Finding Area

EE24BTECH11026 - G. Srihaas

Problem Statement

Find the area of the region $\left\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\right\}$.

Solution

NUMERICAL METHOD

Variable	**Description**
$y^2 = 4x$	Parabola
$4x^2 + 4y^2 = 9$	Circle

The general equation of a parabola with directrix $n^\top x = c$ is given by,

$$g(x) = x^\top Vx + 2u^\top x + f = 0 \tag{1}$$

$$V = \|n\|^2 I - e^2 nn^\top \tag{2}$$

$$u = ce^2 n - \|n\|^2 F \tag{3}$$

$$f = \|n\|^2 \|F\|^2 - c^2 e^2 \tag{4}$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} 1 & 0 \end{pmatrix} x = -1$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$u = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{6}$$

$$f = 0 \tag{7}$$

The given circle can be expressed as conics with parameters

$$V = \frac{1}{4}\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \tag{8}$$

$$u = 0 \tag{9}$$

$$f = -\frac{81}{16} \tag{10}$$

The intersection of two conics with parameters $V_i, u_i, f_i, i = 1, 2$ is defined as,

$$x^\top (V_1 + \mu V_2) x + 2 (u_1 + \mu u_2)^\top x + (f_1 + \mu f_2) = 0 \tag{11}$$

$$\mu = -\frac{4}{9} \tag{12}$$

$$\tag{13}$$

Substituting the values:

- $\mu = -\frac{4}{9}$ - $V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ - $V_2 = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix}$ - $f_1 = 0$ - $f_2 = -\frac{81}{16}$ - $u_1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ - $u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$V_1 + \mu V_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \left(-\frac{4}{9}\right)\begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

Thus,

$$x^\top(V_1 + \mu V_2)x = -x_1^2 \tag{16}$$

$$u_1 + \mu u_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{17}$$

Thus,

$$2(u_1 + \mu u_2)^\top x = -4x_1 \tag{18}$$

The constant term:

$$f_1 + \mu f_2 = \frac{9}{4} \tag{19}$$

Putting everything together:

$$-x_1^2 - 4x_1 + \frac{9}{4} = 0 \tag{20}$$

Multiplying through by 4:

$$-4x_1^2 - 16x_1 + 9 = 0 \tag{21}$$

Multiplying through by -1:

Thus, the solutions for x_1 are:

$$x_1 = \frac{1}{2} \quad \text{or} \quad x_1 = -\frac{9}{2} \tag{27}$$

But $x_1 = -\frac{9}{2}$ cannot be a solution as y is imaginary when it is substituted.Hence $x_1 = \frac{1}{2}$.Substitute it in any one of the equations of circe or parabola.

Given the function

$$g(x) = x^\top Vx + 2u^\top x + f \tag{28}$$

where

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad f = 0, \tag{29}$$

substitute $x = \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix}$ into the function. First, calculate $x^\top Vx$:

$$x^\top Vx = \begin{pmatrix} \frac{1}{2} & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix} = y^2. \tag{30}$$

Next, calculate $2u^\top x$:

$$2u^\top x = 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix} = -2. \tag{31}$$

$$g\left(\frac{1}{2}, y\right) = y^2 - 2 = 0 \tag{32}$$

$$y^2 = 2 \tag{33}$$

The desired area of region is given as

$$= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx\right] \tag{36}$$

$$= 2\left[\frac{4}{3}\sqrt{x^3}\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}} \tag{37}$$

$$= \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) \tag{38}$$

Computational Logic

Using the trapezoidal rule to get the area. The trapezoidal rule is as follows.

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{N} \frac{f(x_{k+1}) + f(x_k)}{2} h \tag{39}$$

where

$$h = \frac{b-a}{N} \tag{40}$$

$\therefore$The difference equation obtained is

$$A = \int_a^b f(x)\, dx \approx h\left(\frac{1}{2}f(a) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right) \tag{41}$$

(42)

$$A = j_n, \text{ where, } j_{i+1} = j_i + h\frac{f(x_{i+1}) + f(x_i)}{2} \tag{43}$$

$$\text{i.e} \tag{44}$$

$$A = h\sum \frac{f(x_{i+1}) + f(x_i)}{2} \tag{45}$$

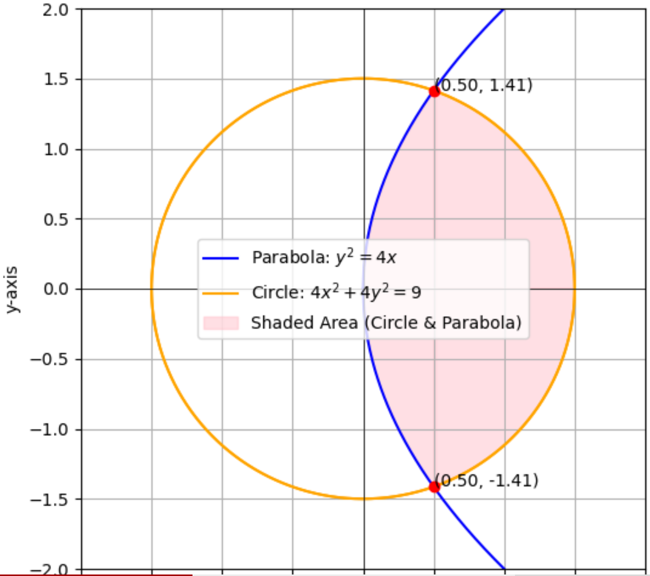
$$x_{i+1} = x_i + h \tag{46}$$

$$\tag{47}$$

where,$a = 0.5$, $b = -0.5$, $h = 0.002$,taking $n = 500$

$$f(x) = \sqrt{\frac{9}{4} - x^2} - 2\sqrt{x} \tag{48}$$

Therefore, Required area =3.0053



2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
y-axis
(0.50, 1.41)
(0.50, -1.41)
Parabola: $y^2 = 4x$
Circle: $4x^2 + 4y^2 = 9$
Shaded Area (Circle & Parabola)

CODES

https://github.com/Srihaas15/EE1003/blob/e8c9f7178b44ad3d69c5fb531f62c
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