Electrostatics

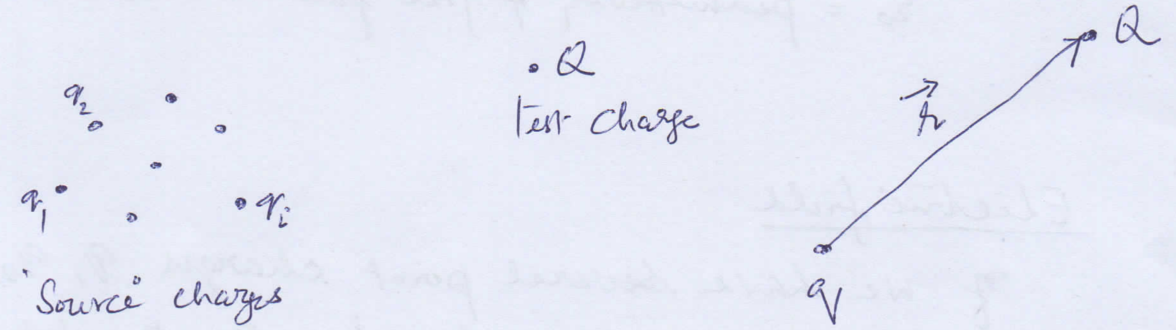
Ref Intro to Electrodynamics Chapter 2

Electric field

Electrodynamics tries to solve is to find the force experienced by a charge Q (test charge) due to a collection of charges q_1, q_2, q_3 --- (Source charges)

Positions of Source charges are given (as function of time) trajectory of test particle is to be calculated.

Generally both test charges and Source charges are in motion.



The Solution is found using the "principle of Superposition" which states that the interaction between any two charges is completely unaffected by the presence of others.

So to find the force on Q, we need to find the force $\vec{F_1}$ due to q_1 alone, then compute $\vec{F_2}$ due to q_2 alone and so on.

Vector sum of all the individual forces $\vec{F} = \vec{F_1} + \vec{F_2} + \cdots$

So the problem boils down to computing the force on Q due to a "single" source charge q.

[Note:- In practice the force not only depends on the distance of separation, but also on the velocities, and acceleration of q. Also since electromagnetic waves travel with speed of light, so force depends on Q, its position, velocity at some earlier time)

To begin with we consider a special case of Electrostatics in which all the source charges are stationary (test charge may be moving)

Coulomb's law

What is the force on a test charge Q due to a single point charge q that is at rest at a distance '$\mathscr{r}$' away. From experiments conducted, this force

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} \quad \text{(Coulomb's law)}$$

$$\varepsilon_0 = \text{permittivity of free space} = 8.85 \times 10^{-12}\, C^2 N^{-1} m^{-2}$$

Electric field

If we have several point charges $q_1, q_2 -- q_n$ at distances $\mathscr{r}_1, \mathscr{r}_2 -- \mathscr{r}_n$ from Q, the total force on Q is

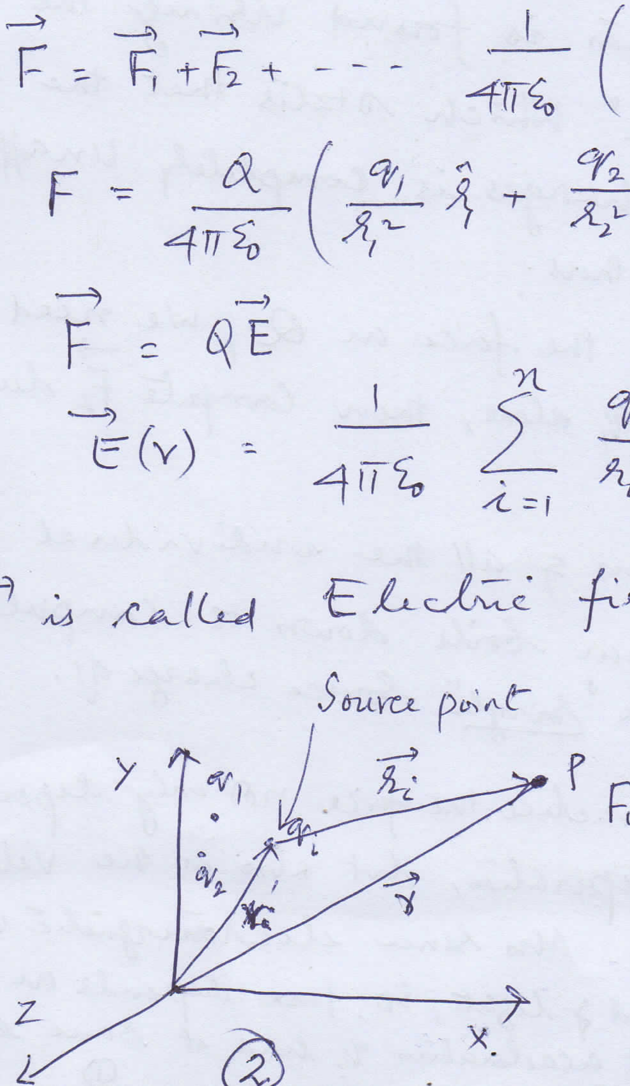
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + --- \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + --\right)$$

$$F = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \frac{q_3}{\mathscr{r}_3^2}\hat{\mathscr{r}}_3 + --\right)$$

$$\vec{F} = Q\vec{E}$$

where $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$

This $\vec{E}$ is called Electric field of the source charges.



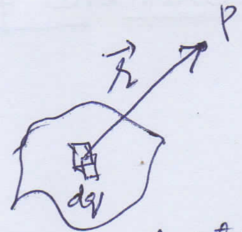
$$\vec{r}_i' + \vec{\mathscr{r}}_i = \vec{r}$$

$$\mathscr{r}_i = \vec{r} - \vec{r}_i'$$

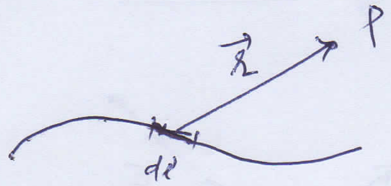
Continous charge distributions

In the earlier definition of electric field, the source charges was a collection of discrete point charges q_i. If the charge is distributed continously over some region, the sum becomes an integral

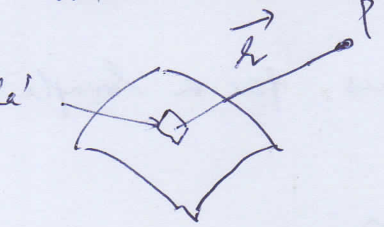
$$E(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,dq$$



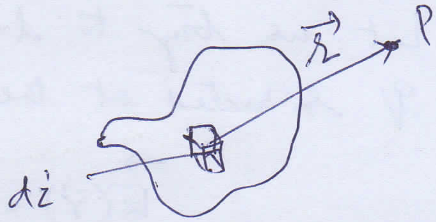
continous distribution



line charge, λ



Surface charge, σ



Volume charge - ρ

Line charge

If the charge is spread out along a line, with charge per unit length λ, then $dq = \lambda\, dl$ is an element of length along the line.

Electric field $\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,dl'$

Surface charge

Charge is smeared out over a surface, with charge per unit area σ, then $dq = \sigma\, da'$ (da' elemental area)

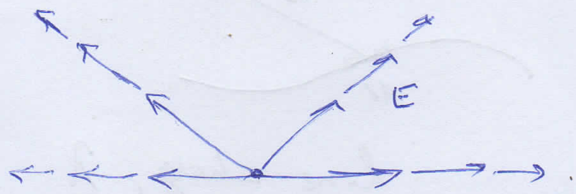
Electric field

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,da'$$

Volume charge :- If the charge fills a volume, the charge per unit volume ρ, then $dq = \rho \, d\tau'$ (where $d\tau'$ is an element of volume)

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{\mathscr{r}^2} \hat{\mathscr{r}} \, d\tau'$$

Divergence and curl of Electrostatic forces

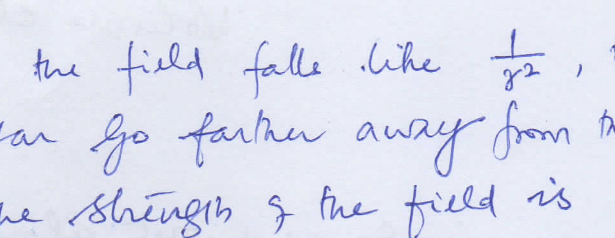


Let us try to draw field lines for a single point charge q situated at the origin

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

Since the field falls like $\frac{1}{r^2}$, the vector gets shorter as you go farther away from the origin.

But the strength of the field is 'missing' in this two dimensional representation. If this diagram is extended to 3D, you will notice that the density of the lines are more near the charge and it is less as you move correctly representing the strength, also in 2D the ~~area~~ circumference of circle is $2\pi r$, so field may appear as follows $\frac{1}{r}$, but in 3D sphere has area $4\pi r^2$, so field correctly follows $\frac{1}{r^2}$ law.



With this model, the flux of E through a surface S is

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a}$$

Flux is the "number of field lines" passing throug S. The dot Product picks out the component of $d\vec{a}$ along the direction of $\vec{E}$.

This suggests that the flux through any closed surface ~~on a positive~~ is a measure of the total charge inside. (charge outside the surface will not contribute to the field).

This is called <u>Gauss's Law</u>

In the case of a point charge 'q' at the origin, the flux of E through a spherical surface of radius r is

$$\oint \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{r} = \frac{1}{\varepsilon_0} q$$

$$\left[\because \int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta \, d\theta \, d\phi = r^2 \right]$$

If we have a collection of charges instead of a single charge 'q', then

$$\oint \vec{E}\cdot d\vec{a} = \sum_{i=1}^{n}\left(\oint \vec{E}_i\cdot d\vec{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\varepsilon_0} q_i\right)$$

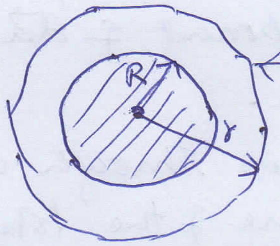
For any closed surface, then

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enclosed}$$

where $Q_{enclosed}$ is the total charge enclosed within the surface.

Applications of Gauss's Law

Find the field outside a uniformly charged ~~Sp~~ solid Sphere of radius R and total charge q



Imagine a Spherical Surface at radius $r > R$ (See Fig), this is Called the Gaussian Surface.

From Gauss's law

$$\oint \vec{E} \cdot \vec{da} = \frac{1}{\epsilon_0} Q_{enclosed}$$

Here $Q_{enclosed} = q$

Because of the Symmetry of the Problem

$$\int_S \vec{E} \cdot \vec{da} = \int_S |E| \, da$$

E points radially outward as does da, so $\theta = 0$, hence $\cos\theta = 1$ so the dot product $\vec{E} \cdot \vec{da} = E da$ and Since magnitude of E is constant it can be taken out of the integral, so

$$\int_S |E| \, da = |E| \int da = |E| \, 4\pi r^2 = \frac{1}{\epsilon_0} q$$

$$|E| = \frac{q}{4\pi \epsilon_0 r^2} \qquad \vec{E} = \frac{q}{4\pi \epsilon_0 r^2} \hat{r}$$

Inference :- The field outside the Sphere is exactly the Same as it would have been if all the charges are concentrated at the Centre.