LinearAlgebra Question 5

Srihari S

***Abstract*—A document implementing solutions to problems using linear algebra.**

Download all python codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/LinearAlgebrafolder/codes
```

Download all LaTeX-Tikz codes from

```
svn co https://github.com/Srihari123456/Summer
    -2020/tree/master/LinearAlgebrafolder/figs
```

1 Question 1.2.5

1.1 Problem

1.1. In $\triangle ABC$ with vertices

$$\mathbf{A} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.1.1}$$

Find the equation and the length of the altitude from vertex $\mathbf{A}$. The following python code computes the length of the altitude $\mathbf{AD}$ in Fig.1.

```
./codes/triangle/q2.py
```

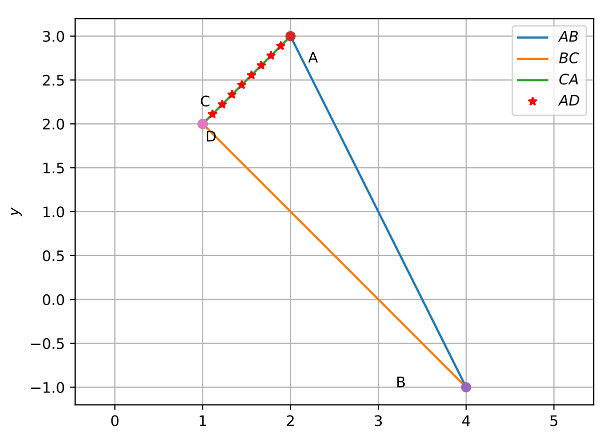

Fig. 1: Triangle of Q.1.2.5

Solution: Let the direction vector $\mathbf{m} = \mathbf{B} - \mathbf{C}$ We define the normal vector

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2}$$

Equation of line $\mathbf{AD}$ is obtained as:

$$\mathbf{m}^{\mathbf{T}}\mathbf{x} = \mathbf{m}^{\mathbf{T}}\mathbf{A} \tag{1.1.3}$$

Equation of line $\mathbf{BC}$ is :

$$\mathbf{n}^{\mathbf{T}}\mathbf{x} = \mathbf{n}^{\mathbf{T}}\mathbf{B} \tag{1.1.4}$$

Since $\mathbf{D}$ is the intersection of lines $\mathbf{AD}$ and $\mathbf{BC}$

$$\begin{pmatrix} \mathbf{m} & \mathbf{n} \end{pmatrix}^T \mathbf{D} = \begin{pmatrix} \mathbf{m}^{\mathbf{T}}\mathbf{A} \\ \mathbf{n}^{\mathbf{T}}\mathbf{B} \end{pmatrix}$$

which is solved to obtain the value of $\mathbf{D} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (1.1.5)

Therefore The length of the altitude is obtained as $\|\mathbf{A} - \mathbf{D}\| = 1.414$

2 Question 2.2.5

2.1 Problem

2.1. Without using distance formula, show that the points

$$\mathbf{A} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{2.1.1}$$

are the vertices of a parallelogram. The following python code computes the area of $\triangle ABC$ in Fig.2.

```
./codes/quadrilateral/q4.py
```

Solution: As the direction vectors,

$$\mathbf{A} - \mathbf{B} = \mathbf{D} - \mathbf{C} \tag{2.1.2}$$

$$\mathbf{A} - \mathbf{D} = \mathbf{B} - \mathbf{C} \tag{2.1.3}$$

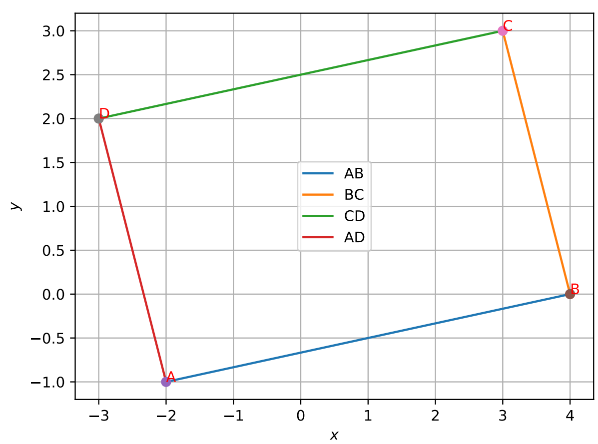

Fig. 2: Parallelogram of Q.2.2.5

$$\implies \mathbf{AB} \parallel \mathbf{CD} \text{ and } \mathbf{AD} \parallel \mathbf{BC}$$

$$\therefore \mathbf{ABCD} \text{ is a parallelogram.} \tag{2.1.4}$$

3 Question 3.2.5

3.1 Problem

3.1. Solve $\mathbf{x} + \mathbf{y} < 5$ graphically. The following python code computes the graphical representation of $\mathbf{x} + \mathbf{y} < 5$ as shown in Fig.3.

```
./codes/lines/q6.py
```

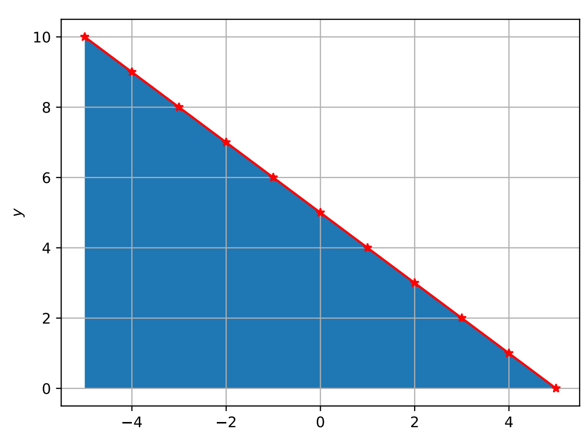

Fig. 3: x+y<5

4 Question 3.3.5

4.1 Problem

4.1. Find the values of a,b,c,x,y and z if

$$\begin{pmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{pmatrix} \tag{4.1.1}$$

Solution: As the two matrices are equal their corresponding entries are also equal. Hence

$$x+3=0 \implies x=-3 \tag{4.1.2}$$
$$z+4=6 \implies z=2 \tag{4.1.3}$$
$$2y-7=3y-2 \implies y=-5 \tag{4.1.4}$$
$$a-1=-3 \implies a=-2 \tag{4.1.5}$$
$$2c+2=0 \implies c=-1 \tag{4.1.6}$$
$$b-3=2b+4 \implies b=-7 \tag{4.1.7}$$

5 Question 3.5.5

5.1 Problem

5.1. Find the angle between the x-axis and the line joining the points

$$\mathbf{A} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}. \tag{5.1.1}$$

The following python code computes the angle which the line in Fig.4 makes with x-axis.

```
./codes/lines/q9.py
```

Solution: The slope of line **AB** is given as:

$$\tan\theta = \frac{B[1]-A[1]}{B[0]-A[0]} \tag{5.1.2}$$

$$\text{which on computing yields} \quad \tan\theta = -1 \tag{5.1.3}$$

$$\therefore \theta = \tan^{-1}(-1) = 135^\circ \tag{5.1.4}$$

6 Question 3.6.5

6.1 Problem

6.1. If the vertices of a parallelogram taken in order are

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 \\ y \end{pmatrix}, \mathbf{C} = \begin{pmatrix} x \\ 6 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\text{find x and y.} \tag{6.1.1}$$

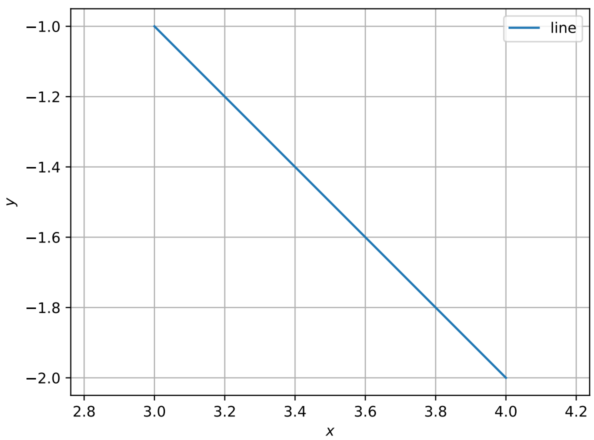

Fig. 4: Line of Q.3.5.5

The following python code computes the value of x and y used in Fig.5.

```
./codes/lines/q10.py
```

Solution: In a parallelogram, the diagonals bisect each other. Hence

$$\frac{\mathbf{A}+\mathbf{C}}{2} = \frac{\mathbf{B}+\mathbf{D}}{2} \tag{6.1.2}$$

$$\therefore \frac{\mathbf{1}+\mathbf{x}}{\mathbf{2}} = \frac{\mathbf{7}}{\mathbf{2}} \text{ and } \frac{\mathbf{8}}{\mathbf{2}} = \frac{\mathbf{y}+\mathbf{5}}{\mathbf{2}} \tag{6.1.3}$$

$$\implies x = 6, y = 3 \tag{6.1.4}$$

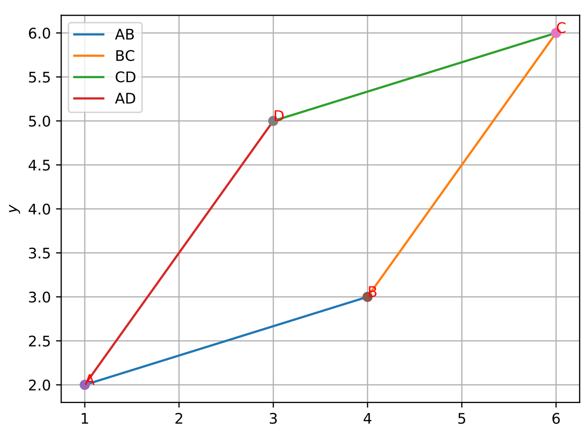

Fig. 5: Parallelogram of Q.3.6.5

7 Question 3.7.5

7.1 Problem

7.1. Draw the graphs of the following equations:

a) $\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0$
b) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 0$
c) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$
d) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = -1$
e) $\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = 4$
f) $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 4$

The following python codes draw the graphs which are represented in Fig.6 and Fig.7.

```
./codes/lines/q11a.py
./codes/lines/q11b.py
```

Solution:

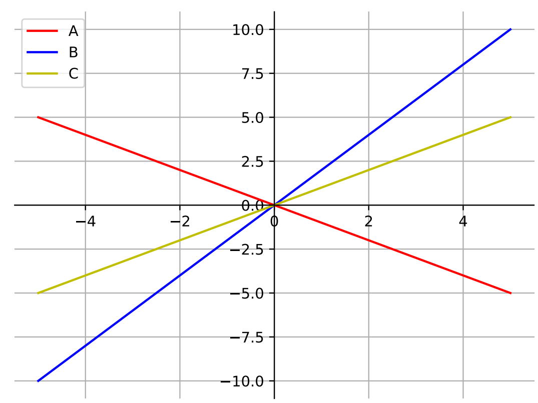

Fig. 6: Lines of Q.3.7.5

8 Question 3.8.5

8.1 Problem

8.1. Rain is falling vertically with a speed of $35ms^{-1}$. A woman rides a bicycle with a speed of $12ms^{-1}$ in east to west direction.What is the direction in which she should hold her umbrella?

The following python code computes the area of $\triangle$ABC in Fig.8.

```
./codes/lines/q12.py
```

Solution: At time t=0 let

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8.1.1}$$

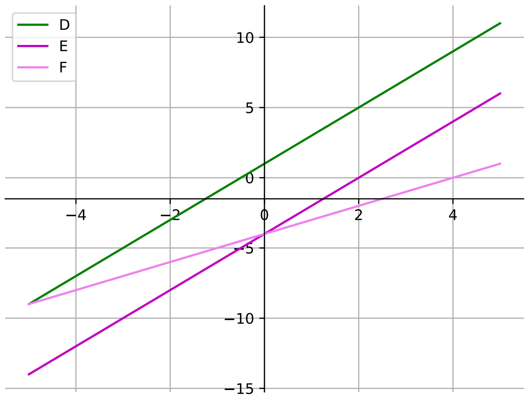

Fig. 7: Lines of Q.3.7.5

denote the position of the woman. Since she rides her bicycle at $12ms^{-1}$ in east to west direction , her position at time t=1 is represented as

$$\mathbf{C} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} \tag{8.1.2}$$

. Let the position of a rain-droplet at time t=0 be

$$\mathbf{A} = \begin{pmatrix} -12 \\ 35 \end{pmatrix} \tag{8.1.3}$$

. The drops which are falling a little ahead of the current position of the woman, will fall on her, because she moves in that direction. To find the direction in which she should hold her umbrella, we need to find $\angle CAB = \theta$.

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 12 \\ -35 \end{pmatrix}$$

$$\mathbf{AC} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 0 \\ -35 \end{pmatrix}$$

$$\mathbf{AB}^T\mathbf{AC} = \|AB\|\,\|AC\|\cos\theta$$

$$\cos\theta = \frac{35}{37}$$

$$\theta = 18.93^\circ \tag{8.1.4}$$

So the cyclist should hold the umbrella at 18.93° to the vertical in the forward direction.

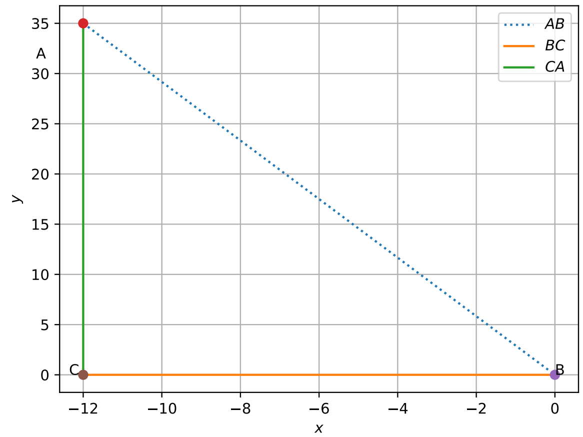

Fig. 8: Figure of Q.3.8.5

9 Question 3.9.5

9.1 Problem

9.1. Construct a a3×4 matrix whose elements are given by:

a) $A_{ij} = \frac{1}{2}|-3i + j|$

b) $A_{ij} = 2i - j$

The following python code computes the required matrix.

```
./codes/lines/q13.py
```

Solution:

a) The matrix $A_{ij} = \frac{1}{2}|-3i + j|$ obtained is

$$\begin{pmatrix} 0 & 0.5 & 1 & 1.5 \\ 1.5 & 1 & 0.5 & 0 \\ 3 & 2.5 & 2 & 1.5 \end{pmatrix} \tag{9.1.1}$$

b) The matrix $A_{ij} = 2i - j$ obtained is

$$\begin{pmatrix} 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{9.1.2}$$

10 Question 3.10.5

10.1 Problem

10.1. Evaluate the determinants

The following python code computes the required determinant value.

```
./codes/lines/q14.py
```

a) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ which on evaluating gives -12

b) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$ which on evaluating gives -46

c) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ which on evaluating gives 0

d) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ which on evaluating gives 5

11 Question 3.11.5

11.1 Problem

11.1. Find all pairs of consecutive odd natural numbers, both of which are greater than 10, such that their sum is less than 40.
The following python code computes the required pairs of consecutive odd natural numbers which satisfy the required condition, shown in Fig.9.

```
./codes/lines/q15.py
```

Solution: Let $\mathbf{x}$ be an odd natural number and $\mathbf{y}$ be the odd natural number consecutive to $\mathbf{x}$.

$$\therefore \mathbf{y} = \mathbf{x} + 2 \tag{11.1.1}$$

We need to find $\mathbf{x}$ and $\mathbf{y}$ such that

$$\begin{aligned} \mathbf{x}, \mathbf{y} > 10 \text{ and } \mathbf{x} + \mathbf{y} < 40 \\ \therefore \mathbf{x} + \mathbf{x} + 2 < 40 \\ 2\mathbf{x} + 2 < 40 \\ \mathbf{x} + 1 < 20 \\ \mathbf{x} < 19 \end{aligned} \tag{11.1.2}$$

Hence the condition is satisfied when $\mathbf{x} > 10$ and $\mathbf{x} < 19$

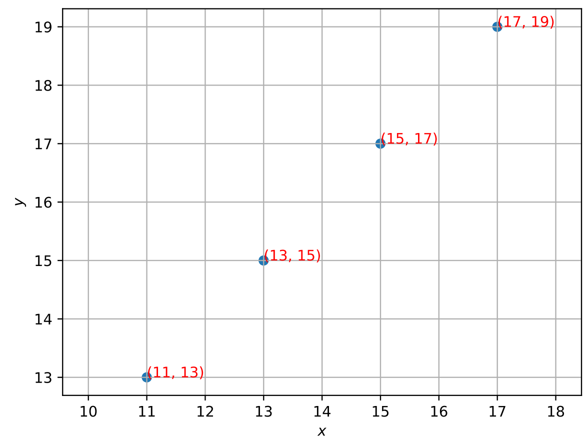

Fig. 9: Triangle of Q.3.11.5

12 Question 4.1.5

12.1 Problem

12.1. Find the area of the region in the first quadrant enclosed by the x-axis, the line $\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0$ and the circle $\|\mathbf{x}\| = 1$. The following python code computes the required area represented in Fig.10.

```
./codes/circle/q17.py
```

Solution: The required area is given by

$$ar(OACB) = ar(OAB) + ar(ACB) \tag{12.1.1}$$

$$ar(OACB) = \int_{0}^{0.707} x dx + \int_{0.707}^{1} \sqrt{1 - x^2} dx \tag{12.1.2}$$

which on computing, we obtain the required area as 0.3924

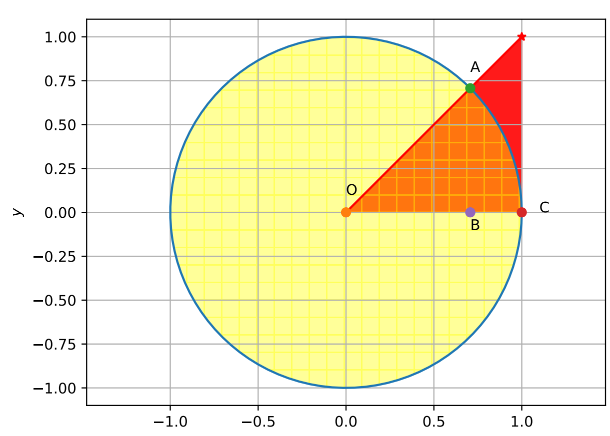

Fig. 10: Figure of Q.4.1.5

13 Question 4.2.5

13.1 Problem

13.1. Sketch circles with equation: The following python codes generate the required circles :

```
./codes/circle/q18a.py
./codes/circle/q18b.py
./codes/circle/q18c.py
./codes/circle/q18d.py
```

a)

$$\left\| x - \begin{pmatrix} 5 \\ -3 \end{pmatrix} \right\| = 36 \text{ represented in Fig:11}$$

$$\text{Center is } \begin{pmatrix} 5 \\ -3 \end{pmatrix} \quad \text{Radius is } 6 \tag{13.1.1}$$

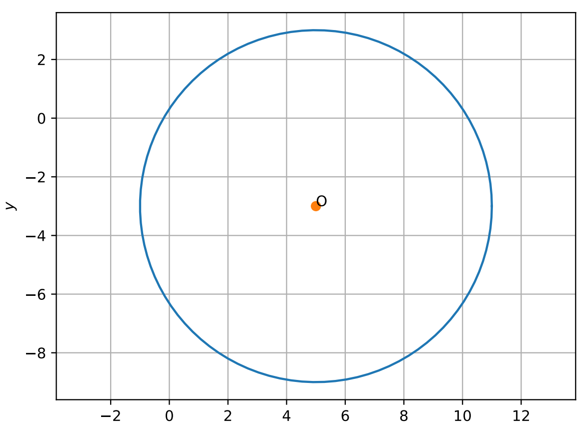

Fig. 11: Circle of Q.4.2.5

b)

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix} 4 \\ 8 \end{pmatrix}\mathbf{x} - 45 = 0 \text{ represented in Fig:12}$$

$$\text{Center is } \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{Radius is } \sqrt{65} \tag{13.1.2}$$

c)

$$\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix} 8 \\ -10 \end{pmatrix}\mathbf{x} - 12 = 0 \text{ represented in Fig:13}$$

$$\text{Center is } \begin{pmatrix} -4 \\ 5 \end{pmatrix} \quad \text{Radius is } \sqrt{53} \tag{13.1.3}$$

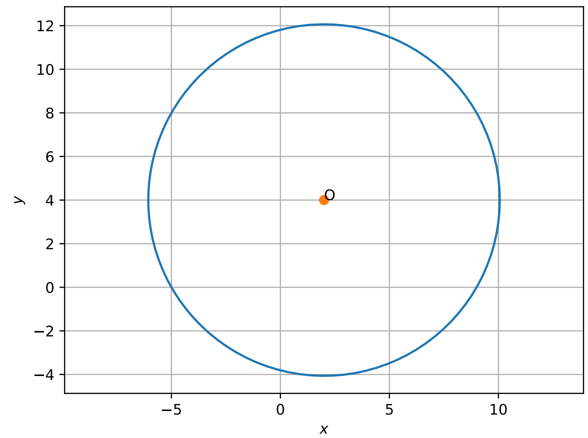

Fig. 12: Circle of Q.4.2.5

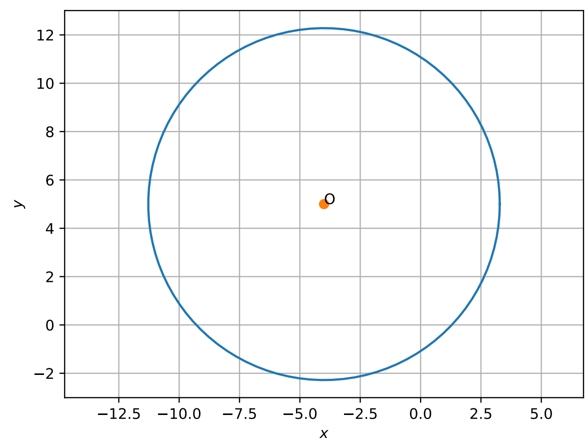

Fig. 13: Circle of Q.4.2.5

d)

$$2\mathbf{x}^{\mathbf{T}}\mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbf{x} = 0 \text{ represented in Fig:14}$$

$$\text{Center is } \begin{pmatrix} 0.25 \\ 0 \end{pmatrix} \quad \text{Radius is } 0.25 \tag{13.1.4}$$

14 Question 5.1.5

14.1 Problem

14.1. Find the roots of the quadratic equation $6x^2 - x - 2 = 0$. The following python code computes roots of the quadratic equation represented in Fig.15.

```
./codes/conics/q19.py
```

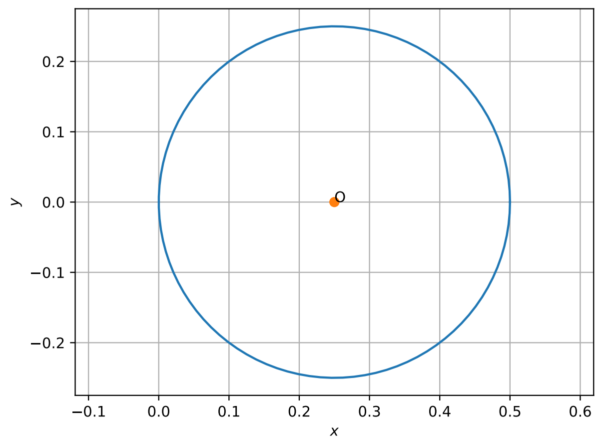

Fig. 14: Circle of Q.4.2.5

Solution: For a general polynomial equation of degree 2,

$$p(x,y) \implies Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

The vector form is

$$\mathbf{x}^T\begin{pmatrix}A & \frac{B}{2}\\ \frac{B}{2} & C\end{pmatrix}\mathbf{x}+\begin{pmatrix}D & E\end{pmatrix}\mathbf{x}+F=0 \quad (14.1.1)$$

Here

$$y = 6x^2 - x - 2 \quad \text{The vector form is} \quad (14.1.2)$$

$$\mathbf{x}^T\begin{pmatrix}6 & 0\\ 0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-1 & -1\end{pmatrix}\mathbf{x}-2=0 \quad (14.1.3)$$

Thus, from 14.1.1

$$y=0 \implies 6x^2-x-2=0 \quad (14.1.4)$$

$$\left(x+\frac{1}{2}\right)\left(x-\frac{2}{3}\right)=0 \quad (14.1.5)$$

$$x=\frac{-1}{2},\frac{2}{3} \quad (14.1.6)$$

15 Question 5.2.5

15.1 Problem

15.1. Find a quadratic polynomial each with the given numbers as the sum and the product of its zeroes.

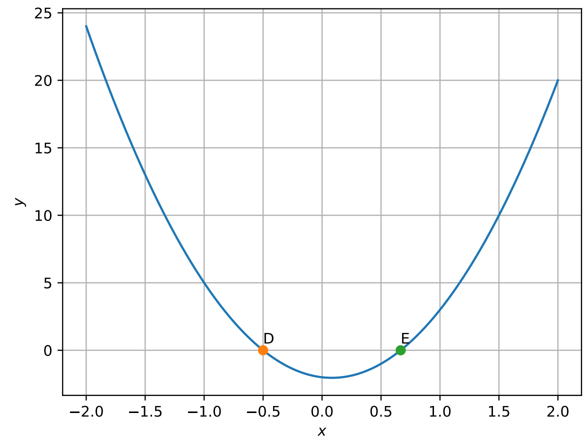

Fig. 15: Parabola of Q.5.1.5

a) -1,$\frac{1}{4}$
Solution: For a general polynomial equation of degree 2,

$$p(x,y) \implies Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

The vector form is

$$\mathbf{x}^T\begin{pmatrix}A & \frac{B}{2}\\ \frac{B}{2} & C\end{pmatrix}\mathbf{x}+\begin{pmatrix}D & E\end{pmatrix}\mathbf{x}+F=0 \quad (15.1.1)$$

Here, sum of zeroes = D = -1
Product of zeroes = F =$\frac{1}{4}$
Substituing the values in 15.1.1,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x}+\frac{1}{4}=0 \quad (15.1.2)$$

$$\implies y=x^2+x+\frac{1}{4} \quad (15.1.3)$$

b) 1,1
Solution: Here, sum of zeroes = D = 1
Product of zeroes = F =1
Substituing the values in 15.1.1,

$$\mathbf{x}^T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{x}+\begin{pmatrix}-1 & -1\end{pmatrix}\mathbf{x}+1=0 \quad (15.1.4)$$

$$\implies y=x^2-x+1 \quad (15.1.5)$$

c) 0, $\sqrt{5}$
Solution: Here, sum of zeroes = D = 0

Product of zeroes = F = $\sqrt{5}$
Substituing the values in 15.1.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} + \sqrt{5} = 0 \quad (15.1.6)$$

$$\implies y = x^2 + \sqrt{5} \quad (15.1.7)$$

d) 4,1
Solution: Here, sum of zeroes = D = 4
Product of zeroes = F = 1
Substituing the values in 15.1.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 & -1 \end{pmatrix} \mathbf{x} + 1 = 0 \quad (15.1.8)$$

$$\implies y = x^2 - 4x + 1 \quad (15.1.9)$$

e) $\frac{1}{4}, \frac{1}{4}$
Solution: Here, sum of zeroes = D = $\frac{1}{4}$
Product of zeroes = F = $\frac{1}{4}$
Substituing the values in 15.1.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\frac{1}{4} & -1 \end{pmatrix} \mathbf{x} + \frac{1}{4} = 0 \quad (15.1.10)$$

$$\implies y = x^2 - \frac{1}{4}x + \frac{1}{4} \quad (15.1.11)$$

f) $\sqrt{2}, \frac{1}{3}$
Solution: Here, sum of zeroes = D = $\sqrt{2}$
Product of zeroes = F = $\frac{1}{3}$
Substituing the values in 15.1.1,

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -\sqrt{2} & -1 \end{pmatrix} \mathbf{x} + \frac{1}{3} = 0 \quad (15.1.12)$$

$$\implies y = x^2 - \sqrt{2}x + \frac{1}{3} \quad (15.1.13)$$