Question 51 Exercise(8.1)

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***Abstract*—A question based on similarity of triangles.**

Download all python codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/geometry/triangle/codes
```

Download all LaTeX-Tikz codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/geometry/triangle/figs
```

1 Question

1.1. **O** is a point in the interior of $\triangle$**ABC**. **D** is a point on **OA**. If **DE** || **OB** and **DF** || **OC**. Show that **EF** || **BC**.

2 Construction

2.1. The figure for a triangle obtained in the question looks like Fig. 1, with sides a,b,c, an arbitrary interior point **O** and a point **D** on line **AO**.

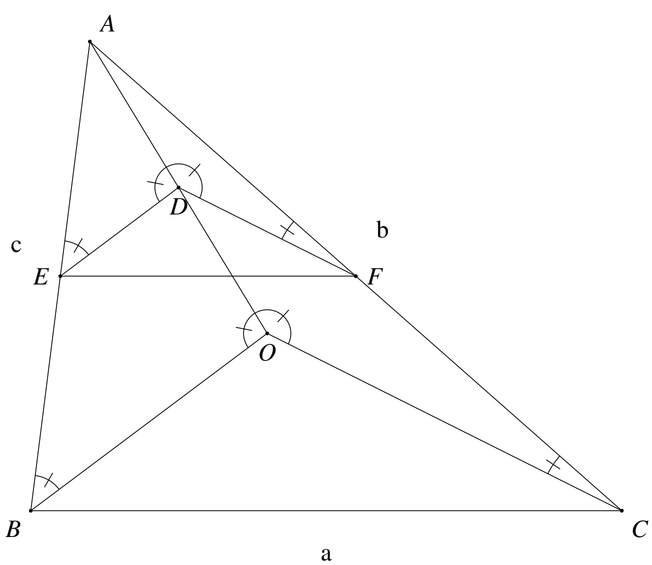

Fig. 1: Triangle by Latex-Tikz

The values used for constructing the triangles in both Python and LaTeX-Tikz is in Table I:

Initial Input Values	
Parameter	**Value**
a	**5**
b	**6**
c	**4**

TABLE I: To construct $\triangle ABC$

2.2. Finding the coordinates of various points of Fig. 1 :
From the information provided in the Table I: let

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mathbf{C} = \begin{pmatrix} a \\ 0 \end{pmatrix} \mathbf{A} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{2.2.1}$$

The derived value of **p** and **q** is available in Table II.

2.3. Given a point **O**, we need to determine whether it lies inside $\triangle ABC$. Consider 3 vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ which are orthogonal to vectors **AB**,**BC** and **CA** which are ordered counterclock-wise.

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} \tag{2.3.1}$$
$$\mathbf{BC} = \mathbf{C} - \mathbf{B} \tag{2.3.2}$$
$$\mathbf{CA} = \mathbf{A} - \mathbf{C} \tag{2.3.3}$$

As $\mathbf{v_1}$ is orthogonal to **AB**, dot product of $\mathbf{v_1}$ and **AB** is 0. This condition is satisfied when

$$\mathbf{v_1} = \begin{pmatrix} \mathbf{AB[1]} \\ -\mathbf{AB[0]} \end{pmatrix} \tag{2.3.4}$$

$$\text{Similarly } \mathbf{v_2} = \begin{pmatrix} \mathbf{BC[1]} \\ -\mathbf{BC[0]} \end{pmatrix} \tag{2.3.5}$$

$$\mathbf{v_3} = \begin{pmatrix} \mathbf{CA[1]} \\ -\mathbf{CA[0]} \end{pmatrix} \tag{2.3.6}$$

Position vector of **O** w.r.t **A** is

$$\mathbf{v_1'} = \mathbf{O} - \mathbf{A} \tag{2.3.7}$$

Position vector of $\mathbf{O}$ w.r.t $\mathbf{B}$ is

$$\mathbf{v}_2' = \mathbf{O} - \mathbf{B} \tag{2.3.8}$$

Position vector of $\mathbf{O}$ w.r.t $\mathbf{C}$ is

$$\mathbf{v}_3' = \mathbf{O} - \mathbf{C} \tag{2.3.9}$$

Now we compute the dot products: $\mathbf{O}$ lies inside $\triangle ABC$ only if dot_1 ,dot_2 and dot_3 are all $\geqslant 0$, where $dot_1 = v_1 \cdot v_1'$ $\quad dot_2 = v_2 \cdot v_2'$ $\quad dot_3 = v_3 \cdot v_3'$.

2.4. Let the arbitrary interior point $\mathbf{O}$ be represented as $\begin{pmatrix} 2 \\ 1.5 \end{pmatrix}$. $\mathbf{D}$ is a point on line $\mathbf{AO}$ such that $\mathbf{DE} \parallel \mathbf{OB}$ and $\mathbf{DF} \parallel \mathbf{OC}$.

2.5. Determination of points D,E and F:
As $\mathbf{DE} \parallel \mathbf{OB}$, by basic proportionality theorem the points $\mathbf{E}$ and $\mathbf{D}$, divide the lines $\mathbf{AB}$ and $\mathbf{AO}$ respectively in the same ratio.
Hence we choose points $\mathbf{E}$ and $\mathbf{D}$ such that

$$\frac{AE}{EB} = \frac{AD}{DO} \tag{2.5.1}$$

Similarly point $\mathbf{F}$ is chosen such that the points $\mathbf{F}$ and $\mathbf{D}$, divide the lines $\mathbf{AC}$ and $\mathbf{AO}$ respectively in the same ratio such that

$$\frac{AF}{FC} = \frac{AD}{DO} \tag{2.5.2}$$

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Derived Values	
Parameter	**Value**
p	**0.5**
q	**3.96**

TABLE II: To construct $\triangle ABC$

2.6. If the point $\mathbf{D}$ divides the line $\mathbf{AO}$ in the ratio x:y, the coordinates of $\mathbf{D}$ is given by section formula as:

$$\mathbf{D} = \frac{y\mathbf{A} + x\mathbf{O}}{x+y} \tag{2.6.1}$$

Similarly the coordinates of points $\mathbf{E}$ and $\mathbf{F}$ is given by

$$\mathbf{E} = \frac{y\mathbf{A} + x\mathbf{B}}{x+y} \tag{2.6.2}$$

$$\mathbf{F} = \frac{y\mathbf{A} + x\mathbf{C}}{x+y} \tag{2.6.3}$$

Let us assume the points divide the respective lines in the ratio 1:1. Then the coordinates of points $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{F}$ is
$\mathbf{D} = \begin{pmatrix} 1.25 \\ 2.73 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 0.25 \\ 1.98 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 2.75 \\ 1.98 \end{pmatrix}$

2.7. To check whether $\mathbf{D}$ lies on line $\mathbf{AO}$:
Let

$$\mathbf{AD} = \mathbf{D} - \mathbf{A} \text{ and} \mathbf{AO} = \mathbf{O} - \mathbf{A} \tag{2.7.1}$$

$\mathbf{D}$ lies on $\mathbf{AO}$ if the below equation is satisfied:

$$\frac{\mathbf{AD[0]}}{\mathbf{AD[1]}} = \frac{\mathbf{AO[0]}}{\mathbf{AO[1]}} \tag{2.7.2}$$

The following Python code generates Fig. 2

```
./codes/similartriangle.py
```

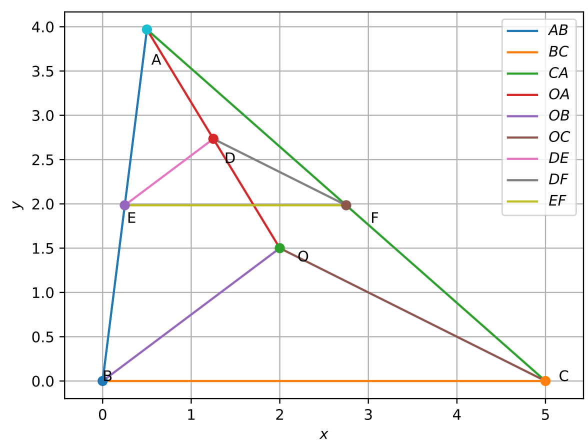

Fig. 2: Triangle generated using python

The equivalent LaTeX- tikz code generating Fig. 1 is

```
./figs/constructionpic.tex
```

The above LaTeX code can be compiled as a standalone document as

```
./figs/constructionpic_standalone.tex
```

To Show:: We need to prove that $\mathbf{EF} \parallel \mathbf{BC}$.

3 Solution

3.1. $\triangle EAD \sim \triangle BAO$ by AAA Similarity:

Since $\mathbf{DE} \parallel \mathbf{OB}$,

a) $\angle DEA = \angle OBA$ (Alternate Interior Angles)
b) $\angle ADE = \angle AOB$ (Alternate Interior Angles)
c) $\angle EAD = \angle BAO$ (Common angle)

Therefore

$$\frac{\mathbf{AE}}{\mathbf{AB}} = \frac{\mathbf{AD}}{\mathbf{AO}} \tag{3.1.1}$$

3.2. Similarly $\triangle FDA \sim \triangle COA$ by AAA Similarity:
Since $\mathbf{DF} \parallel \mathbf{OC}$,

a) $\angle DFA = \angle OCA$ (Alternate Interior Angles)
b) $\angle ADF = \angle AOC$ (Alternate Interior Angles)
c) $\angle FAD = \angle CAO$ (Common angle)

Therefore

$$\frac{\mathbf{AF}}{\mathbf{AC}} = \frac{\mathbf{AD}}{\mathbf{AO}} \tag{3.2.1}$$

3.3. Hence from the above we conclude,

$$\frac{\mathbf{AF}}{\mathbf{AC}} = \frac{\mathbf{AE}}{\mathbf{AB}} = \frac{\mathbf{AD}}{\mathbf{AO}} \tag{3.3.1}$$

As the ratio of the sides is the same, $\triangle$ ABC $\sim$ $\triangle$ AEF, which means $\angle AFE = \angle ACB$ and $\angle AEF = \angle ABC$ as similar triangles have same angles. i.e.

$$\mathbf{EF} \parallel \mathbf{QR} \tag{3.2}$$

Hence Proved.