Random Numbers

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: Download the following files

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/1/1.1.c
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/header.h
```

and compile and execute the C program using

```
$ gcc 1.1.c −lm
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/1/1.2.py
```

It is executed with

```
$ python3 1.2.py.py
```

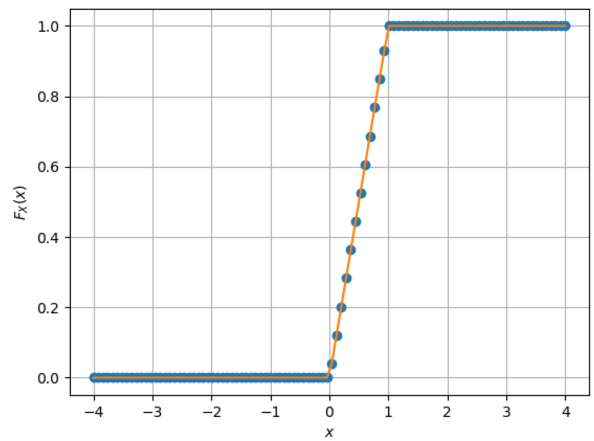

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \le x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.3}$$

c) $x \ge 1$: Put $x = 1$ in (1.3) as U is uniform in [0, 1] to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1.4}$$

This is verified in Figure (1.2)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\mathrm{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: The C program can be downloaded

using

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/1/1.4.c
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/header.h
```

and compiled and executed with

```
$ gcc 1.4.c −lm
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.7}$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.8}$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{1.9}$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \tag{1.10}$$

and

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$

$$= \int_{-\infty}^{\infty} x p_U(x)dx \tag{1.12}$$

$$= \int_0^1 x dx = \frac{1}{2} \tag{1.13}$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{1.14}$$

$$= E\left[U^2 - 2UE\left[U\right] + \left(E\left[U\right]\right)^2\right] \tag{1.15}$$

$$= E\left[U^2\right] - 2\left(E\left[U\right]\right)^2 + \left(E\left[U\right]\right)^2 \tag{1.16}$$

$$= E\left[U^2\right] - \left(E\left[U\right]\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.17}$$

$$\tag{1.18}$$

and this checks out with the empirical variance 0.083301 of the sample data.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/2/2.1.c
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/header.h
```

and compile and execute the C program using

```
$ gcc 2.1.c −lm
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 The required python file can be downloaded using

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/2/2.2.py
```

and executed using

```
$ python3 2.2.py
```

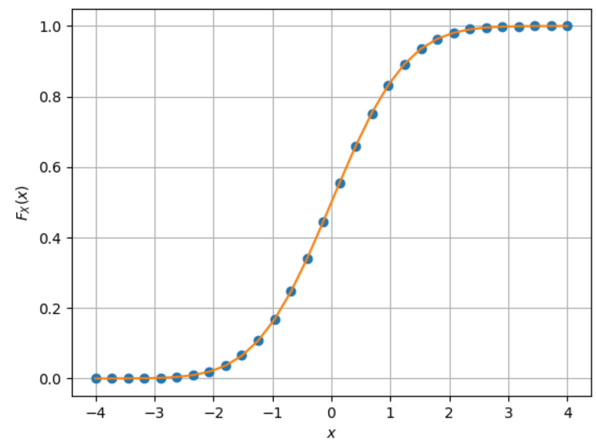

Fig. 2.2: The CDF of X

a) The CDF is non-decreasing
b) It is right-continuous.
c) $\lim_{x\to-\infty} F_X(x) = 0$
d) $\lim_{x\to\infty} F_X(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The

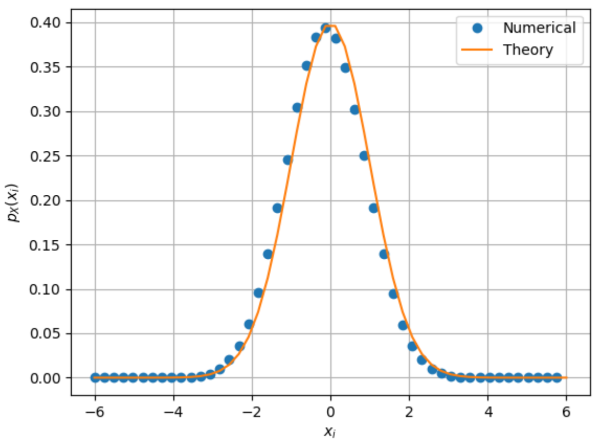

Fig. 2.3: The PDF of X

PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/2/2.3.py
```

The figure is generated using

```
$ python3 2.3.py
```

The properties of a PDF $p_X(x)$ are as follows:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) $\int_{-\infty}^{\infty} p_X(x)dx = 1$
c) For $a < b,\ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \tag{2.3}$$

$$= \int_a^b p_X(x)dx \tag{2.4}$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of X by writing a C program.
Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/2/2.4.c
```

and compiled and executed with the following commands

```
$ gcc 2.4.c −lm
$ ./a.out
```

The calculated mean is 0.000294 and the calculated variance is 0.999561.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.6}$$

as the integrand is odd. This checks out with the empirical mean of 0.000294. The variance is given by

$$\text{var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \tag{2.8}$$

$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}}\sqrt{2t}e^{-t}dt \tag{2.9}$$

$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \tag{2.10}$$

$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1 \tag{2.11}$$

where we have used $t = \frac{x^2}{2}$ and so $dt = xdx$. We have also used the gamma function given as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1}e^{-x}dx \tag{2.12}$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \tag{2.13}$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 0.999561.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files

```
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/3/3.1.c
$ wget https://github.com/SriCharvi/AI1110/
    blob/main/Random_Number/3/3.1.py
```

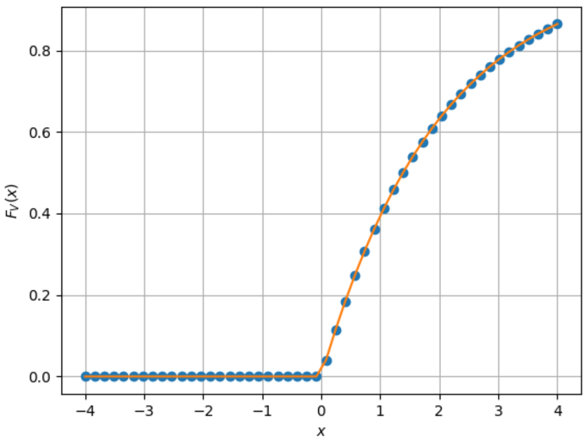

Fig. 3.1: The CDF of V

and can be executed with

```
$ python3 3.1.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Note that the function

$$v = f(u) = -2\ln(1-u) \tag{3.2}$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \tag{3.3}$$

Therefore, from the monotonicity of v, and using (1.4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \tag{3.4}$$

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \tag{3.5}$$