Application of Derivatives

Extreme values of functions on closed Intervals

Absolute Maximum: Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \quad , \text{ for all } x \in D .$$

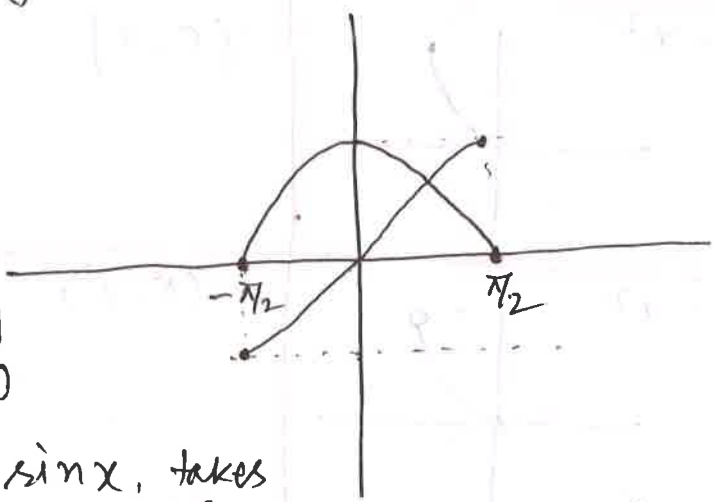
Absolute Minimum: Let f be a function with domain D. Then f has an absolute minimum value on D at a point d if

$$f(x) \geq f(d) \quad , \text{ for all } x \in D .$$

- Maximum & minimum values are called extreme values of the function f.
- Absolute maxima or minima are also referred to as global maxima or minima.

Example:

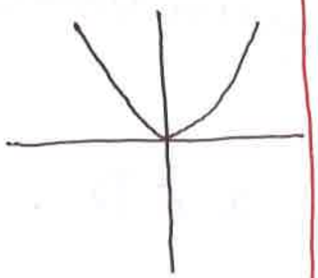
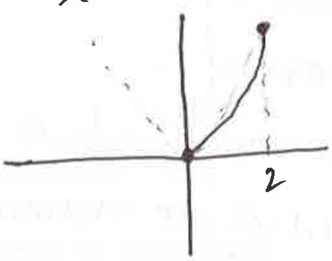
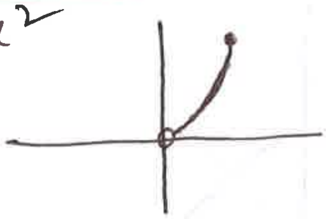
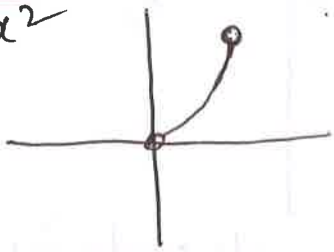
(i) $f(x) = \cos x$, takes on an absolute maximum value 1 (once) in $[-\pi/2, \pi/2]$ & absolute minimum value 0 (twice).



(ii) In $[-\pi/2, \pi/2]$, $f(x) = \sin x$, takes on absolute max value 1 (~~twice~~ once) & absolute min value -1 (once.)

Remark: (i) Functions defined by the same equation or formula can have different extrema (max or min values), depending on the domain.

(ii) A function may not have maximum or minimum if the domain is unbounded or fails to contain an end point.

Equation	Domain	Absolute extrema on D.
$y=x^2$ 	$(-\infty,\infty)$	• No absolute maximum • Absolute minimum at $x=0$ • Abs. min. value $=0$
$y=x^2$ 	$[0,2]$	• Abs. max at $x=2$. • Abs. max value $=4$. • Abs. min at $x=0$ • Abs. min value $=0$
$y=x^2$ 	$(0,2]$	• Abs. max at $x=2$. • Abs max value $=4$. • No abs. min value.
$y=x^2$ 	$(0,2)$.	• No abs max. • No abs min.

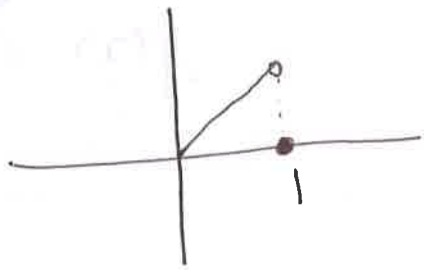
The Extreme Value Theorem

If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$.

i.e. there are numbers $x_1, x_2 \in [a,b]$ with $f(x_1) = m$ & $f(x_2) = M$ and

$$m \le f(x) \le M \quad \text{for all } x \in [a,b].$$

Note: Even a single point of discontinuity can keep a function from having either maximum or minimum value on a closed interval.

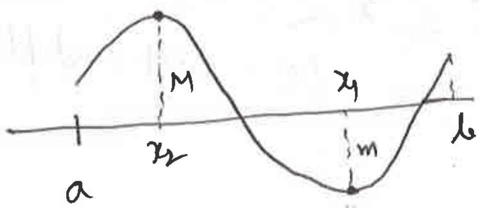


$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

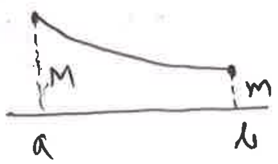
discont at $x = 1$.

- Does not have max. value.

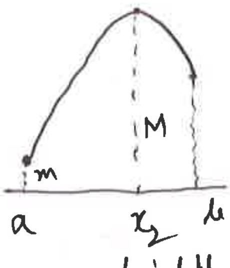
- Some possibilities of cont. fn.s max & min on a closed interval $[a,b]$



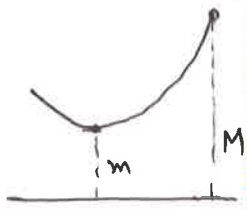
Max & Min at int. points.



Max & min at end points.

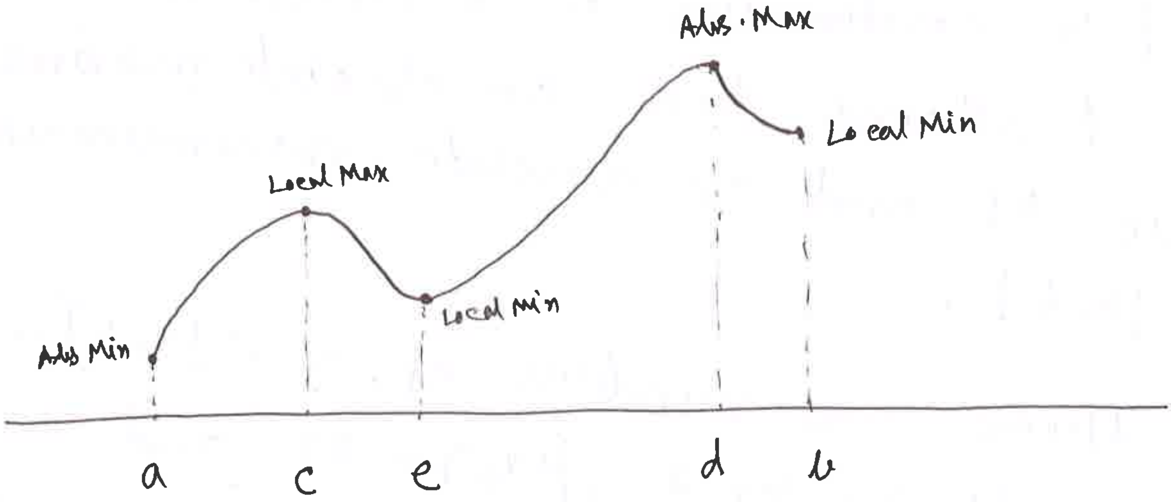


Max at int pt.
Min at end pt.



Min at int pt.
Max at end pt.

Local Maximum & Minimums (Relative Extrema)



Local Maxima: A function f has a local maximum value at a point c within its domain if $f(x) \le f(c)$, for all $x \in D$ lying in some open interval containing c.

i.e. $f(x) \le f(c)$, for $c-\delta < x < c+\delta$, $\delta > 0$.
(c = interior pt.)

for $c-\delta < x \le c$
(c = right end pt.)

for $c \le x < c+\delta$
(c = left end pt.)

Local Minima: A function f has a local minimum value at a point c within its domain if

$$f(x) \geq f(c), \quad \text{for all } x \in D \text{ lying in some open interval containing } c.$$

i.e. $f(x) \geq f(c)$, for $c-\delta < x < c+\delta$, $\delta > 0$ (c = interior pt.)

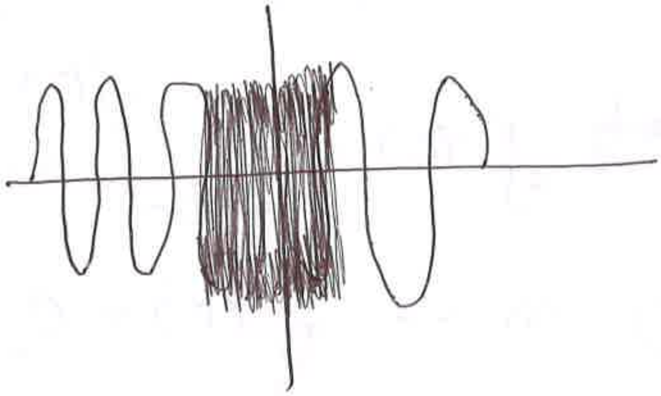
for $c-\delta < x \leq c$ (c = right end pt.)

for $c \leq x < c+\delta$ (c = left. end pt.)

Remark:

Some functions can have infinitely many local extrema, even over a finite interval.

Ex: $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in (0,1]$



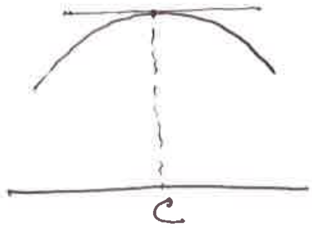
The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

<u>proof</u>: Let f has a local maximum value at $x = c$.

ie. $f(x) - f(c) \leq 0$ for all values near enough to c.



Since f' is defined at c,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$

ie $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$ & $\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$ both exists and are equal.

ie $$f'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \left[\because (x-c) > 0 \text{ \& } f(x) \leq 0 \right] \qquad \text{—— (i)}$$

similarly. $$f'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \left[\because (x-c) < 0 \text{ \& } f(x) \leq 0 \right] \qquad \text{—— (ii)}$$

(i) & (ii) $\Rightarrow$ $f'(c) = 0$

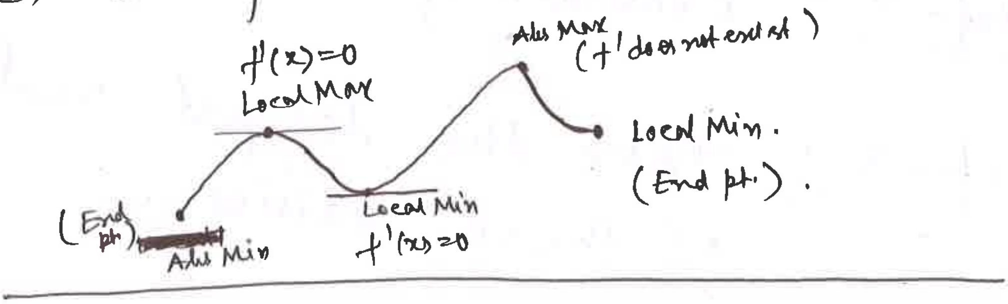
- The proof follows similarly for local minimum values.

Remark: The only points where a function f can possibly have an extreme value (local/global) Abs. are

(i) Interior points where $f' = 0$

(ii) Interior points where f' is undefined.

(iii) Endpoints of the domain of f.

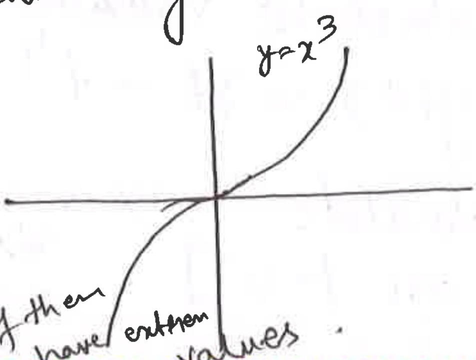


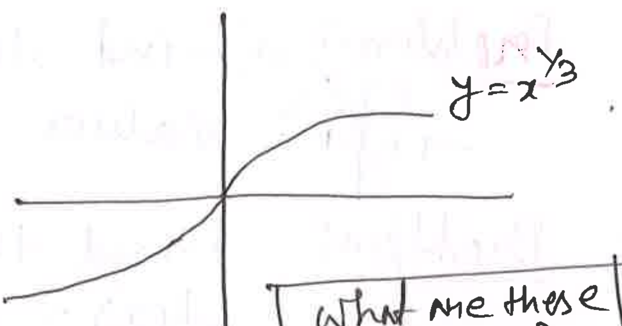
Critical Point: An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

Remark: Only domain points where a function can assume extreme values are critical points and end points.

***__Note:__ A fn. may have a critical point at $x = c$ but may not have local extreme values there.

Ex:





For both $x=0$ is a critical pt. but at $x=0$ none of them have extreme values.

What are these pts. then?

How to find Absolute Extrema of a cont. f^n. f on a finite closed Interval:

Step-1: Identify the end points & then Find all critical points of f on the interval.

Step-2: Evaluate f at all critical pts & endpts.

Step-3: Take the largest and smallest of these values.

Problem: Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$

Solution: The f^n. is diff'ble on $[-2, 1]$.

Step-1: $f'(x) = 0 \Rightarrow x = 0$ is only critical pt.

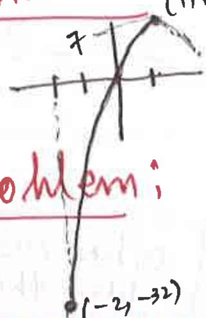
Step-2: Now

$$f(0) = 0$$
$$f(-2) = 4$$
$$f(1) = 1$$

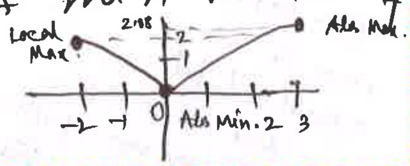
Step-3:

- Abs. Max is at $x = -2$ & Abs. Max value $= 4$
- Abs Min is at $x = 0$ & Abs min value $= 0$

Problem: Find the absolute maximum & minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.



Problem: Find the absolute max & min values of $f(x) = x^{2/3}$ on $[-2, 3]$.



Mean Value Theorem

Rolle's Theorem :

> **proof**: Since f is cont on $[a, b]$, f assumes absolute max & min on $[a, b]$.
> These occurs only when:
> (i) at interior pts where $f' = 0$
> (ii) At int. pts where f' does not exist.
> (iii) At the end pts of the int $(a \& b)$
>
> • Since f is diff'ble on (a, b), this rules out (ii).
> • If max and min occurs at $c \in (a, b)$, then $f'(c) = 0$
> • If both abs max & min occurs at end pts & since $f(a) = f(b)$ $\Rightarrow f$ is const. $\Rightarrow f'(x) = 0$.
> i.e c can be any where in (a, b).

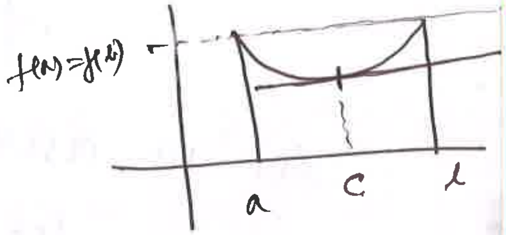
Let a function $f : [a, b] \longrightarrow \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$,

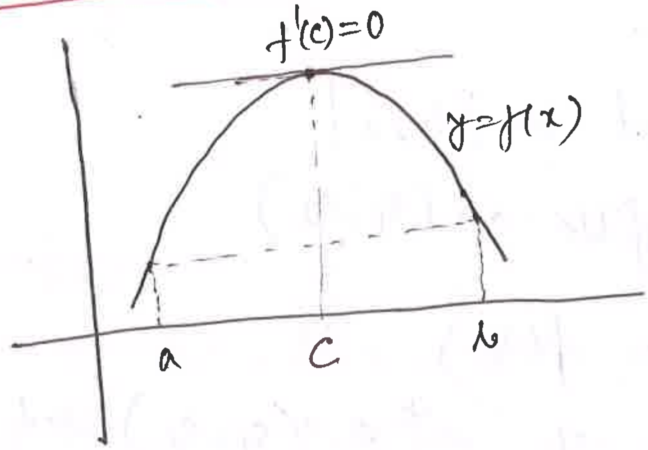
(ii) f is differentiable at every point of (a, b),

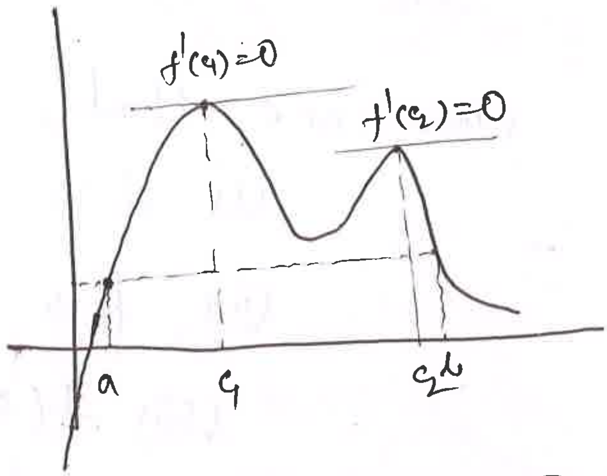
(iii) $f(a) = f(b)$.

Then there exists atleast one point $c \in (a, b)$, such that $f'(c) = 0$.



Geometrical Interpretation:





If a f^n f has a graph which is continuous on $[a, b]$ & the curve has a tangent at every point on it (a, b). & $f(a) = f(b)$. Then $\exists$ one point $c \in (a, b)$ such that the tangent at $(c, f(c))$ is parallel to the x-axis.

Problem: Show that the equation $x^3+3x+1=0$ has exactly one real solution.

Solution: Let $f(x) = x^3+3x+1$

clearly $f(x)$ is a continuous fn.

Now $f(-1) = -3$, $f(0) = 1$. & $f(-1), f(0)$ of opposite sign

By intermediate value theorem, $\exists$ atleast one $c \in (-3, 1)$ s.t

$$f(c) = 0.$$

Let us assume there are two points c_1 & $c_2 \in (-3,1)$ s.t $f(c_1) = 0 = f(c_2)$. (Assumption).

Now see that,

(i) f is cont on $[c_1, c_2]$

(ii) f is diff'ble on (c_1, c_2).

& (iii) $f(c_1) = f(c_2)$

By Rolles theorem $\exists$ $c^* \in (c_1, c_2)$ s.t.

$$f'(c^*) = 0$$

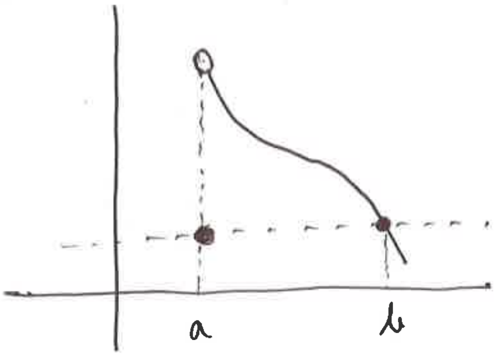
Now $f'(x) = 3x^2+3 = 3(x^2+1)$.

Now we can see that there can not be a $c^* \in (c_1, c_2)$ s.t $f'(c^*) = 0$, (not even $c^* \in \mathbb{R}$).

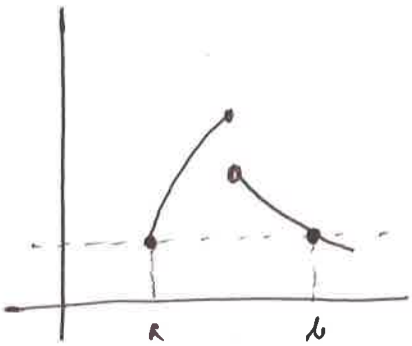
Hence, this is a contradiction to our assumption.

So, $f(x) = 0$ has exactly one real solution.

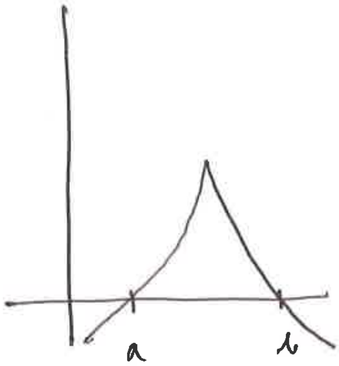
Remark: Both continuity & differentiability are essential in Rolle's Theorem. If they fail at even one point, the graph may not have a horizontal tangent.



Discont. at an endpoint of [a, b]



Discont. at an interior pt. of [a, b]



Cont. on [a, b] but not diff'ble at an interior point.

The Mean Value Theorem (Lagrange's)

Let a function $f : [a, b] \to \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$

(ii) f is differentiable at every point of (a, b).

Then there exist atleast one point $c \in (a, b)$ such that, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

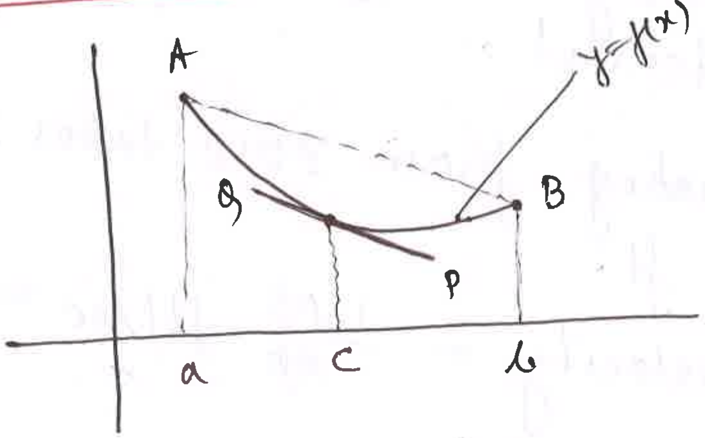
proof

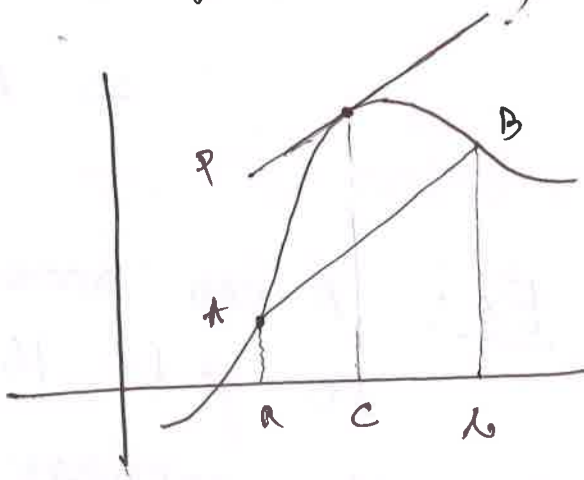
$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$h(x) = f(x) - g(x)$$

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Geometrical Interpretation





Slope of $AB = \dfrac{f(b) - f(a)}{b - a}$.

Slope of PQ at c $= f'(c)$

Slope of $AB = \dfrac{f(b) - f(a)}{b - a}$.

slope of PQ at c $= f'(c)$

If a fn. f has a graph which is continuous on $[a,b]$ & the curve has a tangent at every point in it (a,b) then $\exists$ one point $c \in (a, b)$ such that the tangent to the curve at $(c, f(c))$ is parallel to the line segment joining $(a, f(a))$ & $(b, f(b))$

Physical Interpretation :

$\boxed{\text{Mean} = \text{Average}}$

$\dfrac{f(b)-f(a)}{b-a}$ = Average change in f over $[a, b]$.

& $f'(c)$ = Instantaneous change of f at $x = c$.

MVT says: Instantaneous change at some point in an interval

= Average change over the entire interval.

Ex: A car accelerating from zero takes 80 sec. to go to 1600 ft.

Its average velocity = $\frac{1600}{80}$ ft/sec = 20 ft/sec

The MVT says,

At some point during the acceleration the speedometer must read exactly (20 ft/sec)

Consequences of MVT

Corollary(i): Let a fn. $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

proof: Let $x_1, x_2 \in [a,b]$ & $a \le x_1 < x_2 \le b$.

Then f is cont. on $[x_1, x_2]$ & is diff'ble on (x_1, x_2).

Now by MVT, $\exists$ a pt $c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Now By the given condition

$$f'(c) = 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

ie $$f(x_1) = f(x_2).$$

Since x_1 & x_2 are arbitrary in $[a,b]$, we conclude that f is a constant on $[a,b]$.

Corollary(ii): If two fn.s $f : [a,b] \to \mathbb{R}$ & $g : [a,b] \to \mathbb{R}$ be s.t. $f'(x) = g'(x)$ on (a,b), then $\exists$ a constant c s.t.
$f(x) = g(x) + c \quad \forall x \in (a,b)$

proof: Set $h(x) = f(x) - g(x)$
$h'(x) = f'(x) - g'(x) = 0$. Use above corollary.

Increasing and Decreasing functions

Let f be a function defined on an interval $[a, b]$.

(I) f is said to be (i) increasing on $[a, b]$

if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$.

(ii) strictly increasing on $[a, b]$

if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(II) f is said to be (i) decreasing on $[a, b]$

if $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$.

(ii) strictly decreasing on $[a, b]$

if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Ex:

(1) $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$

(2) $f(x) = x, x \in \mathbb{R}$

(3)

Monotonic Functions and the First Derivative Test

Corollary-(iii): Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).

(i) If $f'(x) > 0$ at each pt. $x \in (a,b)$, then f is increasing on $[a,b]$.

(ii) If $f'(x) < 0$ at each pt $x \in (a,b)$, then f is decreasing on $[a,b]$.

proof: Let $x_1, x_2 \in [a,b]$ be any two arbitrary pts.

& $x_1 < x_2$.

Now f is cont. on $[x_1, x_2]$.

f is diff'ble on (x_1, x_2).

By MVT, $\exists$ $c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1) \quad \text{———} (*)$$

proof of (i) Now since $f'(x) > 0$, $\forall x \in (a,b)$,

$$f'(c) > 0$$

& $(x_2 - x_1) > 0$.

Hence from $(*)$ $f(x_2) - f(x_1) > 0$.

$$\Rightarrow f(x_1) < f(x_2).$$

Hence $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Hence f is increasing on $[a,b]$

proof of (ii)

Now since $f'(x) < 0$, $\forall x \in (a, b)$,

$$f'(c) < 0$$

$$\& \quad x_2 - x_1 > 0$$

Hence from (*)

$$f(x_2) - f(x_1) < 0 \Rightarrow f(x_1) > f(x_2).$$

ie $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Hence f is decreasing on $[a, b]$.

Example: $f(x) = \sqrt{x}$ is increasing on $[0, b]$ for any $b > 0$.

because $f'(x) = \dfrac{1}{2\sqrt{x}}$ is +ve on $(0, b)$.

Note: • $f'(0)$ does not exist but the above corollary still applies.

• Infact $f(x) = \sqrt{x}$ is increasing on $[0, \infty)$.

How to find the intervals where a function f is strictly increasing or strictly decreasing:

Procedure:

Step-1: Find all the critical points of f.

(Let there be three critical pts a, b & c & $a < b < c$. | Arrange them in increasing order.

Step-2: Subdivide the domain of f to create nonoverlapping open intervals on which f' is either +ve or negative

$(-\infty, a), (a, b), (b, c), (c, \infty)$ (note that in these interval f' can not be zero. since a, b, c are only critical pts)

Step-3: Determine the sign of f' at a convenient point in each subinterval.

Step-4: (i) If $f' > 0$, conclude: f is increasing in the corresponding interval.

(ii) If $f' < 0$, conclude: f is decreasing in the corresponding interval.

Problem: Find the critical points of $f(x) = x^3 - 12x - 5$, and identify the open intervals on which f is increasing and on which f is decreasing.

Solution: The fn. $f(x) = x^3 - 12x - 5$ is everywhere cont. & diff'ble.

$f(x) = (x-1)^2(x+2)$

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x+2)(x-2).$$

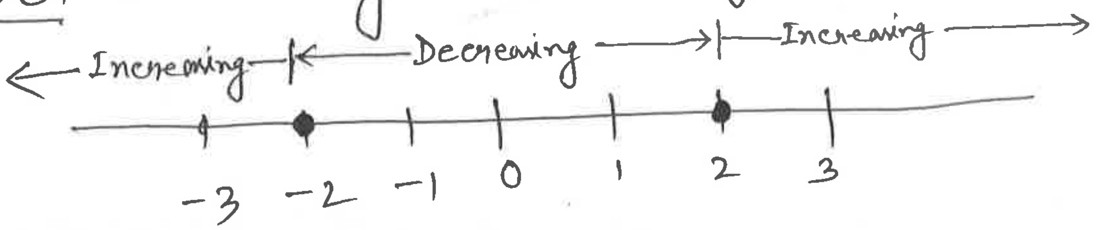
Step-1: critical points are -2 & 2.

Step-2: $(-\infty, -2)$, $(-2, 2)$ & $(2, \infty)$ are the subintervals where f' is either +ve or -ve.

Step-3: choose convenient pts. in those intervals. $-3 \in (-\infty, -2)$, $0 \in (-2, 2)$ & $3 \in (2, \infty)$.

$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$f'(-3) = 15 > 0$	$f'(0) = -12 < 0$	$f'(3) = 15 > 0$
Increasing	Decreasing	Increasing.

Conclude:



First Derivative Test for Local Extrema

Let c be a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself.

Moving accross this interval from left to right,

(i) If f' changes from $-ve$ to $+ve$ at c, then f has local minimum at c.

(ii) If f' changes from $+ve$ to $-ve$ at c, then f has local maximum at c.

(iii) If f' does not change sign at c (i.e. f' is $+ve$ on both sides of c or negative on both sides), then f has no local extremum at c.

Problem: Find the critical points of
$$f(x) = x^{1/3}(x-4).$$
Further, identify the open intervals on which f is increasing and decreasing. Also find the function's local and absolute extreme values.

Solution: $f(x) = x^{1/3}(x-4)$ is cont. for all x, since it is product of two continuous functions.

$$\text{Now} \quad f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right)$$
$$= \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$
$$= \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}.$$

$f'(x) = 0$ when $x = 1$

& $f'(x) =$ undefined when $x = 0$.

- Hence $x = 0$ & $x = 1$ are the critical points.
- There are no end pts of f.
- Hence extreme values occur only at $x = 0$ & $x = 1$.

	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Signs of f':	$-$	$-$	$+$
Behaviours of f:	Decreasing	Decreasing	Increasing.

Value of local min $= f(1) = -3$

Infact this is abs min.

First derivative of local extrema tells that: f does not have an extreme value at $x=0$ but f has local minimum at $x=1$.

Problem: Within the interval $0 \le x \le 2\pi$, find the critical points of

$$f(x) = \sin^2 x - \sin x - 1.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution: The fn. f is cont. over $[0, 2\pi]$ & diff'ble over $(0, 2\pi)$

$$f'(x) = 2\sin x \cos x - \cos x$$
$$= (2\sin x - 1)\cos x.$$

$f'(x) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\cos x = 0$.

$\Rightarrow$ $x = n\pi + (-1)^n \pi/6$ or $x = (2n+1)\pi/2$

$n \in \mathbb{Z}$ (set of all integers)

$n = 0$: $\pi/6$, or $\pi/2$.

$n = 1$: $2\pi/6$, or $3\pi/2$.

$n = 2$: $5\pi/6$, or $5\pi/2$.

$n = 3$: $13\pi/6$, or $7\pi/2$.

⋮

$n = -1$: $-7\pi/6$, $-\pi/2$.

Hence critical points in $[0, 2\pi]$ are:

$x = \pi/6, 5\pi/6$.

& $x = \pi/2, 3\pi/2$.

Interval	$(0, \pi/6)$	$(\pi/6, \pi/2)$	$(\pi/2, 5\pi/6)$	$(5\pi/6, 3\pi/2)$	$(3\pi/2, 2\pi)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behaviour of f	↓	↑	↓	↑	↓

Local Min at $x = \pi/6, 5\pi/6$ (and possibly at $x = 0$ & 2π)

Local Max at $x = \pi/2, 3\pi/2$ (and possibly at $x = 0$ & 2π)

$f(\pi/6) = -5/4$, $f(5\pi/6) = -5/4$ | $f(\pi/2) = -1$, $f(3\pi/2) = 1$ | $f(0) = -1$, $f(2\pi) = -1$

Abs Min in $[0, 2\pi]$ is $-5/4$ occurring at $x = \pi/6$ & $5\pi/6$

Abs Max in $[0, 2\pi]$ is 1 occurring at $x = 3\pi/2$

Concavity

Definition: The graph of a differentiable function $y = f(x)$ is

(i) concave up (or convex) on an open interval I if f' is increasing on I

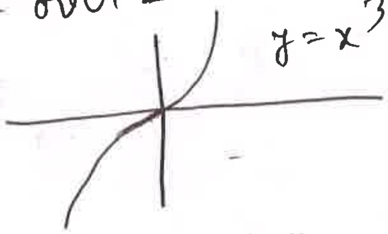
(ii) concave down on an open interval I if f' is decreasing on I.

The second Derivative Test for Concavity:

Let $y = f(x)$ be twice-diff'ble on an interval I.

(i) If $f'' > 0$ on I, the graph of f over I is concave up.

(ii) If $f'' < 0$ on I, the graph of f over I is concave down.

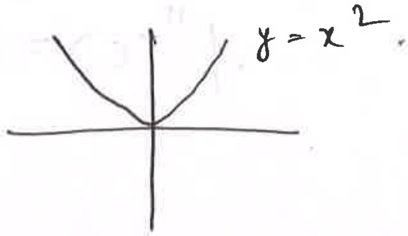


Example: ① The curve $y = x^3$ is

(i) concave down on $(-\infty, 0)$ since $y'' = 6x < 0$

& (ii) concave up on $(0, \infty)$, since $y'' = 6x > 0$

② The curve $y = x^2$ is concave up on $(-\infty, \infty)$ since $y'' = 2 > 0$ on $(-\infty, \infty)$.



Problem: Determine the concavity of
$$y = 3 + \sin x \text{ on } [0, 2\pi]$$

Solution:
$$y = 3 + \sin x.$$
$$\Rightarrow y' = \cos x.$$
$$\Rightarrow y'' = -\sin x.$$

on $(0, \pi)$, $\sin x > 0$
$$\Rightarrow y'' < 0$$
$\Rightarrow y = 3 + \sin x$ is concave down.

on $(\pi, 2\pi)$ $\sin x < 0$
$$\Rightarrow y'' > 0$$
$\Rightarrow y = 3 + \sin x$ is concave up.

Point of Inflection: A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a point of inflection.

At a point of inflection $(c, f(c))$, either
$$f''(c) = 0 \text{ or } f''(c) \text{ fails to exist.}$$

Problem: Determine the concavity and find the inflection points of the function:

$$f(x) = x^3 - 3x^2 + 2$$

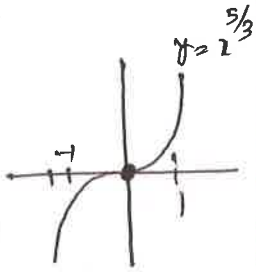
Solution:

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6 = 6(x-1).$$

Points of inflection is $x = 1$.

$-\infty < x < 1$	$x = 1$	$1 < x < \infty$
$f''(x) < 0$	$f''(1) = 0$	$f''(x) > 0$
Concave Down	Pt. of Inflection	Concave up.

Problem: Determine the concavity & find the inflection pts of the function: $f(x) = x^{5/3}$



Solution: $f'(x) = \frac{5}{3}x^{2/3}, \quad f''(x) = \frac{10}{9}x^{-1/3}.$

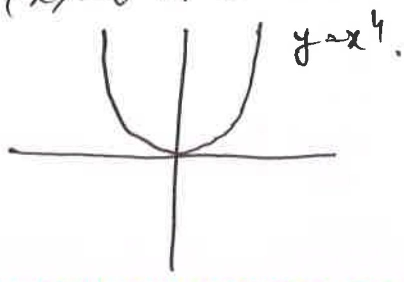
At $x = 0$ $f''(x)$ does not exist.

$-\infty < x < 0$	$x = 0$	$0 < x < \infty$
$f''(x) < 0$	$f''(0)$ does not exist.	$f''(x) > 0$
concave down	Pt. of Inflection	concave up

Problem: Determine the concavity & find the inflection pts of $f(x) = x^4$.

Solⁿ: $f'(x) = \cancel{xx}\, 4x^3, \quad f''(x) = 12x^2, \quad f''(x) = 0$ at $x = 0$.

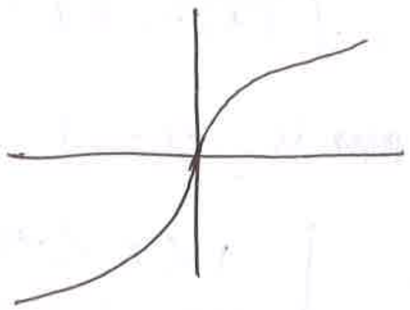
$-\infty < x < 0$	$x = 0$	$0 < x < \infty$
$f''(x) > 0$	$f''(0) = 0$	$f''(x) > 0$
concave up	not a pt of Inflection	Concave up



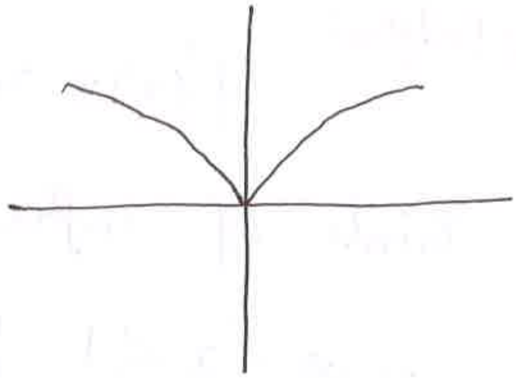
Problem: Determine the concavity and find the inflection points of

$$f(x) = x^{1/3} \quad \& \quad f(x) = x^{2/3}.$$

Solution:



$y = x^{1/3}$



Problem: A particle is moving along a horizontal coordinate line (+ve to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution: velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t-1)(3t-11).$$

& Acceleration is

$$a(t) = s''(t) = 12t - 28 = 4(3t-7).$$

	$0 < t < 1$	$1 < t < 11/3$	$\frac{11}{3} < t < \infty$
Sign of v/s'	+	—	+
Behaviour of s	Increasing	Decreasing	Increasing
Particle Motion	Right	Left	Right.

At $t = 1$ & $11/3$, the particle is at rest.

	$0 < t < 7/3$	$\frac{7}{3} < t < \infty$
Sign of $a = s''$	—	+
Graph of s	Concave down	Concave up.

(i) $f'(x) = (\sin x - 1)(2\cos x + 1)$, $0 \le x \le 2\pi$

(ii) $f'(x) = (\sin x + \cos x)(\sin x - \cos x)$, $0 \le x \le 2\pi$.

Second Derivative Test for local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

(i) If $f'(c) = 0$ & $f''(c) < 0$, then f has a local max at $x = c$.

(ii) If $f'(c) = 0$ & $f''(c) > 0$, the f has a local min at $x = c$.

(iii) If $f'(c) = 0$ & $f''(c) = 0$, then the test fails & the fn. f may have a local max, a local min or neither.

Advantage of second derivative test over 1st derivative test:

Second derivative test requires us to know f'' only at c not ~~on~~ in an interval about c. This makes the test easy to apply.

Disadvantage of second derivative test:

- The test fails when $f''(c) = 0$.
- The test fails when $f''(c)$ does not exist (when this happens use 1st derivative test)

Problem: Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, pts of inflection, and local maxima & minima.

Solution: $f(x) = x^4 - 4x^3$, $f'(x) = 4x^2(x-3)$, $f''(x) = 12x(x-2)$

~~For pts of inflections~~ critical pts: $f'(x) = 0 \Rightarrow x = 0$ & $x = 3$.

Now use 2nd derivative test; $f''(0) = 0$, $f''(3) = 36 > 0$.
$\hookrightarrow f$ has local min at $x = 3$.

Curve Sketching

Together f' and f'' tells us the shape of the function's graph — i.e. (i) where critical pts are located.

(ii) what happens at a critical point.

(iii). where the fn. is increasing.

(iv) where the fn. is decreasing.

(v) How the curve is turning or bending. (by concavity).

These information can be used to sketch a graph of the function that captures its key features.

Problem: Sketch the graph of $f(x) = x^4 - 4x^3 + 10$ using the following steps:

ⓐ Identify where the extrema of f occur.

ⓑ Find the intervals where f is increasing & decreasing.

ⓒ Find where the graph of f is concave up and where it is concave down.

ⓓ Sketch the general shape ~~for~~ of f

ⓔ Plot some specific points, such as local ~~local~~ maximum and minimum points, points of inflection and intercepts. Then sketch the curve.

Solution: $f(x) = x^4 - 4x^3 + 10$.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$.

$f'(x) = 0 \Rightarrow x = 0, 3$.

Domain of f is $(-\infty, \infty)$ & f is cont.