Date, Day: 19-12-2018, Wednesday

Unit-1

Three Dimensional Co-ordinate system

Vectors, Dot Product, Cross product, Lines and Planes

The standard eq for sphere of radius a and centre (x_0 y_0 z_0)

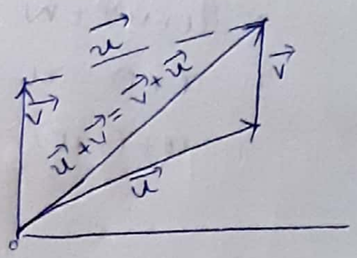
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

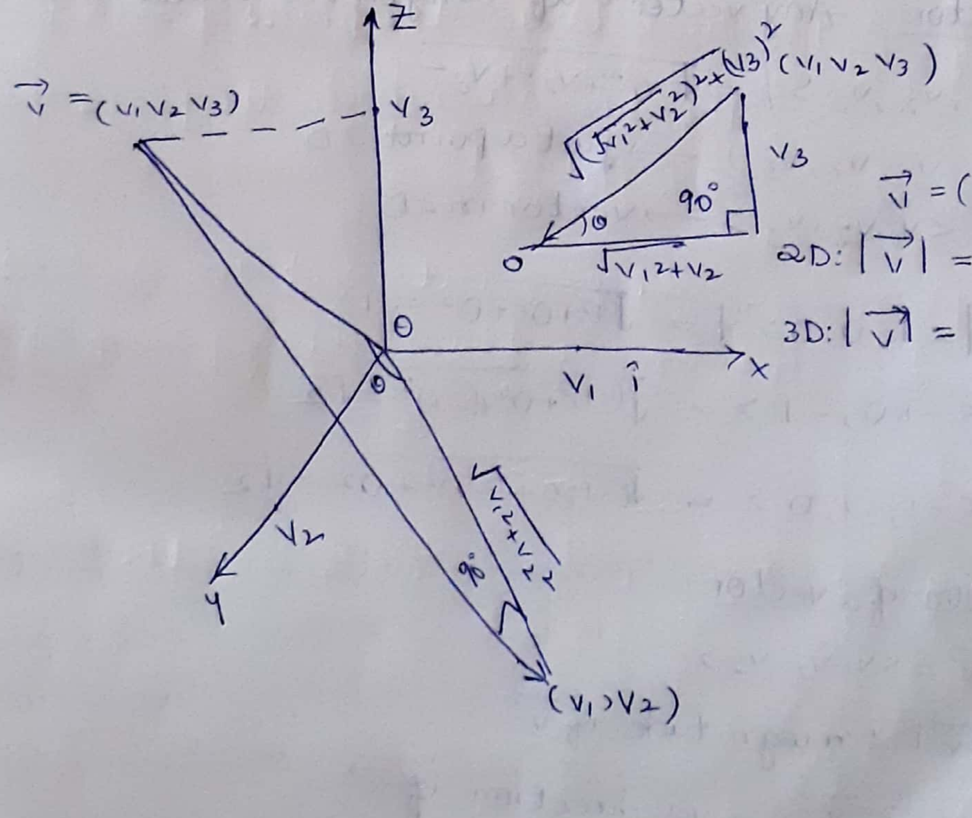
$$(x_0\ y_0\ z_0) = \vec{r_0}$$

$$(x_1\ y_1\ z_1) = \vec{r}$$

$$|\vec{r} - \vec{r_0}| = a^2$$

Commutative law: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$





$\vec{v} = (v_1, v_2, v_3)$

2D: $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$

3D: $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

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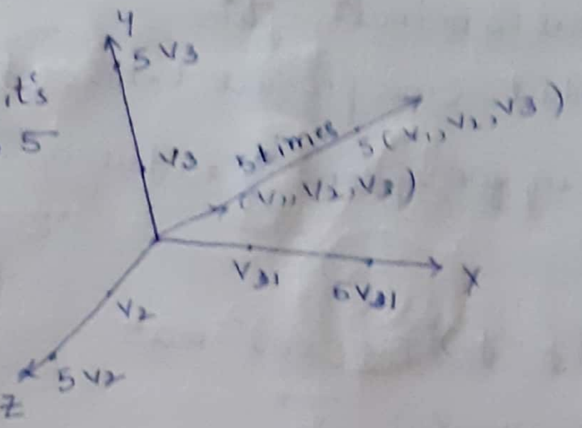
Example $\vec{v} = (v_1, v_2, v_3)$

$k = 5$

$5\vec{v} = (5v_1, 5v_2, 5v_3)$

increases it's Magnitude to 5 times



Properties

1. $u + v = v + u$
2. $(u+v) + w = u + (v + w)$
3. $u + 0 = u$
4. $u + (-u) = 0$
5. $0u = 0$
6. $1u = u$
7. $a(bu) = (ab)u$
8. $a(u+v) = au + av$
9. $(a+b)u = au + bu$

Unit vector: Any vector v of length 1 is called unit vector

$|\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$\vec{v} =$ (v_1, v_2, v_3) ⟶ just a point 3D

$\langle v_1, v_2, v_3 \rangle$ ⟶ vector in 3D.

(A) $|\langle 1, 0, 0 \rangle| = \sqrt{1^2 + 0^2 + 0^2} = 1$

(B) $\langle -1, 0, -1 \rangle = \sqrt{(-1)^2 + 0^2 + (-1)^2} = \sqrt{2}$

(C) $\langle -1, -1, 0 \rangle = \sqrt{(-1)^2 + (-1)^2 + 0^2} = \sqrt{2}$

Direction of a vector:

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$|\vec{v}|$ is magnitude of $\vec{v}$

Now, $\dfrac{\vec{v}}{|\vec{v}|}$ gives direction of $\vec{v}$

$$\left|\frac{\vec{v}}{|\vec{v}|}\right| = \left|\frac{\langle v_1, v_2, v_3 \rangle}{\sqrt{v_1^2 + v_2^2 + v_3^2}}\right| = \frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} |\langle v_1, v_2, v_3 \rangle|$$

$$= \frac{\sqrt{v_1^2 + v_2^2 + v_3^2}}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = 1$$

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$\boxed{\vec{v} = (\text{magnitude})(\text{direction}) = |\vec{v}| \cdot \dfrac{\vec{v}}{|\vec{v}|}}$

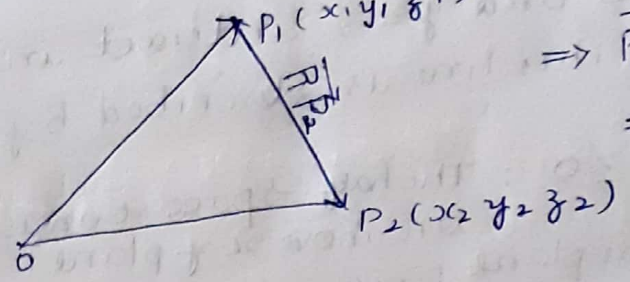
Note: If $\vec{v} \neq \vec{0}$ then

1. $\dfrac{\vec{v}}{|\vec{v}|}$ is unit vector called the direction of $\vec{v}$

2. The standard unit vector are
   (i) $\hat{i} = <1,0,0>$, $\hat{j} = <0\ 1\ 0>$, $\hat{k} = <0,0,1>$

→ The vector from P_1 to P_2 is represented by $\vec{P_1P_2}$ as

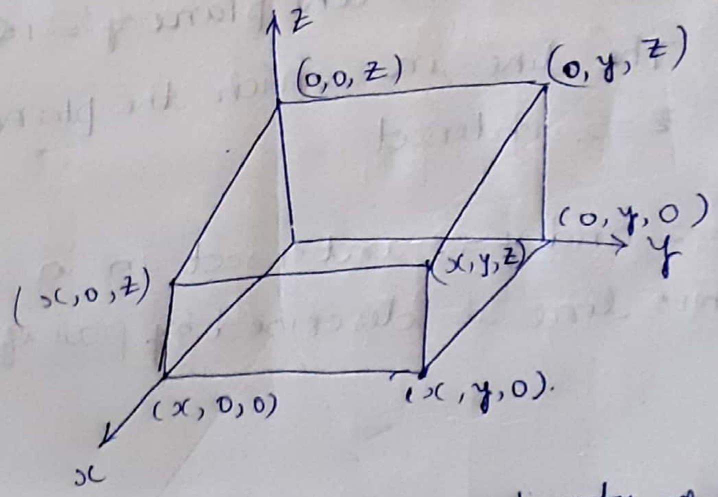
$\vec{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$



$\vec{P_2} = \vec{P_1P_2} + \vec{P_1}$

$\Rightarrow \vec{P_1P_2} = \vec{P_2} - \vec{P_1}$

$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$



Point on x axis have co-ordinates of form (x, 0,0)

on z-axis (0, 0, Z)

on Y-axis (0, Y, 0)

The co-ordinates axis find some plane which are, the

xy plane, where equation is $z = 0$,

yz plane $x = 0$, xz plane $y = 0$

All meet at origin(0,0,0).

Octant

The 3 co-ordinate plane $x = 0, y = 0, z = 0$ divide space into eight cells called octants. The octant in which point co-ordinates are all positive (+ve) is called first octant

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Consider the plane $x = 2$ which is ⊥er to the x-axis at $x = 2$

Similarly the plane $y = 3$ is the plane ⊥er to the y-axis at $y = 3$ and plane $z = 5$ is the plane ⊥er to z-axis at $z = 5$

All the above plane intersect at point $(2, 3, 5)$

The plane $x = 2$ and $y = 3$ intersect in a line parallel to z-axis. This line is described by pair of eq's

Example: (a) $z \geq 0$: The half space consisting of points on above x-y plane.

(b) $y = -3$: The plane perpendicular to y-axis at $y = -3$.

(c) $-1 \leq y \leq 1$: The slab between the plane $y = -1$ & $y = 1$

(d) $y = -2, z = 2$: The line in which the plane $y = -2$ & $z = 2$. intersect

The plane $x = 2$ and $y = 3$ intersect in a line parallel to z-axis. This line is described by pair of equation

$x = 2, y = 3$.

(e) $x^2 + y^2 = 4, z = 3$: This is a circle on the horizontal plane $z = 3$.

Example

Calculate the Dot product of

(a) $\langle 1,-2,-1\rangle \cdot \langle -6,2,-3\rangle$

(b) $(\frac{1}{2}i+3j+k)\cdot(4i-j+2k)$

$$\cos\theta = \frac{uv}{(|u||v|)} = \frac{-6-4+3}{|u||v|} = \underline{-7}$$

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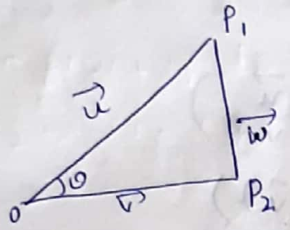
Cosine rule:



$$\left[\begin{aligned} \vec{u}\cdot\vec{v} &= u_1v_1+u_2v_2+u_3v_3 \\ \vec{u}\cdot\vec{u} &= u_1^2+u_2^2+u_3^2 \\ &= |\vec{u}|^2 \end{aligned}\right]$$

$\vec{u} = (u_1, u_2, u_3)$

$\vec{v} = (v_1, v_2, v_3)$.

$|\vec{w}| = |\vec{u}|^2+|\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$.

$|\vec{v}-\vec{u}|^2 = (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})$

$= (v_1-u_1)^2+(v_2-u_2)^2+(v_3-u_3)^2$

$= v_1^2+u_1^2-2v_1u_1+v_2^2+u_2^2-2v_2u_2+v_3^2+u_3^2-2v_3u_3$

$= v_1^2+v_2^2+v_3^2+u_1^2+u_2^2+u_3^2-2(v_1u_1+v_2u_2+v_3u_3)$

$\therefore |\vec{v}-\vec{u}|^2 = |\vec{v}|^2+|\vec{u}|^2-2(\vec{u}\cdot\vec{v})$.

Vector Projection

$$\text{Proj}_{\vec{v}}\vec{u} = \overbrace{\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)}^{\text{Projection component}} \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{Direction}}$$

$\rightarrow \vec{u}\cdot\vec{v} = (u_1,u_2,u_3)\cdot(v_1,v_2,v_3)$

$= u_1v_1+u_2v_2+u_3v_3$.

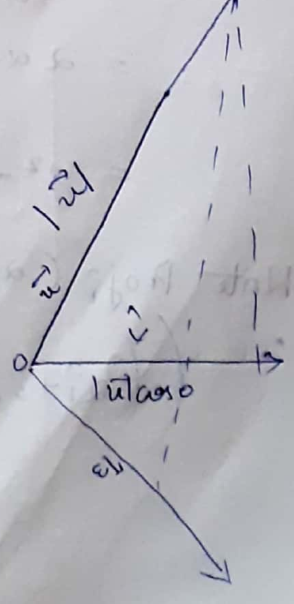
$\rightarrow \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$.

If $\vec{v}$ is unit vector

$|\vec{v}| = 1$

$\vec{u}\cdot\vec{v} = |\vec{u}|\cos\theta$

$= |\vec{u}|\cos\theta$.



Orthogonal Decomposition

For any $\vec{u}$ and $\vec{v} \neq \vec{0}$

$$\vec{u} = (\vec{u} - \text{Proj}_{\vec{v}}\vec{u}) + \text{Proj}_{\vec{v}}\vec{u}$$

→ check whether $(\text{Proj}_{\vec{v}}\vec{u})$ and $(\vec{u} - \text{Proj}_{\vec{v}}\vec{u})$ is orthogonal (or) not.

$$(\text{Proj}_{\vec{v}}\vec{u}) \cdot (\vec{u} - \text{Proj}_{\vec{v}}\vec{u})$$

$$\left(\vec{u} \cdot \frac{\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} \cdot \quad \vec{u} - \text{Proj}_{\vec{v}}\vec{u}$$

$$(\vec{u} \cdot \hat{v})\hat{v} \, (\vec{u} - \text{Proj}_{\vec{v}}\vec{u})$$

$$\Rightarrow (\vec{u} \cdot \hat{v})\hat{v} \, (\vec{u} - (\vec{u}\hat{v})\hat{v}))$$

$$\Rightarrow (\vec{u} \cdot \hat{v})\hat{v} \, (\vec{u} - \vec{u}\hat{v} - 1))$$

$$\Rightarrow (\vec{u}\hat{v} - \hat{v}\hat{v})(\vec{u} - u\hat{v} - 1))$$

$$\Rightarrow (\vec{u}\hat{v} - 1)(\vec{u} - u\hat{v} - 1)) \times$$

Ans: $\left(\frac{\alpha\vec{v}}{|\vec{v}|}\right) \cdot \left(\vec{u} - \alpha\frac{\vec{v}}{|\vec{v}|}\right)$

$$= \alpha\frac{\vec{v}\vec{u}}{|\vec{v}|} - \alpha^2 \frac{\vec{v}}{|\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= \alpha\frac{\vec{v}\vec{u}}{|\vec{v}|} - \alpha^2 \frac{\vec{v} \cdot \vec{v}}{|\vec{v}|^2}$$

$$= \alpha\alpha - \alpha^2 \frac{|\vec{v}|^2}{|\vec{v}|^2}$$

$$= \alpha^2 - \alpha^2 = 0 \text{ (orthogonal)}.$$

$\sqrt{2^2+3^2+1^2}$

$\sqrt{9+4+1}$

Write $\text{Proj}_{\hat{i}}(2\hat{i} + 3\hat{j} - \hat{k})$

$$\left((2\hat{i} + 3\hat{j} - \hat{k})\hat{i}\right)\frac{\hat{i}}{|\hat{i}|} = 2\hat{i}$$

Ex: Decompose

Cross product

For any 2 non-zero vectors $\vec{u}$ and $\vec{v}$ define $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta \, \vec{n}$

Scalar

$\vec{n}$: unit vector orthogonal to $\vec{u}$ and $\vec{v}$

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Properties

1. $(r\vec{u}) \times (s\vec{v}) = rs(\vec{u} \times \vec{v})$
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$
4. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
5. $0 \times \vec{u} = 0$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

① Proof

$$(r\vec{u}) \times (s\vec{v}) = |r\vec{u}||s\vec{v}| \sin\theta \, \hat{n}$$
$$= |r||\vec{u}||s||\vec{v}| \sin\theta \, \hat{n}$$
$$= |rs||\vec{u}||\vec{v}| \sin\theta \, \hat{n}$$
$$= |rs| \, \vec{u} \times \vec{v}$$

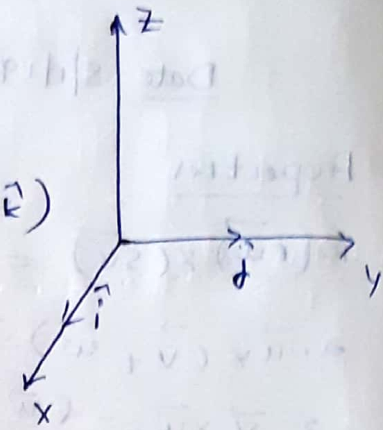
2. $\vec{u} \times (\vec{v} + \vec{w}) = |\vec{u}||\vec{v} + \vec{w}| \sin\theta \, \hat{n}$

$$= \left(|\vec{u}||\vec{v}| + |\vec{u}||\vec{w}|\right) \sin\theta \, \hat{n}$$
$$= \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

Standard unit vector

$$\hat{i}\times\hat{i}=\vec{0},\ \hat{j}\times\hat{j}=\vec{0},\ \hat{k}\times\hat{k}=\vec{0}$$
$$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$$

Right hand Cartesian system



$$\vec{u}\times\vec{v}=(u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\times(v_1\hat{i}+v_2\hat{j}+v_3\hat{k})$$

$$=u_1v_1\hat{i}\times\hat{i}+u_1v_2\hat{i}\times\hat{j}+u_1v_3\hat{i}\times\hat{k}+$$
$$u_2v_1\hat{j}\times\hat{i}+u_2v_2\hat{j}\times\hat{j}+u_2v_3\hat{k}\times\hat{k}$$
$$+u_3v_1\hat{k}\times\hat{i}+u_3v_2\hat{k}\times\hat{j}+u_3v_3\hat{k}\times\hat{k}$$

$$\times| = u_1v_1(\vec{0})+u_1v_2(\hat{k})+u_1v_3(-\hat{j})+u_2v_1(-\hat{k})+u_2v_2(\vec{0})+u_2v_3(\vec{0})$$
$$+u_3v_1(\hat{j}))\times$$

$$=u_1v_1(\vec{0})+u_1v_2(\hat{k})+u_1v_3(-\hat{j})+u_2v_1(-\hat{k})+u_2v_2(\vec{0})+u_2v_3(\hat{i})+$$
$$u_3v_1(\hat{j})+u_3v_2(-\hat{i})+u_3v_3(\vec{0})$$

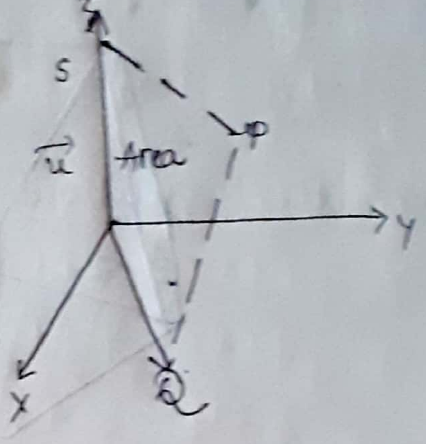
$$=(u_2v_3-u_3v_2)\hat{i}-(u_1v_3-u_3v_1)\hat{j}+(u_1v_2-u_2v_1)\hat{k}$$

$$=\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\hat{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\hat{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\hat{k}$$

$$\Rightarrow \vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

Find the area of Parallelogram from the vectors

$\vec{u} = (0,0,1)$
$\vec{v} = (1,1,0)$



$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$\vec{i}(0-1) - \hat{j}(0-1) + \vec{k}(0-0)$

$-\vec{i} + \vec{j} \quad = \sqrt{(-1)^2 + (1)^2} = |\vec{u} \times \vec{v}|$

$= \sqrt{2}$

$P(1,-1,0)\ Q(2,1,-1)\ R(-1,1,2)$

$\vec{PQ} = \sqrt{(2-1)^2 + (1+1)^2 + (-1-0)^2}$

$= \sqrt{(1)^2 + (2)^2 + (-1)^2}$

$= \sqrt{4} = 2$

$\vec{QR} = \sqrt{(-1-2)^2 + (1-1)^2 + (2+1)^2}$

$= \sqrt{(-3)^2 + (0)^2 + (3)^2}$

$= \sqrt{18}$

$\vec{u} = (-2,2,2)$
$\vec{v} = (1,2,-1)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$= \hat{i}(-2-4) - \hat{j}(2-2) + \vec{k}(-4-2)$

$= \hat{i}(-6) - \hat{j}(0) + \vec{k}(-6)$

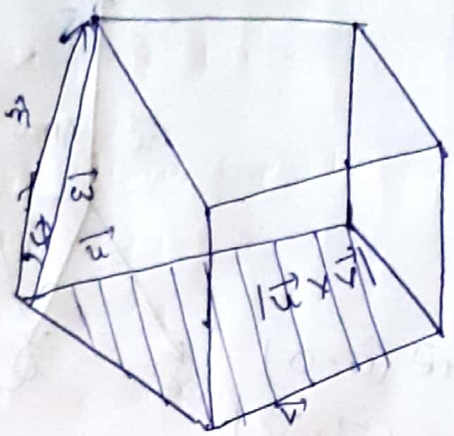
$= -6\hat{i} - 6\hat{k}$

$|\vec{u} \times \vec{v}| = \sqrt{36+36}$

$= \sqrt{72}$

$=$

Find the volume of Parallelpiped formed by sides $\vec{u} = \hat{i}$, $\vec{v} = \hat{j}$, $\vec{w} = \hat{k}$.



Volume of parallepiped = (Base Area) (height).

$$= |\vec{u} \times \vec{v}| \, |\text{proj}_{\hat{n}} \vec{w}|$$

$$= |\vec{u} \times \vec{v}| \, |(\vec{w} \cdot \hat{n})|$$

$$= |\vec{u}||\vec{v}| \sin\theta \, |\vec{w}||\hat{n}| \cos\phi$$

$$= |\vec{u}||\vec{v}| \sin\theta \, |\hat{n}||\vec{w}| \cos\phi$$

$$= |\vec{u} \times \vec{v}| \, |\vec{w}| \cos\phi$$

$$V = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$V = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$= \vec{u} \cdot (\vec{v} \times \vec{w})$$

$$= (\vec{v} \times \vec{w}) \cdot \vec{u}$$

- Box product $= [\vec{u} \;\; \vec{v} \;\; \vec{w}]$

$$\Rightarrow [\vec{v} \;\; \vec{w} \;\; \vec{u}]$$

$$[\vec{w} \;\; \vec{u} \;\; \vec{v}]$$

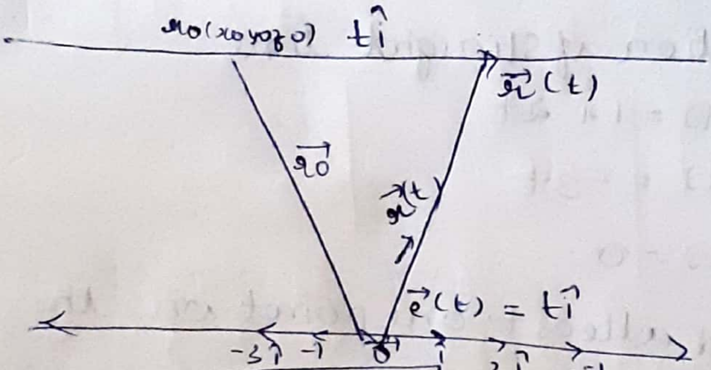
1. $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$, $\vec{w} = \hat{k}$ - Find volume of parallelpiped.

$$V = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\hat{k}.$$

$\Rightarrow 1(-1-0) - 1(1-0)$

$\Rightarrow -1-1 = -2$

Equation of the straight line:



$\boxed{\vec{r}(t) = \vec{r}_0 + t\hat{i}}$ → eq of line passing through $\vec{r_0}$ and Parallel to $\hat{i}$.

Equation of straightline

$\vec{r}(t) = \vec{r_0} + t\vec{v}$

$x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = (x_0\hat{i} + y_0\hat{j} + z_0\hat{k}) + t(v_1\hat{i} + v_2\hat{j}$

Parametric equations for straightline

$x(t) = x_0 + tv_1$

$y(t) = y_0 + tv_2$

$z(t) = z_0 + tv_3$

Eg:- Find direction of straight line

$x(t) = 1 + 2t$

$y(t) = -3t$

$z(t) = 0.$

Also, find atleast one point on the line

Sol: $(v_1, v_2, v_3) = (2, -3, 0)$

$\vec{r_0} = (1, 0, 0)$ (At $t = 0$)

Eg: Parameterize line joining points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Sol: $(3, 2, -3)$ P —— O —— Q $(1, -1, 4)$

$\vec{PQ} = (-2, -3, 7)$ $\frac{4}{2}, \frac{-1}{2}, \frac{1}{2}$

$\vec{OQ} - \vec{OP} = (1, -1, 4) - (3, 2, -3)$

$\vec{r_0} = 3\hat{i} + 2\hat{j} - 3\hat{k}$

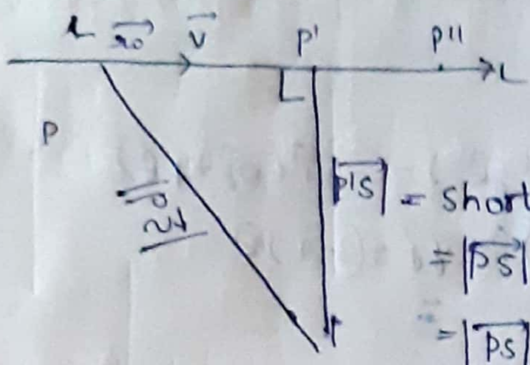
$(1, -1, 4) = \vec{r}t$

$= (3, 2, -3) + t(-2, -3, 7)$

$1 = 3 - 2t$, $-1 = 2 - 3t$

$\cancel{-}2 = \cancel{-}2t$ $\cancel{-}3 = \cancel{-}3t$

$\boxed{t = 1}$ $\boxed{t = 1}$ $0 \leq t \leq 1$



$|\vec{P'S}|$ = shortest distance

$= |\vec{PS}| \sin\theta$

$= \frac{|\vec{PS}||\vec{v}| \sin\theta}{\vec{v}}$

$= \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$

Eg: Find shortest distance from (1,0,0) to y-axis

Sol:- $P(1,0,0)$ y-axis $(0,y,0)$

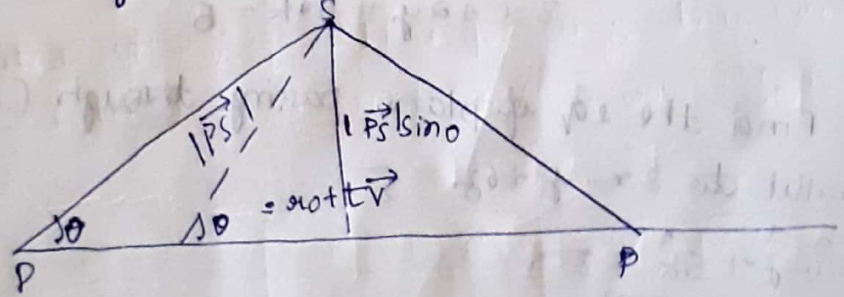
$\vec{r}(t) = \vec{0} + t\hat{j}$

$P = \vec{r_0} = \vec{0}, \vec{v} = \hat{j}$

$\vec{PS} = \vec{OS} - \vec{OP} = \hat{i} - \vec{0} = \hat{i}$

shortest distance $= \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{|\hat{i} \times \hat{j}|}{|\hat{j}|} = \frac{|\hat{k}|}{|\hat{j}|} = \frac{1}{1} = 1$

Q: Find distance from $S(1,1,1)$ to line $\vec{r}(t) = \hat{k} + t(\hat{i}+\hat{j})$



$\vec{PS} = \vec{OS} - \vec{OP}$

$= \hat{i} + \hat{j} + \hat{k} - \hat{k}$

$= \hat{i} + \hat{j}$

$\vec{PS} \times \vec{v} = (\hat{i} + \hat{j}) \times (\hat{i} + \hat{j})$

$= 0$

* Plane Equation: $\vec{n} \cdot \vec{P_0P} = 0$

line Equation: $\vec{r_0} + t\vec{v} = \vec{r}(t)$.

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Eg 6

Find eq for pl

$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$

$= (x\hat{i} + y\hat{j} + z\hat{k}) - (-3\hat{i} + 0\hat{j} + 7\hat{k})$

$= (x+3)\hat{i} + y\hat{j} + (3-7)\hat{k}$

$\overrightarrow{P_0P}$

$= \vec{0}$

Eg

$A(0,0,1)$ $B(2,0,0)$ $C(0,3,0)$ find eq of plane containing A, B, & C

$\overrightarrow{AB} = 2\hat{i} - 1\hat{k}$

$\overrightarrow{AC} = 3\hat{j} - 1\hat{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$= \hat{i}(0+3) - \hat{j}(-2) + \hat{k}(6-0)$

$= 3\hat{i} + 2\hat{j} + 6\hat{k}$ is normal to plane

At point $A(0,0,1)$

$3(x-0) + 2(y-0) + 6(k-1) = 0$

$3x + 2y + 6k = 6$

Problem: Find the eq of plane passing through $(1,0,0)$ and Parallel to $3x - y + 6z = 5$

Sol: $(3\hat{i} - \hat{j} + 6\hat{k} = 5)$

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$(3,-1,6) \cdot (x-1, y-0, z-0) = 0$

$3(x-1) - y + 6z = 0$

$\boxed{3x - y + 6z = 3}$

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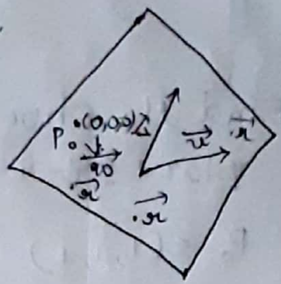
Find the equation of plane containing the vectors $\vec{u} = \hat{i}+\hat{j}$ and $\vec{v} = \hat{i}-\hat{j}$ and passing through origin.



Sol: $\vec{n} = \vec{u} \times \vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(-1-1)$$

$$= -2\hat{k}$$

1. Find the eq of intersecting line for the 2 planes.

$$3x - y + z = 4 \Rightarrow \vec{n_1} \cdot (\vec{r_p} - \vec{r_0}) = 0$$

$$y - z = 2 \rightarrow \vec{n_2} \cdot (\vec{r} - \vec{r_p}) = 0$$

$$\vec{n_1} = 3\hat{i} - \hat{j} + \hat{k}$$

$$\vec{n_2} = 0\hat{i} + \hat{j} - \hat{k}$$

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(1-1) - \hat{j}(-3) + \hat{k}(3)$$

$$= +3\hat{j} + 3\hat{k}$$

Line $\vec{r}(t) = \vec{r_0} + t\vec{v}$

$=$

$$\vec{v} = 3\hat{j} + 3\hat{k}$$

$\vec{r_0}$ = By solving $(2, 2, 0)$

$$\left.\begin{array}{l} 3x - y + z = 4 \\ y - z = 2 \end{array}\right\} \text{solve at } z = 0$$

$$3x - y = 4$$

$$y = 2 \Rightarrow x = 2$$

$$\vec{r}(t) = 2\hat{i} + 2\hat{j} + t(3\hat{j} + 3\hat{k})$$

$$\vec{r}(t) = 2\hat{i} + (2+3t)\hat{j} + 3t\hat{k}$$

$x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ $\quad x(t) \quad y(t) \quad 3t$

$$\boxed{\begin{array}{l} x(t) = 2 \\ y(t) = 2+3t \\ z(t) = 3t \end{array}} \rightarrow \text{Parametric (t) form of line.}$$

Distance $= \left| \operatorname{Proj}_{\vec{n}} \vec{PS} \right| = \left| \left(\frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right) \frac{\vec{n}}{|\vec{n}|} \right|$

$\left| \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right| \hat{v}$ $\quad = \left| \vec{PS} \cdot \frac{\vec{n}}{\vec{n}} \right|$

1. Find the shortest distance between $S(-1, 0, 1)$ to the plane $5x - 4y + 3z = 4$

$P_0 = (1, 1, 1)$

$\vec{PS} = (-1, 0, 1) - (1, 1, 1)$
$= (-2, -1, 0)$

$\vec{n} = (5, -4, 3)$

Date: 8/2/19 Day: Thursday

Limits & Continuity

Ex: $f(x, y) = \sqrt{x^2 + y^2}$

at $(x_0, y_0) = (0, 0)$

$\underset{(x,y) \to (0,0)}{Lt} f(x, y) = 0 = L.$

$\varepsilon > 0$

$|f(x, y) - 1| < \varepsilon \Rightarrow \underset{(x,y) \to (0,0)}{Lt} f(x, y) = 1$

$\left| \sqrt{x^2 + y^2} - 1 \right| < \varepsilon$

$-\varepsilon < \sqrt{x^2 + y^2} - 1 < \varepsilon$

$\boxed{1 - \varepsilon < \sqrt{x^2 + y^2} < 1 + \varepsilon}$

Eg: $f(x, y) = \sqrt{x^2 + y^2}$

at $(x_0, y_0) = (1, 0)$.

$\underset{(x,y) \to (1,0)}{Lt} f(x, y) = 1 = L$

$\varepsilon > 0$

$|f(x, y) - 1| < \varepsilon$ $\quad \cancel{Lt}$

$-\varepsilon < \left| \sqrt{(x-1)^2 + (y-0)^2} - 1 \right| < \varepsilon$

$1 - \varepsilon < \sqrt{x^2 + y^2} < \varepsilon + 1$

$(1-\varepsilon)^2 < x^2+y^2 < (\varepsilon+1)^2$

$1-2\varepsilon+\varepsilon^2 < x^2+y^2 < \varepsilon^2+\varepsilon+1$

$1-2\varepsilon+\varepsilon^2+2x < (x+y)^2 < \varepsilon^2+2\varepsilon+1+2x$

$\boxed{0 < \sqrt{(x-1)^2+(y-0)^2} < \varepsilon^2}$

Eg 3: $f(x,y) = x$

at $(x_0, y_0) = (1,0)$

$\lim_{(x,y)\to(1,0)} f(x,y) = 1 = L$

$|f(x,y) - 1| < \varepsilon$

$\Rightarrow |x-1| < \varepsilon$

$(x-1)^2 < \varepsilon^2$

$0 < (x-1)^2 + y^2 < \varepsilon^2 + y^2$

$< \varepsilon^2 + x^2 + y^2$

$< \varepsilon^2 + (1+\varepsilon)^2 + y^2$

$|y| < \sqrt{x^2+y^2}$

$y^2 < x^2+y^2$

$-\varepsilon < x-1 < \varepsilon$

$1-\varepsilon < x < 1+\varepsilon$

$0 < (x-1)^2 + y^2 < \varepsilon^2 + (1+\varepsilon)^2 + y^2.$

$0 < \sqrt{(x-1)^2+y^2} < \sqrt{\varepsilon^2+(1+\varepsilon)^2+y^2} < \sqrt{\varepsilon^2+(1+\varepsilon)^2+y^2_{max}}$

Pick y_{max} any real number to M

Question: $\lim_{(x,y)\to(0,0)} (xy)^{1/100}$

$\boxed{L>0, \text{ n is ven}}$

$\lim_{x\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

$= 0$

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$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \frac{\cancel{y^2}[4x]}{\cancel{y^2}\left[\frac{x^2}{y^2}+\frac{1}{\cancel{y^2}}\right]}$$

$$\frac{2y(2xy)}{\ }$$

$$= \frac{4x}{\frac{x^2+\cancel{y^2}}{y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

In the direction $y = mx$

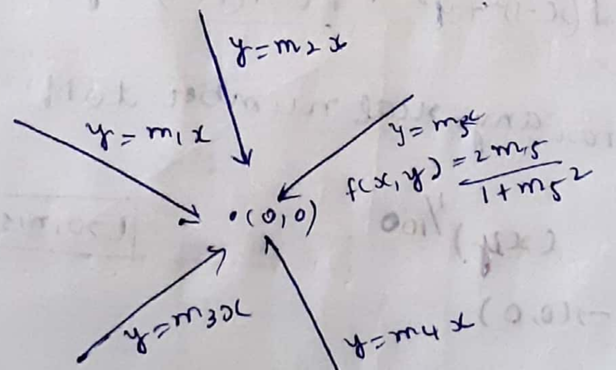
$$f(x,y) = \frac{2xy}{x^2+y^2} = \frac{2x\,mx}{x^2+m^2x^2}$$

$$= \frac{2m}{1+m^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \frac{2x^2\,mx}{x^4+(mx)^2}$$

Q: $\lim_{(x\to y)\to(0,0)} \frac{2xy}{x^2+y^2}$

$y = mx$ will make $f(x,y)$ into $\frac{2x(mx)}{x^2+m^2x^2} = \frac{2m}{1+m^2}$



$f(x,y) = \frac{2xy}{x^2+y^2}$ is it continuous? at $(0,0)$

$f(x_0,y_0) = f(0,0) \to$ undefined

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Q: $f(x,y) = x$ is continuous? at $(x_0, y_0) = (0,0)$.

(i) $f(x_0, y_0) = f(0,0) = 0$

(ii) $\lim_{x \to (0,0)} x = 0$

(iii) $f(x_0, y_0) = \lim = 0$

Q: $f(x,y) = \frac{x}{x+y}$ is it continuous? at $(x_0, y_0) = (0,0)$.

$f(0,0) = $ undefined

$$\lim_{(x_0, y_0) \to (0,0)} \frac{x}{x+y} = \frac{1}{1+m}$$

Q: $f(x,y) = \frac{2x^2 y}{x^4 + y^2}$

let, $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 mx}{x^4 + m^2 x^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{\cancel{x^2}(2mx)}{\cancel{x^2}(x^2 + m^2)}$$

$$\lim_{(x,y)\to(0,0)} \frac{2mx}{x^2 + m^2} = \frac{\cancel{x}(2m)}{\cancel{x}(x + \frac{m^2}{x})} =$$

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^2+xy}{x^2+1}\right) \text{ at } (0,0)$$

(1) $e^{x \cdot y}$ at $(0,0)$

$g(t) = e^t$ — cont

$f(x,y) = x - y$ — cont

$(g \circ f)$ is cont at $(0,0)$

Example :

$$f(x) = x^2 + 3xy + y - 1$$

$$\frac{\partial f}{\partial x} = \left.\frac{df}{dx}\right|_{y \text{ is fixed}}$$

$$= 2x + 3y$$

Example : $f(x,y) = \dfrac{2y}{y+\cos x}$

$$\frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2}$$

$$= \frac{\cancel{2y} + 2\cos x - \cancel{2y}}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

Exercise Find $f'(0)$ for

(1) $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & x \neq 0 \end{cases}$

Sol: $\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$

$\lim_{h \to 0} \dfrac{f(h+0) - f(0)}{h} = \lim_{h \to 0} \dfrac{0-1}{h}$ - undefined.

(2) $f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

$\left.\dfrac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$

$= \lim_{h \to 0} \dfrac{f(0+h) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{1-1}{h} = 0$

$\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h \to 0} \dfrac{f(0, 0+h) - f(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{1-1}{h} = 0.$

26/2/19

Partial Derivatives of higher order

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \rightarrow f_{yy} = (f_y)_x$$

f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$

② $$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$
II I

$w = xy + \frac{e^y}{y^2+1}$ for w is continues

$$\frac{\partial w}{\partial x} = y$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1$$

② $$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}.$$

$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$

$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$

1. $f(x,y) = \sum_{n=0}^{\infty} (xy)^n \; (|xy| < 1)$

find $\frac{\partial f}{\partial x}$; $\frac{\partial f}{\partial y}$

$f(x, y) = \frac{1}{1-xy}$ (G.P) $\quad \frac{df}{dx} = \frac{-1(-y)}{(1-xy)^2} = \frac{y}{(1-xy)^2}$

$f_x = \frac{\partial f}{\partial x} = \frac{y}{(1-xy)^2} \quad \frac{df}{dy} = \frac{x}{(1-xy)^2}$

$f_y = \frac{\partial f}{\partial y} = \frac{x}{(1-xy)^2}$

Chain Rule :-

If $w = f(x)$ & $x = g(t)$

$$\frac{dw}{dt} = \frac{dw}{x} \cdot \frac{dt}{dx}$$

$w = f(x, y)$ & $x = x(t)$; $y = y(t)$

$$\boxed{\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt} + \frac{dw}{dy} \frac{dy}{dt}}$$

Question: Find the local extreme values of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Domain $= \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

Sol:- **Step-1:** Find the Domain and boundary points.

Steps-2: Find all critical points.

Step-3: Find the Second order partial derivatives and evaluate at the critical points.

Sol: $f_x = y - 2x - 2.$

$f_y = x - 2y - 2.$

$y - 2x = 2$

$y = 2x + 2$

$x - 2y = 2$

$x = 2y + 2$

$y - 2x - 2 = 0$

$-2y + 2x - 2 = 0$

$2y - 3x = 0$

$2y = 3x$

$y = -2$

$-2(1)$

Discriminant $= f_{xx} f_{yy} - (f_{xy})^2.$

$= (-2)(-2) - (1)^2$

$= 4 - 1 = 3 > 0.$

$f_{xx} < 0$ for all (x, y)

$(-2,-2) \rightarrow$ local max

(2) $f_x = 10ye^{-y^2} \dfrac{\partial}{\partial x}\left(xe^{-x^2}\right)$

$f_y = 10xe^{-x^2} \dfrac{\partial}{\partial y}\left(y \cdot e^{-y^2}\right)$

$f_x = 10ye^{-y^2}\left(e^{-x^2}((-2x)x + 1)\right)$

$f_y = 10xe^{-x^2}\left(e^{-y^2}((-2y)y + 1)\right)$

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$f_x = 0 \Rightarrow y = 0$ (or) $-2x^2+1=0$

$x = \pm\frac{1}{\sqrt{2}}$

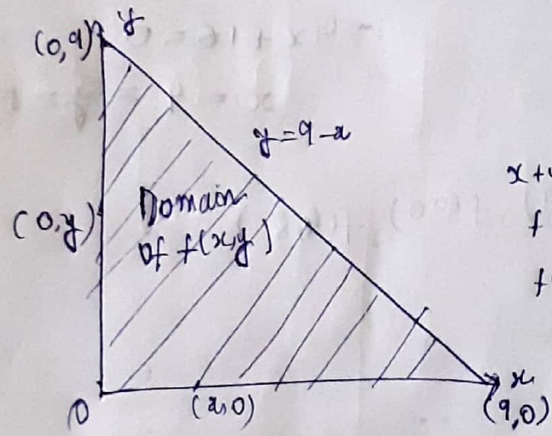
$f_y = 0 \Rightarrow x = 0$ (or) $-2y^2+1=0$

$y = \pm\frac{1}{\sqrt{2}}$

Critical points $\{(0,\frac{1}{\sqrt{2}}),(0,-\frac{1}{\sqrt{2}}),(0,0),(\frac{1}{\sqrt{2}},0),$
$(-\frac{1}{\sqrt{2}},0),(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),$
$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})\}$

Discriminant: $= f_{xx} f_{yy} - (f_{xy})^2$

② 9



$x+y-9=0$

$f(x,y)=0$

$f(0,0) \neq 0$

$1+2-9=-6<0$

$f(x,y) = 2+2x+4y-x^2-y^2$

$f_x = 0$ & $f_y = 0$

$f_x = 2-2x = 0$

$f_y = 4-2y = 0$

$(x,y) = (1,2)$ is the only critical point in the interior of given Domain.

Boundary (i): $\{(x,0) : 0 \le x \le 9\}$

$$f(x,0) = 2+2x-x^2$$

$$f'(x) = 0 \Rightarrow 2-2x = 0$$

$$x = 1$$

$$f(0,0),\ f(9,0),\ f(1,0)$$

Boundary (ii) $\{(0,y) : 0 \le y \le 9\}$

$$f(0,y) = 2+4y-y^2$$

$$f'(0,y) = 0 \Rightarrow y = 2.$$

$$f(0,9),\ f(0,0),\ f(0,2)$$

Boundary (iii) : $y = 9-x$

$$f(x,9-x) = 2+2x+4(9-x) - x^2 - (9-x)^2$$

$$-4x+16 = 0$$

$$x = 4 \Rightarrow y = 5$$

$$f(0,9),\ f(9,0),\ f(4,5)$$

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Lagrange Multiplier

Given $g(x, y, z) = 0$, a condition

Find the max / min of $f(x, y, z)$?

Sol: Solve $\boxed{\vec{\nabla} f = \lambda \vec{\nabla} g}$ → Lagrange multiplier.

Q: Fig 14.56

$f(x, y) = xy$ $\quad \left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) = 0$

$g(x, y) = 0$

$\nabla f = \lambda \nabla g$

$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$.

$\langle y, x \rangle = \lambda \langle x/4, y \rangle$.

$f_x = \lambda g_x$.

$y = \frac{\lambda x}{4}$.

and $x = \lambda y$.

$y = \frac{\lambda \lambda y}{4} \Rightarrow \left(1 - \frac{\lambda^2}{4}\right) y = 0$

$y = 0$ (or) $1 - \frac{\lambda^2}{4} = 0$

$\Rightarrow x = \pm 2\sqrt{2}$ $\Rightarrow \lambda = \pm 2$.

Partition of interval

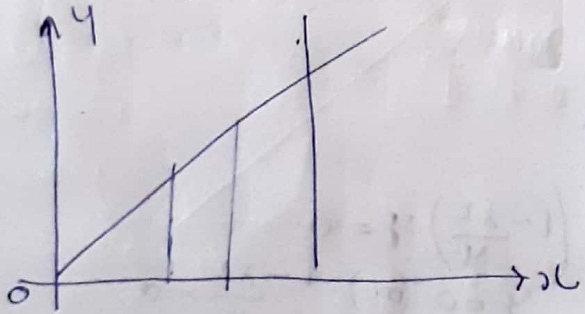
$$\int_{-1}^{1}\int_{-1}^{1} \left(\frac{1}{xy}\right) dx\,dy$$

$$\text{Volume} = \int_{y=0}^{y=1}\int_{x=y}^{x=1} z\,dx\,dy$$

$$= \int_{0}^{1}\int_{x=y}^{1} (3-x-y)\,dx\,dy$$

$$= \int_{0}^{1} \left[3x - \frac{x^2}{2} - yx\right]_0^1 dx\,dy$$

$$= \int_{0}^{1} \left[3(1-y) - \left(\frac{1^2}{2} - \frac{y^2}{2}\right) - y(1-y)\right] dy$$



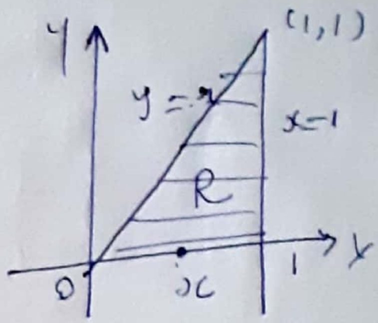
$$= \int \left((3-3y) - \frac{1}{2} + \frac{y^2}{2} - y + y^2 \right) dy$$

$$= \int_{0}^{1} \left(2\tfrac{1}{2} - 4y + \frac{3y^2}{2}\right) dy \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_0^1$$

$$= 1$$

Q: $\iint_R \frac{\sin x}{x}\, dx\, dy$

R: Δ^{le} is x-y-plane bounded by x-axis, the line $y = x$ and the line $x = 1$.



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x}\, dy\, dx$$

$$\Rightarrow \int_{x=0}^{x=1} \frac{\sin x}{x} \left[y \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{\sin x}{x} [x - 0]\, dx$$

$$= \int_0^1 \sin x\, dx = -\cos x \Big|_0^1 = -\cos 1 + 1$$