Spectral method in solving complex fuid flow systems

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1. ABSTRACT

Spectral methods are known for their high accuracy, reaching exponential convergence with increase of point or two into the grid. The use of differential operators to simulate the flow field is an emergent strategy, where machine learning techniques can be applied to obtain a physically correct flow. Their handling of non-periodic boundary problems into the differential operator with ease, the ability to handle non-linear terms more effectively compared to traditional schemes is one more reason why these operator based solving methods needs to be incorporated. In this project, we will demonstrate and implement the power of spectral methods in a variety of non-uniform grids, and prove that they are indeed powerful

2. INTRODUCTION

Chebyshev spectral methods are based on approximating solutions using Chebyshev polynomials. The key advantage of this method is that for a $C^{\inf}$ function, the error decays more rapidly than any power of $\frac{1}{N}$, i.e, the decay is exponential. This makes them ideal for use in high-risk applications like military, aerodynamics, etc.

3. PROBLEM DESCRIPTION

Here, we will solve a one dimensional variation of Sturm-Liouville type problem with Dirichlet boundary conditions on both ends.

$$u'' + k^2 u = f(x) \quad \textbf{(1)}$$

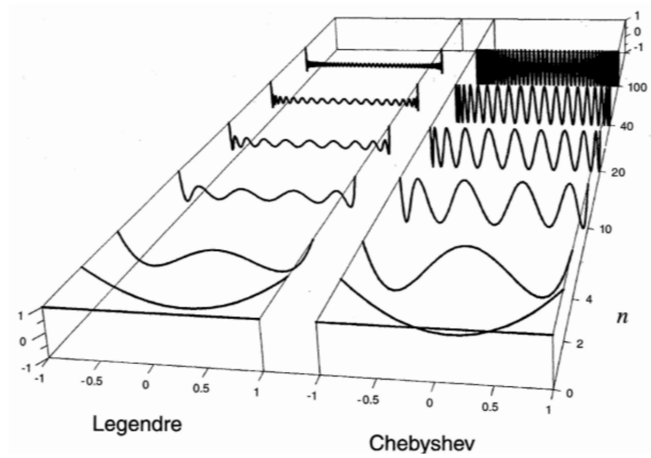

Fig. 1. Chebyshev as Lagrange Polynomials

where

- k = 2π
- f(x) = $2\pi^2 \sin(\pi x)$

This equation is a common equation, used to model wave propagation. Due to the complicated source term, this problem is non-homogeneous, and thus difficult to solve analytically. Thus, the problem is best solved numerically.

Here, we use Chebyshev Polynomials for both interpolation using Trefethen formula, and for grid formation using roots of Chebyshev polynoimials.

The exact solution can still be calculated by eigenvalue method, and if found to be

$$u = sin(\pi x) - xsin(\pi x) \quad \textbf{(2)}$$

There are critical problems which needs to be addressed when implementing this scheme:

- Accurately compute the second derivative matrix using Chebyshev collocation points. Using simple schemes like multiplying D_1 with itself has led to convergence issues and thus had to use a modified scheme. The calculation uses Fronberg weights, to directly calculate D_2 matrix, eliminating aggregation of truncation errors.

4. BRIEF OVERVIEW OF THE METHOD

A. Optimal Approximation Properties

The spectral projection error satisfies:

$$\|u - P_N u\|_{L^2} \leq CN^{-m}\|u^{(m)}\|_{L^2} \quad (3)$$

If u is an analytic function, this equation simplifies to:

$$\|u - P_N u\|_{L^2} \leq Ce^{-cN} \quad (4)$$

B. Error Reduction Mechanisms Per Step

- **Differentiation Matrix Construction**:

  Chebyshev differentiation matrices preserve polynomial relationships:

$$D_2 = D_1^2 + \mathcal{O}(N^{-1}\log N) \quad (5)$$

- **Spectral Filtering**:

  Exponential filter suppresses high-frequency noise:

$$\sigma(\eta) = \exp(-\alpha\eta^p), \quad \eta = k/N \quad (6)$$

- **Iterative Error Reduction**:

  For elliptic problems, the error evolves as:

$$\|\mathbf{e}^{k+1}\| \leq \rho(M)\|\mathbf{e}^k\| \quad (7)$$

  where $\rho(M)$ is the spectral radius of iteration matrix D.

C. Convergence Rates Comparison

Method	Smooth Solution	Analytic Solution
Finite Difference	$\mathcal{O}(N^{-p})$	$\mathcal{O}(N^{-p})$
Spectral	$\mathcal{O}(e^{-cN})$	$\mathcal{O}(e^{-\sigma N})$

D. Stability Enhancement

- **Skew-Symmetric Form**:

  Maintains energy stability:

$$\frac{d}{dt}\|u\|^2 = 0 \quad \text{for } (Du, u) + (u, Du) = 0 \quad (8)$$

- **Aliasing Control**:

  2/3 rule removes aliasing errors:

$$M \geq \frac{3}{2}N + 1 \quad (9)$$

5. ALGORITHM

Algorithm 1. Chebyshev Spectral Method for Helmholtz Eqn.

Input:
- N: Number of Chebyshev-Gauss-Lobatto (CGL) nodes
- $f(x) = 2\pi^2 \sin(\pi x)$: Source term
- $u_{\text{exact}}(x) = (1 - x^2)\sin(\pi x)$: Analytical solution

Output:
- Numerical solution $u_{\text{num}}(x)$
- Error distribution $\epsilon(x) = |u_{\text{num}} - u_{\text{exact}}|$
- Visualizations (4-panel display)

procedure MAIN

Generate CGL nodes: $x_j = -\cos\left(\frac{\pi j}{N}\right)$, $j = 0, ..., N$ [1]

Construct differentiation matrices:

$$D_1(i,j) = \begin{cases} \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, & i \neq j \\ -\sum_{k\neq i} D_1(i,k), & i = j \end{cases}$$

D_2 = Direct calculation with boundary corrections[2]

Form Helmholtz operator:

$$A = D_2 + \pi^2 I \quad \text{(Identity matrix } I\text{)}$$

Apply Dirichlet BCs:

$$A(0,:) \leftarrow 0, \; A(0,0) = 1$$
$$A(N,:) \leftarrow 0, \; A(N,N) = 1$$

Solve linear system:

$$A\mathbf{u} = \mathbf{b} \quad \text{where } b_i = f(x_i)$$

using full-pivoting LU decomposition

Apply spectral filter:

$$\hat{u}_k \leftarrow \hat{u}_k \exp\left(-\alpha\left(\frac{k}{N}\right)^p\right) \text{ [3]}$$

Compute error metrics:

$$\epsilon_\infty = \max_i |u_i - u_{\text{exact}}(x_i)|$$

$$\epsilon_2 = \sqrt{\frac{1}{N}\sum_{i=0}^{N} |u_i - u_{\text{exact}}(x_i)|^2}$$

Visualize results:
- Plot u_{num} vs u_{exact}
- Display error distribution $\epsilon(x)$
- Show spectral coefficients decay
- Plot convergence history

Complexity Analysis:
- Matrix construction: $\mathcal{O}(N^2)$
- LU factorization: $\mathcal{O}(N^3)$
- Filtering: $\mathcal{O}(N)$

Stability Considerations:
- Use Gauss-Lobatto quadrature for integration
- Maintain CFL condition: $\Delta t \sim \mathcal{O}(N^{-2})$
- Regularize ill-conditioned matrices via spectral filtering

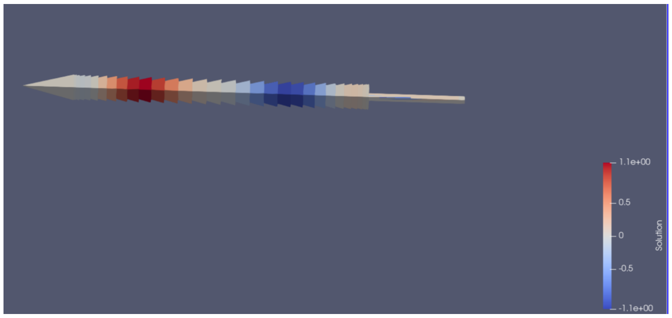

Fig. 2. Observed solution

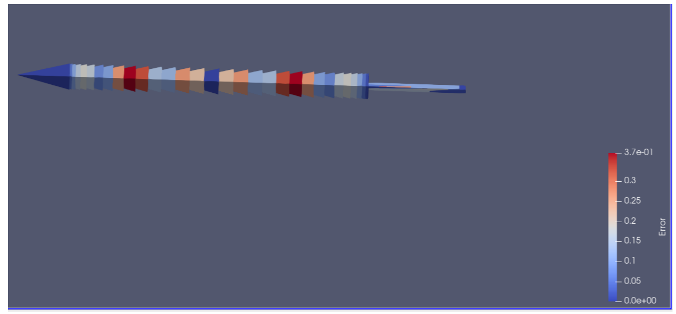

Fig. 3. Observed Error

6. CODE IMPROVEMENTS

- **Direct Second Derivative Calculation**: Replaced $D_2 = D_1^2$ with direct computation using Fornberg's formulation:

$$D_2(i,j) = D_1(i,j)\left(\frac{x_i}{1-x_i^2} - \frac{2}{x_i - x_j}\right)$$

  This avoids error accumulation from matrix squaring [2].

- **Exponential Filtering**: Added frequency-space damping:

$$\hat{u}_k \leftarrow \hat{u}_k \exp\left(-\alpha\left(\frac{k}{N}\right)^p\right)$$

  With $\alpha = 10$, $p = 8$ to suppress high-wavenumber oscillations. [3]. This filter also preserves accuracy while controlling Gibbs phenomena [4].

- The code is available in the Github link, where the procedure to run the code is explained clearly, step to step. Make sure that you download the libraries necessary beforehand to compile the code.

7. OBSERVATIONS

- Even though the decay is supposed to be exponential, the error did'nt converged as expected. This can be seen from the figures below. The order of magnitude of final error array was of -1, which was huge.

- There was considerable checkerboarding of errors, where there are recurring hotspots of error, followed by a sea of small error terms. This could be due to inefficient D2 matrix computation.

- The spectral filter didnt turned out to be helping with the convergence of the solution in this particular problem.

8. FUTURE WORK

- Implementing a variety of improvements like time stepping, iterative solver cascaded in spectral solver, preconditioning etc, to improve convergence.

- Identifying the base cause for less rate of convergence in the existing code, and improving it further.

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