9-3-15

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QUESTION:
Find the area of the region $\left\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\right\}$.
SOLUTION:

Variable	**Description**
$y^2 = 4x$	Parabola
$4x^2 + 4y^2 = 9$	Circle

The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$\begin{aligned} g(\mathbf{x}) &= \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \\ \mathbf{V} &= \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \\ \mathbf{u} &= ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \\ f &= \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \end{aligned}$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} = -1$

$$\begin{aligned} \mathbf{V} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{u} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} \\ f &= 0 \end{aligned}$$

The given circle can be expressed as conics with parameters

$$\begin{aligned} \mathbf{V} &= \frac{1}{4}\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} \\ \mathbf{u} &= 0 \\ f &= -\frac{81}{16} \end{aligned}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^\top (\mathbf{V_1} + \mu\mathbf{V_2})\mathbf{x} + 2(\mathbf{u_1} + \mu\mathbf{u_2})^\top \mathbf{x} + (f_1 + \mu f_2) = 0$$

$$\mu = -\frac{4}{9}$$

Substituting the values:

- $\mu = -\frac{4}{9}$ - $V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ - $V_2 = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix}$ - $f_1 = 0$ - $f_2 = -\frac{81}{16}$ - $u_1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ - $u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$V_1 + \mu V_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \left(-\frac{4}{9}\right)\begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix} \tag{0.1}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.2}$$

Thus,

$$x^\top (V_1 + \mu V_2) x = -x_1^2 \tag{0.3}$$

$$u_1 + \mu u_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{0.4}$$

Thus,

$$2(u_1 + \mu u_2)^\top x = -4x_1 \tag{0.5}$$

The constant term:

$$f_1 + \mu f_2 = \frac{9}{4} \tag{0.6}$$

Putting everything together:

$$-x_1^2 - 4x_1 + \frac{9}{4} = 0 \tag{0.7}$$

Multiplying through by 4:

$$-4x_1^2 - 16x_1 + 9 = 0 \tag{0.8}$$

Multiplying through by -1:

$$4x_1^2 + 16x_1 - 9 = 0 \tag{0.9}$$

Solving for x_1 using the quadratic formula:

$$x_1 = \frac{-16 \pm \sqrt{16^2 - 4(4)(-9)}}{2(4)} \tag{0.10}$$

$$= \frac{-16 \pm \sqrt{256 + 144}}{8} \tag{0.11}$$

$$= \frac{-16 \pm \sqrt{400}}{8} \tag{0.12}$$

$$= \frac{-16 \pm 20}{8} \tag{0.13}$$

Thus, the solutions for x_1 are:

$$x_1 = \frac{1}{2} \quad \text{or} \quad x_1 = -\frac{9}{2} \tag{0.14}$$

But $x_1 = -\frac{9}{2}$ cannot be a solution as y is imaginary when it is substituted.Hence $x_1 = \frac{1}{2}$.Substitute it in any one of the equations of circe or parabola.
Given the function

$$g(x) = x^\top Vx + 2u^\top x + f \tag{0.15}$$

where

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad f = 0, \tag{0.16}$$

substitute $x = \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix}$ into the function. First, calculate $x^\top Vx$:

$$x^\top Vx = \begin{pmatrix} \frac{1}{2} & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix} = y^2. \tag{0.17}$$

Next, calculate $2u^\top x$:

$$2u^\top x = 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ y \end{pmatrix} = -2. \tag{0.18}$$

$$g\left(\frac{1}{2}, y\right) = y^2 - 2 = 0 \tag{0.19}$$

$$y^2 = 2 \tag{0.20}$$

$$y = \pm\sqrt{2} \tag{0.21}$$

Thus, the solutions for y are:

$$y = \sqrt{2} \quad \text{or} \quad y = -\sqrt{2}. \tag{0.22}$$

On solving we get the points of intersection to be $\begin{pmatrix} \frac{1}{2} \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\sqrt{2} \end{pmatrix}$
The desired area of region is given as

$$\begin{aligned}
&= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2\,dx}\right] \\
&= 2\left[\frac{4}{3}\sqrt{x^3}\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}} \\
&= \frac{9\pi}{16} - \frac{1}{2\sqrt{2}} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)
\end{aligned}$$

Computational Logic: Using the trapezoidal rule to get the area. The trapezoidal rule is as follows.

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{N} \frac{f(x_{k+1}) + f(x_k)}{2} h \tag{0.23}$$

where

$$h = \frac{b-a}{N} \tag{0.24}$$

∴The difference equation obtained is

$$A = \int_a^b f(x)\ dx \approx h\left(\frac{1}{2} f(a) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2} f(b)\right) \tag{0.25}$$

$$h = \frac{b-a}{n} \tag{0.26}$$

$$A = j_n, \text{ where, } j_{i+1} = j_i + h\frac{f(x_{i+1}) + f(x_i)}{2} \tag{0.27}$$

$$\rightarrow j_{i+1} = j_i + h\left(x_{i+1}^2 + x_i^2\right) \tag{0.28}$$

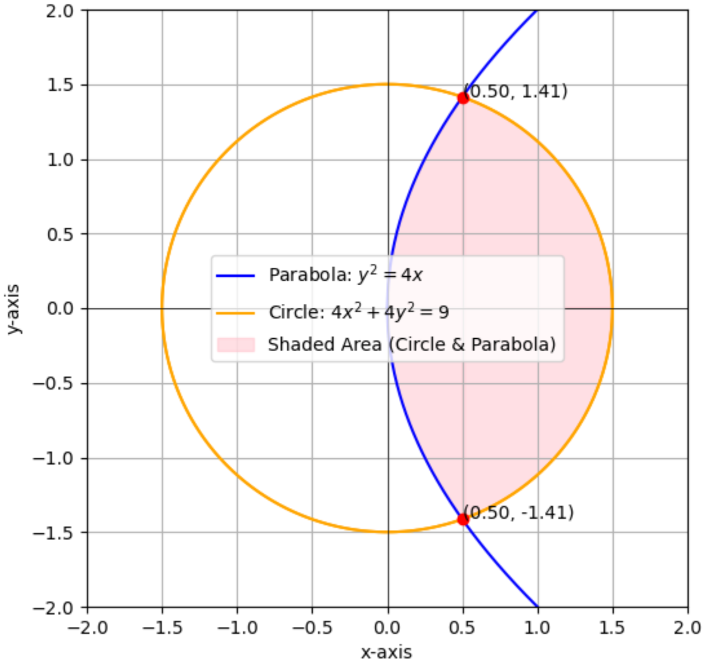
$$x_{i+1} = x_i + h \tag{0.29}$$

$$\tag{0.30}$$

where,$a = 0.5$, $b = -0.5$, $h = 0.002$,taking $n = 500$

$$f(x) = \sqrt{\frac{9}{4} - x^2} - 2\sqrt{x} \tag{0.31}$$

Therefore, Required area =3.0053



(0.50, 1.41)
(0.50, -1.41)
Parabola: $y^2 = 4x$
Circle: $4x^2 + 4y^2 = 9$
Shaded Area (Circle & Parabola)
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
y-axis
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x-axis