COURSE TOPICS

Elements of probability theory, Baye's theorem, Random variable PDF and CDF, Mean, Variance, Markov and Chebshev Inequality, Binomial, Geometric & negative binomial distribution, Poisson & Hyper geometric distribution, uniform, exponential and normal distributions, Linear combination of independent random variables, Central limit theorem, distribution of several random variable, covariance, correlation, Classification of random processes, Random walk, weiner process

Analytic functions of complex variables, Cauchy - Reimann equations, integration of a function of a complex variables, Cauchy's integration M-L inequality, Cauchy's integration formula, Taylor & Laurent expansions, Poles & essential singularities, residues, Cauchy's residue theorem, simple contour integral

BOOKS

1) Ruel V. Churchill and JW Brown: Complex variables and its applications. (McGraw Hill)

2) Jim Pitman - Probability - Springer

PROBABILITY

Probability theory is a mathematical model of uncertainty.
eg. Tossing coin, dice etc.

If a random event can occur large number of times 'n' and the event has occured in 'h' different ways, the probability of an event $= \frac{h}{n}$

eg (i) Toss a coin 1000 times. What is the probability that you get a head? (You have gotten head 532 times)

$$P(\text{head}) = \frac{532}{1000}$$

(ii) Industrial students = 25
Electrical " = 10
Mechanical " = 10
Civil " = 8

What is the probability of selecting an industrial student?

$$P(\text{ind. st.}) = \frac{25}{53}$$

What is the probability of selecting civil or electrical student?

$$P(\text{selecting}) = \frac{18}{53}$$

SPACE

A space consists of all possible outcomes.
A set 'S' that consists of all possible outcomes of a random experiment is called a sample space.

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1) Finite Sample Space
This is the sample space which has finite no. of outcomes

2) Countably infinite sample space
eg. natural numbers $1, 2, \dots \infty$

3) Non countably infinite sample space
eg. $0 \leq x \leq 1$

1) and 2) are discrete sample space. (finite / countably infinite)

3) is a non discrete sample space

ϕ = empty space / null space / impossible event space

AXIOMS

1) $P(A) \geq 0$
$A \rightarrow$ event in class C of events

2) $P(S) = 1$
sum of all probabilities = 1

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
mutually exclusive $\rightarrow$ events cannot occur simultaneously i.e, $P(A \cap B) = 0$

4) If $A_1, A_2, \dots A_n$ are mutually exclusive

$$P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

(Additive rule)

$A - B = A \cap B'$

eg. (i) 2 companies A and B
$C_A = 0.8$
$C_B = 0.6$
Probability of getting into both companies $= 0.5$

what is probability of getting into either company?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

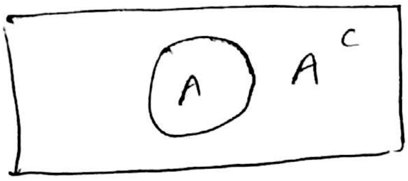
$= 0.8 + 0.6 - 0.5$

$= 0.9$

(ii) A single die has been tossed once. Find probability of 2 or 5 turning up.

$P(2 \cup 5) = P(2) + P(5)$

$= \frac{1}{3}$



$A \cup A^c = S$

$P(S) = 1$

eg.

No. of cars	Probability
3	0.12
4	0.19
5	0.28
6	0.24
7	0.10
8 or more	0.07

What is the probability that he will service atleast 5 cars on his next day at work?

$P(E) = 1 - P(E')$

$= 1 - (0.12 + 0.19)$

$= 0.69$

CONDITIONAL PROBABILITY

$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$

$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$

occurence of B is conditioned on the fact that A has occured where total sample space is A

$P(A \cap B) = P(A)\, P(B/A)$ → multiplication theorem

$= P(A)\, P(B)$ → If A & B are independent

If the events A and B in a probability space S are said to be independent if the occurence of one does not influence occurence of the other.

$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$

eg. (i) Study of rain in July

P(rainy day following a rainy day) = 0.444

P(dry day following a dry day) = 0.724

P(rainy " dry ") = 0.276

P(dry " rainy ") = 0.556)

What is the probability that next two days will be rainy given that today is rainy?

~~$P(E) = (0.4444)^2$~~

$P(A \cap B) = P(A)\, P(B/A)$

$= (0.444)^2$

(ii) A pair of dice is tossed. Find the probability that one of the dice is 2 if sum is 6?

~~$P(A \cap B) = \frac{5}{36} \times \frac{2}{36}$~~

$A = \{\text{sum is } 6\} = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$B = \{$ 2 is the ...

$P(E) = $ ~~$P(A \cap B)$~~ ...

$= \frac{2}{5}$

BAYES THEOR...

$B_1, B_2, \ldots$ collection ...

A → ...

$(B_1 \cap A)$, ...

$P(A \cup B) =$ ...

$P(A) =$ ...

$P(A) =$ ...

$P(B_i/A$ ...

$P(B_i/$ ...

$B = \{ 2$ is there on atleast one die $\}$

E

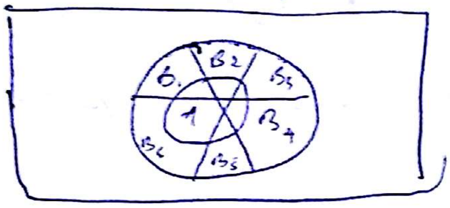
$p(E) = \cancel{p(A \cap B)} = P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{2/36}{5/36}$

$= \dfrac{2}{5}$

BAYES THEOREM

$B_1, B_2, \ldots$ mutually exclusive & collectively exhaustive

$A \rightarrow$ union of mutually exclusive



$(B_1 \cap A), (B_2 \cap A) \ldots\ldots (B_k \cap A)$

$P(A \cup B) = P(A) + P(B)$

$P(A) = P((B_1 \cap A) \cup P(B_2 \cap A) \cup \ldots \cup P(B_k \cap A))$

$= P(B_1 \cap A) + P(B_2 \cap A) \ldots\ldots + P(B_k \cap A)$

$= \sum_{i=1}^{k} P(B_i \cap A)$

$P(A) = \sum_{i=1}^{k} P(B_i) P(A/B_i)$ Total probability theorem

$P(B_i/A) = \dfrac{P(B \cap A)}{P(A)}$

$P(B_i/A) = \dfrac{P(B_i \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \dfrac{P(B_i \cap A)}{\sum_{i=1}^{k} P(B_i) P(A/B_i)}$

of events $B_1, B_2, \dots B_k$ constitute partition of sample space $P(B_i) \neq$

then for any event A, $P(A) \neq 0$

$$P(B_k/A) = \frac{P(B_k \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)}$$

$$= \frac{P(B_k)\, P(A/B_k)}{\sum_{i=1}^{k} P(B_k)\, P(A/B_k)}$$

$P(A \cap B) = P(B/A)\ P(A)$

Q) A manufacturing company adopt 3 plants

Plan		
one	30%	of product
two	20%	"
three	50%	"

$P(D/P_j)$ = Probability of a defective product given a plan j.

$P(D/P_1) = 0.01$

$P(D/P_2) = 0.03$

$P(D/P_3) = 0.02$

If a random product was observed and found to be defective, which plant was most likely & responsible

Ans) we need to find $P(P_j/D)$ for $j = 1, 2, 3$

$P(P_1) = 0.3$

$P(P_2) = 0.2$

$P(P_3) = 0.5$

$P(P_1/D) =$

Q) There ...
4 are de...
drawn ...
the othe...
that all...

Ans) P(fi...

$= \frac{8}{12}$

P(sec...

$= \frac{7}{11}$

P(thi...

$P(E) =$

Q) There ...
fire e...
P(fire ...
P(am...
Find t...
ambul...
avail...
Ans) P(

$P(P_3) = 0.5$

$$P(P_1/D) = \frac{P(P_1)\,P(D/P_1)}{P(P_1)\,P(D/P_1) + P(P_2)\,P(D/P_2) + P(P_3)\,P(D/P_3)}$$

$$= \frac{0.3 \times 0.01}{0.3 \times 0.01 + 0.2 \times 0.03 = 0.5 \times 0.02}$$

$= 0.158$

Q) There are 12 items on which 4 are defective. 3 items are drawn at random one after the other. Find the probability that all three are non defective.

Ans) P(first item is non defective)

$= \frac{8}{12}$

P(second item is non defective)

$= \frac{7}{11}$

P(third is non defective) $= \frac{6}{10}$

$P(E) = \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$,

Q) There is a small town with a fire engine & an ambulance.

P(fire engine is available) = 0.98

P(ambulance ") = 0.92

Find the probability that both ambulance & fire engine is available

Ans) $P(A \cap B) = 0.98 \times 0.92$

RANDOM VARIABLE (X)

~~A function of all possible values when all are equally likely.~~

A random variable is a function that associates a real no.. with each element in the sample space.

The values that a random variable can take is denoted by x.
It can be discrete or continuous.

~~Discrete random variables are a~~

If a sample space contains a finite no. of possibilities or unending sequence with as many as whole nos., it is called discrete random variables.

Continuous random variables - possible outcomes on a continuous scale.

Let $x_1, x_2, x_3, \ldots x_n \in X$
$p(x_1)$, $p(x_2)$, $p(X = x_1)$, $P(X = x_2)$
$\ldots$ $P(X = (B))$

B is a subset of a range of values of X. Probabilities of these range of values must form a distribution which is called distribution function.

$$p(X \in B) = \sum_{x \in B} P(X = x)$$

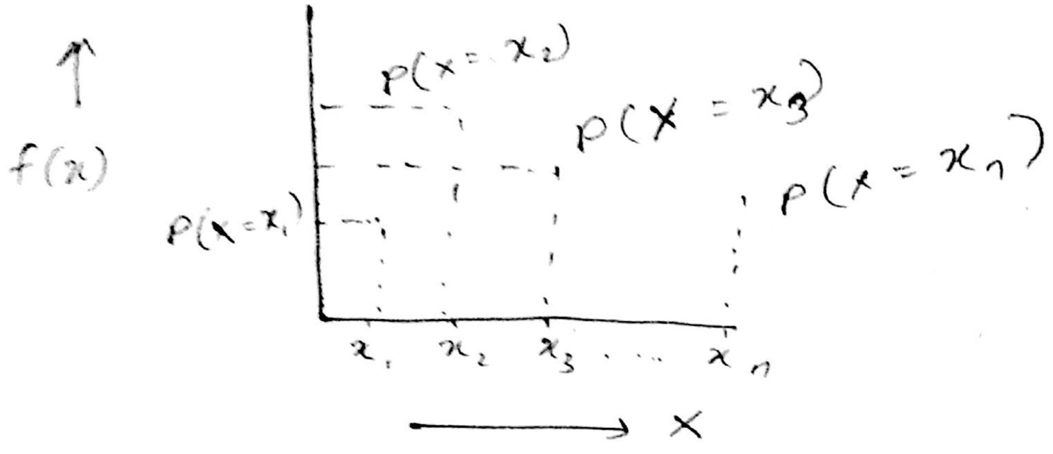
PROBABILITY DISTRIBUTION FUNCTION

1) $p(X = x) = f(x) \rightarrow$ PDF probability mass function

or probability density function

2) $f(x) \geq 0$

3) $\sum_{x} f(x) = 1$



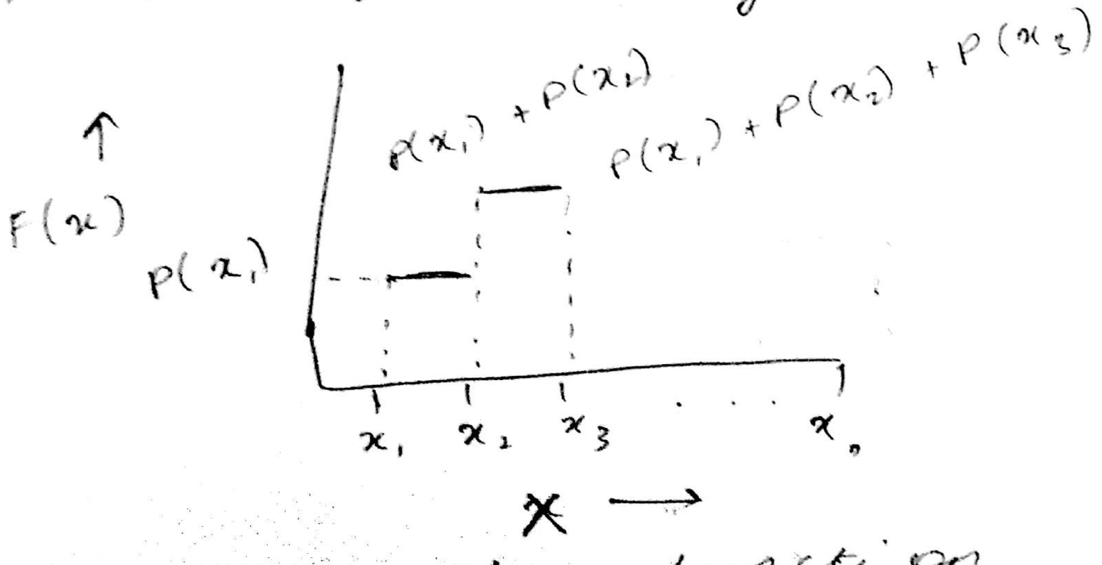
$F(x) = P(X \leq x)$ — CDF Cumulative density function

1) $F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \; -\infty < x < \infty$

Cumulative ~~continuous~~ density function

2) $F(x)$ is a monotonically increasing function, nondecreasing function

i.e, $F(x) \leq F(y)$ if $x \leq y$



This is a step function.

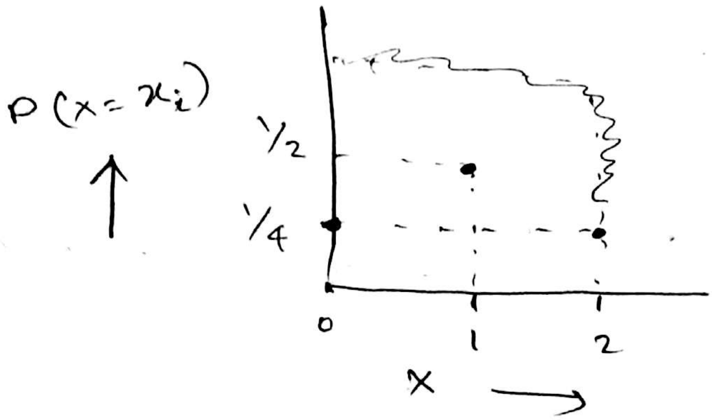
Q) Find the PDF corresponding to

the no. of heads that can come up when you throw a coin twice

Ans) Random variable is the no. of heads

X - r.v the no. of heads that can expect

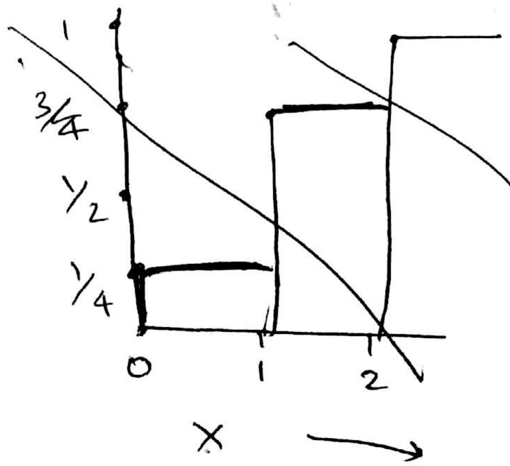
$S = \{TH, HT, HH, TT\}$
$x \in \{0, 1, 2\}$



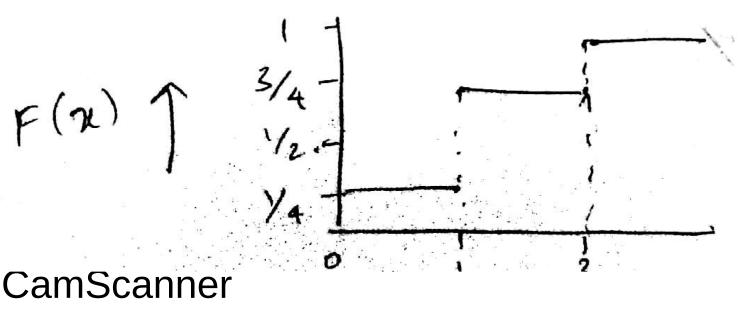
PDF

Sample points	HH	HT	TH	TT
X	2	1	1	0

$P(X = 0) = 1/4$
$P(X = 1) = 1/2$
$P(X = 2) = 1/4$



CDF



CDF

sum of all probabilities upto the point x which is less than or equal to x. is CDF

~~Conti~~

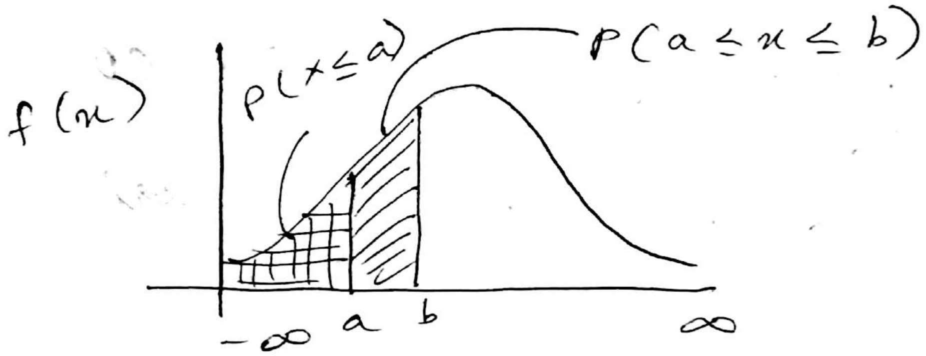
$$F(X) = P(X \le x)$$
$$= \int_{-\infty}^{x} f(x) \cdot dx \quad (-\infty < x < \infty)$$

→ continuous density function

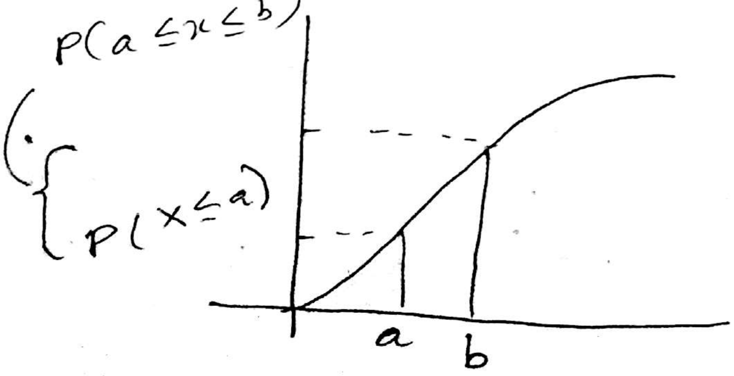
1) $f(x) > 0 \quad , \quad \forall x \in R$

2) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

3) $P(a < x < b) = \int_a^b f(x) \cdot dx$
$$= F(b) - F(a)$$



Area under curve represents the probability.



$$P(X \ge a) = \int_{-\infty}^{\infty} f(x) \cdot dx - \int_{-\infty}^{a} f(x) \cdot dx$$
$$= 1 - P(X \le a)$$

Q) A continuous random variable is defined as

$$f(x) = \begin{cases} x^2/3 & -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

(i) verify $\int_{-\infty}^{\infty} f(x).dx = 1$

(ii) Find $P(0 < x \leq 1)$

Ans) (i) $\int_{-1}^{2} \frac{x^2}{3}.dx = 1$

(ii) $\int_0^1 f(x).dx$

$= \int_0^1 \frac{x^2}{3}.dx = \frac{1}{9}$

$f(x) = \frac{dF(x)}{dx} \longrightarrow$ to find CDF

$F(x) = \int_{-\infty}^{x} f(t).dt$ if derivative exists

Q) Estimate the constant a for the expression

$$f(x) = \begin{cases} ax^2 & 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

~~Ans)~~ What is the probability that a value selected at random from this distribution will be less than 2? What is the probability between 1 and 3? What is probability if it is greater than or equal to 4? What about when it exceeds 6?

Discrete

Probability function / mass

PDF $f(x) =$

CDF $F(x) =$

$=$

CDF is d

values only for di nos.

Q) Defi

$X =$ no

when 3

chosen.

Find th

Ans) P(

Discrete	Continuous
Probability distribution/mass function	Probability density function
PDF $f(x) = P(X = x)$	$f(x) = \int_{-\infty}^{\infty} f(x)\,dx$
CDF $F(x) = P(X \le x)$	
$= \sum P(X \le x)$	$F(x) = \int_{-\infty}^{x} f(x).\,dx$
$x \in R$	

CDF is defined between the discrete values i.e., CDF is assumed not only for discrete values, but all real nos.

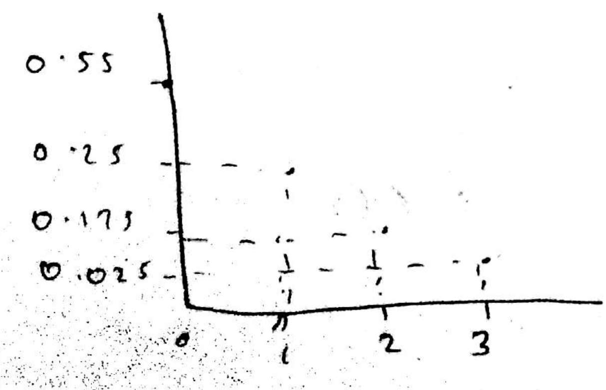
Q) Define a random variable
X = no. of widgets that are defective when 3 widgets are randomly chosen.

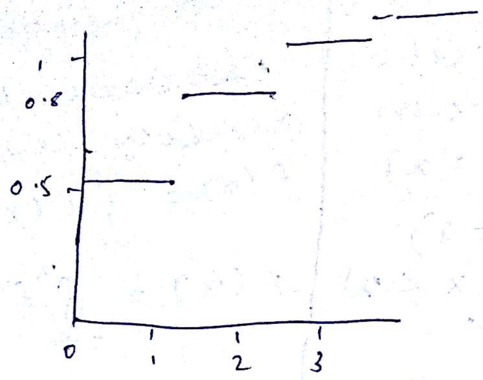
x	$P(X = x)$
0	0.550
1	0.250
2	0.175
3	0.025

Find the probability $P(X \le 2)$?

Ans) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= 0.975$





Q) A random variable X takes the values of $-3, -1, 2$ and 5.

and $P \rightarrow \frac{2k-3}{10}, \frac{k+1}{10}, \frac{k-1}{10}, \frac{k-2}{10}$

Ans) $\frac{2k-3+k+1+k-1+k-2}{10} = 1$

$k = 3$

$P \rightarrow \frac{3}{10}, \frac{4}{10}, \frac{2}{10}, \frac{1}{10}$

Q) $f(x) = ax^2, \quad 0 \le x \le 5$

what is a?

~~Ans)~~ $\cancel{\frac{ax^3}{3}\Big|_0^5 = 1}$

$\cancel{\frac{a \times 125}{3} = 1}$

$\cancel{a = \frac{3}{125}}$

Ans) $\int_{-\infty}^{0} f(x) \cdot dx + \int_{0}^{5} f(x) \cdot dx + \int_{5}^{\infty} f(x) \cdot dx = 1$

$\int_0^5 f(x) \cdot dx = 1$

$$\left.\frac{ax^3}{3}\right|_0^{5} = 1$$

$$a \times \frac{125}{3} = 1$$

$$a = \frac{3}{125}$$

$$P(x \le 2) = F(2) = \frac{3}{125} \times \frac{x^3}{3}$$

$$= \frac{3}{125} \times \frac{8}{3}$$

$$= \frac{8}{125}$$

$$P(1 \le x \le 3) = F(3) - F(1)$$

$$= \frac{26}{125}$$

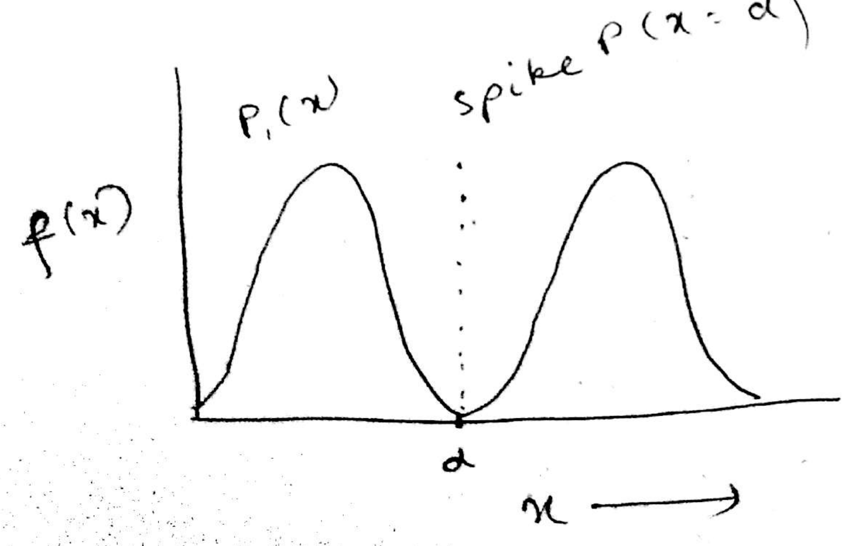
$$P(x \ge 4) = 1 - P(x \le 4)$$

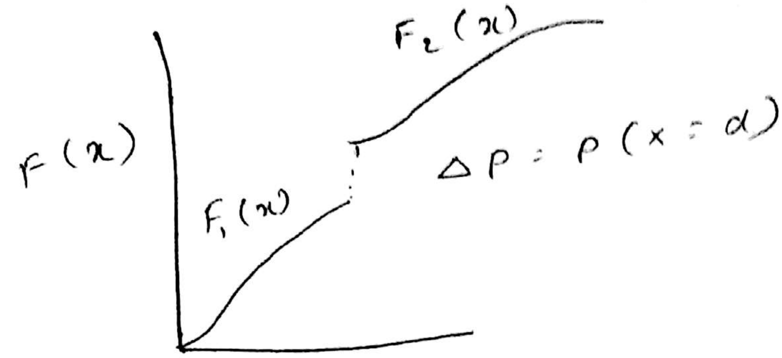
$$= 1 - F(4)$$

$$= 1 - \left(\frac{3}{125} \times \frac{4^3}{3}\right)$$

$$= \frac{61}{125}$$

PIECE WISE CONTINUOUS DISTRIBUTION





$F(X) = F_1(x)$ for $x < d$
$= F_2(x)$ for $x \geq d$.

you differentiate CDF value to get PDF value and take difference between terms to get cumulative.

MULTIVARIATE (JOINT DISTRIBUTION)

X and Y - random variables (discrete)

$f(x,y) = P(X=x, Y=y)$

1) $P(X=x, Y=y) = f(x,y)$

2) $f(x,y) \geq 0$

3) $\sum_x \sum_y f(x,y) = 1$

$$P((x,y) \in A) = \sum_A \sum f(x,y)$$

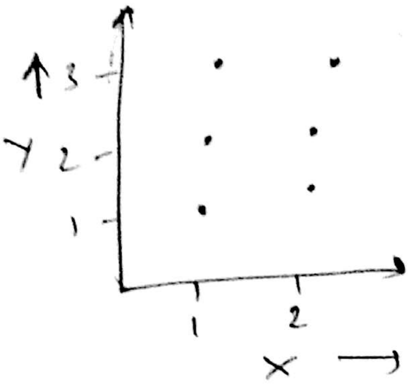
where A is the range.

Q) The joint probability function of 2 discrete random variables is $f(x,y) = c(2x+y)$. x and y can assume all integers such that

~~$f(x,y)$~~ $0 \leq x \leq 2$
$0 \leq y \leq 3$
$f(x,y) = 0$ otherwise

Find value of c.

Ans) $\cancel{\sum_{x=0}^{2} \sum_{y=0}^{3} c(2x+y)} = 1$



Sample space

X/Y →	0	1	2	3
0	0	c	2c	3c
1	2c	3c	4c	5c
2	4c	5c	6c	7c

$42c = 1$

$c = \frac{1}{42}$

Find the probability of $P(x=2, y=1)$

$= \cancel{\frac{8}{42}} \; 5c$

$= 5 \times \frac{1}{42}$

$= \frac{5}{42}$

What is $P(x \geq 1, y \leq 2)$?

$= 2c + 3c + 4c + 5c \; \cancel{\frac{14}{42}} + 4c + 6c$

$= \frac{24}{42}$

X, Y - continuous random variables

joint PDF $f(x, y)$

1) $P[(x,y) \in A] = \iint_A f(x,y)\,dx\,dy$

2) $f(x,y) \geq 0 \;\forall (x,y)$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Q) A candy company distributes boxes of chocolates

X = proportion of light chocolates

Y = " dark "

$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Ans) $\int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$

$= \cancel{\int_0^1 \frac{4}{5}\frac{x^2}{2} + \frac{6}{5}y\,dx}$

$= 1$

Q) $P((X,Y) \in A)$?

$A = \{(x,y) \mid 0 < x < \frac{1}{2},\ \frac{1}{4} < y < \frac{1}{2}\}$

Ans) $\int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\,dx\,dy$

$= \int_{1/4}^{1/2} \left.\frac{2x^2}{5} + \frac{6yx}{5}\right|_{x=0}^{x=1/2} dy$

$= \int_{1/4}^{1/2} \left(\frac{1}{10} + \frac{3y}{5}\right) dy$

$$= \frac{y}{10} + \frac{3y^2}{10} \Big|_{y=1/4}^{y=1/2}$$

$$= \frac{13}{160}$$

Q) $f(x,y) = \begin{cases} cxy, & 0 < x < 4, \ 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$

Ans) $$\int_0^4 \int_1^5 cxy \, dx \, dy = 1$$

$$c = \frac{1}{96}$$

Q) $P(1 < x < 2, \ 2 < y < 3)$

Ans) $$\int_2^3 \int_1^2 cxy \cdot dx \cdot dy$$

$$f(x,y) = \frac{xy}{96}$$

$$\int_1^2 \int_2^3 \frac{xy}{96} \cdot dx \cdot dy = \frac{5}{128}$$

Q) $P(x \geq 3, \ y \leq 2)$

Ans) $$\int_{x=3}^{4} \int_{y=1}^{2} \frac{xy}{96} \cdot dx \cdot dy$$

$$= \frac{7}{128}$$

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joint probability mass function

$P(X=x, Y=y) = f(x,y)$

$P[(X,Y) \in A] = \sum_A \sum f(x,y)$

$f(x,y) \geq 0 \quad \forall\, x\, y$

$\sum_x \sum_y f(x,y) = 1$

Density function

$P[(X,Y) \in A] = \iint_A f(x,y)\, dx\, dy$

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

$X \downarrow$ \ $Y \rightarrow$	Y_1	Y_2	$\ldots$	Y_n	
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	$\ldots$	$f(x_1,y_n)$	$f_1(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	$\ldots$	$f(x_2,y_n)$	$f_1(x_2)$
$\vdots$	$\vdots$				$\vdots$
X_m	$f(x_n,y_1)$	$\ldots$	$\ldots$	$f(x_m,y_n)$	$f_1(x_m)$
	$f_2(y_1)$	$f_2(y_2)$	$\ldots$	$f_2(y_n)$	

(Sum of each column)

~~Marginal~~

$f_1(x_1) = \sum_{i=1}^{n} f(x_1, y_i)$

Given the joint probability distribution $f(x,y)$, probability of 'X' represented as $g(x)$ alone $P(X=x_j) = \sum_{k=1}^{n} f(x_j, y_k)$

|||ly $h(y)$ $\quad P(Y=y_k) = \sum_{j=1}^{m} f(x_j, y_k)$

$$g(x) = \int_{-\infty}^{\infty} f(x,y) \cdot dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y) \cdot dx$$

$g(x)$ is marginal probability of X;
$h(y)$ " of Y

Q) $f(x,y) = c(2x+y)$

$c = \frac{1}{42}$

Ans)

$x \backslash y$	0	1	2	3	
0	0	c	$2c$	$3c$	$6c$
1	$2c$	$3c$	$4c$	$5c$	$14c$
2	$4c$	$5c$	$6c$	$7c$	$22c$
~~3x~~	$6c$	$9c$	$12c$	$16c$	

$\left(\begin{array}{l} x \le 2 \\ y \le 3 \end{array} \right)$

$P(X = x) = g(x)$

$$g(x) = \begin{cases} 6c = 6 \times \frac{1}{42} = \frac{1}{7}, & x = 0 \\ 14c = 1/3, & x = 1 \\ 22c = 11/21, & x = 2 \end{cases}$$

Summation of all these should be 1.

$1/7 + 1/3 + 11/21 = 1$

$$h(y) = \begin{cases} 6c = 1/7, & y = 0 \\ 9c = 3/14, & y = 1 \\ 12c = 2/7, & y = 2 \\ 16c = 8/14, & y = 3 \end{cases}$$

Q) $f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

marginal density?

Ans) $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$= \int_0^1 \frac{2}{5}(2x+3y)\, dy$

$= \int_0^1 \frac{4}{5}x + \frac{6}{5}y \, dy$

$= \frac{4}{5} y \Big|_0^1 + \frac{\cancel{6}^{3}}{5} \frac{y^2}{\cancel{2}} \Big|_0^1$

$= \frac{4x}{5} + \frac{3}{5}, \quad 0 \le x \le 1$

$\cancel{= \frac{7}{5}}$

$\boxed{\text{DON'T FORGET TO MENTION RANGE}}$

$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$= \int_0^1 \frac{4}{5}x + \frac{6}{5}y \, dx$

$= \frac{4x^2}{5\times 2}\Big|_0^1 + \frac{6}{5}y$

$= \frac{2}{5} + \frac{6y}{5}, \quad 0 \le y \le 1$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$X = x,\ Y = y$

$P(Y=y / X=x) = \frac{P(X=x, Y=y)}{P(X=x)}$

Let X,

or cont

$f(x/y)$

$f(y/x$

Q) There

$X \rightarrow$

$Y \rightarrow$

f(x

find

densit

ans) g

h(y)

$$= \frac{f(x,y)}{g(x)}, \quad g(x) \geq 0$$

Let X, Y be discrete, random or continuous random variables

$$f(x/y) = \frac{f(x,y)}{h(y)}$$

$$f(y/x) = \frac{f(x,y)}{g(x)}$$

Q) There is an experiment

X → unit temp. change

Y → proportion of spectrum shift that a certain atomic particle produces.

$$f(x,y) = \begin{cases} 10xy^2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal and conditional density functions

Ans) $g(x) = \int_{-\infty}^{\infty} f(x,y) \cdot dy$

$$= \int_{-\infty}^{\infty} 10xy^2 \cdot dy$$

$$= \frac{10xy^3}{3} \Big|_{x}^{1}$$

$$= \frac{10x}{3} - \frac{10x^2}{3}$$

$$h(y) = \int_{-\infty}^{\infty} 10xy^2 \cdot dx$$

$$= \frac{10x^2 y^2}{2} \Big|_{0}^{y}$$

$$= 5y^2$$

Ans) $g(x) = \int_{-\infty}^{\infty} 10xy^2 \cdot dy$

$= \int_{x}^{1} 10xy^2 \cdot dy$

$= \frac{10xy^3}{3} \Big|_{y=x}^{1}$

$= \frac{10x}{3}(1-x^3), \quad 0<x<1$

$h(y) = \int_{0}^{y} 10xy^2 \cdot dx$

$= \frac{10y^2x^2}{2} \Big|_{0}^{y}$

$= 5y^2(y^2), \quad 0<y<1$

DON'T FORGET TO ADD RANGE

$f(y|x) = \frac{f(x,y)}{g(x)}$

$= \frac{10xy^2}{\frac{10x}{3}(1-x^3)}$

$= \frac{3y^2}{1-x^3}, \quad 0<x<y<1$

Q) Find the probability that the spectrum shifts more than half of the total observation given that temp. is increased to 0.25 unit.

Ans) $f(y|x)$

ans) $P($

Q) $f(x$

ans)

$h(y$

~~Ans) $f(y|x) = \frac{3y^2}{1-x^3}$ = $\frac{3 \cdot 1/4}{1-(0.25)^3}$~~

Ans) $P(Y > 1/2 \,/\, x = 0.25)$

$= \frac{\cancel{10xy^2}}{g(x)}$

$= \frac{\cancel{10xy^2}}{}$

$= \int_{1/2}^{1} \frac{3y^2}{1-x^3} \cdot dy$

$= \int_{1/2}^{1} \frac{3y^2}{1-(0.25)^3} \cdot dy$

$= \frac{8}{9}$

Q) $f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$

Calculate $f(x|y)$

~~Ans~~ $P(\frac{1}{4} < X < 1/2 \,|\, Y = 1/3)$

Ans) $f(x|y) = \frac{f(x,y)}{h(y)}$

$h(y) = \int_0^2 \frac{x(1+3y^2)}{4} \cdot dx$

$= \int_0^2 \left(\frac{x}{4} + \frac{3y^2 x}{4}\right) \cdot dx$

$= \frac{x^2}{8}\Big|_0^2 + \frac{3y^2 x^2}{8}\Big|_0^2$

$$= \frac{1}{2} + \frac{3y^2}{2}(4)$$

$$= \frac{1}{2} + \frac{3y^2}{2}, \quad 0 < y < 1$$

$$f(x/y) = \frac{f(x, y)}{h(y)}$$

$$= \frac{\frac{x(1+3y^2)}{4}}{\frac{1+3y^2}{2}}$$

$$= \frac{x}{2}$$

$$P\left(\tfrac{1}{4} < x < \tfrac{1}{2} \,\middle|\, Y = \tfrac{1}{3}\right)$$

$$= \int_{1/4}^{1/2} \frac{x}{2} \cdot dx$$

$$= \frac{1}{2} \cdot \frac{x^2}{2} \Big|_{1/4}^{1/2}$$

$$= \frac{x^2}{4} \Big|_{1/4}^{1/2}$$

$$= \frac{1}{16} - \frac{1}{64}$$

$$= \frac{3}{64}$$

If X and Y are 2 random variable (discrete or continuous) and $f(x, y)$ are joint PDFs where $g(x)$ & $h(y)$ are marginals of x & y.
if $f(x, y) = g(x) \cdot h(y) \quad \forall (x, y)$
it is statistically idependent.

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$$\frac{f(x,y)}{h(y)} = f(x/y) = g(x)$$

$$\Rightarrow f(x,y) = g(x) \cdot h(y)$$

Let's consider $X_1, X_2, \dots, X_n$ are the multiple variables.

joint PDF $\rightarrow f(x_1, x_2, \dots x_n)$

The marginal distribution of

$$X_1 \rightarrow g(x_1) = \sum_{x_2} \dots \sum_{x_n} f(x_1, x_2, x_3 \dots x_n)$$

$$g(x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, \dots x_n) \cdot \cancel{dx_1}\, dx_2 \cdot dx_3 \dots dx_n$$

marginal distribution of 2 variables

$$g(x_1, x_2) = \begin{cases} \sum_{x_3} \dots \sum_{x_n} f(x_1, x_2, x \dots x_n) \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2, \dots x_n)\, dx_3\, dx_4 \cdot dx_n \end{cases}$$

$$f(x_1, x_2, x_3 / x_4, x_5, \dots x_n)$$

$$= \frac{f(x_1, x_2 \dots x_n)}{g(x_4, x_5, \dots x_n)}$$

↳ joint marginal distribution at $x_4, x_5 \dots x_n$

$$f(x_1, x_2 \dots x_n) = f_1(x_1)\, f_2(x_2) \dots f_n(x_n)$$

in the case of statistically independent

Population is the complete assemblage of all the values representative of a particular random process.
Sample is a subset of measurements selected from the population of interest.

1) Qualitative data - categorical data

2) Quantitative data — Discrete / Continuous

Frequency = no. of measurements in each category.

Relative frequency is the proportion of measurements in each category

$$Rf = \frac{\text{frequency}}{n}$$

where n is total no. of measurements / observations in a set

Range = max. value - min. value

$$\text{width} = \frac{\text{Range}}{\text{Classes}}$$

eg)

Class	Class Boundaries	Tally	Class frequency	Class relative frequency
1	5.6 to <6.1		2	2/30
2	6.1 to <6.6		2	2/30
3	6.6 to <7.1		4	4/30
4	7.1 to <7.6		5	5/30
5	7.6 to <8.1		8	8/30
6	8.1 to <8.6		5	5/30
7	8.6 to <9.1		3	3/30
8	9.1 to <9.6		1	1/30

simple mean $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

median, $\tilde{x} = \begin{cases} x_{(n+1)/2} & \text{if odd} \\ \frac{1}{2}(x_{n/2} + x_{n/2+1}) & \text{even} \end{cases}$

The purpose of sample median is to restrict the central tendency of the sample in such a way that is uninfluenced by extreme values or outliers.

$(x_i - \bar{x})$ is deviation

$(x_i - \bar{x})^2$ is mean deviation

$\sigma^2 = \frac{\Sigma(x_i - \bar{x})^2}{N}$ is variance.

There are $(n-1)$ $(x_i - \bar{x})$ values deviation that make ~~it~~ zero.

$\sigma^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$ for a sample

This is degree of variance.

Sample	Population
n: no. of measurements in the sample	N: no. of measurements in the population
s^2 variance	σ^2
$s^* = \sqrt{s^2}$ = sample standard deviation ~~variance~~	$\sigma = \sqrt{\sigma^2}$ = population std

experiment - two coins 16 times

$x \rightarrow$ no. of heads that can occur

$X \rightarrow 0, 1, 2$

Frequency of each $X \rightarrow 4, 7, 5$

avg. no. of heads of 2 coins

$$= \frac{(0\times4) + (1\times7) + (2\times5)}{4+7+5}$$

$$= \frac{17}{16}$$

$$= 1.06 \qquad = 0\times\frac{4}{16} + 1\times\frac{7}{16} + 2\times\frac{5}{16} = \sum_x x\,f(x)$$

Relative frequency

This is expected value of random variable.

$x_1, x_2, \ldots x_n$ — r.v

$f(x_1), f(x_2), \ldots, f(x_n)$

$\mu = E(x) = \sum_x x f(x) \rightarrow$ Discrete

$\mu = E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \rightarrow$ continuous

Q) In a gambling game, a man is paid 5 dollars if he gets all heads or all tails and he will pay 3 dollars if either 1 or 2 heads show. What is his expected game if 3 coins are tossed?

Ans) S = { HHH, HHT, HTH, THH, HTT, THT, TTH, TTT }

Y - Random variable amount

the gambler can win

$y \rightarrow \$5$ if event $E_1 = \{HHH, TTT\}$

$y \rightarrow -\$3$ if event E_2

$= \$ - E_1$

~~$E(y) = 5 \times \frac{1}{8}$~~

$P(E_1) = \frac{2}{8} = \frac{1}{4}$

$P(E_2) = \frac{3}{4}$

$\mu = 5 \times \frac{1}{4} - \frac{3}{4}(3)$

$= -1$

Q) X is the random variable which denotes the life in hours of a certain electronic device

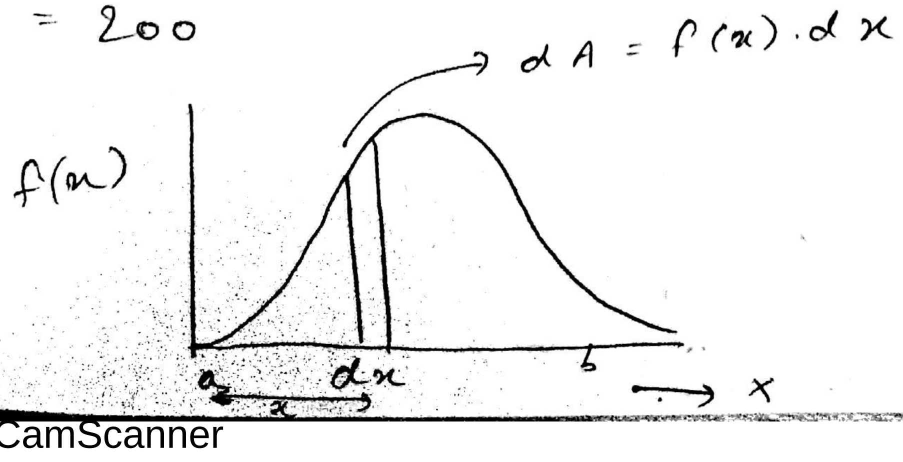
$$f(x) = PDF = \begin{cases} 290000/x^3 & , x > 100 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected life of the device

Ans) $\int_{100}^{\infty} x \cdot \frac{280000}{x^3} \cdot dx$

$= -20000 \; x^{-1} \Big|_{100}^{\infty}$

$= 200$



The first moment of the total area about origin

$$\mu_1 = \int_A x \cdot dA$$

$$dA = f(x) \cdot dx$$

$$\mu_1 = \int_A x f(x) \cdot dx = \int_{-\infty}^{\infty} x f(x) \cdot dx$$

$$\mu_i = \int_{-\infty}^{\infty} x^i f(x) \cdot dx$$

First moment towards mean is variance

i^{th} central moment towards mean $\mu_i = \int_{-\infty}^{\infty} (x-\mu)^i f(x) \cdot dx$

Second moment towards mean – σ^2

$E(x) = \mu,$

$$\begin{aligned} Var(x) &= \sigma^2 = \mu_2 \\ &= E[(x-\mu)^2] \\ &= \sum_x (x-\mu)^2 f(x) \rightarrow \text{Discrete} \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) \cdot dx \rightarrow \text{Continuous} \end{aligned}$$

Q) x are the no. of automobiles used for official purposes.

Company A

$x \rightarrow$	1	2	3
$f(x)$	0.3	0.4	0.3

Company B

$x \rightarrow$	0	1	2	3	4
$f(x) \rightarrow$	0.2	0.1	0.3	0.3	0.1

Which company has more variance in dataset?

Ans) company A

σ^2

$\mu_A = E(A) = 0.3 + 0.8 + 0.9$
$= 2$

$\mu_B = E(B) = 0.1 + 0.6 + 0.9 + 0.4$
$= 2$

$\sigma_A^2 = \sum (x - \bar{x})^2 \cdot f(x)$
$= (1-2)^2 \times 0.3 + (0) + (1 \times 0.3)$
$= 0.6$

$\sigma_B^2 = \cancel{0.2}\ 1.6$

sample

$\bar{x} = \frac{\sum x_i}{n}$

$s^2 = \sum_i \frac{(x_i - \bar{x})^2}{n-1}$

population

μ

$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N}$

PDF = $f(x)$

CDF = $F(x)$

$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$f(x : \theta_1, \theta_2, \ldots \theta_n)$ → population parameter

Estimate parameter $\hat{\theta}_i$

$E[\hat{\theta}] = \theta \rightarrow$ unbiased estimate

$E[\hat{\theta}] \cdot \theta \rightarrow$ gives bias

$E[s^2] = \sigma^2$

$$\sigma^2 = \frac{\Sigma(x_i - \mu)^2}{N}$$

$$\Sigma(x_i - \mu)^2 = \Sigma(x_i - \bar{x} + \bar{x} - \mu)^2$$

$$= \Sigma\left[(x - \bar{x})^2 + (\bar{x} - \mu)^2 + 2(x_i - \bar{x})(\bar{x} - \mu)\right]$$

$$= \Sigma(x_i - \bar{x})^2 + (\bar{x} - \mu)^2 n + 2(\bar{x} - \mu)\,\Sigma(x_i - \bar{x})$$

$\Sigma x_i - \bar{x} = 0$

$$= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

$$= s^2(n-1) + n(\bar{x} - \mu)^2$$

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$$

$$\Sigma(x_i - \bar{x})^2 = s^2(n-1)$$

$$s^2 = \frac{\Sigma(x_i - \mu)^2}{n-1} - \frac{n}{n-1}(\bar{x} - \mu)^2$$

Expected value of s^2?

$$E[s^2] = E\left[\frac{\Sigma(x_i - \mu)^2}{n-1} - \frac{n}{n-1}(\bar{x} - \mu)^2\right]$$

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$$E[s^2] = \frac{1}{n-1} \sum E(x_i - \mu)^2 - \frac{n}{n-1} E(\bar{x}_i - \mu)^2$$

$$= \frac{1}{n-1} \sum var(x_i) - \frac{n}{n-1} var(\bar{x})$$

$$= \frac{n}{n-1} \sigma^2 - \frac{n}{n-1}\left(\frac{\sigma^2}{n}\right)$$

$$\boxed{\begin{aligned} \sigma_{x_i}^2 &= \sigma^2 \\ \sigma_{\bar{x}}^2 &= \frac{\sigma^2}{n} \end{aligned}}$$

$$E[s^2] = \frac{n}{n-1} \sigma^2 - \frac{1}{n-1} \sigma^2$$

$$= \frac{n-1}{n-1} \sigma^2$$

$$= \sigma^2$$

$$\sigma^2 = \sum_x (x - \mu)^2 f(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) \cdot f(x)$$

$$= \sum_x \left[x^2 f(x) - 2\mu x \cdot f(x) + \mu^2 f(x)\right]$$

$$= \sum_x x^2 f(x) - 2\mu \sum_x x f(x) + \mu^2 \sum_x f(x)$$

f(x) is relative frequency

$$\boxed{\sum_x x f(x) = E[x]}$$

$$= \sum_x x^2 f(x) - 2\mu^2 + \mu^2$$

$$= \sum_x x^2 f(x) - \mu^2$$

$$= E[x^2] - \mu^2$$

Q) The weekly demand for cold drink in thousands of litres from a local store chain is a continuous random variable x having probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find mean and variance of x.

Ans) $\int_1^2 2(x-1)$

$= \int_1^2 2x\,dx - \int_1^2 2\,dx$

$= \frac{2x^2}{2}\Big|_1^2 - 2x\Big|_1^2$

$= 3 - 2$

$= 1$

$\mu = E(x) = 2\int_1^2 x(x-1)\,dx = 5/3$

$\sigma^2 = E(x^2) - \mu^2$

$E(x^2) = \int_1^2 x^2 f(x)\,dx = 2\int_1^2 x^2(x-1)\,dx = 17/6$

$\sigma^2 = \frac{1}{18}$

X and Y μ_x μ_y

$\sigma = \sum_x \sum_y (x-\mu_x)(y-\mu_y)\cdot f(x,y)$

$= \sum_x \sum_y (xy - \mu_x y - x\mu_y + \mu_x\mu_y)\cdot f(x,y)$

$= \sum_x \sum_y (xy\,f(x,y)) - \mu_x \sum_x \sum_y y\,f(x,y))$

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$$- \mu_y \sum_x \sum_y x f(x,y) + \mu_x \mu_y \sum_x \sum_y f(x,y)$$

1

$$\boxed{\mu_x = \sum_x x f(x,y)}$$

$$= E(xy) - \mu_x \mu_y - \mu_y \mu_x + \mu_x \mu_y$$

$$\sigma_{xy} = E(XY) - \mu_x \mu_y$$

Correlation

~~$\sigma_{xy} = E(X$~~

Q)

$f(x,y)$	0	1	2	
0	3/28	9/28	3/28	15/28
1	3/14	3/14	0	3/7
2	1/28	0	0	1/28
	5/14	15/28	3/28	

$$\mu_x = \sum \sum x \cdot f(x,y)$$

$$\mu_y = \sum \sum y \, f(x,y)$$

$$E[XY] = \sum_x \sum_y xy \, f(x,y)$$

$$= \frac{3}{14}$$

marginal

$$\mu_x = \sum_y \sum_x x \cdot f(x,y) = \sum_x x \cdot g(x)$$

$$= 0\left(\frac{5}{14}\right) + 1\left(\frac{15}{28}\right) + 2\left(\frac{3}{28}\right) = \frac{3}{4}$$

$$\mu_y = \sum_x \sum_y y f(x,y) = \sum_y y \, h(y)$$

$$= 0\left(\frac{15}{28}\right) + 1\left(\frac{3}{7}\right) + 2\left(\frac{1}{8}\right) = \frac{1}{2}$$

$$E(xy) = \int_{-\infty}\int_{-\infty} xy\, f(x,y)\cdot dx \cdot dy$$

$$f(x,y) = g(x)\cdot h(y) \quad [\text{for uncorrelated}$$

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x)\cdot h(y)\cdot dx\, dy$$

$$= \int_{-\infty}^{\infty} x\cdot g(x)\cdot dx \int_{-\infty}^{\infty} y\cdot h(y)\cdot dy$$

$$= E[x]\, E[Y] \quad [\text{for uncorrelated}$$

$$\sigma_{xy} = E(XY) - \mu_x \mu_y$$

$$= E[X]\, E[Y] - \mu_x \mu_y = 0$$

covariance $(x, y) = 0$

MEAN AND VARIANCE OF LINEAR COMBINATION OF R.Vs

$$g(x) = ax + b$$

$$E[ax+b] = \int_{-\infty}^{\infty} (ax+b)\, f(x)\cdot dx$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\cdot dx$$

$$= a\int_{-\infty}^{\infty} x f(x)\cdot dx + b\int_{-\infty}^{\infty} f(x)\cdot dx$$

$$= a\, E[x] + b$$

Q) $a = 0$, $E[b] = b$

$b = 0$ $E[ax] = a\, E[x]$

Q) $g[x] = 4x + 3$ $\quad X \longrightarrow$ R.V.

$$f(x) = \begin{cases} x^2/3 & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[g(x)]$

Ans) $E[4x+3]$

$= 4E[x] + 3$

$$E[x] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} x \cdot (x^2/3) \cdot dx = \frac{5}{4}$$

$$= \cancel{\left. \frac{x^4}{12} \right|_{-1}^{2}}$$

$$= \cancel{\frac{16}{12} + \frac{1}{12}}$$

$$E[g(x)] = \left(4 \times \frac{5}{4}\right) + 3$$

$= 8$

$$\sigma_{ax+b}^2 = E\left[\left[(ax+b) - \mu_{ax+b}\right]^2\right]$$

$$\mu_{ax+b} = E[ax+b]$$

$= aE[x] + b$

$= a\mu + b$

$= E[(ax+b-a\mu-b)^2]$

$= E[a^2(x-\mu)^2]$

$= a^2 E[(x-\mu)^2]$

$= a^2\sigma^2$

$\sigma_{ax+b}^2 = a^2 \sigma_x^2$

$a = 1 \quad \sigma_{x+b}^2 = \sigma_x^2$

$a = 0, \quad \sigma^2 = 0$

Q) Let x, y denote amt. of 2 impurities

$\sigma_x^2 = 2$

$\sigma_y^2 = 3$

$Z = 3X - 2Y + 5$

Find variance of random variable Z

ans) $\sigma_{ax} = a^2 \sigma_x^2$

$\sigma_Z^2 = 9\sigma_x^2 + 2^2 \sigma_y^2$

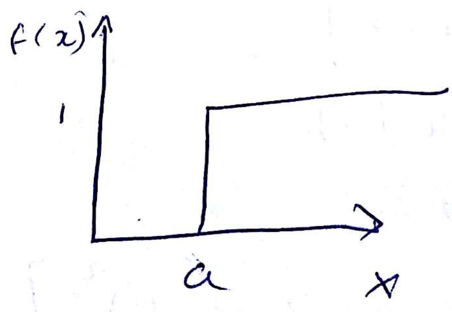
$= \cancel{9x}\,(9 \times 2) + (4 \times 3)$

$= 30$

MARKOV INEQUALITY

$X \rightarrow$ non negative random variable $X \geq 0$

step function:



$P[X \geq a] \leq \frac{E[X]}{a}$

Q,

$E[X] = \int_{-\infty}^{\infty} x . f(x) . dx$

$= \int_{0}^{\infty} x . f(x) . dx$

A $\int_{0}^{a} x . f(x) . dx \geq \int_{a}^{\infty} x . f(x) . dx$

$$\int_a^\infty a f(x)\cdot dx = a\, P(x \geq a) \longrightarrow (2)$$

$$\Rightarrow E[x] \geq a\, P(x \geq a] \quad [\text{from } (1), (2)]$$

$$P(x \geq a) \leq \frac{E[x]}{a}$$

$\cancel{x = (Y-\mu)^2}$

CHEBESHOV INEQUALITY

$x = (Y-\mu)^2 \longrightarrow$ standard deviation

$$\mu_x = E(x) = E[(Y-\mu)^2] = Var[x]$$

$$P[x \geq b^2] \leq \frac{E[x]}{b^2}$$

$$P[(Y-\mu)^2 \geq b^2] \leq \frac{E[x]}{b^2}$$

$$P[(Y-\mu) \geq b] \leq \frac{E[x]}{b^2}$$

$$P[(Y-\mu) \geq b] \leq \frac{var[Y]}{b^2}$$

$$= \frac{\sigma^2}{b^2}$$

$b = k\sigma$ (defining b as a constant) k is not constant

$$P[(Y-\mu) > k\sigma] \leq \frac{\sigma^2}{k^2\sigma^2}$$

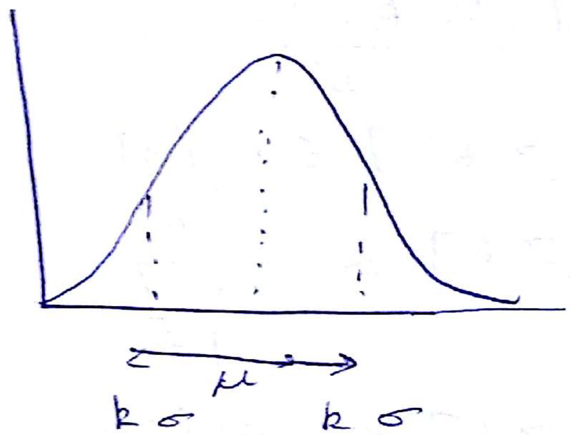
$$P[(Y-\mu] \geq k\sigma] \leq \frac{\sigma^2}{k^2\sigma^2}$$

$$1 - P[(Y-\mu) \leq k\sigma] \leq \frac{1}{k^2}$$

$$P[(Y-\mu) \leq k\sigma] \geq 1 - \frac{1}{k^2}$$

Chebyshev inequality states that a single observation selected at random from any probability distribution will deviate more than k times σ from the mean with a probability less than or equal to $1/k^2$

$$P[\mu - k\sigma < x < \mu + k\sigma] \geq 1 - \frac{1}{k^2}$$



$k = 1$, $1 - \frac{1}{k^2} = 0$

$\mu - \sigma$ to $\mu + \sigma$ is zero probability

$k = 2$, $1 - \frac{1}{k^2} = 3/4$

75% chance of getting in between $\mu - 2\sigma$ to $\mu + 2\sigma$

$k = 3$, $1 - \frac{1}{k^2} = \frac{8}{9}$

$\mu - 3\sigma$ to $\mu + 3\sigma$

Q) x - Random variable

$\mu = 75$ $\quad \sigma = 5$

Determine the interval [a, b] about the mean for which the probability that x lies in the interval is 99%.

Ans) $P[\mu - K\sigma < X < \mu + K\sigma]$

$$\geq \frac{99}{100}$$

$$\frac{99}{100} \geq 1 - \frac{1}{K^2}$$

$$\frac{1}{K^2} = \frac{1}{100}$$

$$K = 10$$

$\mu - 10\sigma$ to $\mu + 10\sigma$

$[75 - 10 \times 5]$ to $[75 + 10 \times 5]$

25 to 125

[25, 125]

DISCRETE PROBABILITY DISTRIBUTIONS

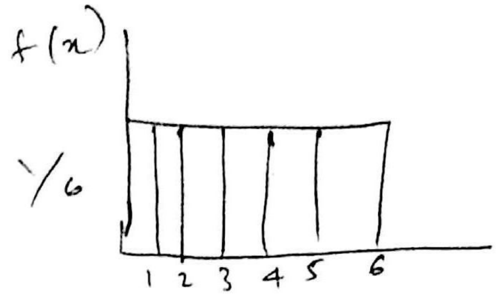
Uniform

$f(x; K) = \frac{1}{K}$ $\quad x = x_1, x_2 \ldots x_k$

K- Parameter of uniform distribution

eg. Rolling a die

$1/6$



$$\mu = E(X) = \sum_{i=1}^{k} x \cdot f(x, k)$$

$$= \sum_{i=1}^{k} \frac{x_i}{k}$$

$$= \frac{1}{k} \sum_{i=1}^{k} x_i$$

$$\sigma^2 = E\left((X-\mu)^2\right)$$

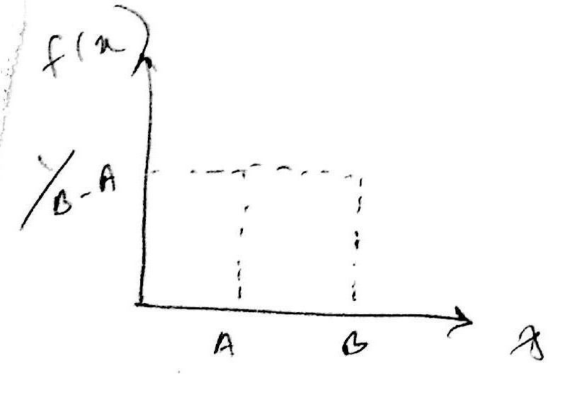
$$= \sum_{i=1}^{k} (x-\mu)^2 f(x, k)$$

$$= \frac{1}{k} \sum_{i=1}^{k} (x_i - \mu)^2$$

$$\mu = \frac{1+2+3+4+5+6}{6} = 3.5$$

$$\sigma^2 = 2.92$$

$$f(x, A, B) = \begin{cases} 1/B-A & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$



Rectangular distribution

rain assumption that probability of falling in an interval of fixed length within (A, B) is constant.

$f(x, A, B)$

Q) $P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$

Ans) $\mu = E[X] = \int_a^b x \, f(x) \, dx$ $\left[f(x) = \frac{d}{dx}\left(\frac{x-a}{b-a}\right) \right]$

$= \int_a^b x \left(\frac{1}{b-a}\right) dx$

$= \frac{1}{b-a} \int_a^b x \, dx$

$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$

$= \frac{a+b}{2}$

$\sigma^2 = E[(X-\mu)^2] = E[X^2] - \mu^2$

$E(X^2) = \int_a^b \frac{1}{b-a} x^2 \, dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$

$= \frac{b^2 + ab + a^2}{3}$

$\sigma^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2$

$= \frac{(b-a)^2}{12}$

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BINOMIAL DISTRIBUTION

eg. 3 years of flow

success : p is $P(X \geq 10,000)$

Failure : q is $P(X \leq 10,000)$

In third year - p
second " - q
first " - q

$\left.\right\}$ pqq

$pqq + qpq + qqp$ } probability of one success and 2 failure
$= 3pq^2$

2 exceedences in 5 years

ppqqq , pqppp

$\binom{5}{2} p^2 q^3$

$$f_X(x : n, p) = \begin{bmatrix} n \\ x \end{bmatrix} p^x q^{n-x}$$

Bernoulli's trial can result in a success with probability p and failure with probability q with probability distribution of binomial random variable. x. The no. of success in n independent trials is ${}^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r}$

~~$p = 1$~~ p - probability of getting 1
~~$q = 0$~~ q - " 0

I → Bernoulli random variable
i^{th} trial I_i

sum of n independent variables

Requirements:

(i) n repeated trials.

(ii) Each repeated trial should result in either success or failure

(iii) Repeated trials are independent

$$X = I_1 + I_2 + \dots + I_n$$

$$E[X] = E[I_1] + E[I_2] + \dots + E[I_n]$$

$$E[I_1] = 0 \times q + 1 \times p = p$$

$$E[X] = p + p + \dots p$$

$$= np$$

$$\sigma_{I_j}^2 = E[(I_j - p)^2]$$

$$= E[I_j]^2 - p^2$$

$$= 0^2 q + 1^2 p - p^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$= pq$$

$$\sigma_X^2 = \sigma_{I_1}^2 + \sigma_{I_2}^2 + \sigma_{I_3}^2 + \dots + \sigma_{I_n}^2$$

$$= pq + pq + \dots$$

$$= npq$$

Q) Find the probability of getting exactly 2 heads in 6 tosses in fair coin.

Ans) $\binom{6}{2} p^2 q^4 = \frac{6!}{2!\,4!} \left(\frac{1}{2}\right)^6 = 15/64$

Q) The probability that a certain kind of component will survive a shock test is 3/4. Find probability that 2 of the next 4 component test will survive.

ans) $^4C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2$

$= \cancel{\frac{4!}{2!\,2!} \left(\frac{1}{16}\right) \left(\frac{9}{16}\right)}$

$= \cancel{\frac{24^6}{4} \left(\frac{1}{16_8}\right) \frac{9}{16}}$

~~Q) b – blue marbles~~
~~r – red ,,~~

GEOMETRIC BINOMIAL DISTRIBUTION

b – blue marble
r – red ,,
x → blue ,, (success) n – trials

p – success of getting a blue marble

$p = \frac{b}{b+r}$, $q = 1-p = 1 - \frac{b}{b+r} = \frac{r}{b+r}$

$$p(X=x) = \begin{bmatrix} n \\ x \end{bmatrix} p^x q^{n-x}$$

$$= \begin{bmatrix} n \\ x \end{bmatrix} \frac{b^x r^{n-x}}{(b+r)^n}$$

$$= \begin{bmatrix} n \\ x \end{bmatrix} \left[\frac{b}{b+r}\right]^x \left[\frac{r}{b+r}\right]^{n-x}$$

$$= \begin{bmatrix} b \\ x \end{bmatrix} \begin{bmatrix} r \\ n-x \end{bmatrix} \Big/ \begin{bmatrix} b+r \\ n \end{bmatrix}$$

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Probability distribution of the hyper geometric random variable x. The no. of success in random sample of size n selected from N items of which k are labeled success

~~of~~

k - successes

$n-k$ - failure

$$P(x : N, n; k) = \left[\binom{k}{x}\binom{N-k}{n-x}\right] / \binom{N}{n}$$

Q) A box contains 6 blue marbles & 4 red marbles.

Marble is chosen at random & it's colour is observed. It is not replaced. Find probability that after 5 trials, 3 blue marbles will have been selected.

Ans) No. of ways of selecting 3 blue marbles from 6 $= \binom{6}{3}$

No. of ways of selecting remaining 2 marbles out of 4 red marbles. $\binom{4}{2}$

Total no. of ways of selecting 5 marbles out of 10 marbles is $\binom{10}{5}$

$$\text{Probability} = \frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}}$$

POISSON DISTRIBUTION

~~(i) The no. of outcomes~~

Experiments aiding numerical values of a random variable with no. of outcomes during a given time interval or a specified region is Poisson process / expt.

(i) The no. of outcomes occuring in one time interval or specified region is independent of any other disjoint time interval or region (no memory)

(ii) Probability that a single outcome will occur over a time interval / specified region $\propto$ length of interval / size of region.

(iii) Probability that more than one outcome ~~will~~ which can occur in that short time interval or space is negligible.

$$P(x, \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

λ is rate of occurence of outcomes
t is space interval (time, volume etc)

$$E(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^x}{(x-1)!} \quad [\text{where } \mu = \lambda t]$$

$$= \mu \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^{x-1}}{(x-1)!} = \mu \quad \left[\text{since } \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^{x-1}}{(x-1)!} = \right.$$

$$= \mu$$

$$= \lambda t$$

$$var[x] = E[x^2] - E[x]^2$$

$$E[x^2] = \sum_x x^2 p(x)$$

$$= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$= \lambda t \sum_{k=1}^{\infty} k \frac{e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!}$$

let $k_1 = k - 1$

$$\therefore E[x^2] = \lambda t \sum_{k_1=0}^{\infty} (k_1+1) \frac{e^{-\lambda t}(\lambda t)^{k_1}}{(k_1)!}$$

$$= \lambda t \left[\sum_{k_1=0}^{\infty} k_1 \frac{e^{-\lambda t}(\lambda t)^{k_1}}{(k_1)!} + \sum_{k_1=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{k_1}}{(k_1)!} \right]$$

$$= \lambda t(\lambda t + 1)$$

$$\therefore var[x] = E[x^2] - E[x]^2$$

$$= \lambda t(\lambda t + 1) - (\lambda t)^2$$

$$= \lambda t$$

mean and variance are the same in Poisson distribution

Exponential distribution

Define random variable

$x \rightarrow$ time to the first Poisson event

$P(T \le t)$ = Probability distribution of time T between the occurrence of events
(nonexcedence)

$= 1 - P(T \ge t)$
(excedence probability)

$P(T \ge t)$ = Probability of no occurrence in time t

$$= P(0, \lambda t)$$

$$= \frac{e^{-\lambda t}(\lambda t)^0}{0!}$$

$$= e^{-\lambda t}$$

$$\Rightarrow P(T \le t) = 1 - P(T \ge t)$$

$$= 1 - e^{-\lambda t}$$

This is CDF of exponential distribution

Q) For exponential distribution,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = 1 - e^{-\lambda t}, \quad x \ge 0, \ \lambda \ge 0$$

$f(x) = \lambda e^{-\lambda t}$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} \cdot dx = \frac{1}{\lambda}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \int_{0}^{\infty} x^2 \lambda e^{-\lambda x} \cdot dx = \frac{1}{\lambda^2}$$

Q) The average no. of traffic accidents on a certain section of highway is 2 per week. What is the probability of no accidents on this section of highway during a one week period?

Ans) Random variable → No. of accidents per week.

Avg. no. of accidents per week $= \mu = 2$

$$P(x, \mu) = \frac{e^{-\mu} \mu^{x}}{x!}$$

$$P(0, \mu) = P(0, 2)$$

$$= \frac{e^{-2} 2^{0}}{0!} = e^{-2}$$

Find the probability of at most 3 accidents on this section of highway during a 2 week period.

Avg. no. of accidents for 2 weeks
$= \lambda t$
$= 2 \times 2$
$= 4$

$P(x \leq 3) = P(0) + P(1) + P(2) + P(3)$

$= \frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} + \frac{4^3 e^{-4}}{3!}$

$= e^{-4} + 4e^{-4} + 8e^{-4} + \frac{32}{3} e^{-4}$

$= 0.43$

Q) In a factory, no. of service for broken machines = 4 per day. What is the probability that 6 machines break down in 2 days?

Ans) μt
$= 4 \times 2$
$= 8$

$P(x = 6) = \frac{8^6 e^{-8}}{6!}$

$= 0.122$

Q) On an average, 9 customers are serviced by one waiter in one hour. What is the probability a customer shall be free within 15 minutes?

Ans) Let time be the random variable

~~A~~ =

$P(x \le 15) = ?$

λ = 9 / 1 hr

t = 15 minutes = 0.25 hrs

$P(x \le 15) = 1 - e^{-(9 \times 0.25)}$

Q) Accidents occur at an average rate of 4 /week. Calculate probability of more than 5 accidents in any one week.

Ans) $P(x \ge 5)$

$= e^{-\lambda t}$

$\lambda = 4$

~~t~~ $\lambda t = 4$

$P(x \ge 5) = e$

$P(x > 5) = 1 - P(x \le 5)$

~~P(0, 4)~~ $= 1 - P(x = 0) - P(x = 1)$

$- P(x = 2) - P(x = 3)$

$- P(x = 4)$

$- P(x = 5)$

~~$= 1 - e^{-4}(4)$~~

P(time between occurrences > 2)

Probability that atleast 2 weeks will elapse between accidents?

$$P(x \geq 2) = \int_2^{\infty} \lambda e^{-\lambda t}\, dt = e^{-2 \times 4}$$

$$= e^{-8}$$

Q) 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability that a given page contains 2 misprints?

~~Ans)~~ $\cancel{\mu = 2}$ $\cancel{\#}$ $\cancel{\frac{300}{500} = \frac{3}{5}}$

$$\cancel{P(x=2,\lambda) = \frac{e^{-3/5}(3/5)^2}{2!}}$$

Ans) $p = \frac{1}{500}$ &

$n = 300$

$n \longrightarrow \infty$
$p \longrightarrow 0$

$\mu = np$
$= \frac{300}{500}$

If x is the binomial random variable with n is large and p close to zero, we can use

Poisson distribution with mean $\mu = np$ to approximate binomical probabilities.

$$\mu = np = \frac{300}{500} = 0.6$$

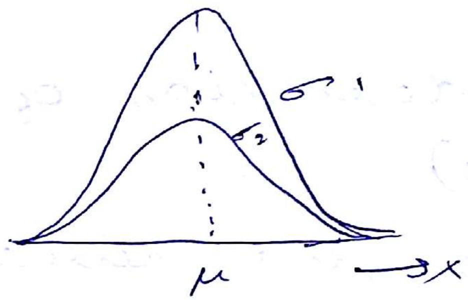
~~$\mu = \lambda t = 0.6x$~~

$$p(x=2, 0.6) = \frac{e^{-0.6}(0.6)^2}{2!}$$

NORMAL DISTRIBUTION

$$N(x, \mu\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty \le x \le \infty$$

This is also gaussian distribution



$\sigma_2 > \sigma$,

(Takes max. value at mean)

i) The mode which is the point on the horizontal axis where the curve is max. occurs at $x = \mu$

ii) Curve is symmetrical about vertical axis (μ) which is mean

iii) The normal curve approaches the horizontal axis asymptotically in the direction from the mean.

iv) The total area under curve, above horizontal access = 1

Inflection points at $x = \mu \pm \sigma$

$$P(a < x < b) = \int_{a}^{b} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

$$P(x_1 \le x \le x_2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \cdot dx$$

$$\left[z = \frac{x-\mu}{\sigma} \right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2}z^2} \cdot dz$$

Q) Find the probability that a normally distributed random variable will fall within the ranges:

(a) one std. deviation of its mean.

$P(-1 \le z \le 1)$

(b) two times std. deviation of mean

$P(-2 \le z \le 2)$

<u>68-95-99 rule</u>

eg. x is a random variable that is normally distributed with mean 10, standard deviation 2. Find probability x lies between 11 and 13.6.

Transform x to a normal z.

$x = 11 \Rightarrow z = \frac{x - \hat{\mu}}{\sigma} = \frac{11-10}{2} = 0.5$

$x = 13.6 \Rightarrow z = \frac{13.6 - 10}{2} = \frac{3.6}{2} = 1.8$

$P(11 \le x \le 13.6) = P(0.5 \le z \le 1.8)$

$$\left.\begin{array}{l} z = 1.8, \quad P = 0.9641 \\ z = 0.5, \quad P = 0.6915 \end{array}\right\} \text{from table}$$

$0.9641 - 0.6915 = 0.2726$

Q) The wt. W of 800 students ~~are~~ follow a normally distributed with mean $= 140$, std. deviation $= 10$.
Find no. of students between wt. 138 and 148.

Ans) $x = 138 \Rightarrow z = \frac{138 - 140}{10} = -0.2$

$x = 148 \Rightarrow z = \frac{148-140}{10} = 0.8$

$P(138 < W < 148) = P(-0.2 \le z \le 0.8)$
$= 0.3674$

multipy with 800,
No. of students between 138 & 148
$= 800 \times 0.3674$
$= 8 \times 36.74$
$= 293.92$
$= 294$

CENTRAL LIMIT THEOREM

n samples drawn with replacement $x_1, x_2, \dots x_n$ are independent.

sample mean $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$

It will have population mean μ and std. deviation σ.

$\bar{x}$ is a random variable. and will have mean $\mu_{\bar{x}}$ and variance $\sigma^2_{\bar{x}}$

$$\mu_{\bar{x}} = \frac{1}{n}(\mu + \mu + \dots + \mu)$$

$$= \mu$$

$$\sigma_{\bar{x}}^2 = \frac{1}{n^2}(\sigma^2 + \sigma^2 + \dots \sigma^2)$$

$$= \frac{\sigma^2}{n}$$

$$= \frac{\sigma}{\sqrt{n}}$$

Suppose x is a random sample with μ and σ defined on some population.

Q,

n is large ($n \geq 30$) and population size is large compared to n, then sample mean $\bar{x}$ is approx. normally distributed with mean $\mu_{\bar{x}} = \mu$ and $\sigma^2_{\bar{x}} = \sigma/\sqrt{n}$

$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

If $\bar{x}$ is th

sample of

std. normal

~~of~~ dis

$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

where n –

Normal di

Q) An elect

light bulb

is approx

with mea

deviatio

Find pr

sample

average

hrs.

Ans) $\mu = \mu$

$\sigma_{\bar{x}} = \frac{\sigma}{n}$

$x = 775$

$z = \frac{-2.5}{}$

$P(z = 2.5)$

Q) A popu

$S = \{4, 7,$

Space. A

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

If $\bar{x}$ is the mean of a random sample of size n, then limiting std. normal ~~of~~ distribution is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

where $n \to \infty$

Normal distribution $N(z, 0, 1)$

Q) An electrical firm manufactures light bulbs with length of life is approx. normally distributed with mean as 800 hrs. and std. deviation 40 hrs

Find probability that a random sample of 16 bulbs will have an average life of less than 775 hrs.

Ans) $\mu = \mu_{\bar{x}} = 800$

$$\sigma_{\bar{x}} = \frac{\sigma^2}{n} = \frac{\sigma}{\sqrt{n}} = \frac{40}{\sqrt{16}} = 10$$

$x = 775$ ~~hr~~

$$z = \cancel{-2.5} \ \cancel{800} \ \cancel{-2.5} = \frac{775 - 800}{10} = -2.5$$

$P(z = 2.5) = 0.0062$

Q) A population consists of set $S = \{4, 7, 10\}$ in equiprobable space. Random sample of 2 are drawn with replacement.

Find a) population mean and std.
b) sample

Ans) $\mu = E[x] = \frac{4}{3} + \frac{7}{3} + \frac{10}{3}$

$= 7$

$\sigma^2 = \Sigma (x-\mu)^2 f(x)$

$= \frac{(4-7)^2}{3} + \frac{(7-7)^2}{3} + \frac{(10-7)^2}{3}$

$= 6$

$\sigma = \sqrt{6}$

sample size = 2

(a, b)	$\bar{x}$
(4, 4)	4
(4, 7)	5.5
(4, 10)	7
(7, 7)	7
(7, 10)	8.5
(10, 10)	10
(7, 4)	5.5
(10, 4)	7
(10, 7)	8.5

$\bar{x}$	$P(\bar{x})$
4	1/9
5.5	2/9
7	3/9
8.5	2/9
10	1/9

$$\mu_{\bar{x}} = \left(4 \times \frac{1}{9}\right) + \left(5.5 \times \frac{2}{9}\right) + \left(7 \times \frac{3}{9}\right) + \left(8.5 \times \frac{2}{9}\right) + \left(10 \times \frac{1}{9}\right)$$

$$= 7$$

$$= \mu$$

$$\sigma^2_{\bar{x}} = \Sigma(\bar{x} - \mu_x)^2 p(\bar{x})$$

$$= (4-7)^2\left(\frac{1}{9}\right) + (5.5-7)^2\left(\frac{2}{9}\right) + (7-7)^2 \frac{3}{9} + (8.5-7)^2 \frac{2}{9} + (10-7) \frac{1}{9}$$

$$= \frac{27}{9}$$

$$\sigma_x = \sqrt{3}$$

$$\frac{\sigma}{\sqrt{n}} = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} = \sigma_x$$

FUNCTION OF RVS

X – R.V. $f(x)$

Y – R.V that is a function of X $u(X)$

find probability distribution of Y.

Q) X is the R.V. $P(X) = \frac{c}{x}$ where $x = 2, 3, 4, 5$

$Y = x^2 - 7x + 12$

Calculate probability distribution function of Y.

Ans)

x	2	3	4	5
y	2	0	0	2

Point to point transformation

$P(x)$	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	$\frac{c}{5}$
$P(y)$	$7c/10$	$7c/12$	$7c/12$	$7c/10$

$$P(Y=2) = P(x=2) + P(x=5)$$
$$= \frac{c}{2} + \frac{c}{5}$$
$$= \frac{7c}{10}$$

$$P(Y=0) = P(x=3) + P(x=4)$$
$$= \frac{7c}{12}$$

Q) $f(x) = e^{-x}, \quad x \geq 0$

$$Y = 2x + 1$$

$$X = \frac{Y-1}{2}$$

$$P(Y \geq 5) = P(2x+1 \geq 5)$$
$$= P(x \geq 2)$$
$$= 1 - P(x \leq 2)$$
$$= 1 - \int_0^2 e^{-x} \cdot dx$$
$$= \cancel{e^{2}} = e^{-2}.$$

Theorem : If x is a discrete random variable with probability distribution $f(x)$, let $Y = u(x)$ define a one to one transformation between the values of x and Y. where $x = w(Y)$

probability distribution $= g(y)$

$= f[w(y)]$

$X \rightarrow$ continuous random variable

$Y \rightarrow$ function of x

Find $g(y)$

~~Append~~ obtain value of $G(Y) \rightarrow$ CDF of Y by finding the event in the range space of X.

$G(Y) = P(Y \leq y)$

2) $f(x) = 2x$, $0 \leq x \leq 1$

$Y = H(x) = 3x + 1$

Find PDF of $H(x)$

Ans) Calculate CDF of Y

$G(Y) = P(Y \leq y)$

$= P(3x + 1 \leq y)$

$= P\left(x \leq \frac{y-1}{3}\right)$

$= \int_0^{\frac{y-1}{3}} 2x \cdot dx$

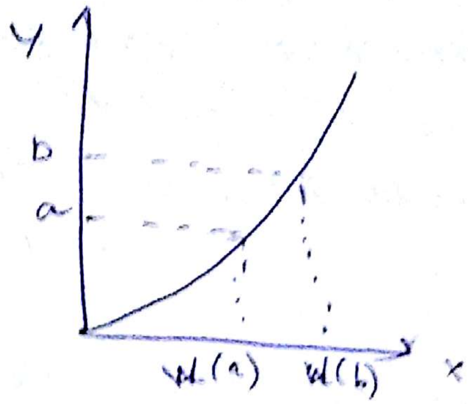
$= \left(\frac{y-1}{3}\right)^2$

$g(y) = \frac{1}{9} \frac{d}{dH} (y-1)^2 = \frac{2y-1}{9}$

$x = 0 \Rightarrow y = 1$

$x = 1 \Rightarrow y = 3 + 1 = 4$

~~$P(Y=2$~~



Monotonically increasing function

$Y = u(x)$ as increasing

$$P(a < Y < b) = P[w(a) < X < w(b)]$$

$$= \int_{w(a)}^{w(b)} f(x) \cdot dx$$

$$X = w(Y)$$

$$dx = w'(y) \cdot dy$$

$$= \int_a^b f(w(y))\, w'(y) \cdot dy$$

$$g(y) = f(w(y))\, w'(y)$$

$$g(y) = f[w(y)]\, |J|$$

J = Jacobian transform

Theorem: If X is a continuous R.V with probability dist. $f(x)$

$Y = u(X)$

Define a one to one correspondence between values of x and y so that the equation $Y = u(x)$ can be uniquely solved for x in terms of Y.

$x = w(Y)$

Then probability dist. of Y

$g(y) = f(w(Y)) \, |J|$

Q) $f(x) = \dfrac{3x^2}{125}$, $0 \leq x \leq 5$

$y = x^2$

Find the probability distribution of Y

ans) ~~$g(y) = f[w(y)]$~~

$$g(y) = f(x) \cdot \left| \frac{dx}{dy} \right|$$

$$x = \sqrt{Y}$$

$$\Rightarrow g(y) = \frac{3x^2}{125} \left| \frac{dx}{dy} \right|$$

$$= \frac{3y}{125} \left| \frac{d\sqrt{y}}{dy} \right|$$

$$= \frac{3y}{125} \times \frac{1}{2} \left(y^{-1/2} \right)$$

$$= \frac{3\sqrt{y}}{250}$$

$x = 0$, $y = 0$

$x = 5$, $y = 25$

$$g(y) = \begin{cases} \dfrac{3\sqrt{y}}{250} & 0 \leq y \leq 25 \\ 0 & \text{elsewhere} \end{cases}$$

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Q) $f(x) = \begin{cases} \frac{x}{12} & 1 < x < 5 \\ 0 & \text{otherwise} \end{cases}$

$Y = 2x - 3$

Ans) $x = \frac{Y+3}{2}$

$$\frac{dx}{dy} = \frac{d}{dy}\left[\frac{Y+3}{2}\right] = \frac{1}{2}$$

$$g(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{y+3}{24}\left(\frac{1}{2}\right)$$

$$= \frac{y+3}{48}$$

$x = 1,\ y = 2x-3 = 1$

$x = 5,\ y = 2 \times 5 - 3 = 7$

$$g(y) = \begin{cases} \frac{y+3}{42}, & -1 < y < 7 \\ 0 & \text{elsewhere} \end{cases}$$

$X, Y - R.V - f(x, y)$

$U = H_1(X, Y)$

6 $V = H_2(X, Y)$

$$g(u, v) = f(x, y)\,|J(u, v)|$$

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \rightarrow \text{determinant}$$

Q) $f(x,y) = \frac{1}{14}\left(5 - \frac{y}{2} - x\right)$

$u = x + y \qquad 0 \le x \le 2$

$v = y/2 \qquad 0 \le y \le 2$

Ans) $x = u - y$

$y = 2v$

$f(x,y) = \frac{1}{14}\left[5 - \frac{2v}{2} - u + y\right]$

$= \frac{1}{14}\left[5 - v - u + 2v\right]$

$= \frac{1}{14}\left[5 - u + v\right]$

$|J| = \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$

$g(u,v) = \frac{1}{14}\left[5 + v - u\right] \times 2$

$= \frac{5 + v - u}{7}$

Find limits of u and v

$\cancel{\frac{\partial x}{\partial u} = 1}$ $\quad \cancel{0 \le v \le 1}$

$y = 2v$

$\cancel{y =}$

$y = 0 \Rightarrow v = 0$

$y = 2 \Rightarrow v = 1$

$0 \le v \le 1$

$x = 0 \Rightarrow u = 2v$

$x = 2 \Rightarrow u = 2v + 2$

$\int_0^1 \int_{2v}^{2v+2} \frac{1}{7}(5 + v - u) = \frac{1}{7} \int_0^1 \left[5u + vu - \frac{u^2}{2}\right]_{2v}^{2v+2}$

$= \frac{1}{7} \int_0^1 (8 - 2v)\, dv$

$= 1$

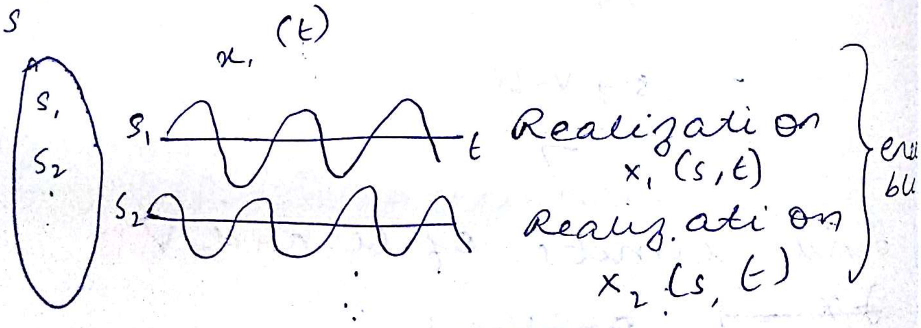
RANDOM PROCESS

Random ~~variable~~ process is a function of not just sample space but also time.

$x(s,t) \rightarrow$ function of sample space, time

A random process is a time varying function that assigns outcome of a random expt to each time instance $x(t)$

Random process is also called stochastic process or time series analysis



Collection of all realizations is ensemble (single time series is called a realization i.e., for a fixed sample space ~~s~~ W, the function $x(s,t)$ is realization

A set of several time series measuring the same variable is called an ensemble.

Q,

The collection of ensemble of realization is a random process.

t - fixed

$F_x(x, t) = P(\{s, t\}, x(s, t) \le X)$ = CDF

for a fixed t

PDF = $f_x(x, t) = \frac{\partial}{\partial x} F_x(x, t)$

PDF of ~~x(t~~ random process describes probabilistic behaviour $x(t)$ over a specified time period t.

$\mu_x(t) = E[x(t)]$ (mean of random process)

$= \int_{-\infty}^{\infty} x \, f_x(x, t) \cdot dx$

variance of random process σ_x^2

$= E[x(t) - \mu_x(t)]^2$

$= E[x^2(t)] - \mu_x^2(t)$

$= \int_{-\infty}^{\infty} (x - \mu_x(t))^2 \, f_x(x, t) \cdot dx$

Q). If a coin is tossed, if head comes, a sine curve

$x_1(t) = \sin(5\pi t)$

$x_2(t) = t$

$t = 0, 1/2, 7/10$

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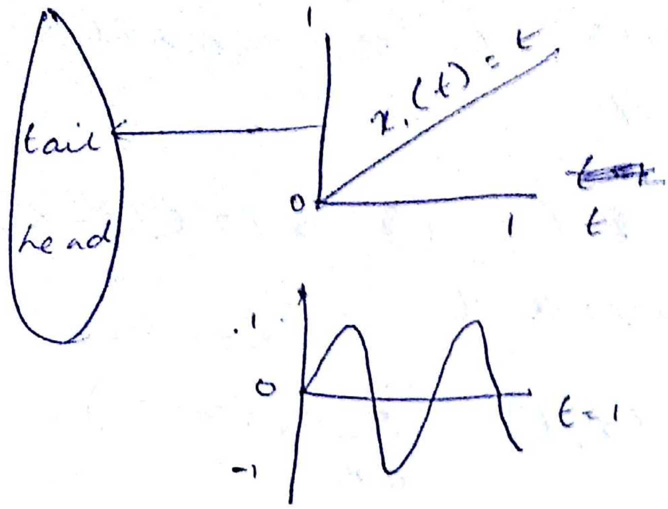
Ans) Sample space



$\mu_x(t) = x_1(t) f(x) + x_2(t) f(x)$

$\mu_x(0) = 0 \times \frac{1}{2} + 0 \times \frac{1}{2} = 0$

~~$\mu_x(1/2) =$~~

$\sigma_x^2 = \cancel{0} \sum [x_i(0) - \mu_x(0)]^2 f(x)$

$= (0-0)^2 \, 1/2 - (0-0)^2 \, 1/2 = 0$

$x_1(t) = x_1(1/2) = \sin(5\pi \times \frac{1}{2}) = 1$

$x_2(1/2) = 1/2$

$\mu_x(1/2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\sigma_x^2 = \sum [x_i(1/2) - \mu_x(1/2)]^2 f(x)$

$= (1/2)(1 - 3/4)^2 + (1/2 - 3/4)^2 (1/2)$

$= (1/16 + 1/16) 1/2$

$= 1/16$

When $t = 7/10$

$x_1(7/10) = \sin(5 \times \pi \times 7/10) = \sin(630)$
$= -1$

$x_2(7/10) = 7/10$

$\mu_x(7/10) = -1 \times 1/2 + 7/10 \times 1/2$

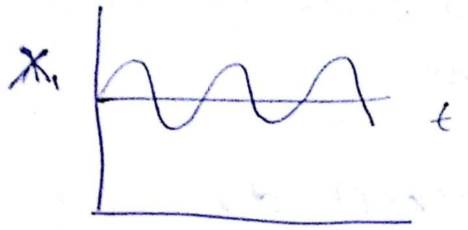
$= -3/20$

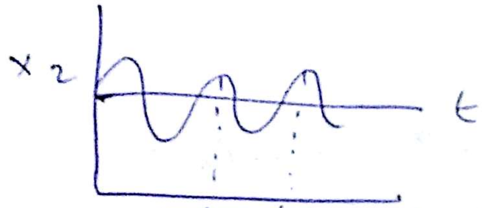
$\sigma_x^2(7/10) = (7/10 + 3/20)^2 \, 1/2$

$+ (-1 + 3/20)^2 \, 1/2$

$= 0.7225$

continuous random process is continuous over time (random stochastic continuous process)





t t+τ (τ is lag/interval)

The properties of a time series can be calculated based on

(i) single realization over a time interval. (time averaged properties)

(ii) several realization at a particular time (ensemble properties)

if time average and ensemble properties are same, random process is ergotic.

$\bar{X}_i = \frac{1}{T}\int_0^T x_i(t)\,dt$

Time average over time interval of i^{th} realization is $\overline{x_i}$

$$\overline{x_i} = \frac{1}{n} \sum_{j=1}^{n} x_i(t_j)$$

$n \rightarrow$ Total no. of equally spaced parts of $x_i(t)$ was observed

$\overline{x_t}$ - Ensemble avg. at time t.

$$\overline{x_t} = \sum_{i=1}^{m} x_i(t) / m$$

m - no. of realizations over a particular time (ensemble)

$$\overline{x_t} = \int_{-\infty}^{\infty} x(t) \, p(x;t) \, dx \rightarrow m \text{ gets large}$$

~~The prop~~

CDF : $F_x(x_1, \ldots x_n, t_1, \ldots t_n)$

$= F_x(x_1, \ldots x_n, t_1 + \tau, t_2 + \tau + \ldots t_n + \tau)$

$$f_x(x_1, \ldots x_n, t_1, \ldots t_n) = f_x(x_1, \ldots x_n, t_1 + \tau, \ldots t_n + \tau)$$

$$F_x(x, t) = F_x(x, t + \tau) = F_x(x)$$

$\longrightarrow$ Random process is ~~the~~ ~~called~~ first order stationary process (independent of time)

If $\overline{x(t)} = \overline{x(t+\tau)}$ - $\forall$ t and τ (ensemble) $\rightarrow$ Stationary in mean of first order stationary

$\text{cov}\,(x(t), x_i(t+\tau))$

$= \frac{1}{m} \sum_{i=1}^{m} [x_i(t) - \overline{x(t)}]\,[x_i(t+\tau) - \bar{x}(t+\tau)]$

$E[x(t) - \bar{x}(t)]\,(x_{t+\tau} - \bar{x}_{t+\tau})$

If $\bar{x}_t = \overline{x_{t+\tau}}$ and $\text{cov}(x_t, x_{t+\tau})$

- same for all → second order stationary

$R_x(t_1, t_2) = E(x(t_1)\, x(t_2))$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_x(x_1, x_2, t_1, t_2)\,.dx_1\,.dx_2$

$R_x(t_1, t_2)$ is autocorrelation

$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x) f(y)\,.dx\,.dy$

$t_1 = t_2 = t$

$R_x(t) = \int_{-\infty}^{\infty} x^2 f_x(x,t)\,.dx$

$= E(x^2_t)$

autocovariance

$C_x(t_1, t_2)$ → covariance between $x_1(t_1)$ and $x_2(t_2)$

$C_x(t_1, t_2) = E\,[(x_1(t_1) - \mu_{x_1}(t_1))\,(x(t_2) - \mu_x(t_2))]$

$= E[x(t_1)\,x(t_2)] - \mu_x(t_1)\,\mu_x(t_2)$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_x(t_1))\,(x_2 - \mu_2(t_2))\, f_x(x_1, x_2, t_1, t_2)\,.dx_1\, dx_2$

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$C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$

$R_x(t_1, t_2) = C_x(t_1, t_2) + \mu_x(t_1)\mu_x(t_2)$

$\sigma_{xy}^2 = E(XY) - \mu_x \mu_y$

$C_x(t_1, t_2) = E[x(t_1)\, x(t_2)] - \mu_x(t_1)\mu_x(t_2)$

$= R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$

$\Rightarrow R_x(t_1, t_2) = C_x(t_1, t_2) + \mu_x(t_1)\mu_x(t_2)$

Normalized autocovariance

$$\rho_x(t_1, t_2) = \frac{C_x(t_1, t_2)}{\sigma_x(t_1)\sigma_x(t_2)}$$

(Random process within time series)

If there are 2 random processes $X(t_1)$ and $Y(t_2)$, we define cross correlation as $R_{XY}(t_1, t_2)$

$= E[X(t_1)\, Y(t_2)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 \, f_{xy}(x_1, y_2, t_1, t_2) \cdot dx_1 \, dy_2$$

If $t_1 = t_2 = t$

$R_{XY}(t) = E[X(t)\, Y(t)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{xy}(x, y, t) \cdot dx \cdot dy$$

Cross covariance

$C_{XY}(t_1, t_2)$

$= E[X(t_1) - \mu_x(t_1),\ Y(t_2) - \mu_y(t_2)]$

$= E[X(t_1)\, Y(t_2) - \mu_x(t_1)\,\mu_y(t_2)]$

$= R_{XY}(t_1, t_2) - \mu_x(t_1)\mu_y(t_2)$

Normalized cross-covariance

$$\rho_{XY}(t_1, t_2) = \frac{C_{xy}(t_1, t_2)}{\sigma_x(t_1)\,\sigma_y(t_2)}$$

Q) $x(t) = A\sin(\omega t + \phi)$ is a random process where A is uniformly distributed R.V where μ_A is mean, σ_A^2 as variance: Find everything.

Ans) $E[x(t)] = E[A\sin(\omega t + \phi)]$

$= \mu_A \sin(\omega t + \phi)$

$\mathrm{var}[x(t)] = E[x^2(t)] - \mu_x^2(t)$

$\cancel{E[x^2(t)]} = E[A^2\sin^2(\omega t + \phi)]$

$- \mu_A^2 \sin^2(\omega t + \phi)$

$= (E[A^2] - \mu_A^2)\sin^2(\omega t + \phi)$

$= \sigma_A^2 \sin^2(\omega t + \phi)$

$R_x(t_1, t_2) = E[x(t_1)\cdot x(t_2)]$

$= E[A\sin(\omega t_1 + \phi)\, A\sin(\omega t_2 + \phi)]$

$= E[A^2\sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)]$

$C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$

$= E[A^2\sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)]$

$- \mu_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)]$

$= \sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)$

$$\rho_x(t_1, t_2) = \frac{C_x(t_1, t_2)}{\sigma_x(t_1)\,\sigma_x(t_2)}$$

$$= \frac{\sigma_A^2 \sin(\omega_1 t + \phi)\sin(\omega_2 t_2 + \phi)}{\sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)}$$

$$= 1$$

Random walks are used to model the situation in which an object moves in a sequence of steps in a randomly chosen direction.

Y_i = R.V representing the jump at the i^{th} step

sequence $[Y_i,\ i = 1, 2, \ldots n]$ independent with $P[Y_i = 1] = p$

$P[Y_i = -1] = q = 1 - p$

$P[Y_i = 0] = 1 - p - q$

$P[Y_i = 0] + p + q = 1$

$$Z_n = \begin{cases} 1 & \text{for success in } n^{th} \text{ trial} \\ -1 & \text{" failure "} \end{cases}$$

$P(Z_n = 1) = p$

$P(Z_n = -1) = 1 - p = q$

If $p = 1/2$, symmetric random walk

$E(Z_n) = 1 \cdot P(X_n = 1) - 1 \cdot P(X_n = -1$

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$= p - (1-p) = 2p - 1$

$\sigma_z^2 = E[x^2] - \mu_n^2$

$E[z_n^2] = 1 \cdot p + 1 \cdot (1-p)$

$= 1$

$\sigma_z^2 = E[z_n)^2 - \mu_{z_n}^2$

$= 1 - (2p-1)^2$

$= 4p(1-p)$

$w_n = \sum_{i=1}^{n} z_i \,, \quad n > 0$

$w_0 = 0$

w_n is position of particle after n jumps

$P(w_n = k) \quad k = 0, \pm 1, \pm 2, \ldots \pm n$

r_1 - positive jumps

r_2 - negative jumps

r_3 - zero jumps

$r_1 + r_2 + r_3 = n$

$k = r_1 - r_2$

$r_3 = n - r_1 - r_2$

$P(w_{n r_1} = k) = \dfrac{n!}{r_1! \, r_2! \, r_3!} p^{r_1} q^{r_2} (1-p-q)^{r_3}$

$\longrightarrow$ multinomial distribution

$k = r_1 - r_2 \Rightarrow r_2 = r_1 - k \longrightarrow r_1 \geq k$

$r_3 = n - r_1 - r_2 = n - r_1 - r_1 - k$

$= n - 2r_1 - k$

$\Rightarrow -2r_1 + k \le -n$

$2r_1 - k \le n$

$2r_1 \le n + k$

$r_1 \le \frac{n+k}{2}$

$$P(W_n r_1 = k) = \frac{n!}{r_1!\,(r_1-k)!\,(n-2r_1+k)!}\, p^{r_1}\, q^{r_1-k}\, (1-p-q)^{n-2r_1+k}$$

$$k \le r_1 \le \left\lfloor \frac{n+k}{2} \right\rfloor$$

$$P(W_n = k) = \sum_{r_1} P(W_n r_1 = k)$$

$$= \sum_{r_1=k}^{\lfloor \frac{n+k}{2} \rfloor} \frac{n!\, p^{r_1}\, q^{r_1-k}\,(1-p-q)^{n-2r_1+k}}{r_1!\,(r_1-k)!\,(n-2r_1+k)!}$$

Q) In a random walk process, $p = 1/2$, $q = 3/10$, determine probability $P(W_n = k)$ being at $k = 3$ in $n = 10$

Ans) $$P(W_{10} = 3) = \sum_{r_1=3}^{6} \left[\cancel{\frac{3+10}{2}}\right] \frac{(1/2)^{r_1}(3/10)^{r_1-3}(10!)(1\ldots}{r_1!\,(r_1-3)!\,(13-2r_1)!}$$

$$k \le r_1 \le \left\lfloor \frac{n+k}{2} \right\rfloor$$

$r_1 = 3, 4, 5, 6$

6 $P[W_{10} = 3] = (3) + (4) + (5) + (6)$

$= 0.135327$

$(x_1, x_2)\ (y_1, \ldots$

$(x, y) = (x \ldots$

$i \cdot i = -1$

$i \cdot i = (0, 1\ldots$

$= -1$

$i \cdot y = (0, \ldots$

$(a + ib)(c \ldots$

① Closure
② Associative
③ Identity exists
③ Inverse

$\bar{e} = e_1 + ie\ldots$

$z = a + ib$

$z\bar{e} = (e_1 a - \ldots$

$\Rightarrow e_1 = 1, e\ldots$

(multiplic…

COMPLEX ANALYSIS

$(x_1, x_2)(y_1, y_2) = (x_1 y_1 - x_2 y_2,\ x_1 y_2 + x_2 y_1)$

$(x, y) = (x, 0) + (0, y)$

$= (x, 0) + y(0, 1)$

$= x + yi$

$i \cdot i = -1$

$i \cdot i = (0,1)(0,1) = (0 \cdot 0 - 1 \cdot 1,\ 0 + 0) = (-1, 0)$

$= -1$

$i \cdot y = (0,1)(y,0)$

$= \cancel{(0, y)}$

$= (0 \cdot y - 1 \cdot 0,\ y \cdot 1 + 0 \cdot 0)$

$= (0, y)$

$= y(0,1)$

$= yi$

$(a+ib)(c+id) = (ac - bd) + i(ad + bc)$

	$(\mathbb{C}, +)$	$(\mathbb{C}, \times)$
① closure	yes	yes
② associative	yes	yes
③ identity exists	$0 + i0$ (origin)	yes, $(1, 0)$
③ inverse	$-z$	$\frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$ (Exists for $\mathbb{C} - \{0\}$)

$\bar{e} = e_1 + ie_2$

$z = a + ib$

$z\bar{e} = (e_1 a - e_2 b,\ e_1 b + e_2 a)$

$\therefore e_1 = 1, e_2 = 0$ if $z\bar{e} = (a, b)$

(multiplicative inverse)

z multiplicative inverse

$z \neq 0$

$z = a + ib \Rightarrow a \neq 0$ or $b \neq 0$

$\Rightarrow a^2 + b^2 \neq 0$

Let $z' = c + id$ (inverse of z

$z z' = z' z = 1$

~~$z' = \frac{1}{z}$~~

$(a + ib)(c + id) = 1$

$\Rightarrow (ac - bd,\ ad + bc) = 1 + 0i$

$\Rightarrow ac - bd = 1,\ ad + bc = 0$

$\Rightarrow c = \frac{a}{a^2+b^2},\ d = \frac{-b}{a^2+b^2}$

$\Rightarrow z = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$

Is z^{-1} unique?

Let z' and z'' be two inverses of z. Then $z' = z' \cdot 1 = z'(z \cdot z'')$

$= 1 \cdot z''$

$= z''$

6

Distributive law:

$z(w + s) = zw + zs$

QUESTIONS)

1) P.T. $\frac{1}{i} =$

2) Find rea[l] part of $\frac{(z}{z}$

3) P.T for c[...] $Re(3z + w)$

ANSWERS

1) $\frac{1 \times i}{i \times i} =$

$\frac{1}{(i+1)(i-1)} (i-1)$

~~2) $\frac{(z+2)}{(z-1)}$~~

2) $\frac{(x + iy)}{x + iy}$

Real : (

Im : $((x-1$

QUESTIONS)

1) P.T. $\frac{1}{i} = -i$ and $\frac{1}{i+1} = \frac{1-i}{2}$

2) Find real and imaginary part of $\frac{(z+2)}{z-1}$ where $z = x + iy$

3) P.T for complex nos. z and w
$Re(z + w) = Re(z) + Re(w)$

ANSWERS

1) $\frac{1 \times i}{i \times i} = -i$

$\frac{1}{(i+1)} \frac{(i-1)}{(i-1)} = \frac{(i-1)}{-2} = \frac{1-i}{2}$

2) ~~$\frac{(z+2)}{(z-1)} \frac{(z+1)}{(z+1)} = \frac{z^2 + 3z + 2}{}$~~

2) $\frac{(x+iy)+2}{x+iy-1} = \frac{(x+2+iy)}{(x-1+iy)} \frac{(x-1-iy)}{(x-1-iy)}$

$= \frac{((x+2)+iy)((x-1)-iy)}{(x-1)^2 + y^2}$

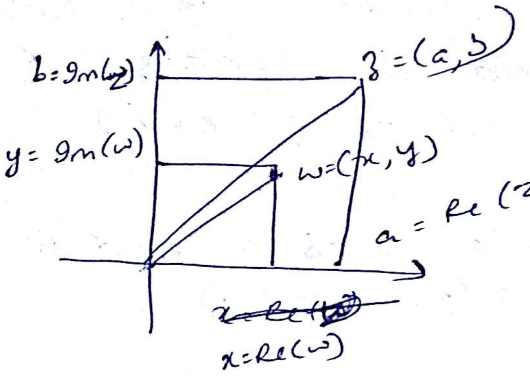
~~$= (x+2)(x-1)$~~

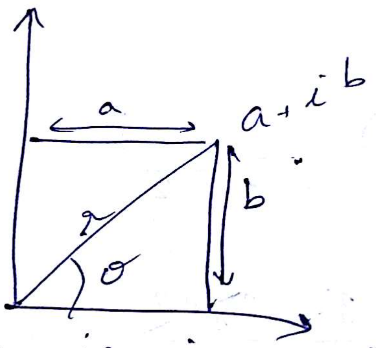
$= \frac{(x+2)(x-1) - (x+2)iy + (x-1)iy + y^2}{(x-1)^2 + y^2}$

Real : $\frac{(x+2)(x-1) + y^2}{(x-1)^2 + y^2}$

Im : $\frac{[(x-1) - (x+2)]y}{}$

Complex field





$a = r \sin\theta$
$b = r \cos\theta$

$z = a + ib = ir\sin\theta + r\cos\theta$

$r = \sqrt{a^2 + b^2}$

The length of the vector z is denoted by $|z|$ is called norm or modulus or absolute value of z.

θ is called argument $\arg(z)$

$\arg z = \{\theta + 2k\pi \mid k \text{ is an integer}\}$

$z_1 = r_1(\cos\theta_1 + \sin\theta_1) \longrightarrow (1)$

$z_2 = r_2(\cos\theta_2 + \sin\theta_2) \longrightarrow (2)$

$z_1 z_2 = r_1 r_2[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$

$+ i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1)]$

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$|z_1 z_2| = |z_1|\,|z_2|$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$[0, 2\pi)$

eg. $z_1 = -1,\ z_2 = -i$

$|z_1|\,|z_2| = 1$

$|z_1 z_2| = 1$

$\arg(z_1) + \arg(z_2) = \cancel{\frac{\pi}{2}}\ \frac{5\pi}{2}$

$\arg(z_1 z_2) = \arg(i) = \frac{\pi}{2} \cancel{= \frac{5\pi}{2}}$

complex multiplication can be viewed as : let $z \in \mathbb{C}$

and define $\varphi_z : \mathbb{C} \to \mathbb{C}$

$\varphi_z(w) = wz$

De moivre's theorem

If $z = r(\cos\theta + i\sin\theta)$ and n is a +ve integer then show that $z^n = r^n(\cos n\theta + i\sin n\theta)$

Prove using induction

$n = 1$

$z = r(\cos\theta + i\sin\theta) \rightarrow$ True

Let it be true uptil k.

For $(k+1)$

$$z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta) \quad (1)$$

$$z^k \cdot z = r^k(\cos k\theta + i\sin k\theta) \cdot r(\cos\theta + i\sin\theta)$$

$$= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)$$

$$= (1)$$

$z^n = \omega$

suppose $\omega = r(\cos\theta + i\sin\theta)$

$$z = \rho(\cos\psi + i\sin\psi)$$

$$\Rightarrow z^n = \rho^n(\cos n\psi + i\sin n\psi)$$

$$= r(\cos\theta + i\sin\theta)$$

$$\left.\begin{aligned} \rho^n &= r = |\omega| \\ n\psi &= \theta + k2\pi \\ \psi &= \frac{\theta}{n} + \frac{k2\pi}{n} \end{aligned}\right\} \quad k = 0, 1, \ldots, n-1$$

$$z = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

This has n roots. $k = 0, 1, \ldots, n-1$

Solve $z^3 = 1$

$$z = \cos\left(\frac{\theta}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{2\pi k}{3}\right)$$

$$= \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right)$$

$k = 0, 1, 2$

$z = 1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$\sqrt{a+ib} = \pm(\alpha + \mu\beta i)$

where $\alpha = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}$

$\beta = \sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$

and $\mu = 1$ if $b \geq 0$

$\mu = -1$ if $b < 0$

$\alpha^2\beta^2 = \frac{(a^2+b^2)-a^2}{4} = \frac{b^2}{4}$

$\alpha\beta = \frac{b}{2}$

If $b < 0$, $\alpha\beta < 0$

$\Rightarrow$ either α is $-ve$ or β is $-ve$

$[(-\alpha, \beta)$ or $(\alpha, -\beta)]$

If $\mu = -1$, $\sqrt{a+ib}$ $\left.\begin{matrix} \alpha - i\beta \\ -\alpha + i\beta \end{matrix}\right\} \begin{matrix} \rightarrow (\alpha, -\beta) \\ \rightarrow (-\alpha, \beta) \end{matrix}$

Q) solve the equation $z^4 + i = 0$

Ans) $z^4 = -i$ $\quad (r = \sqrt{0+1} = 1)$

$z = \cos\left(\frac{3\pi}{8} + \frac{2\pi k}{4}\right) + i\sin\left(\frac{3\pi}{8} + \frac{2\pi k}{4}\right)$

$k = 0, 1, 2, 3$

$z = \cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)$

$\cos(7\pi/8) + i\sin(7\pi/8)$

$\cos(11\pi/8) + i\sin(11\pi/8)$

$= \cos(15\pi/8) + i\sin(15\pi/8)$

$e^{i4\theta} = e^{i\,3\pi/2}$

$\cancel{\Rightarrow e^{i(3\pi/2)(k/4)}}$

$e^{i\theta} = e^{i\left(3\pi k/8\right)}$

$z^4 + i = 0 \Rightarrow z^4 = -i \Rightarrow a = 0,\ b = -1$

$$z = \pm\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}\,i\right)$$

$$= \pm\left(\frac{\sqrt{2-\sqrt{2}}}{2} + \frac{\sqrt{2+\sqrt{2}}}{2}\,i\right)$$

If $i > 0$	If $i < 0$
$\Rightarrow i^2 > 0$	$\Rightarrow -i > 0$
$\Rightarrow -1 > 0$	$\Rightarrow (-i)(-i) > 0$
(not true).	$\Rightarrow i^2 > 0$
	$\Rightarrow -1 > 0$
	(not true)

Q) (i) Solve $z^8 = 1$

(ii) Express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ using De Moivre's theorem

Ans) (ii) $\cos 3\theta = 3\cos\theta - 4\cos^3\theta$

$\cancel{4\cos^3\theta = 3\cos\theta - \cos 3\theta}$

$\cancel{\cos^3\theta}$

$\cos 3\theta = \cancel{3\cos\theta} -$

(ii) Take ...

$(\cos\theta + ...$

Take n ...

$\Rightarrow \cos 3...$

$= (\cos ...$

$= \cos ...$

Equat...

$\cos 3\theta$ ...

(i) $z^8 = ...$

$r = \sqrt{...}$

$z^n = ...$

$\Rightarrow$ s...

$z^8 = ...$

$k = ...$

(ii) Take $r = 1$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Take $n = 3$

$\Rightarrow \cos 3\theta + i\sin 3\theta$

$= (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + i^3\sin^3\theta + 3i\cos^2\theta\sin\theta$

$+ 3i^2\cos\theta\sin^2\theta$

Equating real parts,

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

(i) $z^8 = 1$

$r = \sqrt{1+0} = 1$

$z^n = r(\cos\theta + i\sin\theta)$

$\Rightarrow \sin\theta = 0, \cos\theta = 1$

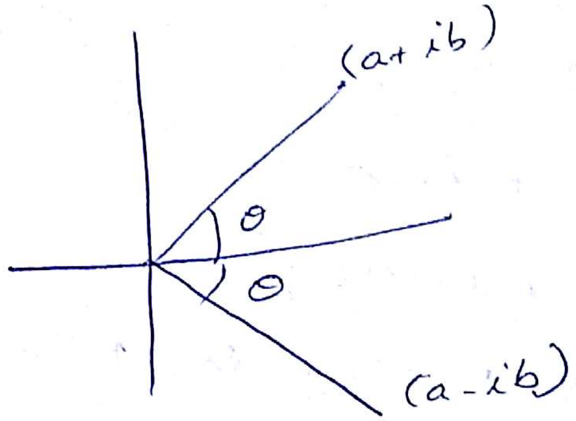
$\Rightarrow \theta = 0$

$z = \sqrt[8]{1}\left[\cos\left(\frac{0}{n} + \frac{2\pi k}{8}\right) + i\sin\left(\frac{2\pi k}{8}\right)\right]$

$= \left(\cos\frac{2\pi k}{4}\right) + \sin\left(\frac{\pi k}{4}\right)$

$k = 0, 1, 2, \ldots 7)$

COMPLEX CONJUGATE



$\bar{z}$ is conjugate

$\bar{z} = a - ib$

$|z| = \sqrt{a^2+b^2}$

$|\bar{z}| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

Propositions :

(i) $\overline{z+z'} = \bar{z} + \bar{z}'$

(ii) $\overline{zz'} = \bar{z}\,\bar{z}'$

(iii) $\overline{z/z'} = \bar{z}/\bar{z}'$

(iv) $z\bar{z} = |z|^2$

(v) $z = \bar{z}$ iff z is real

(vi) $Re(z) = (z+\bar{z})/2$

$Im(z) = (z-\bar{z})/2$

Proofs :

(i) $\overline{(a+ib)+(c+id)} = (a+c) - (b+d)i$

$= a - bi + c - di$

$= \bar{z} + \bar{z}'$

(ii) $\overline{(a+ib)(c+id)} = \overline{ac - bd + a\cancel{x}di + bci}$

$= (ac-ba) - (ad+bc)i$

$\bar{z}\,\bar{z}' = (a-ib)(c-id)$

$= ac - adi - bci - bd$

$= (ac-ba) - (ad+bc)i$

$= \overline{zz'}$

(ii) $\overline{\left(\dfrac{a+ib}{c+id}\right)} = \overline{\left(\dfrac{(a+ib)(c-id)}{c^2+d^2}\right)}$

~~a+b~~ $= \dfrac{ac+bd}{\cancel{c^2+d^2}} \dfrac{-adi+bci}{c^2+d^2}$

$\dfrac{(a-ib)}{(c-id)} = \dfrac{(a\cancel{+}ib)(c+id)}{c^2+d^2}$

$= ac+bd$

(iii) $\overline{z'}\left(\overline{z/z'}\right)$

$= \overline{z'\,z/z'}$ (by (ii))

$= \bar{z}$

$\therefore \overline{z'}\left(\overline{z/z'}\right) = \bar{z}$

$\Rightarrow \overline{z/z'} = \bar{z}/\overline{z'}$

(iv) $(a+ib)(a-ib) = a^2+b^2$

$= |z|^2$

Proposition:

(i) $|zz'| = |z||z'|$

(ii) If $z' \neq 0$, then $|z/z'| = |z|/|z'|$ and

(iii) $-|z| \leq Re(z) \leq |z|$

$-|z| \leq Im(z) \leq |z|$

(iv) $|\bar{z}| = |z|$

(v) $|z + z'| \leq |z| + |z'|$

(vi) $|z - z'| \geq ||z| - |z'||$

Proofs

(i) $|(a+ib)(c+id)|$

$= |(ac-bd) + (ad+bc)i|$

$= \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$

$|(a+ib)||(c+id)|$

$= \sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$

$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$

(ii) $|z'|\ |z/z'|$

$= |z'z/z'|$

$= |z|$

$\Rightarrow |z/z'| = |z|/|z'|$

(iii) $-|z| = -a^2 - b^2$

$Re(z) = a$

$|z| = a^2 + b^2$

$-a^2 - b^2 \leq a \leq a^2 + b^2$

$-a^2 - b^2 \leq b \leq a^2 + b^2$

(iv) $|z| = a^2 + b^2$

$|\bar{z}| = a^2 + b^2$

$|z| = |\bar{z}|$

(v) $|z + z'|$

$= |(a + c) + (b + d)i|$

$= \sqrt{(a+c)^2 + (b+d)^2} = \sqrt{a^2 + c^2 + b^2 + d^2 + 2ac + 2bd}$

$|z| + |z'| = \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$

(v) $|z + z'|^2 = (z + z')\overline{(z + z')}$ $\left\{ \text{since } z\bar{z} = |z|^2 \right\}$

$= (z + z')(\bar{z} + \bar{z}')$

$= |z|^2 + \cancel{z\bar{z}'}\ z\bar{z}' + z'\bar{z} + |z'|^2$

$= |z|^2 + |z'|^2 + 2Re(z\bar{z}')$

$\leq |z|^2 + |z'|^2 + 2|z\bar{z}'|$ (from (iii))

$\leq (|z| + |z'|)^2$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(Show that it converge)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\log(\cancel{1} + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

$$z = x + iy \quad (x, y \in \mathbb{R})$$

$$e^{x+iy} = e^x e^{iy}$$

$$= e^x \left\{ 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} \cdots \right\}$$

$$= e^x \left[\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots \right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots \right) \right]$$

$$= e^x \left[\cos(y) + i \sin(y) \right]$$

Hence, this also converges.

Propositions:

(i) $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$

(ii) e^z is never zero

(iii) If x is real, $e^x > 1$ when $x > 0$ and $0 < e^x < 1$ when $x < 0$

(iv) $|e^{x+iy}| = |e^x| = e^x$

Proofs:

(i) Let $z = x+iy$
$w = c + i d$

$$e^{z+w} = e^{(x+c)+i(y+d)} = e^{(x+c)}[\cos(y+d) + i\sin(y+d)]$$

$$= e^{(x+c)}[\cos y \cos d - \sin y \sin d + i(\sin y \cos d + \cos y \sin d)]$$

$$= e^{(x+c)}[(\cos y + i \sin y)(\cos d + i \sin d)]$$

$$= e^x(\cos y + i \sin y)\, e^c(\cos d + i \sin d)$$

$$= e^z e^w$$

(iv) $|e^{x+iy}| = |e^x \cdot e^{iy}|$

$$= |e^x|\,|e^{iy}|$$

$$= |e^x| \sqrt{\cos^2 y + \sin^2 y}$$

$$= |e^x| = e^x$$

9) (i) Show $e^{\pi i/2} = i$, $e^{\pi i} = 1$, $e^{3\pi/2} = -i$, $e^{2\pi i} = 1$

(ii) e^z is periodic. Each period for e^z has $2\pi n i$ for some integer n

A function $f: \mathbb{C} \to \mathbb{C}$ is called periodic function if there exists $\omega \in \mathbb{C}$ (called a period) such that $f(z+\omega) = f(z)$, $\forall z \in \mathbb{C}$

Ans) (i) $e^{iy} = \cos y + i \sin y$

$$e^{i\pi/2} = \cos \pi/2 + i \sin \pi/2$$

$$= i$$

$$e^{\pi i} = \cos \pi + i \sin \pi$$

$$= 1$$

(ii) If e^z was periodic

$$e^z = e^{z+\omega}$$

$$e^z = e^z . e^\omega$$

$$\Rightarrow e^\omega = 1 = e^{i 2n\pi}$$

$$\Rightarrow \omega = i 2n\pi$$

$$e^{iz} = \cos z + i \sin z \longrightarrow ①$$

$$e^{-iz} = \cos z - i \sin z \longrightarrow ②$$

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

using this,

1) $\sin^2 z + \cos^2 z = 1$

Multiply ① and ②

$1 = (\cos z$ ...

$\Rightarrow \cos^2 z$ ...

2) $\sin(z+$ ...

$\sin z \cos$ ...

$= \left(\frac{e^{iz} -}{2}\right.$ ...

$+ \Big($ ...

$= e^{i(z+\omega}$ ...

$+ e$ ...

$= 2e^{i(}$ ...

$= e^{i(}$ ...

$= \sin($ ...

Q) Let ...
complex ...
$y_0 \le y$ ...

$1 = (\cos z + i\sin z)(\cos z - i\sin z)$

$\Rightarrow \cos^2 z + \sin^2 z = 1$

2) $\sin(z+w) = \sin z \cos w + \cos z \sin w$

$\sin z \cos w + \cos z \sin w$

$$= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{iw} + e^{-iw}}{2}\right) + \left(\frac{e^{iz} + e^{-iz}}{2}\right)\left(\frac{e^{iw} - e^{-iw}}{2i}\right)$$

$$= \frac{e^{i(z+w)} + e^{i(z-w)} - e^{i(w-z)} - e^{-i(z+w)}}{4i}$$

$$+ \frac{e^{i(z+w)} - e^{i(z-w)} + e^{i(w-z)} - e^{-i(z+w)}}{4i}$$

$$= \frac{2e^{i(z+w)} \cancel{+ 2e^{i(z-w)}} - 2e^{-i(z+w)}}{4i}$$

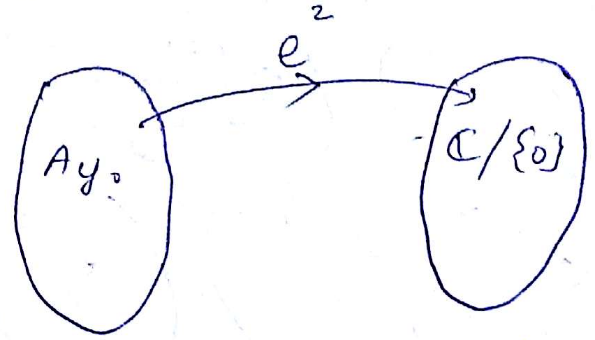
$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i}$$

$= \sin(z+w)$

Q) Let A_{y_0} denote the set of complex nos. $x + iy$ such that $y_0 \le y < y_0 + 2\pi$

$A_{y_0} = \{ x + iy \mid x \in \mathbb{R}$ and $y_0 \le y < y_0 + 2\pi \}$

Then S.T. e^z maps A_{y_0} in one to one manner onto the set $\mathbb{C} / \{0\}$



[we remove 0 from range since e^z can never be zero]

Ans) If $e^{z_1} = e^{z_2}$

$\Rightarrow e^{z_1 - z_2} = 1$

$\Rightarrow z_1 - z_2 = 2\pi n i$ for some integer n

$z_1, z_2 \in A_{y_0}$

max. difference ~~not possible~~ since

$y_0 \le y < y_0 + 2\pi$

$z_1 - z_2 = 2\pi n i$

$\Rightarrow z = 2\pi i n$

$z = x + iy$

$e^z = 1 \Rightarrow e^x = 1 \Rightarrow x = 0$

$[|e^z| = e^x]$

$\Rightarrow z = iy$

$\therefore z_1 = z_2$

Hence 1-1 proved.

Definition: The function $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with the range $y_0 \leq \operatorname{Im}(\log z) < y_0 + 2\pi$

is defined by $\log z = \log |z| + i \arg(z)$

[where $\arg(z)$ takes values in the interval $[y_0, y_0 + 2\pi)$]

Q) If $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, then show that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

Ans) $\log(z_1 z_2) = \log |z_1 z_2| + i \arg(z_1 z_2)$

$= \log(|z_1||z_2|) + i(\arg(z_1) + \arg(z_2))$

up to integral multiple 2π

$= (\log |z_1| + i \arg(z_1)) + (\log |z_2| + i \arg(z_2))$

$= \log z_1 + \log z_2$

$z_1 = -1 - i$, $z_2 = 1 - i$

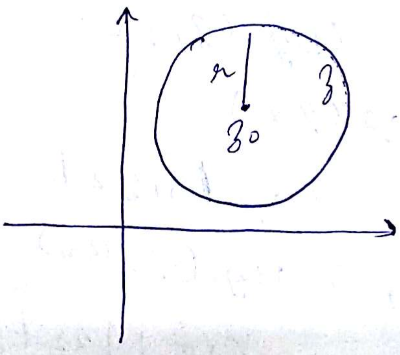
ANALYTIC FUNCTIONS OF COMPLEX NUMBERS

Open set : A set $A \subset \mathbb{C} = \mathbb{R}^2$ is called open when for each point $z_0 \in A$, there is a real no. $\epsilon > 0$ such that $z \in A$ whenever $|z - z_0| < \epsilon$

ϵ depends on what?

For a number $r > 0$, the r-neighbourhood or r-disk around a point z_0 is $\mathbb{C}$ is defined to be a set

$D(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}$



r neighbourhood around z_0

A deleted r-neighbourhood is an r-neighbourhood whose centre point has been removed

$D(z_0, r) \setminus \{z_0\}$

Let $S \subset \mathbb{C}$ be a subset of the complex no. Let $\forall z_0 \in S$, $\exists \epsilon \in \mathbb{R}$, $D(z_0, \epsilon) \subseteq S$. S is an

open set -

i) $\mathbb{C}$ is an

ii) The em
set

iii) Union
open sets

iv) Intersec
of open

MAPPINGS L

Let A be

$f : A \longrightarrow$

function,

~~$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$~~

$f : \mathbb{C} \longrightarrow \mathbb{C}$

$z = x + iy$

$f(z) = u$

$u(x, y) = R$

$v(x, y) =$

$f(x + iy) =$

Let f be
containing
'r'-neigh
say that
as $z \to z_0$

open set -

i) $\mathbb{C}$ is an open set

ii) The empty set ϕ is an open set

iii) Union of any collection of open sets is an open set.

iv) Intersection of any collection of open set is an open set

MAPPINGS LIMIT CONTINUITY

Let A be a subset of $\mathbb{C}$

$f : A \longrightarrow \mathbb{C}$ (f is a complex function)

$\cancel{f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2}$

$f : \mathbb{C} \longrightarrow \mathbb{C}$

$z = x + iy$

$f(z) = u + iv$

$u(x,y) = Re\{f(z)\}$

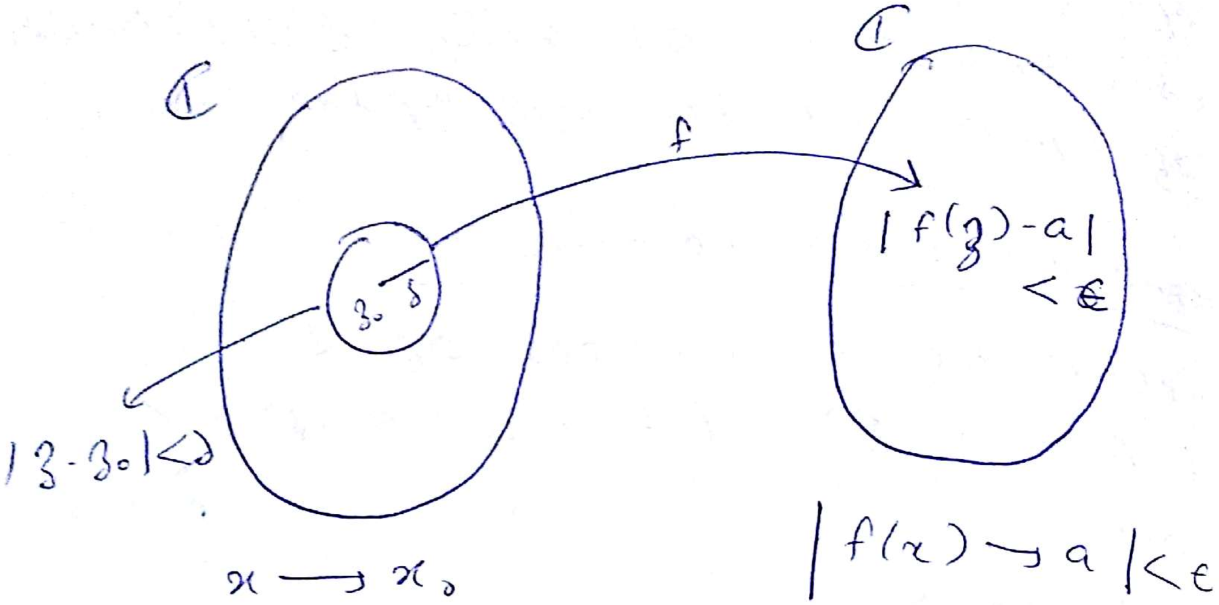
$v(x,y) = Im\{f(z)\}$

$f(x+iy) = u(x,y) + i\,v(x,y)$

Let f be defined on a set containing some deleted 'r'-neighbourhood of z_0. We say that f has the limit a as $z \to z_0$ and write it as : $\longrightarrow$

$$\lim_{z \to z_0} f(z) = a$$

when for every $\epsilon > 0$, $\exists$ a $\delta > 0$ such that for all $z \in D(z_0, \delta)$ and satisfying $|z - z_0| < \delta$, we have $|f(z) - a| < \epsilon$



$x \to x_0$

$|x - x_0| < \delta$

$\lim_{z \to z_0} f(z) = a$ and $\lim_{z \to z_0} f(z) = b$

is not possible.

Suppose $\lim_{z \to z_0} f(z) = a$, $\lim_{z \to z_0} f(z) = b$

Let $\epsilon = \frac{|a-b|}{2}$, $\epsilon > 0$

There is a $\delta > 0$ such that

$0 < |z - z_0| < \delta \Rightarrow |f(z) - a| < \epsilon$ and $|f(z) - b| < \epsilon$

choose a point $z \neq z_0$ [z can't be equal to z_0 as $0 < |z$

$$|a-b| = |a - f(z) + f(z) - b|$$
$$\leq |a - f(z)| + |f(z) - b|$$
(triangle inequality)

$$< \epsilon + \epsilon$$
$$< 2\epsilon$$

But $|a-b| = 2\epsilon$
(contradiction)

If $\lim_{z \to z_0} f(z) = a$, $\lim_{z \to z_0} g(z) = b$

then (i) $\lim_{z \to z_0} [f(z) + g(z)] = a + b$

(ii) $\lim_{z \to z_0} [f(z)\, g(z)] = ab$

(iii) $\lim_{z \to z_0} [f(z) / g(z)] = \frac{a}{b}$ iff $b \neq 0$

Proof:

(i) $|f(z) - a| < \epsilon$

$|g(z) - b| < \epsilon$

$\exists \delta > 0$, $0 < |z - z_0| < \delta$ s.t.

$|f(z) - a| < \epsilon$ and $|g(z) - b| < \epsilon$

$|f(z) + g(z) - (a+b)| < 2\epsilon$

$\cancel{\leq |f(z) - a| + |f(z) - b|}$

$\exists\, \delta_1$ such that $|f(z)-a| < \frac{\epsilon}{2}$

$\Rightarrow 0 < |z - z_0| < \delta_1$

$\Rightarrow$ $\exists\, \delta_2$ such that $|g(z) - b| < \frac{\epsilon}{2}$

$\Rightarrow |z - z_0| < \delta_2$

$$|f(z)-a| + |g(z-b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$|f(z)+g(z) - (a+b)| \le |f(z)-a| + |g(z)-b| < \epsilon$$

$\min(\delta_1, \delta_2)$ satisfies limit condition

(ii) $|f(z)g(z) - ab|$

$$= |f(z)g(z) + f(z)a + f(z)b - ab|$$

$$\le |f(x)g(z) - f(z)b| + |f(z)b - ab|$$

$$\le |f(z)|\,|g(z)-b| + |f(z)-a|\,|b|$$

Let $A \subset \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$ be a function. We say that f is continuous at $z_0 \in A$ iff

$$\lim_{z \to z_0} f(z) = f(z_0)$$

[f is continuous on A if f is continuous at each point z_0 in

Problem: 96 ~~proof~~

h is a fu

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$\Rightarrow |h(f($

$|z - z_0| <$

Problem ~~proof~~: If $\lim_{z \to z_0} f(z) = a$ and h is a function defined on the neighbourhood of 'a' and is continuous at 'a', then show that $\lim_{z \to z_0} h(f(z)) = h(a)$

Given $\epsilon > 0$, there is a $\delta_1 > 0$ such that $|h(w) - h(a)| < \epsilon$, whenever $|w - a| < \delta_1$

and a $\delta > 0$ such that $|f(z) - a| < \delta_1$ whenever $|z - z_0| < \delta$

$\Rightarrow |h(f(z)) - h(a)| < \epsilon$ whenever $|z - z_0| < \delta$

SEQUENCE

$\{z_n\}$, $(n = 1, 2, \dots)$

A sequence of points of $\mathbb{C}$ converges to z_0 iff for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies $|z - z_0| < \epsilon$

~~Property~~

The limit of the sequence is expressed as

$$\lim_{n \to \infty} z_n = z_0 \quad \text{or } z_n \to z_0$$

(1) Limit is unique

(2) If $z_n \to z_0$, $w_n \to w_0$

(i) $z_n + w_n \to z_0 + w_0$

(ii) $z_n w_n \to z_0 w_0$

(iii) $\dfrac{z_n}{w_n} \to \dfrac{z_0}{w_0}$

(3) Also $z_n \to z_0$ iff

$Re(z_n) \to Re(z_0)$

$Im(z_n) \to Im(z_0)$

Proofs :

(1) If $z_n \to z_0$ and $z_n \to z_0'$

$\exists N_1$ such that for all $\epsilon > 0$

$n \geq N_1 \Rightarrow |z - z_0| < \epsilon$ and

$\exists N_2$ such that for all $\epsilon > 0$,

$n \geq$ ...

set $\epsilon =$ ...

$n \geq N \Rightarrow$ ...

$n \geq N \Rightarrow$ ...

$\Rightarrow |z_n - z_0'|$ ...

But $|z$ ...

Hence ...

(4) As z ...

$\exists N_1$ ...

$\exists N_2$ ...

$|z_n - z_0 +$ ...

$n \geq m$ ...

$n > N_2 \Rightarrow |z - z_0'| < \epsilon$

Let $\epsilon = \dfrac{|z_0 - z_0'|}{2}$ and $n = \max\{N_1, N_2\}$

$n \geq N \Rightarrow |z_n - z_0| < \dfrac{|z_0 - z_0'|}{2}$

$n \geq N \Rightarrow |z_n - z_0'| < \dfrac{|z_0 - z_0'|}{2}$

$\Rightarrow |z_n - z_0'| + |z_0 - z_n| < \dfrac{|z_0 - z_0'|}{2} + \dfrac{|z_0 - z_0'|}{2}$

$$< |z_0 - z_0'|$$

But $|z_n - z_0'| + |z_0 - z_n| \geq |z_n - z_0' + z_0 - z_n|$

$$= |z_0 - z_0'|$$

Hence contradiction

(4) As $z_n \to z_0$ and $w_n \to w_0$

$\exists N_1$ such that $n \geq N_1$, $|z_n - z_0| < \epsilon/2$

$\exists N_2$ such that $n \geq N_2$, $|w_n - w_0| < \epsilon/2$

$|z_n - z_0 + w_n - w_0| = |(z_n + w_n) - (z_0 + w_0)|$

$< |z_n - z_0| + |w_n - w_0|$

$< \epsilon/2 + \epsilon/2$

[triangle inequality]

$n \geq \max(N_1, N_2)$

Cauchy sequence: A sequence $\{z_n\}$ is called a cauchy sequence for any $\epsilon > 0$, there exists an integer N such that

$$|z_n - z_m| < \epsilon$$

whenever both $n, m > N$

Cauchy sequence is convergent

$f: A \subset \mathbb{C} \rightarrow \mathbb{C}$
is continuous iff for every convergent sequence $z_n \rightarrow z_0$ of points in A we have
$f(z_n) \rightarrow f(z_0)$

[Image of the function also converges]

Closed set: A subset F of $\mathbb{C}$ is said to be closed if its compliment $\mathbb{C}/F = \{z \in \mathbb{C} \mid z \notin F\}$ is open.

$\Rightarrow$ A set $F \subset \mathbb{C}$ is closed iff whenever $z_1, z_2, z_3 \ldots$ is a sequence of points in F such that $w = \lim_{n \to \infty} z_n$, then $w \in F$

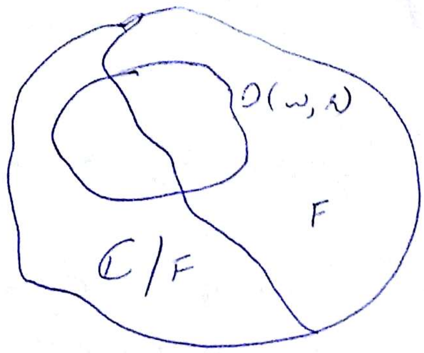
Proof
F is closed
z_n is a sequence of points in F

Let $D(w, r)$ is a r-disk around w whenever $n \geq N$

$z_n \in D(w, r)$

Thus $D(w, r) \not\subset \mathbb{C}/F$

~~open set~~



w is either in $\mathbb{C}/F$ or F

w can never lie in $\mathbb{C}/F$

(since ~~as~~ it is open set, neighbourhood has to be in open set but that's not the case)

Hence $w \in F$

(i) The empty set is closed

(ii) $\mathbb{C}$ is closed

(iii) The intersection of any collection of closed set is closed

(iv) (iii) holds for union of closed sets

(1) If $A \subset \mathbb{C}$, a subset B of A is called open relative to A if $B = A \cap U$ for some open set U.

(2) It is called closed relative to A if $B = A \cap F$ for some closed set F

Q) If $f: \mathbb{C} \Rightarrow \mathbb{C}$, then show that if f is continuous

$\Rightarrow$ The inverse image of every closed set is a closed set

sol) Let $z_1, z_2, \ldots z_n \in f^{-1}(F)$

suppose that $z_n \to w$ | If you are able to prove $w \in f^{-1}(F)$

If f is continuous

$f(z_n) \to f(w)$

[$f(z_n)$ are points in ?]

$f(z_n) \in F$

since F is closed, limiting point of the sequence

$f(w) \in F$

$\Rightarrow w \in f^{-1}(F)$

$\Rightarrow f^{-1}(F)$ is closed set

Definition: Let $f: A \to \mathbb{C}$ where $A \subset \mathbb{C}$ is an open set. The function f is said to be differentiable (in the complex sense) if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists

$= f'(z_0)$ or $\left(\dfrac{\partial f}{\partial z}\right)_{z_0}$

f is said to be analytic / homomorphic.

If f' exists, successive derivatives also exist (unlike in real variables)

Q) If $f'(z_0)$ exists show that f is continuous at z_0

i.e, $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Ans) $\lim\limits_{z \to z_0} [f(z) - f(z_0)]$

$= \lim\limits_{z \to z_0} \left[\dfrac{f(z) - f(z_0)}{z - z_0} (z - z_0) \right]$

$= f'(z_0) \times 0$

$= 0$

Propositions: suppose f and g are analytic on A ($A \subset \mathbb{C} \to$ open set)

then $\to$

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i) $af + bg$ is analytic on A and $(af+bg)'(z) = af'(z) + bg'(z)$

ii) fg is analytic on A $(fg)'(z)$
$= f'(z)g(z) + f(z)g'(z)$

iii) If $g(z) \neq 0$, $\forall z \in A$, then f/g is analytic on A and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}$$

iv) Any polynomial $a_0 + a_1 z + \dots + a_n z^n$ is analytic on all $\mathbb{C}$ with derivative $a_1 + 2a_2 z + \dots + n a_n z^{n-1}$

(ii) Let z_0 be any arbitrary point on A

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \left[\frac{f(z)g(z) - f(z)g(z_0)}{z - z_0} + \frac{f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0} \right]$$

$$= \lim_{z \to z_0} \frac{f(z)[g(z) - g(z_0)]}{z - z_0}$$ (exists because f is analytic)

$$+ \lim_{z \to z_0} \frac{[f(z) - f(z_0)]}{z - z_0} g(z_0)$$

$= f(z_0) g'(z$

$n=2$ $\frac{d(z^2)}{dz} = \frac{d}{dz}$

$= 1 \cdot z$

$= 2z$

say $\frac{d}{dz}($

$\frac{d}{dz}(z^{k+1}) =$

$=$

$=$

$=$

$\frac{d}{dz}(a_0 + a_1 z$

$= a_1 + 2a_2$

Let $f: A$
be analy
and let
$(g \circ f): A$
$(g \circ f)$ $(z$
and $\frac{d}{dz}$

$= f(z_0) g'(z_0) + f'(z_0) g(z_0)$

$$\frac{d(z^2)}{dz} = \frac{d}{dz}(z \cdot z)$$

$$= 1 \cdot z + z \cdot 1$$

$$= 2z$$

say $\frac{d}{dz}(z^k) = k z^{k-1}$

$$\frac{d}{dz}(z^{k+1}) = \frac{d}{dz}(z\, z^k) = 1 \cdot z^k + z \frac{d}{dz}(z^k)$$

$$= z^k + z \cdot k z^{k-1}$$

$$= z^k + k z^k$$

$$= (k+1) z^k$$

$$\frac{d}{dz}(a_0 + a_1 z + \ldots + a_n z^n)$$

$$= a_1 + 2a_2 z + \ldots + n a_n z^{n-1}$$

Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be analytic ($A, B \to$ open sets) and let $f(A) \subset B$. Then ~~(gof)~~ $(g \circ f) : A \to \mathbb{C}$ defined by $(g \circ f)(z) = g(f(z))$ is analytic and $\frac{d}{dz}(g \circ f)(z) = g'(f(z)) f'(z)$

suppose A is an open set in $\mathbb{C}$ and $f : A \to \mathbb{C}$. Then $f'(z_0)$ exists if f is differentiable in the sense of real variables and at $(x_0, y_0) = z_0$, the function u and v ~~is used~~ ~~to~~ satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (and $f(z) = u(x,y) + iv(x,y)$

{ Cauchy Reimann theorem }

$$\cancel{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

Let $f(z) = u(x,y) + i v(x,y)$

$z = x + iy$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{[u(x,y) - u(x_0, y_0)] + i[v(x,y) - v(x_0, y_0)]}{(x - x_0) + i(y - y_0)}$$

when $y = y_0$

$$\lim_{x \to x_0} \frac{[u(x, y_0) - u(x_0, y_0)] + i[v(x, y_0) - v(x_0, y_0)]}{x - x_0}$$

$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$ $\longrightarrow$ ①

$y \to y_0$

$f'(z_0) = -i\, u_y(x_0, y_0) + v_y(x_0, y_0)$ $\longrightarrow$ ②

$u_x = v_y$, $u_y = -v_x$ $\longrightarrow$ (ii)

$\to$ (i)

Differentiate (i) w.r.t x

" (ii) " y

$\hookrightarrow u_{yy} = -v_{yx}$

$\hookrightarrow u_{xx} = v_{xy}$

$u_{xx} + u_{yy} = v_{xy} - v_{yx}$

$= 0$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$ (Laplace equation)

u, v are harmonic conjugates

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

$$f(z) = \frac{z^3 + 2z + 1}{z^3 + 1}$$

COMPLEX INTEGRATION

Consider a complex valued function of a variable 't'

$$f(t) = u(t) + i\,v(t)$$

which is assumed to be piece wise continuous function defined in the closed interval $a \le t \le b$. Then the integral of $f(t)$ from $t = a$ to $t = b$ is defined as

$$\int_a^b f(t)\,dt = \int_a^b u(t).dt + i\int_a^b v(t).dt$$

1) $Re\left[\int_a^b f(t).dt\right]$

$$= \int_a^b Re\, f(t).dt = \int_a^b u(t).dt$$

2) $Im\left[\int_a^b f(t).dt\right] = \int_a^b Im\, f(t)\,dt$

$$= \int_a^b v(t)\,dt$$

3) $\int_a^b [r_1 f_1(t) + r_2 f_2(t)]\,dt$

$$= r_1 \int_a^b f_1(t) + r_2 \int_a^b f_2(t)\,dt$$

($r_1, r_2 \to$ complex)

4) $\left|\int_a^b f(t).dt\right| \le \int_a^b |f(t)|.dt$

$\gamma: [a,b] \longrightarrow \mathbb{C}$, the curve γ will be called piecewise continuous in $\mathbb{C}$

$[a,b] \to [a, a_1]\ (a_1, a_2),\ (a_2, a_3) \ldots$

$[a_{n-1}, \underset{=b}{a_n}]$

Q) suppose α is real, show that

$|e^{2\alpha\pi i} - 1| \leq 2\pi|\alpha|$

Ans) use the property

$$\left|\int_a^b f(t).dt\right| \leq \int_a^b |f(t)|.dt$$

$f(t) = e^{i\alpha t}$

$$\left|\int_0^{2\pi} e^{i\alpha t}.dt\right| \leq \int_0^{2\pi} |e^{i\alpha t}|.dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi$$

$$\left|\int_0^{2\pi} e^{i\alpha t}.dt\right| = \left|\frac{e^{i\alpha t}}{i\alpha}\right|_0^{2\pi}$$

$$= \frac{|e^{i2\pi\alpha} - 1|}{|i\alpha|}$$

$$= \frac{|e^{2\pi\alpha i} - 1|}{|\alpha|}$$

$$\leq 2\pi$$

$\Rightarrow |e^{2\alpha\pi i} - 1| \leq 2\pi|\alpha|$

consider
a set of po
complex plan
$x = x(t)$
where $x(t)$
function of
$z(t) = x(t) + y$
This curve
if $z(t)$ has
$z'(t) \neq 0$ f
the curve.

A contour
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Let $f(z)$ be
defined
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terminal

show that

$|f(t).dt$

$|.dt$

Consider a curve C which is a set of points $z=(x,y)$ in the complex plane defined by

$x = x(t)$, $y = y(t)$, $a \le t \le b$

where $x(t)$, $y(t)$ are continuous function of real parameter t

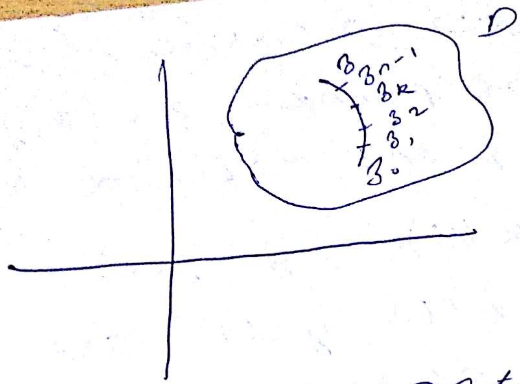
$z(t) = x(t) + y(t)$, $a \le t \le b$

This curve is said to be smooth if $z(t)$ has continuous derivatives $z'(t) \neq 0$ for all points along the curve.

A contour is defined as a curve consisting of finite no. of smooth curves joined end to end.

A contour is said to be a simple closed contour if the initial and final values of $z(t)$ are the same and the contour does not cross itself.

Let $f(z)$ be any complex function defined in a domain D in the complex plane and let C be any contour contained in D with initial point z_0 and terminal point z.



we divide the contour in n subarcs by discrete points $z_0, z_1, z_2, \dots z_{n-1}, z$ = consecutive along direction of increasing t.

Let ξ_k be any arbitrary point in the subarc z_k, z_{k+1}

$$\Delta z_k = z_{k+1} - z_k$$

Let $\lambda = \max_k |\Delta z_k|$

$$\lim_{\substack{\lambda \to 0 \\ n \to \infty}} \sum_{k=0}^{n-1} f(\xi_k)\, \Delta z_k$$

$$= \int_C f(z)\, dz$$

$$= \int_C f(z(t)) \frac{dz(t)}{dt} \cdot dt$$

$$\left[z(t) = x(t) + iy(t) \Rightarrow \frac{dz(t)}{dt} = \frac{dx(t)}{dt} + i\frac{dy(t)}{dt} \right]$$

$f(z) = u$

$dz = dx$

$\Rightarrow \int_C f(z)\, dz$

$\int_{-C} f(z) \cdot$

(move fr z_n to z_0 instea

Q1) $\int_C \frac{1}{z - z_0}$

where C

and is

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directi

Q2) Eva

(i) $\int_C (z$

where

(a) the

point

(b) the a

$f(z) = u(x, y) + iv(x, y) = u + iv$

$dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$

$$= \int (u\,dx - v\,dy) + i\int (u\,dy + v\,dx)$$

$$\int_{-C} f(z)\cdot dz = -\int_C f(z)\cdot dz$$

(move from z_n to z_0 instead)

Q1) $\int_C \frac{1}{z - z_0}\,dz$ (Evaluate)

where C is a circle centred at z_0 and is of any radius. The path is traced out is the anticlockwise direction.

Q2) Evaluate the integral

(i) $\int_C |z|^2 \cdot dz$ (ii) $\int_C \frac{1}{z^2} \cdot dz$

where the contour C is

(a) the line segment with initial point -1 and final point 1

(b) the arc of the unit circle

Im $z \geq 0$ with initial point -1, and final point i.

Ans 1) $z - z_0 = re^{it}$

$0 \leq t \leq 2\pi$

$dz = re^{it} i \, dt$

$$\int_0^{2\pi} \frac{1}{re^{it}} re^{it} i \, dt$$

$= 2\pi i$

Ans 2) (i) (a) $\frac{2}{3}(1+i)$

~~(i) (b) (1+i)~~

(i) (b) $|z| = 1$

& $z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$

$$\int_C |z|^2 . dz = \int_{\pi}^{\pi/2} ie^{i\theta} . d\theta$$

$$= e^{i\theta} \Big]_{\pi}^{\pi/2}$$

$= 1 + i$

(ii) (b) $$\int_{\pi}^{\pi/2} \frac{1}{e^{2i\theta}} ie^{i\theta} . d\theta$$

$$= -e^{-i\theta} \Big]_{\pi}^{\pi/2} = -1 + i$$

point -1

(ii)(a) $-1+i$

In the case of (i), there's a path dependence (same integral gives different values)

In (ii) there's no path dependence

Estimation of the absolute value of a complex integral

The upper bound for the absolute value of a complex integral can be related to the length of the contour C and absolute value of f(z) along C. In fact,

$$\left|\int_C f(z)\cdot dz\right| \leq ML$$

(where M is the upper bound of |f(z)| along C and L is arc length of contour C)

$$\left|\int_C f(z)\cdot dz\right| = \left|\int_a^b f(z(t))\frac{dz}{dt}\,dt\right|$$

$$\leq \int_a^b \left|f(z(t))\right| \left|\frac{dz}{dt}\right| dt$$

$$\leq \int_a^b M\left|\frac{dz}{dt}\right| dt$$

$$= M \int_a^b \left|\frac{dz}{dt}\right| dt$$

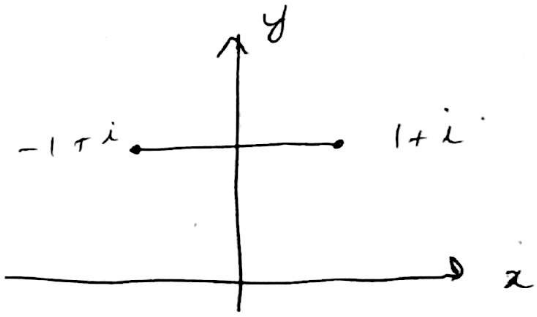
$$= M \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$= ML$ (where L is length of arc)

Q) Show that $\left| \int_C \frac{1}{z^2}\, dz \right| \le z$

where C is the line segment joining $-1+i$ and $1+i$

Ans)



$L = 2$

$z = x + i$ (where $-1 \le x \le 1$)

$|z| = \sqrt{1+x^2}$

$1 \le |z| \le \sqrt{2}$ $\qquad \left|\frac{1}{z^2}\right| = \frac{1}{|z|^2}$

$\frac{1}{2} \le \frac{1}{|z|^2} \le 1$

$M = \max_{z \in C} \left\{ \frac{1}{|z|^2} \right\} = 1$

$$\left| \int_c \frac{1}{z^2} dz \right| \leq ML = 1 \cdot 2 = 2$$

Cauchy integral theorem

Let $f(z) = u(x,y) + iv(x,y)$ be analytic on and inside a simple closed contour C and let $f'(z)$ be also continuous and inside C, then

$$\int_c f(z)\, dz = 0$$

~~If the two real~~

$\cancel{\int_c P\,dx +}$

Green's theorem

If the 2 real functions $P(x,y)$ and $Q(x,y)$ have continuous first order partial derivatives on and inside C, then

$$\int P\,dx + Q\,dy = \iint_D (Q_x - P_y)\, dx\, dy$$

(where D is domain boundary by C)

where $Q_x = \frac{\partial Q}{\partial x}$

$P_y = \frac{\partial P}{\partial y}$

$f(z) = u(x,y) + iv(x,y)$

$z = x + iy$

$$\int_C f(z)\cdot dz = \int_C (u+iv)\, dz$$

$$= \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \iint_D (-v_x - u_y)\, dx\, dy + i\iint_D (u_x - v_y)\, dx\, dy$$

$$= 0$$ (from Cauchy Reimann)

PATH INDEPENDENCE

under what conditions:

$$\int_{C_1} f(z)\cdot dz = \int_{C_2} f(z)\cdot dz$$

where C_1, C_2 are two contours in the domain D near the same initial and final points and $f(z)$ is piece wise continuous inside D.

$\int f(z) \cdot dz = 0$

where $C = C_1 \cup (-C_2)$

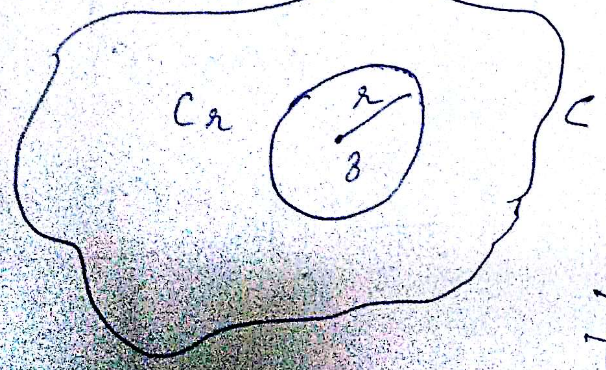
$f(z) = |z|^2$

$f(z) = \frac{1}{z^2}$

Cauchy integral formula

Let function $f(z)$ be analytic on and inside a positively oriented simple closed contour C and z is any point inside C then show that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} \, d\xi$$



$\cancel{\frac{f(\xi)}{\xi - z}}$

In $C - C_r$ is $\frac{f(\xi)}{\xi - z}$ is analytic

$$\int_{C-C_r} \frac{f(\xi)}{\xi - z} d\xi$$

$$= 0$$

$$\frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} dz = \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi)}{\xi - z} d\xi$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi) - f(z)}{\xi - z}$$

$$+ \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{\xi - z} d\xi$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi) - f(z)}{\xi - z}$$

$$+ \frac{f(z)}{2\pi i} \int_{C_r} \frac{1}{\xi - z} d\xi$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi) - f(z)}{\xi - z}$$

$+ f(z)$ $\left\{ \text{since } \int_{C_r} \frac{1}{\xi - z} d\xi = 2\pi i \right.$