Question 39 Exercise(8.5)

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***Abstract*—A question based on intersection of equal chords.**

Download all python codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/geometry/circle/codes
```

Download all LATEX-Tikz codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/geometry/circle/figs
```

1 Question

1) If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the center makes equal angles with the chords.

2 Construction

1) The figure for a circle obtained in the question looks like Fig. 1, with radius **r**, center **O** and equal chords **AB** and **CD** whose point of intersection is **X**.
2) Two chords are said to be equal if their lengths are the same. The length of a chord is given by 2rsin $\frac{\theta}{2}$, where **r** is the radius and θ is the angle subtended by the chord at the center of the circle. Thus in a circle, equal chords subtend equal angles at the center.
3) Let us assume that the two equal chords **AB** and **CD** subtend equal angles of $\theta = 90°$ at the center of the circle. θ_1 and θ_2 are the angles between x-axis and points **B** and **D** respectively.
   The values used for constructing the circle in both Python and LATEX-Tikz is in Table I:
4) Finding the coordinates of various points of Fig. 1, let $\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\mathbf{B} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} r\cos(\theta_1+\theta) \\ r\sin(\theta_1+\theta) \end{pmatrix}$$

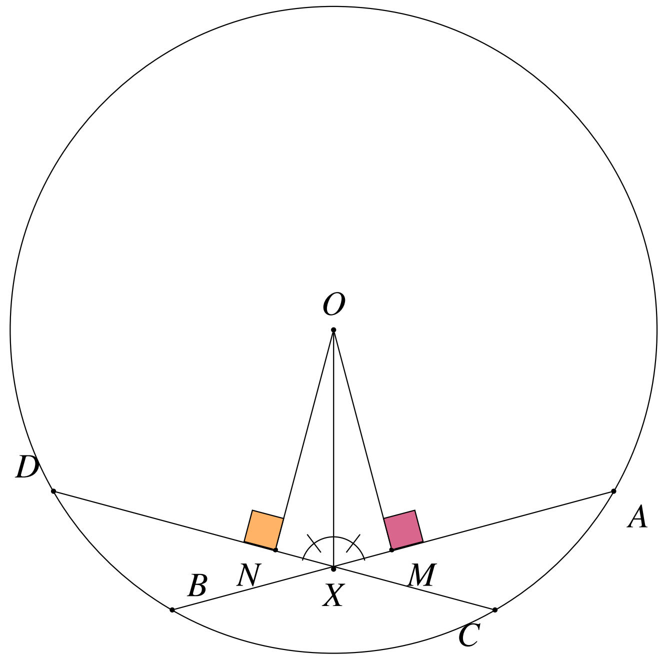

Fig. 1: Circle by Latex-Tikz

Initial Input Values	
Parameters	**Values**
r	**2**
θ	**90°**
θ_1	**-120°**
θ_2	**210°**

TABLE I: To construct circle O

$$\mathbf{D} = \begin{pmatrix} r\cos\theta_2 \\ r\sin\theta_2 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} r\cos(\theta_2+\theta) \\ r\sin(\theta_2+\theta) \end{pmatrix}$$

5) As $\mathbf{OM} and \mathbf{ON}$ are the perpendiculars from the center of the circle to the chord,which are the midpoints of **AB** and **CD**, and are hence represented as:
   $\mathbf{M} = \frac{\mathbf{A+B}}{2} \quad \mathbf{N} = \frac{\mathbf{C+D}}{2}$

   The derived values used is in Table II:

To Show:: We need to prove that $\angle OXD = \angle AXO$.

Derived Values	
Parameter	**Value**
ChordLength	**2.82**
A	$\begin{pmatrix} 1.73 \\ -0.99 \end{pmatrix}$
B	$\begin{pmatrix} -0.99 \\ 1.73 \end{pmatrix}$
C	$\begin{pmatrix} 0.99 \\ -1.73 \end{pmatrix}$
D	$\begin{pmatrix} -1.73 \\ -0.99 \end{pmatrix}$
M	$\begin{pmatrix} 0.36 \\ -1.36 \end{pmatrix}$
N	$\begin{pmatrix} -0.36 \\ -1.36 \end{pmatrix}$

TABLE II: To construct circle O

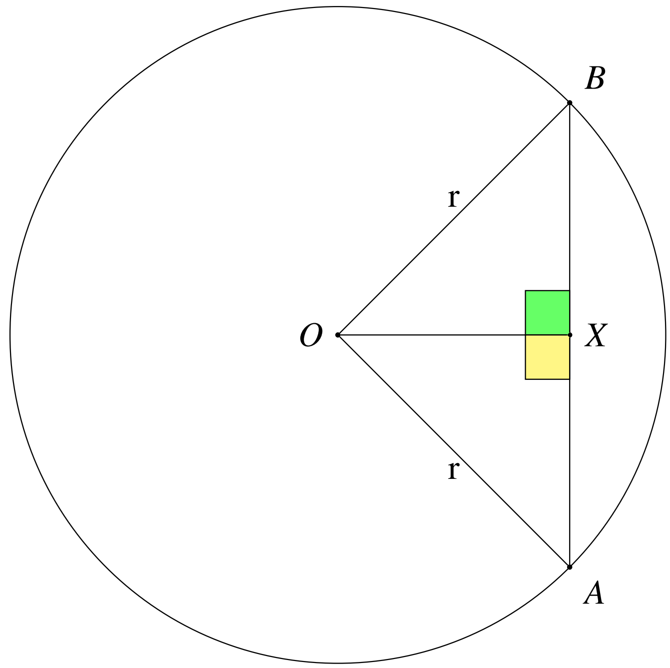

Fig. 2: Perpendicular from the center of a circle to a chord bisects the chord

3 Solution

1) First we need to show that **M** and **N** are the midpoints of **AB** and **CD** respectively. In the given circle of center **O**, let **AB** be an arbitrary chord such that $\mathbf{OX} \perp \mathbf{AB}$. We need to show that $\mathbf{AX} = \mathbf{BX}$.

   a) $\triangle OAX \cong \triangle OBX$by RHS rule as:
      i) $\angle OXA = \angle OXB$ $\{90°\}$
      ii) $\mathbf{OA} = \mathbf{OB}$ {radius of the circle}
      iii) $\mathbf{OX} = \mathbf{OX}$ {Common}

$$Therefore \mathbf{AX} = \mathbf{BX} \tag{3.0.1}$$

   Hence the perpendicular from the center of a circle to a chord bisects the chord.

2) Next we need to show that equal chords of a circle are equidistant from the center.

   In the circle given in Fig:3 whose center is **O**, let**AB** and **CD** be two arbitrary chords such that $\mathbf{OX} \perp \mathbf{AB}$ and $\mathbf{OY} \perp \mathbf{CD}$. We need to show that $\mathbf{OX} = \mathbf{OY}$.

   a) Since $\mathbf{OX} \perp \mathbf{AB}$, $\mathbf{AX} = \mathbf{BX} = \frac{\mathbf{AB}}{2}$ {From:(3.0.1) }
      Similarly $\mathbf{CY} = \mathbf{DY} = \frac{\mathbf{CY}}{2}$

   b) As $\mathbf{AB} = \mathbf{CD}$
      $\frac{\mathbf{AB}}{2} = \frac{\mathbf{CD}}{2}$
      Hence

$$\mathbf{AX} = \mathbf{CY} \tag{3.0.2}$$

   c) In $\triangle AOX$ and $\triangle COY$ as shown in Fig.3
      i) $\angle OXA = \angle OYC$ $\{90°\}$
      ii) $\mathbf{OA} = \mathbf{OC}$ {radius of the circle}
      iii) $\mathbf{AX} = \mathbf{CY}$ {From:(3.0.2)}
      Hence $\triangle AOX \cong \triangle COY$ by RHS rule.

$$Therefore, \mathbf{OX} = \mathbf{OY} \tag{3.0.3}$$

   Hence the equal chords of a circle are equidistant from the center.

   The following Python code generates Fig. 4

./codes/step3.py

   The equivalent LaTeX- tikz code generating Fig. 1 is

./figs/stepthree.tex

   The above LaTeX code can be compiled as a standalone document as

./figs/step3_standalone.tex

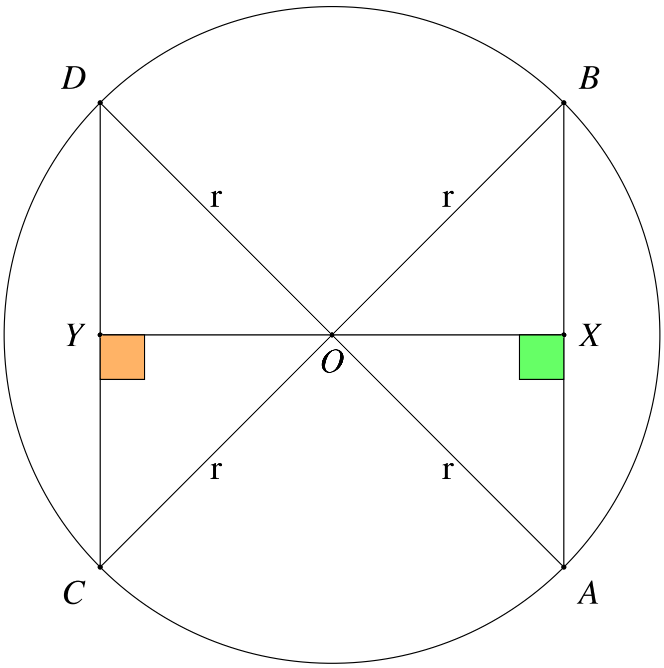

Fig. 3: Equal chords of a circle are equidistant from the center

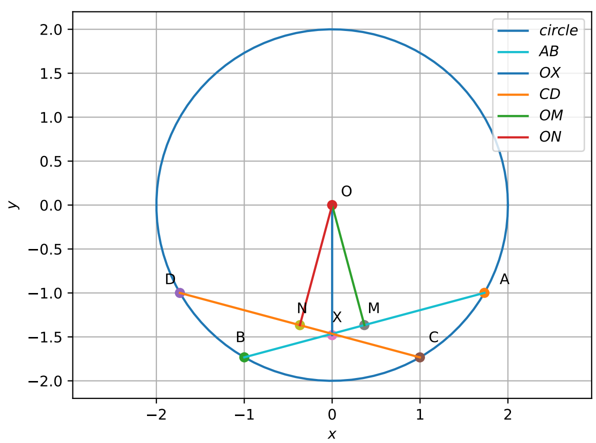

Fig. 4: Circle generated using python

3) In the circle given in Fig.1 we need to show that $\angle OXA = \angle OXD$. We draw **OM** $\perp$ **AB** and **ON** $\perp$ **CD**.
   In $\triangle OMX$ and $\triangle ONX$,
   a) $\angle OMX = \angle ONX$ {90°}
   b) **OM** = **ON** {From:(3.0.3)}
   c) **OX** = **OX** {Common}

   Hence $\triangle OMX \cong \triangle ONX$ by RHS rule.
   Therefore, $\angle OXM = \angle OXN$
   $\implies \angle OXA = \angle OXD$

Hence Proved.