Question 39 Exercise(8.5)

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***Abstract*—A question based on intersection of equal chords.**

Download all python codes from

```
svn co https://github.com/Srihari123456/Summer
    −2020/tree/master/geometry/circle/codes
```

Download all LaTeX-Tikz codes from

```
svn co https://github.com/Srihari123456/Summer
    −2020/tree/master/geometry/circle/figs
```

1 Question

1.1. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the center makes equal angles with the chords.

2 Construction

2.1. The figure for a circle obtained in the question looks like Fig. 1, with radius **r**, center **O** and equal chords **AB** and **CD** whose point of intersection is **X**.

2.2. Two chords **AB** and **CD** are said to be equal if their lengths are the same.Hence we need to show that $\|\mathbf{B}-\mathbf{A}\| = \|\mathbf{D}-\mathbf{C}\|$ to say the two chords are equal.

2.3. Let θ_1, θ_2, θ_3 and θ_4 are the angles between x-axis and points **B**,**A** ,**D** and **C** respectively. The values used for constructing the circle in both Python and LaTeX-Tikz is in Table I:

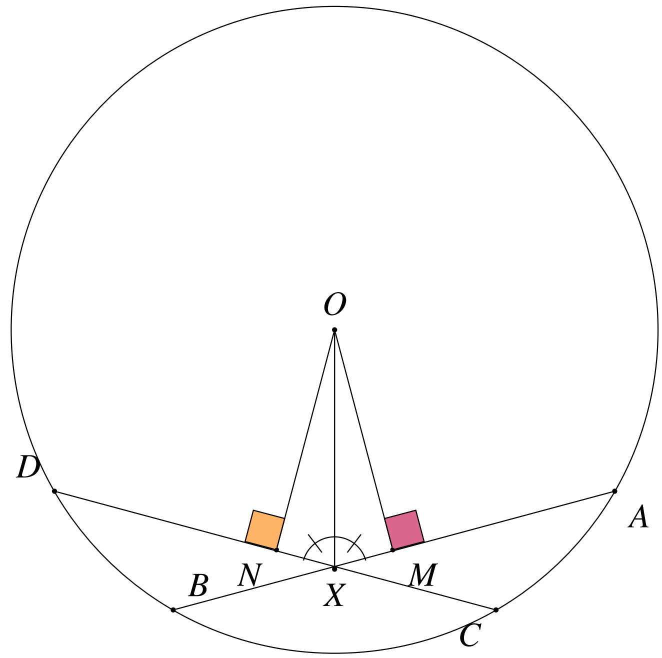

Fig. 1: Circle by Latex-Tikz

Initial Input Values	
Parameters	**Values**
r	**2**
θ_1	**-120°**
θ_2	**-30°**
θ_3	**210°**
θ_4	**300°**

TABLE I: To construct circle O

2.4. Finding the coordinates of various points of Fig. 1, let $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\mathbf{B} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} r\cos\theta_2 \\ r\sin\theta_2 \end{pmatrix}$$
$$\mathbf{D} = \begin{pmatrix} r\cos\theta_3 \\ r\sin\theta_3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} r\cos\theta_4 \\ r\sin\theta_4 \end{pmatrix}$$

2.5. As $\mathbf{OM} and \mathbf{ON}$ are the perpendiculars from the center of the circle to the chord,which are the midpoints of **AB** and **CD**, and are hence represented as:

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{2.5.1}$$
$$\mathbf{N} = \frac{\mathbf{C}+\mathbf{D}}{2} \tag{2.5.2}$$

2.6. To find the point of intersection **X** of the two chords:
Equations of **AB** and **CD** is represented as

$\begin{pmatrix} 0.74 & -2.72 \\ 0.74 & 2.72 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3.97 \\ -3.97 \end{pmatrix}$ On solving we get $\mathbf{x} = \begin{pmatrix} 0 \\ -1.45 \end{pmatrix}$

Thus the coordinates of $\mathbf{X} = \begin{pmatrix} 0 \\ -1.45 \end{pmatrix}$
The derived values used is in Table II:

Derived Values	
Parameter	**Value**
ChordLength	**2.82**
A	$\begin{pmatrix} 1.73 \\ -0.99 \end{pmatrix}$
B	$\begin{pmatrix} -0.99 \\ -1.73 \end{pmatrix}$
C	$\begin{pmatrix} 0.99 \\ -1.73 \end{pmatrix}$
D	$\begin{pmatrix} -1.73 \\ -0.99 \end{pmatrix}$
M	$\begin{pmatrix} 0.36 \\ -1.36 \end{pmatrix}$
N	$\begin{pmatrix} -0.36 \\ -1.36 \end{pmatrix}$
X	$\begin{pmatrix} 0 \\ -1.45 \end{pmatrix}$

TABLE II: To construct circle O

To Show:: We need to prove that $\angle OXD = \angle AXO$.

3 Solution

3.1. First we need to show that **M** and **N** are the midpoints of **AB** and **CD** respectively. In the given circle of center **O**, let **AB** be an arbitrary chord such that $\mathbf{OX} \perp \mathbf{AB}$. We need to show that $\mathbf{AX} = \mathbf{BX}$. Since $\mathbf{OX} \perp \mathbf{AB}$:
$\left(\mathbf{O} - \mathbf{X}\right)^T \left(\mathbf{A} - \mathbf{X}\right) = 0$

$$\implies \begin{pmatrix} -\mathbf{X}[0] & -\mathbf{X}[1] \end{pmatrix} \begin{pmatrix} \mathbf{A}[0] - \mathbf{X}[0] \\ \mathbf{A}[1] - \mathbf{X}[1] \end{pmatrix} = 0 \tag{3.1.1}$$

This condition is satisfied only when $\mathbf{X} = \frac{\mathbf{A}+\mathbf{B}}{2}$ i.e. **X** must be the midpoint of **AB**.

$$\therefore \mathbf{AX} = \mathbf{BX} \tag{3.1.2}$$

Hence Perpendicular from the center of a circle to a chord bisects the chord.

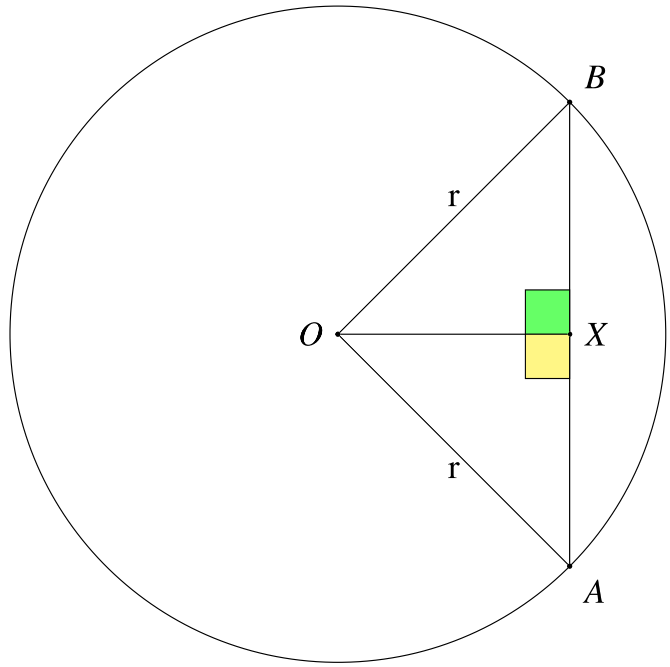

Fig. 2: Perpendicular from the center of a circle to a chord bisects the chord

3.2. Next we need to show that equal chords of a circle are equidistant from the center.

In the circle given in Fig:3 whose center is **O**, let **AB** and **CD** be two arbitrary chords such that $\mathbf{OX} \perp \mathbf{AB}$ and $\mathbf{OY} \perp \mathbf{CD}$. We need to show that $\mathbf{OX} = \mathbf{OY}$.

a) Since $\mathbf{OX} \perp \mathbf{AB}$, $\mathbf{AX} = \mathbf{BX} = \frac{\mathbf{AB}}{2}$ {From:(3.1.2) }
Similarly $\mathbf{CY} = \mathbf{DY} = \frac{\mathbf{CY}}{2}$

b) As $\mathbf{AB} = \mathbf{CD}$
$\frac{\mathbf{AB}}{2} = \frac{\mathbf{CD}}{2}$
Hence

$$\mathbf{AX} = \mathbf{CY} \tag{3.2.1}$$

c) In $\triangle AOX$ and $\triangle COY$ as shown in Fig.3 using baudhayana theorem

$$\|\mathbf{O} - \mathbf{Y}\|^2 + \|\mathbf{Y} - \mathbf{C}\|^2 = \|\mathbf{O} - \mathbf{X}\|^2 + \|\mathbf{X} - \mathbf{A}\|^2 \tag{3.2.2}$$

Using (3.2.1),

$$\|\mathbf{O} - \mathbf{Y}\|^2 = \|\mathbf{O} - \mathbf{X}\|^2 \tag{3.2.3}$$
$$\implies \|\mathbf{O} - \mathbf{Y}\| = \|\mathbf{O} - \mathbf{X}\| \tag{3.2.4}$$

$$There fore, \mathbf{OX} = \mathbf{OY} \tag{3.2.5}$$

Hence the equal chords of a circle are equidistant from the center.

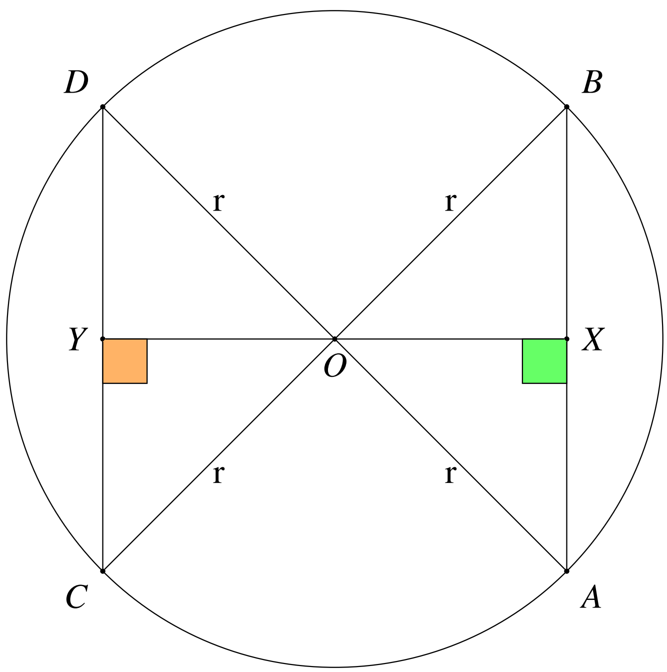

Fig. 3: Equal chords of a circle are equidistant from the center

The following Python code generates Fig. 4

./codes/step3.py

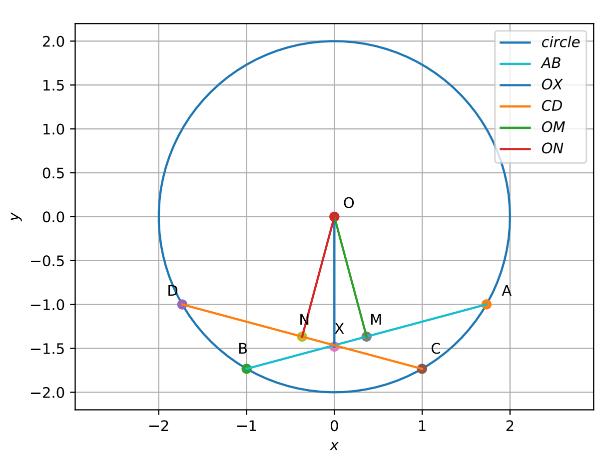

Fig. 4: Circle generated using python

The equivalent LaTeX- tikz code generating Fig. 1 is

./figs/stepthree.tex

The above LaTeX code can be compiled as a standalone document as

./figs/step3_standalone.tex

3.3. In the circle given in Fig.1 we need to show that $\angle OXA = \angle OXD$. We draw $\mathbf{OM} \perp \mathbf{AB}$ and $\mathbf{ON} \perp \mathbf{CD}$.
In $\triangle OMX$ and $\triangle ONX$,using baudhayana theorem

$$\|\mathbf{N} - \mathbf{O}\|^2 + \|\mathbf{X} - \mathbf{N}\|^2 = \|\mathbf{M} - \mathbf{O}\|^2 + \|\mathbf{X} - \mathbf{M}\|^2 \tag{3.3.1}$$

Using (3.2.5), $\|\mathbf{X} - \mathbf{N}\|^2 = \|\mathbf{X} - \mathbf{M}\|^2$

$$\implies \|\mathbf{NX}\| = \|\mathbf{MX}\| \tag{3.3.2}$$

In $\triangle OMX$ and $\triangle ONX$,using cosine formula

$$\cos \angle OXN = \frac{\|\mathbf{NX}\|^2 + \|\mathbf{OX}\|^2 - \|\mathbf{ON}\|^2}{2\,\|\mathbf{NX}\|\,\|\mathbf{OX}\|} \tag{3.3.3}$$

$$\cos \angle OXM = \frac{\|\mathbf{MX}\|^2 + \|\mathbf{OX}\|^2 - \|\mathbf{OM}\|^2}{2\,\|\mathbf{MX}\|\,\|\mathbf{OX}\|} \tag{3.3.4}$$

Using (3.3.2) and (3.2.5)

$$\cos \underline{/OXN} = \cos \angle OXM \tag{3.3.5}$$

$$There fore, \angle OXM = \angle OXN \tag{3.3.6}$$

$\implies \angle OXA = \angle OXD$
Hence Proved.