9-3-15

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QUESTION:
Find the area of the region $\left\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\right\}$.
SOLUTION:

Variable	Description
$y^2 = 4x$	Parabola
$4x^2 + 4y^2 = 9$	Circle

The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$
$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$
$$f = 0$$

The given circle can be expressed as conics with parameters

$$\mathbf{V} = \frac{1}{4}\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$
$$\mathbf{u} = 0$$
$$f = -\frac{81}{16}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^\top \left(\mathbf{V_1} + \mu \mathbf{V_2}\right) \mathbf{x} + 2\left(\mathbf{u_1} + \mu \mathbf{u_2}\right)^\top \mathbf{x} + \left(f_1 + \mu f_2\right) = 0$$
$$\mu = -\frac{4}{9}$$

On solving we get the points of intersection to be $\left(\frac{1}{2} \atop \sqrt{2}\right), \left(\frac{1}{2} \atop -\sqrt{2}\right)$

The desired area of region is given as

$$= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx\right]$$

$$= 2\left[\frac{4}{3}\sqrt{x^3}\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

Computational Logic: Using the trapezoidal rule to get the area. The trapezoidal rule is as follows.

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{N} \frac{f(x_{k+1}) + f(x_k)}{2} h \tag{0.1}$$

where

$$h = \frac{b-a}{N} \tag{0.2}$$

$\therefore$The difference equation obtained is

$$A = \int_a^b f(x)\ dx \approx h\left(\frac{1}{2} f(a) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2} f(b)\right) \tag{0.3}$$

$$h = \frac{b-a}{n} \tag{0.4}$$

$$A = j_n,\ \text{where},\ j_{i+1} = j_i + h\frac{f(x_{i+1}) + f(x_i)}{2} \tag{0.5}$$

$$\rightarrow j_{i+1} = j_i + h\left(x_{i+1}^2 + x_i^2\right) \tag{0.6}$$

$$x_{i+1} = x_i + h \tag{0.7}$$

$$\tag{0.8}$$

where,$a = 0.5$, $b = -0.5$, $h = 0.002$,taking $n = 500$

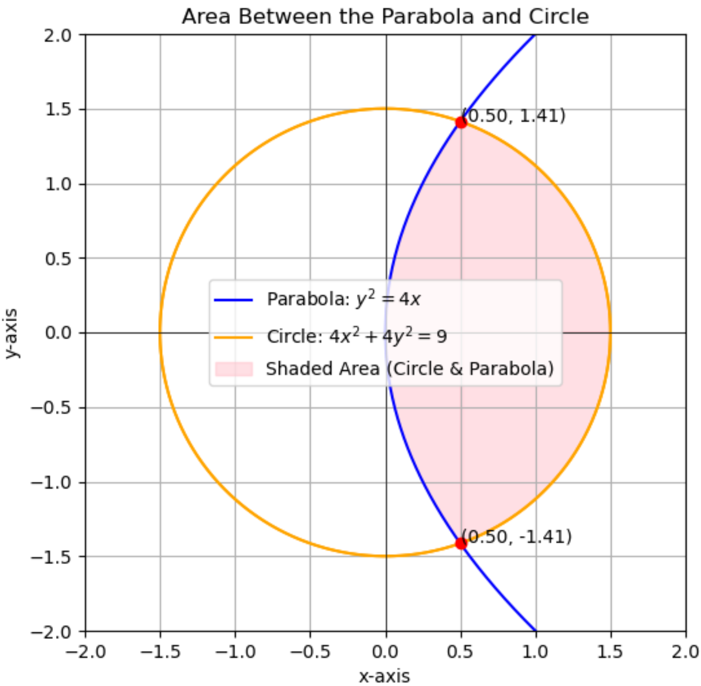

Fig. 0.1: A plot of the given question.