Question 51 Exercise(8.1)

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***Abstract*—A question based on similarity of triangles.**

Download all python codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/geometry/triangle/codes
```

Download all LATEX-Tikz codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/geometry/triangle/figs
```

1 Question

1) **O** is a point in the interior of $\triangle$**ABC**. **D** is a point on **OA**. If **DE** || **OB** and **DF** || **OC**. Show that **EF** || **BC**.

2 Construction

1) The figure for a triangle obtained in the question looks like Fig. 1, with sides a,b,c, an arbitrary interior point **O** and a point **D** on line **AO**.

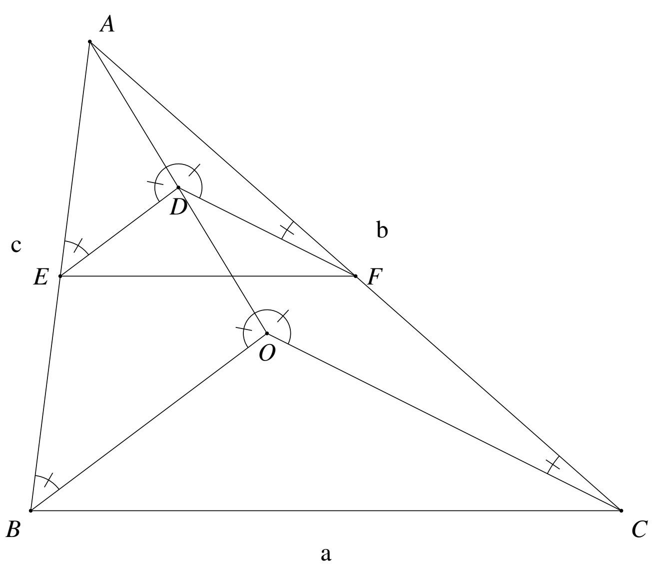

Fig. 1: Triangle by Latex-Tikz

The values used for constructing the triangles in both Python and LATEX-Tikz is in Table I:

Initial Input Values	
Parameter	**Value**
a	**5**
b	**6**
c	**4**

TABLE I: To construct $\triangle ABC$

2) Finding the coordinates of various points of Fig. 1 :
From the information provided in the Table I: let
$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix}$
The derived value of **p** and **q** is available in Table II.
3) Given a point **O**, we need to determine whether it lies inside $\triangle ABC$. Consider 3 vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ which are orthogonal to vectors **AB**,**BC** and **CA** which are ordered counterclock-wise.
Let $\mathbf{A} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \quad \mathbf{O} = \begin{pmatrix} x \\ y \end{pmatrix}$
$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$
$\mathbf{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}$
$\mathbf{CA} = \mathbf{A} - \mathbf{C} = \begin{pmatrix} x_1 - x_3 \\ y_1 - y_3 \end{pmatrix}$
Equation of **AB** is
$$(y_2-y_1)x+(-x_2+x_1)y = x_1(y_2-y_1)-y_1(x_2-x_1) \tag{2.0.1}$$
As $\mathbf{v_1}$ is orthogonal to **AB**, equation of $\mathbf{v_1}$ is
$$(-x_2 + x_1)x - (y_2 - y_1)y = d \tag{2.0.2}$$
where d is some constant.Hence $\mathbf{v_1}$ is represented as $\begin{pmatrix} y_2 - y_1 \\ -x_2 + x_1 \end{pmatrix}$

Similarly $\mathbf{v_2}$ is represented as $\begin{pmatrix} y_3 - y_2 \\ -x_3 + x_2 \end{pmatrix}$
$\mathbf{v_3}$ is represented as $\begin{pmatrix} y_1 - y_3 \\ -x_1 + x_3 \end{pmatrix}$
Position vector of **O** w.r.t **A** is $\mathbf{v_1'} = \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}$
Position vector of **O** w.r.t **B** is $\mathbf{v_2'} = \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix}$
Position vector of **O** w.r.t **C** is $\mathbf{v_3'} = \begin{pmatrix} x - x_3 \\ y - y_3 \end{pmatrix}$
Now we compute the dot products: **O** lies inside $\triangle ABC$ only if dot1 ,dot2 and dot3 are all $\geqslant 0$, where $dot_1 = v_1 \cdot v_1' \quad dot_2 = v_2 \cdot v_2' \quad dot_3 = v_3 \cdot v_3'$.

4) Let the arbitrary interior point **O** be represented as $\begin{pmatrix} 2 \\ 1.5 \end{pmatrix}$. **D** is a point on line **AO** such that **DE** $\parallel$ **OB** and **DF** $\parallel$ **OC**.

5) Determination of points D,E and F:
As **DE** $\parallel$ **OB**, by basic proportionality theorem the points **E** and **D**, divide the lines **AB** and **AO** respectively in the same ratio.
Hence we choose points **E** and **D** such that

$$\frac{AE}{EB} = \frac{AD}{DO} \tag{2.0.3}$$

Similarly point **F** is chosen such that the points **F** and **D**, divide the lines **AC** and **AO** respectively in the same ratio such that

$$\frac{AF}{FC} = \frac{AD}{DO} \tag{2.0.4}$$

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Derived Values	
Parameter	**Value**
p	**0.5**
q	**3.96**

TABLE II: To construct $\triangle ABC$

6) If the point **D** divides the line **AO** in the ratio x:y, the coordinates of **D** is given by section formula as:

$$\mathbf{D} = \frac{y\mathbf{A} + x\mathbf{O}}{x + y} \tag{2.0.5}$$

Similarly the coordinates of points **E** and **F** is given by

$$\mathbf{E} = \frac{y\mathbf{A} + x\mathbf{B}}{x + y} \tag{2.0.6}$$

$$\mathbf{F} = \frac{y\mathbf{A} + x\mathbf{C}}{x + y} \tag{2.0.7}$$

Let us assume the points divide the respective lines in the ratio 1:1. Then the coordinates of points **D**, **E** and **F** is
$\mathbf{D} = \begin{pmatrix} 1.25 \\ 2.73 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0.25 \\ 1.98 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 2.75 \\ 1.98 \end{pmatrix}$

7) To check whether **D** lies on line **AO**, substituting the values of the x and y co-ordinate of **D** must satisfy the equation of line **AO**. Equation of line joining two points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \tag{2.0.8}$$

The following Python code generates Fig. 2

./codes/similartriangle.py

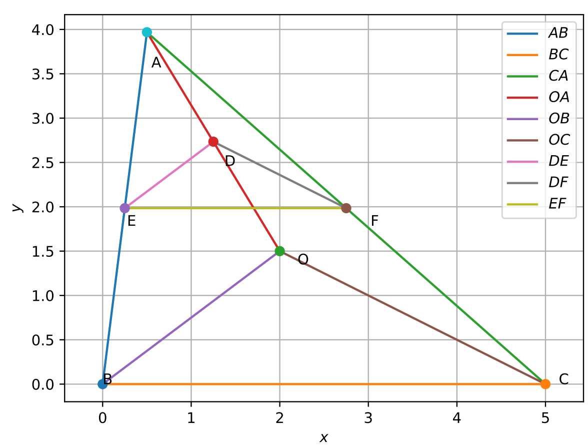

Fig. 2: Triangle generated using python

The equivalent LaTeX- tikz code generating Fig. 1 is

./figs/constructionpic.tex

The above LaTeX code can be compiled as a standalone document as

./figs/constructionpic_standalone.tex

To Show:: We need to prove that **EF** $\parallel$ **BC**.

3 Solution

1) $\triangle EAD \sim \triangle BAO$ by AAA Similarity:
   Since $\mathbf{DE} \parallel \mathbf{OB}$,
   a) $\angle DEA = \angle OBA \quad \{Alternate\ Interior\ Angles\}$
   b) $\angle ADE = \angle AOB \quad \{Alternate\ Interior\ Angles\}$
   c) $\angle EAD = \angle BAO \quad \{Common\ angle\}$

   Therefore
   $$\frac{\mathbf{AE}}{\mathbf{AB}} = \frac{\mathbf{AD}}{\mathbf{AO}} \tag{3.0.1}$$
2) Similarly $\triangle FDA \sim \triangle COA$ by AAA Similarity:
   Since $\mathbf{DF} \parallel \mathbf{OC}$,
   a) $\angle DFA = \angle OCA \quad \{Alternate\ Interior\ Angles\}$
   b) $\angle ADF = \angle AOC \quad \{Alternate\ Interior\ Angles\}$
   c) $\angle FAD = \angle CAO \quad \{Common\ angle\}$

   Therefore
   $$\frac{\mathbf{AF}}{\mathbf{AC}} = \frac{\mathbf{AD}}{\mathbf{AO}} \tag{3.0.2}$$
3) Hence from the above we conclude,
   $$\frac{\mathbf{AF}}{\mathbf{AC}} = \frac{\mathbf{AE}}{\mathbf{AB}} = \frac{\mathbf{AD}}{\mathbf{AO}} \tag{3.0.3}$$

As the ratio of the sides is the same, $\triangle$ ABC $\sim$ $\triangle$ AEF, which means $\angle AFE = \angle ACB$ and $\angle AEF = \angle ABC$ as similar triangles have same angles. i.e.

$$\mathbf{EF} \parallel \mathbf{QR} \tag{3.0.4}$$

Hence Proved.