Chapter - 2 2.1 Sets

Eg 23: (a) Let the domain be

Z^+ = the set of all positive integers

Let $P(x)$ be $|x| = 1$

Find the truthset of P.

$\{ x \in Z^+ \mid P(x) \text{ is true} \}$

$= \{ x \in Z^+ \mid |x| = 1 \}$

$= \{ 1 \}$

This is singleton

23(b) If domain is Z

Find the truth set of above P.

Answer: $\{-1, 1\}$

23(c) If $Q(x)$ is $x^2 \leq 1$ and if domain is $\mathbb{R}$,

find the truth set of Q.

Ans: $\{ x \in \mathbb{R} \mid Q(x) \text{ is true} \}$

$= \{ x \in \mathbb{R} \mid x^2 \leq 1 \}$

$= [-1, 1]$

23(d) Find the truth set of $R(x)$ where $R(x)$ means "$x = |x|$".

Domain $\mathbb{R}$

Answer: $[0, \infty)$

$= \{ x \in \mathbb{R} \mid x \geq 0 \}$

Relation on a set S is a subset of S×S.

Eg 21: How many elements are there in relations

$\{0,1,2,3\}$

$S = \{0,1,2,3\}$

$S \times S = \{0,1,2,3\} \times \{0,1,2,3\}$

$= \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),$

$(1,2),(1,3),(2,0),(2,1),(2,2),(2,3),$

$(3,0),(3,1),(3,2),(3,3)\}$ ∴ $|S| = 16$

$\{(0,0),(0,1),(0,2)(0,3),(1,1),(1,2),(1,3)$

$(2,2)\ (2,3)\ (3,3)\}$

Answer: This set has 10 elements.

21(b) Write the relation < on $\{0,1,2,3\}$ as a set

Answer: $\{(0,1),(0,2)(0,3)\ (1,2)\ (1,3)\ (2,3)\}$

21(c): Find the cardinality of the above two sets.

Cardinality of S = Number of elements in S = $|S|$ notation

$|\{0,1,2,3\}| = 4$

$\{(0,1)(0,2)(0,3)\ (1,2)\ (1,3)\ (2,3)\} = 6$

Eg 20(b) : Find $|A^3|$ if $A = \{1, 2\}$

$A^3 = \{(1,1,1)\ (1,1,2)\ (1,2,1)\ (1,2,2)\ (2,1,1)$

$(2,1,2)\ (2,2,1)\ (2,2,2)\}$

$|A^3| = 8$

Express $|A \times B|$ in terms of $|A|$ and $|B|$

Ans: $|A \times B| = |A| \cdot |B|$

Example 17 : Write the set $\{1,2\} \times \{a,b,c\}$ in the "listing" notation

Ans: $\{(1,a), \ldots\} \times$

Two formulae (on cardinality)

$|A \times B| = |A| \cdot |B|$

$|P(S)| = 2^{|S|}$

Example: If a Set S has 4 elements how many subsets does S have?

Ans: $2^4 = 16$

12-2-20 **2.2 Set Operations**

- Union of 2 sets A & B denoted by A∪B. It is $=\{x \mid x \in A$ (or) $x \in B\}$
- Intersection
- Relative complement
- Complement

Intersection of 2 sets A & B is denoted by A∩B.

It is $=\{x \mid x \in A$ and $x \in B\}$

Relative complement of two sets A and B is denoted by A−B. It is $\{x \mid x \in A$ and $x \notin B\}$

complement is denoted by $\overline{A}$. It is $\{x \mid x$ is not in A$\}$

Example1: The union of $\{1,3,5\}$ and $\{1,2,3\}$

Example3: Find the intersection of $\{1,3,5\}$ and $\{1,2,3\}$

Example 5: Find $\{1,2,3\} - \{1,3,5\}$

It is not equal to $\{1,3,5\} - \{1,2,3\}$ shows this

Example 6: Find the complement of V = {vowels} in English alphabet.

1 Ans:- $\{1,2,3,5\}$

2 Ans:- $\{1,3\}$

3 Ans:- $\{2\}$

$\{5\} \neq \{2\}$

6 Ans:- {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}

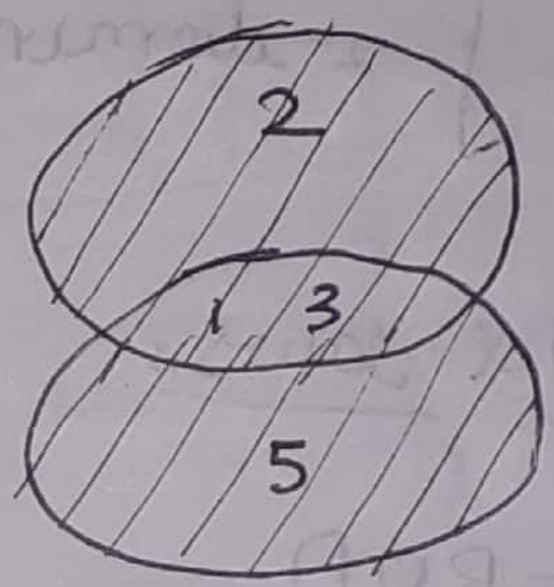
Two sets A and B are said to be disjoint if $A \cap B$ is empty.

$$|A \cup B| = |A| + |B|$$

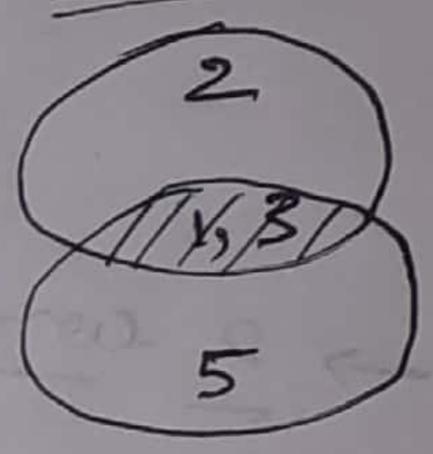
there are no common elements for A and B.

Venn diagram for example 1.

$A \cup B$

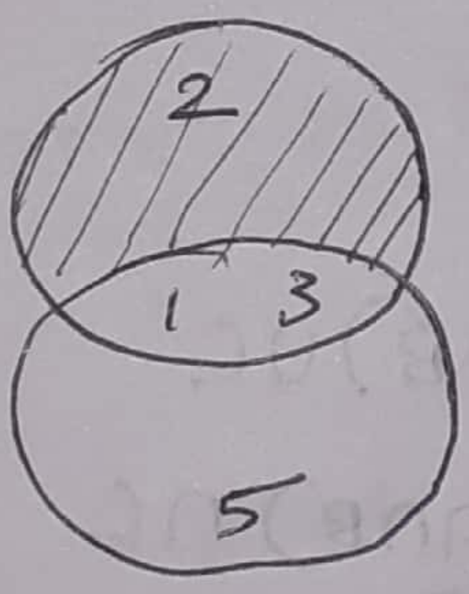


Eg 3: $A \cap B$

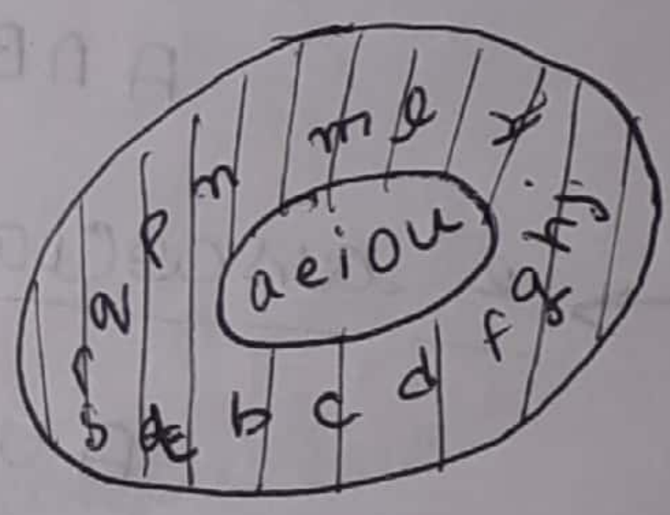


Eg 5:

$A - B$



Eg 6:



The laws of logic

Demorgan's law:

$$(p \vee q) = \neg p \wedge \neg q$$

It's analogous law in set theory

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Set Identities

$A \cup A = A$

$A \cap A = A$ } idempotent laws

Universal Set S

empty Set ϕ

$S \cup A = S$

$S \cap A = A$

$\phi \cup A = A$

$\phi \cap A = \phi$ } 2 identity laws and 2 domination laws.

→ 2 commutative laws:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

→ 2 associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

→ 2 absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

→ 2 Distributive laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

→ Double complement law:

$$(\overline{\overline{A}}) = A.$$

Prove the De-morgan's law:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Method 1: membership table

A	B	$A \cup B$	$\overline{A \cup B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cap \bar{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

By membership table both are equal.

Method 2: $\overline{A \cup B} = \{x \mid x \in \overline{A \cup B}\} =$

$\{x \mid \neg(x \in A \cup B)\} = \{x \mid \neg(x \in A \vee (x \in B))\}$

$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$ By

Demorgan's law

$= \{x \mid x \in \bar{A} \text{ and } x \in \bar{B}\}$

$= \{x \mid x \in \bar{A} \cap \bar{B}\}$

$= \{x \mid x \in \bar{A} \cap \bar{B}\}$

$= \bar{A} \cap \bar{B}$

Method 3: We first prove: every element of $\overline{A \cup B}$ is in $\bar{A} \cap \bar{B}$.

Next we prove:

every element of $\bar{A} \cap \bar{B}$ is in $\overline{A \cup B}$.

Problems from 2.1 & 2.2

Pg 131:

Example 1: Find the union of $\{1,3,5\}$ and $\{1,2,3\}$

Answer: $\{1,3,5\} \cup \{1,2,3\} =$

Prove: $\overline{A \cup (B \cap C)} = \overline{A} \cap (\overline{B} \cup \overline{C})$

Recall two De-morgan's law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Now $\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$ by first De-morgan's law

$= \overline{A} \cap (\overline{B} \cup \overline{C})$ by second Demorgan's law

Prove the Distributive law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$\cap \rightarrow 1,(-1)$

$\cup \rightarrow 0,0 \rightarrow 0$

method 1: (by membership table).

A	B	C	B∩C	A∪(B∩C)	A∪B	A∪C	(A∪B)∩(A∪C)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

② Find $A \cup B \cup C$ and $A \cap B \cap C$

if $A = \{0, 2, 4, 6, 8\}$

$B = \{0, 1, 2, \}$

$C = \{0, 3, 6, 9\}$ etc

Ans: $A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$

$A \cap B \cap C = \{0\}$

$A \cup (B \cap C) = \{0, 2, 4, 6, 8\}$

Example 1 by Set theoretical theory:

Proof:- Two ~~sets~~ S and T are equal for every element if S is in T and vice-versa

First part:

Let $x \in A \cup (B \cap C)$

We shall prove: $x \in (A \cup B) \cap (A \cup C)$

Second part: Let $x \in$

We shall prove: $x \in A \cup (B \cap C)$

Proof of first part

$x \in A \cup (B \cap C)$

this means $x \in A$ (or) $x \in B \cap C$

Consider 2 cases:

Case1: Let $x \in A$

obviously $x \in A \cup B$

and similarly $x \in A \cup C$

$\therefore x \in (A \cup B) \cap (A \cup C)$

Case 2 : Let $x \in B \cap C$

$\therefore x \in B$ and $x \in C$

$\therefore x \in A \cup B$ ($\because x \in B$)

and $x \in A \cup C$ ($\because x \in C$)

$\therefore x \in (A \cup B) \cap (A \cup C)$

Proof of second part

Let $x \in (A \cup B) \cap (A \cup C)$

to prove $x \in A \cup (B \cap C)$

Consider two cases

Case 1 : Let $x \in A$

obviously $x \in A \cup$ (another set)

$\therefore x \in A \cup (B \cap C)$

Case 2 : Let $x \notin A$

$x \in A \cup B$ from (A)

$\therefore x \in B$ — (1)

Similarly, because $x \in A \cup C$,

$x \in C$ ($\because x \notin A$) — (2)

(A∩B) C in ≠ AB

AB ≠

Notation : $\bigcup_{i=1}^{\infty} A_i$

union of $A_1, A_2, A_3 \ldots$

Eg :- Let $A_i = \{i, i+1, i+2 \ldots\} \forall i \in Z^+$

$Z^+ = \{1, 2, 3 \ldots\}$

Find $\bigcup_{i=1}^{\infty} A_i$ and find $\bigcap_{A=1}^{\infty} A_i$

Ans: The union of Z^+
the intersection is ϕ

Eg: Let $S_i = \{1, 2, 3 \ldots i\} \forall\ i \in Z^+$

$S_1 = \{1\}$

$S_2 = \{1, 2\}$

$S_3 = \{1, 2, 3\}$

Find $\bigcup_{i=1}^{\infty} S_i$ and $\bigcap_{i=1}^{\infty} S_i$

Ans :- Union is Z^+

Intersection is $\{1\}$

2.3 Functions

Eg :- $f : Z \to Z$ (domain $\to$ codomain) is $f(x) = x^2 \ \forall x \in Z$

find $f(-3)$ $f(-3) = (-3)^2 = 9$

Find $f(3)$ $f(3) = (3)^2 = 9$

Find the range of f Answer: $\{0, 1, 4, 9, 16\}$

Is this function one-to-one? NO

One-to-one means:

only one element in the domain goes to a given element in the co-domain

There are two distinct in the domain that go to same element

Namely, -3 and

If $f(x) = f(y)$

then $x = y$

f is on-to means every element in the co-domain is in the range.

codomain = Range

$\forall x \in$ codomain $\exists y \in$ domain st

$f(y) = x$

counter example $x = 3$

Give examples of functions $f: \mathbb{Z} \to \mathbb{Z}$

Example 1: f is one-one and onto

Example 2: f is one-one but not onto.

Example 3: f is not one-to-one, but onto

Example 4: f is neither one-to-one nor onto

(1) $f(x) = x \quad \forall x \in \mathbb{Z}$

(2) $f(x) = x^3 \quad \forall x \in \mathbb{Z}$

one to one ? ✓

onto ? ✗

(3)

(4) $f(x) = x^2$

17/2/20 Unit 2 : Sets and Functions

Set-theoritic notation	Logic notation
$A \subseteq B$	$\forall x (x \in A \rightarrow x \in B)$ A is a subset of B
$A = B$	$\forall x (x \in A \leftrightarrow x \in B)$ A and B are equal
$A \subset B$	$\forall x (x \in A \rightarrow x \in B)$ A is proper subset of B
$A \cap B = \phi$	$\forall x (x \in A \rightarrow \neg(x \in B))$ A and B are disjoint (or) $\neg \exists x (x \in A \wedge x \in B)$
$A \cup B = C$	$\forall x (x \in C \leftrightarrow (x \in A) \vee (x \in B))$

$A \cup (B \cap C) =$

Notation : $A_1 \cup A_2 \cup \cdots$

$$= \bigcup_{n=1}^{\infty} A_n$$

$$A_1 \cup A_2 \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

De morgais law,

$$\left(\bigcup_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} A_i$$

Another De-morgan's law

for finite intersections.

$$\bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \overline{A_i}$$

- complement of intersection = union of complements

$$\bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \overline{A_n}$$

one more.

Eg 18 : Let

$U = \{0, 1, 2, \ldots, 9\}$

Let $A = \{$odd integer in $U\}$

Write A in the form of a bit string

(asequence of 0's and 1's)

i^{th} term is 1 if i^{th} element belong to A

0 1 0 1 0 1 0 1 0 1 represents A

length of string is = 10

(18b) Let $B = \{$ even integers in $U\}$

Find the bit-string representing B

Ans: 1 0 1 0 1 0 1 0 1 0

(18c) Let $C = \{$ integers in U that are $\leq 5\}$

Its bit-string is

11111100 00

Eg 19: Let A, B, C, U be as in Eg 18.

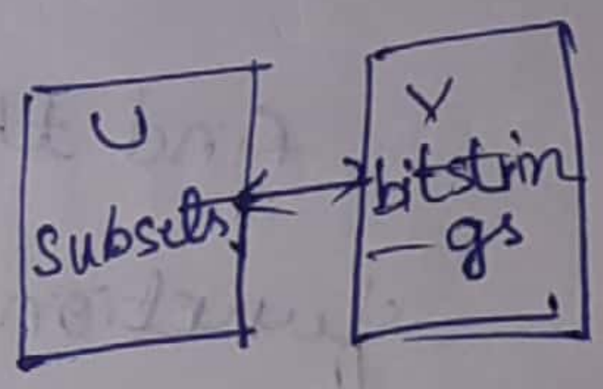
The bit-string of $A \cup C$ is =

$(01010\ 10101) \vee (11111\ 00000)$

$= (11111\ 10101)$

Subset of V corresponding to a bit-string 0110011001

is $\{1, 2, 3, 6, 9\}$



Bit string of $A \cap B$ is = (bitstring of A) $\wedge$ (bitstring of B)

Bit string of $\overline{A}$ is = (bit string of A)

$A \cup C = \{0, 1, 2, 3, 4, 5, 7, 9\}$

Its bit string is 11111 10101

Let $f: Z \to Z$ be defined by

$f(x) = x - 1$ if $x \geq 0$

and $f(x) = x + 1$ if $x < 0$

Is f one-to-one?

$f(-1) = 0$

$f(1) = 0$ ∴ if is not one-one

Is f onto? Yes

Given $m \in \mathbb{Z}$ find x s.t $f(x) = m$

If $m \geq 0$ take $x = m + 1$

If $m < 0$ take $x = m - 1$

2.3 Example 1: The set of all pairs like

(Gautam, A), (Siva sai, A+) etc...;

find the domain, co-domain, range of this function.

$f(\text{Gautam}) = A$

$f(\text{Siva sai}) = A^+$

domain = {Students in this class}

Codomain = {Grades}

= {O, A+, A, B+, B, C, D, F}

Range = {O, A+, A, B+, B, C, D}

If every student has passed

2.3 Functions

Inverse of a Function
Composite of two functions
Graph of a function
Floor and Ceiling Functions.

Example: Let $f(a) = 3$

$f(b) = 2$

$f(c) = 1$

Find the inverse of f

Ans: We note that f is from $\{a, b, c\}$ to $\{1, 2, 3\}$
It's inverse is from $\{1, 2, 3\}$ to $\{a, b, c\}$
Denoted by f^{-1}
(read as f inverse)

$f^{-1}(1) = c$

$f^{-1}(2) = b$

$f^{-1}(3) = a$

Eg: Let $f : \mathbb{Z} \to \mathbb{Z}$ be

$f(x) = x + 1$

Find the inverse of f.

Ans: Let us call $f(x)$ as y

We have $y = x + 1$

$x = y - 1$

$\therefore f^{-1}(y) = y - 1$ ($\because f^{-1}(y)$ is x)

$(f^{-1}(x) = x - 1$ is also correct)

Example: For $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$, inverse is?

Ans: Inverse does not exist.

x^2 is not one one

Inverse exists only for bijections
= one-to-one onto maps

Eg: For $f(x) = x^2$ from $\mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ inverse?

This is one-one but not onto

NO inverse

Eg: $f(x) = x^2$ from $\mathbb{R}^+$ to $\mathbb{R}^+$

$f^{-1}(y) = \sqrt{y} \ \forall \ y \in \mathbb{R}^+$

(Note: $f(\sqrt{y}) = y$)

Identity function

Identity function on a set S $f(x) = x \ \forall x$ in S

Prove: If $f: A \longrightarrow B$ and

if $f^{-1}: B \longrightarrow A$ exists,

then $f(f^{-1}(b)) = b \ \forall \ b$ in B

$f^{-1}(b)$ is that a for which $f(a) = b$

If f and g are two functions, sometimes we define their composite fog

$f: A \to B$

$g: C \to A$

If $c \in C$, $g(c) \in A$, $f(g(c)) \in B$,

$(fog)(c) = f(g(c))$

Example: $g(a)=b$ $f(a)=3$

$g(b)=c$ $f(b)=2$

$g(c)=a$ $f(c)=1$

Find fog

$g: \{a,b,c\} \longrightarrow \{a,b,c\}$

$f: \{a,b,c\} \longrightarrow \{1,2,3\}$

$(fog)(a) = f(g(a)) = f(b) = 2$

$(fog)(b) = f(g(b)) = f(c) = 1$

$(fog)(c) = f(g(c)) = f(a) = 3$

(Inverse is X)

Eg 21: $f: Z \longrightarrow Z$

$f(x) = 2x+3$

$g: Z \longrightarrow Z$

$g(x) = 3x+2$

Find fog

$(fog)(x) = f(g(x))$

$= f(3x+2)$

$= 2(3x+2)+3$

$= 6x+7$

Eg: Find gof where f, g are as in previous example.

$(gof)(x) = 6x+11$

$g(f(x)) = g(2x+3)$

$= 3(2x+3)+2$

$= 6x+9+2 = 6x+11$

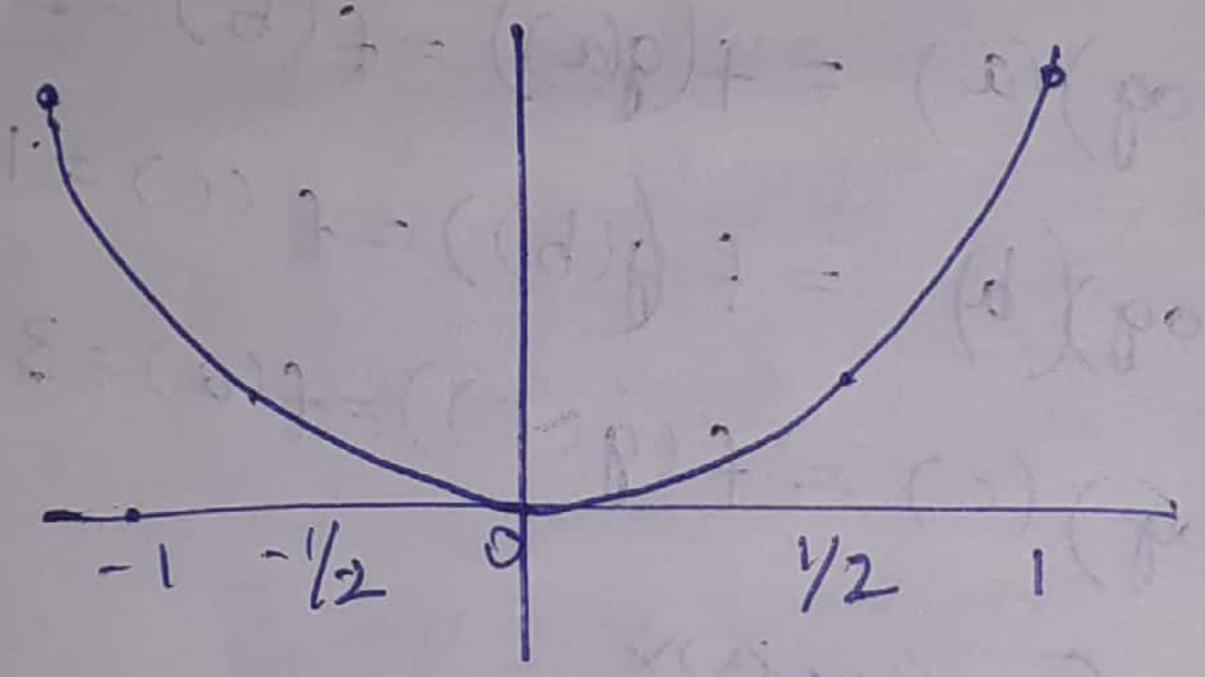
Graph of $f: A \longrightarrow B$.

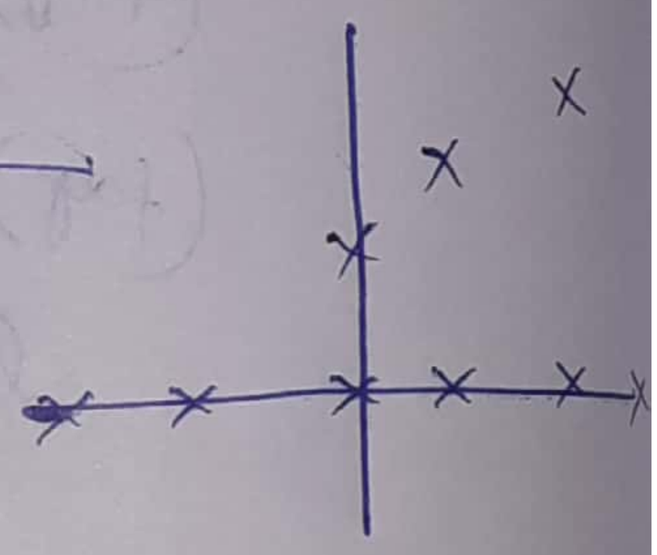
This is subset of $A \times B$.

$\{(a, f(a)) \mid a \in A\}$

Sketch the graph of x^2 on $\mathbb{R}$

Say on $[-1, 1]$





$f(x) = x + 1 \quad \mathbb{Z} \longrightarrow \mathbb{Z}$

(Eg: $f(x) = x^2$ from $\mathbb{R}^+$ to $\mathbb{R}^+$. This is a bijection

$f^{-1}(y) = \sqrt{y} \; \forall y \in \mathbb{R}^+$ (Note

Lastly, for $x \in \mathbb{R}$

$\lfloor x \rfloor$ = greatest integer $\leq x$.

(read as floor of x)

$\lceil x \rceil$ = smallest integer $\geq x$

(read as ceiling of x)

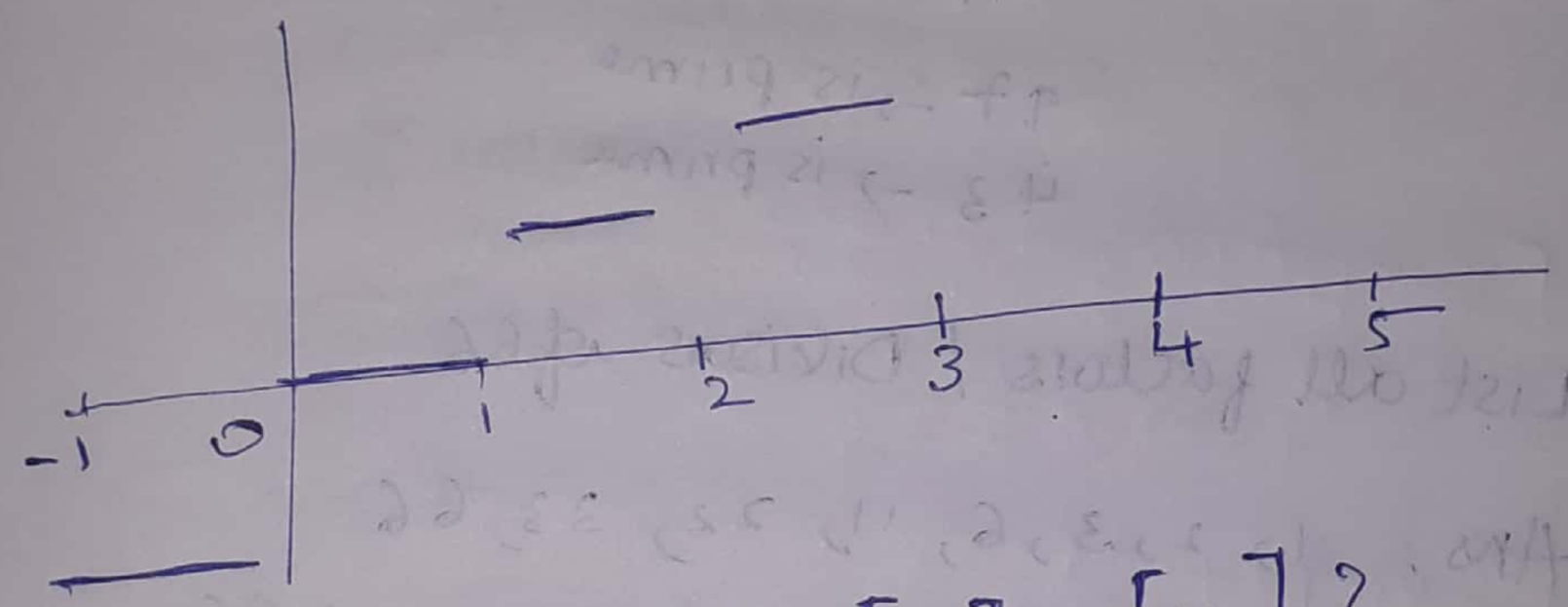
Eg 24 : Find $\lfloor \frac{1}{2} \rfloor = 0$

$\lceil \frac{1}{2} \rceil = 1$

$\lfloor -\frac{1}{2} \rfloor = -1$

$\lceil -\frac{1}{2} \rceil = 0$

Eg 25 : sketch the graph $\lfloor x \rfloor$ on $[-1, 5]$



Eg 28 : Is $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$?

$\lceil 0.5 + 0.5 \rceil = \lceil 0.5 \rceil + \lceil 0.5 \rceil$

$\lceil 1 \rceil = \lceil 1 \rceil + \lceil 1 \rceil$

Numbers and Algorithms

Prime number : (There are no fa) × A positive integer n is called a prime number if there are no factors other than 1 and n

A list of prime numbers	A list of composite numbers
2	$4 = 2 \times 2$
3	$6 = 2 \times 3$
5	$= 2 \times 2 \times 2$
7	8
11	$9 = 3 \times 3$
13	$10 = 2 \times 5$
17	$12 = 2 \times 2 \times 3$
	$14 = 2 \times 7$

methods to verify if a given number is prime or not.

Example : Find which of the following are prime?

21 → 3 × 7 Not a prime

71 → prime

111 → Not prime

29 → is prime

97 → is prime

43 → is prime

List all factors / Divisors of 66

Ans: 1, 2, 3, 6, 11, 22, 33, 66

These eight are divisors of 66.

Eg :- List all divisors of 143

Ans: 1, 11, 13, 143

Eg :- List all divisors of 144

Ans: 1, 2, 3, 4, 6, 8, 12, 9, 16, 18, 24,

14
11
13

36, 48, 72, 144

Eg: List all common divisors of 66 and 143

1, 11 → greatest common factor

26-02-20 Wednesday

Chapter - 3

Pg 215

Eg 2: How many positive integrs are there not exceeding n that are divisible by d?

Ans:- Let $n = 100$

$d = 11$

Divide 100 by 11

Quotient is 9.

$$11 \overline{)100} \quad (9)$$
$$99$$
$$\overline{1}$$

Answer = 9

(11, 22, 33, 44, 55, 66, 77, 88, 99)

$Z^+ \quad |\{x \in Z^+ \mid d|x \text{ and } x \le n\}| = \left\lfloor \frac{n}{d} \right\rfloor$

$100 = 9 \times 11 + 1$

$n = q \times d + 1$

For any 2 positive integrs n and d, there exist positive integrs q and r such that

$n = qd + r$ and such that $0 \le r < d$

Domain is Z^+

$\forall n \forall d \exists q \exists r$ s.t. $n = qd + r$

q and r are in W.

NOTATIONS : If $n = qd + r$

and if $0 \leq r < d$

then $q = n \text{ div } d$

$r = n \pmod d$

DIVISION ALGORITHM

Domain is Z: form

Example 3: Find 101 div 11

and 101 mod 11

Ans: 101 div 11 is = 9

101 mod 11 is = 2

Example 4: Find -11 div 3

and -11 mod 3

$-11 = (-4) \times (3) + 1$

-11 div by 3 is -4

-11 mod 3 is 1

Pg 217:

Example 5: Is 17 congruent to 5 modulo 6?

Ans: Is 17-5 divisible by 6?

Yes. $6 \mid (17-5)$

$a \equiv b \pmod d$ if $a-b$ is divisible by d

if $d \mid a-b$

- a and b leave the same remainder when divided by d.

5(b): Are 24 and 14 congruent modulo 6?

Yes: Does 6 divides 24-14?

No, $24 \not\equiv 14 \pmod 6$

True or False

(i) If $a|b$ and if $c|d$ then,

$a+c \mid b+d$.

Take example: $a|b \Rightarrow 16|32$

$c|d \Rightarrow 12|24$

then $a+c|b+d \Rightarrow 16+12|32+24$

↓

True

16 32
12 24
28 56
×2 ↑
56

(ii) If $a|b$ and if $a|d$ then $a|b+d$.

This is True

$a|b$

$b = m \cdot a$ for some $m \in Z$

$a|c$

$\therefore c = n \cdot a$ for some $n \in Z$

Adding $b+c = ma+na = (m+n)a$

$\therefore a|b+c$

If $a \mid b$

then $a \mid kb \quad \forall k \in \mathbb{Z}$

Because $a \mid b$, $b = ma$ for some $m \in \mathbb{Z}$

$\therefore kb = k(ma)$

$= (km)a$

$\therefore a \mid kb$

Example: If $a \equiv b \pmod{n}$

and if $c \equiv d \pmod{n}$

then $a + c \equiv (b + d) \pmod{n}$

- If $a \mid b$ and

  if $a \mid c$

  then

  $a \mid mb + nc \quad \forall m, n \in \mathbb{Z}$

Integer Representation

A number can be represented in many ways.

2020 two thousand and twenty.

10^1 10^2 10^3

$0+2\times10+0\times100+2\times100$

Decimal representation, Powers of 10.

Binary Representation

= Base 2 - representation

Use powers of 2

$2020 = ; +2, +2^2 x, +2^3 \ldots$

$9 = 1+0\times2+0\times4+1\times8$

$(1001)_2$

Example 1: For the number

101011111 in base 2;

Find the decimal representation.

$101011111 = 1+1\cdot2+1\times2^2+1\times2^3+1\times2^4+$

$0\times2^5+1\times2^6+0\cdot2^7+1\times2^8$

$= 1+2+4+8+16+64+256$

$= 351$

$(101011111)_2 = (351)_{10}$

Integer Representation

Base	Name of representation
2	Binary
8	Octal
10	Decimal
16	Hexadecimal

octal -

Ex 2: Find the decimal rep — representation.

$(7016)_8$

$(7016)_8 = 6 + 1\times8 + 0\times8^2 + 7\times8^3$

$= 6 + 8 + 0 + 7\times512$

$= 14 + 3584$

$= (3598)_{10}$

Eg 3: Find the decimal representation

$(2AE0B)_{16}$

Ans: $(2AE0B)_{16} = 11 + 0 + 14\times16^2 + 10\times16^3 + 2\times16^4$

$= 11 + 14\times256 + 10\times4096 + 2\times65536$

$= (175627)_{10}$

Eg 4: $(12345)_{10} = (?)_5$

Successive division by 8:

$12345 = 8 \times 1543 + 1$

$1543 = 8 \times 192 + 7$

$192 = 8 \times 24 + 0$

$24 = 8 \times 3 + 0$

3

$(30071)_5$

Eg 5: Find the hexadecimal representation of $(177130)_{10}$

We use successive division by 16

$177130 = 16 \times 11070 + 10$ A

$11070 = 16 \times 691 + 14$ E

$691 = 16 \times 43 + 3$

$43 = 16 \times 2 + 11$ B

2

$(2B3EA)_{16}$

Eg 6: $(241)_{10} = (?)_2$

By successive division by 2

$241 = 2 \times 120 + 1$

$120 = 2 \times 60 + 1$

$60 = 2 \times 30 + 0$

$30 = 2 \times 15 + 0$

$15 = 2 \times 7 + 1$

$7 = 2 \times 3 + 1$

$3 = 2 \times 1 + 1$

1

$(11110011)_2$

Eg7: $(1111\,0101\,1100)_2$

Find its octal representation

Note: $8 = 2^3$

Grouping method:

(Remaining DIY)

Eg8: Find the hexadecimal representation of same number

Note: $16 = 2^4$

We group into blocks of length of

$1100 = 1\times2^3 + 1\times2^3 = 12 = C$

$1011 = 1\times2\times8 = 11 = B$

$1110 = 0\times2 + 4\times8 = 14 = E$

$0011 = 1+2 = 3$

Ans: $(3EBC)_{16}$

Eg9: Add $(1112)_2 + (1011)_2$

```
  11
  1110
  1011
------
 11001
```

$1+1+0 = 2$

$= 0$ with carry over 1

$1+1+1 = 3$ $\quad 0+1 = 1$

$= 1$ with carry.

$1+1 = 2$ modulo 2

0 with 1 carry over

over 1

Prove that x is real no, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Proof: Let $x = n + \epsilon$ where n is an integer and $0 \le \epsilon < 1$

Case 1: Let $0 \le \epsilon < 1/2$

Then $\lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor, 0 \le 2\epsilon < 1$

$= 2n$

$\lfloor x \rfloor = \lfloor n + \epsilon \rfloor, 0 \le \epsilon < 1/2$

$= n$

$\lfloor x + 1/2 \rfloor = \lfloor n + \epsilon + 1/2 \rfloor, \frac{1}{2} \le \epsilon + \frac{1}{2} < 1$

$= n$

In this case $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Case 2: $\frac{1}{2} \le \epsilon < 1$

Then $\lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor, 1 \le 2\epsilon < 2$

$= 2n + 1$

$\lfloor x \rfloor = \lfloor n + \epsilon \rfloor, \frac{1}{2} \le \epsilon < 1$

$= n$

$\lfloor x + 1/2 \rfloor = \lfloor n + \epsilon + 1/2 \rfloor, 1 \le \epsilon < 3/2$

$= n + 1/2$

Hence in this case also

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Division algorithm

$$a = dq + r, \quad 0 \le r < d$$

$101 \bmod 9 = 2$

$101 = 9 \times 11 + 2$

$-11 \bmod 3 = 1$

$-11 = 3(-4) + 1$

Theorem : There are infinitely many primes

Proof : We will prove this using contradiction method

We assume that there are finitely many primes which are $p_1, p_2 \cdots p_n$

Let $Q = p_1 \times p_2 \times \cdots \times p_n + 1$

By FTA, either Q is prime as there is a prime which will divides Q

However, none of the prime p_j divide Q, for if p_j / Q,

then $p_j / (Q - p_1 \times p_2 \times \cdots \times p_n)$

i.e $p_j / 1$, which is a contraction

Hence there is a prime which is not in the list $P_1, P_2 \cdots P_n$.

Hence, there are infinitely many primes.

Example 10: Multiply $(110)_2$ by $(101)_2$

Sol:
$$\begin{array}{r} 110 \times 101 \\ \hline 110 \\ 000 \\ 110 \\ \hline 11110 \end{array}$$

Answer: $(11110)_2$

First row is 110×1.
Second row is 110×0.
Third row is 110×1.

Eg 11: Find 3^{664} modulo 4.

(i.e Find the remainder when 3^{64} is divided by 4)

We use modular arithmetic

known: If $a \equiv b \pmod{d}$.

and if $x \equiv y \pmod{d}$.

then $ax \equiv by \pmod{d}$.

$a = qd + b$, where q is the quadrant

$m = sd + n$

multiply,

$am = (qd + b)(sd + n)$

$$= qsd^2 + qdm + bsd + bn$$

$$= d(qsd + qn + bs) + bn$$

$\therefore am \equiv bn \mod d.$

using this in this problem,

$$3^2 \equiv 1 \mod 4$$

$$3^{664} = 3^{2 \times 332} = (3^2)^{332}$$

$$= 1^{332} \mod 4$$

$$= 1 \mod 4.$$

Extra:

Find the unit digit of 9!

$$9! = 1 \times 2 \times 3 \times (4 \times 5) \times 6 \times 7 \times 8 \times 9$$

20

has 0 as unit digit

In our language

$$9! = 0 \mod 10$$

Find the remainder when 9! is divided by 11.

Answer: $9! = 24 \times 5 \times 6 \times 7 \times 8 \times 9$

$$= 2 \times 8 \times 1 \times 9 \pmod{11}.$$

$$= 5 \times 9 \pmod{11}$$

$$= 1 \pmod{11}$$

Eg13 : Are 10, 17, 21 pairwise relatively prime?

gcd (10, 17) = 1

gcd (17, 21) = 1

gcd (21, 10) = 1

Yes, they are pairwise relatively prime

No because gcd (10, 24) = 2 ≠ 1

Eg 15: Find gcd of 120 and 500 in 2 ways.

Method 1: Prime factorization method.

$120 = 2 \times 60$

$= 2^2 \times 30$

$= 2^3 \times 15$

$= 2^3 \times 3 \times 5$

$500 = 2 \times 250$

$= 2^2 \times 125$

$= 2^2 \times 5 \times 25$

$= 2^2 \times 5^3$

The 2 numbers are

$2^3 \times 3 \times 5$

$2^2 \times 5^3$

Their gcd $2^2 \times 5 = 20$.

Method 2: $500 = 4 \times 120 + 20$

$120 = 6 \times 20 + 0$

conclude: 20 is gcd.

Remaining topics for mid - II

1. Applications of congruence $a \equiv b \bmod$
2. Induction.

Mathematical Induction

$\mathbb{N} = \{1, 2, 3 \cdots\}$.

Eg 1: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}$.

the sum of first n positive integers.

Let $P(n)$ denote the statement.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

We want to prove: $P(n)$ is true $\forall n$ is

Basics step: When $n = 1$, this becomes.

$$1 = \frac{1(1+1)}{2}$$

This is $P(1)$

$P(1)$ is true because

$$1 = \frac{1 \times 2}{2} = 1$$

Assume $P(k)$ is true for some arbitory k in $\mathbb{N}$.

This is called induction hypothesis.

$$1+2+\cdots+k = \frac{k(k+1)}{2}$$

Inductive Proof

We shall prove $P(k+1)$

To prove:

$$1+2+\cdots+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$$

Left side $= 1+2+\cdots+k+(k+1)$

$= (1+2+\cdots+k)+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ by (1)

$= (k+1)\left(\frac{k}{2}+1\right)$ by

$= (k+1)\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2}$

$= R.S$

By the principle of mathematical induction $P(n)$ is true for all n in $\mathbb{N}$

If $P(n)$ is true for $n = 1$, and if $P(k+1)$ is true whenever $P(k)$ is true then $P(n)$ is for all positive integers n.

(By the principle of mathematical induction.

$P(n)$ is true for all n in $\mathbb{N}$)

$$P(1) \wedge (\forall k\, P(k) \longrightarrow P(k+1)) \longrightarrow$$

$$P(n)\ \forall\ n \in N.$$

Eg2: (a) Guess a formula for

$1+3+5+\ldots+(2n+1)$ sum of first n odd positive integers.

(b) Prove this by induction.

$1 = 1$

$1+3 = 4$

$1+3+5 = 9$

$1+3+5+9 = 16$.

Guess: The answer is n^2.

(b) Let $P(n)$ be the statement.

$$1+3+\ldots+(2n-1) = n^2$$

Basic Step: $P(1)$ is

$$1 = 1^2$$

this is True

Induction thypothesis:

Assume P(K) is true;

$1+3+\cdots+(2k+1) = k^2$ —— (1)

Induction proof;

1A3A

$1-2+\cdots+(2k-1)+(2(k+1)-1) = (k+1)^2$

to prove this,

$S = (1+3+\cdots+2k-1)+(2k+1)$

$= k^2 + (2k+1)$ by i.h. (1)

$= (k+1)^2 = R.S$

Prove $P(k) \rightarrow P(k+1)$

By PMI, P(n) is true $\forall n$,

Eg3 : Prove $1+2+2^2+\cdots+2^n = 2^{n+1}-1$

for all integers $n=0$

Proof: Let P(n) be the statement

$1+2+\cdots+2^n = 2^{n+1}-1$

Basic Step: P(0) is

$1 = 2^{0+1}-1$

$2^1-1 = 2-1 = 1$.