**SRM University-AP, Amaravati**

Department of Mathematics

Discrete Mathematics(MAT141)

**Assignment 1**:The Foundations: Logic and Proofs

1. Which of these sentences are propositions? what are the truth values of those that are propositions?

   (a) Mumbai is the capital of India.

   (b) $2 + 3 = 5$.

   (c) $5 + 7 = 10$.

   (d) $x + 2 = 11$.

   (e) Answer this question.

   (f) What time is it?

   (g) The moon is made of green cheese.

   (h) $4 + x = 5, \forall x \neq 1$ and domain is the set of real number.

   (i) $4 + x = 5, \forall x = 1$ and domain is the set of real number.

   (j) $4 + x = 6, \forall x \neq 1$ and domain is the set of real number.

   (k) $2^n \geq 100$.

2. Let $p$, $q$, and $r$ be the propositions

   $p$ : You have the flu.

   $q$ : You miss the final examination.

   $r$ : You pass the course.

   Express each of these propositions as an English sentence.

   (a) $p \to q$

   (b) $\neg q \leftrightarrow r$

   (c) $q \to \neg r$

   (d) $p \vee q \vee r$

   (e) $(p \to \neg r) \vee (q \to \neg r)$

   (f) $(p \wedge q) \vee (\neg q \wedge r)$

3. Determine whether these biconditionals are true or false.

   (a) $2 + 2 = 4$ if and only if $1 + 1 = 2$.

   (b) $1 + 1 = 2$ if and only if $2 + 3 = 4$.

   (c) $1 + 1 = 3$ if and only if monkeys can fly.

   (d) $0 > 1$ if and only if $2 > 1$

4. Determine whether each of these conditional statements is true or false.

   (a) If $1 + 1 = 2$, then $2 + 2 = 5$.

   (b) If $1 + 1 = 3$, then $2 + 2 = 4$.

   (c) If $1 + 1 = 3$, then $2 + 2 = 5$.

   (d) If monkeys can fly, then $1 + 1 = 3$.

5. State the converse, contrapositive, and inverse of each of these conditional statements.

   (a) If it snows tonight, then I will stay at home.

   (b) I go to the beach whenever it is a sunny summer day.

   (c) When I stay up late, it is necessary that I sleep until noon.

6. Construct the truth table for each of these compound propositions.

   (a) $p \wedge \neg p$

   (b) $p \vee \neg p$

   (c) $(p \vee \neg q) \to q$

   (d) $(p \vee q) \to (p \wedge q)$

   (e) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

   (f) $(p \to q) \to (q \to p)$

7. Construct a truth table for each of these compound propositions.

   (a) $(p \vee q) \to (p \oplus q)$

   (b) $(p \oplus q) \to (p \wedge q)$

   (c) $(p \vee q) \oplus (p \wedge q)$

   (d) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

   (e) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$

   (f) $(p \oplus q) \to (p \oplus \neg q)$

8. Construct a truth table for each of these compound propositions.

   (a) $p \to (\neg q \vee r)$

   (b) $\neg p \to (q \to r)$

   (c) $(p \to q) \vee (\neg p \to r)$

   (d) $(p \to q) \wedge (\neg p \to r)$

   (e) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

   (f) $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

9. Use truth tables to verify these equivalences.

   (a) $p \wedge T \equiv p$

   (b) $p \wedge F \equiv F$

   (c) $p \vee p \equiv p$

   (d) $p \vee F \equiv p$

   (e) $p \vee T \equiv T$

   (f) $p \wedge p \equiv p$

10. Use truth tables to verify commutative laws

   (a) $p \vee q \equiv q \vee p$

   (b) $p \wedge q \equiv q \wedge p$

11. Use truth tables to verify associative laws

   (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

12. Show that each of these conditional statements is a tautology by using truth tables.

(a) $[\neg p \wedge (p \vee q)] \rightarrow q$

(b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

(c) $[p \wedge (p \rightarrow q)] \rightarrow q$

(d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

13. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

14. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

15. Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent.

16. Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.

17. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology.

18. Translate these statements into English, where $C(x)$ is “x is a comedian” and $F(x)$ is “x is funny” and the domain consists of all people.

(a) $\forall x(C(x) \rightarrow F(x))$

(b) $\exists x(C(x) \rightarrow F(x))$

(c) $\forall x(C(x) \wedge F(x))$

(d) $\exists x(C(x) \wedge F(x))$

19. Let $Q(x)$ be the statement “$x + 1 > 2x$”. If he domain consists of all integers, what are these truth values?

(a) $Q(0)$

(b) $Q(-1)$

(c) $Q(1)$

(d) $\exists x Q(x)$

(e) $\forall x Q(x)$

(f) $\exists x \neg Q(x)$

(g) $\forall x \neg Q(x)$

20. Determine the truth value of each of these statements if the domain consists of all real numbers.

(a) $\exists x(x^3 = -1)$

(b) $\exists x(x^4 < x^2)$

(c) $\forall x((-x)^2 = x^2)$

(d) $\forall x(2x > x)$

21. What are the negations of the statement $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$ ?

22. Let $S(x)$ be the predicate “x is a student”, $F(x)$ the predicate “x is a faculty member”, and $A(x, y)$ the predicate “x has asked y a question”, where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

(a) Lois has asked professor Michaels a question.

(b) Every student has asked Professor Gross a question.

(c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.

(d) Some students has not asked any faculty member a question.

(e) There is a faculty member who has never been asked a question by a student.

(f) Some student has asked every faculty member a question.

(g) There is a faculty member who has asked every other faculty member a question.

(h) Some student has never been asked a question by a faculty member.

23. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

(a) The product of two negative integers is positive.

(b) The average of two positive integers is positive.

(c) The difference of two negative integers is not necessarily negative.

(d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

24. Translate each of these nested quantifications into an English statement that express a mathematical fact. The domain in each case consists of all real numbers.

(a) $\exists x \forall y (xy = y)$

(b) $\forall x \forall y (((x \geq 0) \wedge (y < 0)) \rightarrow (xy > 0))$

(c) $\forall x \forall y \exists z (x + y = z)$

25. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

(a) $\forall x \exists y (x^2 = y)$

(b) $\exists x \forall y (xy = 0)$

(c) $\forall x (x \neq 0 \rightarrow \exists y (xy = 1))$

(d) $\exists x \forall y (y \neq 0 \rightarrow xy = 1)$

(e) $\forall x \exists y (x + y = 1)$

(f) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$

(g) $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$

(h) $\forall x \forall y \exists z (z = (x + y)/2)$

(i) $\forall x \exists y (x = y^2)$

(j) $\exists x \exists y (x + y \neq y + x)$

26. Rewrite each of these statements so that negations appear only within predicates(that is, so that no negation is outside a quantifier or an expression involved logical connectives).

(a) $\neg\exists y\exists x P(x,y)$

(b) $\neg\exists y(Q(y) \wedge \forall x \neg R(x,y))$

(c) $\neg\exists y(\exists x R(x,y) \vee \forall x S(x,y))$

(d) $\neg\exists y(\forall x\exists z T(x,y,z) \vee \exists x\forall z U(x,y,z))$

(e) $\neg\forall x\exists y P(x,y)$

27. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

(a) $\forall x\forall y(x^2 = y^2 \to x = y)$

(b) $\forall x\exists y(y^2 = x)$

(c) $\forall x\forall y(xy \geqslant x)$

28. Establish the validity of the following argument forms using the rules of inferences.

(a)
$$\begin{array}{l} p \to q \\ \neg q \\ \neg r \\ \hline \therefore \neg(p \vee r) \end{array}$$

(b)
$$\begin{array}{l} p \vee q \\ \neg r \\ \underline{\neg p \vee r} \\ \therefore q \end{array}$$

(c)
$$\begin{array}{l} p \vee q \\ p \to r \\ r \to s \\ \hline \therefore q \vee s \end{array}$$

(d)
$$\begin{array}{l} \neg q \\ \neg r \\ \underline{\neg p \vee q \vee r} \\ \therefore \neg p \end{array}$$

(e)
$$\begin{array}{l} (\neg p \vee q) \to r \\ r \to (s \vee t) \\ \neg s \wedge \neg u \\ \underline{\neg u \to \neg t} \\ \therefore p \end{array}$$

(f)
$$\begin{array}{l} p \to q \\ \neg r \vee s \\ p \vee r \\ \hline \therefore \neg q \to s \end{array}$$

29. Use a direct proof to show that the product of two odd numbers is odd.

30. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

31. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers,then $x \geq 1$ or $y \geq 1$.

32. Prove that if $m$ and $n$ are integers and $mn$ is even,then $m$ is even or $n$ is even.

33. Show that if $n$ is an integer and $n^3 + 5$ is odd,then n is even using

    a) a proof by contraposition.

    b) a proof by contradiction.

34. Prove the proposition $P(0)$,where $P(n)$ is the proposition "If $n$ is a positive integer greater than 1, then $n^2 > n$". What kind of proof did you use?

35. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

36. Use a proof by contradiction to show that there is no rational number $r$ for which $r^3 + r + 1 = 0$. [Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms.obtain an equation involving integers by multiplying by $b^3$. then look at whether a and b are each odd or even.]

37. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

38. Prove that these four statements about the integers $n$ are equivalent: (i)$n^2$ is odd, (ii)1-n is even, (iii) $n^3$ is odd, (iv) $n^2 + 1$ is even.