Towards deep learning with spiking neurons in energy based models with contrastive Hebbian plasticity

Thomas Mesnard
École Normale Supérieure
Paris, France
thomas.mesnard@ens.fr

Wulfram Gerstner
École Polytechnique Fédéral de Lausanne
1015 Lausanne, Switzerland
wulfram.gerstner@epfl.ch

Johanni Brea
École Polytechnique Fédéral de Lausanne
1015 Lausanne, Switzerland
johanni.brea@epfl.ch

Abstract

In machine learning, error back-propagation in multi-layer neural networks (deep learning) has been impressively successful in supervised and reinforcement learning tasks. As a model for learning in the brain, however, deep learning has long been regarded as implausible, since it relies in its basic form on a non-local plasticity rule. To overcome this problem, energy-based models with local contrastive Hebbian learning were proposed and tested on a classification task with networks of rate neurons. We extended this work by implementing and testing such a model with networks of leaky integrate-and-fire neurons. Preliminary results indicate that it is possible to learn a non-linear regression task with hidden layers, spiking neurons and a local synaptic plasticity rule.

1 Introduction

Error back-propagation is an efficient and vastly used algorithm to train deep neural networks [7]. Although the architecture of deep learning was inspired by biological neural networks, adaptation of this powerful algorithm for training deep <u>spiking</u> neural networks started to appear only recently [1, 4].

There are several reasons why we are still waiting for a full "integration of deep learning and neuroscience" [8]. In standard deep learning with feed-forward networks, rate-based neurons with real-valued, non-negative output are layer-wise sequentially activated to compute the prediction of the network given a certain input. After comparing this prediction with a target value, errors are back-propagated in the reversed order through the layers, using the transposed of the feed-forward weight matrices, but not the non-linearity of the neurons used in the forward pass to compute the prediction. The update of weights and biases depends both on the neural activations of the forward pass and the back-propagated errors. There is no notion of physical time in standard deep learning. In contrast, biological neural networks consist of spiking neurons with binary outputs that work in continuous time. It is unclear how such a network composed of real spiking neurons could switch between non-linear forward passes and linear backward passes to implement error back-propagation.

Equilibrium propagation was recently proposed by Scellier and Bengio [14] as one way to introduce physical time in deep learning and remove the necessity of different dynamics in forward and backward pass. Their work combines related ideas like recurrent back-propagation [13], infinitesimal perturbation of the outputs [10, 5] or contrastive Hebbian learning [9, 16] and moves these concepts

closer to biology. In a recurrent network of rate-based neurons, whose dynamics is defined by following the negative gradient of an energy function, Scellier and Bengio propose to relax the network to a minimum energy state in the forward phase while fixing the rate of the input neurons at a given value. The rate of the output neurons at the fixed point of the forward phase corresponds to the prediction in standard deep learning. Moving the rate of the output neurons in direction of the target value in the backward phase while keeping the input rates fixed, perturbs also the rate of the hidden neurons, if backward connections exist. The key result of Scellier and Bengio is that supervised learning with this network is possible with a simple contrastive Hebbian plasticity mechanism that subtracts the correlation of firing rates at the fixed point of the forward phase from the correlation of the firing rates after perturbation of the output.

Our contribution is to implement equilibrium propagation in a network of integrate-and-fire neurons and test it on a regression problem. Our model differs slightly from the rate-based model in that neither the input rates get explicitly fixed nor the output rates explicitly moved in direction of the target. Instead, the input neurons receive a constant input current during both phases and the output neurons are treated as two-compartment neurons that receive an extra somatic input current in the backward phase [15]. The contrastive Hebbian plasticity mechanism is implemented with an estimate of the firing rate by low-pass filtering the spike history with a large time constant.

2 Derivation of the learning rule for rate-based neurons

For completeness, we reproduce here the learning rule derivation described by Scellier and Bengio [14].

Let us define the dynamics of a neural network by an energy function

$$2E(s; \hat{s}, \theta, \beta_x, \beta_y) = \sum_i s_i^2 - \sum_{i \neq j} w_{ij} \rho(s_i) \rho(s_j) - 2 \sum_i b_i \rho(s_i) +$$
$$+ \beta_x \sum_{i \in \mathcal{X}} (\hat{s}_i - s_i)^2 + \beta_y \sum_{i \in \mathcal{Y}} (\hat{s}_i - s_i)^2 , \tag{1}$$

where s_i is the state of neuron i, $\rho(s_i)$ its firing rate for some non-linear function ρ, parameters $\theta = (w, b)$, with symmetric connection strengths $w_{ij} = w_{ji}$ and biases b_i, $\mathcal{Y}$ and $\mathcal{X}$ are disjoint subsets of neurons that may receive external input $\hat{s}_i$, if $\beta_x > 0$ or $\beta_y > 0$. Note that the network does not need to have all-to-all connections. By setting some of the weights w_{ij} to zero, a multi-layer architecture can be achieved.

The neural dynamics is given by

$$\tau \dot{s}_i = -\frac{d}{ds_i} E(s; \hat{s}, \theta, \beta_x, \beta_y)$$
$$= -s_i + \rho'(s_i) \left(\sum_j w_{ij} \rho(s_j) + b_i \right) + \mathbb{I}_{i \in \mathcal{X}} \beta_x (\hat{s}_i - s_i) + \mathbb{I}_{i \in \mathcal{Y}} \beta_y (\hat{s}_i - s_i) , \tag{2}$$

with time constant τ and we used the indicator function $\mathbb{I}_{i \in \mathcal{Y}} = 1$ if i is in set $\mathcal{Y}$ and 0 otherwise. Note that for the rectified-linear function, i.e. $\rho(s) = s$ if $s > 0$ and $\rho(s) = 0$ otherwise, the derivative $\rho'(s) = 1$ for $s > 0$. Since for negative s the derivative $\rho'(s) = 0$, s remains non-negative all the time.

The lowest energy state s^* given by

$$\frac{dE}{ds_i}(s^*; \hat{s}, \theta, \infty, 0) = 0, \forall i \tag{3}$$

defines a map $x \mapsto y$, with $x = (s_{i_1}^*, \ldots, s_{i_{N_x}}^*)_{i_k \in \mathcal{X}}$ and $y = (s_{i_1}^*, \ldots, s_{i_{N_y}}^*)_{i_k \in \mathcal{Y}}$. We would like to have a rule for changing the parameters w and b to implement an arbitrary map from x to y.

Let us define a cost function for a single pair of points $\hat{s}$ and s

$$2C(\hat{s}, s) = \sum_{i \in \mathcal{Y}} (\hat{s}_i - s_i)^2 \tag{4}$$

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and a total cost function $C(\hat{s}^1, s^1, \ldots, \hat{s}^N, s^N) = \frac{1}{N} \sum_{\mu=1}^{N} C(\hat{s}^\mu, s^\mu)$.

To find a learning rule, we look at the constraint optimization problem

$$\min_\theta C(\hat{s}^1, s^1, \ldots, \hat{s}^N, s^N) \text{ subject to } \frac{dE}{ds_i}(s^\mu; \hat{s}^\mu, \theta, \infty, 0) = 0, \forall \mu, i \tag{5}$$

and define the Lagrangian for a single data point $\hat{s}$ by

$$\mathcal{L}(s, \lambda, \theta; \hat{s}) = C(\hat{s}, s) + \sum_i \lambda_i \frac{dE}{ds_i}(s; \hat{s}, \theta, \infty, 0) \tag{6}$$

We minimize this Lagrangian by setting the derivatives with respect to s and λ to zero

$$\frac{d\mathcal{L}}{d\lambda_i}(s^*, \lambda^*, \theta; \hat{s}) = 0, \forall i \tag{7}$$

$$\frac{d\mathcal{L}}{ds_i}(s^*, \lambda^*, \theta; \hat{s}) = 0, \forall i \tag{8}$$

and performing stochastic descent on the total cost by changing the parameters according to

$$\Delta \theta_i = -\eta \frac{\partial \mathcal{L}}{\partial \theta_i}(s^*, \lambda^*, \theta; \hat{s}) \tag{9}$$

$$= -\eta \sum_j \lambda_j^* \frac{\partial dE}{\partial \theta_i ds_j}(s^*; \hat{s}, \theta, \infty, 0) \tag{10}$$

where η is a learning rate.

Solving Equation 7, we find that the state s^* is simply given by the lowest energy state (c.f. Equation 3), i.e. it can be obtained by running the dynamics with input x ($\beta_x = \infty$) and no target y ($\beta_y = 0$).

Equation 8 is a bit harder to solve for λ^*. Expanding the definition of the Lagrangian it reads

$$\frac{dC}{ds_i}(\hat{s}, s^*) + \sum_j \lambda_j^* \frac{d^2 E}{ds_i ds_j}(s^*; \hat{s}, \theta, \infty, 0) = 0 \,. \tag{11}$$

The nice idea of Scellier and Bengio [14] is, to look at the fixed point s^β of the dynamics with weak target input $\beta_y = \beta > 0$ given by

$$\frac{dE}{ds_i}(s^\beta; \hat{s}, \theta, \infty, \beta) = 0 \,, \forall \beta \,. \tag{12}$$

Since the left-hand side of this equation is a constant function in β, its derivative with respect to β_y is also zero and evaluated at $\beta_y = 0$ we get

$$0 = \frac{d^2 E}{d\beta_y ds_i} = \frac{\partial dE}{\partial \beta_y ds_i} + \sum_j \frac{ds_j^\beta}{d\beta_y} \frac{d^2 E}{ds_i ds_j} \tag{13}$$

$$= \frac{dC}{ds_i}(\hat{s}, s^*) + \sum_j \frac{ds_j^\beta}{d\beta_y} \frac{d^2 E}{ds_i ds_j}(s^*; \hat{s}, \theta, \infty, 0) \,, \tag{14}$$

which has the same form as Equation 11 if we identify $\lambda_j^* = \frac{ds_j^\beta}{d\beta_y}$.

Using this result in Equation 10, we can write

$$\sum_k \lambda_k^* \frac{\partial dE}{\partial w_{ij} ds_k} = \sum_k \frac{ds_k^\beta}{d\beta_y} \frac{d\partial E}{ds_k \partial w_{ij}} = \frac{d\partial E}{d\beta_y \partial w_{ij}} = -\frac{d}{d\beta_y}\big(\rho(s_i)\rho(s_j)\big) \tag{15}$$

and approximate this by $\frac{d}{d\beta_y}\big(\rho(s_i)\rho(s_j)\big) \approx \frac{1}{\beta}\big(\rho(s_i^\beta)\rho(s_j^\beta) - \rho(s_i^*)\rho(s_j^*)\big)$ which leads to the contrastive Hebbian update rule

$$\Delta w_{ij} = \eta\big(\rho(s_i^\beta)\rho(s_j^\beta) - \rho(s_i^*)\rho(s_j^*)\big) \,, \tag{16}$$

where the factor $1/\beta$ is absorbed into the learning rate η.

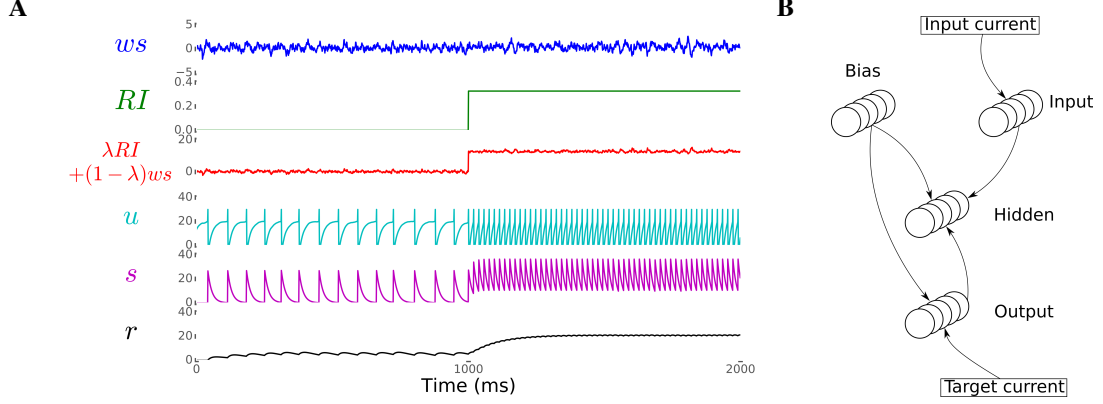This learning rule can be implemented by iterating through:

Figure 1: **Neuron model and network architecture.** **A** Traces of the relevant quantities for a target neuron. The synaptic weights are updated according to $\Delta w_{ij} = \eta(r_i^+ r_j^+ - r_i^- r_j^-)$, where $r_i^+ = r_i(1200\,\mathrm{ms})$ and $r_i^- = r_i(600\,\mathrm{ms})$, which can be implemented online by decreasing the weights appropriately at 600 ms and increase them at 1200 ms. **B** Network architecture. Arrows indicate all-to-all connectivity.

1. Select a data sample $\hat{s}$ and relax the system to the lowest energy state with $\beta_y = 0$ to obtain y^* (Forward phase).

2. Subtract $\eta\rho(s_i^*)\rho(s_j^*)$ from the weights.

3. Set $\beta_y = \beta > 0$ and let the system evolve for some time (Backward phase).[1]

4. Add $\eta\rho(s_i^\beta)\rho(s_j^\beta)$ to the weights.

3 Implementation with leaky integrate-and-fire neurons

To replace rate-based neurons with leaky integrate-and-fire neurons we introduce the somatic membrane potential u_i of neuron i that evolves below a threshold θ as

$$\tau \dot{u}_i(t) = -u_i(t) + u_0 + \big(1 - \lambda_i(t)\big) \sum_j w_{ij} s_j(t) + \lambda_i(t) R I_i(t) \tag{17}$$

with time constant τ, resting potential u_0, post-synaptic response s_j given by the dynamics $\tau_s \dot{s}_j = -s_j + u_{psp} x_j$, with presynaptic spike trains $x_j(t) = \sum_{t_j^{(f)}} \delta(t - t_j^{(f)})$, where $t_j^{(f)}$ are the spike times of neuron j, membrane resistance R and additional current input $I_i(t)$, used in the backward phase to nudge the firing rate of the neuron in direction of the target firing rate. The nudging factor

$$\lambda_i(t) = \frac{R I_i(t)}{R I_i(t) + \sum_j w_{ij} s_j(t)}\,, \tag{18}$$

is important at the end of learning, when the predictions by the network match almost the target inputs and additive instead of convex combination of network input and target input would lead to run-away dynamics. The nudging factor can be motivated with divisive normalization [2] or an argument involving conductance-based synapses [15]. The neuron spikes when its membrane potential reaches threshold θ. The membrane potential is then set to a reset value u_r and kept at this value for a refractory period Δ. Afterwards the dynamics in Eq. 17 determines again the membrane potential.

To implement the contrastive Hebbian learning rule in Equation 16, the pre- and postsynaptic firing rates are estimated in each synapse with hypothesized processes r_i that low-pass filter the spike trains with a large time constant τ_r, i.e.

$$\tau_r \dot{r}_i(t) = -r_i(t) + s_i(t)\,. \tag{19}$$

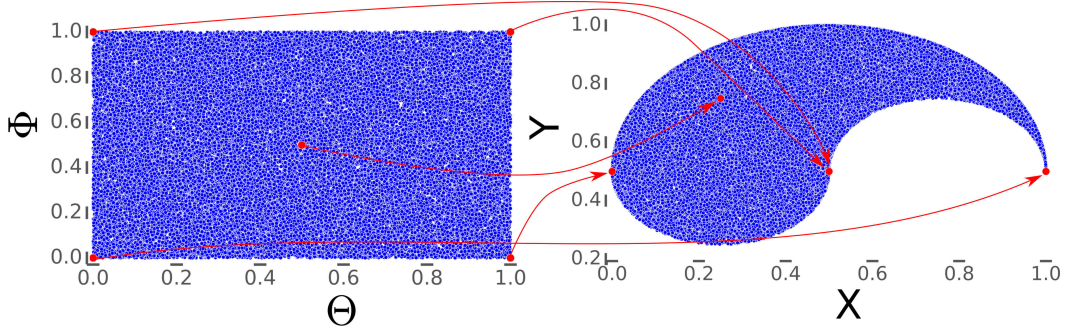[1]Scellier and Bengio [14] observe that relaxation to the fixed-point is not necessary in this second phase.

Figure 2: **A non-linear regression task.** Input values $\theta \in [0, 1]$ and $\varphi \in [0, 1]$ are mapped to target values $x = \big(\cos(\pi\varphi) + \cos(\pi(\varphi + \theta)) + 2 \big)/4$ and $y = \big(\sin(\pi\varphi) + \sin(\pi(\varphi + \theta)) + 2 \big)/4$.

The negative part of the weight update in Equation 16 is applied just before the network receives target input to the output neurons. The positive part of the weight update is applied when the network evolved for some time under the influence of the target input.

Figure 1A shows a trace of the relevant quantities for one target neuron.

The firing rate f of a leaky integrate-and-fire neuron with dynamics $\tau\dot{u} = -u + v$, constant drive v, reset potential u_r, threshold θ and refractory period Δ is given by

$$f(v) = \frac{1}{\tau \log\left(\frac{v - u_r}{v - \theta}\right) + \Delta} \quad \text{if} \quad v > \theta, \quad f(v) = 0 \quad \text{otherwise}. \tag{20}$$

The rate model "closest" to the spiking model uses this non-linearity, i.e. $\rho(v_i) = f(v_i)$ with $v_i = u_0 + \big(1 - \lambda_i(t)\big) \sum_j w_{ij} s_j(t) + \lambda_i(t) R I_i(t)$. Even though the derivative of this non-linearity is not piecewise constant as the one of the rectified-linear function, we did not see a significant difference in simulations when using $\rho'(v) = 1$ for $v > 0$ and $\rho'(v) = 0$ otherwise together with the leaky integrate-and-fire non-linearity f.

For the experiments described in the next section we used a network architecture with one hidden layer (see Figure 1). We used only one hidden layer because the task is learnable with one, but nothing prevents the use of multiple hidden layers. Bias terms are implemented with weights from bias neurons that are active at a constant firing rate. Instead of clamping the input neurons to the input values with $\beta_x \to \infty$, we did not allow any feedback from the other layers to the input layer and provided the input as input currents I_i to the neurons in the input layer.

3.1 Results

In contrast to Scellier and Bengio [14], who tested the model on MNIST classification, we choose a non-linear regression task (see Figure 2). Since energy-based models, like the Hopfield model [6], are well known with parameter settings that create multiple local minima, we were curious to see, if learning is also possible when local minima are undesirable.

3.1.1 Rate-based neurons

In Figure 3, we see that the task can be learned with a single layer of 400 rate-based neurons. Learning is a bit slower when the firing rate f of the leaky integrate-and-fire neuron (c.f. Equation 20) is used instead of the rectified linear function. As expected, the task cannot be learned without this hidden layer.

In these simulations, weights were updated all in once after the backward phase by following Equation 16. With small learning rates, the same results are expected, if the weights are updated online, i.e. after the forward phase by subtracting $\eta\rho(s_i^*)\rho(s_j^*)$ and after the backward phase by adding $\eta\rho(s_i^\beta)\rho(s_j^\beta)$.
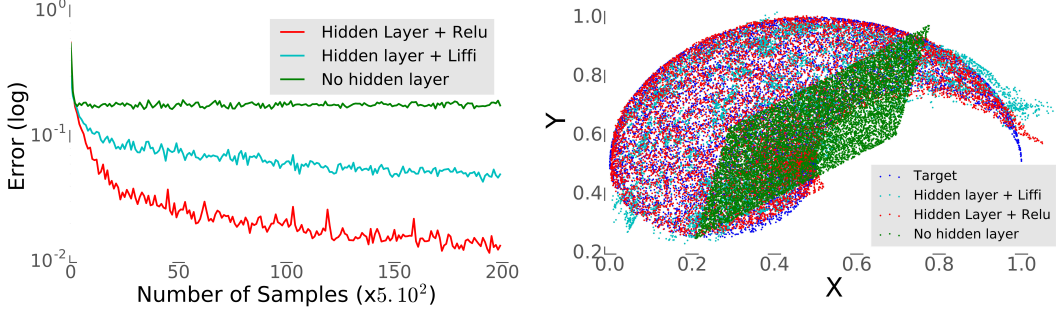
Figure 3: **Learning the regression task with networks of rate-based neurons.** The error is the estimated average Euclidian distance between prediction and target. The differential equations were integrated with Euler method, time steps 1 ms, $\tau = 15$ ms, $\tau_s = 10$ ms, $\tau_r = 300$ ms, $u_0 = 20$, $u_r = 0$, $\theta = 20$, $u_{\text{psp}} = 400$, $R = 40$, $\Delta = 5$ ms, $I_i \in [0, 1]$, duration of forward and backward phase 600 ms, constant learning rates $\eta_i = 0.1/\sqrt{\text{indegree}_i}$, where indegree_i is the number of presynaptic neurons of neuron i. Weights are randomly initialized to small values without symmetry constraints. Note that the learning rule Equation 16 tends to symmetrize the weight matrix. We tested two non-linearities: the rectified linear function (green and red) and the Liffi function defined in Equation 20 (cyan). Preliminary results (not shown) indicate that learning with Liffi is slower, even when learning rates are optimized for each non-linearity.

3.1.2 Spiking neurons

In the spiking implementation each real-valued input and output dimensions was represented by 20 neurons that received identical current inputs and the hidden layer is composed of 300 neurons.

Figure 4 shows a typical result with spiking neurons. Even though the target is not perfectly learned, the predictions are much more accurate than without a hidden layer (c.f. Figure 3). In the simulations with rate-based neurons we noticed the importance of converging close to the fixed point in the forward phase. The poorer results with spiking neurons could be partially due to not having a good enough estimate of the fixed point in the forward phase.
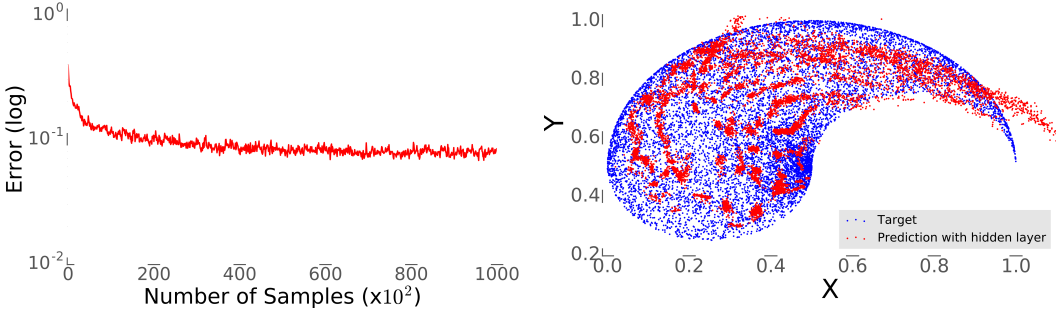

Figure 4: **Learning the regression task with a network of leaky integrate-and-fire neurons.** The output values x and y are given by the average firing rate of 20 neurons. The differential equations were integrated with Euler method, time steps 1 ms, $\tau = 15$ ms, $\tau_s = 15$ ms, $\tau_r = 100$ ms, $u_0 = 20$, $u_r = 0$, $\theta = 20$, $u_{\text{psp}} = 400$, $R = 40$, $\Delta = 5$ ms, $I_i \in [0, 1]$, duration of forward and backward phase 1000 ms, constant learning rates $\eta_i = 5.10^{-5}/\sqrt{\text{indegree}_i}$, where indegree_i is the number of presynaptic neurons of neuron i.

4 Discussion

We have implemented Equilibrium Propagation [14] with a multilayer network of leaky integrate-and-fire neurons and demonstrated that the network can learn a non-linear regression task. The results with the spiking networks are not as convincing as with the rate-based networks, which could be due to non-optimal estimates of the fixed point in the forward phase.

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The locality of the contrastive Hebbian plasticity rule and the error back-propagation by natural recurrent dynamics make this approach appealing from the perspective of biological plausibility. The plasticity mechanism requires quite precisely timed induction of anti-Hebbian and Hebbian plasticity. Potential processes to implement this involve theta waves [11] or neuromodulators [12, 3], known for modulating and even sign-reversing synaptic plasticity.

Equilibrium propagation is not the only attempt to implement error back-propagation in a more biologically plausible way. A recent and compelling approach [4] postulates an implementation of error back-propagation using the computational richness of multi-compartment neurons.

It is exciting to see the theoretical approaches that try to integrate deep learning and neuroscience. Ultimately, experiments are needed to support or falsify these hypotheses.

Acknowledgments

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