Antiderivatives:

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

- The process of recovering a function $F(x)$ from its derivative $f(x)$ is called antidifferentiation.

Ex: $f(x) = 2x$, $g(x) = \cos x$, $h(x) = \sec^2 x + \frac{1}{2\sqrt{x}}$.

Antiderivatives are:
$F(x) = x^2$, $G(x) = \sin x$, $H(x) = \tan x + \sqrt{x}$.

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

Problem: Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution: General antiderivative of $f(x)$ is.

$$F(x) = x^3 + C \quad , \quad C = \text{const.}$$

Now since $F(1) = -1$,

$$-1 = 1 + C \Rightarrow C = -2.$$

Hence $F(x) = x^3 - 2$.

Antiderivative formulas: (k = nonzero constants)

Function	General Antiderivative
x^n	$\frac{1}{n+1}x^{n+1} + C, \ n \neq -1$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
$\operatorname{cosec}^2 kx$	$-\frac{1}{k}\cot kx + C$
$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
$\operatorname{cosec} kx \cot kx$	$-\frac{1}{k}\operatorname{cosec} kx + C$.

Indefinite Integrals

Definition: The collection of all antiderivatives of f is called the indefinite integral of f with respect to x, and is denoted by

$$\int f(x)\, dx .$$

The symbol $\int$ is an integral sign. The function f is the integrand of the integraland x is the variable of integration.

Example: Evaluate $\int (x^2-2x+5)\, dx$.

$f(x) = x^2-2x+5$.

An antiderivative of $f(x)$ is $\left(\frac{x^3}{3}-x^2+5x\right)$.

Hence $$\int (x^2-2x+5)\, dx = \underbrace{\frac{x^3}{3}-x^2+5x}_{\text{Antiderivative}} + C.$$

Problem: Find the general antiderivative or indefinite integral.

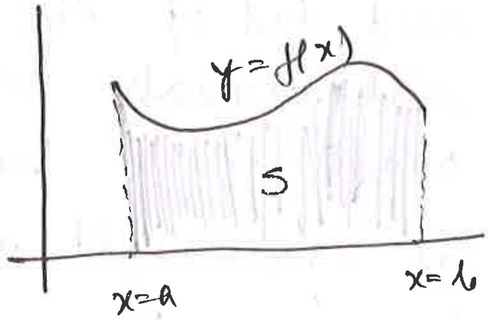
(i) $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

(ii) $\int \left(\frac{4+\sqrt{t}}{t^3}\right) dt$ (iii) $\int \frac{\operatorname{cosec}\theta \cot\theta}{2}\, d\theta$

(iv) $\int \frac{2}{5} \sec\theta \tan\theta \, d\theta$ (v) $\int (4\sec x \tan x - 2\sec^2 x)\, dx$

Integrals

The area problem: Find the area of the region S that lies under the curve $y=f(x)$ from $x=a$ to $x=b$.

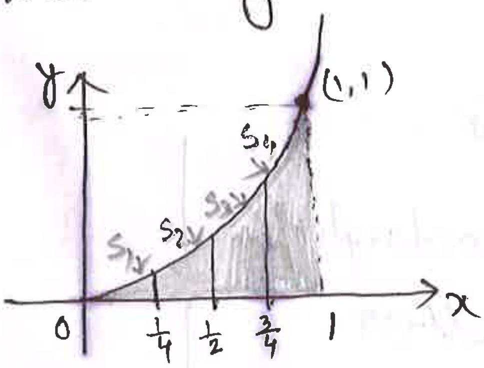


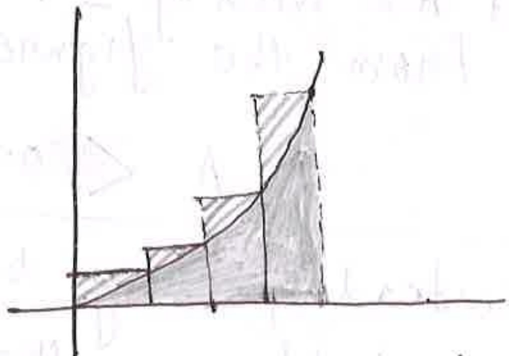
i.e. to find the area S bounded by the graph of a cont. f^n. $f(x)$, (where $f(x) \geq 0$) the vertical lines $x=a$ & $x=b$ & the x-axis.

One way to estimate the area;

Example: Use rectangles to estimate the area under the parabola $y=x^2$ from 0 to 1.

Solution:





Suppose we divide the area into four strips S_1, S_2, S_3 & S_4 by drawing the vertical lines $x=\frac{1}{4}$, $x=\frac{1}{2}$, $x=\frac{3}{4}$.

We can approximate each strip by a rectangle that has the same base as the strip and whose height is same as the right edge of the strip.

Basically, we have divided the interval $[0,1]$ into four subintervals as

$$\left[0,\tfrac{1}{4}\right], \left[\tfrac{1}{4},\tfrac{1}{2}\right], \left[\tfrac{1}{2},\tfrac{3}{4}\right] \& \left[\tfrac{3}{4},1\right].$$

and put up the rectangles over these subintervals where each rectangle has width $\frac{1}{4}$ & heights are the functional values at the right end pt of the subintervals

ie. $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2, \left(\frac{3}{4}\right)^2, \& (1)^2.$ & (Right end pts)

Since we have taken 4 subintervals, we denote the estimated area by R_4 &

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot(1)^2.$$

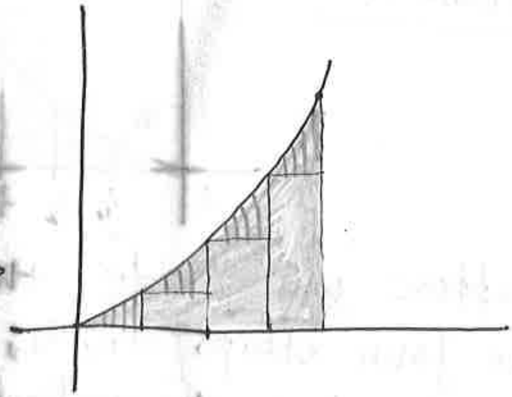
$$= \frac{15}{32} = 0.46875$$

Let A = area of S.

From the figure,

$$A < 0.46875.$$

Instead of using bigger rectangles we could use the smaller rectangles as



whose heights are the functional values of f at left endpts of the subintervals.

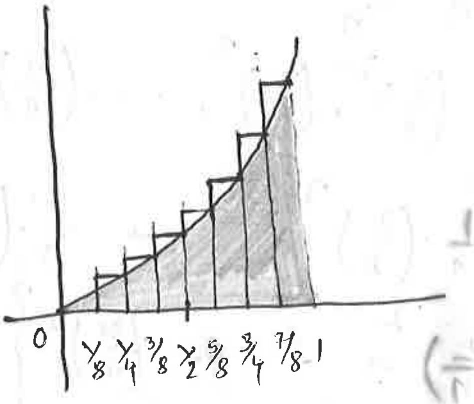
$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = 0.21875.$$

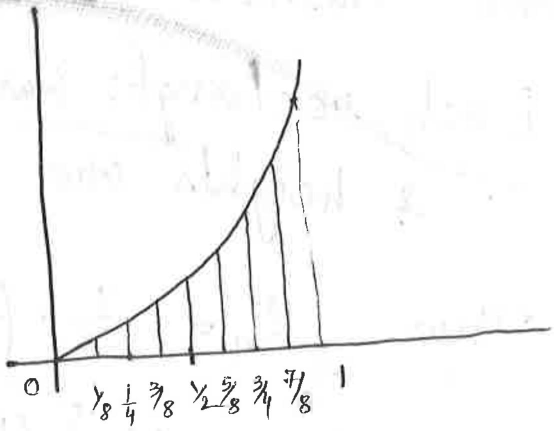
From the figure $A > 0.21875$.

Hence

$$0.21875 < A < 0.46875.$$

- We can repeat this procedure with a larger number of strips.





By a similar way,

$$0.2734375 < A < 0.3984375.$$

which is a better lower & upper estimates for A.

- In this way, we can obtain better estimates by increasing the number of strips.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

$A \approx 0.3328335$

- Yet another estimate can be obtained by using rectangles whose heights are values of f at the mid pts of the rectangles.

In this case, it is not clear whether it overestimates or underestimates the true area.

For the above example, show that

$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$

Solution: For n rectangles, i.e. we have divided the interval $[0,1]$ into n subintervals.

Each rectangle has width $\frac{1}{n}$.

& heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \cdots \left(\frac{n}{n}\right)^2$.

Thus $$R_n = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\cdot\left\{\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right\}$$

$$= \frac{1}{n}\cdot\frac{1}{n^2}\cdot\left\{1^2 + 2^2 + \cdots + n^2\right\}.$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}.$$

Thus $$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n^2}\right)\left(2+\frac{1}{n^2}\right).$$

$$= \frac{1}{6}\cdot 1\cdot 2 = \frac{1}{3}$$

Similarly, it can be shown that

$$\lim_{n\to\infty} L_n = \frac{1}{3}.$$

Riemann Sums :

- Let us begin with an arbitrary bounded fn. f on $[a,b]$. & the fn. may have +ve as well as -ve values.
- We divide the interval $[a,b]$ into subintervals, not necessarily of equal widths.

Let us take $(n-1)$ pts $\{x_1, x_2, \ldots x_{n-1}\}$ between a & b, that are s.t.

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

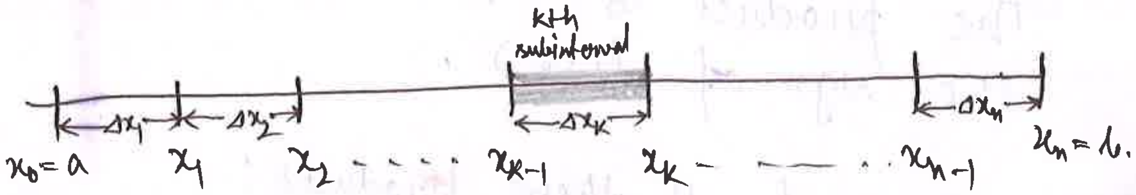
Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

- The set of all pts,

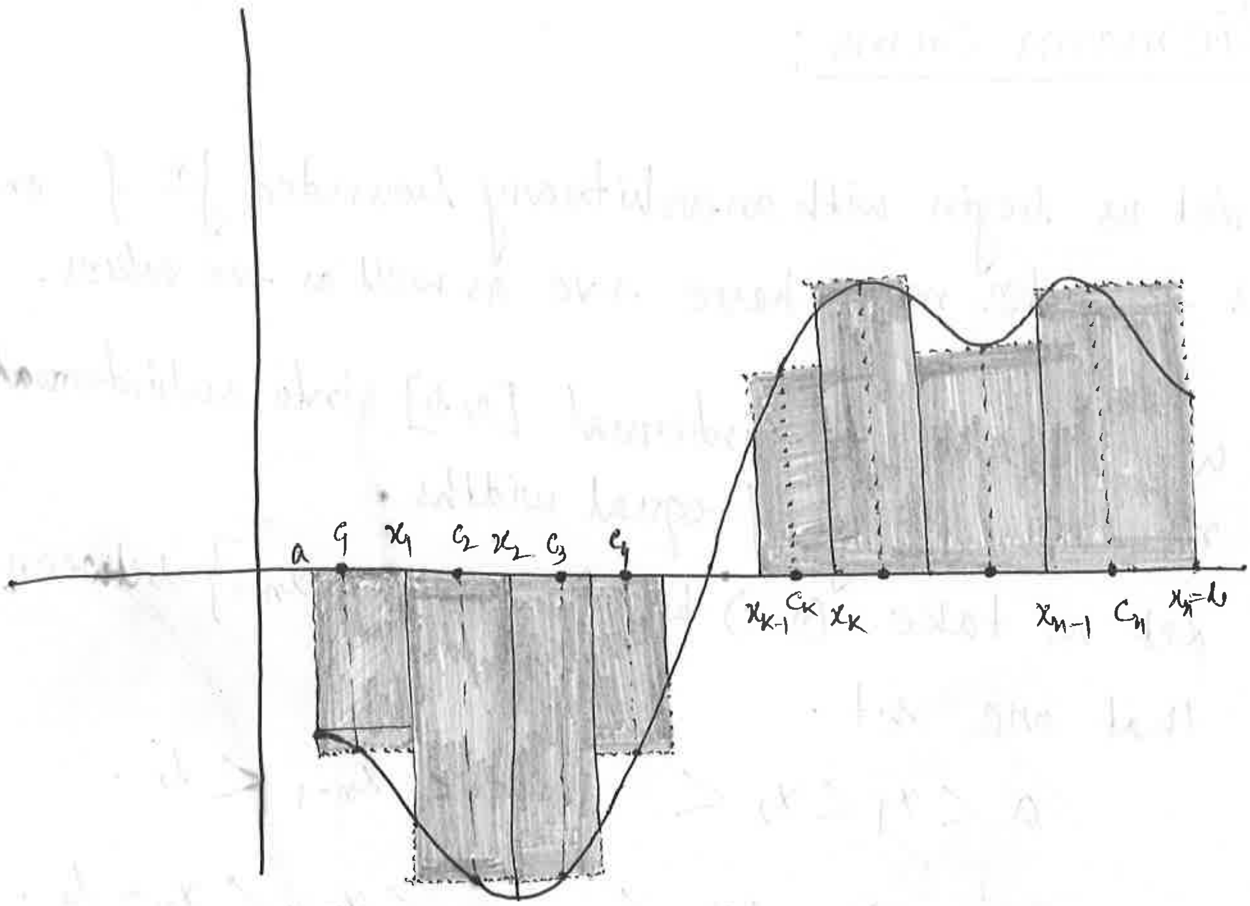
$$P = \{x_0, x_1, x_2, \ldots x_n\}$$ is called a partition of $[a,b]$.

- Subintervals are s.t.

$$[a,b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n].$$



- Let $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, $\ldots$ $\Delta x_k = x_k - x_{k-1}$, $\ldots$, $\Delta x_n = x_n - x_{n-1}$
- If all n subintervals have equal length, then $\Delta x = \frac{b-a}{n}$.
- In each subinterval, select some point. The pt. chosen in kth subinterval $[x_{k-1}, x_k]$ is called c_k.



- On each subinterval, we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is +ve or −ve or on the x-axis if $f(c_k)=0$.
- On each subinterval, form the product $f(c_k) \cdot \Delta x_k$. The product is +ve, −ve or zero, depending on the sign of $f(c_k)$.
- Sum of all these products:

$$S_p = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k .$$

- The sum S_p is called a Riemann sum for f on the interval $[a, b]$.

Note: There are many such sums, depending on the partition P we choose, and

(i) the partition P we choose, and
(ii) the choices of the points c_k in the subintervals.

- When subintervals have equal width $\Delta x = \frac{b-a}{n}$, we can make them thinner by simply increasing their number n.

In this case, if we choose c_k to be the right end pt. of each subinterval, the Rimann sum,

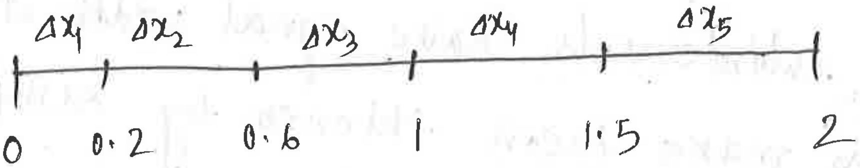
$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

— similar formulas can be obtained if instead we choose c_k to be the left-hand end pt or mid point, of each subinterval.

- When a partition has subintervals of varying widths, we can ensure, they are all thin by controlling the width of the widest (longest) subinterval.

Norm of a partition: Norm of a partition P, denoted as $\|P\|$, is defined as, to be the longest of all the subinterval widths.

<u>Example</u>: The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0,2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$.



The length of the subintervals are

$\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, $\Delta x_5 = 0.5$.

So $\|P\| = 0.5$.

The Definite Integral

Definition: Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that a number J is the definite integral of f over $[a,b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k$ if the following conditions are satisfied;

Given $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \varepsilon .$$

- In easier words;

When the limit exists, we write

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

and we say that the definite integral exists.

- The limit of any Riemann sum is always taken as the norm of the partitions $\to 0$ & number of subintervals goes to infinity & the same limit J must be obtained no matter what choices we make for the points c_k.

Leibniz Notation:

$$\int_a^b f(x)\,dx.$$

A formula for the Riemann Sum with equal-width subintervals:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left[a + k\frac{(b-a)}{n}\right] \times \left(\frac{b-a}{n}\right).$$

Theorem:

Integrability of continuous functions:

If a function f is continuous over the interval $[a,b]$, or if f has atmost finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a,b]$.

Problem: Show that The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

is not integrable over $[0,1]$.

Solution: Let $P = \{x_0, x_1, \ldots x_n\}$ be a partition of $[0,1]$.

Subintervals are,

$[x_0, x_1]$, $[x_1, x_2]$, $\ldots [x_{k-1}, x_k]$, $\ldots [x_{n-1}, x_n]$.

In each of these subintervals there are rational pts as well as irrational points.

choice-1 of c_k's: c_k's are all rational members.

then $\lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k = \lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} 1 \cdot \Delta x_k = \lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} \Delta x_k = 1.$

choice-2 of c_k's: c_k's are all irrational numbers.

then $\lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} f(c_k)\Delta x_k = 0.$

Thus for different choices for the pts c_k, we get different limits for Riemann sums. Hence, the definite integral does not exist.

Properties of Definite Integrals:

when f & g are integrable over $[a,b]$.

(i) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

(ii) $\int_a^a f(x)\,dx = 0$

(iii) $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$.

(iv) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

(v) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

(vi) If f has a maximum value M & minimum value m on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

(vii) If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq 0$.

Problem: Show that $\int_0^1 \sqrt{1+\cos x}\, dx \leq \sqrt{2}$.

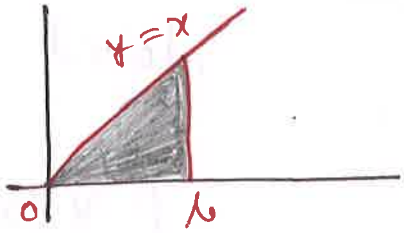
Solution: Find maximum value of

$$f(x) = \sqrt{1+\cos x} \quad \text{over } [0,1].$$

Area under the graph of a Non-negative Function

Definition: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx .$$



Example: Compute $\int_a^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Solution: We compute the area in two ways.

(a) By computing Riemann sums, $\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k$.

- Consider the partition P of $[0, b]$ of n subintervals of equal width $\Delta x = \frac{b-0}{n}$ & choose c_k to be the right endpt. of each subinterval.

$$P = \left\{ 0, \frac{b}{n}, \frac{2b}{n}, \ldots, \frac{(n-1)b}{n}, \frac{nb}{n} \right\}.$$

& $c_k = \frac{kb}{n}$, $k = 1, 2 \ldots n$.

Now $$\sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^{n} k$$

$$= \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{b}{2}\left(1 + \frac{1}{n}\right).$$

Now as $n \to \infty$ & $\|P\| \to 0$, $\int_0^b x\,dx = \frac{b^2}{2} =$

(ii) $A =$ Area of the triangle

$$= \frac{1}{2} \cdot b \cdot b = \frac{b^2}{2}.$$

Note: This example can be generalized to any closed interval $[a, b]$, $0 < a < b$.

$$\int_a^b x\,dx = \int_a^0 x\,dx + \int_0^b x\,dx$$

$$= -\int_0^a x\,dx + \int_0^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}.$$

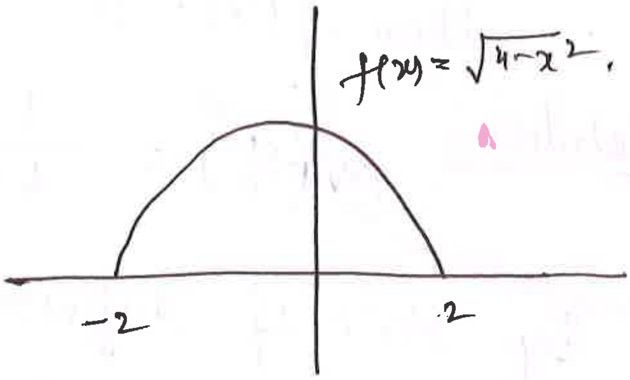
Average Value of a continuous Function Revisited:

Definition: If f is integrable on $[a,b]$, then its average value on $[a,b]$, which is also called its mean, is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problem: Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2,2]$.

Solution:



$$\text{Area} = \frac{1}{2}\pi r^2$$

$$= \frac{1}{2}\pi \times 4 = 2\pi$$

$$av(f) = \frac{1}{2-(-2)} \times \int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{4} \times 2\pi$$

$$= \pi/2.$$

Mean Value theorem for Definite Integrals

Theorem: If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Problem: Show that if f is continuous on $[a,b]$, $a \neq b$ & if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ at least once in $[a,b]$.

Solution:

$$Av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

By MVT of definite integral, there is some pt. $c \in [a,b]$. s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = 0$$

$$\Rightarrow \quad f(c) = 0.$$

Fundamental Theorem of Integral Calculus:

Part-1: If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Part-2: If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Problem: Use the Fundamental Theorem to find $\frac{dy}{dx}$ if

(i) $y = \int_a^x (t^3+1)\,dt$ (ii) $y = \int_x^5 3t\sin t\,dt$.

(iii) $y = \int_1^{x^2} \cos t\,dt$ (iv) $y = \int_{1+3x^2}^{4} \frac{1}{2+t}\,dt$.

Solution:

(i) $\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1$.

(ii) $\frac{dy}{dx} = \frac{d}{dx}\left[\int_x^5 3t\sin t\,dt\right] = \frac{d}{dx}\left[-\int_5^x 3t\sin t\,dt\right]$

$= -3x\sin x$.

(iii) $y = \int_1^{x^2} \cos t\,dt$.

Let $y = \int_1^u \cos t\,dt$ & $u = x^2$.

Now $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}\left(\int_1^u \cos t\,dt\right)\cdot\frac{du}{dx}$

$= \cos u \cdot 2x = \cos(x^2)\cdot 2x = 2x\cos(x^2)$.

(iv) $y = \int_{1+3x^2}^{4} \frac{1}{2+t}\,dt$.

$= -\int_4^{1+3x^2} \frac{1}{2+t}\,dt$

Let $y = -\int_1^u \frac{1}{2+t}\,dt$ & $u = 1+3x^2$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{1}{2+u}\cdot\frac{du}{dx}$

$= -\frac{1}{2+(1+3x^2)}\cdot 6x = \frac{-2x}{1+x^2}$

Problem: Find

(i) $\displaystyle\int_0^{\pi} \cos x \, dx$ (ii) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x \, dx$

(iii) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

Solution: (i) $\displaystyle\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi} = \sin \pi - \sin 0 = 0.$

(ii) $\displaystyle\int_{-\pi/4}^{0} \sec x \cdot \tan x \, dx = \sec x \Big|_{-\pi/4}^{0} = \sec 0 - \sec(-\pi/4)$
$= 1 - \sqrt{2}.$

(iii) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = x^{3/2} + \frac{4}{x}\Big|_1^4 = \underline{4}$

Remark: Relationship between Integration & Differentiation.

(i) $\displaystyle\frac{d}{dx}\int_a^x f(t)\, dt = f(x).$

(ii) $\displaystyle\int_a^x F'(t)\, dt = F(x) - F(a).$

Total Area: Area is always a non-negative quantity.

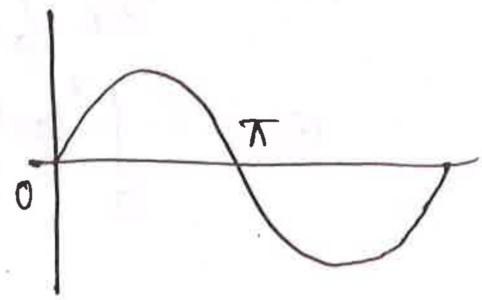
- Riemann Sum $= \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$.
- $f(c_k) \cdot \Delta x_k$ gives the area of a rectangle when $f(c_k)$ is +ve.
- $f(c_k) \cdot \Delta x_k$ gives the negative of the area of a rectangle when $f(c_k)$ is −ve.
- To obtain total area, we have to take absolute values of all the areas and add.

Problem: For the function $f(x) = \sin x$, $x \in [0, 2\pi]$, compute (i) the definite integral of $f(x)$ over $[0, 2\pi]$, (ii) the area between the graph of $f(x)$ & the x-axis over $[0, 2\pi]$.

Solution: (i) $\int_0^{2\pi} \sin x \, dx = -\cos x \Big|_0^{2\pi}$

$= -(\cos 2\pi - \cos 0)$

$= -(1-1) = 0$



(ii) Area = Absolute value of $\int_0^{\pi} \sin x \, dx$ + Absolute value of $\int_{\pi}^{2\pi} \sin x \, dx$

$= \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right| = 2 + 2 = \underline{\underline{4}}$

How to find the area between the graph of $y = f(x)$ & the x-axis over $[a, b]$:

(i) Subdivide $[a, b]$ at the zeros of f.

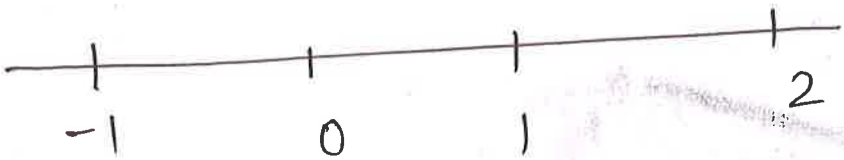
(ii) Integrate f over each subinterval.

(iii) Add the absolute values of the integrals.

Problem: Find the area of the region between the x-axis & the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution: $f(x) = 0$

$$\Rightarrow x(x^2 - x - 2) = 0 \Rightarrow x = 0, -1, 2.$$



$$\text{Area} = \left| \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx \right| + \left| \int_{0}^{1} (x^3 - x^2 - 2x)\, dx \right| + \left| \int_{1}^{2} (x^3 - x^2 - 2x)\, dx \right|.$$

Indefinite Integrals and the Substitution Method:

The substitution Rule: If $u = g(x)$ is a diff'ble function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du .$$

The method: To evaluate $\int f(g(x))\, g'(x)\, dx$.

Step-1: Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain $\int f(u)\, du$

Step-2: Integrate w.r.to u.

Step-3: Replace u by $g(x)$.

Problem: Find (i) $\int \sec^2(5x+1) \cdot 5\,dx$

(ii) $\int \sqrt{2x+5}\; dx$

(iii) $\int \cos(7\theta+3)\,d\theta$

(iv) $\int x^2 \cos x^3\, dx$

(v) $\int x\sqrt{2x+1}\; dx$

(vi) $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

(vii)

Definite Integrals & the substitution method:

Theorem: If g' is cont. on $[a,b]$ & f is cont on the range of $g(x)=u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du .$$

Problem: Evaluate (i) $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$. (Ans. $\frac{4\sqrt{2}}{3}$)

(ii) $\int_{\pi/4}^{\pi/2} \cot\theta \,\mathrm{cosec}^2\theta \, d\theta$ (Ans: $\frac{1}{2}$)

(iii) $\int_0^{\pi/2} \frac{2\sin x\cos x}{(1+\sin^2 x)}\,dx$ (Ans. $\frac{3}{8}$)

Definite Integrals of Symmetric functions:

Theorem: Let f be continuous on the symmetric interval $[-a,a]$.

(i) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

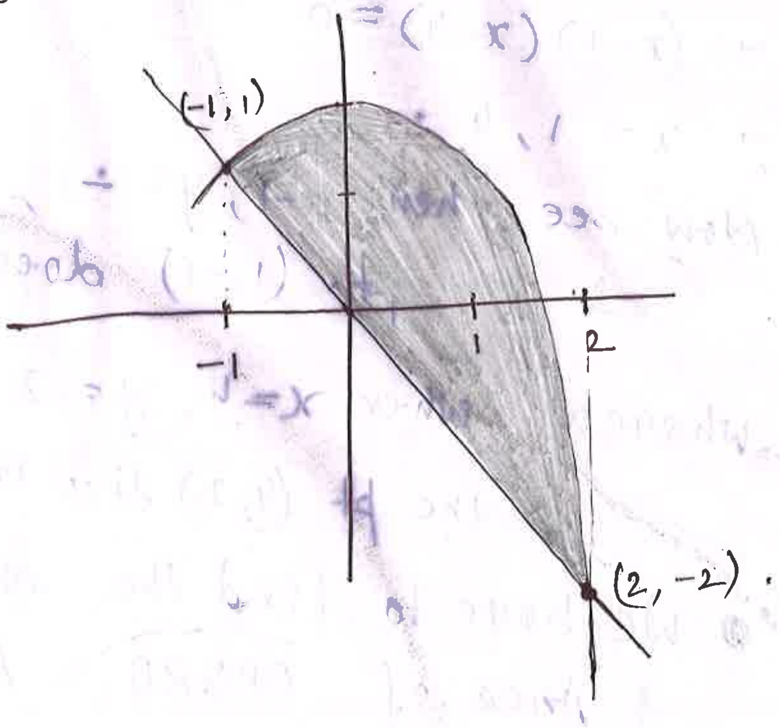
Area Between Curves:

Definition: If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ & $y = g(x)$ from a to b is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Problem: Find the area of the region enclosed by the parabola $y = 2 - x^2$ & the line $y = -x$.

Solution: • First sketch the two curves.



- The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$2 - x^2 = -x.$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1, 2$$

- Area between curves is

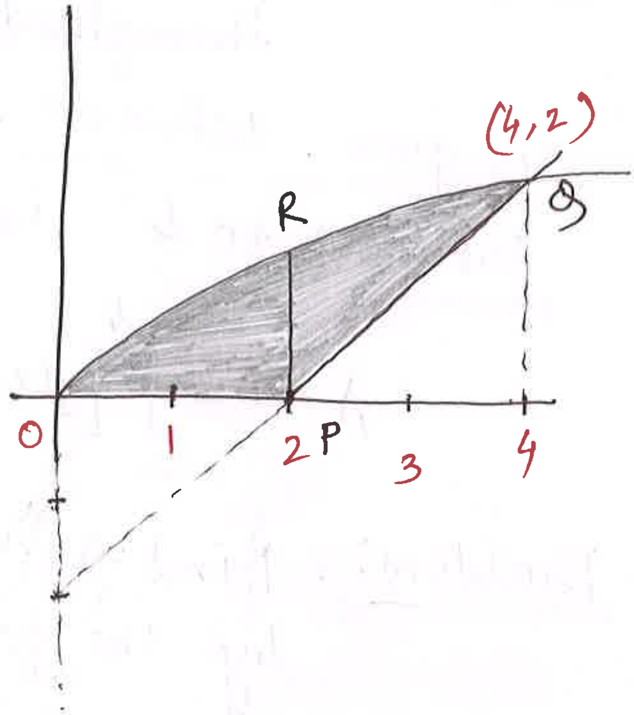
$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^{2} (2 - x^2 + x)\, dx = \cdots = 9/2.$$

Problem: Find the area of the region in the first quadrant that is bounded above by $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$.

Solution:

- $y = x-2 \Rightarrow x-y=2$
  $\Rightarrow \frac{x}{2}+\frac{y}{-2}=1$

  $y=\sqrt{x}$.



- $\sqrt{x} = x-2$.

$\Rightarrow x = (x-2)^2$.

$\Rightarrow x^2-5x+4=0$

$\Rightarrow (x-1)(x-4)=0$

$\Rightarrow x = 1, 4$

Now see when $x=1$, $y=-1$, ~~...~~
& the pt. $(1,-1)$ does not lie in 1st quadrant.

whereas when $x=4$, $y=2$.
& the pt $(4,2)$ lies in 1st quadrant.

- We have to find the area of the shaded region - $OPQRO$
& Area of $\overline{OPQRO}$ = Area of $\overline{OPRO}$ + Area of $\overline{PQRP}$

Finding the area $\overline{OPRO}$; ~~...~~
ie when $0 \le x \le 2$. In this case $f(x)=\sqrt{x}$ & $g(x)=0$

$$\text{Area} = \int_0^2 [f(x)-g(x)]\,dx = \int_0^2 \sqrt{x}\,dx$$

Finding area of $\overline{PQRP}$: (ie when $2 \le x \le 4$).

In this case $f(x) = \sqrt{x}$ & $g(x) = x - 2$.

$$\text{Area} = \int_2^4 (\sqrt{x} - x + 2)\, dx.$$

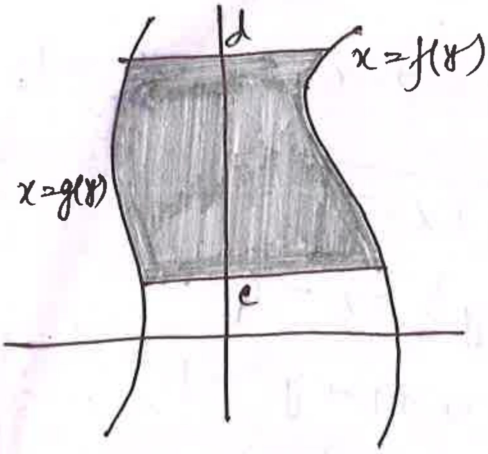
Hence total Area of the shaded region

$$= \int_0^2 \sqrt{x}\, dx + \int_2^4 (\sqrt{x} - x + 2)\, dx$$

$$= 10/3.$$

Integration w.r.to y:

If a regions bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical



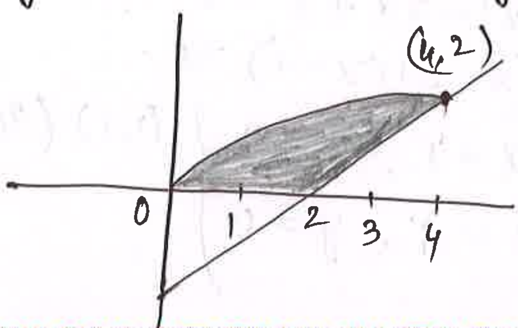
$$\text{Area} = \int_c^d [f(y) - g(y)]\, dy.$$

In this eqn. f always denotes the right hand curve & g the left-hand curve, so $f(y) - g(y)$ is non-ve.

Problem: Find the area of the region in the 1st quadrant that is bounded by $y = \sqrt{x}$, x-axis & $y = x - 2$.

Solution: $f(y) = y + 2$

$g(y) = y^2$.



$y + 2 = y^2 \Rightarrow y = -1, 2.$