_n = \frac{1}{n!}\left[\frac{d^n}{dz^n}(e^z)\right]_{z=i\pi} = \frac{e^{i\pi}}{n!} = \frac{-1}{n!}$$

and thus

$$e^z = -\left[1 + (z - i\pi) + \frac{(z - i\pi)^2}{2!} + \cdots\right]$$

The function e^z can of course also be expanded around the point $i\pi$ by writing

$$e^z = (e^{z-i\pi})e^{i\pi} = -e^{z-i\pi}$$

which gives the same result. This is an illustration of the fact that the power-series expansion of an analytic function must be unique, because of the identity theorem for power series proved in Sec. 1-3.

In expanding the function $f(z) = \sin(\sin z)$ around the origin, Eq. (2-28) can again be used, though it leads to tedious computations. Alternatively, the known expansion for sin z can be employed,

$$f(z) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right) - \frac{1}{3!}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)^3 + \frac{1}{5!}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)^5 - \cdots$$

and terms of the same order collected. Notice that in an example like this it is useful to have the lowest order of terms increase from one expression to the next, so that only a finite number of terms need be used to find the coefficient of each power of z. Thus, for $f(z) = \sin(\cos z)$, it would be advantageous to write

$$\begin{aligned}\sin(\cos z) &= \sin[(\cos z - 1) + 1] \\ &= \sin(\cos z - 1)\cos 1 + \cos(\cos z - 1)\sin 1\end{aligned}$$

before using an expansion similar to that for $\sin(\sin z)$.

It may also be useful to differentiate or integrate a given function before power-series expansion; thus, if $f(z) = \arctan z$, $f'(z) = (1 + z^2)^{-1}$, which is easily expanded, and the result can then be integrated term by term to give the expansion for $f(z)$ itself.

If $f(z)$ is analytic at z_0 but is multiple-valued, then each of its branches is a locally single-valued analytic function which may be expanded about z_0. For example, to expand $f(z) = \ln z$ about $z_0 = 1$, we write

$$f' = \frac{1}{z} = \frac{1}{(z-1)+1} = 1 - (z-1) + (z-1)^2 - \cdots$$

so that $$f(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} \cdots + \text{const}$$

Setting the constant equal to zero, we obtain the expansion for the principal value of $\ln z$ (since then $\ln 1 = 0$); that for any other branch is obtained by setting the constant equal to an appropriate multiple of $2\pi i$. It would not have been possible to obtain an expansion for $\ln z$ about the origin, since no branch of $\ln z$ is analytic there.

Further Properties Deduced From Taylor Series

If $f(z)$ is analytic at z_0 (and so also in a neighborhood of z_0, by the definition of analyticity), and if $f(z_0) = 0$, then the power-series expansion (2-27) will have $a_0 = 0$. More generally, it may happen that all of $a_0, a_1, \ldots, a_{p-1}$ vanish, but a_p does not; in that case, all of $f(z_0), f'(z_0), \ldots, f^{(p-1)}(z_0)$ vanish, and $f(z)$ is said to have a *p*th-order zero at z_0. If $f(z)$ has a zero of order p at z_0, then, given any positive integer $m \leq p$, the function

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^m} & z \neq z_0 \\ a_m & z = z_0 \end{cases}$$

is analytic at z_0. Note that the usual L'Hospital rule holds, in that, if two functions $w(z)$ and $h(z)$ both vanish at z_0,

$$\lim_{z \to z_0} \frac{w(z)}{h(z)} = \frac{w'(z_0)}{h'(z_0)}$$

if this quantity exists; if still indeterminate, second derivatives may be taken, and so on.

Next, let $f(z)$ be analytic and nonconstant in a region R, and consider any closed curve C lying wholly in R. Then, inside C, $f(z)$ can attain any given complex value α at most a finite number of times, for otherwise there would be an infinite number of such points, having a limit point z_1, say. But an expansion of $f(z)$ in a power series about z_1 would have to agree at each of the α points with the power series $g(z) = \alpha + 0$ and so by the identity theorem would have everywhere the value α, which is contrary to hypothesis. A consequence of this result is that if $f(z)$ is analytic at a point z_0, and non-constant, then it is possible to find a small circle around z_0 such that inside that circle $f(z)$ does not take on the value $f(z_0)$ at any point other than z_0.

There is an interesting Parseval-type formula relating the values of the coefficients a_n in a Taylor-series expansion of $f(z)$ to the values of $f(z)$ on a circle surrounding z_0. Let $f(z)$ be analytic in the circle $|z - z_0| \leq r$; then, setting $z - z_0 = re^{i\theta}$ on the boundary of the circle gives

$$\begin{aligned} f(z) &= a_0 + a_1 re^{i\theta} + \cdots + a_n r^n e^{in\theta} + \cdots \\ f^*(z) &= a_0^* + a_1^* re^{-i\theta} + \cdots + a_n^* r^n e^{-in\theta} + \cdots \end{aligned}$$

so that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|^2 \, d\theta = |a_0|^2 + |a_1|^2 r^2 + \cdots + |a_n|^2 r^{2n} + \cdots \tag{2-30}$$

We already have, from Eqs. (2-14) and (2-28), the inequality

$$|a_n| \leq \frac{M(r)}{r^n}$$

where $M(r)$ is the maximum value of $|f|$ on the circle $|z - z_0| = r$. Equation (2-30) clearly allows us to sharpen this inequality, to the extent to which $f(z_0)$, $f'(z_0)$, $f''(z_0)$, etc., are known.

The existence of a power-series expansion for an analytic function implies that an arbitrary real harmonic function has a similar property. For let $u(x,y)$ be harmonic in a neighborhood of (x_0,y_0), and let $f(z)$ be that analytic function whose real part it is. Then, if the expansion of $f(z)$ is given by Eq. (2-27), where $z = x + iy$, and if $a_j = \alpha_j + i\beta_j$, where α_j and β_j are real, we have

$$\begin{aligned} u(x,y) = \alpha_0 + \alpha_1(x - x_0) - \beta_1(y - y_0) + \alpha_2(x - x_0)^2 \\ - 2\beta_2(x - x_0)(y - y_0) - \alpha_2(y - y_0)^2 + \cdots \end{aligned}$$

The coefficients can be obtained either from the expansion of $f(z)$, as above, or by computing the various partial derivatives of u at the point (x_0,y_0). If we choose polar coordinates (r,θ) so that $z = z_0 + re^{i\theta}$, the expansion of a harmonic function is a little simpler. For then write $a_j = r_j \exp(i\theta_j)$, and we have

$$f(z) = r_0e^{i\theta} + r_1r\exp[i(\theta + \theta_1)] + r_2r^2\exp[i(2\theta + \theta_2)] + \cdots$$

so that

$$u(r,\theta) = u(0,0) + r_1r\cos(\theta + \theta_1) + r_2r^2\cos(2\theta + \theta_2) + \cdots$$

This is of course a Fourier series for the function $u(r,\theta)$, where the Fourier expansion is with respect to θ. It has, however, the interesting property that the dependence of the coefficients on r is very simple.

EXERCISES

1. Obtain the power-series expansions for the following functions around the indicated points. Where the function is multiple-valued, give the results for all possible branches.

(*a*) $(1 + z + z^2)^{-1}$; $z_0 = 0,\ 1 + i$
(*b*) $\sin^2 z$; $z_0 = 0,\ -1$
(*c*) $z^{1/2}$; $z_0 = 1,\ i\pi$
(*d*) $(z^2 - 1)^{1/2}$; $z_0 = 0$
(*e*) $(z - \pi)/(\sin z)$; $z_0 = \pi$
(*f*) $\ln[iz + (1 - z^2)^{1/2}]$; $z_0 = 0,\ i$
(*g*) $\arctan z$; $z_0 = 1$

2. In terms of the Bernoulli numbers B_n of Eq. (1-7), show that

$$(a)\quad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$(b)\quad \cot z = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}B_{2n}}{(2n)!} z^{2n}$$

$$(c)\quad \tanh z = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)B_{2n}}{(2n)!} z^{2n-1}$$

[*Hint:* tanh z may be expressed as a linear combination of coth z and coth $2z$].

3. Let $f(z)$ be analytic inside the unit circle, with $f(0) = 0$, and with $|f(z)| \leqq 1$ inside and on the circle. Show that $|f(z)| \leqq |z|$ for any point z inside the circle and that if equality holds at any interior point then $f(z) = e^{i\alpha}z$ everywhere, with α some real constant (*Schwarz's lemma*).

4. Obtain a series solution for the Dirichlet problem for the interior of the circle $|z| = 2$, if $u(2e^{i\theta}) = \min(\theta, 2\pi - \theta)$ for $0 \leqq \theta \leqq 2\pi$.

5. The *hypergeometric series* is defined by

$$F(a,b;c;z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \cdots$$

Here, a, b, and c are arbitrary complex numbers, except that c is not to be a negative integer. Show that the series converges inside the unit circle and diverges outside. Using the criterion of Sec. 1-3, show that for $|z| = 1$ it converges absolutely if Re $(a + b - c) < 0$. Prove the following results:

(*a*) $z(1 - z)F'' + [c - (a + b + 1)z]F' - abF = 0$ (*the hypergeometric equation*)

(*b*) $(1 + z)^a = F(-a,b;b;-z)$

(*c*) $\arcsin z = zF(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)$

(*d*) $\ln(1 + z) = zF(1,1;2;-z)$

6. Show that:

(*a*) The power-series expansion about the origin of an even analytic function contains only even powers of z.

(*b*) If $f(z)$ is entire, with $|f|/|z|^n$ bounded as $|z| \to \infty$, where n is some positive integer, then $f(z)$ must be a polynomial of degree $\leqq n$.

2-7 *Laurent Series*

If $f(z)$ is analytic inside some circle centered on z_0, then it possesses a Taylor-series expansion about that point. It may happen, however, that $f(z)$ is analytic outside rather than inside the circle, or, more generally, that $f(z)$ is analytic only in some annular region contained between two concentric circles centered on z_0. If $f(z)$ is analytic and single-valued in such an annular region, then a generalization of the Taylor series, called a *Laurent series*, may be used to obtain a series expansion of $f(z)$ about z_0. The generalization consists in permitting negative as well as positive powers of $z - z_0$ to appear in the expansion.

Let $f(z)$ be analytic and single-valued in the annular region R defined by

$$r_1 \leqq |z - z_0| \leqq r_2$$

where r_1 and r_2 are the radii of two circles C_1 and C_2 centered on z_0, with of course $r_1 < r_2$. By Cauchy's integral formula, the value of $f(z)$ at any point z in R is given by

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

Here the first integral is taken counterclockwise, the second clockwise, in accordance with our standard convention for multiply connected regions. Rewriting the equation as

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z_0}\left(1 - \frac{z - z_0}{\zeta - z_0}\right)^{-1} d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{z - z_0}\left(1 - \frac{\zeta - z_0}{z - z_0}\right)^{-1} d\zeta$$

expanding the integrands by use of the binomial theorem, and integrating term by term (justified by uniform convergence; note that $|\zeta - z_0| < |z - z_0|$ in the second integrand), we obtain

$$\begin{aligned} f(z) &= \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \\ &= \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \end{aligned} \tag{2-31}$$

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

Here C can be *any* curve in R enclosing the inner circle and traversed counterclockwise. The inner circle may be contracted and the outer expanded, until they are respectively as small or as large as they can be without encountering any singularities of $f(z)$; the *Laurent series* (2-31) is clearly valid in this maximal region. Moreover, the expansion is unique, for if there were two series yielding $f(z)$, we would have

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$$

throughout R. Multiply this equation by $(z - z_0)^k$, where k is any positive or negative integer, and integrate around any closed curve C enclosing C_1; then since

$$\frac{1}{2\pi i}\int_C (z - z_0)^{n+k}\,dz = \begin{cases} 0 & \text{if } n + k \neq -1 \\ 1 & \text{if } n + k = -1 \end{cases}$$

we obtain $a_{-k-1} = b_{-k-1}$ for all k, thus proving the identity of the two Laurent series.

The Laurent series divides naturally into two series, the first of which contains all the positive powers (including zero) and the second of which contains all the negative powers. The first series converges everywhere inside the outer circle, and the second series converges everywhere outside the inner circle. If $f(z)$ is analytic everywhere inside the outer circle, then

Cauchy's theorem shows that $a_n = 0$ for n negative so that the Laurent series reduces to a Taylor series for this case.

As an example, the function $f(z) = [(z - 1)(z + 2)]^{-1}$ has three different expansions about the origin, each of which is most easily obtained by separation into partial fractions and use of the binomial theorem,

$$(a)\ |z| < 1: \qquad f(z) = -\frac{1}{6}\sum_{n=0}^{\infty} [2^{n+1} + (-1)^n]\left(\frac{z}{2}\right)^n$$

$$(b)\ 1 < |z| < 2: \quad f(z) = \frac{1}{3}\sum_{n=1}^{\infty} z^{-n} + \frac{1}{6}\sum_{n=0}^{\infty} (-1)^{n+1}\left(\frac{z}{2}\right)^n$$

$$(c)\ |z| > 2: \qquad f(z) = \frac{1}{6}\sum_{n=0}^{\infty} [2^{-n} - (-1)^n]\left(\frac{2}{z}\right)^{n+1}$$

If $f(z)$ is multiple-valued, it may still happen that a particular branch is single-valued in the annular region of interest; a Laurent series may then be obtained for that branch. This is possible, for example, with each branch of $[(z - 1)(z + 2)]^{1/2}$ in the region $|z| > 2$; the appropriate expansion in powers of z is easily found by use of the binomial theorem. On the other hand, $\ln z$ does not have a single-valued branch in any annular region centered on the origin, so that it has no Laurent-series expansion in any such region.

Exercises

1. Expand $1/\sin z$ in powers of z for $0 < |z| < \pi$ and also for $\pi < |z| < 2\pi$.

2. In what annular regions centered on $-i$ could expansions in powers of $z + i$ be obtained for each of $(z^2 - 1)^{1/2}$, $(z^2 + 1)^{1/2}$, $\ln [(z + 1)/(z - 1)]$, $[z(z^2 - 1)]^{1/2}$, $[z(z - i)(z^2 - 1)]^{1/2}$? Obtain an expansion for the last of these, valid for large z.

3. Expand in powers of z the function $\sin (z + 1/z)$ in whatever annular region is closest to the origin. Express the coefficients as simple trigonometric integrals.

4. What can be deduced from the Laurent-series theorem concerning the possibility of series expansions in annular regions for certain harmonic functions?

Isolated Singularity of a Single-valued Function

Suppose now that $f(z)$ is analytic and single-valued (e.g., some chosen branch of a multiple-valued function) everywhere in some neighborhood $|z - z_0| < r$ of a point z_0, except at z_0 itself. Then $f(z)$ is said to have an *isolated singularity* at z_0. For example, each of $1/z$ and $\sin (1/z)$ has an isolated singularity at the origin. Each branch of the function