# 14.7 Extreme Values and Saddle Points

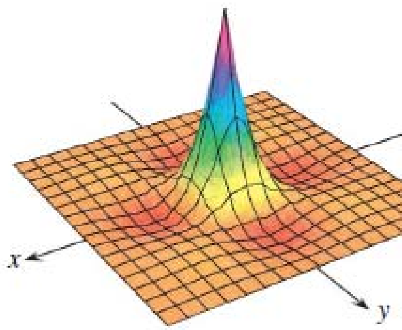



**FIGURE 14.41** The function

$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$$

has a maximum value of 1 and a minimum value of about $-0.067$ on the square region $|x| \le 3\pi/2$, $|y| \le 3\pi/2$.

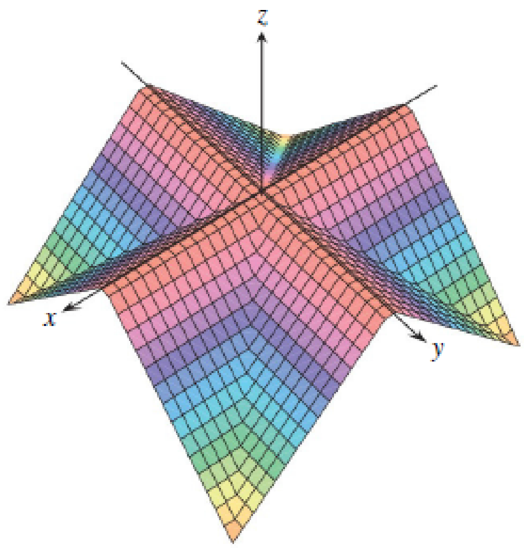



**FIGURE 14.42** The "roof surface"

$$z = \frac{1}{2}\left(\left||x| - |y|\right| - |x| - |y|\right)$$

has a maximum value of 0 and a minimum value of $-a$ on the square region $|x| \le a$, $|y| \le a$.

Now let us try to define what is a local minimum and local maximum in more than one variable /dimension case.

**DEFINITIONS** Let $f(x, y)$ be defined on a region $R$ containing the point $(a, b)$. Then

1. $f(a, b)$ is a **local maximum** value of $f$ if $f(a, b) \geq f(x, y)$ for all domain points $(x, y)$ in an open disk centered at $(a, b)$.
2. $f(a, b)$ is a **local minimum** value of $f$ if $f(a, b) \leq f(x, y)$ for all domain points $(x, y)$ in an open disk centered at $(a, b)$.

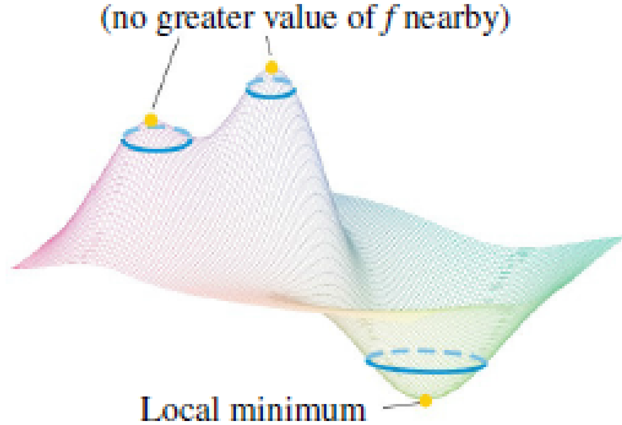



**FIGURE 14.43** A local maximum occurs at a mountain peak and a local minimum occurs at a valley low point.

## Derivative Tests for Local Extreme Values

**THEOREM 10—First Derivative Test for Local Extreme Values**
If $f(x, y)$ has a local maximum or minimum value at an interior point $(a, b)$ of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

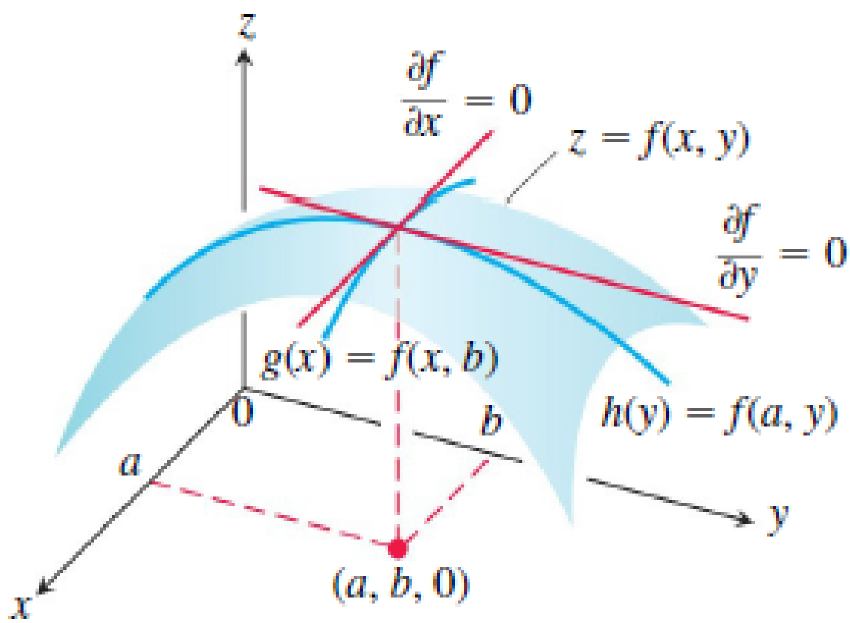



**FIGURE 14.44** If a local maximum of $f$ occurs at $x = a, y = b$, then the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero.

## Critical Point :

**DEFINITION** An interior point of the domain of a function $f(x, y)$ where both $f_x$ and $f_y$ are zero or where one or both of $f_x$ and $f_y$ do not exist is a **critical point** of $f$.

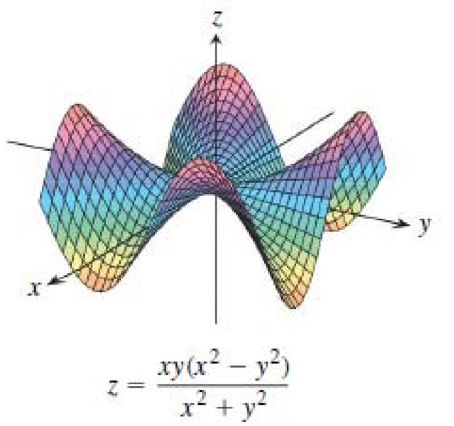
**EXAMPLE 1** Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

**Solution** The domain of $f$ is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$, where the value of $f$ is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point $(0, 2)$ gives a local minimum (Figure 14.46).

## Saddle Point :



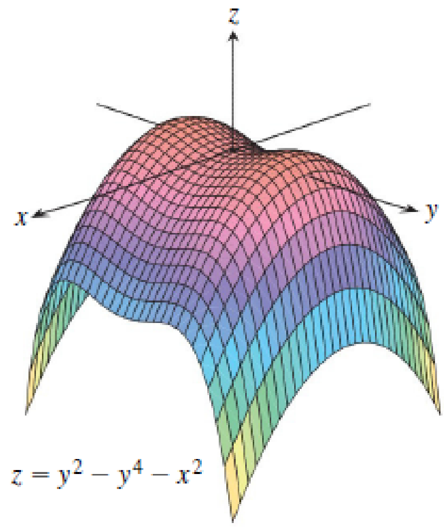



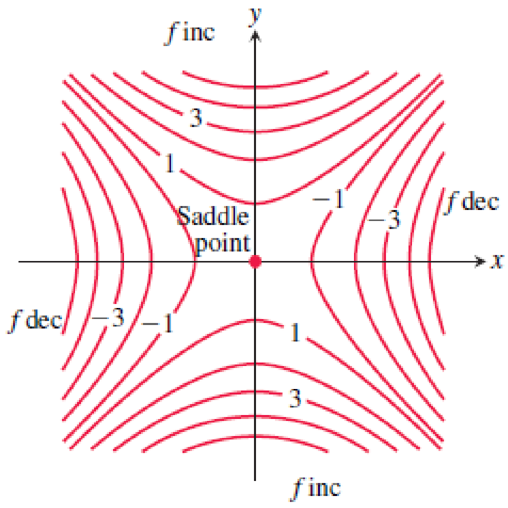

**FIGURE 14.45** Saddle points at the origin.

**EXAMPLE 2** Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

**Solution** The domain of $f$ is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$ where $f_x = 0$ and $f_y = 0$. Along the positive $x$-axis, however, $f$ has the value $f(x, 0) = -x^2 < 0$; along the positive $y$-axis, $f$ has the value $f(0, y) = y^2 > 0$. Therefore, every open disk in the $xy$-plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin and no local extreme values (Figure 14.47a). Figure 14.47b displays the level curves (they are hyperbolas) of $f$, and shows the function decreasing and increasing in an alternating fashion among the four groupings of hyperbolas. ■

**THEOREM 11—Second Derivative Test for Local Extreme Values**
Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at $(a, b)$ and that $f_x(a, b) = f_y(a, b) = 0$. Then

**i)** $f$ has a **local maximum** at $(a, b)$ if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b)$.
**ii)** $f$ has a **local minimum** at $(a, b)$ if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b)$.
**iii)** $f$ has a **saddle point** at $(a, b)$ if $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at $(a, b)$.
**iv)** **the test is inconclusive** at $(a, b)$ if $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at $(a, b)$. In this case, we must find some other way to determine the behavior of $f$ at $(a, b)$.

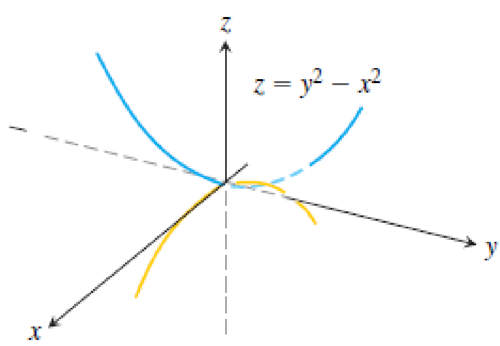



(b)

**FIGURE 14.47** (a) The origin is a saddle point of the function $f(x, y) = y^2 - x^2$. There are no local extreme values (Example 2). (b) Level curves for the function $f$ in Example 2.

The expression $f_{xx}f_{yy} - f_{xy}^{\ 2}$ is called the **discriminant** or **Hessian** of $f$. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^{\ 2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$




Theorem 11 says that if the discriminant is positive at the point $(a, b)$, then the surface curves the same way in all directions: downward if $f_{xx} < 0$, giving rise to a local maximum, and upward if $f_{xx} > 0$, giving a local minimum. On the other hand, if the discriminant is negative at $(a, b)$, then the surface curves up in some directions and down in others, so we have a saddle point.

**EXAMPLE 3** Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

**Solution** The function is defined and differentiable for all $x$ and $y$, and its domain has no boundary points. The function therefore has extreme values only at the points where $f_x$ and $f_y$ are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where $f$ may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of $f$ at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that $f$ has a local maximum at $(-2, -2)$. The value of $f$ at this point is $f(-2, -2) = 8$. ■

**EXAMPLE 4** Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

**Solution** Since $f$ is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for $y$ into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$.

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point (0, 0) we see that the value of the discriminant is the negative number $-72$, so the function has a saddle point at the origin. At the critical point (2, 2) we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point (2, 2) gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■

**EXAMPLE 5** Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

**Solution** First we find the partial derivatives $f_x$ and $f_y$ and set them simultaneously to zero in seeking the critical points:

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1 - 2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0,$$
$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1 - 2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$f_{xx} = -20xy(1 - 2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2x^2)e^{-(x^2+y^2)},$$
$$f_{xy} = f_{yx} = 10(1 - 2x^2)e^{-(x^2+y^2)} - 20y^2(1 - 2x^2)e^{-(x^2+y^2)} = 10(1 - 2x^2)(1 - 2y^2)e^{-(x^2+y^2)},$$
$$f_{yy} = -20xy(1 - 2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3 - 2y^2)e^{-(x^2+y^2)}.$$

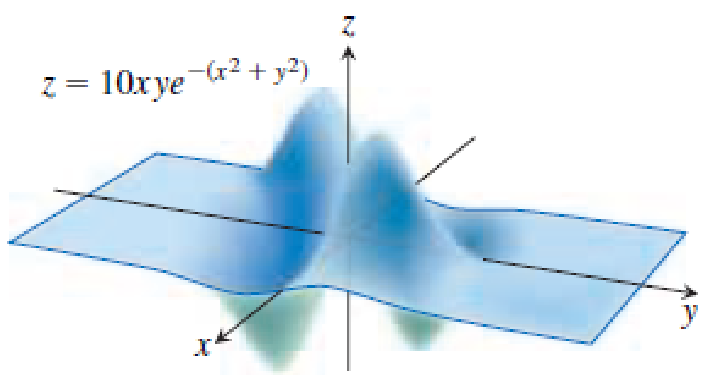



**FIGURE 14.49** A graph of the function in Example 5.

The following table summarizes the values needed by the Second Derivative Test.

| Critical Point | $f_{xx}$ | $f_{xy}$ | $f_{yy}$ | Discriminant $D$ |
|---|---|---|---|---|
| $(0, 0)$ | $0$ | $10$ | $0$ | $-100$ |
| $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ | $-\frac{20}{e}$ | $0$ | $-\frac{20}{e}$ | $\frac{400}{e^2}$ |
| $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ | $\frac{20}{e}$ | $0$ | $\frac{20}{e}$ | $\frac{400}{e^2}$ |
| $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ | $\frac{20}{e}$ | $0$ | $\frac{20}{e}$ | $\frac{400}{e^2}$ |
| $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ | $-\frac{20}{e}$ | $0$ | $-\frac{20}{e}$ | $\frac{400}{e^2}$ |

From the table we find that $D < 0$ at the critical point $(0, 0)$, giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$, each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

## Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region $R$ into three steps.

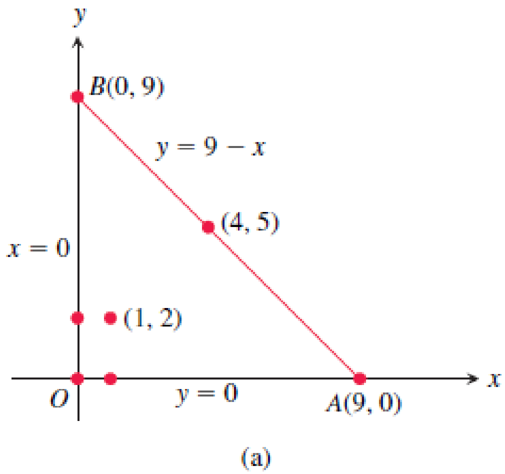
1. *List the interior points of* $R$ where $f$ may have local maxima and minima and evaluate $f$ at these points. These are the critical points of $f$.
2. *List the boundary points of* $R$ where $f$ has local maxima and minima and evaluate $f$ at these points. We show how to do this in the next example.
3. *Look through the lists* for the maximum and minimum values of $f$. These will be the absolute maximum and minimum values of $f$ on $R$.

**EXAMPLE 6** Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

**Solution** Since $f$ is differentiable, the only places where $f$ can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary (Figure 14.50a).



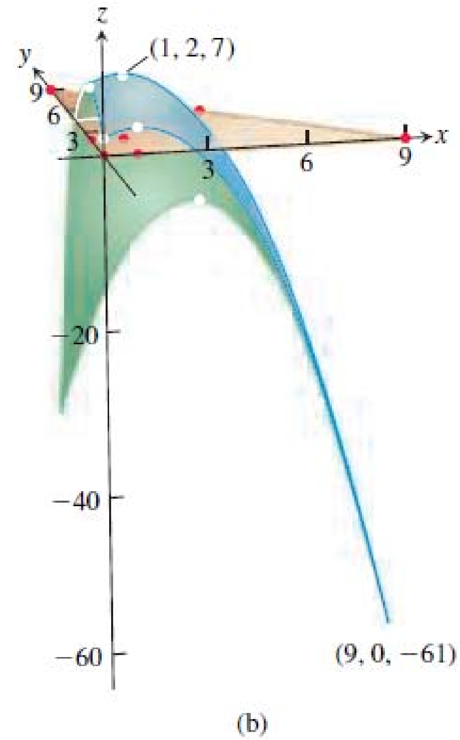




**FIGURE 14.50** (a) This triangular region is the domain of the function in Example 6. (b) The graph of the function in Example 6. The blue points are the candidates for maxima or minima.

**(a) Interior points.** For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$

yielding the single point $(x, y) = (1, 2)$. The value of $f$ there is

$$f(1, 2) = 7.$$

**(b) Boundary points.** We take the triangle one side at a time:

**i)** On the segment $OA$, $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of $x$ defined on the closed interval $0 \leq x \leq 9$. Its extreme values (as we know from Chapter 4) may occur at the endpoints

$$x = 0 \quad \text{where} \quad f(0, 0) = 2$$
$$x = 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

or at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

**ii)** On the segment $OB$, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 4y - y^2.$$

As in part i), we consider $f(0, y)$ as a function of $y$ defined on the closed interval $[0, 9]$. Its extreme values can occur at the endpoints or at interior points where $f'(0, y) = 0$. Since $f'(0, y) = 4 - 2y$, the only interior point where $f'(0, y) = 0$ occurs at $(0, 2)$, with $f(0, 2) = 6$. So the candidates for this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -43, \qquad f(0, 2) = 6.$$

**iii)** We have already accounted for the values of $f$ at the endpoints of $AB$, so we need only look at the interior points of the line segment $AB$. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

Setting $f'(x, 9 - x) = 16 - 4x = 0$ gives

$$x = 4.$$

At this value of $x$,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$

**Summary** We list all the function value candidates: $7, 2, -61, 3, -43, 6, -11$. The maximum is 7, which $f$ assumes at $(1, 2)$. The minimum is $-61$, which $f$ assumes at $(9, 0)$. See Figure 14.50b.

**EXAMPLE 7** A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

**Solution** Let $x$, $y$, and $z$ represent the length, width, and height of the rectangular box, respectively. Then the girth is $2y + 2z$. We want to maximize the volume $V = xyz$ of the box (Figure 14.51) satisfying $x + 2y + 2z = 108$ (the largest box accepted by the delivery company). Thus, we can write the volume of the box as a function of two variables:

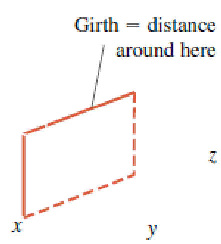



**FIGURE 14.51** The box in Example 7.

$$\begin{aligned} V(y, z) &= (108 - 2y - 2z)yz \\ &= 108yz - 2y^2z - 2yz^2. \end{aligned}$$

$V = xyz$ and
$x = 108 - 2y - 2z$

Setting the first partial derivatives equal to zero,

$$\begin{aligned} V_y(y, z) &= 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0 \\ V_z(y, z) &= 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0, \end{aligned}$$

gives the critical points $(0, 0)$, $(0, 54)$, $(54, 0)$, and $(18, 18)$. The volume is zero at $(0, 0)$, $(0, 54)$, and $(54, 0)$, which are not maximum values. At the point $(18, 18)$, we apply the Second Derivative Test (Theorem 11):

$$V_{yy} = -4z, \qquad V_{zz} = -4y, \qquad V_{yz} = 108 - 4y - 4z.$$

Then

$$V_{yy}V_{zz} - V_{yz}{}^2 = 16yz - 16(27 - y - z)^2.$$

Thus,

$$V_{yy}(18, 18) = -4(18) < 0$$

and

$$\left(V_{yy}V_{zz} - V_{yz}{}^2\right)\Big|_{(18,\,18)} = 16(18)(18) - 16(-9)^2 > 0,$$

so $(18, 18)$ gives a maximum volume. The dimensions of the package are $x = 108 - 2(18) - 2(18) = 36$ in., $y = 18$ in., and $z = 18$ in. The maximum volume is $V = (36)(18)(18) = 11{,}664\ \text{in}^3$, or $6.75\ \text{ft}^3$.

### Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of $f$

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where $f_x$ or $f_y$ fails to exist)

If the first- and second-order partial derivatives of $f$ are continuous throughout a disk centered at a point $(a, b)$ and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**

ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive**

**39.** Find two numbers $a$ and $b$ with $a \leq b$ such that

$$\int_a^b (6 - x - x^2)\, dx$$

has its largest value.

**40.** Find two numbers $a$ and $b$ with $a \leq b$ such that

$$\int_a^b (24 - 2x - x^2)^{1/3}\, dx$$

has its largest value.

**41. Temperatures** A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point $(x, y)$ is

$$T(x, y) = x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

Hint: (41) Use Polar coordinates

<u>Optional Study for Statistical fitting of data :</u>

**67. Least squares and regression lines** When we try to fit a line $y = mx + b$ to a set of numerical data points $(x_1, y_1)$, $(x_2, y_2), \ldots, (x_n, y_n)$, we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of $m$ and $b$ that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \qquad (1)$$

(See the accompanying figure.) Show that the values of $m$ and $b$ that do this are

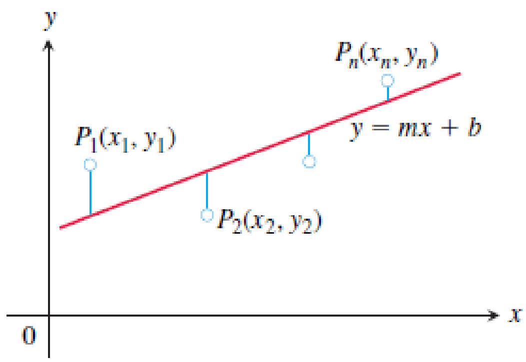
$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n \sum x_k y_k}{\left(\sum x_k\right)^2 - n \sum x_k{}^2}, \qquad (2)$$

$$b = \frac{1}{n}\left(\sum y_k - m \sum x_k\right), \qquad (3)$$

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find $m$ and $b$ with only a few keystrokes after you have entered the data.

The line $y = mx + b$ determined by these values of $m$ and $b$ is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of $y$ for other, experimentally untried values of $x$,
3. handle data analytically.




# 14.8 Lagrange Multipliers

## Constrained Maxima and Minima

**EXAMPLE 1** Find the point $p(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

**Solution** The problem asks us to find the minimum value of the function

$$|\overrightarrow{OP}| = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

Since $|\overrightarrow{OP}|$ has a minimum value wherever the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

has a minimum value, we may solve the problem by finding the minimum value of $f(x, y, z)$ subject to the constraint $2x + y - z - 5 = 0$ (thus avoiding square roots). If we regard $x$ and $y$ as the independent variables in this equation and write $z$ as

$$z = 2x + y - 5,$$

our problem reduces to one of finding the points $(x, y)$ at which the function

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$$

has its minimum value or values. Since the domain of $h$ is the entire $xy$-plane, the First Derivative Test of Section 14.7 tells us that any minima that $h$ might have must occur at points where

$$h_x = 2x + 2(2x + y - 5)(2) = 0, \qquad h_y = 2y + 2(2x + y - 5) = 0.$$

This leads to

$$10x + 4y = 20, \qquad 4x + 4y = 10,$$

which has the solution

$$x = \frac{5}{3}, \qquad y = \frac{5}{6}.$$

We may apply a geometric argument together with the Second Derivative Test to show that these values minimize $h$. The $z$-coordinate of the corresponding point on the plane $z = 2x + y - 5$ is

$$z = 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}.$$

Therefore, the point we seek is

$$\text{Closest point:} \qquad P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The distance from $P$ to the origin is $5/\sqrt{6} \approx 2.04$. ■

**<u>The following problem is a very tricky one if you follow the above method :: There is an easy approach to this using Lagrangian X's</u>**

EXAMPLE 2 Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

Solution 1 The cylinder is shown in Figure 14.52. We seek the points on the cylinder closest to the origin. These are the points whose coordinates minimize the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{Square of the distance}$$

subject to the constraint that $x^2 - z^2 - 1 = 0$. If we regard $x$ and $y$ as independent variables in the constraint equation, then

$$z^2 = x^2 - 1$$

(Do the rest of the steps on your own in the same lines as above method and realize, there is a hitch.)

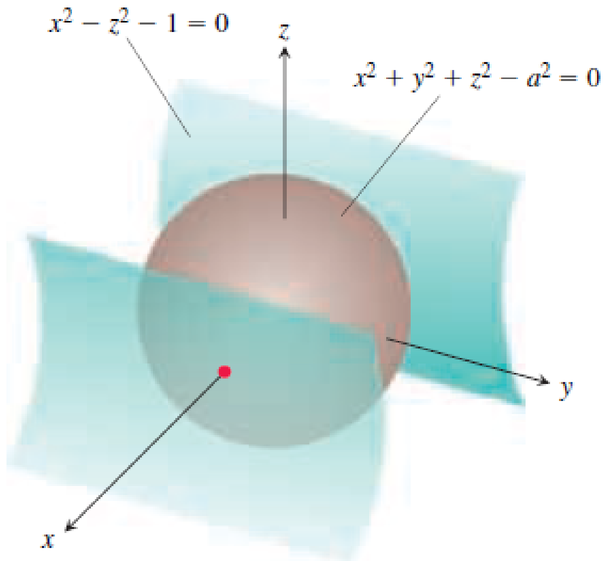



**FIGURE 14.54** A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).

**Solution 2** Another way to find the points on the cylinder closest to the origin is to imagine a small sphere centered at the origin expanding like a soap bubble until it just touches the cylinder (Figure 14.54). At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \quad \text{and} \quad g(x, y, z) = x^2 - z^2 - 1$$

equal to 0, then the gradients $\nabla f$ and $\nabla g$ will be parallel where the surfaces touch. At any point of contact, we should therefore be able to find a scalar $\lambda$ ("lambda") such that

$$\nabla f = \lambda \nabla g,$$

or

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2z\mathbf{k}).$$

Thus, the coordinates $x$, $y$, and $z$ of any point of tangency will have to satisfy the three scalar equations

$$2x = 2\lambda x, \qquad 2y = 0, \qquad 2z = -2\lambda z.$$

For what values of $\lambda$ will a point $(x, y, z)$ whose coordinates satisfy these scalar equations also lie on the surface $x^2 - z^2 - 1 = 0$? To answer this question, we use our knowledge that no point on the surface has a zero $x$-coordinate to conclude that $x \neq 0$. Hence, $2x = 2\lambda x$ only if

$$2 = 2\lambda, \qquad \text{or} \qquad \lambda = 1.$$

For $\lambda = 1$, the equation $2z = -2\lambda z$ becomes $2z = -2z$. If this equation is to be satisfied as well, $z$ must be zero. Since $y = 0$ also (from the equation $2y = 0$), we conclude that the points we seek all have coordinates of the form

$$(x, 0, 0).$$

What points on the surface $x^2 - z^2 = 1$ have coordinates of this form? The answer is the points $(x, 0, 0)$ for which

$$x^2 - (0)^2 = 1, \qquad x^2 = 1, \qquad \text{or} \qquad x = \pm 1.$$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. ■

## The Method of Lagrange Multipliers

**The Method of Lagrange Multipliers**

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of $f$ subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of $x$, $y$, $z$, and $\lambda$ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable $z$.

**EXAMPLE 3** Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

**Solution** We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of $x$, $y$, and $\lambda$ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

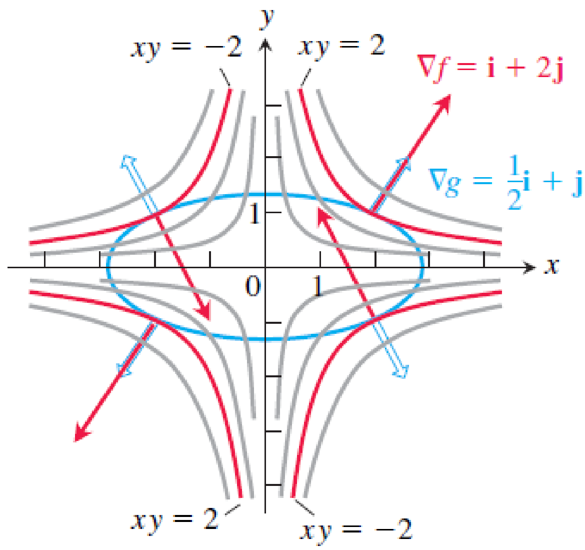



**FIGURE 14.56** When subjected to the constraint $g(x, y) = x^2/8 + y^2/2 - 1 = 0$, the function $f(x, y) = xy$ takes on extreme values at the four points $(\pm 2, \pm 1)$. These are the points on the ellipse where $\nabla f$ (red) is a scalar multiple of $\nabla g$ (blue) (Example 3).

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

**Case 1:** If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

**Case 2:** If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1), (\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

**EXAMPLE 4** Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

**Solution** We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of $x$, $y$, and $\lambda$ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$
$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

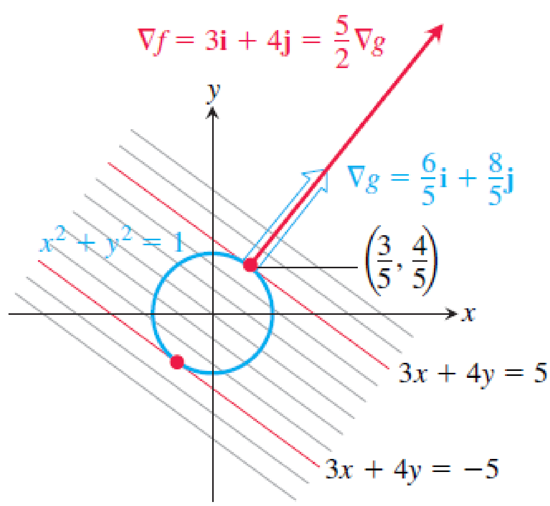



**FIGURE 14.57** The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle $g(x, y) = x^2 + y^2 - 1 = 0$ at the point $(3/5, 4/5)$ and its smallest value at the point $(-3/5, -4/5)$ (Example 4). At each of these points, $\nabla f$ is a scalar multiple of $\nabla g$. The figure shows the gradients at the first point but not the second.

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that $x$ and $y$ have the same sign. With these values for $x$ and $y$, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

**Optional Reading:**

## Lagrange Multipliers with Two Constraints

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \qquad \text{and} \qquad g_2(x, y, z) = 0$$

and $g_1$ and $g_2$ are differentiable, with $\nabla g_1$ not parallel to $\nabla g_2$, we find the constrained local maxima and minima of $f$ by introducing two Lagrange multipliers $\lambda$ and $\mu$ (mu, pronounced "mew"). That is, we locate the points $P(x, y, z)$ where $f$ takes on its constrained extreme values by finding the values of $x, y, z, \lambda$, and $\mu$ that simultaneously satisfy the three equations

$$\boxed{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0 \qquad (2)}$$

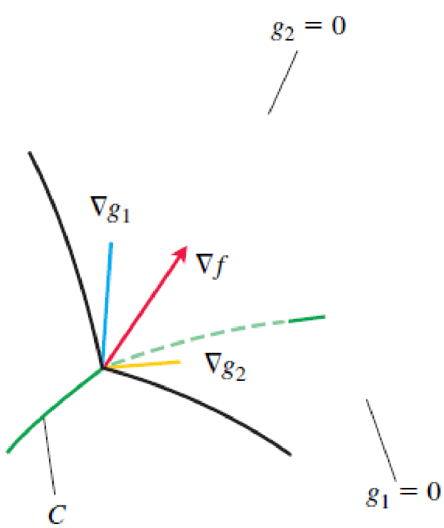



**FIGURE 14.58** The vectors $\nabla g_1$ and $\nabla g_2$ lie in a plane perpendicular to the curve $C$ because $\nabla g_1$ is normal to the surface $g_1 = 0$ and $\nabla g_2$ is normal to the surface $g_2 = 0$.

**THEOREM 12—The Orthogonal Gradient Theorem**
Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If $P_0$ is a point on $C$ where $f$ has a local maximum or minimum relative to its values on $C$, then $\nabla f$ is orthogonal to $C$ at $P_0$.

Equations (2) have a nice geometric interpretation. The surfaces $g_1 = 0$ and $g_2 = 0$ (usually) intersect in a smooth curve, say $C$ (Figure 14.58). Along this curve we seek the points where $f$ has local maximum and minimum values relative to its other values on the curve. These are the points where $\nabla f$ is normal to $C$, as we saw in Theorem 12. But $\nabla g_1$ and $\nabla g_2$ are also normal to $C$ at these points because $C$ lies in the surfaces $g_1 = 0$ and $g_2 = 0$. Therefore, $\nabla f$ lies in the plane determined by $\nabla g_1$ and $\nabla g_2$, which means that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ for some $\lambda$ and $\mu$. Since the points we seek also lie in both surfaces, their coordinates must satisfy the equations $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, which are the remaining requirements in Equations (2).

29. **Hottest point on a space probe** A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point $(x, y, z)$ on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. **Extreme temperatures on a sphere** Suppose that the Celsius temperature at the point $(x, y, z)$ on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

34. **Blood types** Human blood types are classified by three gene forms $A$, $B$, and $O$. Blood types $AA$, $BB$, and $OO$ are *homozygous*, and blood types $AB$, $AO$, and $BO$ are *heterozygous*. If $p$, $q$, and $r$ represent the proportions of the three gene forms to the population, respectively, then the *Hardy-Weinberg Law* asserts that the proportion $Q$ of heterozygous persons in any specific population is modeled by

$$Q(p, q, r) = 2(pq + pr + qr),$$

subject to $p + q + r = 1$. Find the maximum value of $Q$.