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Stable Marriage Problem

Scribes: Dhananjay Raman, Anish Yogesh Kulkarni, Yarabolu Vennela, Yash Virani, Prahasith Pasnoor, PVR Sai Kumar *
Edited by: Sanjeev Kumar

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1 Introduction

In recent years, integrating Artificial Intelligence (AI) has brought significant advancements to matching problems. AI-driven solutions often outperform traditional methods like resource distribution and job allocation. These problems require optimizing pairings for desirable outcomes.

We explore how this technology is harnessed to create dynamic and optimized solutions that revolutionize two-sided markets, ensuring that participants are matched in ways that maximize their mutual satisfaction and utility.

Throughout our discussion, we will also explore the impact of AI on matching problems and how it leverages advanced computational methods to tackle the challenges posed by two-sided markets, ultimately leading to more effective and stable pairings.

2 Two-sided Market

Two-sided markets have emerged as a captivating and influential concept. These markets embody a dynamic exchange where two distinct groups interact, each with its own unique set of preferences and interests. They have a profound impact on various sectors, including the digital realm, where platforms like ride-sharing apps, e-commerce platforms, and dating websites have thrived. Two-sided markets are not merely about connecting buyers and sellers; they involve facilitating interactions that are as much about supply as they are about demand.

In a two-sided market, the equilibrium is achieved when both groups—buyers and sellers—find their ideal match, optimizing their respective objectives. The notion of equilibrium here is not just about economic balance; it is deeply intertwined with notions of satisfaction, compatibility, and stability. As the dynamics unfold in two-sided markets, the question arises: How do we create a system where every participant is content with their pairing?

This brings us to the stable marriage problem, in its simplest form, which involves an equal number of two types of participants, such as men and women, with each group expressing their preferences for potential matches. A stable matching always exists, and the goal of the Deferred Acceptance algorithm is to find one. A matching is a bijection from the elements of one set to the elements of the other set. A matching is considered stable if:

*Equal contributions by all scribes

1. There is no man (A) who prefers a specific woman (B) over the woman he is currently matched with.
2. Simultaneously, the woman (B) also prefers this man (A) over her current match.

In other words, a matching is stable when there are no pairs (A, B) where both individuals prefer each other over their current partners. If such a pair exists, the matching is not stable, implying that these two individuals would prefer to be matched to each other, potentially leaving other participants unmatched.

3 Stable Marriage Problem

We formally define the Stable Marriage Problem as follows:

Suppose there are n men and women denoted by $M = \{m_1, m_2, \ldots, m_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$.
Every man m has a preference order of women $\{w_{x_i}\}_{i=1}^n$ such that $\{x_i\}_{i=1}^n$ is a permutation of $\{i\}_{i=1}^n$, and m prefers to marry w_{x_i} over w_{x_j} (denoted by $w_{x_i} P_m w_{x_j}$) $\forall_{i<j}$.
Similarly, every woman w has a preference order of men $\{m_{x_i}\}_{i=1}^n$ such that $\{x_i\}_{i=1}^n$ is a permutation of $\{i\}_{i=1}^n$, and w prefers to marry m_{x_i} over m_{x_j} (denoted by $m_{x_i} P_w m_{x_j}$) $\forall_{i<j}$.
The set of all preferences for all men and women is denoted by P, as the **Preference Profile**.

A **Matching** μ is a bijective function from M to W.

Over a given P and μ, a pair (m, w) is called a **Blocking Pair** if m prefers w over $\mu(m)$ (denoted by $wP_m\mu(m)$) and w prefers m over $\mu^{-1}(w)$ (denoted by $mP_w\mu^{-1}(w)$).

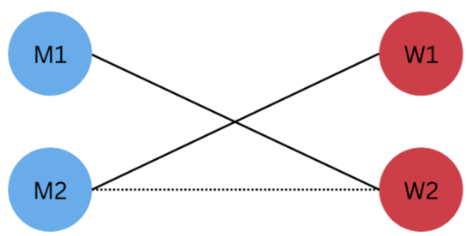


Figure 1: Here the solid lines give a matching between men (represented as M1, M2) and women (represented as W1, W2). The preferences of each person are next to the corresponding circle. The dashed line pairing (M2, W2) is a **blocking pair**

A matching μ is called a **Pairwise Stable Matching** over a preference profile P if there do not exist any blocking pairs in the matching.

The **Stable Marriage Problem** is to find a pairwise stable matching over a given P. There are other formulations of the Stable Marriage Problem, one of which will be discussed later in this report.

4 Deferred Acceptance Algorithm

The Deferred Acceptance Algorithm, often known as the Gale–Shapley Algorithm, is a systematic method for creating stable pairings in scenarios like matchmaking and job allocation. It guarantees that everyone is matched and that the resulting pairs are stable. This algorithm is widely used in real-world applications.

Let us explore how this algorithm works by illustrating its application to the Stable Marriage Problem considering n men and n women along with their preference orders (note that we are writing this algorithm from the men proposing to women approach; equivalently, it can also be written from the women proposing to men approach)

- Initialization: First Round
  - Each man makes proposals to the woman he prefers the most
  - Each woman who has at least one proposal responds by accepting the proposal from her most preferred man and rejecting all other proposals.
  - Temporary pairings are established, with each woman matched with her accepted man.
- Subsequent Rounds
  - In each subsequent round, each man proposes to his most preferred woman among those he hasn't proposed to yet, regardless of her current match status.
  - Each woman, upon receiving a proposal, accepts if she is not currently matched or if she finds the proposing man more preferable than her existing partner.
  - This may lead to rejections of existing matches, causing the previously matched man to become unmatched.
- This process continues through multiple rounds until everyone is matched.

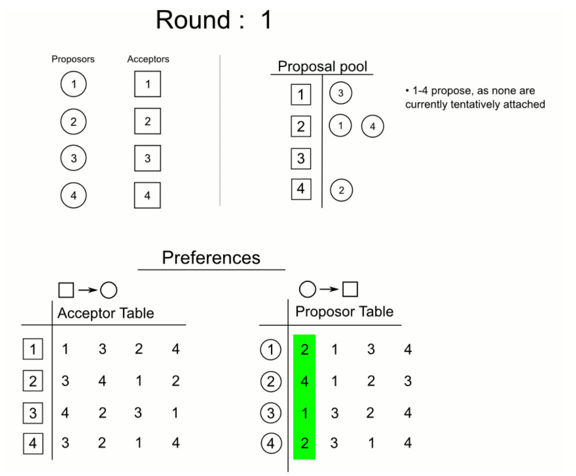
Algorithm 1 Stable Matching between Men and Women

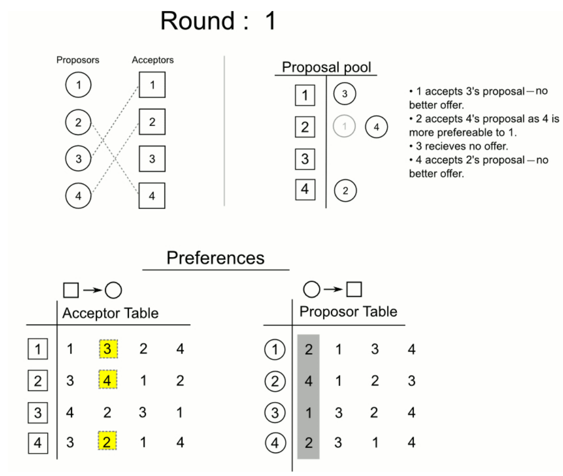
Input: Set of men M and set of women W with preference lists
Output: Stable matching of men and women
Initialize all men and women to be single
while $\exists$ a man m who has a woman w to whom he has not proposed **do**
  for all men m who are currently single **do**
    $w :=$ the first woman on m's list to whom he has not proposed
    m proposes to w
  end for
  for all w who were proposed **do**
    if w is single **then**
      (m, w) become matched where m is w's most preferred man out of all men who proposed to her
    else
      $m' :=$ the current partner of w
      if w prefers m to m' for some man m who proposed her **then**
        m' becomes single
        (m, w) become matched where m is the man w prefers the most out of all men who proposed to her
      end if
    end if
  end for
end while

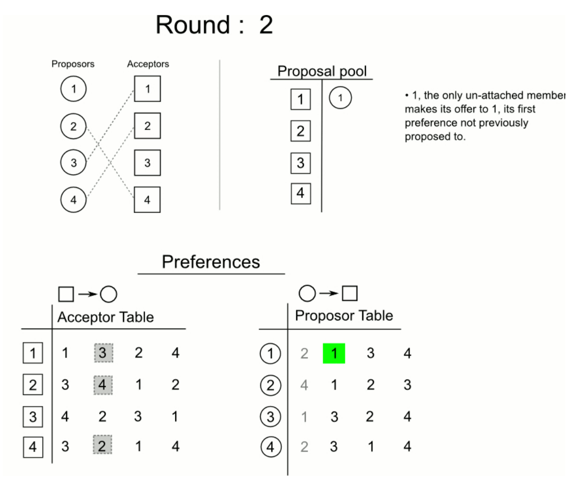
The algorithm ensures a stable match for all participants and operates with a time complexity of $O(n^2)$, where n denotes the number of men or women involved which can be proven based on the following facts:

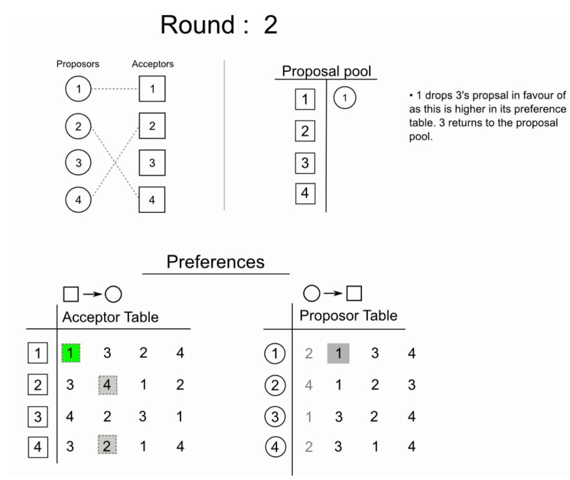
1. In the worst case, each man proposes to each woman, leading to n rounds of proposals (each man making n - 1 proposals).
2. In each round, women may consider and respond to all n proposals.

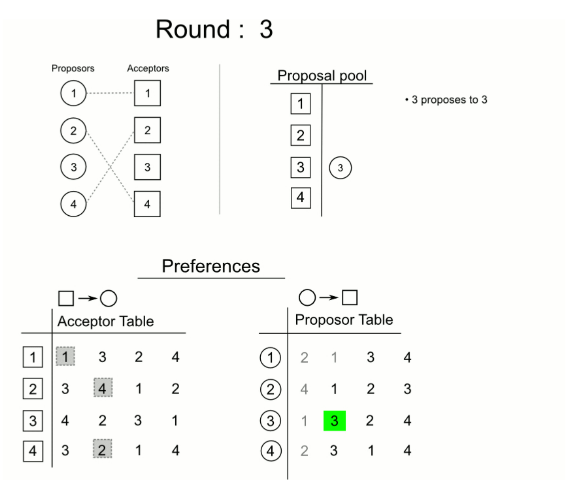
These two factors combine to create a time complexity of $O(n^2)$ since, in the worst-case scenario, there are n rounds, and in each round, n proposals are evaluated. Here is how the algorithm works, considering 4 men and 4 women with their preference list mentioned in the table.

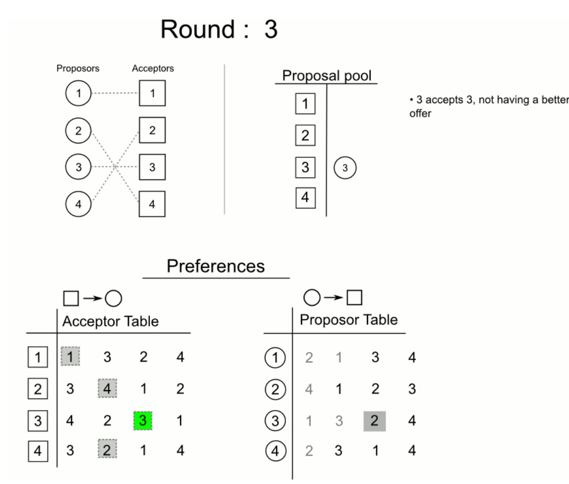

Round : 1
Proposors
Acceptors
Proposal pool
• 1-4 propose, as none are currently tentatively attached
Preferences
Acceptor Table
Proposor Table


Round : 1
Proposors
Acceptors
Proposal pool
• 1 accepts 3's proposal — no better offer.
• 2 accepts 4's proposal as 4 is more prefereable to 1.
• 3 recieves no offer.
• 4 accepts 2's proposal — no better offer.
Preferences
Acceptor Table
Proposor Table


Round : 2
Proposors
Acceptors
Proposal pool
• 1, the only un-attached member makes its offer to 1, its first preference not previously proposed to.
Preferences
Acceptor Table
Proposor Table


Round : 2
Proposors
Acceptors
Proposal pool
• 1 drops 3's propsal in favour of as this is higher in its preference table. 3 returns to the proposal pool.
Preferences
Acceptor Table
Proposor Table


Round : 3
Proposors
Acceptors
Proposal pool
• 3 proposes to 3
Preferences
Acceptor Table
Proposor Table


Round : 3
Proposors
Acceptors
Proposal pool
• 3 accepts 3, not having a better offer
Preferences
Acceptor Table
Proposor Table

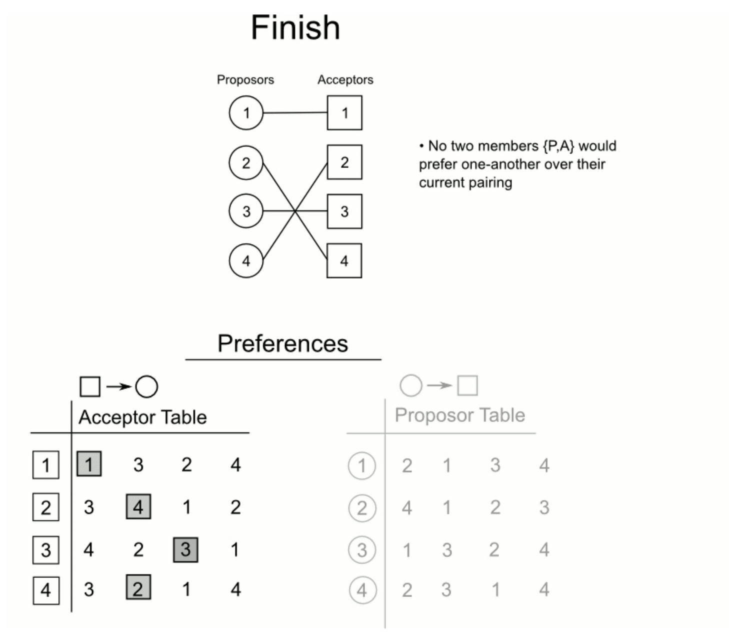


Figure 3: Final Stable matching

4.1 Correctness

For any given instance of the stable marriage problem, the Deferred Acceptance algorithm terminates, and, on termination, the matching pairs constitute a stable matching.

Claim: Deferred Acceptance algorithm will terminate with acceptance time complexity of O(n^2).
Proof: Firstly, we show that no man can be rejected by all women. A woman can reject only when she is paired, and once she is paired she never again becomes free. So the rejection of a man by the last woman on his list would imply that all the women were already paired. But since there are equal numbers of men and women, and no man has two mappings to women, all the men would also be paired, which is a contradiction. Also, each iteration involves one proposal, and no man ever proposes twice to the same woman, so the total number of iterations cannot exceed n^2 (for instance involving n men and n women). Termination is therefore established.

Claim: On termination Deferred Acceptance algorithm always returns a pair-wise stable matching.
Proof: On termination, the mapped pairs specify a matching, which we denote by M. If man m prefers woman w to $p_w(m)$, then w must have rejected m at some point during the execution of the algorithm. But this rejection implies that w was, or became, engaged to a man she prefers to m, and any subsequent change of her partner brings her a still better partner. So w cannot prefer m to $p_m(w)$, and therefore (m, w) cannot block M. It follows that there are no blocking pairs for M, and therefore that M is a stable matching.

Claim: Deferred acceptance algorithm always returns a pair-wise stable matching.
Proof: Suppose there exists a blocking pair in the DA solution. $\mu(m) = w'$ & $\mu(m') = w$ in the DA solution, and (m,w) form a blocking pair. Then m would have proposed to w and w must have rejected it for a better option, but w ended up with worse than m which is not possible. So DA algorithm always gives pair-wise stable matching.

5 Group Stability

Group stability ensures that no group of agents can collectively improve their situations by deviating from the given matching to another matching that respects their preferences. $\forall G \subseteq M + W$, There shouldn't be any other matching which strictly improves the priorities of every person in G and doesn't decrease the priority of every person in $M + W - G$.

5.1 Definition

A set $S \subset M \cup W$ blocks μ, P if there exists another matching μ' such that:

1. $\forall m \in M \cap S, \mu'(m) \in W \cap S$ and $\forall w \in W \cap S, \mu'^{-1}(w) \in M \cap S$
2. $\forall m \in M \cap S, \mu'(m) P_m \mu(m)$ and $\forall w \in W \cap S, \mu'^{-1}(w) P_w \mu^{-1}(w')$

Here $w P_m w'$ means m prefers w over w'.

A matching μ along with preferences P is said to be Group stable if it does not have a blocking set.

5.2 Theorem

Pairwise stability is equivalent to group stability.

Proof -

Part 1 $GS \implies PS$
If possible let there exist blocking pair $w' = \mu(m), w = \mu(m')$ such that $m P_w m'$ and $w P_m w'$.
We shall show that $S = \{m, w\}$ is a blocking set.
Consider μ' such that $\mu'(m) = w$ and $\mu'(m') = w'$ and at other places μ' agrees with μ
Then we have $\mu'(m) P_m \mu(m)$ and $\mu'^{-1}(w) P_w \mu^{-1}(w)$. Hence S is a blocking set.
But this contradicts our assumption of group stability.
Therefore no blocking pair exists and the matching is pairwise stable.

Part 2 $PS \implies GS$
If possible let there exist a blocking set S with corresponding matching μ'.
Let $m \in S$ and $w = \mu'(m), w' = \mu(m), m' = \mu^{-1}(w)$. Then by definition of blocking set $w P_m w'$ and $m P_w m'$.
Hence we have found a blocking pair but this contradicts the assumption of pairwise stability.
Hence no blocking set exists and our matching is group stable.

6 Optimality of a Matching

Till now we have defined an optimal matching as one that does not have any blocking pairs (or blocking set). This is not the only way to define optimality.

To look at it from another perspective, we define the **Rank** (with respect to men) of a matching μ (over P, M, and W) as a tuple $(r_1, r_2, \ldots, r_n)$, where r_i is the (1-indexed) position of $\mu(m_i)$ in the preference order P^{m_i} of m_i. For example, if every man gets matched with their ideal woman, the rank of such matching will be $(1, 1, \ldots, 1)$.

Then, an optimal matching can be one which optimizes one of the properties of the Rank. Some examples are:

- Minimizing the average rank

- Defining a Total Order on the ranks, where $r_1 < r_2$ if (WLOG assume values in r_1 and r_2 are sorted in ascending order) if:

  1. $r_1[0] < r_2[0]$, or
  2. $r_1[0] = r_2[0]$ and $r_1[1] < r_2[1]$, or
  3. $r_1[0] = r_2[0]$ and $r_1[1] = r_2[1]$ and $r_1[2] < r_2[2]$, and so on

  Then we define the optimal matching to be the one that corresponds to the lowest achievable rank according to this order.

Equivalently, we could also have defined the Rank with respect to women as a tuple $(r_1, r_2, \ldots, r_N)$, where r_i is the (1-indexed) position of $\mu^{-1}(w_i)$ in the preference order P^{w_i} of w_i. For example, if every woman gets matched with their ideal man, the rank of such matching will be $(1, 1, \ldots, 1)$.

What property does the (men-proposing) deferred acceptance algorithm optimize? We claim that each man ends up with the "best possible partner" in a concrete sense.

First, we will say that a woman w is a *valid partner* of a man m if there is a stable matching that contains the pair (m, w). We will say that w is the *best valid partner* of w if w is a valid partner of w, and no woman whom m ranks higher than w is a valid partner of his. We will use $best(m)$ to denote the **best valid partner** of m.

Now, let S^* denote the set of pairs $(m, best(m)) : m \in M$. We claim that every execution of the (men-proposing) deferred acceptance algorithm results in the set S^*.[1]

[1]The proof was not discussed during the class, however, it can be found on page 10 in the book *Algorithm Design* by Eva Tardos.