

Extreme values of functions on closed intervals

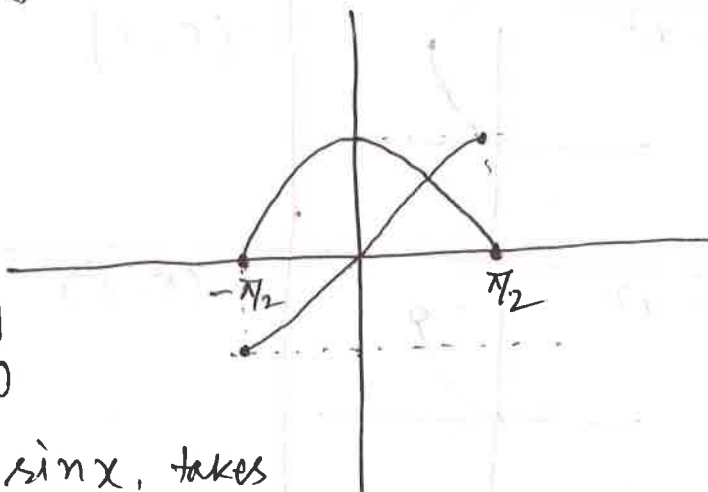
Absolute Maximum: Let f be a function with domain D .
Then f has an absolute maximum value on D
at a point c if
$$f(x) \leq f(c) \text{ , for all } x \in D .$$

Absolute Minimum: Let f be a function with domain D .
Then f has an absolute minimum value on D
at a point d if
$$f(x) \geq f(d) \text{ , for all } x \in D .$$

- Maximum & minimum values are called extreme values of the function f .
- Absolute maxima or minima are also referred to as global maxima or minima.

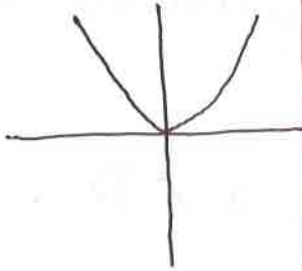
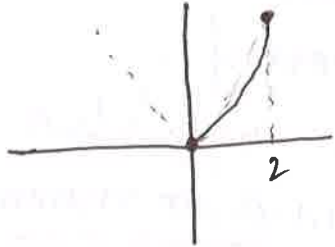
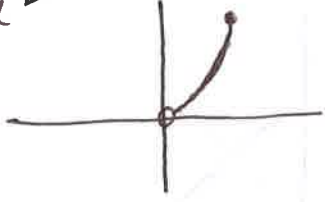
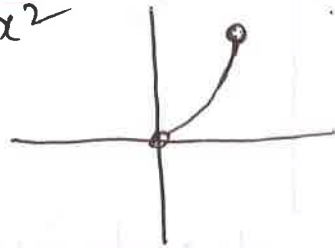
Example:

- (i) $f(x) = \cos x$, takes on an absolute maximum value 1 (once) in $[-\pi/2, \pi/2]$ & absolute minimum value 0 (+twice).
- (ii) In $[-\pi/2, \pi/2]$, $f(x) = \sin x$, takes on absolute max value 1 (once) & absolute min value -1 (once).



Remark: (i) Functions defined by the same equation or formula can have different extrema (max or min values), depending on the domain.

(ii) A function may not have maximum or minimum if the domain is unbounded or fails to contain an end point.

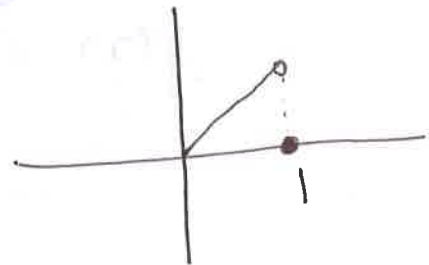
Equation	Domain	Absolute extrema on D.
$y = x^2$ 	$(-\infty, \infty)$	<ul style="list-style-type: none"> No absolute maximum Absolute minimum at $x = 0$ Abs. min. value = 0
$y = x^2$ 	$[0, 2]$	<ul style="list-style-type: none"> Abs. max at $x = 2$ Abs. max value = 4 Abs. min at $x = 0$ Abs. min value = 0
$y = x^2$ 	$[0, 2)$	<ul style="list-style-type: none"> Abs. max at $x = 2$ Abs max value = 4 No abs. min value.
$y = x^2$ 	$(0, 2)$	<ul style="list-style-type: none"> No abs max. No abs min.

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

ie. there are numbers $x_1, x_2 \in [a, b]$ with $f(x_1) = m$ & $f(x_2) = M$ and $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Note: Even a single point of discontinuity can keep a function from having either maximum or minimum value on a closed interval.

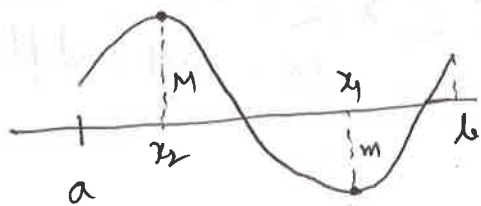


$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

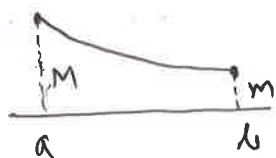
discont at $x = 1$.

- Does not have max. value.

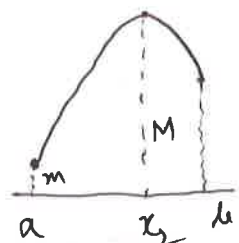
- Some possibilities of cont. fns
max & min on a closed interval $[a, b]$



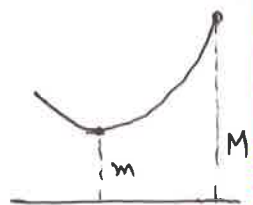
Max & Min at int. points.



Max & min at end points.



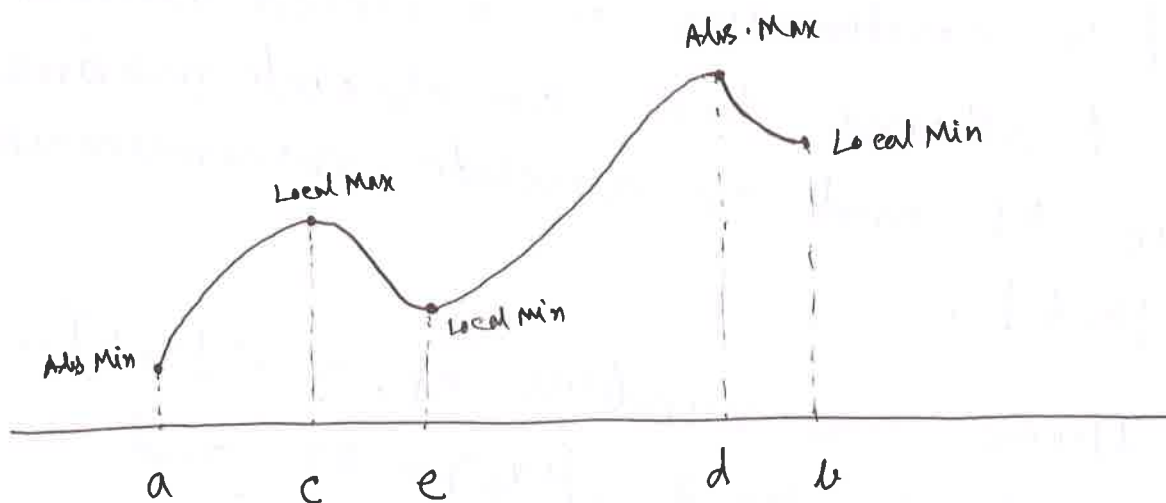
Max at int pt.
Min at end pt.



Min at int pt.
Max at end pt.

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Local Maximum & Minimum (Relative Extrema)



Local Maxima: A function f has a local maximum value at a point c within its domain if

$$f(x) \leq f(c) \quad , \quad \text{for all } x \in D \text{ lying in some open interval containing } c.$$

ie $f(x) \leq f(c) \quad , \quad \text{for } c - \delta < x < c + \delta, \delta > 0.$
 (c = interior pt.).

for $c - \delta < x \leq c$
 (c = right end pt.).

for $c \leq x < c + \delta$
 (c = left end pt.).

Local Minima: A function f has a local minimum value at a point c within its domain if

$f(x) \geq f(c)$, for all $x \in D$ lying in some open interval containing c .

i.e. $f(x) \geq f(c)$, for $c - \delta < x < c + \delta$, $\delta > 0$
(c = interior pt.)

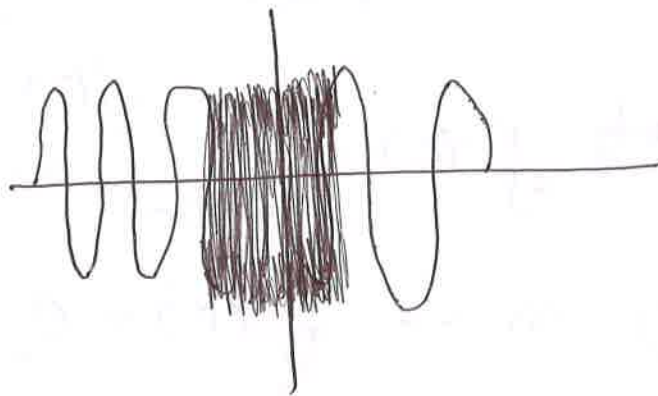
for $c - \delta < x \leq c$.
(c = right end pt.)

for $c \leq x < c + \delta$
(c = left end pt.)

Remark:

Some functions can have infinitely many local extrema, even over a finite interval.

Ex: $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in (0, 1]$



The First Derivative Theorem for local Extreme Values

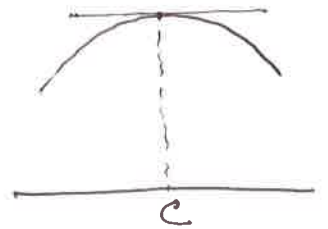
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c , then

$$f'(c) = 0.$$

proof: Let f has a local maximum value at $x=c$.
ie. $f(x) - f(c) \leq 0$ for all values near enough to c .

Since f' is defined at c ,

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$



ie $\lim_{x \rightarrow c+} \frac{f(x) - f(c)}{x - c}$ & $\lim_{x \rightarrow c-} \frac{f(x) - f(c)}{x - c}$ both exist and are equal.

$$\text{ie } f'(c) = \lim_{x \rightarrow c+} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \left[\begin{array}{l} \because (x-c) > 0 \\ \& f(x) \leq f(c) \end{array} \right] \quad \text{(i)}$$

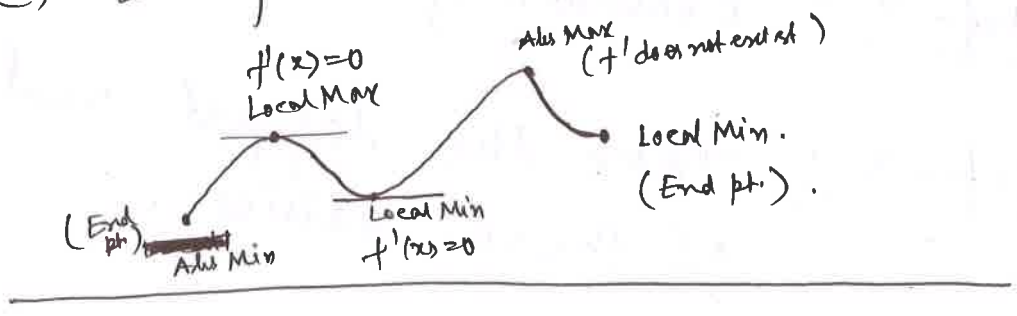
$$\text{similarly, } f'(c) = \lim_{x \rightarrow c-} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \left[\begin{array}{l} \because (x-c) < 0 \\ \& f(x) \leq f(c) \end{array} \right] \quad \text{(ii)}$$

$$(i) \& (ii) \Rightarrow f'(c) = 0$$

● The proof follows similarly for local minimum values.

Remark: The only points where a function f can possibly have an extreme value (local/global) are

- (i) Interior points where $f' = 0$
- (ii) Interior points where f' is undefined.
- (iii) Endpoints of the domain of f .



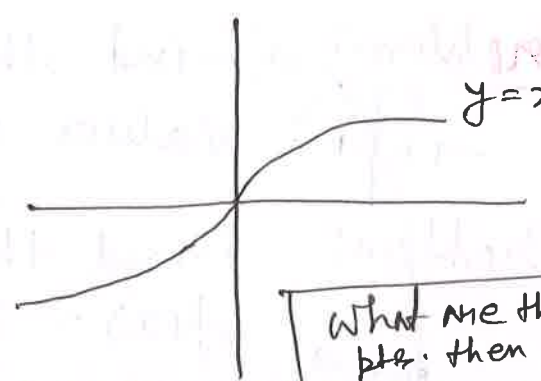
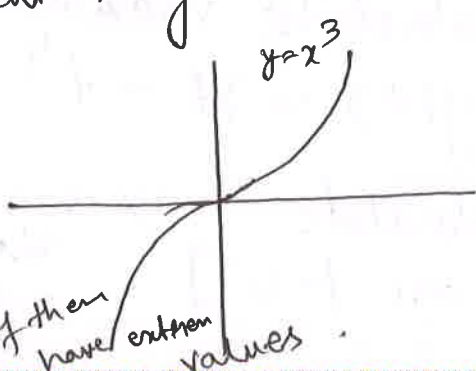
Critical Point: An interior point of the domain of a function f where f' is zero or undefined is a critical point of f .

Remark: Only domain points where a function can assume extreme values are critical points and end points.

*** Note: A fn. may have a critical point at $x=c$ but may not have local extreme values there.

Ex:

For both $x=0$ is a critical pt. but at $x=0$ none of them have extreme values.



What are these pts. then?

How to find Absolute Extrema of a cont. fn. f on a finite closed Interval:

Identify the endpoints & then
 \hookrightarrow step-1: Find all critical points of f on the interval.

step-2: Evaluate f at all critical pts & endpoints.

step-3: Take the largest and smallest of these values.

Problem: Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$

Solution: The fn. is diff ble on $[-2, 1]$.
step-1: $f'(x) = 0 \Rightarrow x = 0$ is only critical pt.

step-2 Now

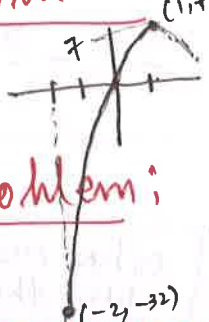
$$\left. \begin{aligned} f(0) &= 0 \\ f(-2) &= 4 \\ f(1) &= 1 \end{aligned} \right\}$$

step-3:

• Abs. Max is at $x = -2$
 & Abs. Max value = 4

• Abs. Min is at $x = 0$
 & Abs. min value = 0

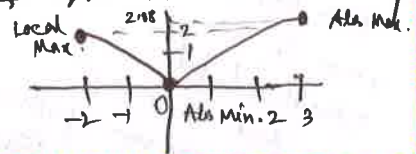
Problem:



Find the absolute maximum & minimum values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Problem:

Find the absolute max & min values of $f(x) = x^{2/3}$ on $[-2, 3]$.



Mean Value Theorem

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Rolle's Theorem:

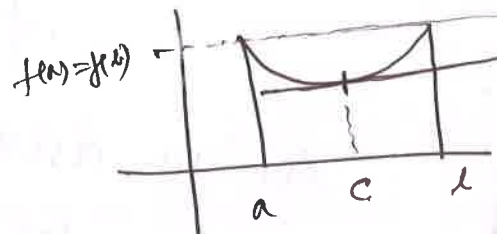
proof: Since f is cont on $[a, b]$,
absolute max & min on $[a, b]$.
These occur only when:
(i) at interior pts where $f' = 0$
(ii) At int. pts where f' does not exist.
(iii) At the end pts of the int. (a, b)

f assumes
• since f is diffble on (a, b) ,
this rules out (ii).
• If max and min occurs
at $c \in (a, b)$, then $f'(c) = 0$.
• If both abs max & min occurs
at end pts & since $f(a) = f(b)$,
at end pts $\Rightarrow f$ is const.
 $\Rightarrow f'(x) = 0$
i.e. c can be any
where in (a, b) .

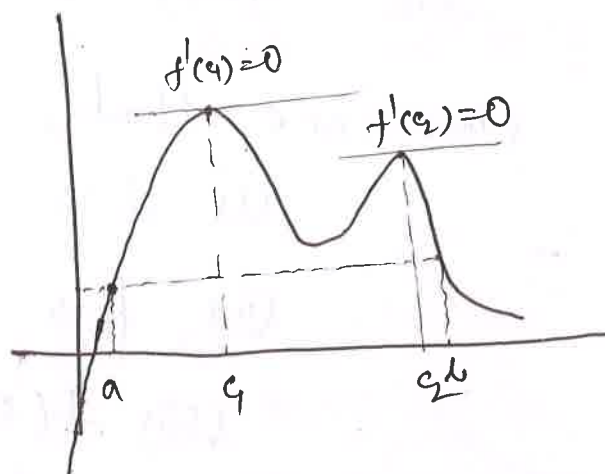
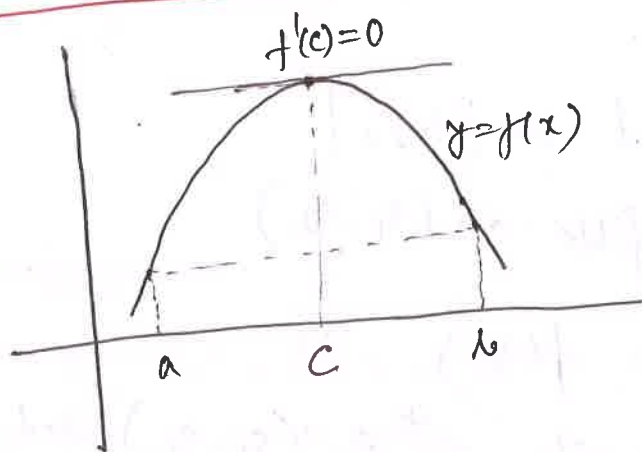
Let a function $f: [a, b] \rightarrow \mathbb{R}$ be such that

- (i) f is continuous on $[a, b]$,
- (ii) f is differentiable at every point of (a, b) ,
- (iii) $f(a) = f(b)$.

Then there exists atleast one point $c \in (a, b)$,
such that $f'(c) = 0$.



Geometrical Interpretation:



If a fcn f has a graph which is continuous on $[a, b]$
& the curve has a tangent at every point on it (a, b) .
& $f(a) = f(b)$. Then \exists one point $c \in (a, b)$.
such that the tangent at $(c, f(c))$ is
parallel to the x-axis.

Problem: Show that the equation

$$x^3 + 3x + 1 = 0 \text{ has exactly one real solution.}$$

Solution: Let $f(x) = x^3 + 3x + 1$

clearly $f(x)$ is a continuous fⁿ.

Now $f(-1) = -3$, $f(0) = 1$. & $f(-1), f(0)$ of opposite sign

By intermediate value theorem, \exists at least one
 $c \in (-3, 1)$ s.t.

$$f(c) = 0.$$

Let us assume there are two points $c_1 + c_2 \in (-3, 1)$

s.t $f(c_1) = 0 = f(c_2)$. (Assumption).

Now see that,

(i) f is cont on $[c_1, c_2]$

(ii) f is diff'ble on (c_1, c_2) .

$$\& \text{(iii)} \quad f(c_1) = f(c_2)$$

By Rolle's theorem $\exists c^* \in (c_1, c_2)$ s.t.

$$f'(c^*) = 0$$

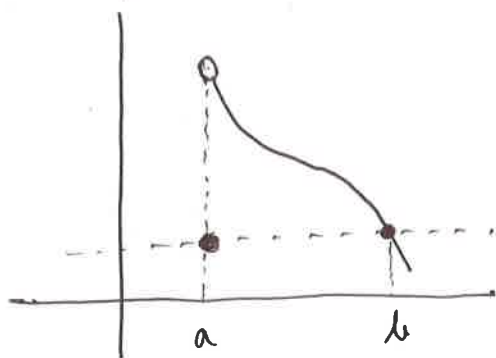
$$\text{Now } f'(x) = 3x^2 + 3 = 3(x^2 + 1)$$

Now we can see that there can not be a $c^* \in (c_1, c_2)$
s.t $f'(c^*) = 0$, (not even $c^* \in \mathbb{R}$).

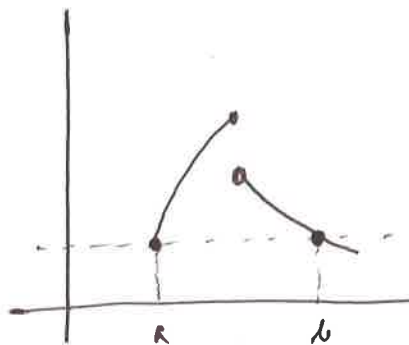
Hence, this is a contradiction to our assumption.
So, $f(x) = 0$ has exactly one real solution.

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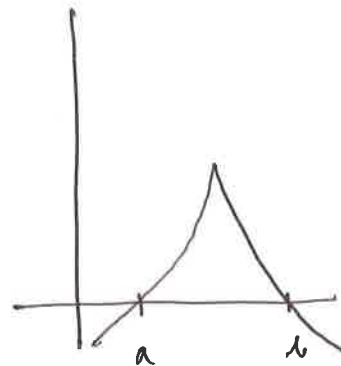
Remark: Both continuity & differentiability are essential in Rolle's Theorem. If they fail at even one point, the graph may not have a horizontal tangent.



Discont. at an endpoint of $[a, b]$



Discont. at an interior pt. of $[a, b]$



Cont. on $[a, b]$ but not diffble at an interior point.

The first part of the paper is devoted to a discussion of the
 various methods of determining the rate of reaction. It is shown
 that the most reliable method is the one which involves the
 measurement of the change in concentration of one of the
 reactants or products. This method is applicable to all reactions
 in which the concentration of one of the reactants or products
 can be measured.



The rate of reaction is defined as the change in concentration of one of the reactants or products per unit time. It is a measure of the speed at which the reaction proceeds.

The rate of reaction is affected by many factors, including the concentration of the reactants, the temperature, and the presence of a catalyst.

The rate of reaction is a function of the concentration of the reactants. It is found that the rate of reaction increases as the concentration of the reactants increases.

The Mean Value Theorem (Lagrange's)

Let a function $f: [a, b] \rightarrow \mathbb{R}$ be such that

(i) f is continuous on $[a, b]$

(ii) f is differentiable at every point of (a, b) .

then there exist at least one point $c \in (a, b)$ such

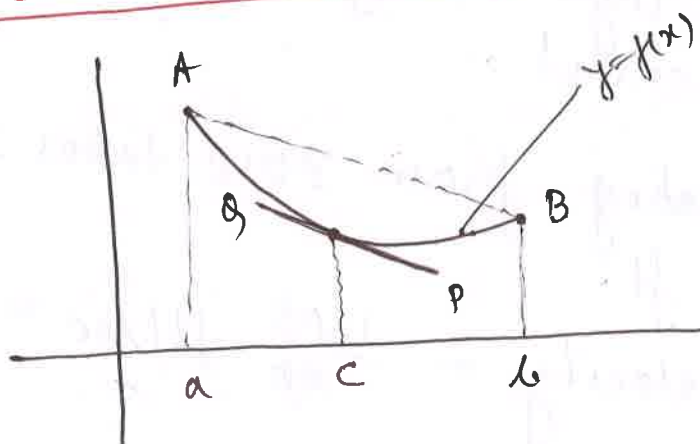
that, $f'(c) = \frac{f(b) - f(a)}{b - a}$

proof
 $g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$

$$h(x) = f(x) - g(x)$$

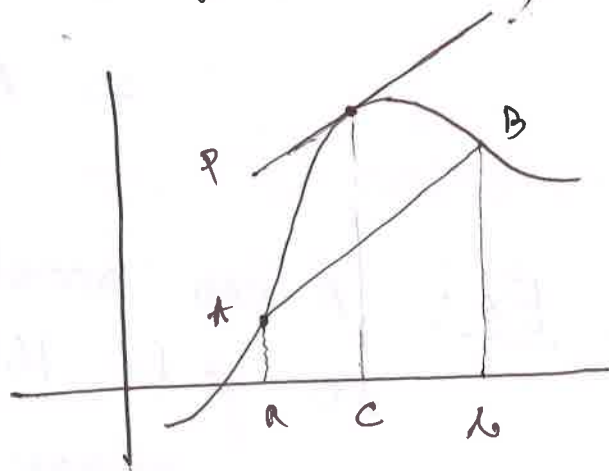
$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Geometrical Interpretation



slope of $AB = \frac{f(b) - f(a)}{b - a}$

slope of $PQ = f'(c)$
at C



slope of $AB = \frac{f(b) - f(a)}{b - a}$

slope of PQ at C
 $= f'(c)$

If a fcn. f has a graph which is continuous on $[a, b]$ & the curve has a tangent at every point in (a, b) then \exists one point $c \in (a, b)$ such that the tangent to the curve at $(c, f(c))$ is parallel to the line segment joining $(a, f(a))$ & $(b, f(b))$

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Physical Interpretation:

Mean = Average

$$\frac{f(b) - f(a)}{b - a} = \text{Average change in } f \text{ over } [a, b].$$

$$\& \quad f'(c) = \text{Instantaneous change of } f \text{ at } x=c.$$

MVT says: Instantaneous change at some point in an interval
= Average change over the entire interval.

Ex: A car accelerating from zero takes 80 sec.
to go to 1600 ft.
Its average velocity = $\frac{1600}{80} \text{ ft/sec} = 20 \text{ ft/sec}$.

The MVT says:
At some point during the acceleration
the speedometer must read exactly (20 ft/sec) .

Consequences of MVT

Corollary(i): Let a fcn. $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) .
If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

proof: Let $x_1, x_2 \in [a, b]$ & $a \leq x_1 < x_2 \leq b$.

Then f is cont. on $[x_1, x_2]$ & is diff'ble on (x_1, x_2) .

Now by MVT, \exists a pt $c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Now By the given condition $f'(c) = 0$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

$$\text{i.e. } f(x_1) = f(x_2).$$

Since x_1 & x_2 are arbitrary in $[a, b]$, we conclude that f is a constant on $[a, b]$.

Corollary(ii): If two fcn.s $f: [a, b] \rightarrow \mathbb{R}$ & $g: [a, b] \rightarrow \mathbb{R}$ be s.t. $f'(x) = g'(x)$ on (a, b) , then \exists a constant C s.t.
 $f(x) = g(x) + C$ $\forall x \in (a, b)$

proof: Let $h(x) = f(x) - g(x)$
 $h'(x) = f'(x) - g'(x) = 0$.
Use above Corollary.



THEOREM 1.1

Let f be a function defined on a set S . Then f is continuous at $a \in S$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in S$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

Proof: Suppose f is continuous at a . Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that for all $x \in S$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. This is exactly the definition of continuity at a .

Conversely, suppose that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in S$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$. Then f is continuous at a .

$$|f(x) - f(a)| < \epsilon$$

$$|f(x) - f(a)| < \epsilon$$

Let f be a function defined on a set S . Then f is continuous at $a \in S$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in S$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.

1897 1898 1899 1900 1901 1902 1903 1904 1905 1906 1907 1908 1909 1910 1911 1912 1913 1914 1915 1916 1917 1918 1919 1920 1921 1922 1923 1924 1925 1926 1927 1928 1929 1930 1931 1932 1933 1934 1935 1936 1937 1938 1939 1940 1941 1942 1943 1944 1945 1946 1947 1948 1949 1950 1951 1952 1953 1954 1955 1956 1957 1958 1959 1960 1961 1962 1963 1964 1965 1966 1967 1968 1969 1970 1971 1972 1973 1974 1975 1976 1977 1978 1979 1980 1981 1982 1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023 2024 2025 2026 2027 2028 2029 2030 2031 2032 2033 2034 2035 2036 2037 2038 2039 2040 2041 2042 2043 2044 2045 2046 2047 2048 2049 2050 2051 2052 2053 2054 2055 2056 2057 2058 2059 2060 2061 2062 2063 2064 2065 2066 2067 2068 2069 2070 2071 2072 2073 2074 2075 2076 2077 2078 2079 2080 2081 2082 2083 2084 2085 2086 2087 2088 2089 2090 2091 2092 2093 2094 2095 2096 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Increasing and Decreasing functions

Let f be a function defined on an interval $[a, b]$.

① f is said to be (i) increasing on $[a, b]$

if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$.

(ii) strictly increasing on $[a, b]$

if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

② f is said to be (i) decreasing on $[a, b]$

if $f(x_1) \geq f(x_2)$ whenever $x_1 \leq x_2$.

(ii) strictly decreasing on $[a, b]$

if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Ex:

$$\textcircled{1} f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

② $f(x) = x, x \in \mathbb{R}$

③

Monotonic Functions and the First Derivative Test

Corollary (iii): Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

(i) If $f'(x) > 0$ at each pt. $x \in (a, b)$, then f is increasing on $[a, b]$.

(ii) If $f'(x) < 0$ at each pt. $x \in (a, b)$, then f is decreasing on $[a, b]$.

proof: Let $x_1, x_2 \in [a, b]$ be any two arbitrary pts.
 $\& \quad x_1 < x_2$.

Now f is cont. on $[x_1, x_2]$.
 f is diff'ble on (x_1, x_2) .

By MVT, $\exists c \in (x_1, x_2)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1) \quad (*)$$

proof of (i) Now since $f'(x) > 0, \forall x \in (a, b)$,

$$f'(c) > 0$$

$$\& (x_2 - x_1) > 0$$

Hence from (*) $f(x_2) - f(x_1) > 0$.

$$\Rightarrow f(x_1) < f(x_2)$$

Hence $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Hence f is increasing on $[a, b]$.

proof of (ii)

Now since $f'(x) < 0$, $\forall x \in (a, b)$,

$$f'(c) < 0$$

$$\& x_2 - x_1 > 0$$

Hence from (*)

$$f(x_2) - f(x_1) < 0 \Rightarrow f(x_1) > f(x_2).$$

ie $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Hence f is decreasing on $[a, b]$.

Example: $f(x) = \sqrt{x}$ is increasing on $[0, b]$ for any $b > 0$.

because $f'(x) = \frac{1}{2\sqrt{x}}$ is +ve on $(0, b)$.

Note: • $f'(0)$ does not exist but the above corollary still applies.

• In fact $f(x) = \sqrt{x}$ is increasing on $[0, \infty)$.

How to find the intervals where a function f is strictly increasing or strictly decreasing:

Procedure:

Step-1: Find all the critical points of f .

(Let there be three critical pts a, b, c .
 $a < b < c$. | Arrange them in increasing order.

Step-2: Subdivide the domain of f to create nonoverlapping open intervals on which f' is either +ve or negative

$(-\infty, a), (a, b), (b, c), (c, \infty)$ (note that in these interval f' can not be zero. Since a, b, c are only critical pts)

Step-3: Determine the sign of f' at a convenient point in each subinterval.

Step-4: (i) If $f' > 0$, conclude: f is increasing in the corresponding interval.

(ii) If $f' < 0$, conclude: f is decreasing in the corresponding interval.

Problem: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution: The fn. $f(x) = x^3 - 12x - 5$ is everywhere cont. & diff'ble.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

$$f(x) = (x-1)^2(x+2)$$

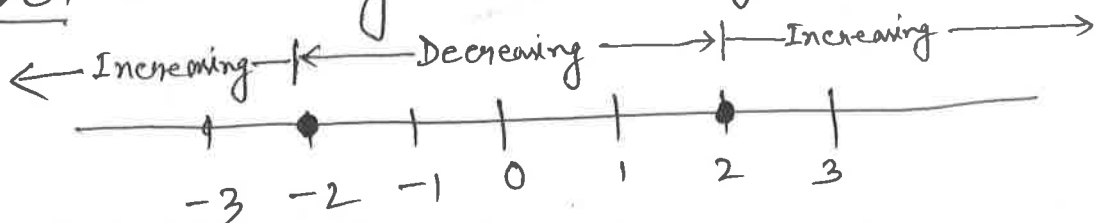
step-1: critical points are -2 & 2 .

step-2: $(-\infty, -2)$, $(-2, 2)$ & $(2, \infty)$ are the subintervals where f' is either +ve or -ve.

step-3: choose convenient pts. in those intervals.
 $-3 \in (-\infty, -2)$, $0 \in (-2, 2)$ & $3 \in (2, \infty)$.

$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
> 0	< 0	> 0

Conclude: Increasing Decreasing Increasing.



First Derivative Test for Local Extrema

Let c be a critical point of a continuous function f , and that f is differentiable at every point in some interval containing c except possibly at c itself.

Moving across this interval from left to right,

- (i) If f' changes from -ve to +ve at c , then f has local minimum at c .
- (ii) If f' changes from +ve to -ve at c , then f has local maximum at c .
- (iii) If f' does not change sign at c (i.e. f' is +ve on both sides of c or negative on both sides), then f has no local extremum at c .

Problem: Find the critical points of

$$f(x) = x^{1/3}(x-4).$$

Further, identify the open intervals on which f is increasing and decreasing. Also find the function's local and absolute extreme values.

Solution: $f(x) = x^{1/3}(x-4)$ is cont. for all x , since it is product of two continuous functions.

$$\begin{aligned} \text{Now } f'(x) &= \frac{d}{dx} (x^{1/3} - 4x^{1/3}) \\ &= \frac{1}{3}x^{-2/3} - \frac{4}{3}x^{-2/3} \\ &= \frac{1}{3}x^{-2/3}(x-4) = \frac{x-4}{3x^{2/3}} \end{aligned}$$

$$f'(x) = 0 \text{ when } x = 4$$

& $f'(x)$ is undefined when $x = 0$.

Hence $x = 0$ & $x = 4$ are the critical points.

There are no endpoints of f .

Hence extreme values occur only at $x = 0$ & $x = 4$.

	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Signs of f' :	—	—	+
Behaviors of f :	Decreasing	Decreasing	Increasing.

First derivative of local extrema tells that: f does not have an extreme value at $x = 0$ but f has local minimum at $x = 4$.

value of local min
= $f(4)$
= -3
Infact this is abs min.

Problem: Within the interval $0 \leq x \leq 2\pi$, find the critical points of

$$f(x) = \sin^2 x - \sin x - 1$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution: The fn. f is cont. over $[0, 2\pi]$ & diff'ble over $(0, 2\pi)$

$$\begin{aligned} f'(x) &= 2 \sin x \cos x - \cos x \\ &= (2 \sin x - 1) \cos x \end{aligned}$$

Hence critical points in $[0, 2\pi]$

are:

$$x = \pi/6, 5\pi/6$$

$$\text{& } x = \pi/2, 3\pi/2$$

Interval	$(0, \pi/6)$	$(\pi/6, \pi/2)$	$(\pi/2, 5\pi/6)$	$(5\pi/6, 3\pi/2)$	$(3\pi/2, 2\pi)$
Sign of f'	—	+	—	+	—
Behaviour of f	↓	↑	↓	↑	↓

Local Min at $x = \pi/6, 5\pi/6$ (and possibly at $x = 0$ & 2π)

Local Max at $x = \pi/2, 3\pi/2$ (and possibly at $x = 0$ & 2π)

$$\begin{aligned} f(\pi/6) &= -5/4 & f(\pi/2) &= -1 & f(0) &= -1 \\ f(5\pi/6) &= -5/4 & f(3\pi/2) &= 1 & f(2\pi) &= -1 \end{aligned}$$

$$f'(x) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \text{ or } \cos x = 0$$

$$\Rightarrow \text{or } x = (2n+1)\pi/2$$

$$x = n\pi + (-1)^n \pi/6$$

$n \in \mathbb{Z}$
(set of all integers)

$$n = 0 : \pi/6 \text{ or } \pi/2$$

$$n = 1 : 2\pi/6 \text{ or } 3\pi/2$$

$$n = 2 : 5\pi/6 \text{ or } 5\pi/2$$

$$n = 3 : 7\pi/6 \text{ or } 7\pi/2$$

⋮

$$n = -1 : -7\pi/6, -\pi/2$$

Thus Min in $[0, 2\pi]$ is $-5/4$ occurring at $x = \pi/6, 5\pi/6$

Thus Max in $[0, 2\pi]$ is 1 occurring at $x = 3\pi/2$

Concavity

Definition: The graph of a differentiable function $y = f(x)$ is

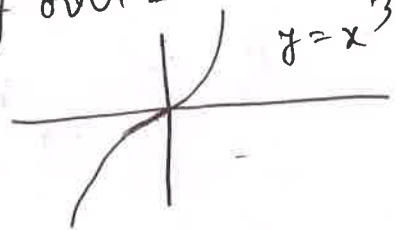
- (i) concave up (or convex) on an open interval I if f' is increasing on I
- (ii) concave down on an open interval I if f' is decreasing on I .

The second Derivative Test for Concavity:

Let $y = f(x)$ be twice-diff'ble on an interval I .

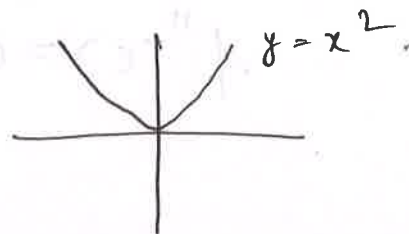
- (i) If $f'' > 0$ on I , the graph of f over I is concave up.
- (ii) If $f'' < 0$ on I , the graph of f over I is concave down.

Example: ① The curve $y = x^3$ is



- (i) concave down on $(-\infty, 0)$ since $y'' = 6x < 0$.
- (ii) concave up on $(0, \infty)$, since $y'' = 6x > 0$.

② The curve $y = x^2$ is concave up on $(-\infty, \infty)$ since $y'' = 2 > 0$.



Problem: Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$

Solution:

$$y = 3 + \sin x.$$

$$\Rightarrow y' = \cos x.$$

$$\Rightarrow y'' = -\sin x.$$

$$\text{on } (0, \pi), \quad \sin x > 0$$

$$\Rightarrow y'' < 0$$

$\Rightarrow y = 3 + \sin x$ is concave down.

$$\text{on } (\pi, 2\pi), \quad \sin x < 0$$

$$\Rightarrow y'' > 0$$

$\Rightarrow y = 3 + \sin x$ is concave up.

Point of Inflection: A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a point of inflection.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Problem: Determine the concavity and find the inflection points of the function:

$$f(x) = x^3 - 3x^2 + 2$$

Solution:

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6 = 6(x-1)$$

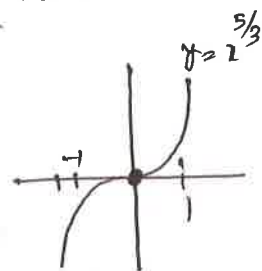
Points of inflection ~~are~~ is $x = 1$.

$-\infty < x < 1$	$x = 1$	$1 < x < \infty$
$f''(x) < 0$	$f''(1) = 0$	$f''(x) > 0$
Concave Down	Pt. of Inflection	Concave up.

Problem: Determine the concavity & find the inflection pts of the function: $f(x) = x^{5/3}$

Solution: $f'(x) = \frac{5}{3}x^{2/3}, \quad f''(x) = \frac{10}{9}x^{-1/3}$

At $x = 0$ $f''(x)$ does not exist.

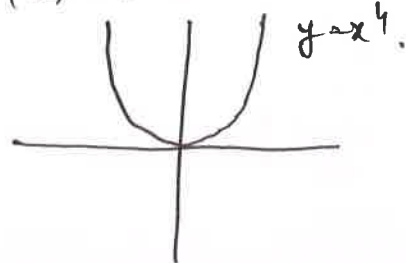


$-\infty < x < 0$	$x = 0$	$0 < x < \infty$
$f''(x) < 0$	$f''(0)$ does not exist.	$f''(x) > 0$
concave down	Pt. of Inflection	concave up

Problem: Determine the concavity & find the inflection pts of $f(x) = x^4$.

Soln: $f'(x) = 4x^3, \quad f''(x) = 12x^2, \quad | \quad f''(x) = 0 \text{ at } x = 0$

$-\infty < x < 0$	$x = 0$	$0 < x < \infty$
$f''(x) > 0$	$f''(0) = 0$	$f''(x) > 0$
concave up	Not a pt. of Inflection	Concave up



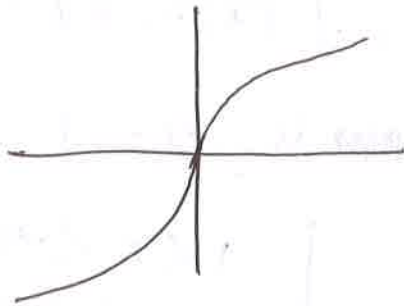
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Problem: Determine the concavity and find the inflection points of

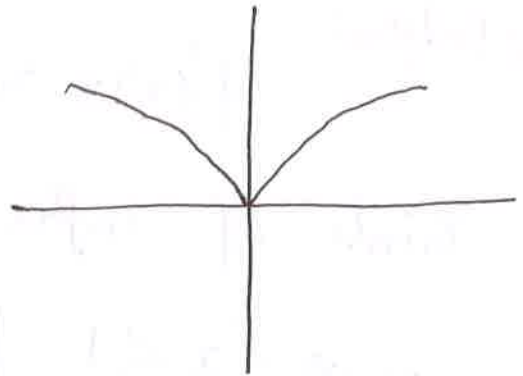
$$f(x) = x^{1/3}$$

$$\& f(x) = x^{2/3}$$

Solution:



$$y = x^{1/3}$$



Problem:

A particle is moving along a horizontal coordinate line (+ve to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution:

velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t-1)(3t-11).$$

& Acceleration is

$$a(t) = s''(t) = 12t - 28 = 4(3t - 7).$$

	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t < \infty$
Sign of v/s'	+	-	+
Behaviour of s	Increasing	Decreasing	Increasing
Particle Motion	Right	Left	Right.

At $t = 1$ & $11/3$, the particle is at rest.

	$0 < t < 7/3$	$7/3 < t < \infty$
Sign of $a = s''$	-	+
Graph of s	Concave down	Concave up.

$$(i) f'(x) = (\sin x - 1)(2\cos x + 1), \quad 0 \leq x \leq 2\pi$$

$$(ii) f'(x) = (\sin x + \cos x)(\sin x - \cos x), \quad 0 \leq x \leq 2\pi$$

Second Derivative Test for local Extrema

Suppose f'' is continuous on an open interval that contains $x=c$.

- (i) If $f'(c)=0$ & $f''(c)<0$, then f has a local max at $x=c$.
- (ii) If $f'(c)=0$ & $f''(c)>0$, then f has a local min at $x=c$.
- (iii) If $f'(c)=0$ & $f''(c)=0$, then the test fails & the f may have a local max, a local min or neither.

Advantage of second derivative test over 1st derivative test:

Second derivative test requires us to know f'' only at c not in an interval about c . This makes the test easy to apply.

Disadvantage of second derivative test:

- The test fails when $f''(c)=0$.
- The test fails when $f''(c)$ does not exist.
(when this happens use 1st derivative test)

Problem: Discuss the curve $y=x^4-4x^3$ with respect to concavity, pts of inflection, and local maxima & minima.

Solution: $f(x)=x^4-4x^3$, $f'(x)=4x^2(x-3)$, $f''(x)=12x(x-2)$

critical pts
For pts of inflection: $f'(x)=0 \Rightarrow x=0$ & $x=3$.

Now use 2nd derivative test: $f''(0)=0$, $f''(3)=36>0$.
 $\hookrightarrow f$ has local min at $x=3$.

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Curve sketching

Together f' and f'' tells us the shape of the function's graph — i.e. (i) where critical pts are located.

(ii) what happens at a critical point.

(iii) where the f^n is increasing.

(iv) where the f^n is decreasing.

(v) How the curve is turning or bending (by concavity).

① These ^{information} can be used to sketch a graph of the function that captures its key features.

~~Problem~~ ~~Sketch the graph of~~

Problem: Sketch the graph of $f(x) = x^4 - 4x^3 + 10$ using the following steps:

- Identify where the extrema of f occur.
- Find the intervals where f is increasing & decreasing.
- Find where the graph of f is concave up and where it is concave down.
- Sketch the general shape of f .
- Plot some specific points, such as local local maximum and minimum points, points of inflection and intercepts. Then sketch the curve.

Solution:

$$f(x) = x^4 - 4x^3 + 10$$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f'(x) = 0 \Rightarrow x = 0, 3$$

Domain of f is

$(-\infty, \infty)$

& f is cont.